Discrete math

Data Structures
C++ for C Coders

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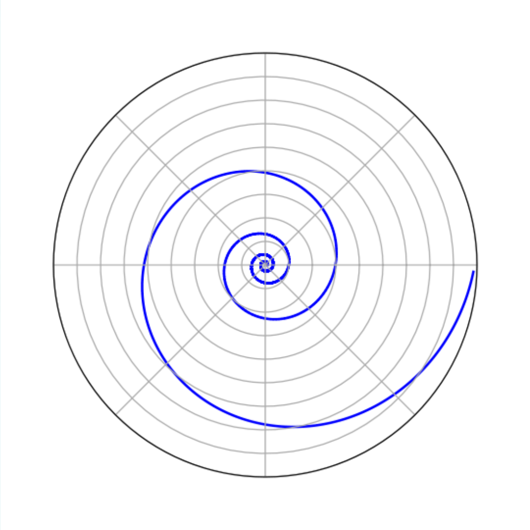


- *Logarithmic spiral*
- *miraculous spiral*
- *Spira mirabilis [Latin]*

1. Logarithms

- **Exponents:**
  - X^Y , or "**X to the Y^th power**";
    X multiplied by itself Y times

- **Some useful identities:**
  - $X^0 = 1$, provided $x \neq 0$.
  - $X^A X^B = X^{A+B}$
  - $X^A / X^B = X^{A-B}$
  - $X^{-B} = \frac{1}{X^B}$
  - $X^{1/n} = \sqrt[n]{X}$
  - $X^N + X^N = 2X^N$
  - $2^N + 2^N = 2^{N+1}$

1. Logarithms

- **Logarithms**
  - ***definition***: $\boldsymbol{X^A = B}$ if and only if $\boldsymbol{log_X B = A}$
  - ***intuition:*** $\boldsymbol{log_X B}$ means: the power $\boldsymbol{X}$ must be raised to, to get $\boldsymbol{B}$
  - This is the same as asserting $\boldsymbol{X^{log_X B} = B}$.
- Examples:
  - $log_X 1 = 0$
  - $log_X X = 1$
  - $\log_2 16 = 4$
  - $\log_{10} 1000 = 3$

- Most people use base 10, written as log_{10} or log, base e or ln .

- In computer science, we typically use base 2, written as $\boldsymbol{log_2}$ or lg.
  Between us, however, we simply use $\boldsymbol{log}$ instead of log_2 or $\boldsymbol{lg}$.

1. Logarithms

Exercise:

- How many bits does it take to encode $1{,}000{,}000$ different values?
  - Each bit can take on one of two values (0 or 1).
  - Therefore, n bits can represent 2^n values. (ex. 8bits: 0~255)
  - So, encoding 1,000,000 values will require $log_2 n = 20$ bits.
    To be exact, $\lceil log_2(1{,}000{,}000) \rceil$ = 20. // "a little under 20"

- ⌈ x ⌉ → **Ceiling** function: the smallest integer ≥ x. 올림
  - Ex.
- ⌊ x ⌋ → **Floor** function: the largest integer ≤ x. 버림
  - Ex.

$$\lceil 2.3 \rceil = 3 \qquad \lceil -2.3 \rceil = -2 \qquad \lceil 2 \rceil = 2$$

$$\lfloor 2.7 \rfloor = 2 \qquad \lfloor -2.7 \rfloor = -3 \qquad \lfloor 2 \rfloor = 2$$

1. Logarithms

Powers of 2:

- A bit is 0 or 1 (just two different "letters" or "symbols")
- A sequence of *n* bits can represent 2^n distinct things
  - For example, the numbers 0 through 2^n-1
- 2^{10} is 1024 ("about a thousand", kilo in CSE speak)
- 2^{20} is "about a million", mega in CSE speak
- 2^{30} is "about a billion", giga in CSE speak

Java: an **`int`** is 32 bits and signed, so "max int" is "about 2 billion"
a **`long`** is 64 bits and signed, so "max long" is $2^{63}-1$

1. Logarithms

Examples:

- If we have an alphabetically **sorted list of 100 names**, how many records do we need to look at to find a given individual?
  - Since the list is sorted, we can use **binary search**.
  - Look at the middle element: if it’s after than the name we’re looking for, search the first half of the list. If it’s before the name we’re looking for, look at the second half of the list.
  - Each check cuts the size of the list in half;
    how many times can we do this?

