KAIST
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Homework

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HW 2.1

Let $\mathbf{a}_j \in \mathbb{F}_p^n$ (respectively, $\mathbf{b}_j$) be the $\mathbb{F}_p$-coefficient characteristic vector of A_j (respectively, B_j) for all $1 \le j \le m$. If we define n-variable polynomials

$$f_j(x_\bullet) := \sum_{l \in L} \left(x_\bullet \cdot \mathbf{b}_j - \bar{l} \right)$$

then we have an invertible lower triangular matrix $\left(f_j\left(\mathbf{a}_{j'}\right)\right)_{(j,j')} \in U_n(\mathbb{F}_p)$, because

$$f_j\left(\mathbf{a}_{j'}\right) = \begin{cases} \sum_l \left(\overline{|A_j \cap B_j|} - \bar{l}\right) \in \mathbb{F}_p^\times & j = j' \\ \sum_l \left(\overline{|A_{j'} \cap B_j|} - \bar{l}\right) = 0 & j > j' \end{cases}$$

holds. Let's consider induced polynomials $\bar{f}_j(x_\bullet) \in \mathbb{F}_p[x_i]_{1 \le i \le n}$ which obtained by changing x_i^2 to x_i for each term of the polynomial $f_j(x_\bullet)$ for all $1 \le i \le n$, as in the lecture. Then, it is clear that $\left(f_j\left(\mathbf{a}_{j'}\right)\right) = \left(\bar{f}_j\left(\mathbf{a}_{j'}\right)\right)$ is invertible, in particular, the set of polynomials $\left\{\bar{f}_j(x_\bullet)\right\}_{1 \le j \le m}$ is $\mathbb{F}_p$-linearly independent. Since every term of $\bar{f}_j(x_\bullet)$ is square-free, and $\deg \bar{f}_j(x_\bullet) \le |L| = s$ holds, so we can deduce that $f_j(x_\bullet) \in \langle x_\bullet^{\alpha_\bullet} \rangle_{|\alpha_\bullet|_\infty \le 1}$. By linear algebra, we can obtain that

$$m = \left| \{f_j(x_\bullet)\}_{1 \le j \le m} \right| \le \dim \langle x_\bullet^{\alpha_\bullet} \rangle_{|\alpha_\bullet|_\infty \le 1} = \sum_{i=0}^{s} \binom{n}{i}$$

This completes the proof. $\square$

HW 2.2

We can label $\mathscr{F} = \{A_j\}_{1\leq j\leq m}$ to be $|A_1| \leq \cdots \leq |A_m|$ holds. Let $\mathbf{a}_j \in \mathbb{R}^n$ be the $\mathbb{R}$-coefficient characteristic vector of A_j, and we define n-variable polynomials

$$f_j(x_\bullet) := \sum_{l\in L\,;\,l<|\mathbf{a}_j|_1} (x_\bullet \cdot \mathbf{a}_j - l)$$

for all $1 \leq j \leq m$. Then, we have an invertible upper triangular matrix $\left(f_j\left(\mathbf{a}_{j'}\right)\right)_{(j,j')} \in U_n(\mathbb{R})$, because

$$f_j\left(\mathbf{a}_{j'}\right) = \begin{cases} \sum_{l<|\mathbf{a}_j|_1} (|\mathbf{a}_j|_1 - l) \in \mathbb{R}^\times & j = j' \\ \sum_{l<|\mathbf{a}_j|_1} \left(|A_{j'} \cap A_j| - l\right) = 0 & j > j' \end{cases}$$

Also, we consider polynomials

$$g_I(x_\bullet) := \left(\prod_{i=1}^{r} (|x_\bullet|_1 - k_i)\right) \cdot \prod_{i\in I} x_i$$

for all $I \subseteq \{1, 2, \cdots, n\}$ satisfying $|I| \leq s - r$. Let's consider an indexing $\{I\}_{|I|<s-r} = \{I_\bullet\}$ such that $|I_1| \leq |I_2| \leq \cdots$. Then, the matrix $\left(g_{I_j}\left(\chi_{I_{j'}}\right)\right)_{(j,j')}$ is invertible upper triangular matrix where χ is the characteristic function, since

$$g_{I_j}\left(\chi_{I_{j'}}\right) \begin{cases} \in \mathbb{R}^\times & j = j' \text{ because } k_i > s - r \text{ for all } i. \\ = 0 & j > j' \text{ because } \prod_{i\in I_j} \chi_{I_{j'}}(i) = 0. \end{cases}$$

