# Institutional Asset Pricing with Segmentation and Household Heterogeneity*

Goutham Gopalakrishna† Zhouzhou Gu‡ Jonathan Payne§

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**Abstract**

How does the organization of the financial sector impact different households? To answer this question, we build a heterogeneous-agent macro-finance model with households facing asset market participation constraints, banks providing deposits, funds providing insurance/pension products, and endogenous asset price volatility. We solve the model globally by developing a new deep learning methodology for macro-finance models and calibrate the model to asset pricing dynamics and household portfolio choices. Counterfactual experiments reveal distinct trade-offs. Regulatory interventions increase stability but at the expense of lower growth and higher wealth inequality. Demographic shifts compress inequality but reduce aggregate wealth, showing how regulation and demographics jointly shape the interaction of asset prices, stability, and the wealth distribution.

Keywords: Market Segmentation, Asset Pricing, Heterogeneous Agent Macroeconomic Models, Deep Learning, Inequality.

JEL: C63, C68, E27, G12, G21, G22, G23

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†University of Toronto, Rotman School of Management.
Email: goutham.gopalakrishna@rotman.utoronto.ca

‡Princeton, Department of Economics. Email: zg3990@princeton.edu

§Princeton, Department of Economics. Email: jepayne@princeton.edu

# 1 Introduction

Since the 2007-9 financial crisis, a large literature has documented the economic importance of frictions within the financial sector. They amplify business cycles (e.g. Gertler and Kiyotaki (2010), Brunnermeier and Sannikov (2014)), explain asset pricing spreads (e.g. Koijen and Yogo (2019, 2023), Vayanos and Vila (2021)), and constrain investment (e.g. Ottonello and Winberry (2020)). This research has led to extensive discussion about the best way to organize and regulate financial intermediaries. However, many questions about how different households are affected have been left unanswered. Are poorer agents more exposed to financial sector risk because they are more dependent on banking, pension, and insurance products? Can wealthier agents take advantage of regulations and earn higher returns? Do these considerations affect investment? This paper studies these macroeconomic connections and shows how policy makers face additional important and subtle tradeoffs between managing financial stability, economic growth, and household inequality.

We make three main contributions. First, we develop a novel quantitative heterogeneous-agent macro-finance (HAMF) model that embeds financial intermediaries and endogenous volatility into a continuous-time heterogeneous agent real business cycle environment. This model nests key macro-finance environments and so allows us to bridge the different intermediary asset pricing and macroeconomics literatures. Second, we develop a deep-learning solution method to characterize global solutions to our model. This overcomes the major technical challenge for heterogeneous-agent macro-finance that standard numerical techniques cannot handle environments like ours with aggregate risk, constrained portfolio choice, and a non-degenerate distribution of household wealth. Finally, we calibrate the model to the post-financial crisis period and conduct counterfactual experiments that examine alternative regulatory and demographic regimes.

The details of our baseline model are outlined in Section 2. The economy is populated by overlapping generations of price-taking households who retire at random times (modeled as exit from the productive economy). Households face two frictions: they incur costs when they participate directly in asset markets and they cannot contract among each other to insure against retirement shocks. There are two types of financiers who provide services to help households overcome these frictions: bankers and fund managers. Bankers issue risk-free short-term deposits while the fund managers issue contracts that payout when agents retire, which we refer to as "pensions". Both types of financiers face equity raising constraints that prevent them from quickly recapitalizing after negative wealth shocks. Finally, there is a government that issues long-term bonds, raises wealth taxes, and potentially places regulatory portfolio restrictions on the financial intermediaries. The combined model features lead to an economy with heterogeneous asset market access: both bankers and fund managers can invest in the underlying long term assets in the economy, capital stock and government bonds, while households can hold deposits, pensions, and capital stock (subject to the participation costs). The result is portfolio choice heterogeneity with wealthier households primarily holding direct claims to capital stock and poorer households primarily holding deposits and pensions. This household portfolio heterogeneity ultimately generates a non-degenerate household wealth distribution.

