MONASH University

Information Technology

FIT1008/2085
Algorithms and Algorithmic Complexity

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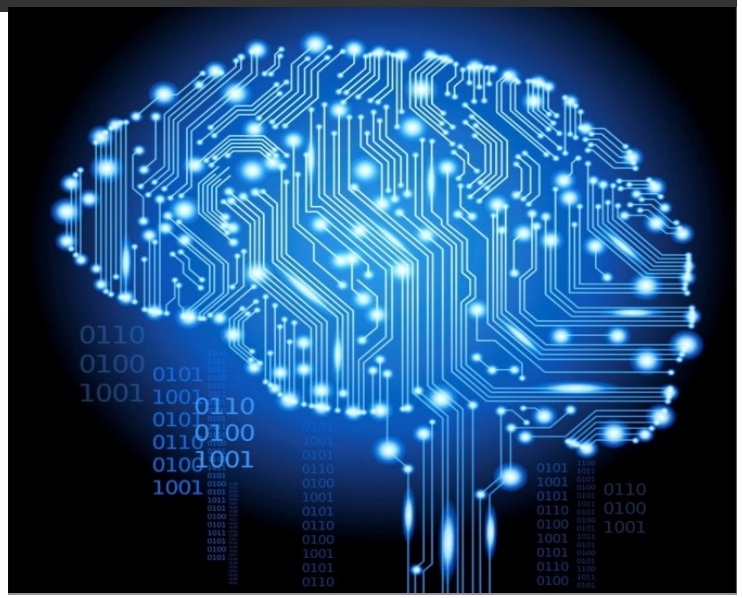

Objectives for this lecture

- **To understand the concept of an algorithm**
- **To understand algorithmic complexity**
  - See how complexity is defined (as dependent on problem size)
  - Understand the basics of asymptotics
  - Study the Big-O notation and its basic properties
- **Get familiar with common complexity "classes":**
  - $O(1)$, $O(n)$, $O(\log n)$, $O(n^2)$, etc.
- **In the following lessons, we will use this knowledge to understand the complexity of sorting algorithms**
  - Best-case complexity
  - Worst-case complexity

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Algorithm

What is an algorithm and when is it needed?

- **General computational problem:**
  - Well-specified but this is *subjective*
  - Defines input-to-output relationship

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  - Given integers a and b, find their greatest common divisor
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  - Given a sequence of numbers $\langle a_1, \ldots, a_n \rangle$, produce a sorted list $\langle a'_1, \ldots, a'_n \rangle$ s.t. $a'_1 \leq a'_2 \leq \cdots \leq a'_n$

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- **Example algorithms:**
  - Euclid's algorithm for GCD
  - Dijkstra's algorithm for shortest path
  - Bubble Sort algorithm

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Example:

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bin(10) = 0b1010
 bin(7) = 0b111
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bin(10) = 0b1010
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Hint: *we need to check the right-most digit in the binary representation of k.*

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k = 0b101

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Looks hard! Can we do better?

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def is_even2(k):
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Note: Both algorithms solve the same problem but one runs faster than the other!

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```
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Two `return` statements.

`int(True) == 1` and `int(False) == 0`

Running example 2 - algorithms

- **Naïve approach:**
  - Test every value from 0 to $k-1$
  - Stop when the correct one is found

```
def naive(k):
    for b in range(k):
        if alice_replies(b) == 0:
            break
    return b

def alice_replies(b):
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This algorithm makes k iterations in the worst case.

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- **A computer scientist's approach:**
  - Halve the search space at each step
    - *We will cover binary search later!*

```
def clever(k):
    lb, ub = 0, k - 1
    while lb <= ub:
        b = (lb + ub) // 2
        reply = alice_replies(b)
        if reply == 0:
            return b
        elif reply < 0:
            lb = b + 1
        elif reply > 0:
            ub = b - 1
    return lb
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Depending on the outcome, we either
(1) stop, or
(2) consider interval $[b+1, ub]$, or
(3) consider interval $[lb, b-1]$

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Two algorithms solve the same problem but run differently!

