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SWCON253: Machine Learning

Lecture 15
Probability & Information Theory

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1. Probability Review

1. Probability
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References

- "*Schaum's Outline of Probability, Random Variables, and Random Processes*," by Hwei P. Hsu

Probability

Random Experiment

- *experiment*: any process of observation
- *outcomes*: the results of an observation
- *random experiment*: if outcome cannot be predicted with certainty

Sample Space (S) and Event Space (E)

- *sample space S*: the set of all possible outcomes
- *event*: any subset of the sample space S
  - ★ Note that Ø and S are also events.
- *event space E*: the set of all possible events

Probability Space (S, E, P)

- *probability measure P*: a function defined over the event space E
- *probability space*: the triplet (S, E, P)

Example: *Rolling a Dice*

- *sample space S:*
- *event space E:*
- *probability measure P:*

Probability (cont'd)

Axiomatic Definition of *Probability*

- Consider a probability space (S, E, P).
- The probability $P(A)$ of an event $A \in E$ is defined as a real number assigned to A which satisfies the following three axioms:

1. $P(A) \geq 0$

2. $P(S) = 1$

3. $P(A \cup B) = P(A) + P(B)$ *if* $P(A \cap B) = \emptyset$ (disjoint)

Properties of Probability

★ $P(A^c) = 1 - P(A)$

★ $P(\emptyset) = 0$

★ $P(A) \leq P(B)$ *if* $A \subset B$

★ $P(A) \leq 1$

★ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Conditional Probability & Bayes' Theorem

Conditional Probability

- The *conditional probability* of an event A given event B, $P(A|B)$, is defined as:

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}} \qquad P(B) > 0$$

★ $P(A \cap B)$ is the *joint probability* of A and B.

★ Note that $A|B$ is not a set (i.e., not an event).
'$|B$' is just a notation saying that event B has occurred already.

Bayes' Rule

- Note that $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$.
- Thus, we can obtain the following *Bayes' Rule*:

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

Example: *Rolling a Dice*

Assume all outcomes are equally likely.
And let A={1, 2, 3, 4} and B={4, 5, 6}.

- P(A) =
- P(B) =
- P(A∩B) =
- P(A|B) =

Conditional Probability & Bayes' Theorem (cont'd)

Bayes' Theorem

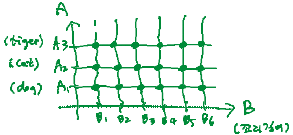
- Suppose the events $A_1, A_2, \dots, A_n$ are a **partition** of S, i.e.,
  - ★ $A_i \cap A_j = \emptyset$ for $\forall i \neq j$: *mutually exclusive (disjoint)*
  - ★ $\bigcup_{i=1}^{n} A_i = S$

- Let B be any event in S. Then we can obtain $P(B)$ by:

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$: *the total probability*

- Using Bayes' Rule, we obtain ***Bayes' Theorem***:

$$\boxed{P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}}$$

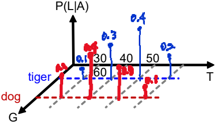


★ Sometimes, we call each component:

- $P(A_i|B)$: a ***posteriori*** probability → 꼬리길이를 알때 각 동물의 확률
- $P(B|A_i)$: a ***likelihood/conditional*** probability → 각 동물의 꼬리길이 분포
- $P(A_i)$: a ***priori*** probability → dog, cat, tiger의 개체수 비율

ML (Maximum Likelihood) vs. MAP (Maximum A Posteriori)

Example



- Animal class: S_A={*dog*, *tiger*}, P(*dog*)=0.8, P(*tiger*)=0.2
- Tail length: S_L={30, 40, 50, 60}
- Joint Sample Space: S= S_A x S_L ={(dog,30), (tiger,30), (dog,40),…,(tiger,60)}
- Conditional Probability: P(*L*|*dog*) and P(*L*|*tiger*) are given as the figure.

- Now, suppose we want to classify animals based only on the observed tail length of the animal.

★ ***ML test*: P(*L*|*dog*) ⋛ P(*L*|*tiger*)**

- P(*L*=50|dog) = 0.3 **<** P(L=50|tiger) = 0.4 → *tiger!*

★ ***MAP test*: P(*dog*|L) ⋛ P(*tiger*|*L*)**

→ P(L|dog)P(dog) ⋛ P(L|tiger)P(tiger)

- P(L=50|dog)P(*dog*) = 0.24 **>** P(L=50|tiger)P(*tiger*) = 0.08 → *dog!*

ML & MAP Classification

Problem Definition

- 입력 샘플 $\mathbf{x}_{new}$를 K개의 class $C = \{c_1, c_2, \cdots, c_K\}$ 중 하나로 분류하는 문제를 생각해 보자.

