Isotropic Diffusion via Heat Equation

import library

In [1]:
```
import numpy as np
import matplotlib.image as img
import matplotlib.pyplot as plt
import matplotlib.colors as colors
```

load image

In [2]:
```
I_color = img.imread('222.jpg')   # load color image
```

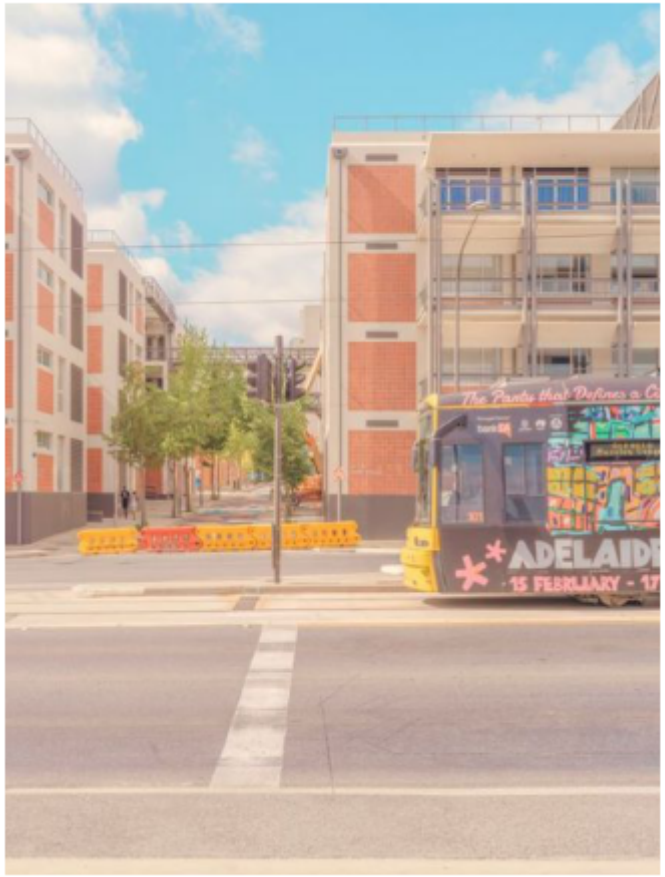
check the size of image and convert 3D to 2D if necessary

In [3]:
```
def rgb2gray(rgb):
    r, g, b = rgb[:,:,0], rgb[:,:,1], rgb[:,:,2]
    gray = 0.3333* r + 0.3333 * g + 0.3333 * b
    return gray


n_dimension = I_color.shape[2]        # number of dimension of data
print(I_color.shape[2])

if n_dimension == 3:
    I = rgb2gray(I_color)

n_dimension = 2    # number of dimension of the convered data
n_row       = I.shape[1]    # number of rows of data
n_column    = I.shape[0]    # number of columns of data


print("number of dimension = ", n_dimension)
print("number of rows = ", n_row)
print("number of columns = ", n_column)

plt.figure(figsize=(16,8))

plt.subplot(121)
plt.imshow(I_color)
plt.title('input image in color')
plt.axis('off')

plt.subplot(122)
plt.imshow(I, cmap = 'gray')
plt.title('input image in grey')
plt.axis('off')
```

```
3
number of dimension =  2
number of rows =  433
number of columns =  577
```

Out[3]:
```
(-0.5, 432.5, 576.5, -0.5)
```

input image in color



input image in grey

```
git commit -a -m "load image and check its size"

git push origin master
```

normalize input image so that the range of image is $[0, 1]$

In [4]:
```
I /= 255
```

```
git commit -a -m "normalise data"

git push origin master
```

define a function to compute the first-order derivatives with Neumann boundary condition

In [5]:
```
print(I)
D = np.pad(I, (1,1), 'constant', constant_values=0)

print(D)

derivative = np.roll(D ,1, axis=0 ) -D

print(derivative)
```

```
[[0.89664235 0.89664235 0.89664235 ... 0.73848824 0.74502353 0.74502353]
 [0.89664235 0.89664235 0.89664235 ... 0.74633059 0.75155882 0.75286588]
 [0.8888     0.8888     0.8888     ... 0.75678706 0.76070824 0.75940118]
 ...
 [0.85350941 0.85350941 0.84958824 ... 0.86527294 0.86919412 0.86658   ]
 [0.86135176 0.86135176 0.85743059 ... 0.86396588 0.86788706 0.86396588]
 [0.86527294 0.85743059 0.85350941 ... 0.85743059 0.85743059 0.85220235]]
[[0.         0.         0.         ... 0.         0.         0.        ]
 [0.         0.89664235 0.89664235 ... 0.74502353 0.74502353 0.        ]
 [0.         0.89664235 0.89664235 ... 0.75155882 0.75286588 0.        ]
 ...
 [0.         0.86135176 0.86135176 ... 0.86788706 0.86396588 0.        ]
 [0.         0.86527294 0.85743059 ... 0.85743059 0.85220235 0.        ]
 [0.         0.         0.         ... 0.         0.         0.        ]]
[[ 0.          0.          0.         ...  0.          0.
   0.        ]
 [ 0.         -0.89664235 -0.89664235 ... -0.74502353 -0.74502353
   0.        ]
 [ 0.          0.          0.         ... -0.00653529 -0.00784235
   0.        ]
 ...
 [ 0.         -0.00784235 -0.00784235 ...  0.00130706  0.00261412
   0.        ]
 [ 0.         -0.00392118  0.00392118 ...  0.01045647  0.01176353
   0.        ]
 [ 0.          0.86527294  0.85743059 ...  0.85743059  0.85220235
   0.        ]]
```

In [6]:
```
def compute_derivative_first_order(data, axis, scheme, boundary):
    if boundary == 'neumann':
        D = np.pad(data , ((1,1),(1,1)), 'constant', constant_values=0)   # use numpy.pad function
    if axis == 'x':
        if scheme == 'forward':
            derivative = np.roll(D ,1, axis=1 ) - D  # use numpy.roll function
        elif scheme == 'backward':
            derivative = np.roll(D,-1, axis=1 ) - D # use numpy.roll function
        else:   # scheme == central
            derivative = np.roll(D,-1, axis=1 ) - D   # use numpy.roll function
    elif axis == 'y':
        if scheme == 'forward':
            derivative = np.roll(D,1, axis=0 ) - D   # use numpy.roll function
        elif scheme == 'backward':
            derivative = np.roll(D,-1, axis=0 ) - D   # use numpy.roll function
        else:   # scheme == central
            derivative = np.roll(D,-1, axis=0 ) - D   # use numpy.roll function
    dD = derivative    # remove the first and the last rows and columns for the boundary condition
    return(dD)
```

```
git commit -a -m "define a function for computing the first-order derivative"
```

```
git push origin master
```

define a function to compute the second-order derivatives with Neumann boundary condition

