Week 10 Preparation

(Solutions)

Useful advice: The following solutions pertain to the preparation problems. You are strongly advised to attempt the problems thoroughly before looking at these solutions. Simply reading the solutions without thinking about the problems will rob you of the practice required to be able to solve complicated problems on your own. You will perform poorly on the exam if you simply attempt to memorise solutions to these problems. Thinking about a problem, even if you do not solve it will greatly increase your understanding of the underlying concepts.

Problems

Problem 1. Consider the following circulation with demands problem presented in Figure 1. Does it have a feasible solution?

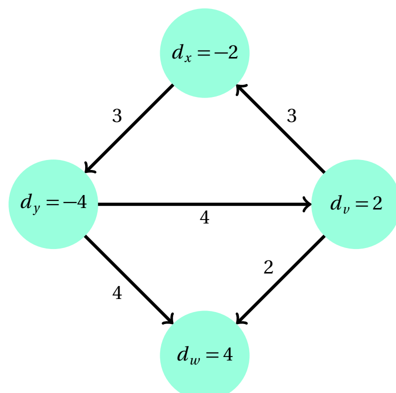

Figure 1: An instance of the circulation with demands problem. The demand is indicated in each vertex, and the capacity in each edge.

Solution

The idea is to create a supergraph G' of the original graph G, solve a max-flow problem in G' to obtain a solution f'_{max}, and translate the solution f'_{max} of the max-flow problem in G' to a solution to the circulation with demands $\{d_u\}$ problem in G. The supergraph G' is created from G as follows:

- Add a super-source vertex s.
- For each vertex u such that $d_u < 0$, add an edge (s, u) with capacity $-d_u$.
- Add a super-sink vertex t.
- For each vertex u such that $d_u > 0$, add an edge (u, t) with capacity d_u.

Figure 2 presents the supergraph G' obtained from G presented in Figure 1.

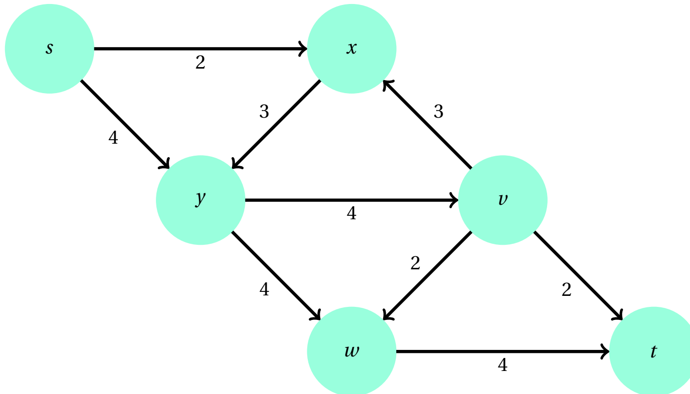

Figure 2: The supergraph G' obtained from G presented in Figure 1 so that a max-flow problem can be solved.

We now solve the max-flow problem on G' to obtain the maximum flow f'_{max} between s and t in G'. A solution of the max-flow problem in our example is presented in Figure 3.

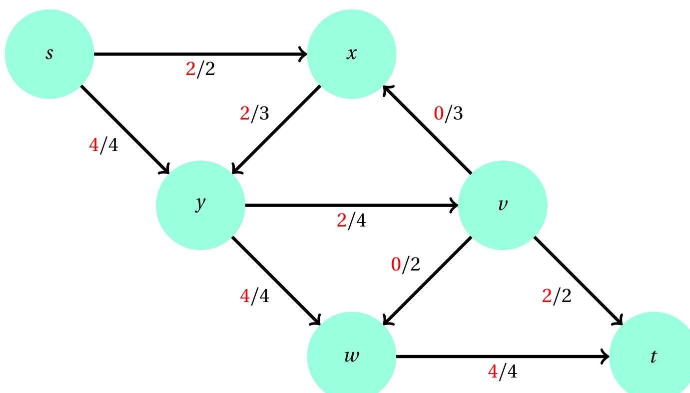

Figure 3: A solution of the max-flow problem presented in Figure 2.

As all the outgoing edges of the source and incoming edges of the sink are saturated, the circulation with demands problem has a feasible solution, and one feasible solution is then presented in Figure 4 (obtained by simply removing the additional vertices and edges of the supergraph G').