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MONASH University

Algorithmic complexity

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alternatively, **computational complexity**

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- **Basically, there are two types:**

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    - How much time does an algorithm spend solving the corresponding problem?
      - *Usually measured as the number of "elementary operations" performed*

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    - How much time does an algorithm spend solving the corresponding problem?
      - *Usually measured as the number of "elementary operations" performed*

  - **Space complexity**
    - How much space does an algorithm spend solving the corresponding problem?
      - Usually measured as the amount of "memory" occupied simultaneously

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alternatively, **computational complexity**

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  - **Time complexity**
    - How much time does an algorithm spend solving the corresponding problem?
      - *Usually measured as the number of "elementary operations" performed*
  - **Space complexity**
    - How much space does an algorithm spend solving the corresponding problem?
      - Usually measured as the amount of "memory" occupied simultaneously
- **Both types of complexity are measured wrt. the input size!**

Algorithmic complexity

alternatively, **computational complexity**

- **Basically, there are two types:**
  - **Time complexity**
    - How much time does an algorithm spend solving the corresponding problem?
      - *Usually measured as the number of "elementary operations" performed*
  - **Space complexity**

Space complexity is out of scope of FIT1008!

    - How much space does an algorithm spend solving the corresponding problem?
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- **Both types of complexity are measured wrt. the input size!**

Time complexity as a function of size

- **Algorithm's time complexity can be measured as a function of input size**
  - Given an input of size n, we can calculate the value $T(n)$…
  - … as the number of *"elementary" steps* (instructions) performed by the algorithm on the input of size n
    - each elementary step contributes a 1 to the total value $T(n)$

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- **Another question: *"what is n?"***

Sometimes it is clear, sometimes not – ***let's see next!***

What is input size?

- **It depends on the type of input:**
  - **Numeric input:**
    - Length of its binary representation
- **Example problem:**
  - *"is integer n even?"*

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  - `n = 125 = 0b1111101`

Size equals 7

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  - in general, size equals $\lceil \log_2(n+1) \rceil$

$\approx \log_2 n$

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- **It depends on the type of input:**
  - **Numeric input:**
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  - **Collection (array, list, set, stack):**
    - Number of elements in collection

- **Example problem:**
  - *"sort an input array"*
  - size of `[1, 2, 3, 5, 8]` is 5

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  - **Collection (array, list, set, stack):**
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  - **String input:**
    - Number of characters

- **Example problem:**
  - *"is a given string a palindrome?"*
  - size of "`hello world!`" is 12

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  - **$n \times m$ matrix:**
    - Number of rows n
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  - "multiply two matrices"

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$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

Number of columns equals m

Number of rows equals n

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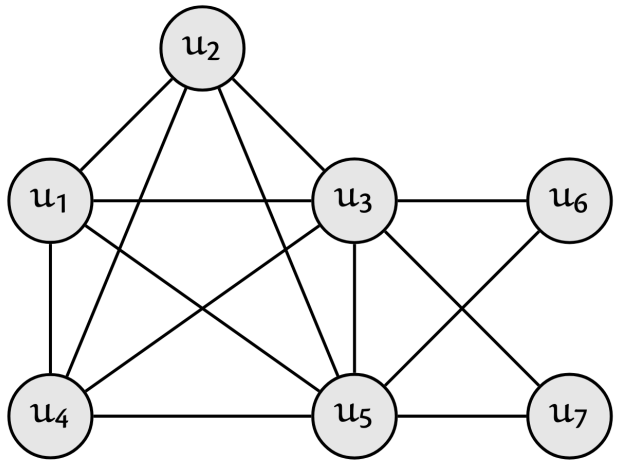
  - **Graph or tree:**
    - Number of nodes
  - **Etc.**

- **Example problem:**
  - *"find a maximum clique in a graph"*
  - The size of this graph equals 7:



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In other cases, input size can be defined similarly.

It is normally clear from the context!