Our environment is constructed to nest a collection of key macro-finance models. For example, if we restrict household preferences so that households only consume at retirement, then we recover the inelastic "preferred habitat" demand in Vayanos and Vila (2021). Alternatively, if we eliminate retirement and household participation frictions, then we recover Brunnermeier and Sannikov (2014). Moreover, if we take the household participation friction to infinity, then we recover a segmented market banking model similar to Gertler and Kiyotaki (2010). The flexibility (and complexity) in our household choice problem allow us to incorporate these different models into one framework.

In Section 3, we outline our algorithm for solving the model. General equilibrium for our economy can be characterized by three blocks: (1) a collection of high dimensional PDEs capturing agent *optimization*, (2) a law of motion for the *distribution* of wealth shares and other aggregate state variables, and (3) a set of conditions that ensure the price processes are *consistent* with equilibrium. The technical challenge is that, unlike for other macro-finance models, we need a solution approach that can handle complexity in all three blocks. We do this by drawing upon and expanding recent advances in the continuous time deep learning macroeconomics literature (e.g. Duarte, Duarte and Silva (2024); Gopalakrishna (2021); Fernández-Villaverde, Hurtado and Nuño (2023b); Sauzet (2021); Gu, Laurière, Merkel and Payne (2023)). This involves using neural networks to approximate derivatives of value functions, the price volatility of long-term assets, portfolio choices, and other equilibrium equilibrium objects. We then use stochastic gradient descent to train the neural network to minimize the error in the "master" equations that characterize the equilibrium blocks for the system. The novel challenge is that the combination of complicated portfolio choice and a non-generate household distribution leads to sharp curvature in the policy functions and a stochastic Kolmogorov Forward Equation.

In Section 4, we calibrate our baseline model to the post-financial crisis period (2010Q1 to 2024Q4) and show that the model does a good job of matching asset pricing, macroeconomic, and household portfolio moments. We set regulatory constraints and other financial intermediary frictions rates to target the leverage of the banking sector and the combined pension-insurance sector. We set household portfolio constraints to target median household portfolio shares. We do three main “out-of-sample” tests on the external validity of our calibration. First, we show that we match the investment rate. Second, we show that we can approximately match the risk exposure of combined pension-insurance sector, as documented by Koijen and Yogo (2022). Finally, we show that we can match both household portfolio choices and exposure to risk in the financial sector.

In Sections 5, we use our calibrated model to revisit proposed changes to Basel III regulations: (1) tightening bank leverage requirements to capture suggestions that current restrictions remain too lax (e.g. (Admati and Hellwig, 2013)) and (2) imposing bank-style leverage restrictions on funds respond to observations that the pension-insurance sector has taken on additional risk under Basel III (e.g. Koijen and Yogo (2022), Begenau, Liang and Siriwardane (2024)). Both counterfactual experiments lead to higher Sharpe ratios, lower household consumption volatility, lower investment, *and* higher inequality. This suggest that higher stability comes at the cost of both lower investment and higher inequality. All changes are amplified when funds are further restricted compared to when banks are further restricted. The difference is that tightening bank restrictions leads to lower bank leverage and higher fund leverage while increasing fund restrictions actually leads to both lower bank and lower fund leverage.

A collection of mechanisms are important for understanding these results. First, funds have act as risk absorbers in the economy by purchasing capital stock from banks and households during recessions when banks and households attempt to take on less risk. This behavior arises because fund liabilities (pensions) fall in value during recessions, while banks liabilities (deposits) value remain unaffected. As a result, all else equal, funds gain wealth share during recessions, which allows them to purchase assets from other agents and stabilize the capital market. This means that restricting fund leverage actually crowds out bank risk taking because risk taking banks need the funds to buy their assets during recessions. Conversely, restricting bank leverage actually crowds in fund risk taking because the funds no longer indirectly provide insurance to banks.

