이항계수의 합

N, M이 주어질 때,

$$\sum_{i=0}^{N}\sum_{j=0}^{M}\binom{i+j}{i} \qquad (0 \le N, M \le 1,000,000)$$

의 합을 빠르게 계산하는 프로그램 작성. 계산해야 하는 값이 너무 커질 수 있기 때문에, $\underbrace{1,000,000,007}_{\text{소수}}$로 나눈 나머지 출력.

빠르게 계산하기 위해 식을 간단히 정리해야 한다.

- 이항계수의 성질 이용
- 모듈러 연산의 성질 및 정리 이용.

$$\sum_{i=0}^{N}\sum_{j=0}^{M}\binom{i+j}{i} = \binom{N+M+2}{N+1} - 1$$

pf) $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ ($\because$ 파스칼의 삼각형)

$$\Rightarrow \binom{n}{r} = \binom{n+1}{r+1} - \binom{n}{r+1}$$

$$\sum_{i=0}^{N}\sum_{j=0}^{M}\binom{i+j}{i} = \sum_{i=1}^{N}\sum_{j=1}^{M}\binom{i+j}{i} + \sum_{j=0}^{M}\binom{j}{0} + \sum_{i=0}^{N}\binom{i}{i} - \binom{0+0}{0}$$

$$= \underline{\sum_{i=1}^{N}\sum_{j=1}^{M}\binom{i+j}{i}} + M+N+1$$

$$\sum_{i=1}^{N}\sum_{j=1}^{M}\binom{i+j}{i} = \sum_{i=1}^{N}\sum_{j=1}^{M}\left\{\binom{i+j+1}{i+1} - \binom{i+j}{i+1}\right\}$$

$$= \sum_{i=1}^{N}\Big(\cancel{{}_{i+2}C_{i+1}} - {}_{i+1}C_{i+1} + \cancel{{}_{i+3}C_{i+1}} - \cancel{{}_{i+2}C_{i+1}} +$$

$$\cdots {}_{i+m+1}C_{i+1} - \cancel{{}_{i+m}C_{i+1}}\Big)$$

$$= \sum_{i=1}^{N} \left({}_{i+m+1}C_{i+1} - {}_{i+1}C_{i+1} \right)$$

$$= \sum_{i=1}^{N} \left({}_{i+m+1}C_{i+1} - 1 \right) = \sum_{i=1}^{N} \binom{i+m+1}{i+1} - N \quad \cdots (1)$$

by (1), $$\sum_{i=1}^{N} \sum_{j=1}^{m} \binom{i+j}{i} + m + N + 1 = \underline{\sum_{i=1}^{N} \binom{i+m+1}{i+1}} + m + 1$$

$$\sum_{i=1}^{N} {}_{i+m+1}C_{i+1} = \sum_{i=1}^{N} {}_{i+m+1}C_{m}$$

$$= \sum_{i=1}^{N} {}_{i+m+2}C_{m+1} - {}_{i+m+1}C_{m+1}$$

$$= \cancel{{}_{m+3}C_{m+1}} - {}_{m+2}C_{m+1}$$

$$+ \cancel{{}_{m+4}C_{m+1}} - \cancel{{}_{m+3}C_{m+1}}$$

$$\vdots$$

$$+ {}_{m+N+2}C_{m+1} - \cancel{{}_{m+N+1}C_{m+1}}$$

$$= {}_{m+N+2}C_{m+1} - {}_{m+2}C_{m+1}$$

$$= {}_{m+N+2}C_{m+1} - {}_{m+2}C_{1}$$

$$= {}_{m+N+2}C_{m+1} - (m+2) \cdots (2)$$

by (2), $$\sum_{i=1}^{N} \binom{i+m+1}{i+1} + m + 1 = \binom{m+N+2}{m+1} - 1 \;//$$

모듈러 연산의 곱셈 성질

$AB \pmod C \equiv A \pmod C \times B \pmod C$ ...

페르마의 소정리

p: prime, a: integer

$$a^p \equiv a \pmod p$$

if $p \nmid a$, (a가 p의 배수가 아니면)

$$a^{p-1} \equiv 1 \pmod p \quad (a \neq 0)$$

let $A = (N+M+2)!$, $B = (N+1)!\,(M+1)!$, $P = 1{,}000{,}000{,}007$

$$\Rightarrow \binom{N+M+2}{N+1} = \frac{A}{B}$$

$$\Rightarrow \binom{M+N+2}{N+1} - 1 = \frac{A}{B} - 1$$

$$\frac{A}{B} - 1 \pmod p \equiv \left\{ \frac{A}{B} \pmod p - 1 \pmod p \right\} \pmod p$$

$$\equiv \left[\{A \pmod p \cdot B^{-1} \pmod p\} \pmod p - 1 \right] \pmod p$$

모듈러 연산의 결합법칙은 나눗셈에 닫혀있지 않으므로,
곱셈의 역원을 이용.

By Fermat's little thm,

$$B^{p-1} \equiv 1 \pmod p$$

$$\Rightarrow B^{p-2} \equiv B^{-1} \pmod p \quad \cdots (3)$$

by (3), $(A \% P \cdot B^{-1} \% P - 1) \% P = (A \% P \cdot B^{p-2} \% P - 1) \% P$ //

B^{P-2} % P 효율적으로 구하는 방법

- P-2를 이진수로 나타낸다.

ex) $B=2$, $P=13$.

$2^{11} = 2^8 \cdot 2^2 \, 2^1$ $(\because 11 = 1011_{(2)})$

$2^{11} \equiv 2^8 \cdot 2^2 \cdot 2^1 \pmod{P}$

$\equiv \{2^8 \pmod{P} \cdot 2^2 \pmod{P} \cdot 2 \pmod{P}\} \pmod{P}$