Introduction to Commutative Algebra

— Solution of Homework 5 —

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1 Solutions

Solution of Problem 1. We need a lemma.

Lemma 1.1. *Let $\langle a \rangle \trianglelefteq \mathbb{Z}$ be a non-zero ideal. If $a = \prod_{i=1}^{m} p_i^{e_i}$ is the prime factorization of a where $p_\bullet$ are distinct prime numbers and $e_\bullet \geq 1$ are integers, then the radical of $\langle a \rangle$ is $\langle \prod_{i=1}^{m} p_i \rangle$.*

Proof. Simple calculation shows that

$$\sqrt{\langle a \rangle} = \sqrt{\prod_i \langle p_i \rangle^{e_i}} = \bigcap_i \sqrt{\langle p_i \rangle^{e_i}} = \bigcap_i \langle p_i \rangle$$

Since $p_\bullet$ are distinct, they are coprime pairwisely. Hence, the intersection becomes the produce, so we can conclude that

$$\bigcap_i \langle p_i \rangle = \prod_i \langle p_i \rangle = \left\langle \prod_i p_i \right\rangle$$

This completes a proof of the lemma. $\square$

Let $Q \trianglelefteq \mathbb{Z}$ be a non-zero primary ideal of $\mathbb{Z}$, and let a be a generator of the ideal. If we write the prime factorization of a as $\prod_{i=1}^{m} p_i^{e_i}$ where $p_\bullet$ are distinct prime numbers and $e_\bullet \geq 1$ are integers, then the radical $\langle \prod_i p_i \rangle$ must be prime because Q is primary. Thus, $m = 1$ and $Q = \langle p_1^{e_1} \rangle$ and this completes the proof. $\square$

Solution of Problem 2. It is enough to show that the inverse image of a closed set by the map ϕ^* is closed, or equivalently, $(\phi^*)^{-1}(V(I))$ is closed for all ideal $I \trianglelefteq R$. I'll show that $(\phi^*)^{-1}(V(I)) = V(S \cdot \phi(I))$

Let's consider the direct inclusion, i.e., $\subseteq$. Let $P \in (\phi^*)^{-1}(V(I))$, or equivalently, $\phi^*(P) = \phi^{-1}(P) \supseteq I$. This implies that $P \supseteq \phi(\phi^{-1}(P)) \supseteq \phi(I)$. By considering ideals generated by each of them, we have a relation $P \supseteq S \cdot \phi(I)$. Thus, $P \in V(S \cdot \phi(I))$.

Conversely, let's assume that $P \in V(S \cdot \phi(I))$, or equivalently, $P \supseteq S \cdot \phi(I) \supseteq \phi(I)$. By taking the inverse image, we have a relation $\phi^*(P) = \phi^{-1}(P) \supseteq \phi^{-1}(\phi(I)) \supseteq I$, hence $\phi^*(P) \in V(I)$ holds. This implies that $P \in (\phi^*)^{-1}(V(I))$, and this completes the proof. $\square$

Solution of Problem 3. First, let's show that the first condition implies the second one. We need a lemma.

Lemma 1.2. *Let* $\phi : A \to B$ *be a ring map. For a prime ideal* $\mathfrak{p} \trianglelefteq A$*, it is in the image of the induced map* $\phi^* : \operatorname{Spec} B \to \operatorname{Spec} A$ *if and only if* $\phi^{-1}(B \cdot \phi(\mathfrak{p})) = \mathfrak{p}$.

Proof. For the direct implication, let's assume that $\mathfrak{p} = \phi(\mathfrak{q})$ where $\mathfrak{q} \trianglelefteq B$ is a prime. Since $\phi^{-1}(B \cdot \phi(\mathfrak{p})) \supseteq \mathfrak{p}$, it is enough to show the reversed inclusion. Let $a \in \phi^{-1}(B \cdot \phi(\mathfrak{p}))$, then there are elements $b_i \in B$ and $a_i \in \mathfrak{p}$ such that $\phi(a) = \sum_i b_i \phi(a_i)$. Since $a_i \in \mathfrak{p} = \phi^{-1}(\mathfrak{q})$, we can deduce that $\phi(a) \in \mathfrak{q}$, i.e., $a \in \phi^{-1}(\mathfrak{q})$. Hence, the equality $\phi^{-1}(B \cdot \phi(\mathfrak{p})) = \mathfrak{p}$ holds.

Conversely, let's assume that $\phi^{-1}(B \cdot \phi(\mathfrak{p})) = \mathfrak{p}$. Let's consider a localization at $\mathfrak{p}$, then we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\ A_{\mathfrak{p}} \arrow[r, "\phi_{\mathfrak{p}}"] & B_{\mathfrak{p}} \end{tikzcd}$$

where the vertical maps are natural and $\phi_{\mathfrak{p}}$ is the induced map by the universal property. It is not too hard to show that the module $B_{\mathfrak{p}}$ can be identified with $S^{-1}B$ under the identification $\frac{x}{s} \leftrightsquigarrow \frac{x}{\phi(s)}$, where $S = \phi(\mathfrak{p}^c)$, in particular, we can consider $B_{\mathfrak{p}}$ as a localized ring. Moreover, $\phi_{\mathfrak{p}}$ forms a ring map because this satisfies that