In [7]:
```
def compute_derivative_second_order(data, axis):

    D_forward   = compute_derivative_first_order(data, axis, 'forward', 'neumann')
    D_backward  = compute_derivative_first_order(data, axis, 'backward', 'neumann')

    dDdD  = D_forward - D_backward   # use the above D_forward and D_backward

    return(dDdD)
```

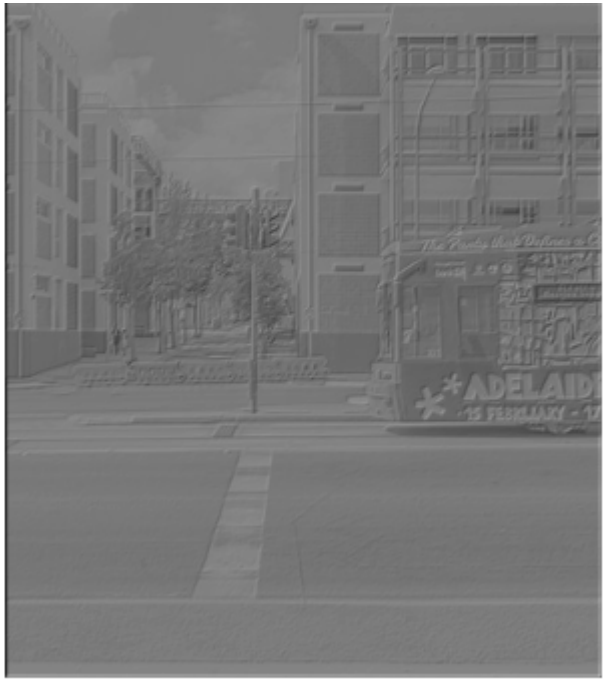
```
git commit -a -m "define a function for computing the second-order derivative"

git push origin master
```

define a function to compute heat equation

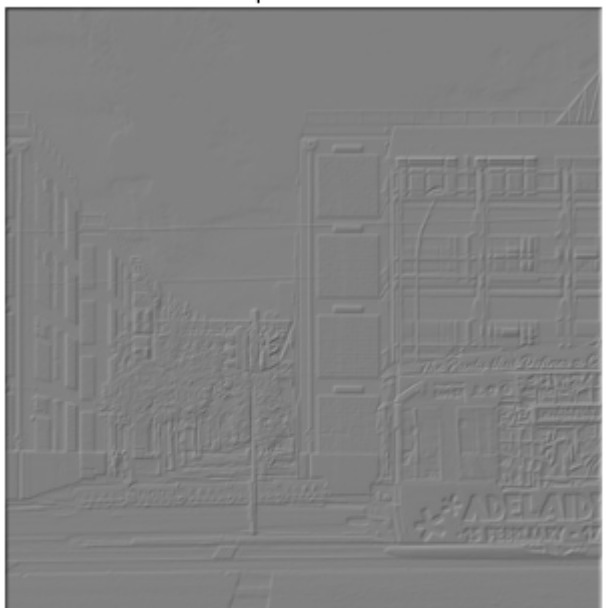
In [25]:
```
def heat_equation(data, delta_t, number_iteration):

    u = np.pad(data , ((1,1),(1,1)), 'constant', constant_values=0)   # initialisation

    for t in range(number_iteration):

        laplace = compute_derivative_second_order(data, axis='x') +compute_derivative_second_order(data, axis='y')  # use the abo
        u       = u +  delta_t*laplace  # compute the heat equation u(t+1) = u(t) + delta_t * laplace(u)

    return(u)
```

In [26]:
```
delta_t = 0.1   # delta t in the computation of heat equation

u10     = heat_equation(I, delta_t, 10)     # compute heat equation with 10 iterations

u100    = heat_equation(I, delta_t, 100)    # compute heat equation with 100 iterations

u1000   = heat_equation(I, delta_t, 1000)   # compute heat equation with 1000 iterations

plt.figure(figsize=(16,16))

# visualise the original image
plt.subplot(221)
plt.imshow(I, cmap='gray')
plt.title('input image')
plt.axis('off')

# visualise the results of heat equation with 10 iterations
plt.subplot(222)
plt.imshow(u10, cmap='gray')
plt.title('heat equation t = 10')
plt.axis('off')

# visualise the results of heat equation with 100 iterations
plt.subplot(223)
plt.imshow(u100, cmap='gray')
plt.title('heat equation t = 100')
plt.axis('off')

# visualise the results of heat equation with 1000 iterations
plt.subplot(224)
plt.imshow(u1000, cmap='gray')
plt.title('heat equation t = 1000')
plt.axis('off')
```

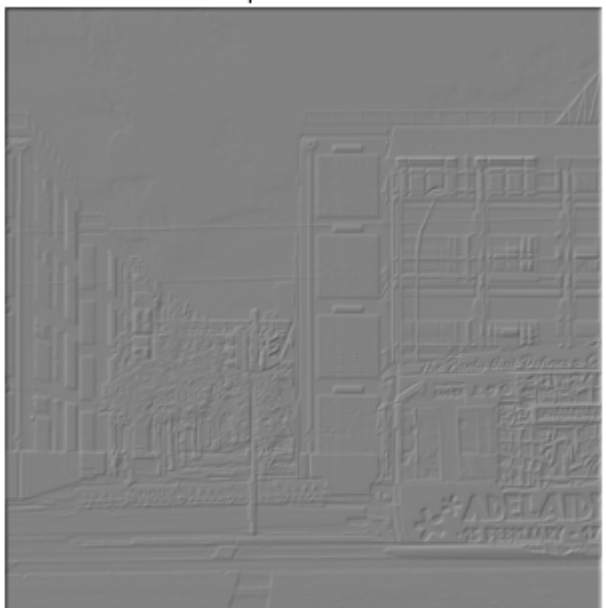
Out[26]: (-0.5, 434.5, 578.5, -0.5)

