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SWCON253: Machine Learning

Lecture 02

Review of Linear Algebra

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Contents

1. Notation & Basic Operations
2. Linear Independence & Inverse
3. Norm, Orthogonality & Projection
4. Determinant, Decomposition, Quadratic Forms

References

- *Linear Algebra Review* by J. Zico Kolter (<http://www.cs.cmu.edu/~zkolter/course/linalg/>)
- *Review of Linear Algebra* by Xinkun Nie & Fereshte Khani (<http://cs229.stanford.edu/syllabus-fall2020.html>)
- *Linear Algebra for Machine Learning* by Sargur N. Srihari (<https://cedar.buffalo.edu/~srihari/CSE574/>)



1. Notation & Basic Operations

1. Vector & Matrix
2. Special Matrices
3. Additions & Multiplications
4. Transpose
5. Trace



Vector & Matrix

- By $x \in \mathbb{R}^n$, we denote a vector with n entries.

1-D array of numbers

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

$$x^T = [x_1 \ x_2 \ \dots \ x_n]$$

- By $A \in \mathbb{R}^{m \times n}$ we denote a matrix with m rows and n columns, where the entries of A are real numbers.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} | & | & & | \\ a^1 & a^2 & \cdots & a^n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ - & \vdots & - \\ - & a_m^T & - \end{bmatrix}.$$

2-D array of numbers

a set of column vectors

a set of row vectors



Tensor

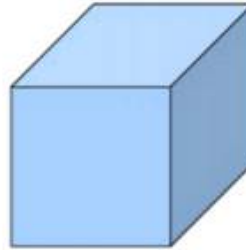
Multi-dimensional array of numbers



1d-tensor



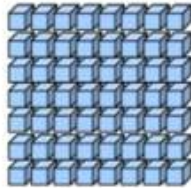
2d-tensor



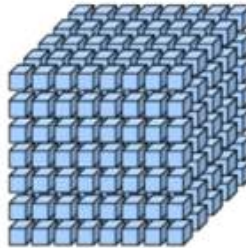
3d-tensor



4d-tensor

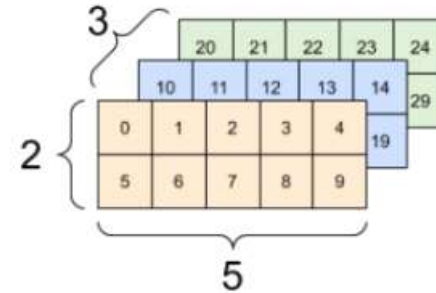


5d-tensor



6d-tensor

```
tf.Tensor(  
[[[ 0  1  2  3  4]  
 [ 5  6  7  8  9]]  
  
[[10 11 12 13 14]  
 [15 16 17 18 19]]  
  
[[20 21 22 23 24]  
 [25 26 27 28 29]]], shape=(3, 2, 5)
```



Product of Vectors

"row x column"

inner product or dot product

WHY

$$x^T y \in \mathbb{R} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^n x_i y_i = y^T x$$

outer product

WHY

$$x y^T \in \mathbb{R}^{m \times n} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_m y_1 & x_m y_2 & \cdots & x_m y_n \end{bmatrix} = (y x^T)^T$$



Special Matrices – Zero

The Zero Matrix

$$0 \in \mathbb{R}^{m \times n} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Special Matrices – *Identity*

The Identity Matrix

$$I \in \mathbb{R}^{n \times n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

~ collection of **unit vectors** in \mathbb{R}^n

Has the property that for any $A \in \mathbb{R}^{m \times n}$

$$AI = A = IA$$



Special Matrices – *Diagonal*

Diagonal Matrices

For $d \in \mathbb{R}^n$

$$\text{diag}(d) \in \mathbb{R}^{n \times n} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

- For example, the identity is given by $I = \text{diag}(1)$

$$1 \in \mathbb{R}^n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Scalar Operations: $aA+b$

A scalar can be added to a matrix.

A matrix can be multiplied by a scalar.

$$B = aA + b \rightarrow B_{i,j} = aA_{i,j} + b$$

1.7.2. 각 행 더하기가 곱하면 모든 행분에 대해서
하면 됨

Matrix Addition: $A+B$

- For two matrices *of the same size and type*,
 $A, B \in \mathbb{R}^{m \times n}$ addition is just sum of
corresponding elements

$$A + B = C \in \mathbb{R}^{m \times n} \iff C_{ij} = A_{ij} + B_{ij}$$

Addition is *undefined* for matrices of different
sizes $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$

(Note: Numpy's *broadcasting* allows addition of different sized arrays)



Matrix-Vector Multiplication: Ax

- If we write A by rows, then we can express Ax as,

$$y = Ax = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_m^T x \end{bmatrix}.$$

- If we write A by columns, then we have:

$$y = Ax = \begin{bmatrix} | & | & & | \\ a^1 & a^2 & \cdots & a^n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a^1 \end{bmatrix} x_1 + \begin{bmatrix} a^2 \end{bmatrix} x_2 + \cdots + \begin{bmatrix} a^n \end{bmatrix} x_n.$$

y is a linear combination of the *columns* of A .



Vector-Matrix Multiplication: $x^T A$

- If we write A by columns, then we can express $x^T A$ as,

$$y^T = x^T A = x^T \begin{bmatrix} \begin{array}{|c|} a^1 \end{array} & \begin{array}{|c|} a^2 \end{array} & \cdots & \begin{array}{|c|} a^n \end{array} \end{bmatrix} = \begin{bmatrix} x^T a^1 & x^T a^2 & \cdots & x^T a^n \end{bmatrix}$$

- expressing A in terms of rows we have:

$$\begin{aligned} y^T = x^T A &= \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix} \begin{bmatrix} \text{---} & a_1^T & \text{---} \\ \text{---} & a_2^T & \text{---} \\ & \vdots & \\ \text{---} & a_m^T & \text{---} \end{bmatrix} \\ &= x_1 \begin{bmatrix} \text{---} & a_1^T & \text{---} \end{bmatrix} + x_2 \begin{bmatrix} \text{---} & a_2^T & \text{---} \end{bmatrix} + \cdots + x_m \begin{bmatrix} \text{---} & a_m^T & \text{---} \end{bmatrix} \end{aligned}$$

y^T is a linear combination of the *rows* of A .



