Introduction to Commutative Algebra

— Solution of Homework #7 —

Jeongwoo Park

Student ID : 20190262

Department of Mathematical Sciences,

Korea Advanced Institute of Science and Technology

1 Solutions

Solution of Problem 1. For a formal power series $f(x) = \sum_{i\geq 0} a_i x^i$, we define $\operatorname{Supp} f(x) := \{i\,;\, a_i \neq 0\}$, $\operatorname{ord} f(x) := \inf\left(\operatorname{Supp} f(x)\right)$, and $\ell\left(f(x)\right) = a_{\operatorname{ord} f(x)}$, where a_∞ is considered as 0. Also, we define $\pi_d\left(f(x)\right) = a_d$. Let $I \trianglelefteq A[\![x]\!]$ be an ideal, then $\ell(I) \trianglelefteq A$ because

1. It contains 0.
2. For any $f(x), g(x) \in I$, a relation $\ell\left(f(x)+g(x)\right) \in \left\{\ell\left(f(x)\right), \ell\left(g(x)\right), \ell\left(f(x)+g(x)\right)\right\}$ holds, i.e., $\ell(I)$ is closed under the addition.
3. For any $f(x) \in I$, the equality $-\ell\left(f(x)\right) = \ell\left(-f(x)\right)$ holds, i.e., $\ell(I)$ forms an additive subgroup.
4. The equality $\ell\left(f(x)\right) \cdot \ell\left(g(x)\right) = \ell\left(f(x)\cdot g(x)\right)$ holds, i.e., $\ell(I)$ is closed under the constant multiplication.

Let $f_i(x) \in A[\![x]\!]$ be non-zero formal power series such that $a_i := \ell\left(f_i(x)\right)$ generates $\ell(I)$. We can assume that $1 \leq i \leq m$ for some natural number m, since A is Noetherian.

Let $g(x) \in A[\![x]\!]$ be a formal power series of order at least $d := \min_i \left(\operatorname{ord} f_i(x)\right)$. Let $g_d(x) = g(x)$, then $\pi_d\left(g_d(x)\right) \in \ell(I)$ because it is either $\ell\left(f(x)\right)$ or zero, so there are $c_{d,i} \in A$ such that $\ell\left(f(x)\right) = \sum_i c_{d,i} \cdot a_i$. Let $\hat{g}_d(x) := \sum_i c_{d,i} \cdot x^{d-\operatorname{ord} f_i(x)} f_i(x)$, then the order of $g_{d+1}(x) := g_d(x) - \hat{g}_d(x)$ is at least $d+1$ because it is clearly at least d, and $\pi_d\left(g_d(x) - \hat{g}_d(x)\right) = 0$ holds. Similarly, there are $c_{d+1,i} \in A$ such that $\ell\left(g_{d+1}(x)\right) = \sum_i c_{d+1,i} \cdot a_i$, and if we define $\hat{g}_{d+1}(x) = \sum_i a_i \cdot x^{d+1-\operatorname{ord} f_i(x)} f_i(x)$, then the order of $g_{d+1}(x) - \hat{g}_{d+1}(x)$ is at least $d+1$. Recursively, one can find $\hat{g}_n(x) \in \sum_i A[\![x]\!] \cdot f_i(x)$ such that $\operatorname{ord} g_n(x) \geq n$ and the order of $g_n(x) - \hat{g}_n(x)$ is at least $n+1$, for any $n \geq d$. The formal power series $\hat{g}(x) := \sum_n \hat{g}_n(x) \in \sum_i A[\![x]\!] \cdot f_i(x)$ is well-defined since $\lim_{n\to\infty} \operatorname{ord}\left(\hat{g}_n(x)\right) = \infty$. Moreover, one can deduce that, for any natural number n,

$$
\begin{aligned}
\pi_n\left(g(x) - \hat{g}(x)\right) &= \pi_n\left(g_d(x) - \left(\hat{g}_d(x) + \cdots + \hat{g}_n(x)\right)\right) \\
&= \pi_n\left(g_{n+1}(x)\right) \\
&= 0
\end{aligned}
$$

because $\sum_{i \geq n+1} \hat{g}_i(x)$ is of order at least $n+1$. Hence, every formal power series in I of order at least d is an element of $\sum_i A[\![x]\!] \cdot f_i(x)$.