  ★ 앞의 예에서 x는 꼬리길이, c_1=dog, c_2=tiger
- **x**에 관한 확률분포(probability density)를 이미 알고 있다면, ML이나 MAP을 이용하여 입력 샘플을 분류할 수 있다.

Maximum Likelihood (ML) Classification

- If we know the ***class conditional distributions*** (i.e., the ***likelihoods of*** c_k) $P(\mathbf{x}|c_k)$ for all $k = 1 \ldots K$, then we can classify a new sample $\mathbf{x}_{new}$ by :

$$\boxed{k^* = \arg\max_{k=1..K} P(\mathbf{x}_{new}|c_k)}$$

Maximum A Posteriori (MAP Classification

- If we also know the ***prior distribution*** $P(c_k)$ for all $k = 1 \ldots K$, then we can classify a new sample $\mathbf{x}_{new}$ by :

$$\boxed{k^* = \arg\max_{k=1..K} P(\mathbf{x}_{new}|c_k)P(c_k)}$$

이 확률분포들을 어떻게 구하지?
→ Density Estimation (분포추정)

Random Variables

Definition

- A ***random variable*** *X* is a function that assigns a *real number* to each *sample point* (i.e., outcome) of *S*.

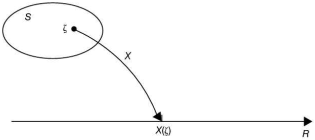


Fig. 2-1 Random variable *X* as a function.

Independence

Independent Events

- $P(A \cap B) = P(A)P(B)$
- $P(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} P(A_i)$

Independent Random Variables

- Concept: $P(X = x, Y = y) = P(X = x)P(Y = y)$ for any x and y
- Discrete: $p_{XY}(x_i, y_j) = p_X(x_i)p_Y(y_j)$ for any x_i and y_j
- Continuous: $f_{XY}(x, y) = f_X(x)f_Y(y)$ for any x and y

Cf.) Naive Bayes Classifiers

◆ *Naive Bayes Assumption*

- The ***features are conditionally independent*** given the class label
  - ★ Called “naive” since we do not expect the features to be independent, even conditional on the class label

$$\boxed{P(\mathbf{x}|c_k) = \prod_{d=1}^{D} P(x_d|c_k)}$$

where $\mathbf{x} = [x_1, \dots, x_D]^{\mathrm{T}}$

- 각 샘플벡터(**x**)들의 발생 확률은 통상 독립으로 가정한다: $P(\mathbb{X}|c_k) = P(\mathbf{x}_1, \dots, \mathbf{x}_n|c_k) = \prod_{i=1}^{n} P(\mathbf{x}_i|c_k)$
- 그러나 각 샘플벡터의 원소(feature)들의 발생 확률은 독립으로 가정하기 어렵다.
  Naive Bayes는 (naive하게도) 이걸 독립이라고 가정한다: $P(\mathbf{x}|c_k) = P(x_1, \dots, x_D|c_k) = \prod_{d=1}^{D} P(x_i|c_k)$

- Note: even if the naive Bayes assumption is not true, it often results in classifiers that work well
  - ★ One reason for this is that the model is quite simple (it only has $O(CD)$ parameters, for C classes and D features), and hence it is relatively immune to overfitting.

Expectations

Mean (Expectation) of a Random Variable

The *mean* (or *expected value*) of a r.v. X, denoted by μ_X or $E(X)$, is defined by

$$\mu_X = E(X) = \begin{cases} \sum_k x_k p_X(x_k) & X\text{: discrete} \\ \int_{-\infty}^{\infty} x f_X(x)\,dx & X\text{: continuous} \end{cases}$$

The *variance* of a r.v. X, denoted by σ_X^2 or $\operatorname{Var}(X)$, is defined by

$$\sigma_X^2 = \operatorname{Var}(X) = E\{[X - E(X)]^2\}$$

$$= \begin{cases} \sum_k (x_k - \mu_X)^2 p_X(x_k) & X\text{: discrete} \\ \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\,dx & X\text{: continuous} \end{cases}$$

$$\operatorname{Var}(X) = E(X^2) - [E(X)]^2$$

[Note] Mean & Variance from Samples

- Sample Mean (empirical mean)
- Sample Variance (empirical variance)

