Harthshorne Reading Study

at KAIST Mathematical Problem Solving Group

20190262 Jeongwoo Park

Period: 2022 December - 2023 January

Section II.1

Exercise (II.1.8). For any open subset $U \subset X$, show that the functor $\Gamma(U, \_)$ from sheaves on X to abelian groups is a left-exact functor. The functor $\Gamma(U, \_)$ need not be exact.

First Solution (for beginners). The sequence

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \xrightarrow{g} \mathscr{F}''$$

is exact means $\mathscr{F}$ is the kernel object of the second map g, i.e, there is an isomorphism $\alpha : \mathscr{F}' \to \ker g$ such that

$$\begin{tikzcd} \mathscr{F}' \arrow[r] \arrow[dr, "\sim"'] & \mathscr{F} \arrow[r, "g"] & \mathscr{F}'' \\ & \ker g \arrow[u, hook] & \end{tikzcd}$$

commutes. Taking the global section gives a commutative diagram

$$\begin{tikzcd} \mathscr{F}'(X) \arrow[r] \arrow[dr, "\sim"'] & \mathscr{F}(X) \arrow[r, "g_X"] & \mathscr{F}''(X) \\ & (\ker g)(X) \arrow[u, hook] & \end{tikzcd}$$

Since the global section functor and the kernel commutes, we know that $(\ker g)(X) = \ker g_X$. Hence, $\mathscr{F}'(X)$ is a kernel object of the map g_X, so the sequence

$$0 \longrightarrow \mathscr{F}'(X) \longrightarrow \mathscr{F}(X) \xrightarrow{g_X} \mathscr{F}''(X)$$

is exact. □

Second Solution (keep in mind). A functor is exact if and only it it commutes with the kernel. □

Third Solution (for those who are interested in category theory).

1. The global section functor PSh → Ab is exact.

2. The forgetful functor Sh → PSh is left-exact. Indeed, it is right adjoint to the sheafification functor PSh → Sh.

By combining these two results, the global section functor Sh → Ab is left-exact. □

[[Exercise II.1.16 solution]]

Problems

Section II.1

1. Let
$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$
be a sequence of sheaves.
    (a) Explain how's different the exactness of the sequence in $\underline{\mathrm{Sh}}$ and in $\underline{\mathrm{PSh}}$. Do the exactness in one category implies one in the another category?
    (b) (With some knowledge in cohomology theory) Why the exactness of the global section is guaranteed only by the first term $\mathscr{F}'$?
    (c) Why the sequence
$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathscr{H} \longrightarrow \mathscr{H}^* \longrightarrow 0$$
is <u>not exact</u> after taking the global section functor? How this fact is related with the sheaf cohomology?
(Here, $\mathscr{H} \in \underline{\mathrm{Sh}}(\mathbb{C})$ is the sheaf of holomorphic functions, and $\mathscr{H}^* \in \underline{\mathrm{Sh}}(\mathbb{C})$ is one of <u>invertiable</u> holomorphic functions.)

Properties in Philosophy

Proposition 1. *(adjoint functors) Following pairs are adjoint.*

1. *Sheafification* $\dashv$ *Forgetful functor* : $\underline{PSh} \rightleftarrows \underline{Sh}$
2. *Constant presheaf functor* $\dashv$ *Global section functor* : $\underline{Ab} \rightleftarrows \underline{PSh}$

1+2. *Constant sheaf functor* $\dashv$ *Global section functor* : $\underline{Ab} \rightleftarrows \underline{Sh}$

3. *Forgetful functor* $\dashv$ *Reduction functor* : $\underline{Sch} \rightleftarrows \underline{Sch}_{\mathrm{red}}$

[[adjunction and pullback/pushout? (co)limit? commutes... Gluing? Refuction of schemes?]]

Section II.1

1. An abelian functor is left-exact (resp. right-exact) if and only if it commutes with the kernel (resp. cokernel).