I'll show the equality

$$
I = \left(\sum_{i=1}^{m} A[\![x]\!] \cdot f_i(x)\right) + A[\![x]\!] \cdot \left(I \cap \frac{1}{x^d} \cdot \left(\sum_{i \geq d} A \cdot x^i \cap \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq d-1}} A \cdot x^j f_i(x)\right)\right) \tag{1}
$$

The reversed inclusion can be shown easily, since each term is contained in I. So, let's focus on the direct inclusion. Let $g(x) \in I$, then $x^d \cdot g(x) \in I$ is of degree at least d. Hence, there are formal power series $c_i(x)$ such that $x^d \cdot g(x) = \sum_i c_i(x) \cdot f_i(x)$. Let $c_i'(x)$ is the sum of all terms of $c_i(x)$, of degree at most $d-1$, and $c''(x) = c_i(x) - c_i'(x)$. Then, the order of $c''(x)$ is at least x^d, so we have the equality

$$
g(x) - \sum_i \frac{c''(x)}{x^d} \cdot f_i(x) = \frac{1}{x^d} \cdot \sum_i c'(x) \cdot f_i(x) \in I \cap \frac{1}{x^d} \cdot \left(\sum_{i \geq d} A \cdot x^i \cap \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq d-1}} A \cdot x^j f_i(x)\right)
$$

One can deduce that $g(x)$ is an element of the right hand side of (1).

Let's conclude the proof. Of course, the module $\sum_{i=1}^m A[\![x]\!] \cdot f_i(x)$ is finite. For the another term, note that $\sum_{i,j} A \cdot x^j f_i(x)$ is Noetherian A-module, because it is a finitely generated, and A is Noetherian. Hence, the submodule $\sum_{i \geq d} A \cdot x^i \cap \sum_{i,j} A \cdot x^j f_i(x)$ is also Noetherian. However, the module

$$
\frac{1}{x^d} \cdot \left(\sum_{i \geq d} A \cdot x^i \cap \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq d-1}} A \cdot x^j f_i(x)\right)
$$

is isomorphic to $\sum_{i\geq d} A\cdot x^i \cap \sum_{i,j} A\cdot x^j f_i(x)$ since x^d is a non-zero divisor, i.e., a map $\mu_{x^d} : A[[x]] \to A[[x]], f(x) \mapsto x^d\cdot f(x)$ is injective, so the restriction

$$\mu_{x^d}\Bigg|_{\frac{1}{x^d}\cdot\left(\sum_{i\geq d} A\cdot x^i\cap\sum_{\substack{1\leq i\leq m\\1\leq j\leq d-1}} A\cdot x^j f_i(x)\right)} : \frac{1}{x^d}\cdot\left(\sum_{i\geq d} A\cdot x^i \cap \sum_{\substack{1\leq i\leq m\\1\leq j\leq d-1}} A\cdot x^j f_i(x)\right) \to \sum_{i\geq d} A\cdot x^i \cap \sum_{\substack{1\leq i\leq m\\1\leq j\leq d-1}} A\cdot x^j f_i(x)$$

is an isomorphism. Thereform, the submodule

$$I\cap\frac{1}{x^d}\cdot\left(\sum_{i\geq d} A\cdot x^i \cap \sum_{\substack{1\leq i\leq m\\1\leq j\leq d-1}} A\cdot x^j f_i(x)\right)$$

is Noetherian A-module, in particular, a finite A-module. This implies that the second term

$$A[[x]]\cdot\left(I\cap\frac{1}{x^d}\cdot\left(\sum_{i\geq d} A\cdot x^i \cap \sum_{\substack{1\leq i\leq m\\1\leq j\leq d-1}} A\cdot x^j f_i(x)\right)\right)$$

is finite $A[[x]]$-module, and this completes the proof. □

Solution of Problem 2. 1. We need a lemma.

Lemma 1.1. *Every Hausdorff Noetherian space is finite and discrete.*

Proof of Lemma 1.1. Let X be a Hausdorff Noetherian space. If $X = \emptyset$, then there is nothing to prove, so let's assume that X is non-empty. By **Problem 2**(3,4), every subspace of X is compact, in particular, any subset of X is closed, i.e., X is discrete.

Note that every subset of X is clopen, since its complement is closed. One can consider a non-empty collection of open sets $\{X'\}_{X'\subseteq X \text{ is finite.}}$, then there is a maximal element X'. If X is infinite, then clearly there is a finite subset of X which properly containing X', but this contradicts to the maximality of X'. Hence, X must be finite, and this completes the proof of the lemma. □

Hence, the closed interval $[0,1]$ is not a Noetherian space, since it is Hausdorff and not finite. Also, every sphere S^n and torus T^n are not Noetherian provided with $n\geq 1$, because they are Hausdorff spaces that are not finite.

2. Let's consider a non-empty collection of closed sets $\{V(I_i)\}_{i\in\Sigma}$ where each I_i is a radical ideal of A. Since A is Noetherian, there is a maximal element I of the non-empty collection of ideals $\{I_i\}_{i\in\Sigma}$. Since V is contravariant, $V(I)$ must be the minimal element of $\{V(I)\}_{i\in\Sigma}$, and this completes the proof.