Running example 1 – *checking if a number is even*

- **Traverse the bits from left to right:**

```
def is_even1(k):
    res = None
    while k:
        b = pop_left(k)
        res = (b == 0)
    return res
```

- **Get the value of right-most bit:**

```
def is_even2(k):
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How many elementary steps given the **size of input k**?

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$n \approx \log k$

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2 (res = (b == 0))

1 (return res)

$T(n) = 1 + n \times (2 + 2)$

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Linear function on the input size!

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```
def is_even2(k):
    b = pop_right(k)
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```

2 · 2 · 1

$T(n) = 2 + 2 + 1 = 5$

Constant function on the input size!

How many elementary steps given the **size of input n**?

Recall that size n of k equals $\log k$

Running example 2 – *finding a guessed number*

- **Naïve approach:**

```
def naive(k):
    for b in range(k):
        if alice_replies(b) == 0:
            break
    return b



def alice_replies(b):
    if b < HIDDEN_VALUE_A:
        return -1
    else:
        return int(b > HIDDEN_VALUE_A)
```

- **A computer scientist's approach:**

```
def clever(k):
    lb, ub = 0, k - 1
    while lb <= ub:
        b = (lb + ub) // 2
        reply = alice_replies(b)
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$T(n) =$

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$T(n) = 2^n \times$

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$T(n) = 2^n \times (4$

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$T(n) = 3 + 1 +$

How many elementary steps here?

Recall that the size n of input is $\log k$.

Running example 2 – *finding a guessed number*

- **Naïve approach:**

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def naive(k):
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at most $k = 2^{\log k} = 2^n$

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$T(n) = 2^n \times (4 + 1 + 1) + 1 = 2^n \times 6 + 1$

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Initially, there are k possibilities.
Each iteration halves it,
thus decrementing $\log k$ by 1.

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Linear on input size, i.e. on n.

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Important: *the picture changes if we change the perspective.*

Consider the number of steps wrt. **input *k* itself**.

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Logarithmic on k.

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Logarithmic on k.

Always be clear about the input size when analyzing complexity!

MONASH University

Asymptotics and Big-O Notation

Rationale

- **In practice, we do not count the exact number of elementary steps**
  - It is difficult
  - It is time-consuming
  - It is *error-prone*
- **Instead, we determine the most "expensive" parts of the code**
  - You will master this skill with (lots of) practice!
- **How badly do they impact the "rate of growth" of algorithmic complexity?**

This is when the Big-O notation comes in handy!

A bit of history…

Sometimes called **Bachmann-Landau notation.**

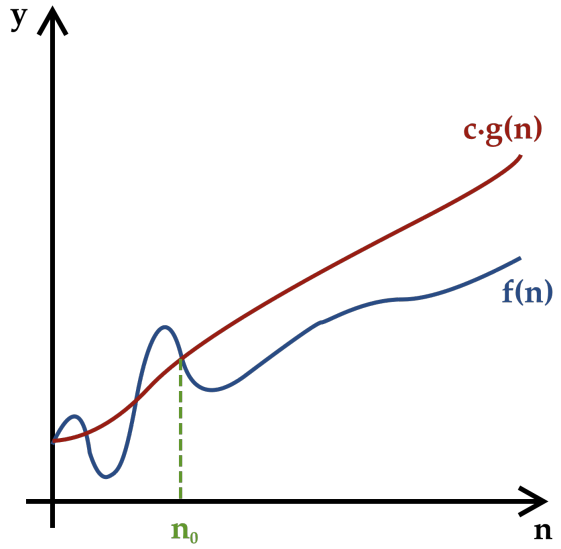
- **The original concept and the corresponding O-notation was introduced by German mathematician *Paul Bachmann* in 1894**
  - The name is determined by the German word *"ordnung"*
  - i.e. *"order of approximation"*
- **Popularised in mathematics by German mathematician *Edmund Landau***
  - *Mathematical analysis, differential equations, etc.*
- **Adapted by Donald Knuth for algorithmic complexity analysis in CS**
  - *This is what we are using!*

What is O()?

- **Assume we are given two functions $f(n)$ and $g(n)$ defined on $n \in \mathbb{N}$.**
- **$f(n) = O(g(n))$ if the following holds:**
  - there are two positive constants n_0 and c such that for all $n \geq n_0$