The second intuition is that the evolution of the household wealth distribution is driven by the return spreads in the economy. Wealthy households choose a portfolio that is more weighted towards capital and so are more able to earn the risk premium in the economy. Poorer households choose a portfolio that is more weighted towards deposits and pensions and so pay the fees to the fund managers for the insurance products. When the government introduces regulations, it open up risk premia and fund manager fees, both of which allow wealthy agents to accumulate wealth more quickly. Put another way, introducing regulations makes financial intermediation difficult. This does ultimately reduce wealth volatility. However, it also gives wealthy agents the opportunity to circumvent the regulations and invest directly into capital markets to earn high returns.

**Literature Review:** We contribute to several strands of literature. The first is the macro-finance literature studying how financial institutions generate endogenous risk in dynamic general equilibrium models (Brunnermeier and Sannikov, 2014; He and Krishnamurthy, 2013; Krishnamurthy and Li, 2020; Maxted, 2023; Gertler, Kiyotaki and Prestipino, 2019). These papers highlight the role of financial intermediaries in amplifying aggregate shocks, typically focusing on sectoral heterogeneity between households and intermediaries. More recent work introduces heterogeneity across intermediaries themselves (Kargar, 2021; Coimbra and Rey, 2024), emphasizing the dynamics of financial cycles. Our contribution relative to this literature is to combine intermediary heterogeneity with household wealth distribution, thereby producing rich cross-sectional outcomes for households and sector-level outcomes between the intermediaries.

Secondly, we are part of an active literature studying how asset pricing can impact the household wealth distribution (recent examples include Gomez (2017), Cioffi (2021), Gomez and Gouin-Bonenfant (2024), Fagereng, Gomez, Gouin-Bonenfant, Holm, Moll and Natvik (2022), Basak and Chabakauri (2024), Fernández-Villaverde and Levintal (2024), Irie (2024) amongst many others). This work builds on ample empirical evidence documenting the heterogeneity in household portfolio choices and asset returns (Bach, Calvet and Sodini (2020), Fagereng, Guiso, Malacrino and Pistaferri (2020), Catherine, Miller, Paron and Sarin (2022), Bricker, Volz and Hansen (2018), etc.). Our model extends this literature by endogenizing both capital market participation and price volatility in a heterogeneous-agent macro-finance environment with multiple intermediaries and portfolio choice over long-lived assets. In doing so, we connect to the literature on limited financial market participation affecting asset prices (Gomes and Michaelides (2007), Guvenen (2009), Favilukis (2013),Lansing (2015), Vayanos and Vila (2021), Khorrami (2021), Gaudio, Petrella and Santoro (2023), Gaudio (2024), etc.). A smaller but growing line of work incorporates pensions into macro-finance models. For instance, Coimbra and Rey (2024) embeds defined-benefit pension funds in an incomplete-markets framework to explain the equity premium. Our contribution is to integrate pension and insurance sectors directly into a DSGE model with financial amplification, thereby linking regulatory regimes to asset pricing and distributional outcomes. This connects to empirical work showing that regulation of financial institutions and institutional investors influences asset prices (Greenwood and Vissing-Jorgensen, 2018).

Finally, we speak to the computational economics literature employing deep learning to solve complex heterogeneous-agent models that challenge traditional solution techniques (Azinovic, Gaegauf and Scheidegger, 2022; Han, Yang and E, 2021; Maliar, Maliar and

Winant, 2021; Kahou, Fernández-Villaverde, Perla and Sood, 2021; Bretscher, Fernández-Villaverde and Scheidegger, 2022; Fernández-Villaverde, Marbet, Nuño and Rachedi, 2023a; Han, Jentzen and E, 2018; Huang, 2022; Duarte et al., 2024; Gopalakrishna, 2021; Fernández-Villaverde et al., 2023b; Sauzet, 2021; Gu et al., 2023). While several papers apply deep learning to heterogeneous-agent models with portfolio choice between short-term risky assets (Fernández-Villaverde, Hurtado and Nuno, 2023; Huang, 2023), very few tackle long-term asset pricing in general equilibrium. For example, Azinovic and Žemlička (2023) solve a discrete-time model with long-lived assets by encoding equilibrium conditions directly into neural networks, employing low-dimensional approximations of the wealth distribution in the spirit of Kubler and Scheidegger (2018). A central challenge is that pricing long-term assets requires solving simultaneously for equilibrium allocations and individual choices, as emphasized by Guvenen (2009). Our contribution is to demonstrate that in continuous time, working directly in the wealth share space, these equilibrium objects can be determined globally in a unified framework. Methodologically, we show how to solve macro-finance models with heterogeneous intermediaries, long-lived assets, and rich wealth distributions without resorting to restrictive distribution and portfolio choice approximations.