Matrix Multiplication: AB

- For two matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, their product is

$$AB = C \in \mathbb{R}^{m \times p} \iff C_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$C = AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$C_{11} = 1 \times 1 + 3 \times 2 + 5 \times 3 = 22$$

Multiplication is undefined when number of columns in A doesn't equal number of rows in B (one exception: cA for $c \in \mathbb{R}$ taken to mean scaling A by c)



Matrix Multiplication – *Different Views*

1. As a set of vector-vector products

$$C = AB = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} | & | & \dots & | \\ b^1 & b^2 & \dots & b^p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} a_1^T b^1 & a_1^T b^2 & \dots & a_1^T b^p \\ a_2^T b^1 & a_2^T b^2 & \dots & a_2^T b^p \\ \vdots & \vdots & \ddots & \vdots \\ a_m^T b^1 & a_m^T b^2 & \dots & a_m^T b^p \end{bmatrix}.$$

2. As a sum of outer products

$$C = AB = \begin{bmatrix} | & | & \dots & | \\ a^1 & a^2 & \dots & a^n \\ | & | & & | \end{bmatrix} \begin{bmatrix} - & b_1^T & - \\ - & b_2^T & - \\ & \vdots & \\ - & b_n^T & - \end{bmatrix} = \sum_{i=1}^n a^i b_i^T.$$



Matrix Multiplication – *Different Views (cont'd)*

3. As a set of matrix-vector products.

$$C = AB = A \left[\begin{array}{c|c|c|c} b^1 & b^2 & \dots & b^p \\ \hline \end{array} \right] = \left[\begin{array}{c|c|c|c} Ab^1 & Ab^2 & \dots & Ab^p \\ \hline \end{array} \right]. \quad (2)$$

Here the i th column of C is given by the matrix-vector product with the vector on the right, $c_i = Ab_i$. These matrix-vector products can in turn be interpreted using both viewpoints given in the previous subsection.

4. As a set of vector-matrix products.

$$C = AB = \left[\begin{array}{c|c|c} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{array} \right] B = \left[\begin{array}{c|c|c} - & a_1^T B & - \\ - & a_2^T B & - \\ & \vdots & \\ - & a_m^T B & - \end{array} \right].$$



Matrix Multiplication – *Diagonal Matrix*

- Multiplying $A \in \mathbb{R}^{m \times n}$ by a diagonal matrix $D \in \mathbb{R}^{n \times n}$ on the right scales the *columns* of A

$$AD = \begin{bmatrix} | & | & & | \\ d_1 a_1 & d_2 a_2 & \cdots & d_n a_n \\ | & | & & | \end{bmatrix}$$

행렬 곱셈

→ column scalar
(diagonal on the right)

- Multiplying by a diagonal matrix $D \in \mathbb{R}^{m \times m}$ on the left scales the *rows* of A

$$DA = \begin{bmatrix} - & d_1 a_1^T & - \\ - & d_2 a_2^T & - \\ & \vdots & \\ - & d_m a_m^T & - \end{bmatrix}$$

행렬 곱셈

→ row scalar
(diagonal on the left)

Matrix Multiplication – *Important Properties*

- Associative: ($A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{p \times q}$)

$$A(BC) = (AB)C$$

- Distributive: ($A \in \mathbb{R}^{m \times n}$, $B, C \in \mathbb{R}^{n \times p}$)

$$A(B + C) = AB + AC$$

- *NOT* commutative: (the dimensions might not even make sense, but this doesn't hold even when the dimensions are correct)

$$AB \neq BA$$



Transpose ~ flipping

The *transpose* of a matrix results from "flipping" the rows and columns. Given a matrix $A \in \mathbb{R}^{m \times n}$, its transpose, written $A^T \in \mathbb{R}^{n \times m}$, is the $n \times m$ matrix whose entries are given by

$$(A^T)_{ij} = A_{ji}.$$

– The mirror image across a diagonal line

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} A_{1,1} & A_{2,1} & A_{3,1} \\ A_{1,2} & A_{2,2} & A_{3,2} \\ A_{1,3} & A_{2,3} & A_{3,3} \end{bmatrix}$$

The following properties of transposes are easily verified:

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
- $(A + B)^T = A^T + B^T$



Special Matrices – Symmetric

Symmetric Matrices

- Symmetric matrix: $A \in \mathbb{R}^{n \times n}$ with $A = A^T$
- Arise naturally in many settings

Symmetric matrix examples

- distance matrix
- covariance matrix

- For $A \in \mathbb{R}^{m \times n}$, $A^T A \in \mathbb{R}^{n \times n}$ is symmetric

Gram matrix

$$A = \begin{bmatrix} | & | & | \\ a_1 & a_2 & \dots & a_n \\ | & | & | \end{bmatrix} \quad (m \times n) \quad \rightarrow \quad A^T = \begin{bmatrix} -a_1^T & - \\ -a_2^T & - \\ \vdots & \\ -a_n^T & - \end{bmatrix} \quad (n \times m)$$

$$A^T A = \begin{bmatrix} a_1^T a_1 & a_1^T a_2 & \dots & a_1^T a_n \\ a_2^T a_1 & a_2^T a_2 & \dots & a_2^T a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^T a_1 & a_n^T a_2 & \dots & a_n^T a_n \end{bmatrix} \quad (n \times n)$$

$$(A^T A)^T = A^T A \quad \therefore \text{symmetric}$$

$$(A A^T)^T = A A^T \quad \therefore \text{symmetric}$$

Trace ~ *sum of diagonals*

The **trace** of a square matrix $A \in \mathbb{R}^{n \times n}$, denoted $\text{tr}A$, is the sum of diagonal elements in the matrix:

$$\text{tr}A = \sum_{i=1}^n A_{ii}.$$

The trace has the following properties:

- For $A \in \mathbb{R}^{n \times n}$, $\text{tr}A = \text{tr}A^T$.
- For $A, B \in \mathbb{R}^{n \times n}$, $\text{tr}(A + B) = \text{tr}A + \text{tr}B$.
- For $A \in \mathbb{R}^{n \times n}$, $t \in \mathbb{R}$, $\text{tr}(tA) = t \text{tr}A$.
- For A, B such that AB is square, $\text{tr}AB = \text{tr}BA$.
- For A, B, C such that ABC is square, $\text{tr}ABC = \text{tr}BCA = \text{tr}CAB$, and so on for the product of more matrices.