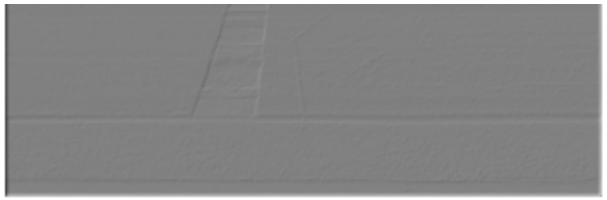


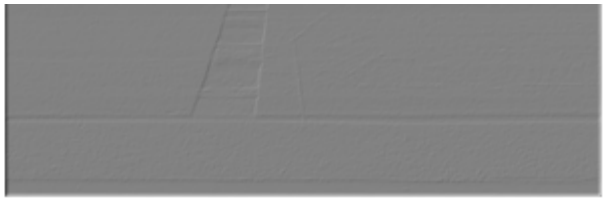
heat equation t = 100



heat equation t = 1000





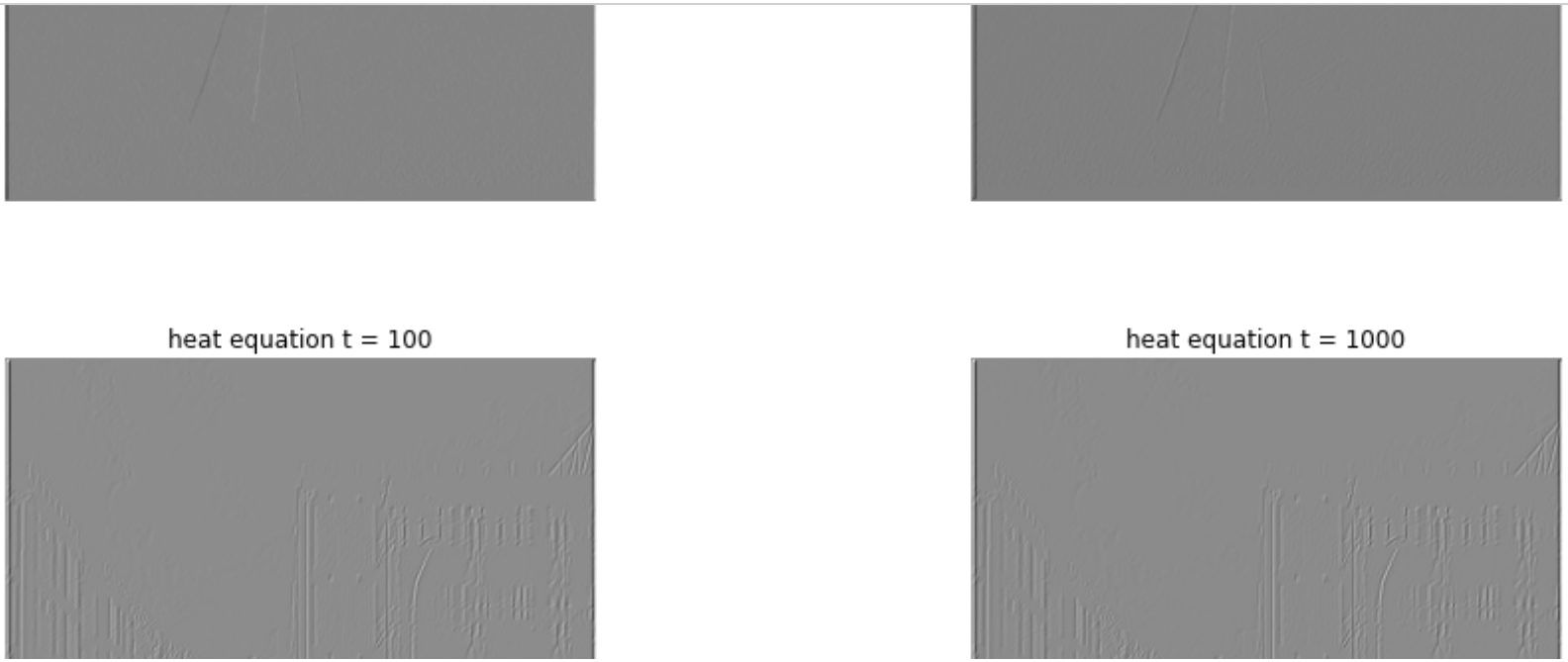


```
git commit -a -m "compute the heat equation"

git push origin master
```

compute the (forward) first-order derivative of the solutions obtained by heat equation

In [10]:
```
Ix      = compute_derivative_first_order(I, 'x', 'forward', 'neumann')   # compute I_x using forward scheme
Iy      = compute_derivative_first_order(I, 'y', 'forward', 'neumann')   # compute I_y using forward scheme

u10_x   = compute_derivative_first_order(u10, 'x', 'forward', 'neumann')   # compute u(10)_x using forward scheme
u10_y   = compute_derivative_first_order(u10, 'y', 'forward', 'neumann')   # compute u(10)_y using forward scheme

u100_x  = compute_derivative_first_order(u100, 'x', 'forward', 'neumann')   # compute u(100)_x using forward scheme
u100_y  = compute_derivative_first_order(u100, 'y', 'forward', 'neumann')    # compute u(100)_y using forward scheme

u1000_x = compute_derivative_first_order(u1000, 'x', 'forward', 'neumann')   # compute u(1000)_x using forward scheme
u1000_y = compute_derivative_first_order(u1000, 'y', 'forward', 'neumann')    # compute u(1000)_y using forward scheme

plt.figure(figsize=(16,16))

# visualise the first-order derivative in x-axis of the original image
plt.subplot(221)
plt.imshow(Ix, cmap='gray')
plt.title('input image')
plt.axis('off')

# visualise the first-order derivative in x-axis of the results of heat equation with 10 iterations
plt.subplot(222)
plt.imshow(u10_x, cmap='gray')
plt.title('heat equation t = 10')
plt.axis('off')

# visualise the first-order derivative in x-axis of the results of heat equation with 100 iterations
plt.subplot(223)
plt.imshow(u100_x, cmap='gray')
plt.title('heat equation t = 100')
plt.axis('off')

# visualise the first-order derivative in x-axis of the results of heat equation with 1000 iterations
plt.subplot(224)
plt.imshow(u1000_x, cmap='gray')
plt.title('heat equation t = 1000')
plt.axis('off')

plt.figure(figsize=(16,16))

# visualise the first-order derivative in y-axis of the original image
```

```
plt.subplot(221)
plt.imshow(Iy, cmap='gray')
plt.title('input image')
plt.axis('off')

# visualise the first-order derivative in y-axis of the results of heat equation with 10 iterations
plt.subplot(222)
plt.imshow(u10_y, cmap='gray')
plt.title('heat equation t = 10')
plt.axis('off')

# visualise the first-order derivative in y-axis of the results of heat equation with 100 iterations
plt.subplot(223)
plt.imshow(u100_y, cmap='gray')
plt.title('heat equation t = 100')
plt.axis('off')

# visualise the first-order derivative in y-axis of the results of heat equation with 1000 iterations
plt.subplot(224)
plt.imshow(u1000_y, cmap='gray')
plt.title('heat equation t = 1000')
plt.axis('off')
```