$$\phi_{\mathfrak{p}}(1) = 1$$

$$\begin{aligned} \phi_{\mathfrak{p}}\left(\frac{a_1}{s_1} + \frac{a_2}{s_2}\right) &= \phi_{\mathfrak{p}}\left(\frac{s_2 a_1 + s_1 a_2}{s_1 s_2}\right) \\ &= \frac{\phi(s_2 a_1 + s_1 a_2)}{s_1 s_2} \\ &= \frac{s_2 \phi(a_1) + s_1 \phi(a_2)}{s_1 s_2} \\ &= \frac{\phi(a_1)}{s_1} + \frac{\phi(a_2)}{s_2} \\ &= \phi_{\mathfrak{p}}\left(\frac{a_1}{s_1}\right) + \phi_{\mathfrak{p}}\left(\frac{a_2}{s_2}\right) \end{aligned}$$

and

$$\begin{aligned} \phi_{\mathfrak{p}}\left(\frac{a_1}{s_1} \cdot \frac{a_2}{s_2}\right) &= \frac{\phi(a_1 a_2)}{s_1 s_2} \\ &= \phi_{\mathfrak{p}}\left(\frac{a_1}{s_1}\right) \cdot \phi_{\mathfrak{p}}\left(\frac{a_2}{s_2}\right) \end{aligned}$$

Because image, inverse image, sum commute with the localization, we have the identity

$$(\phi_{\mathfrak{p}})^{-1}(B_{\mathfrak{p}} \cdot \phi_{\mathfrak{p}}(\mathfrak{p}_{\mathfrak{p}})) = \left(\phi^{-1}(B \cdot \phi(\mathfrak{p}))\right)_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}$$

Hence, there must be a maximal ideal $\mathfrak{m} \trianglelefteq B_{\mathfrak{p}}$ containing $B_{\mathfrak{p}} \cdot \phi_{\mathfrak{p}}(\mathfrak{p}_{\mathfrak{p}})$, and the prime ideal $\phi_{\mathfrak{p}}^{-1}(\mathfrak{m}) \supseteq (\phi_{\mathfrak{p}})^{-1}(B_{\mathfrak{p}} \cdot \phi_{\mathfrak{p}}(\mathfrak{p}_{\mathfrak{p}})) = \mathfrak{p}_{\mathfrak{p}}$ must be the maximal ideal $\mathfrak{p}_{\mathfrak{p}}$. By the correspondence theorem, there must be a prime ideal of B whose inverse image is $\mathfrak{p}$, and this completes the proof. $\square$

The implication follows immediately from the lemma above.

Let's consider the implication from (2) to (3). By the assumption, there is a prime ideal $\mathfrak{p} \trianglelefteq B$ such that $\phi^{-1}(\mathfrak{p}) = \mathfrak{m}$ holds. Thus, we have

$$\phi(\mathfrak{m}) \cdot B = \phi(\phi^{-1}(\mathfrak{p})) \cdot B \subseteq \mathfrak{p} \cdot B = \mathfrak{p} \subsetneq B$$

and the is what we want to show.

Now, let's show that the third one is a sufficient condition of the fourth one. Let's pick a non-zero element $x \in M$, then there is a proper ideal $I \trianglelefteq A$ such that $A \cdot x \cong A/I$ as A-modules since $A \cdot x$ is a non-zero simple module. Let $\mathfrak{m} \trianglelefteq A$ be a maximal ideal containing I, then we have a surjective map $B \otimes_A (A/I) \twoheadrightarrow B \otimes_A (A/\mathfrak{m})$ which is obtained by tensoring on the surjective map $A/I \twoheadrightarrow A/\mathfrak{m}$. Because $B \otimes_A (A/\mathfrak{m}) \cong B/(\mathfrak{m} \cdot B)$ is non-zero, hence $B \otimes_A (A/I) \cong B \otimes_A (A \cdot x)$ is non-zero. Because B is flat, an injection $A \cdot x \hookrightarrow M$ induces a monomorphism $B \otimes_A (A \cdot x) \hookrightarrow B \otimes_A M$, thus the module $B \otimes_A M$ is non-zero. Hence, the third condition implies the fourth one.

Let's prove that the fourth property deduces the fifth one. Since B is flat and the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "x \mapsto 1 \otimes x"', "\cong"] & B \otimes_A M \\ A \otimes_A M \arrow[ur] & \end{tikzcd}$$

commutes, it is enough to show that the map $\phi : A \to B$ is injective. If $B \otimes_A \ker \phi = 0$, then so is $\ker \phi$ by the assumption, and this will show the injectivity of the map ϕ. Because the tensoring by B commute with kernel, we have to show that the map $B \otimes_A A \to B \otimes_A B, b \otimes a \mapsto b \otimes \phi(a)$ is injective. We have a commutative diagram

$$\begin{tikzcd} B \otimes_A A \arrow[r] \arrow[d, "b \otimes a \mapsto ab"', "\cong"] & B \otimes_A B \\ B \arrow[ur, "b \mapsto b \otimes 1"'] & \end{tikzcd}$$

so it is enough to show that the map $B \to B \otimes_A B, b \mapsto b \otimes 1$ is injective. However, there is a map $B \otimes_A B \to B, b_1 \otimes b_2 \mapsto b_1 b_2$ by the universal property of tensor product because a product is bilinear. The composition

$$B \xrightarrow{b \mapsto b \otimes 1} B \otimes_A B \xrightarrow{b_1 \otimes b_2 \mapsto b_1 b_2} B$$

is the identity, in particular, the map at the left is injective. This is what we want to show.

$\square$