$$0 \leq f(n) \leq c \cdot g(n)$$

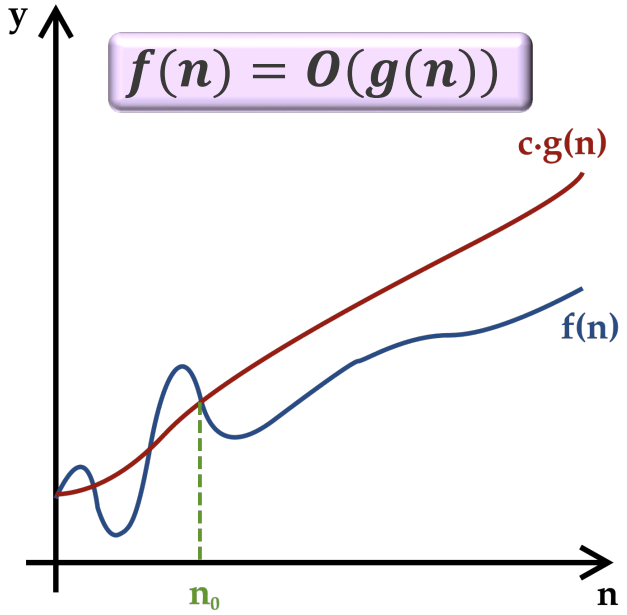
  - *$g(n)$ is referred to as asymptotic upper bound for $f(n)$, i.e. as n approaches $+\infty$*
- **Geometric interpretation:**



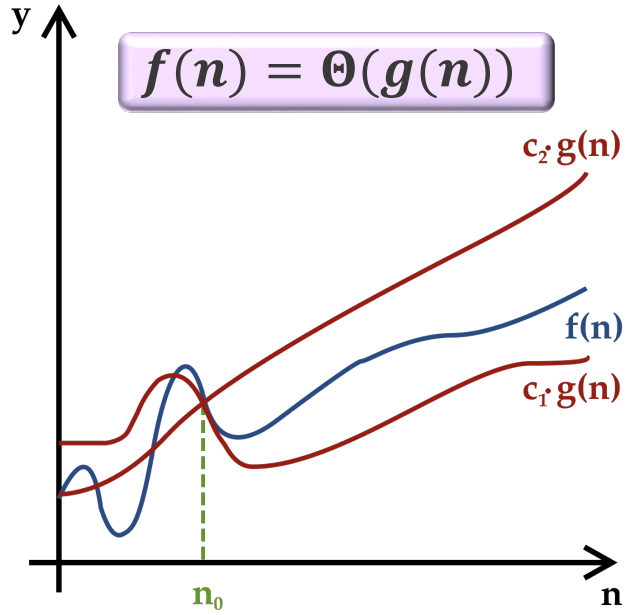
This is what asymptotic analysis means.