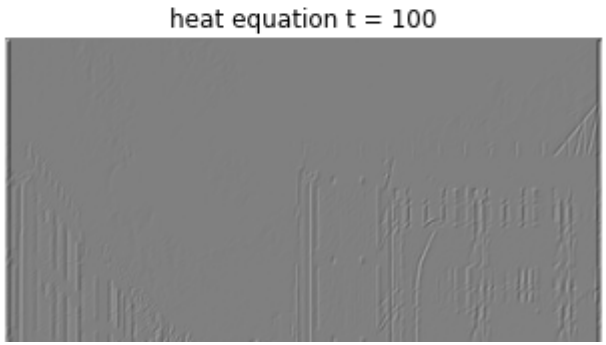


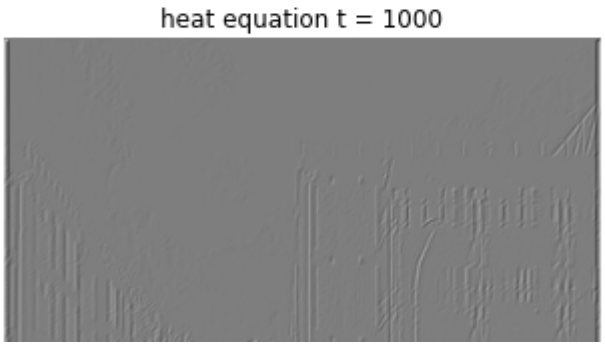
```
git commit -a -m "compute the first-order derivative of the solution from the heat equations"

git push origin master
```

compute the second-order derivative of the solutions obtained by heat equation

In [19]:
```
Ixx         = compute_derivative_second_order(I, 'x' )  # compute second derivative I_xx in x-direction
Iyy         = compute_derivative_second_order(I, 'y' )  # compute second derivative I_yy in y-direction

u10_xx      = compute_derivative_second_order(u10, 'x' )   # compute second derivative u(10)_xx in x-direction
u10_yy      = compute_derivative_second_order(u10, 'y' )   # compute second derivative u(10)_yy in y-direction

u100_xx     = compute_derivative_second_order(u100, 'x' )   # compute second derivative u(100)_xx in x-direction
u100_yy     = compute_derivative_second_order(u100, 'y')   # compute second derivative u(100)_yy in y-direction

u1000_xx    = compute_derivative_second_order(u1000, 'x' )   # compute second derivative u(1000)_xx in x-direction
u1000_yy    = compute_derivative_second_order(u1000, 'y')   # compute second derivative u(1000)_yy in y-direction

plt.figure(figsize=(16,16))

# visualise the second-order derivative in x-axis of the original image
plt.subplot(221)
plt.imshow(Ixx, cmap='gray')
plt.title('input image')
plt.axis('off')

# visualise the second-order derivative in x-axis of the results of heat equation with 10 iterations
plt.subplot(222)
plt.imshow(u10_xx, cmap='gray')
plt.title('heat equation t = 10')
plt.axis('off')

# visualise the second-order derivative in x-axis of the results of heat equation with 100 iterations
plt.subplot(223)
plt.imshow(u100_xx, cmap='gray')
plt.title('heat equation t = 100')
plt.axis('off')

# visualise the second-order derivative in x-axis of the results of heat equation with 1000 iterations
plt.subplot(224)
plt.imshow(u1000_xx, cmap='gray')
plt.title('heat equation t = 1000')
plt.axis('off')

plt.figure(figsize=(16,16))

# visualise the second-order derivative in y-axis of the original image
```

```
plt.subplot(221)
plt.imshow(Iyy, cmap='gray')
plt.title('input image')
plt.axis('off')

# visualise the second-order derivative in y-axis of the results of heat equation with 10 iterations
plt.subplot(222)
plt.imshow(u10_yy, cmap='gray')
plt.title('heat equation t = 10')
plt.axis('off')

# visualise the second-order derivative in y-axis of the results of heat equation with 100 iterations
plt.subplot(223)
plt.imshow(u100_yy, cmap='gray')
plt.title('heat equation t = 100')
plt.axis('off')

# visualise the second-order derivative in y-axis of the results of heat equation with 1000 iterations
plt.subplot(224)
plt.imshow(u1000_yy, cmap='gray')
plt.title('heat equation t = 1000')
plt.axis('off')
```





```
git commit -a -m "compute the second-order derivative of the solution from the heat equations"

git push origin master
```

define functions for the visualization

In [12]:
```
def plot_image_color(I, title = 'title'):

    plt.figure(figsize=(8,8))
    plt.imshow(I)
    plt.title(title)
    plt.axis('off')

def plot_image_gray(I, title = 'title'):

    plt.figure(figsize=(8,8))
    plt.imshow(I, cmap='gray')
    plt.title(title)
    plt.axis('off')

def plot_image_gray_2x1(I1, I2, title1 = 'title 1', title2 = 'title 2'):

    plt.figure(figsize=(16,8))

    plt.subplot(121)
    plt.imshow(I1, cmap = 'gray')
    plt.title(title1)
    plt.axis('off')

    plt.subplot(122)
    plt.imshow(I2, cmap = 'gray')
    plt.title(title2)
    plt.axis('off')

def plot_image_gray_2x2(I1, I2, I3, I4, title1 = 'title 1', title2 = 'title 2', title3 = 'title 3', title4 = 'title 4'):

    plt.figure(figsize=(16,16))

    plt.subplot(221)
    plt.imshow(I1, cmap = 'gray')
    plt.title(title1)
    plt.axis('off')

    plt.subplot(222)
    plt.imshow(I2, cmap = 'gray')
    plt.title(title2)
    plt.axis('off')
```

```
    plt.subplot(223)
    plt.imshow(I3, cmap = 'gray')
    plt.title(title3)
    plt.axis('off')

    plt.subplot(224)
    plt.imshow(I4, cmap = 'gray')
    plt.title(title4)
    plt.axis('off')
```

```
git commit -a -m "define functions for the visualization"

git push origin master
```