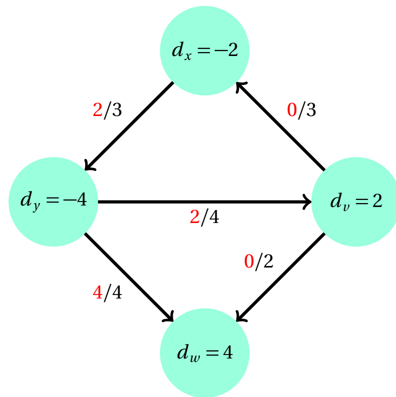

Figure 4: A solution for the above instance of the circulation with demands problem, with the flows along each edge indicated in red.

Problem 2. Consider the following circulation with demands and lower bounds problem presented in Figure 5. Does it have a feasible solution?

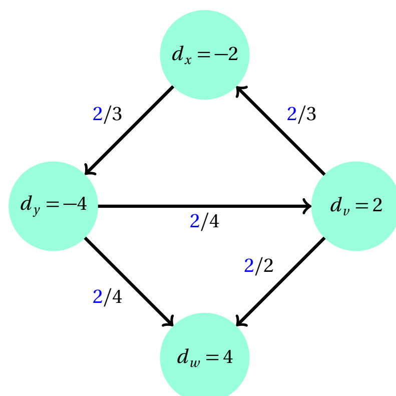

Figure 5: An instance of the circulation with demands and lower bounds problem. The demand is indicated in each vertex, and in each edge its capacity is in black and its lower bound in blue.

Solution

The idea is that we will reduce this problem to the problem of circulation with demands but no lower bounds.

We first define a flow f^ℓ by setting, for each edge, $f^\ell(u,v) = \ell(u,v)$. And then we figure out if there is exists a flow f^* such that, for $f = f^\ell + f^*$, f is a feasible solution to the circulation with demands $\{d_u\}$ and lower bounds problem on G. Note that f^ℓ already takes care that the flow f is at least equal to the lower bound $\ell(u,v)$ on each edge. Now we just have to consider the related graph G^* with adjusted capacities and demands:

- the vertices of G^* are the same as in G, but they now have demand $d^*_u = d_u - \sum_{v \in V} f^\ell(v,u) + \sum_{v \in V} f^\ell(u,v)$;
- the edges of G^* are the same as in G, but they now have capacities $c^*(u,v) = c(u,v) - \ell(u,v)$ and no lower bounds;

and solve the problem of circulation with demands $\{d^*_u\}$ - without lower bounds - in G^*.

The flow f^ℓ meeting the lower bound requirements is presented in Figure 6, and the adjusted graph G^* in Figure 7.

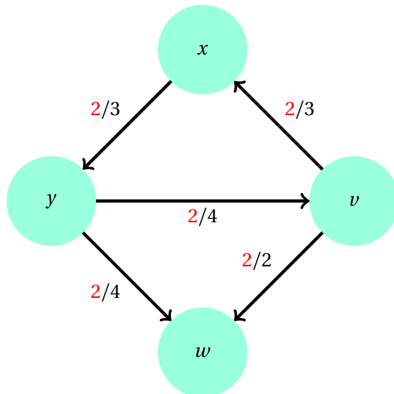

Figure 6: The fixed flow f^ℓ (in red) corresponding to our instance of the circulation with demands and lower bounds problem.

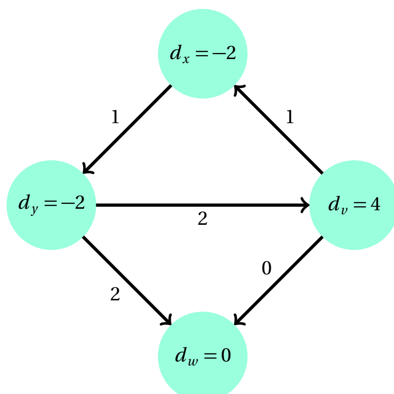

Figure 7: The adjusted graph G^* for the circulation with demands $\{d_u^*\}$ problem (without lower bounds) corresponding to our instance.

It is clear that the demand of node v in this adjusted graph cannot be met, as its only incoming edge can provide a flow of at most 2. You can solve the corresponding max-flow problem and check the fact that not all outgoing edges of the source would be saturated.

Therefore the original circulation with demands and lower bounds problem also does not have a feasible solution.