3. Let X' be a subspace of a Noetherian space X, and let $\{X'\cap C_i\}_{i\in I}$ be a non-empty collection of closed subsets of X', where each C_i is a closed subset of X. Since a collection $\{C_i\}_{i\in I}$ is non-empty, there is a minimal element C. Of course, $X'\cap C$ must be a minimal element of $\{X'\cap C_i\}_{i\in I}$, since a map $X'\cap -$ is covariant. Hence, X' is Noetherian, and this completes the proof.

4. Let X be a Noetherian space, and let $\{C_i\}_{i\in I}$ be a collection of closed sets such that $\bigcap_{i\in I} C_i = \varnothing$. One can consider a non-empty collection of closed sets $\{\bigcap_{i\in I'} C_i\}_{I'\subseteq I \text{ is finite.}}$, then there is a minimal element $\bigcap_{i\in I'} C_i$ where $I' \subseteq I$ is finite. If this intersection is non-empty, then there is $i' \in I$ such that $C_{i'} \not\supseteq \bigcap_{i\in I'} C_i$ holds, because $\bigcap_{i\in I} C_i = \varnothing$ holds. However, this implies the $\bigcap_{i\in I'\cup\{i'\}} C_i \subsetneq \bigcap_{i\in I'} C_i$ holds, and this contradicts to the minimality of $\bigcap_{i\in I'} C_i$. Hence, $\bigcap_{i\in I'} C_i$ must be empty, and this shows the compactness of the space X. This is what we want to show.

□

Solution of Problem 3. Assume that X is a Noetherian space, but it can't be represented as a finite union of closed irreducible subspaces. By the assumption, X itself have to be reducible, so there are proper closed subsets C_1 and C_1' such that $X = C_1 \cup C_1'$. Again by the assumption, at least one of C_1 and C_1' is reducible, and we can suppose that C_1 is reducible, i.e., there are proper closed subsets C_2 and C_2' (which are also closed in X) such that $C_1 = C_2 \cup C_2'$. Again by the assumption, we can suppose that C_2 is reducible, and so there are proper closed subsets C_3 and C_3' satisfying the property. By repeating this process, one can obtain a chain of closed subsets

$$C_1 \supsetneq C_2 \supsetneq C_3 \supsetneq \cdots$$

but this contradicts to the fact that X is Noetherian. Hence, X can be represented as a finite union of irreducible closed subsets. □

Solution of Problem 4. We can consider a composition

$$k \longrightarrow k[x_i]_{1\leq i\leq n} \longrightarrow \frac{k[x_i]_i}{M}$$

Because this composition is finite by the weak nullstellensatz, and k is algebraically closed, so it must be an isomorphism. Hence, there is $a_i \in k$ such that $x_i + M = a_i + M$ for each i, that means, the ideal $\langle x_i - a_i\rangle_i \trianglelefteq k[x_i]_i$ is contained in M. To conclude the proof, it is enough to show that the ideal $\langle x_i - a_i\rangle$ is maximal.

Note that $k[x_i]_i = k[x_i - a_i]_i$ holds. One consider a surjective ring map

$$\phi : k[x_i]_i \to k;\ x_i \mapsto a_i$$

Every polynomial in $k[x_i]_i = k[x_i - a_i]_i$ can be represented as $\sum_\alpha c_\alpha \cdot (x-a)^\alpha$, where $x = (x_i)_i$ and $a = (a_i)_i$. This polynomial mapsto c_0 by the map ϕ, and this value is zero if and only if $\sum_\alpha c_\alpha \cdot (x-a)^\alpha$ is an element of $\langle x_i - a_i\rangle_i$, i.e., $\ker\phi = \langle x_i - a_i\rangle_i$ holds. By the first isomorphism theorem, one can conclude that

$$\frac{k[x_i]_i}{\langle x_i - a_i\rangle_i} \cong k$$

so the ideal $\langle x_i - a_i\rangle_i$ is maximal. This finishes the proof. □

Solution of Problem 5. Let's consider an ideal $\left\langle x_1^2+1, x_i\right\rangle_{i\geq 2}$, then one can conclude that

$$\frac{\mathbb{R}[x_i]_i}{\left\langle x_1^2+1, x_i\right\rangle_{i\geq 2}} = \frac{(\mathbb{R}[x_1])\,[x_i]_{i\geq 2}}{\left\langle \left\langle x_1^2+1\right\rangle, x_i\right\rangle_{i\geq 2}} \cong \frac{\mathbb{R}[x_1]}{\left\langle x_1^2+1\right\rangle} \cong \mathbb{C}$$

Thus, the ideal $\left\langle x_1^2+1, x_i \right\rangle_{i \geq 2}$ is maximal. However, it is not of the form $\langle x_i - a_i \rangle_i$, because the residue field corresponds to this maximal ideal is $\mathbb{R}$, but not $\mathbb{C}$, by the solution of **Problem 4**. This gives the result. $\square$