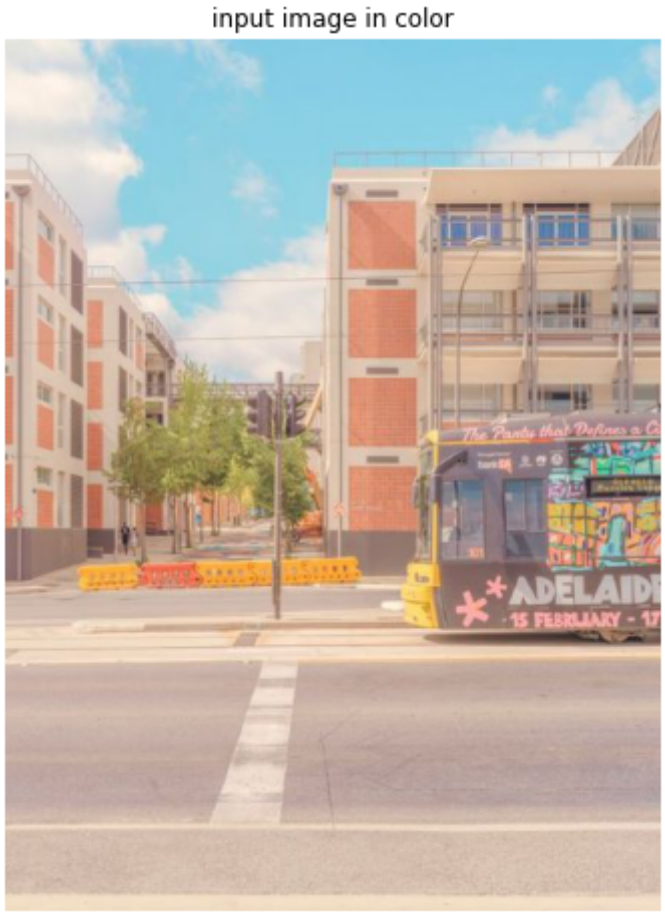
##
#

results

##
#

01. plot the input image in color

In [13]:
```
plot_image_color(I_color, 'input image in color')
```

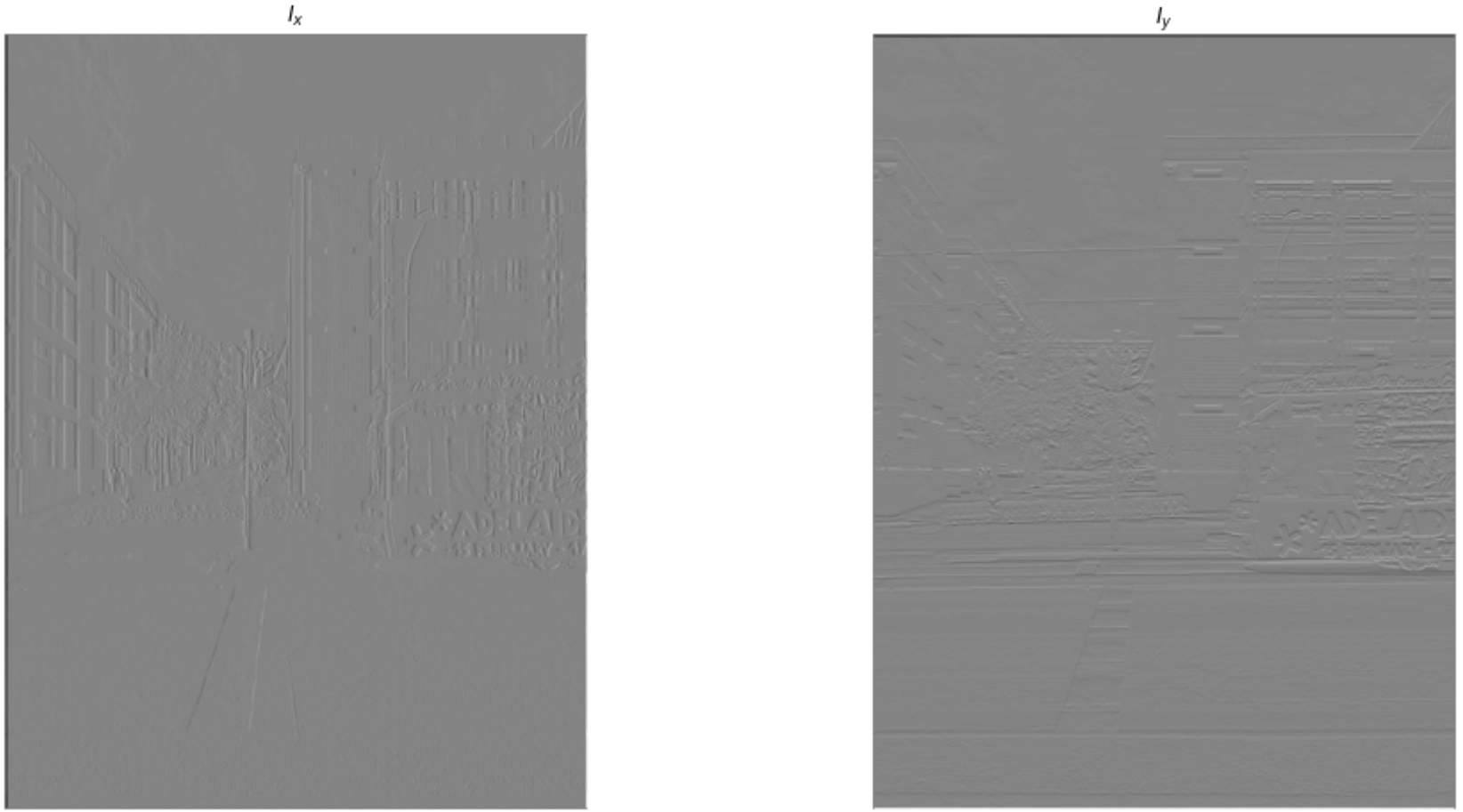


 # 02. plot the input image in gray

In [14]:
```
plot_image_gray(I, 'input image in grey')
```

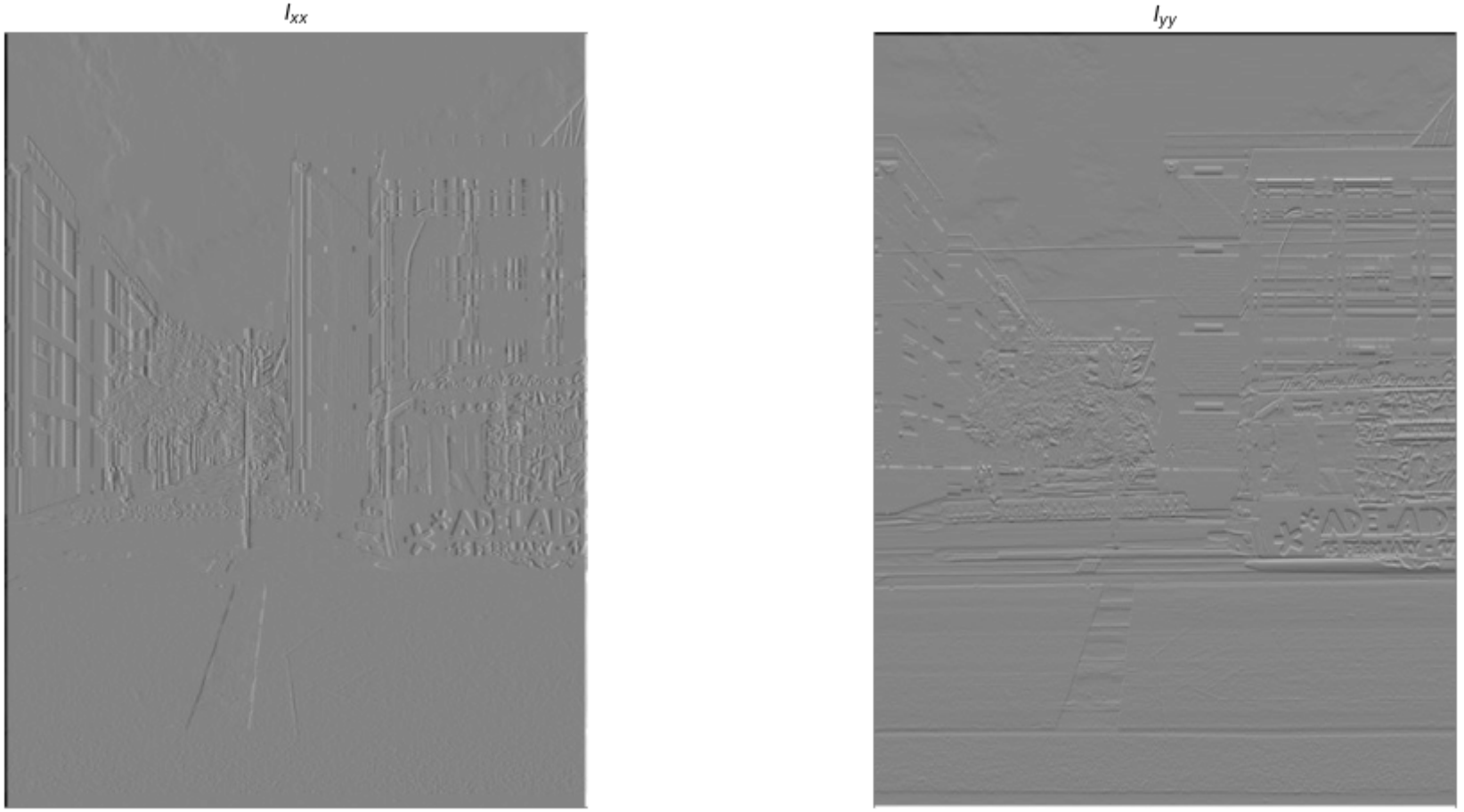