1. Logarithms

Examples:

- Let's suppose that we begin with a value N, divide it by 2, then the result that we divide it by 2, and so on, until reaching 1 or less.
  - N, N/2, N/4, ……, 4, 2, 1

- **Question:** How many times did we divide before reaching 1 or less?
  - Think of it from the other direction: How many times do I have to multiply by 2 to reach N?
  - 1, 2, 4, …., N/4, N/2, N
    Call this **k** number of times, then $N = 2^{k}$, or $k = \lg(N)$.

- **Exercise:** How is this related to the idea of binary search?
  - (I leave this one for you to think about it, as an exercise.)

1. Logarithms

Logarithmic Operators:

- $\log ab = \log a + \log b$
- $\log \frac{a}{b} = \log a - \log b$
- $\log a^b = b \log a$
- $log_a n = \frac{log_b n}{log_b a} = \frac{\log n}{\log a}$
  (this is used to change bases.)
- $log_a a = 1, for\ all\ a > 0$
- $log_a 1 = 0, for\ all\ a > 0$

- Evaluate $log_4 4 + log_2 2 + log_{10} 1$
- Evaluate $log_2 \frac{1}{2}$
- Plot $y = log_2 x$

Example: Solve for x. $x^{x^{x^{\cdots}}} = 2$

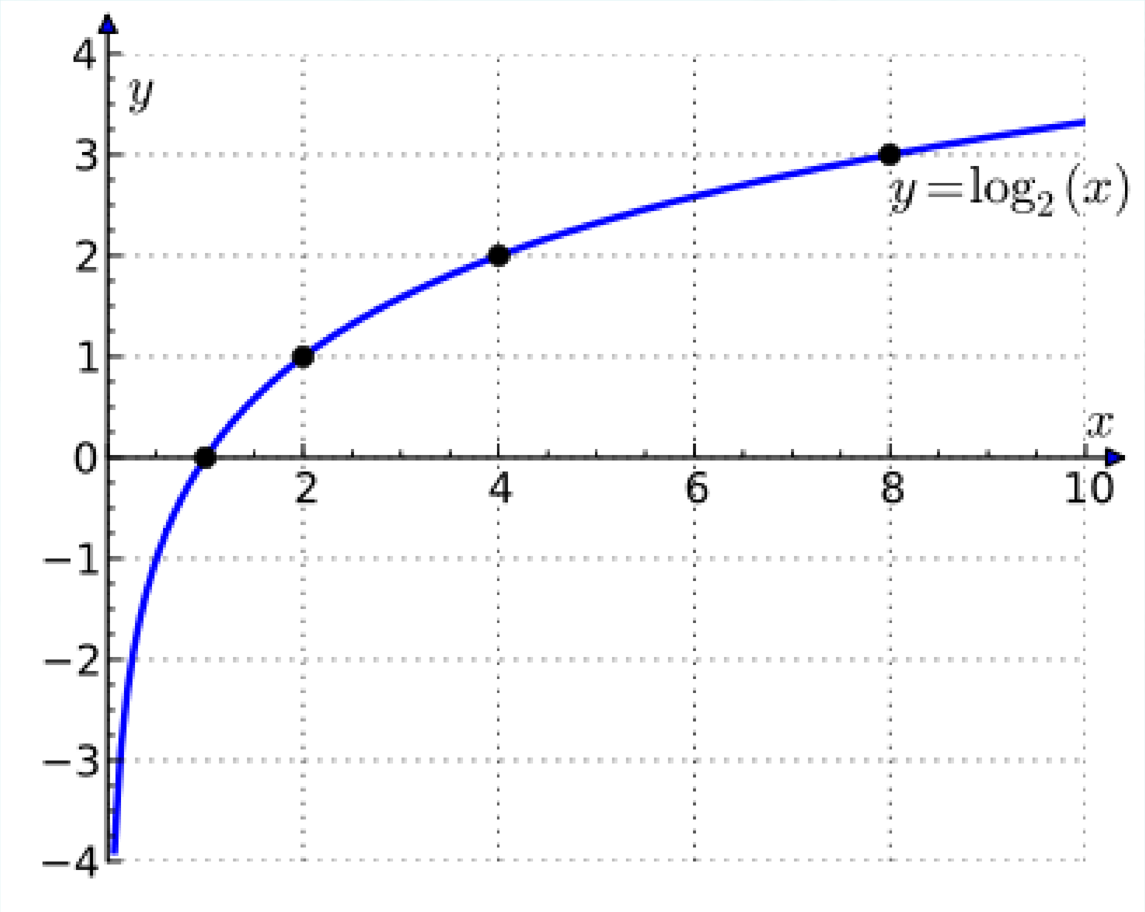
$$x^{x^{x^{\cdots}}} = 2$$

log x^x^x… = log 2
x^x.. log x = log 2
2 log x = log 2

logx = 1/2 log 2

1. Logarithms

- Plot $y = log_2 x$



1. Logarithms

Logarithmic Operators:

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- $log_a n = \frac{log_b n}{log_b a}$ (this is used to change bases.)

- Evaluate $log_4 2 + log_4 32$
  - $log_4 2 + log_4 32 = log_4 2 \ * 32 = log_4 64 = (log_4 4^3) = 3$
- Evaluate $log_2 400$ (all most all calculators don't do base 2…. wow!)
  - $log_2 400 = \frac{log_e 400}{log_e 2} = \frac{5.991}{0.69} = 8.68$