Big-Omega and Big-Theta – similar notation

Out of scope of FIT1008

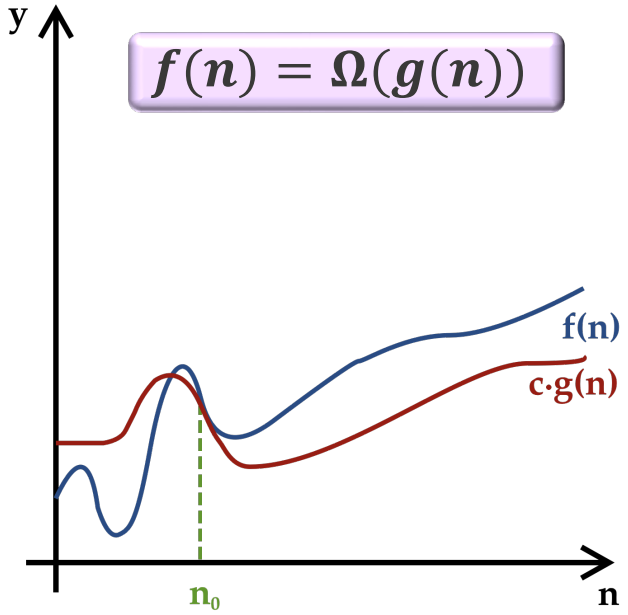


$g(n)$ - asymptotic upper bound for $f(n)$



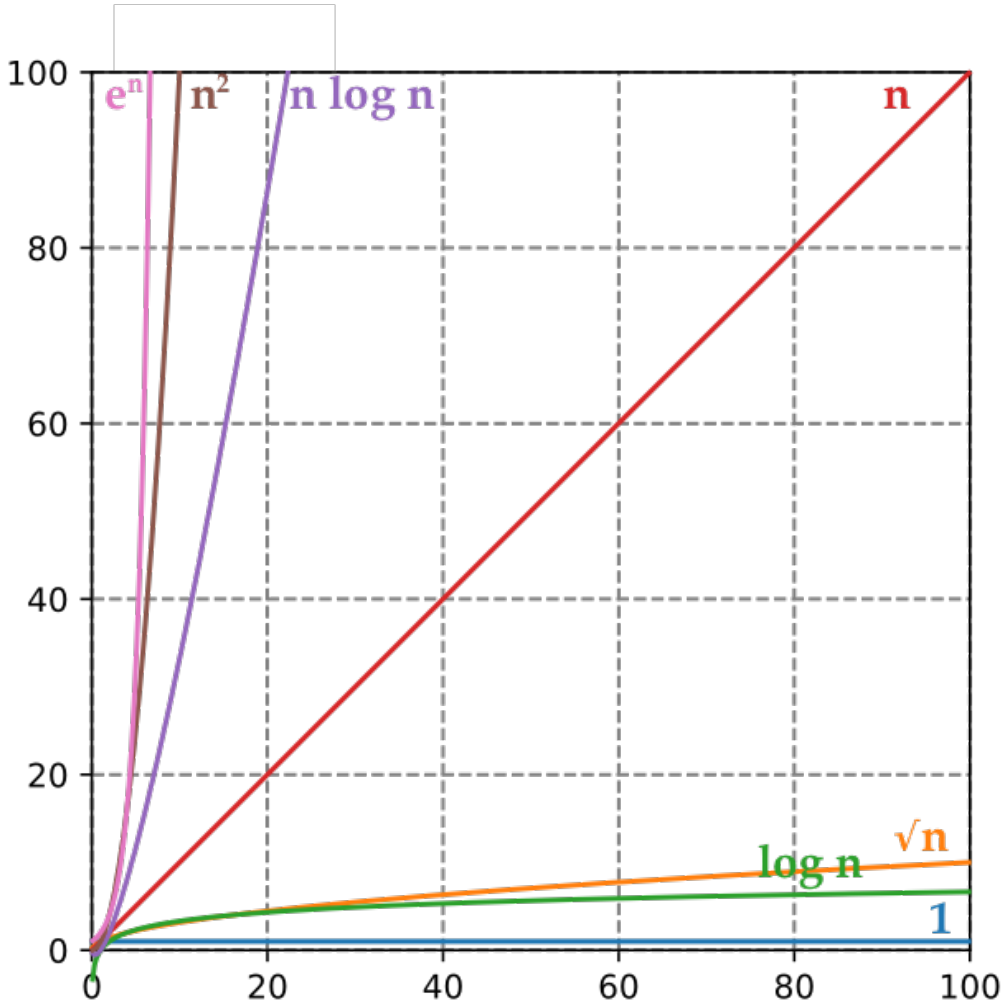
$g(n)$ - asymptotic tight bound for $f(n)$

both lower and upper!



$g(n)$ - asymptotic lower bound for $f(n)$

Common function growth rates



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There are **no such constants c' and n_0'** to make the definition of Big-O hold for $O(n^2 \cdot \log^2 n)$!

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  - We say that such a component **dominates** all the others.

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  - For instance, $4n^2$ dominates $1000n + 100$ while n^3 dominates $n^2 \cdot \log^2 n$.
  - See the previous slide for the growth rate of common functions.

The same applies to the analysis of algorithmic time complexity!

Basic properties of Big-O

- **Constant growth rate:**
  - If c is any positive constant then $c = O(1)$

The **dominating** one.

Recall: pick the fastest growing component!

- **Sum of O():**
  - If $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$ then $f_1(n) + f_2(n) = O(\max(g_1(n), g_2(n)))$
- **Product of O():**
  - If $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$ then $f_1(n) \cdot f_2(n) = O(g_1(n) \cdot g_2(n))$
    - In particular, if $f_1(n) = O(g_1(n))$ and $f_2(n) = O(1)$ then $f_1(n) \cdot f_2(n) = O(g_1(n))$

Thus, we ignore the constant!