The rest of this paper is structured as follows. Section 2 outlines our economic model. Section 3 introduces our numerical algorithm. Section 4 describes the calibration of the baseline model. Section 5 and 6 presents counterfactual experiments before concluding.

# 2 Baseline Economic Model

In this section, we outline the economic model used throughout the paper. We study a stochastic production economy with heterogeneous households who face retirement shocks and asset market participation constraints. The households are serviced by bankers who issue deposits and a fund managers who offers pensions to insure the retirement shocks.

## 2.1 Environment

*Setting:* The model is in continuous time with an infinite horizon. There is a perishable consumption good and a durable capital stock. The economy is populated by a unit continuum of price-taking households ($h$), indexed by $i \in [0, 1]$, a unit continuum of price-taking bankers ($b$), and a unit continuum of price-taking funds ($f$) that we interpret as the combined pension and insurance sector. The banking and fund sectors will each aggregate to pseudo representative agents but the household sector will not. The economy has the following assets: short-term bank deposits, fund pension contracts, bank loans, capital stock, and government bonds.

*Production:* There is a production technology that creates consumption goods according to the linear production function $Y_t = e^{z_t}k_t$ where $k_t$ is the capital used at time $t$ and $z_t$ is aggregate productivity that evolves according to:

$$dz_t = \beta_z(\bar{z} - z_t)dt + \sigma_z dW_t,$$

where $W_t$ denotes the aggregate Brownian motion process. All agents can create capital stock using an investment technology that converts $\iota_t k_t$ goods into $\phi(\iota_t)k_t$ capital, where $\iota_t$ is referred to as the investment rate. Capital depreciates at rate $\delta > 0$. So, an agent's physical capital stock evolve according to:

$$dk_t = (\phi(\iota_t)k_t - \delta k_t)dt.$$

*Households:* There is a unit measure of households who follow a life cycle. Households are born as "young" agents and then transition to "retired" agents at rate $\lambda_h$. While young, agents have discount rate $\rho_h$ and get flow utility $u(c_{h,t}) = \beta c_{h,t}^{1-\gamma}/(1-\gamma)$ from flow consumption $c_{h,t}$. When households retire, they disengage from the productive sector and consume using their accumulated wealth. We follow the macro-finance literature and model this by imposing that households receive a lump sum of utility $\mathcal{U}(\mathcal{C}_{h,t}) = (1-\beta)\mathcal{C}_{h,t}^{1-\Gamma}/(1-\Gamma)$ from consuming a lump sum of consumption $\mathcal{C}_{h,t}$ at retirement. In Appendix A.1 we provide some options for micro-founding this expression as the present discounted value of consumption during retirement. After one household retires, it is immediately replaced by a new young household who receives initial wealth $\underline{a}_{h,t} = \phi_h A_t$, where $A_t$ is the total wealth in the economy.