- What is x?
  - $2^x = 2^{10} + 2^{10}$
  - x lg 2 = lg ($2*2^{10}$), x = lg 2 + lg 2^{10} , x = 1 + 10 = 11

Logarithmic Operators:

- Logarithms can also be very useful for comparing very large or small numbers.
- **Exercise**: $10^{100} > 2^{256}$ is true or not?
  Neither number can be calculated directly without risking overflow.

- Hint:
  Since the logarithm function is monotonically increasing, if $a < b$, then $\log a < \log b$.
-
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Exercise: Compute the order of *growth rate b* in $T(n) \cong a\, n^b$ of the running time as a function of n using Selection Sort of which the time complexity is $O(n^2)$.

n	time
100	0.000023
200	0.000079
300	0.000173
400	0.000299
500	0.000477
600	0.000660
700	0.000904
800	0.001174
900	0.001468
1000	0.001818

As input size changes. the *growth rate* **b** of the execution time would be

$$\left(\frac{n_2}{n_1}\right)^{\boldsymbol{b}} = \frac{t_2}{t_1}$$

When input size increases twofold , it would be

$$(2)^{\boldsymbol{b}} = \frac{t_2}{t_1}$$

In case of the $O(n^2)$ algorithm, the growth rate should be close to 2.0.
Let's pick up n_1 = 500 to n_2 = 1000, *then* t_1 = 0.000477 to t_2 = 0.001818, respectively.
The growth rate of this algorithm is

$$b = \log\left(\frac{t_2}{t_1}\right) = \log\left(\frac{0.001818}{0.000477}\right) = \log(3.81) = \mathbf{1.93}$$

실제로 빅오가 n^2가 맞는지 확인해보라
계산할 때 걸린 시간을 체크해서 비율을 계산해보면 됨

1. Logarithms

Exercise: Compute the order of *growth rate b* in $T(n) \cong a\,n^b$ of the running time as a function of n using Selection Sort of which the time complexity is $O(n^2)$.

