Real Analysis

— Solution of Homework 1 —

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1 Solutions

Solution of the Exercise 1.4. Let $U_{i,j}$ be the jth open interval removed at the ith stage where $0 \le j \le 2^{i-1} - 1$, and let $C_{i,j}$ be the jth closed interval at the ith stage where $0 \le j \le 2^i - 1$. Of course, these intervals are non-empty. Then, we have properties

1. The disjoint union $[0,1] \setminus \widehat{\mathcal{C}} = \coprod_{i\ge1} \coprod_{j=0}^{2^{i-1}-1} U_{i,j}$ and $C_{i,j} = C_{i+1,2j} \amalg U_{i+1,j} \amalg C_{i+1,2j+1}$ holds.
2. The length or the Lebesgue measure of $U_{i,j}$ is ℓ_i, and the length of $C_{i,j}$ converges to 0 as i goes to ∞.

(a) Since every intervals are measurable, we have

$$m(\widehat{\mathcal{C}}) = m\left([0,1] \setminus \coprod_{i\ge1} \coprod_{j=0}^{2^{i-1}-1} U_{i,j}\right) = m([0,1]) - \sum_{i\ge1} \sum_{j=0}^{2^{i-1}-1} \mu(U_{i,j}) = \sum_{i\ge1} = 1 - \sum_{i\ge1} 2^{i-1}\ell_i$$

This completes the proof of the identity.

(b) Let $x \in \widehat{\mathcal{C}}$, then for each $i \ge 1$, there is $0 \le j_i \le 2^i - 1$ such that $x \in C_{i,j_i}$. Note that

$$\bigcap_{i\ge1} C_{i,i_j} = \{x\}$$

since the diameter of C_{i,i_j} converges to 0. By the property 1, we can know that $j_{i+1} \in \{2j_i, 2j_i+1\}$. Now, pick any element $x_i \in U_{i-1,j_i} \subseteq C_{i,j_i}$, then it must converge to x. Also, U_{i,j_i} are subintervals of $\widehat{\mathcal{C}}$, and their length converges to 0. This completes the proof.

(c) Of course, $\widehat{\mathcal{C}}$ is closed since it is a complement of an open set in the space $[0,1]$. Also, there is no isolated point because if we take $x_i \in C_{i,j_i} \setminus x$ in the proof of the part b, then it must converge to x. Hence, $\widehat{\mathcal{C}}$ is perfect.

The space $\widehat{\mathcal{C}}$ is interior-empty because the length of $C_{i,j}$ converges to 0 as i goes to infinity.

(d) We need a lemma.

Lemma 1.1. *Every non-empty perfect complete metric space is uncountable.*

Proof of Lemma 1.1. Let (X, d) be a non-empty perfect complete metric space. By the Baire category theorem, (X, d) must be a Baire space. Assume that the space X is countable, and let's write $X = \{x_i\}_{i\in\mathbb{N}}$. Of course, X can't be finite because every point in a finite Hausdorff space is isolated. Since there is no isolated point, open sets $U_j := \{x_i\}_{i \geq j}$ is dense. However, the intersection of all U_j's gives an empty set, which is not dense in X. This contradicts to the fact that X is a Baire space, and this completes the proof of the lemma. $\square$

Since $\widehat{\mathcal{C}}$ is a closed subset of a complete metric space $\mathbb{R}$, it is a complete metric space. Thus, $\widehat{\mathcal{C}}$ is a non-empty perfect complete metric space, so we can deduce that this space is uncountable. This completes the proof.