2. Linear Independence & Inverse

1. Linear Independence
2. Span & Linear Transform
3. Rank & Inverse
4. Solving Linear Equations



Linear Independence

선형 독립

A set of vectors $\{x_1, x_2, \dots, x_n\} \subset \mathbb{R}^m$ is said to be *(linearly) dependent* if one vector belonging to the set can be represented as a linear combination of the remaining vectors; that is, if

$$x_n = \sum_{i=1}^{n-1} \alpha_i x_i$$

for some scalar values $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{R}$; otherwise, the vectors are *(linearly) independent*.

Example:

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad x_2 = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} \quad x_3 = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

are linearly dependent because $x_3 = -2x_1 + x_2$.



Span

The **span** of a set of vectors $\{x_1, x_2, \dots, x_n\}$ is the set of all vectors that can be expressed as a linear combination of $\{x_1, \dots, x_n\}$. That is,

$$\text{span}(\{x_1, \dots, x_n\}) = \left\{ v : v = \sum_{i=1}^n \alpha_i x_i, \alpha_i \in \mathbb{R} \right\}.$$

만들 수 있는 모든 선형 결합의 집합

- Note that the **span** of n **linearly independent** vectors is \mathbb{R}^n .

Linear Transform: $v = Ax$

◆ $v = Ax$

- An $m \times n$ matrix A maps an n -dimensional vector x into an m -dimensional vector v .

$$x \in \mathbb{R}^n \xrightarrow[A \in \mathbb{R}^{m \times n}]{v = Ax} v \in \mathbb{R}^m$$

- If the mapping is **linear**, it is called a **linear transform**.

For $\forall \alpha, \beta \in \mathbb{R}$ and $\forall x_1, x_2 \in \mathbb{R}^n$,

$$A(\alpha x_1 + \beta x_2) = \alpha(Ax_1) + \beta(Ax_2)$$

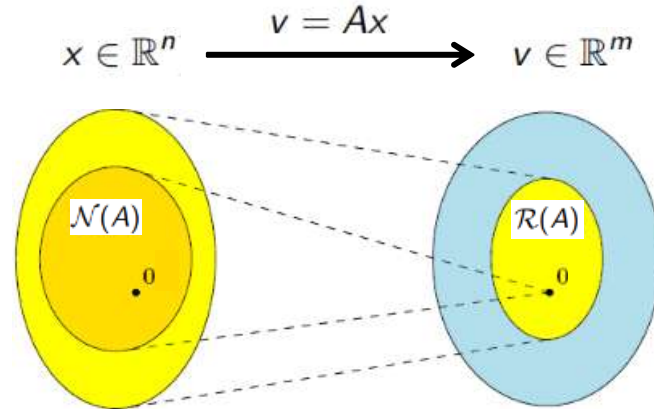
Range & Null Space: $\mathcal{R}(A)$ & $\mathcal{N}(A)$

The *range* or the columnspace of a matrix $A \in \mathbb{R}^{m \times n}$, denoted $\mathcal{R}(A)$, is the the span of the columns of A .

$$\mathcal{R}(A) = \{v \in \mathbb{R}^m : v = Ax, x \in \mathbb{R}^n\}. \quad (\text{image})$$

The *nullspace* of a matrix $A \in \mathbb{R}^{m \times n}$, denoted $\mathcal{N}(A)$ is the set of all vectors that equal 0 when multiplied by A , i.e.,

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = 0\}. \quad (\text{kernel})$$



Rank ~ # of lin. ind. columns (rows)

The **column rank** of a matrix $A \in \mathbb{R}^{m \times n}$ is the size of the largest subset of columns of A that constitute a linearly independent set.

The **row rank** is the largest number of rows of A that constitute a linearly independent set.

For any matrix $A \in \mathbb{R}^{m \times n}$, it turns out that the column rank of A is equal to the row rank of A (prove it yourself!), and so both quantities are referred to collectively as the **rank** of A , denoted as $\text{rank}(A)$.

Properties of the Rank

- For $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) \leq \min(m, n)$. If $\text{rank}(A) = \min(m, n)$, then A is said to be **full rank**.
- For $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) = \text{rank}(A^T)$.
- For $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$.
- For $A, B \in \mathbb{R}^{m \times n}$, $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.



Inverse

The *inverse* of a square matrix $A \in \mathbb{R}^{n \times n}$ is denoted A^{-1} , and is the unique matrix such that

$$A^{-1}A = I = AA^{-1}.$$

A^{-1} exists $\iff Ax \neq 0$ for all $x \neq 0$

- We say that A is invertible or non-singular if A^{-1} exists and non-invertible or singular otherwise.
- In order for a square matrix A to have an inverse A^{-1} , then A must be full rank.

Properties (Assuming $A, B \in \mathbb{R}^{n \times n}$ are non-singular):

- $(A^{-1})^{-1} = A$
 - $(AB)^{-1} = B^{-1}A^{-1}$
 - $(A^{-1})^T = (A^T)^{-1}$. For this reason this matrix is often denoted A^{-T} .
- $$(A + B)^{-1} \neq A^{-1} + B^{-1}$$



Solving Linear Equations: *Using Inverse*

- ◆ If exists, *the inverse can be used* to solve a system of linear equations

- Two linear equations

$$\begin{aligned} 4x_1 - 5x_2 &= -13 \\ -2x_1 + 3x_2 &= 9 \end{aligned}$$

- In vector form, $Ax = b$, with

$$A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} -13 \\ 9 \end{bmatrix}$$

- Solution using inverse

$$\begin{aligned} Ax &= b \\ A^{-1}Ax &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

$$Ax = b \iff A^T Ax = A^T b \iff x = (A^T A)^{-1} A^T b$$

- ◆ Note that the inverse may not exist
- ◆ Even exists, *the inverse is not used* in practice
 - It cannot be represented with sufficient precision
- ◆ Gaussian elimination also has disadvantages
 - It has numerical instability (division by small no.)
 $O(n^3)$ for $n \times n$ matrix
- ◆ Iterative algorithms are used instead
 - Many different decompositions can be used depending on the characteristics of A , x , and b



Solving Linear Equations: *Using Gaussian Elimination*

1. Find a particular solution to $Ax = b$
2. Find all solutions to $Ax = 0$
3. Combine the solutions from 1. and 2. to the general solution.