Households face two financial frictions. First, they face a "participation friction" that holding capital stock incurs the flow cost:

$$\Psi_{h,k}(k_{h,t}, a_{h,t}) = \psi_{h,k,t}\Xi_{h,t}a_{h,t}, \text{ where } \psi_{h,k,t} = \psi_h(\theta^k_{h,t}, \eta_{h,t}) = \frac{\bar{\psi}_k}{2\eta_{h,t}}\left(\theta^k_{h,t}\right)^2$$

where $\bar{\psi}_k$ is the severity of the constraint, $\theta^k_{h,t}$ is the household's share of wealth in capital, $\eta_{h,t}$ is the household's share of wealth in the economy, and $\Xi_{h,t}$ is the equilibrium stochastic discount factor of the household[1]. The constraint imposes that wealthier agents have better direct access to production opportunities and is interpreted as capturing the fixed costs, ed-

[1]We normalize by the equilibrium stochastic discount factor to remove marginal utility effects from the constraint.

ucation differences, and/or behavioral constraints associated with direct entrepreneurship. We will show this creates demand for intermediated access to capital markets. Second, households cannot write contracts with each other to insure against retirement shocks. We will show this generates a "preferred-habitat" style need for financial intermediaries that can provide pension/insurance products that payout with average maturity $\lambda_h$.

*Financial intermediaries:* There are two types of financial intermediaries servicing households: bankers ($b$) and fund managers ($f$). Each type of intermediary $j \in \{b, f\}$ has discount rate $\rho_j$ and gets flow utility $u_j(c_{j,t}) = c_{j,t}^{1-\gamma_j}/(1-\gamma_j)$ from consuming $c_{j,t}$ flow goods. The intermediaries differ in what type of liabilities they issue. Bankers only issue risk-free short-term deposits that pay a deposit rate $r_t^d$. Fund managers only sell contracts to households that pay one good to the holder when they die so it can interpreted as a combination of a life-insurance and a pension product. For convenience, we refer to them as pensions. On the asset side of the balance sheet, both banks and fund managers can hold capital and government bonds subject to any regulatory frictions described below. Financial intermediaries of type $j \in (b, f)$ exit at rate $\lambda_j$ and are replaced by new financial intermediaries with initial wealth $\underline{a}_{f,t} = \phi_f A_t$.[2] Ultimately, both banker and fund manager policies will be independent of wealth so we can replace the continuum of bankers and funds by a representative banker and fund.

*Government:* The government issues zero coupon bonds that mature at rate $\lambda_m$ and pay 1 unit of the consumption good at maturity. We impose that bond supply scales with capital stock so total bond supply is $M_t = \mathcal{M}k_t$. The government raises flow wealth taxes to finance the issuance of debt. For convenience, we also assume that the government raises taxes to redistribute wealth to the new entrants in the economy.[3] We let $\tau_j$ denote the total proportional flow wealth tax on an agent of type $j \in \{h, b, f\}$. The details of tax and transfer accounting are discussed further in Appendix A.2.

In addition to its fiscal policy, the government also imposes regulatory constraints on the financial sector. Financial intermediaries of type $j \in \{b, f\}$ face the flow portfolio penalties:

$$\Psi_{j,l}(k_{j,t}, a_{j,t}) = \psi_{j,l,t}\Xi_{j,t}a_{j,t}, \text{ where } \psi_{j,l,t} = \frac{\psi_j^l}{2}\left[\max\{0, \theta_{j,t}^l - \bar{\theta}_{j,t}^l\}^2 + \min\{0, \theta_{j,t}^l\}^2\right],$$

for holding asset $l \in \{k, m\}$ where $\bar{\theta}_{j,l}$ is the regulatory target portfolio share for agent

[2] This can be interpreted as a rate of recapitalization.

[3] This could equivalently be decentralized as an inheritance system or equity re-balancing as Gertler and Kiyotaki (2010).

$j$ in asset $l$, $\psi_j^l$ is the tightness of the regulatory requirement, and $\Xi_{j,t}$ is the equilibrium stochastic discount factor of intermediary $j$. The first term in the constraint is the regulatory restriction that restricts portfolio allocation to different asset classes. The second term is a penalty on short-selling that captures the inability of the financial sector produce real capital stock or government debt.