 # 03. plot the (forward) first derivative I_x and I_y of input gray image I with Neumann boundary condition

In [15]:
```
plot_image_gray_2x1(Ix, Iy, '$I_x$','$I_y$')
```



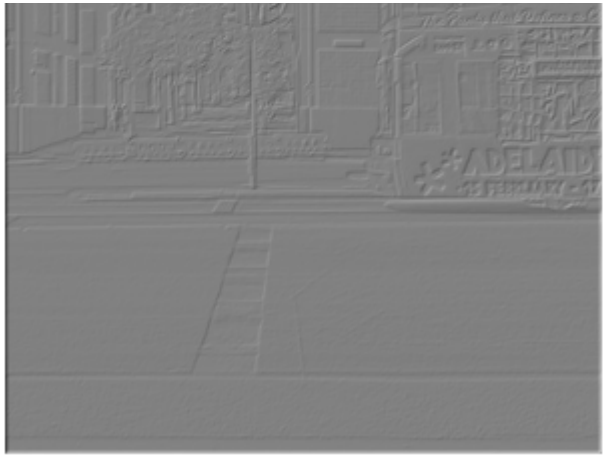
 # 04. plot the second derivative I_{xx} and I_{yy} of input gray image I with Neumann boundary condition

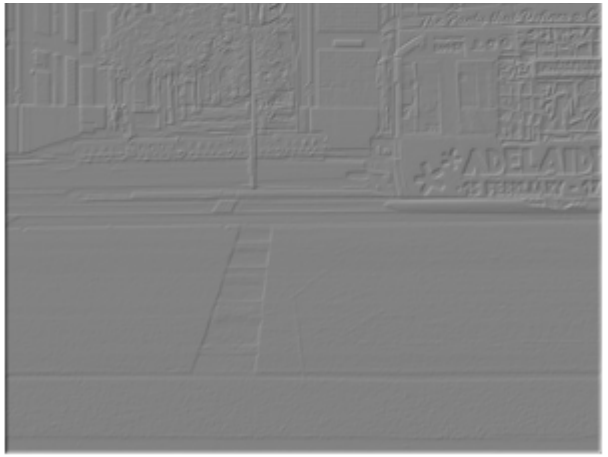
In [20]:
```
plot_image_gray_2x1(Ixx, Iyy, '$I_{xx}$', '$I_{yy}$')
```