Let's pick up
n_1 = 4096, n_2 = 8192, $then$ t_1 = 0.022022, t_2 = 0.09075, respectively.
The ***measured*** *growth rate b* of the algorithm is

$$b = \log\left(\frac{t_2}{t_1}\right) = \log\left(\frac{0.09075}{0.022022}\right) = \log(4.12) = \mathbf{2.04}$$

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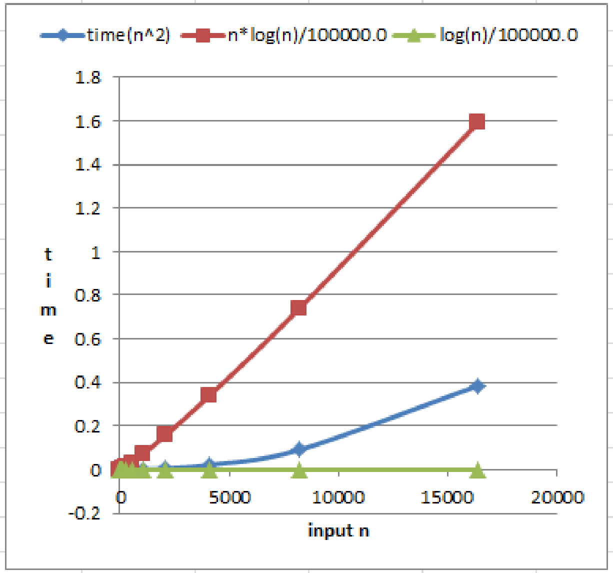
	A	B	C	D
1	n	time(n^2)	n*log(n)/100000.0	log(n)/100000.0
2		SelectionSort	QuickSort(average)	?
3	1	0	0	0
4	2	0	0.000014	0.000007
5	4	0	0.000055	0.000014
6	8	0	0.000166	0.000021
7	16	0	0.000444	0.000028
8	32	0.000002	0.001109	0.000035
9	64	0.000006	0.002662	0.000042
10	128	0.000023	0.006211	0.000049
11	256	0.000089	0.014196	0.000055
12	512	0.000364	0.03194	0.000062
13	1024	0.001427	0.070978	0.000069
14	2048	0.005464	0.156152	0.000076
15	4096	0.022022	0.340696	0.000083
16	8192	0.09075	0.738174	0.00009
17	16384	0.384667	1.589913	0.000097
18				



2. Summations

- In analyzing a program's performance, we'll need to add up the number of times an operation is taken.
- It is typically written as:

$$\sum_{i=1}^{n} f(i)$$

a closed form

which is equivalent to $f(1) + f(2) + \ldots + f(n)$

- **Example:**
  - summation of 1 … 10?
  - summation of 1 … n?

2. Summations

- We will be particularly interested in the following sum:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- We can easily see the solution if we add the sequence twice with one of the sequences written in inverse order.