$\square$

Solution of Exercise 1.7. First, I'll show that $m^*(\delta \cdot E) = \delta_1 \delta_2 \cdots \delta_d \cdot m^*(E)$. Note that δ induces a bijection

$$\{\text{Hypercubes containing } E\} \xrightarrow{\delta \cdot \bullet} \{\text{Hypercubes containing } \delta \cdot E\}$$

Hence, we can conclude that

$$\begin{aligned} m^*(\delta \cdot E) &= \inf\{|\delta \cdot E| \, ; \, E \text{ is a hypercube containing } E\} \\ &= \inf\{\delta_1 \delta_2 \cdots \delta_d \cdot |E| \, ; \, E \text{ is a hypercube containing } E\} \\ &= \delta_1 \delta_2 \cdots \delta_d \cdot m^*(E) \end{aligned}$$

Now, the remainder is to show that the set $\delta \cdot E$ is measurable. By the definition, for every $\varepsilon > 0$, there is an open set $U \supseteq E$ such that $m^*(U \setminus E) < \frac{\varepsilon}{\delta_1 \delta_2 \cdots \delta_d}$. Also, $\delta \cdot U$ is an open set containing E, and we have an inequality

$$m^*(\delta \cdot U \setminus \delta \cdot E) = m^*(\delta(U \setminus E) = \delta_1 \delta_2 \cdots \delta_d \cdot m^*(U \setminus E) < \varepsilon$$

Hence, $\delta \cdot E$ must be measurable. This completes the proof. $\square$

Solution of Exercise 1.24. We define sets $A_i := \left\{ n \in \mathbb{Z}_{>0} \, ; \, 2^i \| n \right\}$[1] and $B_i := \mathbb{Q} \cap \left(B_{\frac{1}{i-1}}(e) \setminus B_{\frac{1}{i}}(e) \right)$ where e is any irrational number, and we define $B_{\frac{1}{0}} := \mathbb{R}$. Then, the A_i's (respectively, B_i's) forms a partition of $\mathbb{Z}_{>0}$ (respectively, $\mathbb{Q}$). Also, every A_i and B_i is countably infinite, so we can find a bijection $f_i : A_i \to B_i$. By gluing all f_i's, we can define an bijection $f : \mathbb{Z}_{>0} \to \mathbb{Q}$ satisfying $f|_{A_i} = f_i$. In particular, $|f(n) - e| > \frac{1}{n}$ holds, because there is i such that $A_i \ni n$, and we have $n \geq 2^i$ because $2^i | n$, and so $|f(n) - e| > \frac{1}{i} > \frac{1}{n}$ holds. Hence, $e \notin \left(f(n) - \frac{1}{n}, f(n) + \frac{1}{n} \right)$ for all n, and so $e \notin \bigcup_{n \geq 1} \left(f(n) - \frac{1}{n}, f(n) + \frac{1}{n} \right)$. This completes the proof. $\square$

Solution of Exercise 1.35. We need a lemma.

Lemma 1.2. *Every set $A \subseteq \mathbb{R}$ of positive outer measure has a non-measurable subset.*

Proof of Lemma 1.2. We can assume that A is measurable, and moreover bounded because

$$\lim_{R \to \infty} m^*([-R, R] \cap A) = m^*(A)$$

implies that there is $R > 0$ such that $m^*([-R, R] \cap A) > 0$, and every subset of $[-R, R] \cap A$ is also a subset of A. Assume that A is bounded by $R > 0$.

[1]The relation $\|$ denotes the exact divisibility, i.e., for any prime number p and natural numbers k and n, the relation $p^k \| n$ means that $p^k \mid n$ but $p^{k+1} \nmid n$.

Let's consider the equivalence relation on A inherited from the equivalence relation on $\mathbb{R}$ defined by the natural $\mathbb{Q}$-action on $\mathbb{R}$, as in the construction of a Vitali set. Pick a set of representatives $A' \subseteq A$, then we can obtain that

$$A \subseteq \coprod_{q \in [-2R, 2R] \cap \mathbb{Q}} q + A' \subseteq [-3R, 3R]$$

If A' is measurable, then the inequality

$$0 < m(A) \leq \sum_{q \in [-2R, 2R] \cap \mathbb{Q}} m(A') \leq 6R$$

holds. However, if $m(A') = 0$, this contradicts to the left inequality, and if $m(A') > 0$, then this contradicts to the right inequality. Thus, $A' \subseteq A$ is non-measurable. This completes the proof. $\square$

We consider a continuous function $f : x \mapsto x + c(x)$ where c is the Cantor–Lebesgue function. Then, $f^{-1}([0,1] \setminus \mathcal{C}) = 2 \cdot \left([0,1] \setminus \widehat{\mathcal{C}}\right)$ where $\widehat{\mathcal{C}}$ is the Cantor-like set defined by the sequence $\ell_i = \frac{1}{2 \cdot 3^i}$ as in the **Exercise 1.4**. By a simple calculation, $m\left(f^{-1}(\mathcal{C})\right) = 1$. By the lemma above, there is a non-measurable subset A of $f^{-1}(\mathcal{C})$, but $f(A) \subseteq \mathcal{C}$ is measurable because it must be a null-set. Now, we consider the composition

$$[0,1] \xrightarrow{f} [0,2] \xrightarrow{\chi_{f(A)}} [0,1]$$

This composition is not measurable because if we consider an inverse image of the set $\{1\} \subseteq [0,1]$, then we have the non-measurable set $f^{-1}(f(A)) = A$ since f is injective. This completes the proof. $\square$

Solution of Problem 1.4. (a) It is enough to show that the set given in the problem is closed because every closed subset of a compact set is compact. In the other word, it is enough to show that the set

$$A_\varepsilon^c = \{c \in J \,;\, \operatorname{osc}(f, c) < \varepsilon\}$$

is open. Let $c \in A_\varepsilon^c$, then there is a small $\delta > 0$ such that $\operatorname{osc}(f, c, r) < \varepsilon$ for all $0 < r < \delta$. By definition of the oscillation function, for any $c' \in]c - \delta, c + \delta[$, we have $\operatorname{osc}(f, c', r') \leq \operatorname{osc}(f, c, r) < \varepsilon$ for all $0 < r' < \min\{|c' - c - \delta|, |c' - c + \delta|\}$. Since the function $\operatorname{osc}(f, c, \bullet)$ is non-increasing function, we can conclude that $]c - \delta, c + \delta[\subseteq A_\varepsilon^c$, hence A_ε^c is open.

(b) Suppose that the function is bounded by $M > 0$, and let $\varepsilon > 0$. We can take finitely many disjoint open intervals I_i's satisfying $A_{\frac{\varepsilon}{m(J)}} \subseteq \bigcup_{i=1}^n I_i$ and $\sum_{i=1}^n m(I_i) < \frac{\varepsilon}{4M}$ since A_ε is a null-set because it is a subset of a collection of all the discontinuities, and A_ε is compact. For point c outside of I_i's, we define $r_c > 0$ as a number such that $\operatorname{osc}(f, c, r_c) < \frac{\varepsilon}{2m(J)}$. Of course, $\{I_i\}_{1 \leq i \leq n} \cup \{B_{r_c}(c)\}_{c \notin \bigcup_i I_i}$ is an open covering of the compact interval J, so there is a countable subcovering. If we define a partition of J as the set of all boundary points of the countable subcovering, then we can conclude that

$$U(f, P) - L(f, P) \leq \sum_{I \,;\, I \subseteq \bigcup_i I_i} 2M \cdot m(I) + \sum_{I \,;\, I \subsetneq \bigcup_i I_i} \frac{\varepsilon}{2m(J)} \cdot m(I) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

where I runs over the subinterval of the partition P. This completes the proof.

(c) Note that the set of all the discontinuities is $\bigcup_{n=1}^\infty A_{\frac{1}{n}}$. Let P be a partition of the interval J such that $U(f, P) - L(f, P) < \frac{\varepsilon}{n}$ where $\varepsilon > 0$, and n is any positive integer. Then, we can conclude that

$$\frac{\varepsilon}{n} > U(f,P) - L(f,P) \geq \sum_{I\,;\,I\cap A_{\frac{1}{m}} \neq \varnothing} \operatorname{osc}\left(f, c \in I \cap A_{\frac{1}{n}}\right) \cdot m(I) \geq \frac{m\left(A_{\frac{1}{n}}\right)}{n}$$

or equivalently, $m\left(A_{\frac{1}{n}}\right) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we can conclude that $m\left(A_{\frac{1}{n}}\right) = 0$, hence the set of all the discontinuities is a null-set.

$\square$