KAIST
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Homework

Jeongwoo Park
University: KAIST
Department: Mathematical Sciences
Student ID: 20190262

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HW 3.1

First, let's consider the left inequality. The inequality is clear when $n = 1$, and if $n = 2$, then the regular 5-gon gives an inequality $m(2) \geq 5$, thus the inequality holds in these cases. Now, let's think about cases for $n \geq 3$. Note that it doesn't matter if we consider arbitrary sphere of dimension $n-1$, instead of the unit sphere S^{n-1}. Also, we can identify $\mathbb{R}^n$ with the hyperplane $H := \left\{(a_i)_i \in \mathbb{R}^{n+1}\,;\, \sum_i a_i = 1\right\}$ equipped with the distance inherited from $\mathbb{R}^{n+1}$. Let $S \subseteq H$ be an $(n-1)$-dimensional sphere with radius $\sqrt{\frac{1}{2} - \frac{1}{n}}$, centered at $\left(\frac{1}{n}\right)_i \in H$. Let $X = \left\{\frac{e_i+e_j}{2}\right\}_{1 \leq i < j \leq n+1}$, then this set is a subset of S due to the equality

$$d\left(\frac{e_i+e_j}{2}, \left(\frac{1}{n}\right)_i\right) = \sqrt{2 \cdot \left(\frac{1}{2} - \frac{1}{n}\right)^2 + \frac{n-2}{n^2}} = \sqrt{\frac{1}{2} - \frac{1}{n}}$$

Moreover, the set X is a 2-distance set because

$$d\left(\frac{e_i+e_j}{2}, \frac{e_{i'}+e_{j'}}{2}\right) = \begin{cases} 1 & |i, i', j, j'| = 4 \\ \frac{1}{\sqrt{2}} & |i, i', j, j'| = 3 \end{cases}$$

holds where $i < j, i' < j'$ are integers in $[1, n+1]$. Hence, $m(n) \geq |X| = \binom{n+1}{2} = \frac{n(n+1)}{2}$ holds.

For the second inequality, note that the equality $d(x, y) = 1 - x \cdot y$ holds for all unit vectors x and y, by the elementary Euclidean geometry. So, if we define two numbers $\hat{a} = 1 - a, \hat{b} = 1 - b$, then the condition "$\|x - y\| \in \{a, b\}$ for all different $x, y \in X$" is equivalent to "$x \cdot y \in \left\{\hat{a}, \hat{b}\right\}$ for all different $x, y \in X$". Here, we can suppose that $a, b \neq 0$, or equivalently, $\hat{a}, \hat{b} \neq 1$. Let's use an index $X = \{a_i\}_{1 \leq i \leq m}$ with $m = |X|$. Now, we define n-variable polynomials

$$f_{i'}(x_\bullet) = \frac{\left(x_\bullet \cdot a_{i'} - \hat{a}\right)\left(x_\bullet \cdot a_{i'} - \hat{b}\right)}{\left(1 - \hat{a}\right)\left(1 - \hat{b}\right)} \in \mathbb{R}[x_i]_{1 \leq i \leq n}$$

for all $1 \leq i' \leq m$. Note that the equality $f_{i'}(a_i) = \delta_{i,i'}$ holds by the construction, in particular, the the set $\{f_{i'}(x_\bullet)\}_{i'}$ is linearly independent.

If one of $\hat{a}$ and $\hat{b}$ is zero, then $f_{i'}(x_\bullet) \in \langle x_i x_j, x_i \rangle_{1 \leq i < j \leq n+1}$ holds since there is no constant term. By using the dimension argument, we can conclude that

$$m = |\{f_{i'}(x_\bullet)\}_{i'}| \leq \dim \langle x_i x_j, x_i \rangle_{i,j} = n + \binom{n}{2} + n = \frac{n(n+3)}{2}$$

Let's consider the case for $\hat{a}$ and $\hat{b}$ are non-zero. If we define a new polynomial $f_0(x_\bullet) := 1 - x_\bullet \cdot x_\bullet$ and a vector $a_0 = 0$, then the equalities $f_0(x_i) = \delta_{i,0}$ and $f_{i'}(x_0) = \frac{\hat{a}\hat{b}}{(1-\hat{a})(1-\hat{b})} \cdot \delta_{0,i'}$ hold. By the usual technique of linear independence, we can deduce that the set $\{f_{i'}(x_\bullet)\}_{0 \leq i' \leq m}$ is linearly independent, because the number $\frac{\hat{a}\hat{b}}{(1-\hat{a})(1-\hat{b})}$ is non-zero. By a relation $f_{i'}(x_\bullet) \in \langle x_i x_j, x_i, 1 \rangle_{1 \leq i < j \leq n}$, we can conclude that