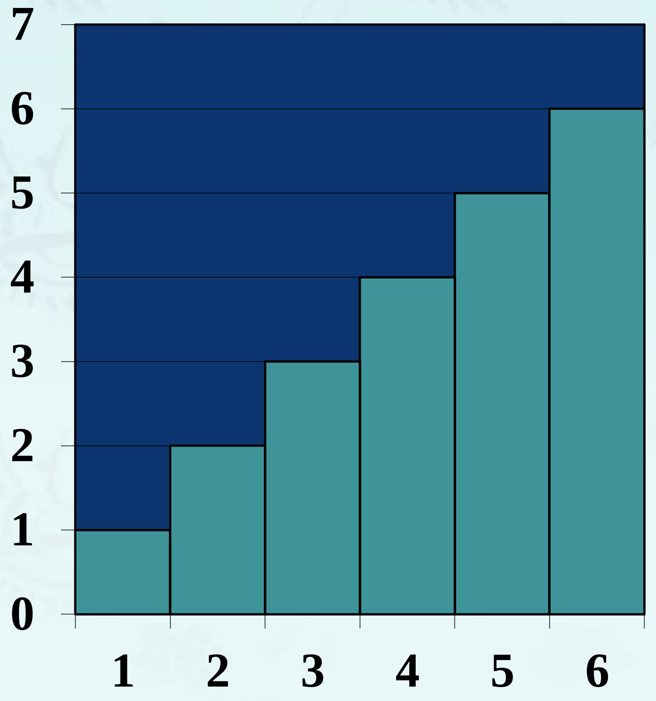
$$S = 1 + 2 + 3 + \ldots\ldots + (n-1) + n$$
$$S = n + (n-1) + (n-2) + \ldots\ldots + 2 + 1$$

- By adding two sequences, each of the pairs adds to $(n+1)$. There are n of them.
- $2S = (1+n) + (1+n) + (1+n) + \ldots\ldots + (n+1) + (n+1)$
- $\quad = n(n+1)$
- Therefore, $S = n(n+1)/2$.

2. Summations – Arithmetic Sum

- There is a simple visual proof of this fact for the following sum:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$



2. Summations – Arithmetic Sum

- Some common summations and their closed-form solutions:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + n}{6}$$

2. Summations – Geometric Sum

We are also be interested in the sum, it is so called **geometric** sum;

$$\sum_{i=0}^{n} a^i = 1 + a^1 + a^2 + \ldots\ldots + a^{n-1} + a^n$$

- **Proof:** Let's use S to denote the sum:
- $\boldsymbol{S} = 1 + a^1 + a^2 + \ldots\ldots + a^{n-1} + a^n$
- $\boldsymbol{aS} = a^1 + a^2 + \ldots\ldots + a^{n-1} + a^n + a^{n+1}$

$$= S + a^{n+1} - 1$$

From $aS = S + a^{n+1} - 1$, **we solve for $\boldsymbol{S}$, obtaining:**

$$\boldsymbol{S} = \sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}$$

2. Summations – Geometric Sum

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$$\sum_{i=0}^{n} a^i = 1 + a^1 + a^2 + \ldots\ldots + a^{n-1} + a^n = \frac{a^{n+1} - 1}{a - 1}$$

Exercise: $\sum_{i=0}^{n-1} a^i =$

Exercise: $\sum_{i=0}^{n} 2^i =$

2. Summations – Geometric Sum

Infinite geometric series….

$$\sum_{i=0}^{n} a^i = 1 + a^1 + a^2 + \;\dots\dots + a^{n-1} + a^n = \sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}$$

When $a < 1$ **and** n **goes to infinity**, the sum becomes

$$\sum_{i=0}^{\infty} a^i = \frac{\mathbf{1}}{\mathbf{1} - \boldsymbol{a}}$$

Exercise:

Compute the infinite "geometric" series intuitively and using arithmetic sum.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \;\dots\dots$$

Infinite geometric series….

$$\sum_{i=0}^{n} a^i = 1 + a^1 + a^2 + \;\ldots\ldots + a^{n-1} + a^n = \sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}$$

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$=$

$=$

2. Summations – Geometric Sum

Infinite geometric series….

$$\sum_{i=0}^{n} a^i = 1 + a^1 + a^2 + \ \dots\dots + a^{n-1} + a^n = \sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}$$

When $\boldsymbol{a} < \mathbf{1}$ **and $\boldsymbol{n}$ goes to infinity**, the sum becomes

$$\sum_{i=0}^{\infty} a^i = \frac{\mathbf{1}}{\mathbf{1} - \boldsymbol{a}}$$

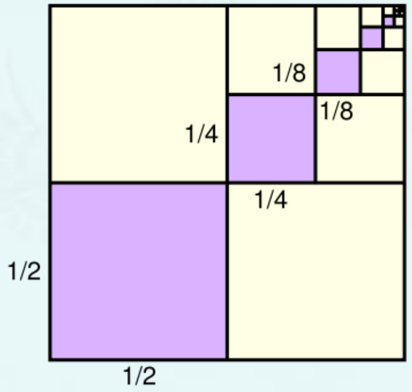
Exercise: Compute the sum of the areas of the purple squares.