Expectations (cont'd)

Conditional Expectation

- Two random variables: *X* and *Y*

$$E(Y|X) = \begin{cases} \sum_{k} y_k \, p(y_k|X) & Y\text{: discrete} \\ \int_{-\infty}^{\infty} y \, f(y|x) \, dy & Y\text{: continuous} \end{cases}$$

Expectation of a *Function* of a Random Variable

- *Y* = g(*X*)

$$E(g(X)) = \begin{cases} \sum_{K} g(x_k) \, p(x_k) & X\text{: discrete} \\ \int_{-\infty}^{\infty} g(x) \, f(x) \, dx & X\text{: continuous} \end{cases}$$

$$\text{cf. } E(Y) = \begin{cases} \sum_{k} y_k \, p(y_k) & Y\text{: discrete} \\ \int_{-\infty}^{\infty} y \, f(y) \, dy & Y\text{: continuous} \end{cases}$$

Correlation & Covariance

Two Random Variables: X and Y

- **Correlation**: $E(XY)$
  - ★ orthogonal: $E(XY) = 0$
  - ★ uncorrelated: $E(XY) = E(X)E(Y)$
- **Covariance**: $Cov(X,Y) = \sigma_{XY} = E[(X - E(X))(Y - E(Y))]$

★음수도 될수 있음에 유의!

$$= E(XY) - E(X)E(Y)$$

  - ★ uncorrelated: $\sigma_{XY} = 0$
- **Correlation Coefficient**: a normalized covariance

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \qquad |\rho_{XY}| \le 1$$

- Note
  - ★ Independence implies uncorrelatedness - (1)
  - ★ Uncorrelatedness does NOT imply independence - (2)

independent

(1) $$E(XY) = \sum_{y_j}\sum_{x_i} x_i y_j \underline{p_{XY}(x_i, y_j)} = \sum_{y_j}\sum_{x_i} x_i y_j \underline{p_X(x_i) p_Y(y_j)}$$

$$= \left[\sum_{x_i} x_i p_X(x_i)\right]\left[\sum_{y_j} y_j p_Y(y_j)\right] = E(X)E(Y)$$

(2) $$p_{XY}(x_i, y_j) = \begin{cases} \frac{1}{3} & (0,1),(1,0),(2,1) \\ 0 & \text{otherwise} \end{cases}$$

$$\left\{\begin{array}{l} E(X) = \sum_{x_i} x_i p_X(x_i) = (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) + (2)\left(\frac{1}{3}\right) = 1 \\ E(Y) = \sum_{y_j} y_j p_y(y_j) = (0)\left(\frac{1}{3}\right) + (1)\left(\frac{2}{3}\right) = \frac{2}{3} \end{array}\right.$$

$$E(XY) = \sum_{y_j}\sum_{x_i} x_i y_j p_{XY}(x_i, y_j)$$

$$= (0)(1)\left(\frac{1}{3}\right) + (1)(0)\left(\frac{1}{3}\right) + (2)(1)\left(\frac{1}{3}\right) = \frac{2}{3}$$

$E(XY) = E(X)E(Y)$

uncorrelated

$$p_{XY}(0,1) = \frac{1}{3} \ne p_X(0) p_Y(1) = \frac{2}{9} \Rightarrow \text{NOT independent}$$

Correlation & Covariance (cont'd)

Correlation Coefficient & Linear Dependence

Let $Y = aX + b$.

(*a*) Find the covariance of X and Y.

(*b*) Find the correlation coefficient of X and Y.

(*a*) By Eq. (4.131), we have

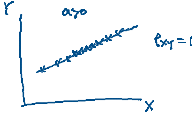
$$E(XY) = E[X(aX+b)] = aE(X^2) + bE(X)$$
$$E(Y) = E(aX+b) = aE(X) + b$$

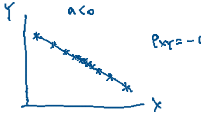
Thus, the covariance of X and Y is [Eq. (3.51)]

$$\begin{aligned}\text{Cov}(X,Y) = \sigma_{XY} &= E(XY) - E(X)E(Y)\\ &= aE(X^2) + bE(X) - E(X)[aE(X)+b]\\ &= a\{E(X^2) - [E(X)]^2\} = a\sigma_X^2\end{aligned}$$

(*b*) By Eq. (4.130), we have $\sigma_Y = |a|\,\sigma_X$. Thus, the correlation coefficient of X and Y is

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{a\sigma_X^2}{\sigma_X|a|\sigma_X} = \frac{a}{|a|} = \begin{cases} 1 & a > 0 \\ -1 & a < 0 \end{cases}$$





Correlation & Covariance (cont'd)

Covariance Matrix of a Random Vector

- *random vector*: an array of random variables

$$\mathbf{X} = [X_1 \quad \dots \quad X_n]^T$$

- *covariance matrix* of $\mathbf{X}$:

$$K_X = \begin{bmatrix} \sigma_{11} & \dots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \dots & \sigma_{nn} \end{bmatrix} \quad \text{where } \sigma_{ij} = Cov(X_i, X_j)$$

★ If X_i's are *uncorrelated*, then K becomes a diagonal matrix since $\sigma_{ij} = 0 \;\; for \; \forall i \neq j$.

$$K_X = \begin{bmatrix} \sigma_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_{nn} \end{bmatrix}$$

Correlation & Covariance (cont'd)

Estimating Mean & Covariance from a Dataset

- Consider *n* training samples of *d*-dimensional data:

$$\mathbb{X} = \{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)}\}, \quad \mathbf{x}^{(k)} = \begin{bmatrix} x_1^{(k)} & \ldots & x_d^{(k)} \end{bmatrix}^T$$

- The *mean* of each component can be estimated from the given dataset:

$$\mu_i \equiv E[x_i] \approx \frac{1}{n}\sum_{k=1}^{n} x_i^{(k)} \quad (1 \le i \le d)$$

or we can collectively estimate the ***mean vector*** by:

$$\boldsymbol{\mu} = E[\mathbf{x}] = [\mu_1 \quad \ldots \quad \mu_d]^T \approx \frac{1}{n}\sum_{k=1}^{n} \mathbf{x}^{(k)}$$

- The *covariance* of each pair of data components (i.e., feature components) is:

$$\sigma_{ij} \equiv E\left[(x_i - \mu_i)(x_j - \mu_j)\right] \approx \frac{1}{n}\sum_{k=1}^{n} \left(x_i^{(k)} - \mu_i\right)\left(x_j^{(k)} - \mu_j\right) \quad (1 \le i, j \le d)$$

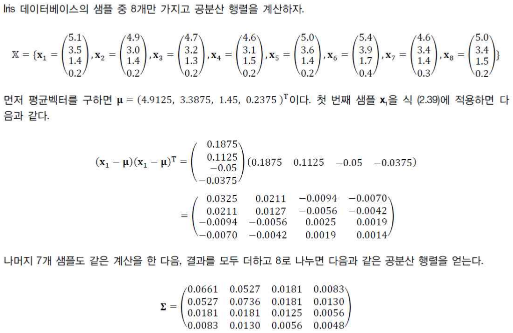
or we can collectively estimate the ***covariance matrix*** by:

$$K \equiv [\sigma_{ij}] \approx \frac{1}{n}\sum_{k=1}^{n}\left(\mathbf{x}^{(k)} - \boldsymbol{\mu}\right)\left(\mathbf{x}^{(k)} - \boldsymbol{\mu}\right)^T$$

$$\because\ (\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^T = \begin{bmatrix} (x_1 - \mu_1) \\ \vdots \\ (x_d - \mu_d) \end{bmatrix} [(x_1 - \mu_1) \ \ldots \ (x_d - \mu_d)] = \begin{bmatrix} (x_1 - \mu_1)(x_1 - \mu_1) & \cdots & (x_1 - \mu_1)(x_d - \mu_d) \\ \vdots & \ddots & \vdots \\ (x_d - \mu_d)(x_1 - \mu_1) & \cdots & (x_d - \mu_d)(x_d - \mu_d) \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1d} \\ \vdots & \ddots & \vdots \\ \sigma_{d1} & \cdots & \sigma_{dd} \end{bmatrix}$$

Correlation & Covariance (cont'd)

◆ 평균 벡터와 공분산 행렬 예제



Iris 데이터베이스의 샘플 중 8개만 가지고 공분산 행렬을 계산하자.

$$\mathbb{X} = \{\mathbf{x}_1 = \begin{pmatrix} 5.1 \\ 3.5 \\ 1.4 \\ 0.2 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 4.9 \\ 3.0 \\ 1.4 \\ 0.2 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} 4.7 \\ 3.2 \\ 1.3 \\ 0.2 \end{pmatrix}, \mathbf{x}_4 = \begin{pmatrix} 4.6 \\ 3.1 \\ 1.5 \\ 0.2 \end{pmatrix}, \mathbf{x}_5 = \begin{pmatrix} 5.0 \\ 3.6 \\ 1.4 \\ 0.2 \end{pmatrix}, \mathbf{x}_6 = \begin{pmatrix} 5.4 \\ 3.9 \\ 1.7 \\ 0.4 \end{pmatrix}, \mathbf{x}_7 = \begin{pmatrix} 4.6 \\ 3.4 \\ 1.4 \\ 0.3 \end{pmatrix}, \mathbf{x}_8 = \begin{pmatrix} 5.0 \\ 3.4 \\ 1.5 \\ 0.2 \end{pmatrix}\}$$

먼저 평균벡터를 구하면 $\boldsymbol{\mu} = (4.9125,\ 3.3875,\ 1.45,\ 0.2375\ )^{\mathrm{T}}$이다. 첫 번째 샘플 $\mathbf{x}_1$을 식 (2.39)에 적용하면 다음과 같다.

$$(\mathbf{x}_1 - \boldsymbol{\mu})(\mathbf{x}_1 - \boldsymbol{\mu})^{\mathrm{T}} = \begin{pmatrix} 0.1875 \\ 0.1125 \\ -0.05 \\ -0.0375 \end{pmatrix} (0.1875 \quad 0.1125 \quad -0.05 \quad -0.0375)$$

$$= \begin{pmatrix} 0.0325 & 0.0211 & -0.0094 & -0.0070 \\ 0.0211 & 0.0127 & -0.0056 & -0.0042 \\ -0.0094 & -0.0056 & 0.0025 & 0.0019 \\ -0.0070 & -0.0042 & 0.0019 & 0.0014 \end{pmatrix}$$

나머지 7개 샘플도 같은 계산을 한 다음, 결과를 모두 더하고 8로 나누면 다음과 같은 공분산 행렬을 얻는다.

$$\boldsymbol{\Sigma} = \begin{pmatrix} 0.0661 & 0.0527 & 0.0181 & 0.0083 \\ 0.0527 & 0.0736 & 0.0181 & 0.0130 \\ 0.0181 & 0.0181 & 0.0125 & 0.0056 \\ 0.0083 & 0.0130 & 0.0056 & 0.0048 \end{pmatrix}$$

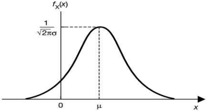
Gaussian Distribution

Univariate

- A r.v. X is called a *normal* (or *Gaussian*) r.v. if its pdf is given by

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}}$$

$$\mu_X = E(X) = \mu$$
$$\sigma_X^2 = \mathrm{Var}(X) = \sigma^2$$



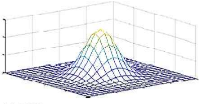
Bivariate

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y(1-\rho^2)^{1/2}} \exp\left[-\frac{1}{2}q(x,y)\right] \qquad q(x,y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]$$

- If the correlation coefficient $\rho = 0$ (i.e., *uncorrelated*), *then* X and Y are *independent*.

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y}\exp\left\{-\frac{1}{2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_X}\exp\left[-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right]\frac{1}{\sqrt{2\pi}\sigma_Y}\exp\left[-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right] = f_X(x)f_Y(y)$$



Gaussian Distribution (cont'd)

Multivariate

- Consider an n-dimensional *random vector* $\mathbf{X} = [X_1 \quad \dots \quad X_n]^T$.
- The random vector is called an ***n-variate normal*** if its joint pdf is given by

$$\boxed{f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\,|\det K|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T K^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \boldsymbol{\mu} = E[X] = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} = \begin{bmatrix} E(X_1) \\ \vdots \\ E(X_n) \end{bmatrix} \qquad K = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix} \qquad \sigma_{ij} = \mathrm{Cov}(X_i, X_j)$$

- Note that $f_{\mathbf{X}}(\mathbf{x})$ stands for $f_{X_1 \cdots X_n}(x_1, \dots, x_n)$.

$\cdot$ ※ If X_i's are uncorrelated, then $K = \begin{bmatrix} \sigma_{11} & & 0 \\ & \ddots & \\ 0 & & \sigma_{nn} \end{bmatrix}$ and $|\det K| = \left|\prod_{i=1}^{n} \sigma_{ii}\right|$ and $f_{\underline{X}}(x) = \prod_{i=1}^{n} f_{X_i}(x_i)$.

Gaussian Distribution (cont'd)

http://cs229.stanford.edu/section/gaussians.pdf

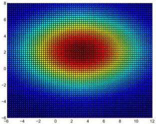
The Diagonal Covariance Matrix (i.e., *Uncorrelated* Gaussian)

- Consider the simple case where n = 2 (i.e., bivariate):

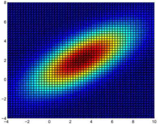
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

$$\begin{aligned}
p(x;\mu,\Sigma) &= \frac{1}{2\pi \begin{vmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{vmatrix}^{1/2}} \exp\left(-\frac{1}{2} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}^T \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}^{-1} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix} \right) \\
&= \frac{1}{2\pi(\sigma_1^2 \cdot \sigma_2^2 - 0 \cdot 0)^{1/2}} \exp\left(-\frac{1}{2} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 \\ 0 & \frac{1}{\sigma_2^2} \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix} \right), \\
&= \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{1}{2} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\sigma_1^2}(x_1-\mu_1) \\ \frac{1}{\sigma_2^2}(x_2-\mu_2) \end{bmatrix} \right) \\
&= \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{1}{2\sigma_1^2}(x_1-\mu_1)^2 - \frac{1}{2\sigma_2^2}(x_2-\mu_2)^2 \right) \\
&= \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{1}{2\sigma_1^2}(x_1-\mu_1)^2 \right) \cdot \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left(-\frac{1}{2\sigma_2^2}(x_2-\mu_2)^2 \right).
\end{aligned}$$

$$\Sigma = \begin{bmatrix} 25 & 0 \\ 0 & 9 \end{bmatrix}$$



$$\Sigma = \begin{bmatrix} 10 & 5 \\ 5 & 5 \end{bmatrix}$$



- In general, an *n-dimensional Gaussian* with mean μ ∈ $\mathbb{R}^n$ & diagonal covariance matrix $\Sigma = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_n^2)$ is the same as the *product of n independent Gaussian* with mean μ_i and variance σ_n^2, respectively.
  - ★ Gaussian의 경우는 uncorrelated면 independent!

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References

- "*Schaum's Outline of Probability, Random Variables, and Random Processes*," by Hwei P. Hsu
- "*기계학습*" by 오일석

Information

$$\boxed{I(x_i) = \log_b \frac{1}{P(x_i)} = -\log_b P(x_i)} \qquad \boldsymbol{X} \in \{x_i\}_{i=1..m}$$

$$\begin{array}{lll} I(x_i) = 0 & \text{for} & P(x_i) = 1 \\ I(x_i) \geq 0 & & \\ I(x_i) > I(x_j) & \text{if} & P(x_i) < P(x_j) \\ I(x_i x_j) = I(x_i) + I(x_j) & & \text{if } x_i \text{ and } x_j \text{ are independent} \end{array}$$

Entropy

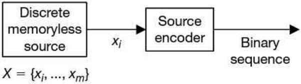
$$\boxed{\begin{aligned} H(X) = E[I(x_i)] &= \sum_{i=1}^{m} P(x_i) I(x_i) \\ &= -\sum_{i=1}^{m} P(x_i) \log_2 P(x_i) \quad \text{b/symbol} \end{aligned}}$$

$0 \le H(X) \le \log_2 m$ (*m*: the number of symbols of the source *X*)

Source Coding Theorem

◆ Source Coding

A conversion of the output of a DMS into a sequence of binary symbols (binary code word) is called *source coding*. The device that performs this conversion is called the *source encoder* (Fig. 10-7).



An objective of source coding is to minimize the average bit rate required for representation of the source by reducing the redundancy of the information source.