$$m + 1 = |\{f_{i'}(x_\bullet)\}_{i'}| \leq \dim \langle x_i x_j, x_i, 1 \rangle_{i,j} = n + \binom{n}{2} + n + 1 = \frac{n(n+3)}{2}$$

or equivalently,

$$m \leq \frac{n(n+3)}{2}$$

holds. This completes the proof. $\square$

HW 3.2

Let $N = \left|\bigcup_{i,j} A_{i,j}\right|$, then we can assume that $A_{i,j} \subseteq [N]$ for all i and j. We define sets

$$P_i := \left\{\sigma \in \mathrm{Sym}(N)\,;\, \text{Every element of } A_{i,k} \text{ appears before every element of } A_{i,k'} \text{ for all } k < k'.\right\}$$

By the elementary combinatorics, we can deduce an identity $|P_i| = \binom{N}{\sum_i r_i} \cdot (\prod_i r_i!) \cdot (N - \sum_i r_i)! = \frac{N!}{\binom{\sum_i r_i}{r_1, r_2, \cdots, r_k}}$.

Claim 2.1. *These sets P_i are pairwisely disjoint.*

Proof of the Claim. Let's assume that $\sigma \in P_i \cap P_{i'}$ for some different i and i'. By the property in the problem, there are $j_1 < j_2$ and $j_1' < j_2'$ such that $A_{i,j_1} \cap A_{i',j_2'} \neq \varnothing$ and $A_{i,j_2} \cap A_{i',j_1'} \neq \varnothing$. If we pick an element a_1 from the first set and an element a_2 from the second set, then a_1 appears before a_2 in σ since $a_1 \in A_{i,j_1}$ and $a_2 \in A_{i,j_2}$ hold, by the property of $\sigma \in P_i$. However, the fact that $a_1 \in A_{i',j_2'}$ and $a_2 \in A_{i',j_1'}$ implies that a_1 appears after a_2 in σ, by the property of $\sigma \in P_{i'}$, and this makes a contradiction. This proves the claim. $\square$

In conclusion, we can deduce that

$$\begin{aligned} N! &= |\mathrm{Sym}(N)| \\ &\geq \left|\coprod_i P_i\right| \\ &= \sum_i |P_i| \\ &= m \cdot \frac{N!}{\binom{\sum_i r_i}{r_1, r_2, \cdots, r_k}} \end{aligned}$$

or equivalently, $m \leq \binom{\sum_i r_i}{r_1, r_2, \cdots, r_k}$ holds. This completes the proof. $\square$

HW 3.3

We need some definitions and lemmas.

Definition 3.1 (tensor product of vector spaces)**.** Let V_i's be real vector spaces, where $1 \leq i \leq n$. Define a tensor product of V_i's as the dual vector space of the space of all multilinear maps $L(V_1, \cdots, V_n; \mathbb{R})$. We denote this vector space as $\bigotimes_{i=1}^n V_i$.

Definition 3.2 (tensor)**.** Let V_i's be real vector spaces, and $v_i \in V_i$. We define a tensor $\otimes_i v_i \in \bigotimes_i V_i$ as a map sending a multilinear map $L \in L(V_1, \cdots, V_n, \mathbb{R})$ to $L(v_1, \cdots, v_n)$.

Lemma 3.1. *Let V_i's be d_i-dimensional real vector spaces, where $1 \leq i \leq n$. Then, the dimension of the tensor product $\bigotimes_i V_i$ is $\prod_i d_i$.*

Proof of Lemma 3.1. Let $\{w_{i,j}\}_{1\leq j\leq d_i}$ be a basis of V_i, and let $\{w^*_{i,j}\}_{1\leq j\leq d_i}$ be the dual basis of it. We define $\otimes_i w^*_{i,j_i} \in L(V_1, \cdots, V_n; \mathbb{R})$[1] as a map sending $(v_i)_i$ to $\prod_i w^*_{i,j_i}(v_i)$. It is enough to show that the set $B = \{\otimes_i v_{i,j_i}\}_{i,j}$ forms a basis of the tensor product $\bigotimes_i V_i$.

First, let's show that B generated the tensor product. Let $\varphi \in \bigotimes_i V_i$, and $c_{j_\bullet} = \varphi\left(\otimes_i w^*_{i,j_i}\right)$. Then, we can conclude that $\varphi = \sum_{j_\bullet} c_{j_\bullet} \cdot (\otimes_i w_{i,j_i})$, since $L(V_1, \cdots, V_n; \mathbb{R})$ is generated by tensors $\{\otimes_i w_{i,j_i}\}_{j_\bullet}$, by the elementary linear algebra.