 # 05. plot the original image and its solution of the heat equation with 10, 100, 1000 iterations

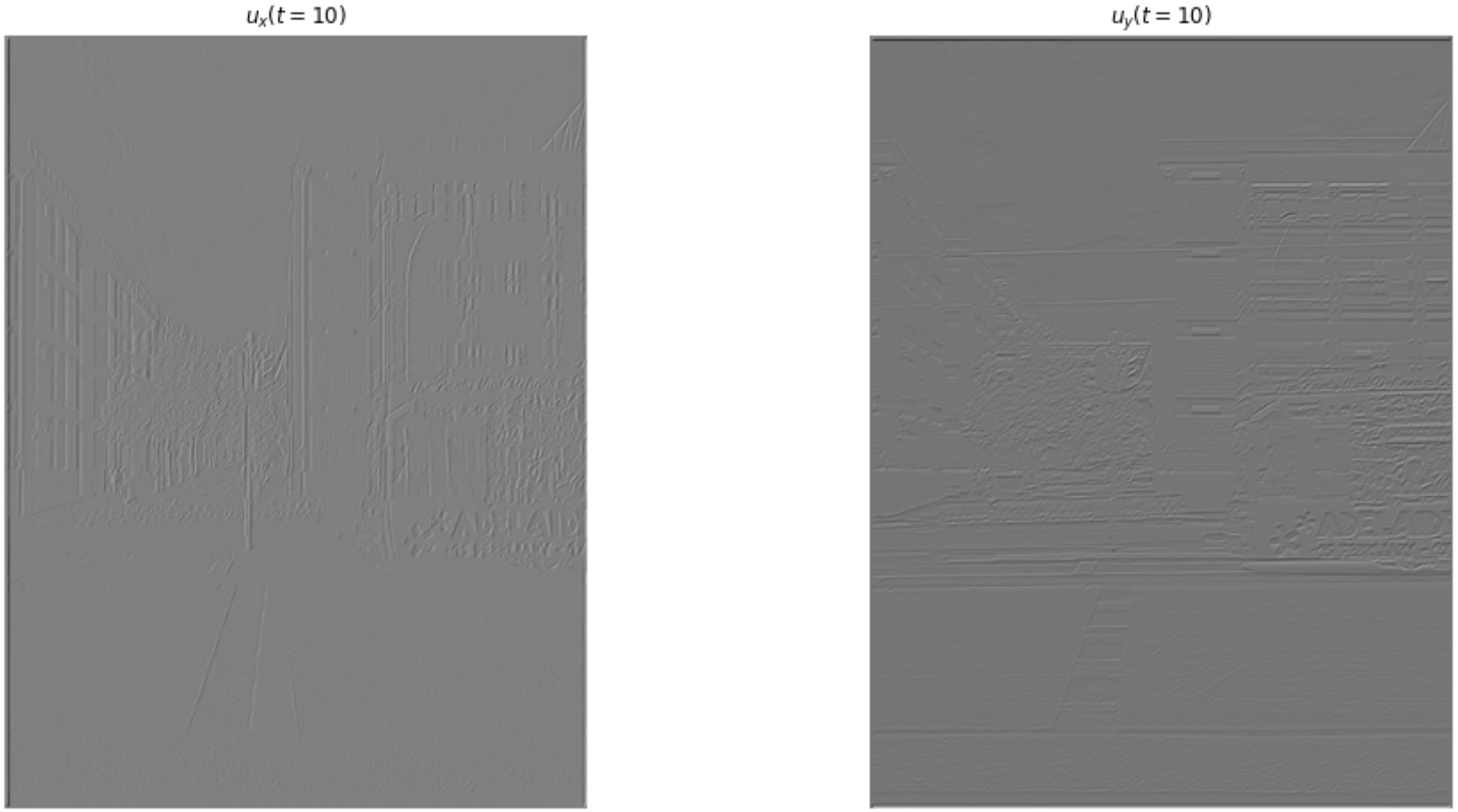
In [21]:
```
plot_image_gray_2x2(I, u10, u100, u1000, 'original $I$', '$u(t=10)$', '$u(t=100)$', '$u(t=1000)$')
```





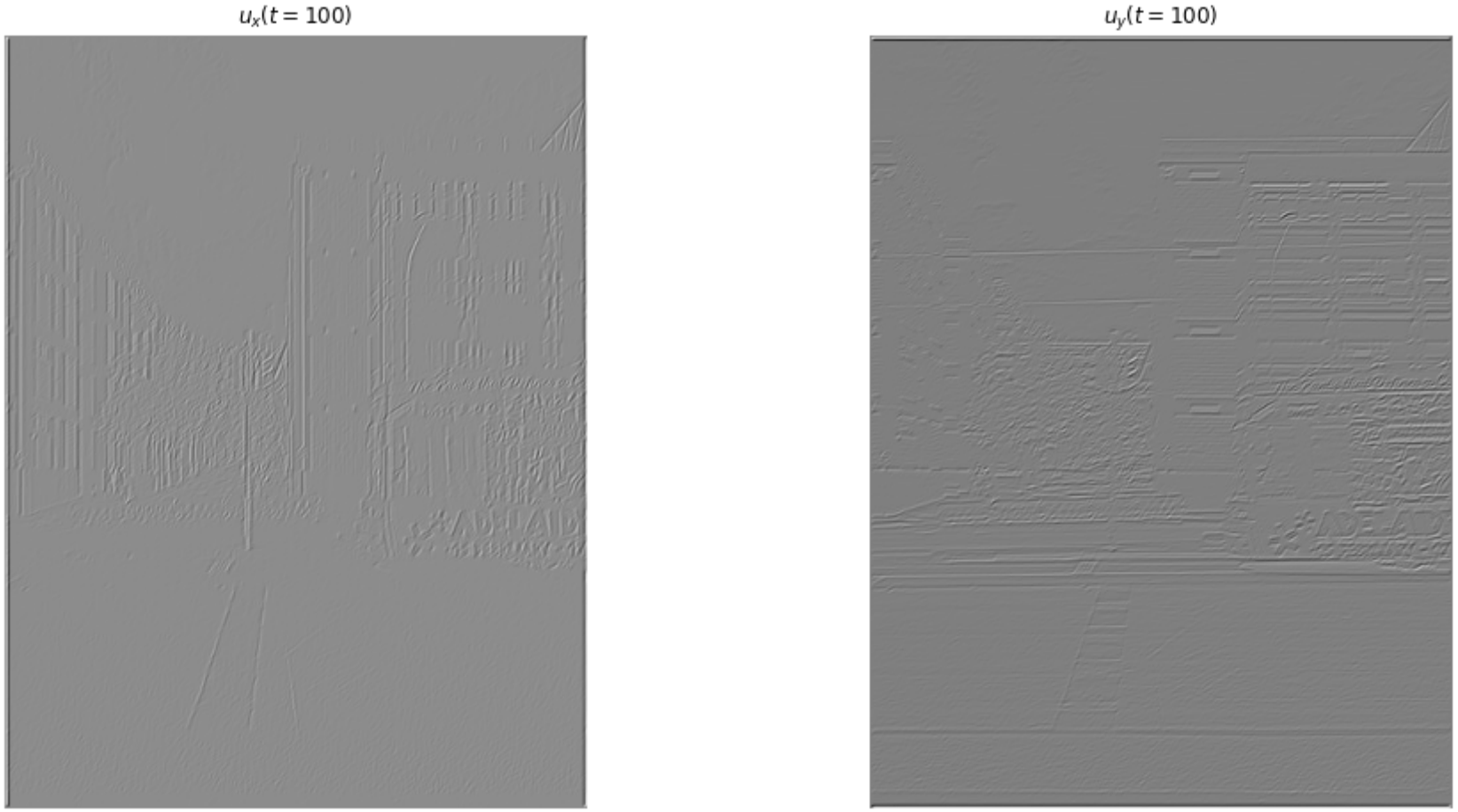
 # 06. plot the (forward) first derivative of the solution of the heat equation with 10 iterations