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Source Coding Theorem (cont'd)

Average Code Length

Let X be a DMS with finite entropy $H(X)$ and an alphabet $\{x_1, \ldots, x_m\}$ with corresponding probabilities of occurrence $P(x_i)(i = 1, \ldots, m)$. Let the binary code word assigned to symbol x_i by the encoder have length n_i, measured in bits. The length of a code word is the number of binary digits in the code word. The average code word length L, per source symbol, is given by

$$\boxed{L = \sum_{i=1}^{m} P(x_i) n_i}$$

The source coding theorem states that for a DMS X with entropy $H(X)$, the average code word length L per symbol is bounded as (Prob. 10.39)

Source Coding Theorem

$$\boxed{L \geq H(X)} = -\sum_{i=1}^{m} P(x_i) \log_2 P(x_i)$$

lower bound

Source Coding Theorem (cont'd)

Example:

- FLC (Fixed Length Coding) vs. VLC (Variable Length Coding)

X	*a*	*b*	*c*	*d*	*e*	*f*	*g*
P(X)	24/32	2/32	2/32	1/32	1/32	1/32	1/32
I(X)	0.42	4	4	5	5	5	5
FLC (n_X)	000 (3)	001 (3)	010 (3)	011 (3)	100 (3)	101 (3)	110 (3)
VLC (n_X)	0 (1)	10 (2)	110 (3)	1110 (4)	11110 (5)	111110 (6)	1111110 (7)

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Cross-Entropy & KL Divergence

◆ 교차 엔트로피와 상대 엔트로피

- DMS $X = \{x_1, \dots, x_m\}$에 대한 두 개의 확률분포 p(X)와 q(X)를 생각하자.

  ★ *p*: true pdf, *q*: our guess or approximation

- 이때, 확률분포 p에 대한 확률분포 q의 **교차 엔트로피**를 다음과 같이 정의 한다.

$$\boxed{H(p,q) = E_p[I_q(X)] = E_p[-\log(q(X))] = -\sum_{i=1}^{m} p(x_i)\log(q(x_i))}$$

- 이때, 확률분포 q에서 p로의 **Kullback-Leibler (KL) Divergence (상대 엔트로피)**는 다음과 같이 정의 된다.

$$\boxed{D_{KL}(p||q) = \sum_{i=1}^{m} p(x_i)\log\left(\frac{p(x_i)}{q(x_i)}\right) = H(p,q) - H(p,p) \geq 0}$$

  ★ 교차 엔트로피와의 관계 증명:

$$H(p,q) - H(p,p) = \sum_{i=1}^{m} p(x_i)\log(q(x_i)) - \sum_{i=1}^{m} p(x_i)\log(p(x_i)) = \sum_{i=1}^{m} p(x_i)\log\left(\frac{p(x_i)}{q(x_i)}\right) = D_{KL}(p||q)$$

Cross-Entropy & KL Divergence (cont'd)

Example

X	*a*	*b*	*c*	*d*	*e*	*f*	*g*
p(X)	24/32	2/32	2/32	1/32	1/32	1/32	1/32
l_p(X)	0.42	4	4	5	5	5	5
q(X)	16/32	4/32	4/32	4/32	2/32	1/32	1/32
l_q(X)	1	3	3	3	4	5	5

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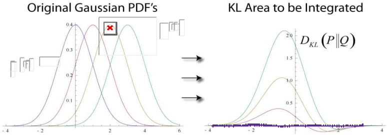
Cross-Entropy & KL Divergence (cont'd)


Original Gaussian PDF's

KL Area to be Integrated

https://en.wikipedia.org/wiki/Kullback%E2%80%93Leibler_divergence#/media/File:KL-Gauss-Example.png

Cross-Entropy & KL Divergence (cont'd)

$$\begin{aligned} H(p,q) &= -p(x_1)\log\left(q(x_1)\right) - p(x_2)\log\left(q(x_2)\right) \\ &= -p(x_1)\log\left(q(x_1)\right) - (1-p(x_1))\log\left(1-q(x_1)\right) \end{aligned}$$

$$H(y,h(x)) = -y\log\left(h(x)\right) - (1-y)\log\left(1-h(x)\right)$$

- **For Multi-class Classification (# classes=K)**

$$H(p,q) = -\sum_{i=1}^{K} p(x_i)\log(q(x_i))$$

$$H(y_i,h(x_i)) = -\sum_{i=1}^{K} y_i \log(h(x_i))$$

★ $y_i \in \{0,1\}$: true label (*true probability*)

★ $h(x_i)$: our *predicted probability* ($0 \le h(x_i) \le 1$)