$A(m \times n)$: m equations with n unknowns

- Under-determined: $m < n$
- Over-determined: $m > n$

$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 42 \\ 8 \end{bmatrix} \quad \text{Reduced Row Echelon Form}$$

Remark (Gaussian Elimination). Gaussian elimination is an algorithm that performs elementary transformations to bring a system of linear equations into reduced row echelon form. \diamond

1. a solution is $[42, 8, 0, 0]^T$

2.
$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \begin{pmatrix} \lambda_1 \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix} \end{pmatrix} = \lambda_1(8e_1 + 2e_2 - e_3) = 0$$

$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \begin{pmatrix} \lambda_2 \begin{bmatrix} -4 \\ 12 \\ 0 \\ -1 \end{bmatrix} \end{pmatrix} = \lambda_2(-4e_1 + 12e_2 - e_4) = 0$$



3.
$$\left\{ x \in \mathbb{R}^4 : x = \begin{bmatrix} 42 \\ 8 \\ 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 \\ 12 \\ 0 \\ -1 \end{bmatrix}, \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

non-pivot columns are expressed as a combination of the *pivot* columns

3. Norm, Orthogonality & Projections

1. Norm
2. Angle
3. Orthogonality
4. Projection & Least Squares



Norm ~ *length of a vector*

A **norm** of a vector $\|x\|$ is informally a measure of the “length” of the vector.

More formally, a norm is any function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ that satisfies 4 properties:

1. For all $x \in \mathbb{R}^n$, $f(x) \geq 0$ (non-negativity).
2. $f(x) = 0$ if and only if $x = 0$ (definiteness).
3. For all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $f(tx) = |t|f(x)$ (homogeneity).
4. For all $x, y \in \mathbb{R}^n$, $f(x + y) \leq f(x) + f(y)$ (triangle inequality).

▪ Note: $\|x - y\|$ is a measure of the “**distance**” of the two vectors.



p-Norms

- ℓ_2 norm (Euclidean norm)

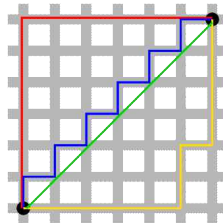
$$\|x\|_2 = \sqrt{x^T x} = \sqrt{\sum_{i=1}^n x_i^2}$$

- ℓ_1 norm (Manhattan norm)

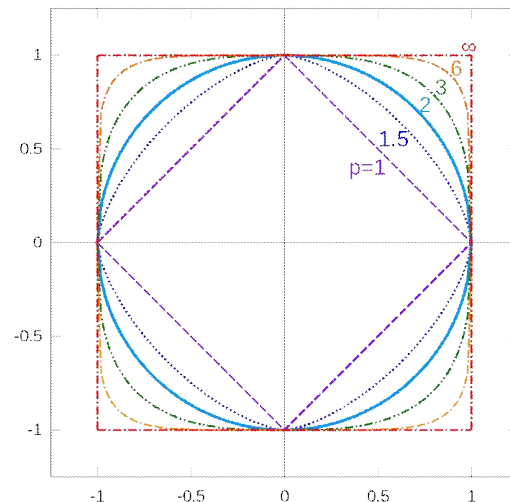
$$\|x\|_1 = \sum_{i=1}^n |x_i|$$

- ℓ_∞ norm (Max norm)

$$\|x\|_\infty = \max_{i=1, \dots, n} |x_i|$$



Illustrations of unit circles in \mathbb{R}^2
(length-one vectors from the origin)



In fact, all three norms presented so far are examples of the family of ℓ_p norms, which are parameterized by a real number $p \geq 1$, and defined as

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}.$$

Above figure is from:
<https://commons.wikimedia.org/w/index.php?curid=17428655>

Frobenius Norm ~ a norm for matrices

$$\mathbb{R}^{m \times n} \rightarrow \mathbb{R}$$

$$\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n A_{ij}^2} = \sqrt{\text{tr}(A^T A)}.$$

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix} = \begin{bmatrix} | & | & \dots & | \\ A_1 & A_2 & \dots & A_n \\ | & | & \dots & | \end{bmatrix}$$

(m x n)

$$A_1^T A_1 = A_{11}^2 + A_{21}^2 + \dots + A_{m1}^2$$

$$A_2^T A_2 = A_{12}^2 + A_{22}^2 + \dots + A_{m2}^2$$

$$\vdots$$

$$A_n^T A_n = A_{1n}^2 + A_{2n}^2 + \dots + A_{mn}^2$$

$$\rightarrow A^T A = \begin{bmatrix} A_1^T A_1 & A_1^T A_2 & \dots & A_1^T A_n \\ A_2^T A_1 & A_2^T A_2 & \dots & A_2^T A_n \\ \vdots & \vdots & \ddots & \vdots \\ A_n^T A_1 & A_n^T A_2 & \dots & A_n^T A_n \end{bmatrix}$$

$$\rightarrow \text{tr}(A^T A) = \sum_{i=1}^m \sum_{j=1}^n A_{ij}^2$$

Cf. Frobenius norm is used for

$$\text{reg} \begin{matrix} \dots \\ \dots \end{matrix} J_R = J + \lambda \sum_{i=1}^L \|W^{(i)}\|_F$$



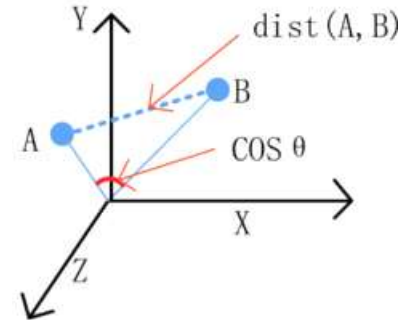
Angle *between* Vectors

Dot product of two vectors can be written in terms of their L^2 norms and angle θ between them

$$x^T y \Rightarrow \|x\|_2 \|y\|_2 \cos \theta$$

Cosine between two vectors is a measure of their (*orientation*) **similarity**

$$\cos \theta = \frac{x^T y}{\|x\|_2 \|y\|_2}$$



Orthogonal Vectors

- Two vectors $x, y \in \mathbb{R}^n$ are *orthogonal* if

$$x^T y = 0$$

$$(\cos(\theta) = 0 \text{ or } \theta = \pi/2)$$

$$x \perp y$$

- They are *orthonormal* if, in addition,

$$\|x\|_2 = \|y\|_2 = 1$$

A vector $x \in \mathbb{R}^n$ is ***normalized*** if $\|x\|_2 = 1$.



Orthogonal *Matrices*

- A matrix $U \in \mathbb{R}^{n \times n}$ is orthogonal if all it's columns are orthonormal, i.e.,

$$U^T U = I = U U^T \quad \Leftrightarrow \quad \text{all its columns are orthogonal to each other}$$

$U^T = U^T$ (linearly independent)

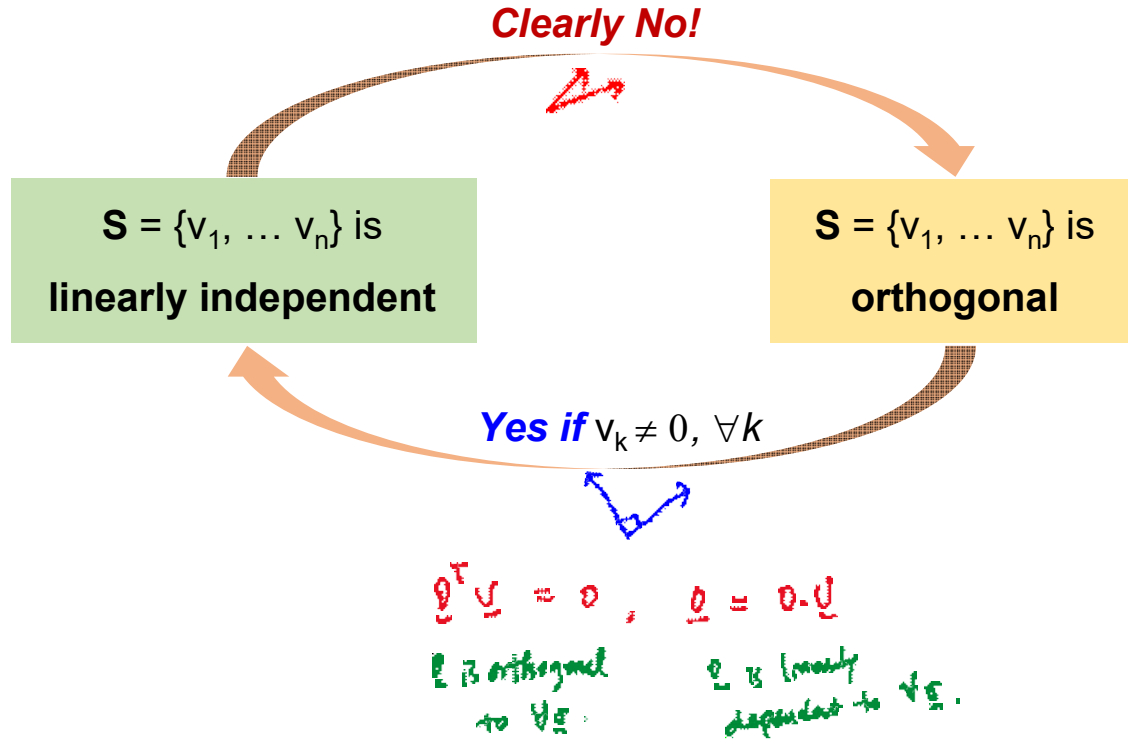
Transformations with orthogonal matrices preserve Euclidean **distances** and **angles**!

$$\|Ax\|^2 = (Ax)^T (Ax) = x^T A^T Ax = x^T I x = x^T x = \|x\|^2$$

$$\cos \omega = \frac{(Ax)^T (Ay)}{\|Ax\| \|Ay\|} = \frac{x^T A^T Ay}{\sqrt{x^T A^T A x y^T A^T A y}} = \frac{x^T y}{\|x\| \|y\|}$$

Linear Independence vs. Orthogonality

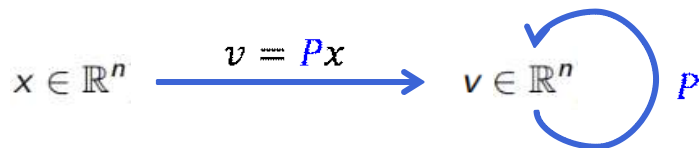
Consider a set of vectors $\mathbf{S} = \{v_1, \dots, v_n\}$



Projection

◆ Definition: Idempotence

- A **projection matrix** P is a linear transformation from a vector space to itself such that $P^2 = P$.
- Such mapping is called a **projection**.



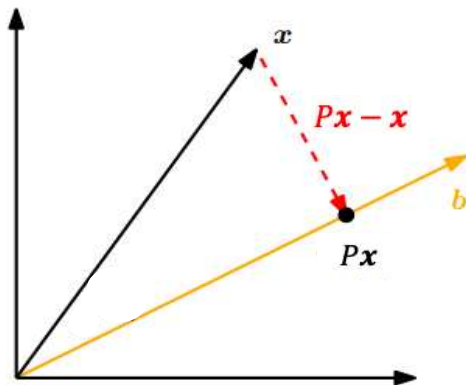
◆ Examples

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 \\ \alpha & 1 \end{bmatrix}$$

Projection onto 1-D Subspaces

Assume we are given a line (1-dimensional subspace) through the origin with basis vector $\mathbf{b} \in \mathbb{R}^n$. The line is a one-dimensional subspace $U \subseteq \mathbb{R}^n$ spanned by \mathbf{b} . When we project $\mathbf{x} \in \mathbb{R}^n$ onto U , we seek the vector $\mathbf{v} = P\mathbf{x} \in U$ that is closest to \mathbf{x} .



(a) Projection of $\mathbf{x} \in \mathbb{R}^2$ onto a subspace U with basis vector \mathbf{b} .

The projection $\mathbf{v} = P\mathbf{x}$ is closest to \mathbf{x} .

$\Rightarrow \|\mathbf{Px} - \mathbf{x}\|$ is minimal.

$\Rightarrow (\mathbf{Px} - \mathbf{x})$ is orthogonal to \mathbf{b} .

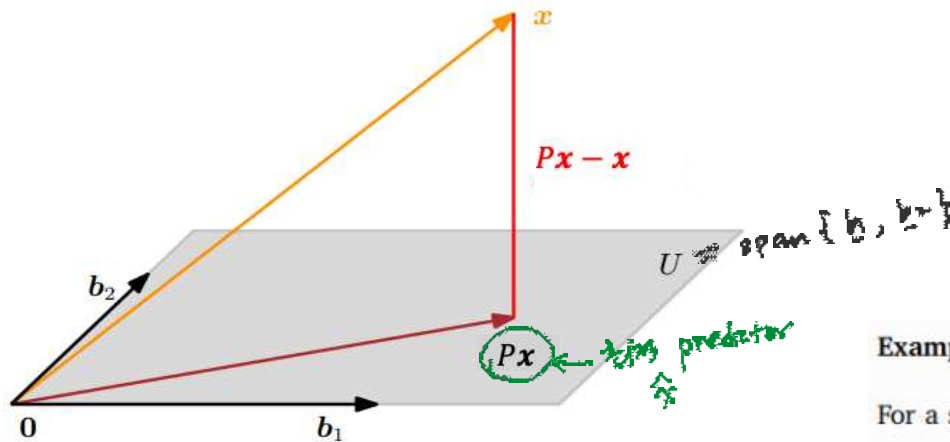
$$\Rightarrow P = \frac{\mathbf{b}\mathbf{b}^T}{\|\mathbf{b}\|^2}$$

Projection onto General Subspaces

$$x \in \mathbb{R}^n$$

lower-dimensional subspaces $U \subseteq \mathbb{R}^n$ with $\dim(U) = m$

Assume that (b_1, \dots, b_m) is an ordered basis of U .



$$B = [b_1, \dots, b_m] \in \mathbb{R}^{n \times m},$$

$$P = B(B^T B)^{-1} B^T.$$

Example 3.11 (Projection onto a Two-dimensional Subspace)

For a subspace $U = \text{span}\left[\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right] \subseteq \mathbb{R}^3$ and $x = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3$

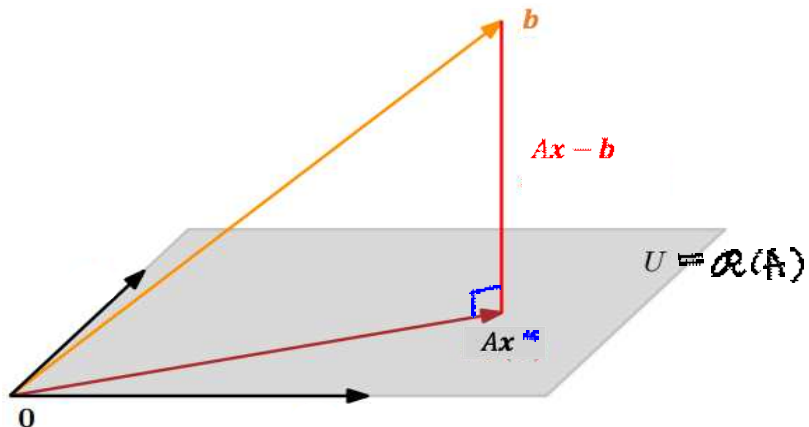
$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\hat{x} = Px = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}.$$

Projection & Least Squares

Using projections, we can find approximate solutions to linear equations $Ax = b$.

- Suppose b does not lie in the span of A . Given that the linear equation cannot be solved exactly.
- We can find an approximate solution by computing the **orthogonal projection** of b onto the span of A .



Orthogonality: $(Ax)^T (Ax - b) = 0$

$$\Rightarrow \mathbf{x}^* = (A^T A)^{-1} A^T \mathbf{b}$$

$$A\mathbf{x}^* \perp \mathbf{b}$$

This problem arises often in practice, and the solution is called the **least-squares solution** of an over-determined system.

Squared Error

$$\|Ax - b\|^2 = (Ax - b)^T (Ax - b)$$

$$= x^T A^T A x - b^T A x - x^T A^T b + b^T b$$

$$\frac{\partial}{\partial x} \|Ax - b\|^2 = 2A^T A x - 2A^T b = 0$$

$$\Rightarrow \mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}$$

Least Square Solution
(min ||Ax - b||)

$$\begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix} \mathbf{x} = \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix}$$

4. Determinant, Decomposition, Quadratic Forms

1. Determinant
2. Eigenvector & Eigenvalue
3. Eigendecomposition
4. Quadratic Forms
5. Positive Definite
6. Singular Value Decomposition (SVD)



Determinant ~ volume

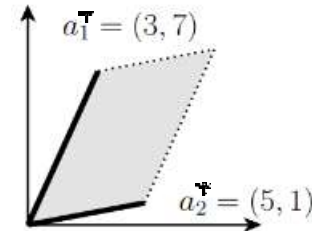
The **determinant** of a square matrix $A \in \mathbb{R}^{n \times n}$,
is a function det: $\mathbb{R}^{n \times n} \rightarrow \mathbb{R}$, and is denoted $|A|$ or $\det A$.

Measures how much multiplication by
the matrix expands or contracts space

Given a matrix $\begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_n^T & - \end{bmatrix}$, consider the set of points $S \subset \mathbb{R}^n$ as follows:

$$S = \{v \in \mathbb{R}^n : v = \sum_{i=1}^n \alpha_i a_i \text{ where } 0 \leq \alpha_i \leq 1, i = 1, \dots, n\}.$$

The absolute value of the determinant of A ,
is a measure of the “volume” of the set S



$$A = \begin{bmatrix} 3 & 7 \\ 5 & 1 \end{bmatrix}$$

$|\det A|$ is the area of the parallelogram

Determinant: *Definition*

- Can be formally defined by three properties

1. Determinant of identity is one: $\det I = 1$

2. Multiplying a row by scalar $t \in \mathbb{R}$ scales determinant:

$$\det \begin{bmatrix} - & ta_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_n^T & - \end{bmatrix} = t \det A$$

3. Swapping rows negates determinant:

$$\det \begin{bmatrix} - & a_2^T & - \\ - & a_1^T & - \\ & \vdots & \\ - & a_n^T & - \end{bmatrix} = -\det A$$

In case you are wondering, it is not immediately obvious that a function satisfying the above three properties exists. In fact, though, such a function does exist, and is unique (which we will not prove here).



Determinant: *Properties*

- Important properties

For $A, B \in \mathbb{R}^{n \times n}$,

- $\det A = \det A^T$
- $\det AB = \det A \det B$
- $\det A = 0 \Leftrightarrow A$ singular (non-invertible)
- $\det A^{-1} = 1 / \det A$



Determinant: Formula

Let $A \in \mathbb{R}^{n \times n}$, $A_{\setminus i, \setminus j} \in \mathbb{R}^{(n-1) \times (n-1)}$ be the *matrix* that results from deleting the i th row and j th column from A .

The general (recursive) formula for the determinant is

$$\begin{aligned} |A| &= \sum_{i=1}^n (-1)^{i-j} a_{ij} |A_{\setminus i, \setminus j}| \quad (\text{for any } j \in 1, \dots, n) \\ &= \sum_{j=1}^n (-1)^{i-j} a_{ij} |A_{\setminus i, \setminus j}| \quad (\text{for any } i \in 1, \dots, n) \end{aligned}$$

with the initial case that $|A| = a_{11}$ for $A \in \mathbb{R}^{1 \times 1}$. If we were to expand this formula completely for $A \in \mathbb{R}^{n \times n}$, there would be a total of $n!$ (n factorial) different terms. For this reason, we hardly ever explicitly write the complete equation of the determinant for matrices bigger than 3×3 .

However, the equations for determinants of matrices up to size 3×3 are fairly common, and it is good to know them:

$$\begin{aligned} |[a_{11}]| &= a_{11} \\ \left| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right| &= a_{11}a_{22} - a_{12}a_{21} \\ \left| \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right| &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}$$

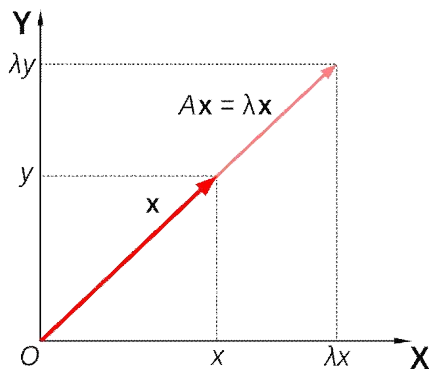


Eigenvector & Eigenvalue

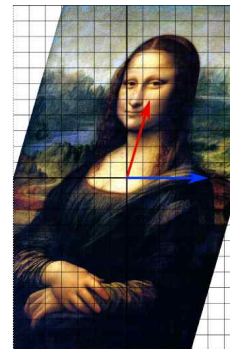
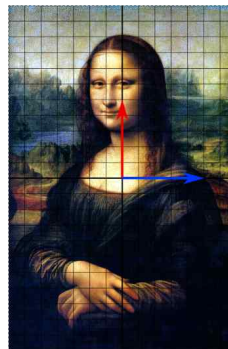
Given a square matrix $A \in \mathbb{R}^{n \times n}$, we say that $\lambda \in \mathbb{C}$ is an *eigenvalue* of A and $x \in \mathbb{C}^n$ is the corresponding *eigenvector* if

$$Ax = \lambda x, \quad x \neq 0.$$

Intuitively, this definition means that multiplying A by the vector x results in a new vector that points in the same direction as x , but scaled by a factor λ .



Matrix A acts by stretching the vector x , not changing its direction, so x is an eigenvector of A .



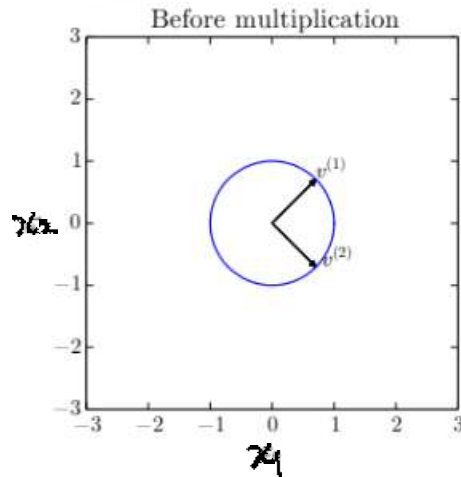
In this shear mapping, the red arrow changes direction, but the blue arrow does not. Thus the blue arrow is an eigenvector of this mapping.

Figures are from https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors

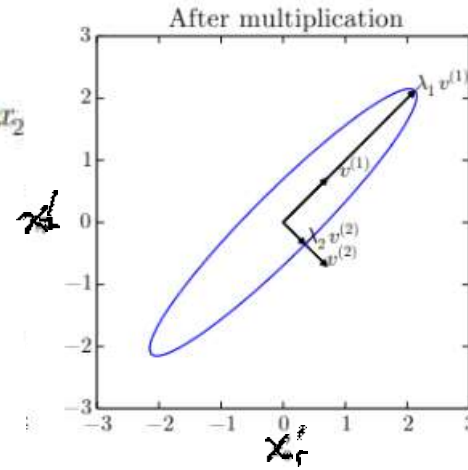
Eigenvector & Eigenvalue (cont'd)

- Example of 2×2 matrix
- Matrix A with two orthonormal eigenvectors
 - $v^{(1)}$ with eigenvalue λ_1 , $v^{(2)}$ with eigenvalue λ_2

Plot of unit vectors $u \in \mathbb{R}^2$
(circle)



Plot of vectors Au
(ellipse)



Characteristic Polynomial

We can rewrite the equation above to state that (λ, x) is an eigenvalue-eigenvector pair of A if,

$$(\lambda I - A)x = 0, \quad x \neq 0.$$

But $(\lambda I - A)x = 0$ has a non-zero solution to x if and only if $(\lambda I - A)$ has a non-empty nullspace, which is only the case if $(\lambda I - A)$ is singular, i.e.,

$$|(\lambda I - A)| = 0. \quad = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda)$$

We can now use the previous definition of the determinant to expand this expression $|(\lambda I - A)|$ into a (very large) polynomial in λ , where λ will have degree n . It's often called the characteristic polynomial of the matrix A .

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad |A - \lambda I| = \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} = 3 - 4\lambda + \lambda^2 \quad \begin{matrix} \lambda=1, \\ \lambda=3 \end{matrix} \quad v_{\lambda=1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, v_{\lambda=3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigendecomposition

- Suppose that matrix A has n linearly independent eigenvectors $\{v^{(1)}, \dots, v^{(n)}\}$ with eigenvalues $\{\lambda_1, \dots, \lambda_n\}$
- Concatenate eigenvectors to form matrix V
- Concatenate eigenvalues to form vector $\lambda = [\lambda_1, \dots, \lambda_n]$
- Eigendecomposition of A is given by

$$A = V \text{diag}(\lambda) V^{-1}$$

$$\begin{cases} A v_1 = \lambda_1 v_1 \\ A v_2 = \lambda_2 v_2 \\ \vdots \\ A v_n = \lambda_n v_n \end{cases}$$

$$A \begin{bmatrix} | & & | \\ v^{(1)} & \dots & v^{(n)} \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1 v^{(1)} & \dots & \lambda_n v^{(n)} \\ | & & | \end{bmatrix}$$

$$AV = V \begin{bmatrix} \lambda_1 & 0 \\ & \ddots \\ 0 & \dots & \lambda_n \end{bmatrix}$$

columns
scalar
(diagonal
on the
right)

$$\Rightarrow A = \boxed{V \Lambda V^{-1}}$$

Properties of Eigenvalues

$$- \text{tr } A = \sum_{i=1}^n \lambda_i$$

$$- \det A = \prod_{i=1}^n \lambda_i$$

$$- \text{rank}(A) = \text{number of non-zero eigenvalues}$$

Matrix is singular \Leftrightarrow any eigenvalue is zero

$$- \text{Eigenvalues of } A^{-1} \text{ are } 1/\lambda_i, i = 1, \dots, n, \\ \text{eigenvectors are the same}$$

$$\begin{aligned} \because \text{tr}(V \Lambda V^{-1}) &= \text{tr}(\Lambda V^{-1} V) = \text{tr}(\Lambda) \\ \because \det(V \Lambda V^{-1}) &= \det V \cdot \det \Lambda \cdot \det V^{-1} \\ &= \det \Lambda \end{aligned} \quad \left(\begin{aligned} \det V^{-1} \\ = \frac{1}{\det V} \end{aligned} \right)$$

$$\begin{aligned} \because Ax &= \lambda x \\ x &= \lambda A^{-1} x \\ \frac{1}{\lambda} x &= A^{-1} x \end{aligned}$$



Eigendecomposition of *Real Symmetric Matrices*

let's assume that A is a symmetric real matrix

- Every real symmetric matrix A can be decomposed into real-valued eigenvectors and eigenvalues

$$A = Q\Lambda Q^T$$

where Q is an orthogonal matrix composed of eigenvectors of A : $\{v^{(1)}, \dots, v^{(n)}\}$

Λ is a diagonal matrix of eigenvalues $\{\lambda_1, \dots, \lambda_n\}$

- By convention order entries of Λ in descending order:
- Decomposition is not unique when two eigenvalues are the same



Quadratic Forms

Given a square matrix $A \in \mathbb{R}^{n \times n}$ and a vector $x \in \mathbb{R}^n$, the scalar value $x^T A x$ is called a quadratic form. Written explicitly, we see that

$$x^T A x = \sum_{i=1}^n x_i (A x)_i = \sum_{i=1}^n x_i \left(\sum_{j=1}^n A_{ij} x_j \right) = \sum_{i=1}^n \sum_{j=1}^n A_{ij} x_i x_j .$$

$$\begin{aligned} x^T A x &= A_{11} x_1^2 + A_{12} x_1 x_2 + A_{13} x_1 x_3 + \dots \\ &+ A_{21} x_2 x_1 + A_{22} x_2^2 + A_{23} x_2 x_3 + \dots \\ &+ A_{31} x_3 x_1 + A_{32} x_3 x_2 + A_{33} x_3^2 + \dots \\ &+ \dots \end{aligned}$$

a **quadratic form** is a polynomial with terms all of degree two

$$4x^2 + 2xy - 3y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$N(x | \mu, \Sigma) = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} \underbrace{(x - \mu)^T}_{x^T} \underbrace{\Sigma^{-1}}_A \underbrace{(x - \mu)}_x \right\}$$

Positive Definite

A symmetric matrix $A \in \mathbb{S}^n$ is:

- **positive definite** (PD), denoted $A \succ 0$ if for all non-zero vectors $x \in \mathbb{R}^n$, $x^T A x > 0$.
- **positive semidefinite** (PSD), denoted $A \succeq 0$ if for all vectors $x^T A x \geq 0$.
- **negative definite** (ND), denoted $A \prec 0$ if for all non-zero $x \in \mathbb{R}^n$, $x^T A x < 0$.
- **negative semidefinite** (NSD), denoted $A \preceq 0$ if for all $x \in \mathbb{R}^n$, $x^T A x \leq 0$.
- **indefinite**, if it is neither positive semidefinite nor negative semidefinite — i.e., if there exists $x_1, x_2 \in \mathbb{R}^n$ such that $x_1^T A x_1 > 0$ and $x_2^T A x_2 < 0$.

One important property of positive definite and negative definite matrices is that they are always full rank, and hence, invertible.

Given any matrix $A \in \mathbb{R}^{m \times n}$ (not necessarily symmetric or even square), the matrix $G = A^T A$ (sometimes called a **Gram matrix**) is always positive semidefinite. Further, if $m \geq n$ and A is full rank, then $G = A^T A$ is positive definite.

$$\begin{aligned}\|Ax\|^2 &\geq 0 \\ &= (Ax)^T Ax \\ &= x^T \underline{A^T A} x\end{aligned}$$



Definiteness & Eigenvalue Signs

1. If all $\lambda_i > 0$, then the matrix A is positive definite
2. If all $\lambda_i \geq 0$, it is positive semidefinite
3. Likewise, if all $\lambda_i < 0$ or $\lambda_i \leq 0$, then A is negative definite or negative semidefinite respectively.
4. Finally, if A has both positive and negative eigenvalues, say $\lambda_i > 0$ and $\lambda_j < 0$, then it is indefinite.



Singular Value Decomposition (SVD)

- SVD is more general than eigendecomposition

- If A is not square, eigendecomposition is undefined
- Every real matrix has a SVD

$$A = UDV^T$$

- U and V are orthogonal matrices
- D is a diagonal matrix not necessarily square
 - Elements of Diagonal of D are called *singular values of A*
 - Columns of U are called left singular vectors
 - Columns of V are called right singular vectors
- Left singular vectors of A are eigenvectors of AA^T
- Right singular vectors of A are eigenvectors of $A^T A$
- Nonzero singular values of A are square roots of eigenvalues of $A^T A$. Same is true of AA^T

Use of SVD in ML

1. SVD is used in generalizing matrix inversion
 - Moore-Penrose inverse (discussed next)
2. Used in Recommendation systems
 - Collaborative filtering (CF)



Moore-Penrose Pseudoinverse

$A^+ = VD^+U^\top$

- Solution to $y = Ax$ (using the pseudoinverse): $x = A^+y$

If the equation has:

- Exactly one solution: this is the same as the inverse.
- No solution: this gives us the solution with the smallest error $\|\mathbf{Ax} - \mathbf{y}\|_2$.
- Many solutions: this gives us the solution with the smallest norm of \mathbf{x} .