Running example 1 – *checking if a number is even*

- **Traverse the bits from left to right:**

```
def is_even1(k):
    res = None
    while k:
        b = pop_left(k)
        res = (b == 0)
    return res
```

$T(n) = 1 + n \times (2 + 2) + 1 = 4n + 2$

Linear function on the input size!

- **Get the value of right-most bit:**

```
def is_even2(k):
    b = pop_right(k)
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$T(n) = 2 + 2 + 1 = 5$

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What is the complexity in terms of $O()$?

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def naive(k):
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$T(n) = 2^n \times (4 + 1 + 1) + 1 = 2^n \times 6 + 1$

4 elementary steps in total

```
def alice_replies(b):
    if b < HIDDEN_VALUE_A:
        return -1
    else:
        return int(b > HIDDEN_VALUE_A)
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def clever(k):
    lb, ub = 0, k - 1
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What is the complexity in terms of $O()$?

Running example 2 – *finding a guessed number*

- **Naïve approach:**

```
def naive(k):
    for b in range(k):
        if alice_replies(b) == 0:
            break
    return b
```

$T(n) = 2^n \times (4 + 1 + 1) + 1 = 2^n \times 6 + 1$

4 elementary steps in total = $\boldsymbol{O(1)}$

```
def alice_replies(b):
    if b < HIDDEN_VALUE_A:
        return -1
    else:
        return int(b > HIDDEN_VALUE_A)
```

- **A computer scientist's approach:**

```
def clever(k):
    lb, ub = 0, k - 1
    while lb <= ub:
        b = (lb + ub) // 2
        reply = alice_replies(b)
        if reply == 0:
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        elif reply < 0:
            lb = b + 1
        elif reply > 0:
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A few more Big-O examples

- **What is the complexity of this?**
  - This function computes the sum of elements in a list:

```
def sum(input_list):
    res = 0
    for item in input_list:
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def sum(input_list):
    res = 0
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  - Again, input size == number of items n
  - Loop iterates at most 10 times – a constant!

Does not depend on input size!

  - Complexity is $O(1)$

Yet another example

- **Analyse the complexity of function:**
  - Given 2 integers k and l, the function computes $k \times (-1)^l$

```
def weird_func(k, l):
    res = k
    for i in range(l):
        res = multiply(res)
    return res

def multiply(k):
    return k * (-1)
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    - Because it is $O(2^m) \times O(1)$

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    - Because it is $O(2^m) \times O(1)$

But it is $O(l)$ if we are interested in complexity wrt. the input numbers k and l themselves.

Once again: it is crucial to understand and be clear about with respect to what the complexity is analysed!

Yet another example, sorry!

- **Analyse the complexity of `func0()`:**

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```
def func0(input_of_size_n):
    for i in range(n):
        ... # constant time operations
        a = b + 2
        ... # constant time operations
        for j in range(100):
            res += func1(res)
        res -= func2(res)
    return res
def func1(something):
    ...
def func2(something):
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```

Assume it is $O(n)$.

Assume it is $O(2^n)$.

Total complexity:

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  - The outer loop runs for n times
    - i.e. it should be $O(n) \times \langle \ldots \rangle$

Total complexity: $O(n) \cdot (\ldots)$.

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  - This part is $O(1)$

Total complexity: $O(n) \cdot (O(1) + \cdots)$.

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Total complexity: $\boldsymbol{O(n)\cdot(O(1)+O(1)\cdot(\dots)+)}$.

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  - This is known to be $O(n)$

Total complexity: $O(n)\cdot(O(1)+O(1)\cdot O(n)+\cdots)$.

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Recall that this part becomes $O(n)$.

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Inside the parentheses, $O(2^n)$ dominates.

Total complexity: $O(n)\cdot\left(O(1)+O(n)+O(2^n)\right)$.

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Total complexity: $O(n)\cdot O(2^n)$.

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Total complexity: $O(n \cdot 2^n)$.

Summary

- **After these lesson you are now able to:**
  - Understand the concept of algorithm and how its time complexity is analysed
  - Understand the **Big-O notation** and its properties:
    - Basics of asymptotic analysis
    - Combination of Big-O's
  - Apply the Big-O notation when reasoning about algorithmic complexity
  - Understand basic algorithmic complexity classes and how they behave
    - Constant, linear, logarithmic, etc.