Intuitively? Using arithmetic sum?

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} \dots\dots$$

$=$

$=$



3. Mathematical Proofs

- The three most common forms of proof are:
  1. Direct proof (sometimes called a constructive proof)
  2. Indirect proof or proof by contradiction
  3. **Inductive proof**
  - It is very similar to recursion.
  - You establish a **base case** that is proved directly.
  - This is followed by an **inductive step** which shows how the **hypothesis** holds for **larger cases**.

- **Inductive proof**
  - In data structures and algorithms, we often want to prove that something holds over a range of values

  - The **base case** will prove the theorem for the initial c values.
  - The **inductive step** will show that, if the theorem holds for $\boldsymbol{n-1}$, then it holds for $\boldsymbol{n}$.
    Alternatively, you may assume that it is true for n which is the inductive hypothesis first and then prove it for $\boldsymbol{n+1}$.
    or for **2n**.

This is so called the inductive hypothesis

3. Mathematical Proofs

Inductive proof Example: prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

- **Base case**: Let $n = 1$. $\frac{1(1+1)}{2} = \frac{2}{2} = 1.$
- **Inductive step**: We state the **inductive hypothesis** for $\boldsymbol{n-1}$ that:

$$\sum_{i=1}^{n-1} i = \frac{(n-1)((n-1)+1)}{2} = \frac{(n-1)(n)}{2}$$

- Assuming this is true, adding the $\boldsymbol{n}$**th term** yields:

$$\sum_{i=1}^{n} i = \frac{(n-1)(n)}{2} + n = \frac{(n-1)(n)}{2} + \frac{2n}{2} = \frac{n(n+1)}{2}$$

Therefore, we have proved it.

Inductive proof Example: prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

- **Base case**: Let $n = 1$. $\frac{1(1+1)}{2} = \frac{2}{2} = 1.$
- **Inductive step**: Assume that it holds for n, then for $2n$; that is :

$$\sum_{i=1}^{2n} i = \frac{2n(2n+1)}{2} = \boldsymbol{n(2n+1)}$$

Using the **induction hypothesis** that the left side above can be rewritten and rearranged algebraically:

$$\sum_{i=1}^{2n} i = \frac{n(n+1)}{2} + [(n+1) + (n+2) + \ldots + (n+n)]$$

$$= \frac{n(n+1)}{2} + n * n + \frac{n(n+1)}{2}$$

$$= n(n+1) + n * n$$

$$= \boldsymbol{n(2n+1)}$$

Therefore, we have proved it.

3. Mathematical Proofs

Inductive proof **Exercise:** prove $\sum_{i=1}^{n} i^2 = \frac{2n^3+3n^2+n}{6}$

- **Base case**: Let $n = 1$. $\frac{2+3+1}{6} = 1$.
- **Inductive step**: We state the **inductive hypothesis** for $\boldsymbol{n-1}$ that:

$$\sum_{i=1}^{n-1} i^2 = \frac{2(n-1)^3+3(n-1)^2+(n-1)}{6}$$

- Assuming this is true, adding the ***n*th term** yields:

$$\sum_{i=1}^{n-1} i^2 + n^2 = \frac{2(n-1)^3+3(n-1)^2+(n-1)}{6} + \frac{6n^2}{6}$$

$$= \frac{2n^3+3n^2+n}{6}$$

Therefore, we have proved it.

Exercise: Prove that $8^n - 3^n$ is divisible by 5 for all $n \geq 1$.

Summary

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