Polynomial Ham Sandwich Theorem and Its Applications in Combinatorics

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— Proof of the Szemerédi–Trotter Theorem —

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Abstract

One of the most basic and powerful tool in combinatorics is *counting case by case*. This article includes a polynomial method for partitioning sets, which can be used to apply case-by-case-counting method. Also, I'll introduce its applications, like the Szemerédi–Trotter theorem.

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1 Brief Introduction

The ham sandwich theorem states that, for any d bounded open subsets of a Euclidean space $\mathbb{R}^d$, there is a *hyperplane* bisecting each of them, in the sense of the *Lebesgue measure*. One can consider *hypersurfaces* instead of hyperplanes. In this case, we can bisect *more than* d bounded open subsets of $\mathbb{R}^d$. Also, there are discrete analogues of them, by replacing the Lebesgue measure to the *counting measure*, and bounded open subsets to finite subsets. We can use this to make a *nice division* of a finite set, which can be applied to prove Szemerédi–Trotter theorem.

2 The Ham Sandwich theorems

Theorem 2.1 (Borsuk–Ulam theorem)**.**

The classical ham sandwich theorem can be stated as follows.

Definition 2.2 (bisection)**.** Let A be a Lebesgue-measurable subset of $\mathbb{R}^d$. We call a hyperplane $H = \left\{x \in \mathbb{R}^d ; x \cdot v = b\right\}$ *bisects* A, if and only if, $m\left(A \cap H^+\right) = m\left(A \cap H^-\right) = \frac{1}{2} \cdot m(A)$, where $H^+ := \left\{x \in \mathbb{R}^d ; x \cdot v > b\right\}$ and $H^- := \left\{x \in \mathbb{R}^d ; x \cdot v < b\right\}$ are two half-spaces.[1]

Theorem 2.3 (classical ham sandwich theorem)**.** *Let* $(O_i)_{1 \le i \le d}$ *be a collection of bounded open subsets of* $\mathbb{R}^d$. *There is a hyperplane* H *bisecting each of* $(O_i)_{1 \le i \le d}$.

Proof. Use Borsuk–Ulam theorem. Standard proofs can be found in (???) □

Theorem 2.4 (polynomial ham sandwich theorem)**.**

Theorem 2.5 (general ham sandwich theorem, cf. [ST42])**.**

Theorem 2.6 (discretized classical sandwich theorem)**.**

Theorem 2.7 (discretized polynomial sandwich theorem)**.**

Definition 2.8 (Veronese embedding)**.**

[1] The well-definedness is not too hard to show, because H^+ and H^- can be considered as the two connected components of $\mathbb{R}^d \setminus H$. However, I prefer the former definition, because it is more fundamental and easy to generalize.

3 Cell Decomposition

Lemma 3.1 (cell decomposition)**.**

Theorem 3.2 ([Tho15])**.**

Lemma 3.3 (Bézout's theorem, cf. [Ful08])**.**

4 Szemerédi–Trotter Theorem

Theorem 4.1 (Szemerédi–Trotter theorem)**.**

[KMS11] [Mun00] [Tho15] [ST42] [Tao11] [Yuv15]

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