Second, let's show that B is linearly independent. If we have an equality $\sum_{j_\bullet} c_{j_\bullet} \cdot (\otimes_i w_{i,j_i}) = 0$, then

$$c_{j'_\bullet} = \left(\sum_{j_\bullet} c_{j_\bullet} \cdot (\otimes_i w_{i,j_i})\right)\left(\otimes_i w^*_{i,j'_i}\right) = 0$$

holds for all $j'_\bullet$. This completes the proof. □

In particular, a tensor of vectors forms a zero map if and only if one of these vectors is zero.

Definition 3.3 (wedge product of a tensor product)**.** Let V_i's be real vector spaces, and let r_i, s_i be natural numbers. We define a wedge product $\wedge : \left(\bigotimes_i V_i^{\wedge r_i}\right) \times \left(\bigotimes_i V_i^{\wedge s_i}\right) \to \left(\bigotimes_i V_i^{\wedge r_i+s_i}\right)$ as $(\otimes_i v_i) \wedge (\otimes_i w_i) := \otimes_i (v_i \wedge w_i)$.

To show the well-definedness of this wedge product, let $\otimes_i v_i = \otimes_i v'_i$ and $\otimes_i w_i = \otimes_i w'_i$, where $v_i, v'_i \in V_i^{\wedge r_i}$ and $w_i, w'_i \in V_i^{\wedge s_i}$. Let $L \in L\left(V_i^{\wedge r_i+s_i}; \mathbb{R}\right)_i$ be a multilinear map, then we can define $L_1 \in L\left(V_i^{\wedge r_i}; \mathbb{R}\right)_i$ and $L_2 \in L\left(V_i^{\wedge s_i}; \mathbb{R}\right)_i$ as $L_1 : (a_i)_i \mapsto L(a_i \wedge w_i)_i$ and $L_2 : (b_i)_i \mapsto L(v'_i \wedge b_i)_i$. From this one, we can deduce that

$$L(v_i \wedge w_i)_i = (\otimes_i v_i)(L_1) = \left(\otimes_i v'_i\right)(L_1) = L(v'_i \wedge w_i)_i$$

and

$$L(v'_i \wedge w_i)_i = (\otimes_i w_i)(L_2) = \left(\otimes_i w'_i\right)(L_2) = L(v'_i \wedge w'_i)_i$$

i.e., the wedge is well-defined.

[1] This is some kind of abuse of notation.

Now, let's start the proof of the problem. Note that we have equality

$$\left(\bigcup_i A_{ij}\right) \cap \left(\bigcup_i B_{ij'}\right) = \bigcup_i \left(A_{ij} \cap B_{ij'}\right)$$

since X_i's are disjoint. Let's consider an inner direct sum $\mathbb{R}^{\sum_i (r_i+s_i)} = \bigoplus_{i=1}^n V_i$ such that $\dim V_i = r_i + s_i$. We can find an infinite subset of V_i with property "every $r+s$ points in this set is linearly independent", and moreover, we can assume that X_i is actually this set, since such a subset doesn't contain 0, an equality $V_i \cap V_{i'} = 0$ holds for all different i and i', and by the displayed equality[2] of sets above. Under this assumption, we define w_I as a wedge product of all elements in I, where the order of wedge product can be chosen arbitrary one. Also, let's consider vectors $v_{i,j} := w_{A_{ij}} \in V_i^{\wedge r_i}$, $w_{i,j} := w_{B_{ij}} \in V_i^{\wedge s_i}$, and $v_j := \otimes_i v_{i,j} \in \bigotimes_i V_i^{\wedge r_i}$ and $w_j := \otimes_i w_{i,j} \in \bigotimes_i V_i^{\wedge s_i}$. By the assumptions that

$$\bigcup_i (A_{ij} \cap B_{ij'}) \begin{cases} = \varnothing & j = j' \\ \neq \varnothing & j < j' \end{cases}$$

and the properties (e.g. a tensor product of vectors is zero if and only if one of these vectors is zero; every $r_i + s_i$ points in X_i is linearly independent; a wedge product of vectors is zero if and only if these vectors are linearly dependent) we have, we can deduce that

$$v_j \wedge w_{j'} = \otimes_i (v_{ij} \wedge w_{ij'}) \begin{cases} \neq 0 & j = j' \\ = 0 & j < j' \end{cases}$$

Therefore, v_j's are linearly independent in the tensor product $\bigotimes_i V_i^{\wedge r_i}$, as we did in the lecture. Since the dimension of this tensor product is

$$\dim \left(\bigotimes_i V_i^{\wedge r_i}\right) = \prod_i \dim \left(V_i^{\wedge r_i}\right) = \prod_i \binom{r_i + s_i}{r_i}$$

This completes the proof. $\square$

[2] This guarantees that we can consider the sets X_i's separately.