Also, we have $g_{I_j}\left(\mathbf{a}_{j'}\right) = 0$, hence the matrix

$$\begin{bmatrix} \left(f_j\left(\mathbf{a}_{j'}\right)\right)_{(j,j')} & \left(f_j\left(\chi_{I_{j'}}\right)\right)_{(j,j')} \\ \left(g_{I_j}\left(\mathbf{a}_{j'}\right)\right)_{(j,j')} & \left(g_{I_j}\left(\chi_{I_{j'}}\right)\right)_{(j,j')} \end{bmatrix} = \begin{bmatrix} \left(f_j\left(\mathbf{a}_{j'}\right)\right)_{(j,j')} & \left(f_j\left(\chi_{I_{j'}}\right)\right)_{(j,j')} \\ 0 & \left(g_{I_j}\left(\chi_{I_{j'}}\right)\right)_{(j,j')} \end{bmatrix}$$

is an invertible upper triangular matrix. In particular, the set of polynomials $\{f_j(x_\bullet), g_I(x_\bullet)\}_{\substack{1\leq i\leq m \\ |I|\leq s-r}}$ is $\mathbb{R}$-linearly independent.

Now, let $\bar{f}_j(x_\bullet)$ (respectively, $\bar{g}_I(x_\bullet)$) be the reduction of $f_j(x_\bullet)$ (respectively, $g_I(x_\bullet)$) as in the **Problem 1**. Then, we have the same invertible upper triangular matrix

$$\begin{bmatrix} \left(\bar{f}_j\left(\mathbf{a}_{j'}\right)\right)_{(j,j')} & \left(\bar{f}_j\left(\chi_{I_{j'}}\right)\right)_{(j,j')} \\ \left(\bar{g}_{I_j}\left(\mathbf{a}_{j'}\right)\right)_{(j,j')} & \left(\bar{g}_{I_j}\left(\chi_{I_{j'}}\right)\right)_{(j,j')} \end{bmatrix} = \begin{bmatrix} \left(f_j\left(\mathbf{a}_{j'}\right)\right)_{(j,j')} & \left(f_j\left(\chi_{I_{j'}}\right)\right)_{(j,j')} \\ \left(g_{I_j}\left(\mathbf{a}_{j'}\right)\right)_{(j,j')} & \left(g_{I_j}\left(\chi_{I_{j'}}\right)\right)_{(j,j')} \end{bmatrix}$$

Again, the set $\left\{\bar{f}_j(x_\bullet), \bar{g}_I(x_\bullet)\right\}_{j,I}$ is linearly independent. Since each term of $\bar{f}_j(x_\bullet)$ and $\bar{g}_I(x_\bullet)$ is square-free, and by two inequalities $\deg \bar{f}_j(x_\bullet) \leq s$, $\deg \bar{g}_I(x_\bullet) \leq |I| + r \leq s$, we can obtain that $\bar{f}_j, \bar{g}_I \in \langle x_\bullet^{\alpha_\bullet}\rangle_{|\alpha_\bullet|_\infty\leq 1}$. Therefore, the inequality

$$m + \sum_{i=0}^{s-r} \binom{n}{i} = \left|\left\{\bar{f}_j(x_\bullet), \bar{g}_I(x_\bullet)\right\}_{j,I}\right| \leq \dim\langle x_\bullet^{\alpha_\bullet}\rangle_{|\alpha_\bullet|_\infty\leq 1} = \sum_{i=0}^{s} \binom{n}{i}$$

holds, or equivalently,

$$m \leq \sum_{i=s-r+1}^{s} \binom{n}{i}$$

holds. This completes the proof. $\square$

HW 2.3

Let $A_i = \{i\}$ and $B_i = \{1, 2, \cdots, i-1\}$. Then, this satisfies the conditions (i) and (ii). Also, we have

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i}} = \sum_{i=1}^{m} \frac{1}{i}$$

If we pick sufficiently large m, then we can obtain the inequality

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \geq n$$

and this completes the proof. $\square$

HW 2.4

Let's write $\mathscr{F} = \{A_i\}_{1\leq i\leq m}$, and let $\mathbf{a}_i \in \mathbb{F}_2^n$ be the $\mathbb{F}_2$-coefficient characteristic vector of A_i. Pick $i \in [1, m]$, then there are vectors $b_j \in \mathbb{F}_2^n$ where $1 \leq j \leq k$ such that

1. Each coordinate-wise product $b_i b_{i'}$ is zero when $i \neq i'$, and the sum of all b_i's is the vector consist of only 1 as coordinates.

2. For each i', the equality $a_{i'} \cdot b_j = 1$ holds for all j if and only if $i' = i$ holds.

because of the coloring condition in the problem. By the symmetry, we can assume that the first coordinate of b_1 is 1. In this situation, we define $(n-1)$-variable polynomial

$$f_i(x_\bullet) = \prod_{j=2}^{k}(x'_\bullet \cdot b_j)$$

over $\mathbb{F}_2$, where $x'_\bullet$ is the vector obtained by deleting the first entry of the vector $x_\bullet$. If $i' \neq i$, then the number of a non-zero entry of the coordinate-wise product $a_{i'}b_j$ is either 0 or 2 for some $j \geq 2$, by the pigeonhole principle. In the other word, the equality $f_i(a_{i'}) = \delta_{i,i'}$ holds, where $\delta_{\bullet,\bullet}$ denotes a Kronecker-delta function, i.e., the set $\{f_i(x_\bullet)\}_{1\leq i\leq m}$ is $\mathbb{F}_2$-linearly independent. Note that $f_i(x_\bullet)$ is homogeneous of degree $k-1$ only consist of square-free terms, so $f_i(x_\bullet) \in \langle x'^{\alpha_\bullet}_\bullet \rangle_{|\alpha_\bullet|_1 = k-1}$ holds. Hence, the inequality

$$m = \left|\{f_i(x_\bullet)\}_{1\leq i\leq m}\right| \leq \dim\langle x'^{\alpha_\bullet}_\bullet \rangle_{|\alpha_\bullet|_1=k-1} = \binom{n-1}{k-1}$$

holds, and this completes the proof. $\square$

HW 2.5

Claim 5.1. *Let $\mathscr{F}_1, \mathscr{F}_2$ be two k-uniform families over $[n]$. Then, the equality $\mathbb{E}\left(|\mathscr{F}_1 \cap \sigma(\mathscr{F}_2)|\right) = \frac{|\mathscr{F}_1|\cdot|\mathscr{F}_2|}{\binom{n}{k}}$ holds, where the expectation is with respect to the uniform probability measure on the symmetric group $\mathrm{Sym}(n)$.*

Proof of Lemma 5.1. Easy calculation shows that

$$\begin{aligned}
\mathbb{E}(|\mathscr{F}_1 \cap \sigma(\mathscr{F}_2)|) &= \sum_{A\in\mathscr{F}_1, B\in\mathscr{F}_2} \Pr(A = \sigma(B)) \\
&= \sum_{A\in\mathscr{F}_1, B\in\mathscr{F}_2} \frac{|\mathrm{Sym}(n)_A|}{n!} \\
&= \sum_{A\in\mathscr{F}_1, B\in\mathscr{F}_2} \frac{k!(n-k)!}{n!} \\
&= \frac{|\mathscr{F}_1|\cdot|\mathscr{F}_2|}{\binom{n}{k}}
\end{aligned}$$

where $\mathrm{Sym}(n)_A$ denotes the stabilizer of A, and this proves the claim. □

We have to show that $\mathbb{E}(|\mathscr{F}_1 \cap \sigma(\mathscr{F}_2)|) \leq 1$ if we set $\mathscr{F}_1$ and $\mathscr{F}_2$ as in the problem. It is enough to show that for each $\sigma \in \mathrm{Sym}(n)$, the inequality $|\mathscr{F}_1 \cap \sigma(\mathscr{F}_2)| \leq 1$ holds. If not, let A, B be two different sets in $\mathscr{F}_1 \cap \sigma(\mathscr{F}_2)$ for some σ. Then, both $|A \cap B| \in L_1$ and $|A \cap B| = |\sigma^{-1}(A) \cap \sigma^{-1}(B)| \in L_2$ holds since $\sigma^{-1}(A), \sigma^{-1}(B) \in \mathscr{F}_2$, but this contradicts to the assumption that the given two sets L_1 and L_2 are disjoint. Therefore, the inequality $|\mathscr{F}_1 \cap \sigma(\mathscr{F}_2)| \leq 1$ holds for every σ, and this completes the proof. □