In [22]:
```
plot_image_gray_2x1(u10_x, u10_y, '$u_x(t=10)$', '$u_y(t=10)$')
```



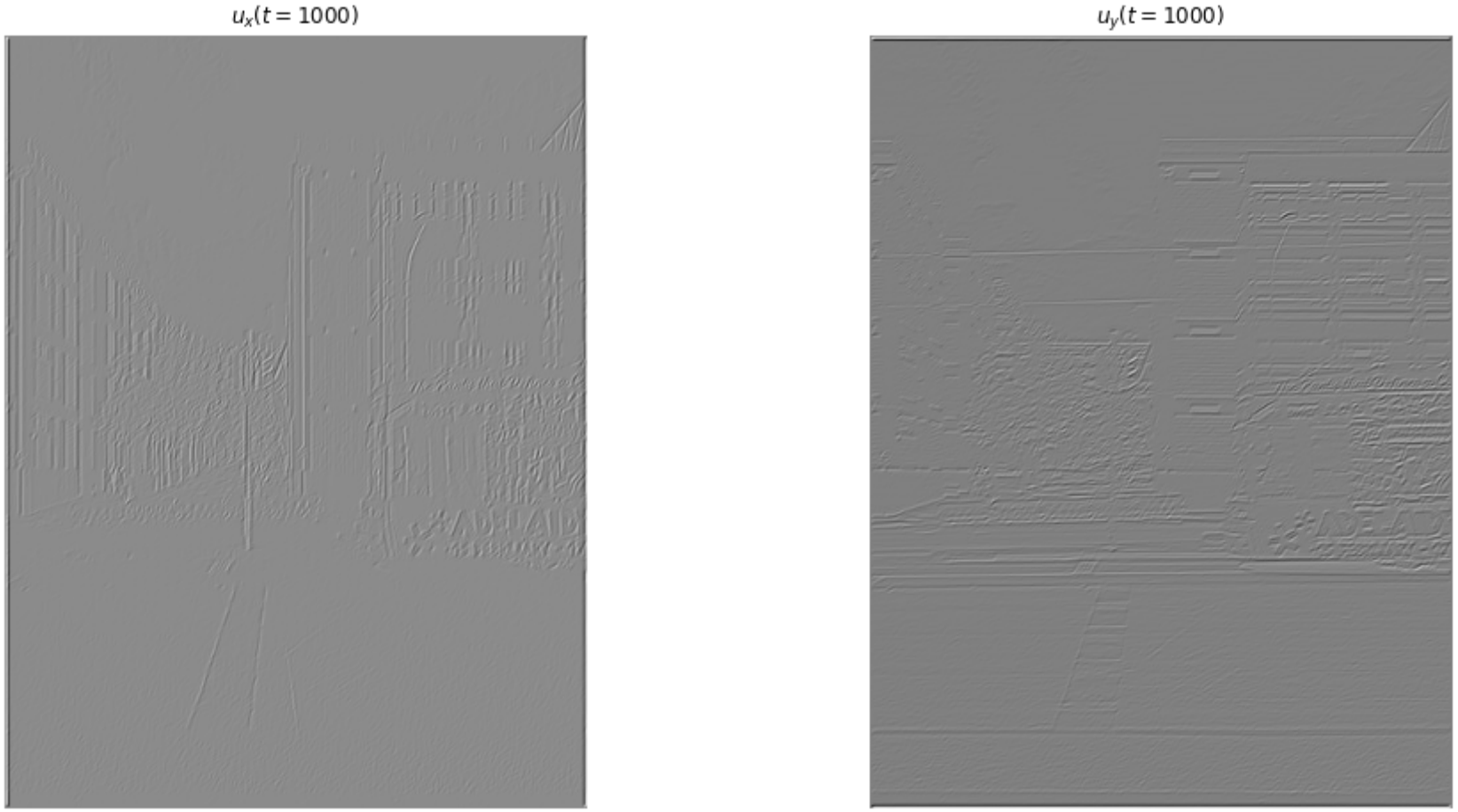
 # 07. plot the (forward) first derivative of the solution of the heat equation with 100 iterations

In [23]:
```
plot_image_gray_2x1(u100_x, u100_y, '$u_x(t=100)$', '$u_y(t=100)$')
```



 # 08. plot the (forward) first derivative of the solution of the heat equation with 1000 iterations

In [24]:
```
plot_image_gray_2x1(u1000_x, u1000_y, '$u_x(t=1000)$', '$u_y(t=1000)$')
```



In [ ]: