Advanced Macroeconomics: Structural Vector Autoregressive Analysis

Winter Examination 2021-2022: Take Home Assignment

Exam Number: 6

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1 The Data

Question 1

This paper considers data set `6.mat` that covers quarterly data covering the period from 1985:1-2021:4. Furthermore, this paper considers a monetary policy scheme for a small open economy that make use of inflation targeting. The quarterly data considered is (i) the natural logarithm of the gross domestic product (GDP), (ii) the natural logarithm of the price level (P), (iii) the interest rate, i.e., the policy rate (I) and (iv) the natural logarithm of exchange rate (E), which is measured as home currency units per unit foreign currency.

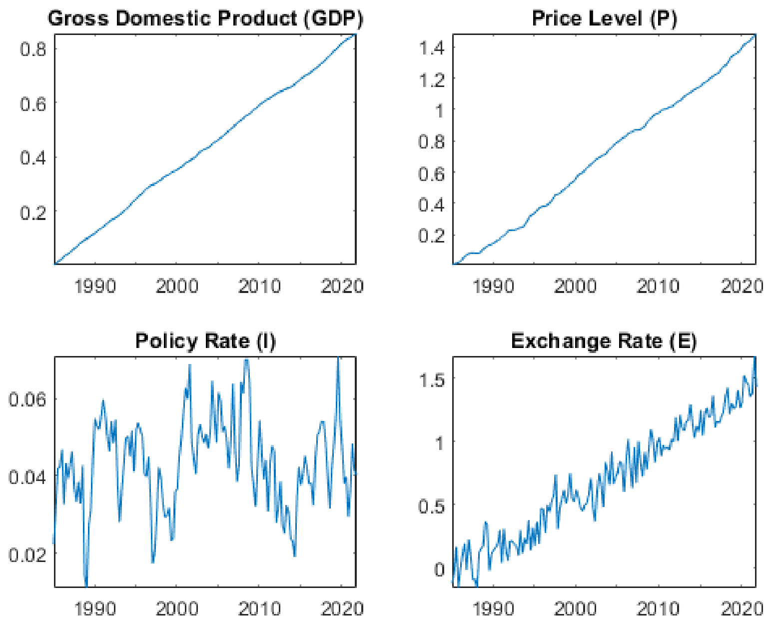

Figure 1: Plot of Data

From Figure 1, it is evident that there are linear trends in the data set. It is clear that GDP, P and E are increasing over time. Furthermore, it seems as if GDP, P and E have the same stochastic trend, i.e., they are unit-root processes, which could indicate that these three variables could co-integrate. Whereas GDP, P and E are increasing over time, the policy rate seems to be stationary. Thus, in conclusion, it seems as if, GDP, P and E are either trend-stationary, and thus, by definition $I(0)$ or integrated of order one $I(1)$, i.e., not trend-stationary, but the first-difference is covariance stationary (Bergman, 2018a). Moreover, it seems as if I is covariance stationary, i.e., $I(0)$.

2 The Vector Error Correction Model

Question 2

To show that the vector autoregressive (VAR) model can be rewritten to the vector error correction (VEC) model, as shown in equation (1). This paper considers the VAR(3), in order to show equation (1), however, this can be rewritten to generalised case, namely, the VAR(p). The VAR(3) can be written as

$$x_t = \nu + A_1 x_{t-1} + A_2 x_{t-2} + A_3 x_{t-3} + u_t \tag{2.1}$$

First-differences are applied to equation (2.1)

$$\Delta x_t = A_1 x_{t-1} - x_{t-1} + A_2 x_{t-2} + A_3 x_{t-3} + \nu + u_t \tag{2.2}$$

Rewriting the first-difference equation by adding and subtracting $A_3 x_{t-2}$, we find

$$\Delta x_t = (A_1 - I_4)x_{t-1} + A_2 x_{t-2} + A_3 x_{t-3} + A_3 x_{t-2} - A_3 x_{t-2} + \nu + u_t$$
$$\Delta x_t = (A_1 - I_4)x_{t-1} + A_2 x_{t-2} + A_3(x_{t-3} - x_{t-2}) + A_3 x_{t-2} + \nu + u_t$$
$$\Delta x_t = (A_1 - I_4)x_{t-1} + (A_3 + A_2)x_{t-2} + A_3(x_{t-3} - x_{t-2}) + \nu + u_t$$

Rewriting the first-difference equation by adding and subtracting $(A_2 + A_3)x_{t-1}$, we find

$$\Delta x_t = (A_1 - I_4)x_{t-1} + (A_3 + A_2)x_{t-2} + A_3(x_{t-3} - x_{t-2}) + (A_2 + A_3)x_{t-1} - (A_2 + A_3)x_{t-1} + \nu + u_t$$
$$\Delta x_t = (A_1 + A_2 + A_3 - I_4)x_{t-1} + (A_3 + A_2)(x_{t-2} - x_{t-1}) + A_3(x_{t-3} - x_{t-2}) + \nu + u_t$$
$$\Delta x_t = -(I_4 - A_1 - A_2 - A_3)x_{t-1} - (A_3 + A_2)(x_{t-1} - x_{t-2}) - A_3(x_{t-2} - x_{t-3}) + \nu + u_t$$

Which we can write as

$$\Delta x_t = \Pi x_{t-1} + \Gamma_1(x_{t-1} - x_{t-2}) + \Gamma_2(x_{t-2} - x_{t-3}) + \nu + u_t$$

Which we can rewrite to

$$\Delta x_t = \nu + \Pi x_{t-1} + \Gamma_1 \Delta x_{t-1} + \Gamma_2 \Delta x_{t-2} + u_t \tag{2.3}$$

In equation (2.3), we have shown the desired VEC model using that $\Pi = -(I_4 - A_1 -, \ldots, - A_3)$ and $\Gamma = -A_i + 1+, \ldots, +A_3$.

Question 3

The conditions that the adjustment coefficients, $\alpha_1 = \begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{bmatrix}'$, must satisfy for the system to have one co-integration vector with the given $\beta_1' = \begin{bmatrix}0 & -1 & 0 & 1\end{bmatrix}$, is that α_2 and α_4 must have opposite signs of β_2 and β_4. Additionally, it must also hold that the at least one of the adjustment coefficients must be different from zero, as this implies that one eigenvalue will be different from zero. We can show that for this specific problem

$$\det\left(\lambda \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} - \left[\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{bmatrix}\begin{bmatrix}0 & -1 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & -\alpha_1 & 0 & \alpha_1\\0 & -\alpha_2 & 0 & \alpha_2\\0 & -\alpha_3 & 0 & \alpha_3\\0 & -\alpha_4 & 0 & \alpha_4\end{bmatrix}\right]\right) \tag{2.4}$$

$$= \det\begin{bmatrix}-\lambda & -\alpha_1 & 0 & \alpha_1\\0 & -\alpha_2 - \lambda & 0 & \alpha_2\\0 & -\alpha_3 & -\lambda & \alpha_3\\0 & -\alpha_4 & 0 & \alpha_4 - \lambda\end{bmatrix} = \lambda^3(\alpha_2 - \alpha_4 + \lambda) \tag{2.5}$$

has solutions $\lambda_1 = -\alpha_2 + \alpha_4$, $\lambda_2 = 0$, $\lambda_3 = 0$ and $\lambda_4 = 0$. From equation (2.4-2.5), it is evident that the roots that solve the equation such that the system has rank equal to one must be $\lambda_1 = -\alpha_2 + \alpha_4$, as this is the only non-zero eigenvalue, this implies there must be only one co-integration vector in the system (Bergman, 2021).

Question 4

The conditions that the adjustment coefficients, $\alpha_2 = \begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{bmatrix}'$, must satisfy for the system to have one co-integration vector with the given $\beta_2' = \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}$, is that α_3 must have opposite sign of β_3. Moreover, it must also hold that the eigenvalue problem

$$\det\left(\lambda\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}-\left[\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{bmatrix}\begin{bmatrix}0&0&1&0\end{bmatrix}=\begin{bmatrix}0&0&\alpha_1&0\\0&0&\alpha_2&0\\0&0&\alpha_3&0\\0&0&\alpha_4&0\end{bmatrix}\right]\right) \tag{2.6}$$

$$=\det\begin{bmatrix}-\lambda&0&\alpha_1&0\\0&-\lambda&\alpha_2&0\\0&0&\alpha_3-\lambda&0\\0&0&\alpha_4&-\lambda\end{bmatrix}=\lambda^3(-\alpha_3+\lambda) \tag{2.7}$$

has solutions $\lambda_1=\alpha_3$, $\lambda_2=0$, $\lambda_3=0$ and $\lambda_4=0$. From equation (2.6-2.7) it is evident that α_3 must have non-zero eigenvalue in order for this system to have rank equal to one (Bergman, 2021).

3 Lag Order Determination and Diagnostic Testing

Question 5

To formulate a well-specified VAR model for, x_t, this paper first determines that appropriate lag length. There are three approaches, which we can deem relevant to determine the lag length these are based on (i) information criteria, (ii) general-to-specific sequence, i.e., top-down and (iii) specific-to-general sequence, i.e., bottom-up (Kilian and Lütkepohl, 2017). For the formulation of the VAR, this paper considers the minimisation of information criteria, namely, SIC, HQC and AIC[1]. The general form for VAR lag-order selection have the form

$$C(m)=\log(\det(\tilde{\Sigma}_u(m)))+c_t\phi(m)$$

where $\tilde{\Sigma}_u=T^{-1}\sum_{t=1}^{T}\hat{u}_t\hat{u}_t'$ is the residual covariance matrix estimator for a reduced-form VAR model of order m based on the least squares residuals $\hat{u}_t$, m is the candidate lag order at which the criterion function is evaluated, $\phi(m)$ is a function of the order m that penalises large lag order and c_T is a sequence of weights that may depend on the sample size (Kilian and Lütkepohl, 2017). Therefore, information criteria balances the objective of model fit and parsimony, and thus, the optimal lag length is obtained by minimising the information criteria, i.e.,

$$p=\arg\min_m(C(m))$$

Moreover, to make use of information criteria one must determine the maximum lag length, $p_{\max}$, to consider. This paper makes use of the maximum lag length suggest by (Schwert, 1989)

$$p_{\max}=\left[12\cdot(\frac{T}{100})^{\frac{1}{4}}\right]=13$$

In Table 1, this paper presents that results from the lag order determination using information criteria using the MATLAB function `pfinduniv.m`. According to Kilian and Lütkepohl (2017), AIC can be shown to be the most reliable in cases where the three information criteria should differ in conclusion. However, this paper finds that all information criteria minimise the criteria function to $p=2$.

Additionally, this paper has also made use of the general-to-specific approach, which also shows $p=2$ assuming that $p_{\max}=13$. Therefore, we conclude that the optimal lag length is equal to two and (cross) lag order determination method robust.

[1] For detailed analysis and interpretation of the specified information criteria, please see Kilian and Lütkepohl (2017).

p	**SIC**	**HQC**	**AIC**
0	-24.2995	-24.3506	-24.3856
1	-37.7194	-37.9749	-38.1498
2	**-38.0081**	**-38.468**	**-38.7829**
3	-37.601	-38.2653	-38.72
4	-37.0764	-37.9451	-38.5398
5	-36.6742	-37.7474	-38.482
6	-36.1871	-37.4646	-38.3391
7	-35.7284	-37.2103	-38.2248
8	-35.3782	-37.0645	-38.2189
9	-34.9518	-36.8425	-38.1368
10	-34.4369	-36.5321	-37.9663
11	-34.0429	-36.3424	-37.9166
12	-33.7046	-36.2086	-37.9227
13	-33.3218	-36.0301	-37.8841

Table 1: Lag Length Determination Procedure

Question 6

To verify whether the model is well-specified assuming the lag length is equal to two, we test for multivariate autocorrelation, heteroskedasticity and normality up to $h = p_{\max}$[2]. First, we have estimated the model using `VARls.m`, where we have allowed for a constant term, but no linear trend. The argument for excluding a linear trend is that VAR in levels, i.e., nominal value, can be rewritten as a VAR in first-differences (and a VEC model) by subtracting y_{t-1} from both sides of the levels, leaving the VAR constant term and the residuals unaffected by the subtraction. Second, before computing the misspecification tests, this paper computes the companion matrix, A, of the aforementioned VAR. This is computed to ensure that the stability of the system is intact. The stability condition states that the eigenvalues of the companion matrix must have modulus less than one (Kilian and Lütkepohl, 2017; Bergman, 2018b). The resulting companion matrix, which is computed using the MATLAB function `comp.m` is

$$A = \begin{bmatrix} 1.5482 & 0.0074 & -0.0050 & -0.0020 & -0.5564 & -0.0001 & 0.0067 & -0.0007 \\ -0.3607 & 1.4053 & 0.0438 & 0.0066 & 0.4317 & -0.4485 & 0.0573 & -0.0023 \\ 0.0002 & 0.1336 & 0.7344 & -0.0148 & -0.0667 & -0.0718 & -0.1358 & -0.0095 \\ -17.4225 & 6.3553 & -1.4891 & -0.1408 & 18.4603 & -5.6054 & -1.0602 & -0.1975 \\ 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The above companion matrix suggests a maximum eigenvalue of $\lambda = 1.005$, which indicates that the system has a unit-root, and hence, the stability condition is not satisfied. Thus, the infinite-order vector moving average representation does not exist (Kilian and Lütkepohl, 2017), which means that the common trends representation derivation could become complicated. However, in this paper, we assume that the stability condition is satisfied and continue.

To compute the test for multivariate autocorrelation, we use the function `Portmanteau.m`, and the results can be found in Table 2.

We cannot reject the null hypothesis of no autocorrelation in the residuals in both tests with test statistics of 168.5461 and 177.5828 and p-values of 0.6433 and 0.4523 respectively. Furthermore, to compute test for multivariate heteroskedasticity, we use the function `march.m`. The results can be found in Table 3.

We cannot reject the null of no ARCH with a test statistic of 213.9245 and a p-value of 0.2376.

[2]For a rigorous analysis and interpretation of the misspecification tests, please see Kilian and Lütkepohl (2017).

Test	**Portmanteau**	**Modified Portmanteau**
test statistic	168.5461	177.5828
p-value	0.64333	0.45239
degrees of freedom	176	176

Table 2: Test for Multivariate Autocorrelation

Test	**Doornik-Hendry**
test statistic:	213.9245
p-value	0.23766
degrees of freedom	200

Table 3: Test for Multivariate Heteroskedasticity

Moreover, we use the function `multnorm.m` to compute the test for multivariate non-normality, and the results can be found in Table 4.

Test	**Doornik-Hansen**	**Lütkepohl**
joint test statistic:	5.2873	7.2677
p-value	0.72647	0.50804
degrees of freedom	8	8
Skewness only	0.94614	4.6961
p-value	0.91785	0.31992
kurtosis only	4.3412	2.5716
p-value	0.3618	0.63187

Table 4: Test for Multivariate Non-Normality

We find that we cannot reject the null hypothesis that the residuals are normally distributed with joint test statistic of 5.2873 and 7.2677 with p-values of 0.7264 and 0.5080 respectively, which, in theory, i.e., the central limit theorem (CLT), states that our model is consistent with regards to asymptotic inference. The individual tests suggest that we cannot reject the null of skewness, but no kurtosis and the null of kurtosis but no skewness, which also indicates that the residuals are normally distributed in the third and fourth order moments.

In addition to the multivariate misspecification tests, this paper has also tested the univariate case of each equation, and found that none of the individual equations that are in the VAR, violate the misspecification tests regarding univariate autocorrelation and heteroskedasticity.

Overall the above misspecification tests suggest that our preferred model with $p = 2$ is well-specified at the five per cent level.

4 Testing for Co-integration

Question 7

Assuming $p = 2$ in the underlying VAR, this implies that the lag length in the VEC is equal to 1. To compute the number of co-integration system, i.e., the rank, r. This paper uses, the built-in MATLAB function `jcitest.m`. This paper presents the following results in Table 5 and 6 for the system allowing for a constant term in the co-integration vector and both a constant and a linear trend in the co-integration vector[3].

[3]One might note that H1 and H* in MATLAB allows for deterministic linear trends in the levels of the data. However, as first-differences are taken this results in a constant term, which - if insignificant - would result in its value being equal to zero.

r	h	stat	cValue	pValue	eigValue
0	1	147.8899	47.8564	0.0010	0.5114
1	1	43.3240	29.7976	0.0010	0.2011
2	0	10.5386	15.4948	0.2642	0.0682
3	0	0.2283	3.8415	0.7376	0.0016

Table 5: Result Summary: Johansen Test - Model H1 (Significance Level 0.05)

r	h	stat	cValue	pValue	eigValue
0	1	154.8933	63.8766	0.0010	0.5129
1	1	49.8863	42.9154	0.0090	0.2033
2	0	16.7052	25.8723	0.4748	0.0733
3	0	5.5904	12.5174	0.5650	0.0376

Table 6: Result Summary: Johansen Test - Model H* (Significance Level 0.05)

Using the approach suggest by Johansen and Juselius (1990), this paper has implemented a trace test to determine the co-integrating rank of the VAR system. This approach suggests that we consider the null hypothesis that the rank is equal to zero, i.e., $r = 0$, first and increase until $r = K$, where $K - 1$ denotes the number of variables in the system. This paper finds that the null hypotheses that the rank is equal to zero and one is rejected at the one per cent level for both co-integration system. Increasing the rank to two, we find that this paper cannot reject the null that the rank is equal to two at the five per cent level in both systems. Additionally, we see that we cannot reject the the null that rank is equal to three at the five per cent level. However, this paper notes that the eigenvalue decreases by substantially more from $r = 1$ to $r = 2$ than from $r = 2$ to $r = 3$, which supports choosing $r = 2$ over $r = 3$, i.e., full rank.

In addition to this, we present a sensitivity analysis on the importance of lag length when determining the co-integrating rank. The lag length is based on the VEC, i.e., $p = 1$. In Table 7, we present our sensitivity analysis, which suggest that our findings are somewhat delicate, as the preferred rank, $r = 2$, changes when the number of lags are increased to four and three for respectively H1 and H*. However, we know from Johansen and Juselius (1990) that when relying on asymptotic results careful study is needed when the sample size is small.

p	r_{H1}	r_{H*}
1	2	2
2	2	2
3	2	1
4	1	1
5	1	0
6	0	0
7	1	0
8	0	0
9	0	0
10	0	0
11	0	0
12	1	1
13	1	2

Table 7: Sensitivity Analysis: Rank Determination Based on Lag Length

Question 8

In our above model, namely, H*, we have allowed for a linear trend in the co-integration vector. In theory, we distinguish between five different cases depending on the assumption about the deterministic components in the data generating process (DGP). In our case, it is evident that a constant term is necessary (from the graphical analysis), as it will allow us to control the unit measurement of the data, and hence, we keep an unrestricted VEC. However, it is not evident if we should include a linear trend in the co-integration. Therefore, we test the null hypothesis that there is no linear trend in the co-integration vectors. To test this, this paper uses the function `jcontest.m`, with the linear constraints on $\beta = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}'$, i.e., constraining the linear trend to take on value equal to zero. The results provide us with a likelihood ratio test statistic of value 0.8368 and with corresponding p-value of 0.6581. Thus, we cannot reject the null that the linear trend in the co-integrating vectors have value equal to zero.

Question 9

We impose our preferred rank, i.e., $r = 2$ and specification of deterministic component, namely, H1, and test the hypotheses on stationarity, exclusion and weak exogeneity. First, the test for stationarity tests whether variable, i, is stationary. For example, $\beta = (\beta_1,\beta_2,\beta_3)' = (0,1,0)'$, where we test whether the unit vector with "1" in position i is a co-integrating vector. Then, we find that:

$$\beta_1 y_{1,t} + \beta_2 y_{2,t} + \beta_3 y_{3,t} = 0 \cdot y_{1,t} + 1 \cdot y_{2,t} + 0 \cdot y_{3,t} = y_{2,t} \ I(0)$$

Thus, $y_{2,t}$, is stationary.

Secondly, the test for exclusion is conducted. This test tests if a variable, i, can be excluded from all co-integrating relationships. The test is conducted by testing if the coefficient belonging to variable, i, is equal to zero, for example, $\beta_1 + \beta_2 + \beta_3 \sim I(0)$. If we cannot reject the null hypothesis of $\beta_i = 0$, we conclude that it can be excluded from the co-integrating vector.

Third, we test for weak exogeneity, which tests whether variable i depend on the co-integrating relationship, namely:

$$\alpha\beta' y_{t-1} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \\ \alpha_{31} & \alpha_{32} \end{pmatrix} \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \\ y_{3,t-1} \end{pmatrix}$$

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \\ \alpha_{31} & \alpha_{32} \end{pmatrix} \begin{pmatrix} \beta_{11} + y_{1,t-1} & \beta_{12} + y_{2,t-1} & \beta_{13} + y_{3,t-1} \\ \beta_{21} + y_{1,t-1} & \beta_{22} + y_{2,t-1} & \beta_{23} + y_{3,t-1} \end{pmatrix}$$

This test has the purpose of indicating whether a given variable, i, has a speed of adjustment coefficient, generally denoted α different from zero. In case we are not able to reject the hypothesis of an $\alpha_i = 0$, we cannot reject the hypothesis that the variable, i, is weakly exogenous, implying it does not move towards the long-run equilibrium, denoted by the cointegrating vector, β, when outside and could be excluded from the co-integrating relationship.

These tests are using the MATLAB function `SEWE.m`, which implements the `jcontest.m` with option `Bvec`, `Bcon` and `Acon` for respectively, stationarity, exclusion and weak exogeneity. The results can be found in Table 8 In the above Table, we find that we cannot reject that the policy rate is stationary, which is in line with our assumption from Question 1. Furthermore, we find that we cannot reject that GDP can be excluded from the co-integrating relationship, which is concur with our assumption from Question 1, i.e., GDP is most likely to be considered stationary around a linear trend, implying that GDP does not contribute to the co-integrating relationship. Moreover, we find that we cannot reject that GDP is weakly exogenous, i.e., it does not error correct when in disequilibrium, which also coincides with the fact that we cannot reject that we can exclude GDP from the co-integrating relationship.

Stationarity	**Exclusion**	**Weak exogeneity**
0.00	0.18	0.09
0.00	0.00	0.01
0.18	0.00	0.00
0.00	0.00	0.00

Table 8: Result Summary: SEWE Tests (Significance Level 0.05)

Question 10

In this paper, we estimate the unrestricted (normalised) co-integration vectors. The first co-integration vector, we have normalised w.r.t. the purchasing power parity (PPP) PPP $\beta_1' = \begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix}$, i.e., normalising w.r.t. the fourth element in β_1'. We find this estimated co-integration vector to be

$$\beta_{e1}' = \begin{bmatrix} -0.0331 & -0.9747 & 3.5115 & 1 \end{bmatrix} \tag{4.1}$$

The second co-integration vector, we have normalised w.r.t.the Fisher relation $\beta_2' = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$, i.e., normalising w.r.t. the third element in β_2'. We find this estimated co-integration vector to be

$$\beta_{e2}' = \begin{bmatrix} 0.3129 & -0.1503 & 1 & -0.0247 \end{bmatrix} \tag{4.2}$$

First, we find somewhat plausible values for the estimated normalised beta w.r.t. the PPP relation, the first element is close to zero, the second element is close to the fourth element with opposite as theorised by the PPP relation co-integration vector. However, the third element does seem off, as this should zero. Tentatively (without any analysis of the significance level), this indicates that the coefficient to the policy rate is not equal zero, and thus, that the policy rate has somewhat of an influence on the PPP relation in the long-run, which is contradictory to the theory. Second, we again find somewhat plausible values for the estimated normalised beta w.r.t. the Fisher relation, as the second and fourth element are close to zero, which is theorised. However, again we see that one element, namely, the first element is does seem off, as this also should be zero. Again, with reservations, this indicates that GDP has somewhat of an effect on the long-run equilibrium between the nominal and real interest rates under the effect of inflation.

Question 11

In order to test whether the theoretical co-integration vectors concur with the information from the DGP, this paper uses the built-in MATLAB function `jcontest.m`. The results are presented in Table 9.

	LR-test statistic	**Conclusion(1=rejected)**	**p-value**
PPP	2.5145	0	0.2844
Fisher Relation	3.3906	0	0.1835
PPP & Fisher Relation	3.6950	0	0.6581

Table 9: Result Summary: Testing PPP and Fisher Relation (Significance Level 0.05)

In the above table, we find that we cannot reject PPP, the Fisher relation and PPP and the Fisher relation tested simultaneously. In regard to exclusion, we earlier found (Question 9) that GDP could not be rejected to be excluded from the co-integrating system. This coincides with our theorised co-integration vector where the coefficient to GDP is equal to zero and that we cannot reject any of our theorised co-integration vectors. Moreover, concerning the stationarity test, we find that we cannot reject that the policy rate is stationary (Question 9). This accords with priors regarding the theorised co-integration vectors, where in the PPP relation the policy rate is set to have no effect in the long-run and in the Fisher relation the policy rate by definition must be stationary.

Question 12

In Figure 2, this paper plots the unrestricted (normalised) and the theoretical co-integration vectors.

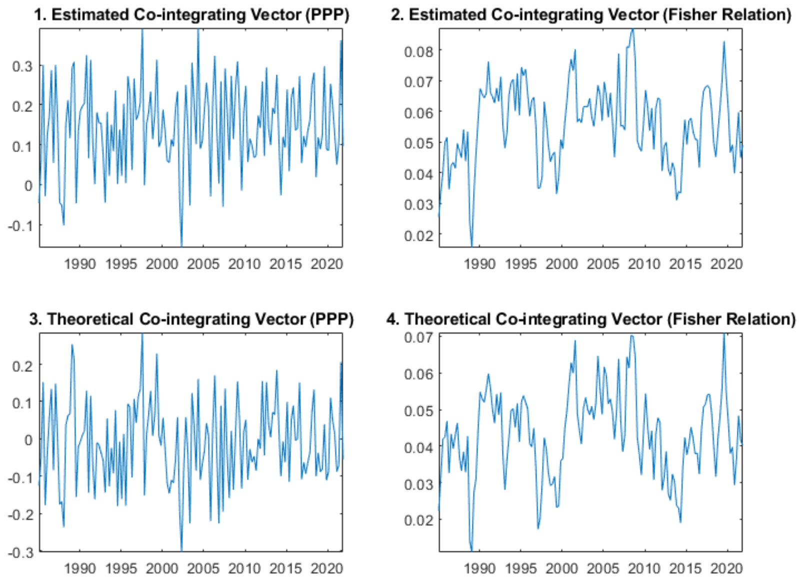

Figure 2: Plot of Unrestricted and Theoretical Co-integration Vectors

Interpreting the above plots, we quickly see that the co-integration vectors from the DGP are almost indistinguishable, which coincides with our earlier findings. We note that all plots seem stationary, especially the plots regarding PPP. Therefore, we conclude that estimated and theorised beta values are chosen such that $\beta' x_t$ is not only as stationary as possible, but stationary.

5 Identification of Structural Model

Question 13

Imposing $r = 2$, which we have already suggested and estimating the model using the full sample and lag length equal to two. This paper computes the adjustment coefficients, α_i, for the PPP, Fisher relation and the PPP and Fisher relation. The adjustment coefficients are presented in the following equations

$$\alpha_1 = \begin{bmatrix} -0.0029 & -0.0000 \\ \mathbf{0.0090} & -0.0011 \\ -0.0364 & 0.0038 \\ \mathbf{-1.3784} & 0.0233 \end{bmatrix} \tag{5.1}$$

$$\alpha_2 = \begin{bmatrix} -0.0002 & 0.0025 \\ 0.0003 & 0.0957 \\ -0.0016 & \mathbf{-0.3906} \\ -0.0879 & -2.6584 \end{bmatrix} \tag{5.2}$$

$$\alpha_3 = \begin{bmatrix} -0.0028 & 0.0024 \\ \mathbf{0.0051} & 0.0958 \\ -0.0251 & \mathbf{-0.3923} \\ \mathbf{-1.3246} & -2.6979 \end{bmatrix} \tag{5.3}$$

Where α_1, α_2 and α_3 denotes the adjustment coefficients to PPP, Fisher relation and the PPP and Fisher relation respectively. We find that the signs of the adjustment coefficients coincides with earlier assumptions (Question 3 and 4), which states that the adjustment coefficients must have opposite signs w.r.t. their corresponding beta coefficient in order for the PPP and the Fisher relation to exhibit mean-reversion.

Question 14

In the present VAR model, we have four variables, K, and thus, four structural shocks. Additionally, we have established a rank equal to two, $r = 2$, which indicates that there are two common trends or permanent shocks, $K - r$. Moreover, there are two transitory shocks, as this is equal to the co-integration rank (Kilian and Lütkepohl, 2017).

As mentioned earlier, we are working with a small open economy with inflation targeting. This suggests that we do not have any permanent effects from the policy rate on the variables in the system, and thus, its effects on the system is only transitory. Furthermore, the exchange rate can also be considered a transitory variable, as we have not been able to reject the PPP in our system. Therefore, exchange rate shocks will have no effect in the long-run on the variables in the system. This also coincides with the literature, where we know that shocks from nominal variables do affect real variables in the long-run (Bergman, 2021). This implies that we have two permanent shocks, namely, a productivity shock and a price shock. In theory, we would assume a positive productivity shock has a positive permanent effect on GDP and a negative permanent effect on price level. Moreover, we would assume that a positive price shock is associated with no long-run effects on other variables than it self. In conclusion, we have two transitory shocks, namely, a monetary policy shock and a exchange rate shock. Furthermore, we have two permanent shocks, i.e., a productivity shock and a price shock. Thus, we must impose one long-run and one short-run restriction to identify the shocks. Given the ordering of our VAR, we impose that the first element in the second column in long-run multiplier, Υ is equal to zero. This implies that a price level shock does not have long-run effects on GDP. Additionally, we must impose one short-run restriction in order to identify the transitory shocks. We impose the restriction that the first element in the second column in $B_0^{-1} = 0$. This implies that a monetary policy shock does not have any contemporaneous effects on GDP.

6 Impulse Responses and Forecast Error Variances

Question 15

Implementing the identification scheme suggested earlier (Question 14), this paper make use of the MATLAB solver function. Computing B_0^{-1}, Υ and Σ, i.e., variance-covariance matrix of the identified structural shocks, we find that solver function provides valid identification, as the variance-covariance matrix is an identity matrix, i.e., $\Sigma = I_4$ and that $B_0^{-1}B_0^{-1\prime} - \Sigma_u = 0_{4\times 4}$. The identification of B_0^{-1} and Υ for the theoretical co-integration vectors results in the following

$$B_0^{-1} = \begin{bmatrix} 0.0010 & 0.0000 & -0.0000 & 0.0003 \\ 0.0003 & 0.0040 & -0.0013 & 0.0012 \\ 0.0030 & 0.0020 & 0.0052 & -0.0044 \\ -0.0439 & 0.0038 & 0.0480 & 0.0813 \end{bmatrix}$$

$$\Upsilon = \begin{bmatrix} 0.0029 & 0.0000 & 0.0000 & -0.0000 \\ -0.0001 & 0.0087 & 0.0000 & 0.0000 \\ 0 & 0 & 0 & 0 \\ -0.0001 & 0.0087 & 0.0000 & 0.0000 \end{bmatrix}$$

The above is in line with the restrictions implemented on B_0^{-1} and Υ, which can be found in the Appendices (Appendix A). Moreover, we have ensured that the diagonal in B_0^{-1} is positive, which ensures that we can interpret the shocks as positive shocks.

Question 16

Using the above MATLAB solver, we have computed the impulse responses (IRF)s and forecast error variance decomposition (FEVD)s and computed standard errors using a standard residual based recursive design bootstrap with 500 trials (using a 95 per cent significance level corresponding to two standard deviations) and Efron's percentile intervals for the FEVDs. Moreover, we check that each identification is valid in each replication by ensuring that the maximum eigenvalue in the companion matrix in each iteration is less than one, which was also implemented when checking the original VAR model. In Figure 3 and 4, we present the effects of a monetary policy shock to each variable and their corresponding FEVDs.

When examining Figure 3, we first notice that all IRFs converge towards zero in the long-run. This concurs with our restrictions, as we have restricted that a monetary policy shock has no long-run effects on the variables in the system. However, we do notice that a positive monetary policy shock does have different contemporaneous effects on the variables in the system. For example, we find that a monetary policy shock to GDP has no contemporaneous effects, which is also what we have restricted, i.e., the impulse response function starts at zero, but negative short-run restrictions. Moreover, we find that a monetary policy shock has a negative contemporaneous effect on price level. This coincides with economic intuition, as a positive monetary policy shock insinuates a contractionary monetary policy, which has been known to reduce output and lower prices in the short-run. Moreover, we see that a positive monetary policy shock has a positive contemporaneous effect on the policy rate, which is concur with the fact that a central bank increase the interest rate in the short-run. Furthermore, we see that positive monetary policy shock has a positive effect in *very* short, as an increase in interest rate will decrease the price level, and thus, depreciate the domestic currency, however, this is quickly reversed, as depreciation of domestic currency will favour foreign investments, which will increase the price level, and thus, appreciate domestic currency (Bergman, 2021).

When examining Figure 4, we see the variance of the forecast error given by the positive monetary policy shock. We find that the FEVD of a positive monetary policy shock does not explain much of the variance in GDP and price level. However, we do find that a monetary policy shock seems to explain a significant share of the exchange rate forecast error variance, which does not seem to decrease over time.

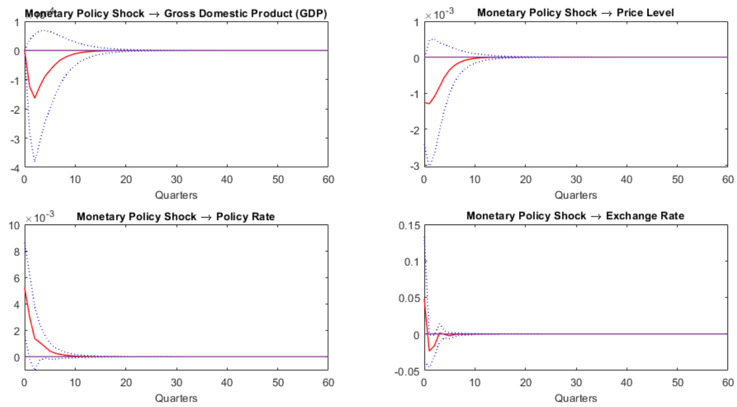

Figure 3: IRF: Positive Monetary Shock to Variables

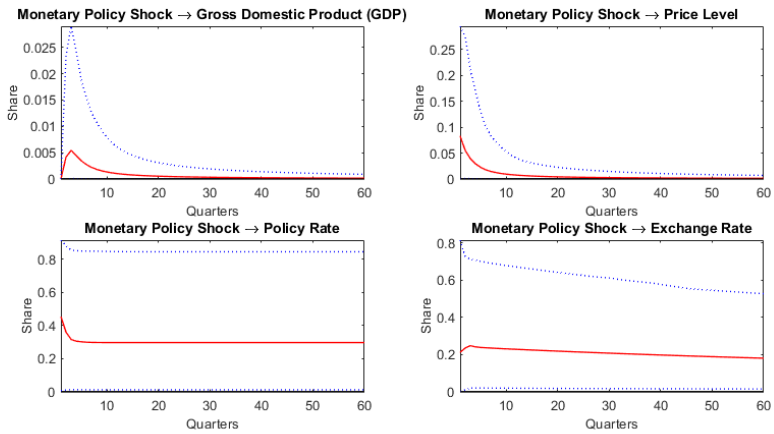

Figure 4: FEVD: Positive Monetary Shock to Variables

Question 17

In this section, we present the IRFs for the remaining three shocks, which can be found in Figure 5. Additionally, we present the FEVDs to the three remaining shocks in the Appendices (Appendix B).

When examining Figure 5, we find that a positive productivity shock a positive long-run effect on GDP, negative long-run effect on price level and no long-run effect on policy rate and exchange rate. However, in the short-run, we do see that a positive productivity shock has a positive effect on the policy rate and a negative effect on the exchange rate. These effects coincides with economic theory (AS-AD), which states that a productivity shock will drive price down, however, the central bank will react due to its monetary policy and inflation targeting, and thus, the price level will converge back to a new steady state, which is just below the original, as the productivity shock results constantly higher GDP, whilst effect of the productivity shock wears off on the policy rate (Bergman, 2021). Additionally, we see that a productivity shock has a negative contemporaneous effect on the exchange rate, which is not in line with the fact that the price level is decreasing in the short-run and constant in the long-run. Moreover, this does not concur with the fact that the exchange rate is defined as home currency units per unit foreign currency. Thus, PPP in the short-run does not hold, whilst in the long-run it is unclear.

Moreover, we find that a positive price shock has positive contemporaneous effects on all variables. In the long-run, we find that a positive price shock has transitory effects on GDP and the policy rate. This coincides with economic theory, which postulate that an increase in prices will drive productivity due to an increase in aggregate demand, however, this only transitory. Additionally, an increase in price level will force the central bank to react in contrast to the increase, which will increase the policy rate, again, only in the short-run. Moreover, we see that a positive price shock has positive contemporaneous effects on the exchange rate, which will reach a new steady state. This is again not in line with PPP in the short-run. However, we do notice that both price level and exchange rate are constant in the long-run suggesting that PPP holds in the long-run.

Furthermore, we find that a positive exchange rate shock has a positive contemporaneous effect on all variables, however, insignificant on price level. Additionally, we find that all effects from the exchange rate shock are transitory. This coincides with economic theory, as an increase in the exchange rate will increase GDP through increased foreign investments due to increased competitiveness. The increase in competitiveness comes from the fact that foreign investors have increased their purchasing power. From the increase in GDP, the price level will increase, and thus, we will see a increase in the policy rate.

Overall, we find that the impulse responses are consistent with economic intuition, albeit some IRFs do concur with PPP. In general, the shocks are insignificant at the five per cent level when looking at the confidence bands.

Question 18

Implementing the identification scheme suggest earlier (Question 14), but with the normalised co-integration vectors, which are computed to be

$$\beta_{e1}' = \begin{bmatrix} -0.0331 & -0.9747 & 3.5115 & 1.0000 \end{bmatrix}$$
$$\beta_{e2}' = \begin{bmatrix} 0.3129 & -0.1503 & 1.0000 & -0.0247 \end{bmatrix}$$

instead of the theoretical co-integration vectors, we find the following B_0^{-1} and Υ

$$B_0^{-1} = \begin{bmatrix} 0.0010 & 0.0001 & -0.0000 & 0.0002 \\ 0.0004 & 0.0037 & -0.0019 & 0.0011 \\ 0.0019 & 0.0034 & 0.0061 & -0.0027 \\ -0.0369 & -0.0064 & 0.0207 & 0.0941 \end{bmatrix}$$

$$\Upsilon = \begin{bmatrix} 0.0028 & 0.0000 & 0.0000 & -0.0000 \\ 0.0002 & 0.0082 & 0.0000 & 0.0000 \\ -0.0008 & 0.0013 & -0.0000 & 0.0000 \\ 0.0030 & 0.0034 & 0.0000 & -0.0000 \end{bmatrix}$$

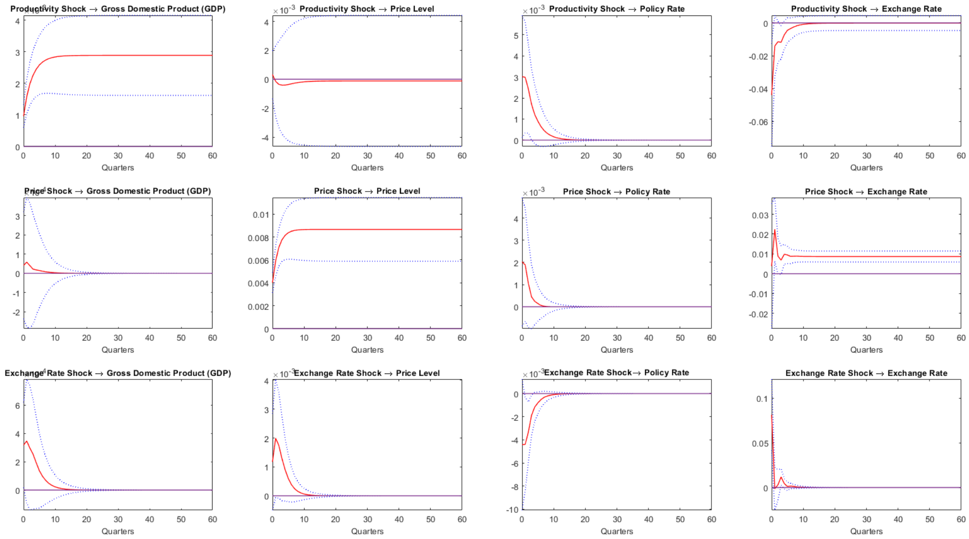

Figure 5: IRF: Remaining Positive Shocks to Variables

The above coincides with the restrictions implemented on B_0^{-1} and Υ, which can be found in the Appendices (Appendix A). Moreover, we present IRFs of the DGP in regard to a positive monetary policy shock with confidence bands based on a standard residual based recursive design bootstrap with 500 trials (using a 95 per cent significance level corresponding to two standard deviations). The IRFs can be found in Figure 6.

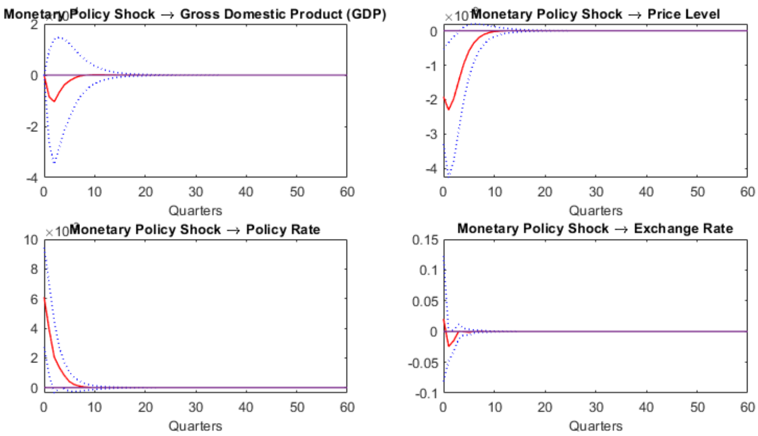

Figure 6: IRF: Positive Monetary Shock to Variables (Estimated Co-integration Vectors)

When examining Figure 6, we find that they are almost identical to the IRFs computed using the theoretical co-integration vectors. This confirms that the DGP coincides with theory regarding the co-integration vectors defined as PPP and the Fisher relation. Moreover, FEVDs for the IRFs computed using the estimated co-integrations vectors can be found in the Appendices (Appendix D).

7 Extensions

Question 19

As suggested by Warne (1993), this paper presents a close-form solution to compute B_0^{-1} from the structural form common trends model (CT). First, we decompose

$$B_0^{-1} = \begin{bmatrix} F_k \\ F_r \end{bmatrix}$$

Where F_k identifies the permanent shocks and F_r the transitory shocks under the normalisation that $\Sigma_w = I_K$. Following Warne (1993), we know that we need to define Υ_0, such that $\beta'\Upsilon_0 = 0$. This implies that the freely estimated parameters in Υ are lumped into the $k \times k$ matrix π. It is possible to make use of the theoretical or the estimated co-integration vectors to solve for Υ_0. In this paper, we use the theoretical co-integration vectors. Given that $\Upsilon\Upsilon' = \Xi\Sigma_u\Xi'$, we find that using a Cholesky decomposition

$$\pi\pi' = (\Upsilon_0'\Upsilon_0)^{-1}\Upsilon_0'\Xi\Sigma_u\Xi'\Upsilon_0(\Upsilon_0'\Upsilon_0)^{-1}$$

Where we note that given our choice of Υ_0 and the estimated VEC model the expression given above is known. This implies that we can compute π, which hence implies that we can compute $\Upsilon = \Upsilon_0\pi$. Thus, the permanent shocks are identified, as

$$F_k = (\Upsilon'\Upsilon)^{-1}\Upsilon'\Xi$$

Similarly, we can identify the transitory shocks. First, we define a matrix, U,

$$U = \begin{bmatrix} 0_{r\times K-r} & I_r^+ \end{bmatrix}$$

Moreover, we define $\varepsilon = \alpha(U\alpha)^{-1}$. Second, we compute a matrix, Q, which is a Cholesky decomposition of $\varepsilon'\Sigma_u^{-1}\varepsilon$, and thus, the transitory shocks are identified using the matrix

$$F_r = Q^{-1}\varepsilon'\Sigma_u^{-1}$$

The above is equal to F_r in B_0^{-1}. In our case, we have two theoretical co-integration vectors. Thus, we define

$$\Upsilon_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

such that

$$\beta'\Upsilon_0 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{7.1}$$

Furthermore, as we have two transitory shocks, namely, the policy rate and the exchange rate, we impose restrictions on U

$$U = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Such that the first element in the third column is equal to zero.

Question 20

In this section, we present the closed-form solution and the implied B_0^{-1} using the Warne (1993) approach.

$$B_0^{-1} = \begin{bmatrix} 0.000960680772470 & 0.000041284502895 & -0.000000000000000 & 0.000317066104368 \\ 0.000324526337224 & 0.004015217695535 & -0.001260856456173 & 0.001174100447646 \\ 0.003017807714740 & 0.002027988501840 & 0.005208666589232 & -0.004417137676550 \\ -0.043918905950840 & 0.003779241882458 & 0.048004136557541 & 0.081304796765347 \end{bmatrix}$$

Comparing this to B_0^{-1} found using the solver function

$$B_0^{-1} = \begin{bmatrix} 0.000960680927795 & 0.000041284491528 & -0.000000000000001 & 0.000317066116224 \\ 0.000324526267910 & 0.004015217688357 & -0.001260856456160 & 0.001174100492008 \\ 0.003017807348355 & 0.002027988685037 & 0.005208666589178 & -0.004417137843615 \\ -0.043918900633062 & 0.003779238916195 & 0.048004136556655 & 0.081304799788204 \end{bmatrix}$$

Thus, we have shown the closed-form solution is identical to the one found by the solver at least 9-10 decimal points. The code for the identification using the Warne (1993) approach can be found in the appendices (Appendix D).

8 References

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9 Appendices

Appendix A - Identifying Restrictions

```
% restrictions.m
function q=restrictions(B0inv)
global GAMMA SIGMA alpha beta alpha_perp beta_perp Xi p so
K=size(B0inv,1);
THETA1=Xi*B0inv;
% This is Upsilon
F=vec(B0inv*B0inv'-so(1:K,1:K));
% Long run and short run restrictions
q=[F; B0inv(1,3); THETA1(1,2); THETA1(1,3);
THETA1(1,4); THETA1(2,3); THETA1(2,4);
THETA1(3,3); THETA1(3,4); THETA1(4,3); THETA1(4,4)];
q'+1;
```

Appendix B - FEVDs to Remaining Shocks

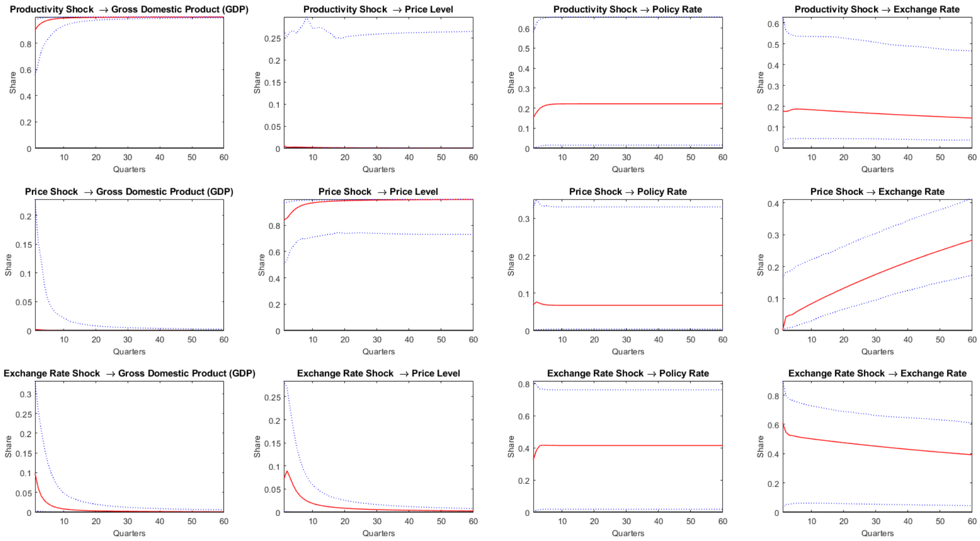

Figure 7: FEVD: Remaining Positive Shocks to Variables

Appendix C - FEVDs to a Positive Monetary Policy Shock (Estimated Co-integration Vectors)

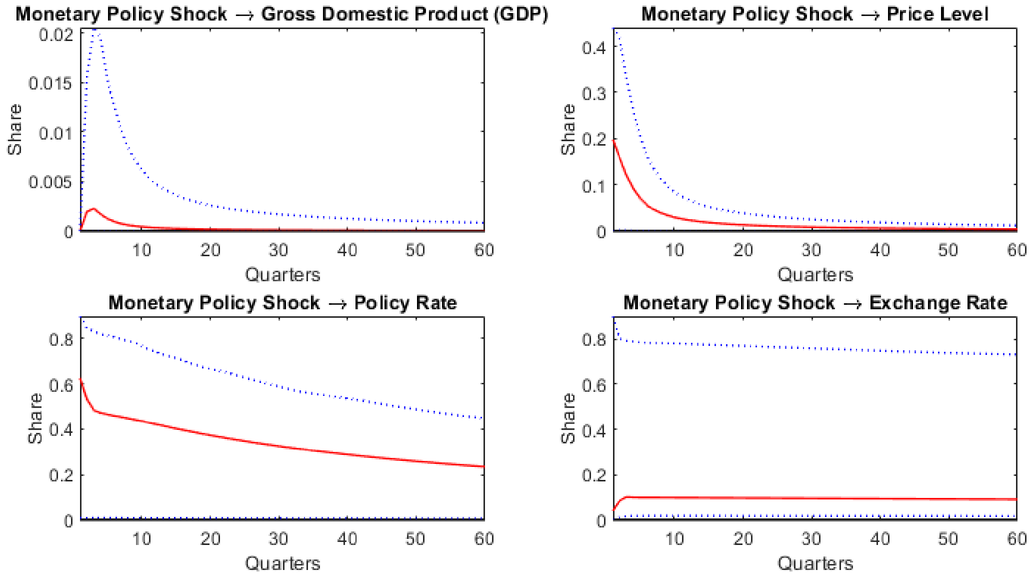

Figure 8: FEVD: Positive Monetary Policy Shock (Estimated Co-integration Vectors)

Appendix D - The Warne Approach

```
%% Problem 20
format long
% Identification of transitory shock
% This process is not automatic
r=2;
p=2;
% Note: cointegration vector is not normalized
beta=[0 -1 0 1; 0 0 1 0]';
K=4;

sigmahat = so;
% Impose restrictions to identify permanent shock
% Upsilon_0

Upsilon0 = [1 0; 0 1; 0 0; 0 1];

% Impose restrictions to identify transitory shocks
% Code below use Umat = TID
% If you want to use an automatic selection,
% set TID=0.

TID = [0 0 0 1; 1 0 0 0];

%------ No need to change code below --------
MHLP=inv(Upsilon0'*Upsilon0)*Upsilon0'*Xi;
pipit=MHLP*sigmahat*MHLP';
pimat=chol(pipit)';
Upsilon=Upsilon0*pimat;
Fk=inv(Upsilon'*Upsilon)*Upsilon'*Xi;
display(Fk,'Fk matrix');

% Identification transitory shocks

Umat=zeros(r,K);

% If TID=0, use automatic Umat, otherwise use TID defined above
if TID==0;
i=1;
while i<=r;
 Umat(i,K-i+1)=1;
 i=i+1;
end;
else
    Umat = TID;
end

% Check that identification of transitory shocks is valid
if det(Umat*alpha1)==0
   display('Identification of transitory shock is invalid');
else
   display('Identification of transitory shocks is valid');
end
xi=alpha1*inv(Umat*alpha1);
i=1;
```

```
while i<=K;
    j=1;
    while j<=r;
        if abs(xi(i,j))<=1E-12; % just to make sure that elements are = 0
            xi(i,j)=0;
        else
        end
        j=j+1;
    end
    i=i+1;
end

qr=chol(xi'*inv(sigmahat)*xi)';
Fr=inv(qr)*xi'*inv(sigmahat);
display(Fr,'Fr matrix');

% Putting it all together to compute B0inv

invB0 = inv([Fk;Fr]);

% Display result and compare to solver solution

display(invB0,'B0^{-1} matrix');
display(B0inv,'Compare to solver');


% Check that identification is valid

display(beta'*Xi,'(1) beta*Xi should be zero');
display(beta'*Upsilon0,'(2) beta*Upsilon_0 should be zero');
display(-Xi*invB0,'(3) C(1)*B0^{-1} should be Upsilon~zeros(K,r)');
display(Upsilon,'where Upsilon');
display(inv(invB0)*so*inv(invB0)','
(4) Covariance matrix of structural shocks w_t should be I_K');
display(inv(qr)*xi'*inv(so)*xi*inv(qr'),'Should be diagonal');
```