HW 3.4

Let V_1 (respectively, V_2) be a $(a+b+c)$-dimensional (respectively, $(b+c)$-dimensional) real vector space, and let $X_1 \subseteq V_1$ (respectively, $X_2 \subseteq V_2$) is an infinite subset such that every $a+b+c$ (respectively, $b+c$) points in the subset are linearly independent. Moreover, we can assume that V_1 and V_2 together form an inner direct sum of a real vector space V. Let $i_1 : \bigcup_{i,j} A_{i,j} \to X_1$ and $i_2 : \bigcup_{i,j;j\neq 1} A_{i,j} \to X_2$ be injective maps. Again, we use the notation w_I to denote the wedge of all elements in I, and the order of the wedge product doesn't matter. Consider vectors $v_{i,j} := w_{i_1(A_{i,j})} \in V_1^{\wedge a_j}, w_{i,j} := w_{i_2(A_{i,j})} \in V_2^{\wedge a_j}$ where $(a_1, a_2, a_3) = (a, b, c)$. Then, we have (in)equalities

$$v_{i,1} \wedge v_{i',2} \wedge v_{i',3} \wedge w_{i,2} \wedge w_{i',3} \begin{cases} \neq 0 & i = i' \\ = 0 & i < i' \end{cases}$$

by the property of sets X_i's and $A_{i,j}$'s, because the wedge product is non-zero if and only if the vectors $v_{i,1}, v_{i',1}, v_{i',3}, w_{i,2}, w_{i',3}$ are linearly independent, and it is equivalent to that each of two sets $v_{i,1}, v_{i',2}, v_{i',3}$ and $w_{i,2}, w_{i',3}$ is linearly independent, because V_1 and V_2 together form an inner direst sum. By the usual argument, we can conclude that the vectors $v_{i,1} \wedge w_{i,2}$ are linearly independent in the vector space $V_1^{\wedge a} \wedge V_2^{\wedge b}$. However, if e_i's (respectively, $e'_{i'}$'s) form a basis of $V_1^{\wedge a}$ (respectively, $V_2^{\wedge b}$) then $e_i \wedge e'_{i'}$ generates the wedge $V_1^{\wedge a} \wedge V_2^{\wedge b}$. In conclusion, we can obtain the inequality

$$m \leq \dim\left(V_1^{\wedge a} \wedge V_2^{\wedge b}\right) \leq \dim\left(V_1^{\wedge a}\right) \cdot \dim\left(V_2^{\wedge b}\right) = \binom{a+b+c}{a} \cdot \binom{b+c}{b} = \binom{a+b+c}{a, b, c}$$

This completes the proof. $\square$

HW 3.5

If $|\mathbb{F}| \leq d+1$, then we can deduce that

$$\dim L \leq \dim \mathbb{F}^{\mathbb{F}^n} = |\mathbb{F}^n| \leq (d+1)^n$$

so we are done. If not, let $S \subseteq \mathbb{F}$ be a subset of $d+1$ elements, and we define a linear map $\varphi : L \to \mathbb{F}^{S^m}, f \mapsto (f(s_i)_i)_{(s_i)_i \in S^n}$. It is enough to show that the map φ is injective, or equivalently, $f \in \ker \varphi$ must be zero. Let's denote the zero set of f as $V(f)$. By the definition of φ, we have a relation $V(f) \supseteq S^n$. By the property given in the problem, we can deduce that $V(f) \supseteq \mathbb{F} \times S^{n-1}$, since $f(t, s_2, \cdots, s_n)$ has $d+1$ solutions for each $(s_2, \cdots, s_n) \in S^{n-1}$. Again by the property, we can obtain that $V(f) \supseteq \mathbb{F}^2 \times S^{n-2}$, and by repeating this process, we can finally conclude that $V(f) \supseteq \mathbb{F}^n$, i.e., $f = 0$. Therefore, the linear map φ is injective, hence the inequality $\dim L \leq \dim \mathbb{F}^{S^n} = |S^n| = (d+1)^n$ holds, and this completes the proof. $\square$

$\left\{\mathbb{R}^3\right\}, \{\mathbb{R}^3\}$