*Assets, markets, and financial frictions:* Each period, there are competitive markets for goods and capital trading. We use goods as the numeraire. Let $r_t^d$ denote the interest rate on deposits and let $\boldsymbol{q}_t := (q_t^k, q_t^n, q_t^m)$ denote a vector with the price of capital, pensions, and government bonds respectively. We guess and verify that for each asset $l \in \{k, n, m\}$ the long-term price process satisfies:

$$\frac{dq_t^l}{q_t^l} = \mu_{q^l,t}dt + \sigma_{q^l,t}dW_t,$$

where $\mu_{q^l,t}$ and $\sigma_{q^l,t}$ are the drift and volatility for asset $l \in \{k, n, m\}$. We also express the return processes by the notation:

$$\begin{aligned}
dR_t^k &= r_t^k dt + \sigma_{q^k,t}dW_t, & r_t^k &:= \mu_{q^k,t} + \Phi(\iota) - \delta + \frac{e^{z_t} - \iota}{q_t^k}, \\
dR_{h,t}^n &= r_{h,t}^n dt + \sigma_{q^n,t}dW_t + \frac{1}{q_t^n}dN_{h,t}, & r_{h,t}^n &:= \mu_{q^n,t} \\
dR_{f,t}^n &= r_{f,t}^n dt + \sigma_{q^n,t}dW_t, & r_{f,t}^n &:= \mu_{q^n,t} + \left(\frac{1}{q_t^n} - 1\right)\lambda_h \\
dR_t^m &= r_t^m dt + \sigma_{q^m,t}dW_t & r_t^m &:= \mu_{q^m,t} + \left(\frac{1}{q_t^m} - 1\right)\lambda_m
\end{aligned}$$

where pensions have different flow returns for the household, $R_{h,t}^n$, and fund, $R_{f,t}^n$, because the fund aggregates across a continuum of households.

**Discussion of key environmental features:** This environment has been constructed to nest a collection of models and economic forces commonly studied in the macro-finance literature. We discuss these connections below:

(i) *Preferred habitat literature:* If we set $\beta = 0$ so the households only care about consumption at retirement and set $\mathcal{U}(\cdot)$ to be either the Type I or Type II agents from the Appendix in Vayanos and Vila (2021), then our households reduce to the "preferred habitat agents" in Vayanos and Vila (2021) who only demand maturities with average maturity $1/\lambda_h$.[4] For the general setup with $\beta > 0$, our model has an important extension compared to the preferred habitat literature—we integrate the preferred habitat demand into a standard portfolio choice problem so that overall household demand is a combination of the "preferred-habitat" component and a standard portfolio choice problem that balances risk and return. This allows us to understand how risk and inelastic demand interact in a general equilibrium model.

(ii) *Perpetual youth literature:* If we set $\beta = 1$ so the households only care about consumption while young, then we recover the perpetual youth model from Blanchard (1985). In this case, households demand annuities that pay until they die, which the households could recreate synthetically by shorting the life insurance products offered by the funds and purchasing bonds. In this sense, the two extreme parameterizations, $\beta \in \{0, 1\}$, nest the two most commonly used models of demand for pension/insurance products: preferred habitat and perpetual youth. Our model can be viewed as an intermediate model that nests these two forces. Throughout the paper we focus on parametrizations where the households take long positions in the fund contracts. However, the model could just as easily be solved for the case where some households end up shorting the pension products.

(iii) *Participation constraint models:* We have set up the household participation penalty so that households increase their fraction of wealth in capital as get older. In this sense, as household wealth becomes small (or the participation penalty becomes infinitely large), the model becomes the Basak and Cuoco (1998) environment in which households cannot participate in the capital market. However, as household wealth becomes large (or the participation penalty becomes zero), the agents become unconstrained like in the Brunnermeier and Sannikov (2014) environment where households can freely participate in the capital market. At either extreme, household portfolio choices become homogeneous and the household sector aggregates. By introducing heterogeneous household portfolio choice, we allow our model to intermediate between these extremes. However, this is also what leads to many of the technical difficulties in solving the model.

(iv) *Different type models (e.g. Chan and Kogan (2002), Gomez (2017)):* There is an important collection of models in which households have different types ex-ante (e.g. because they have heterogeneous risk aversion) but all agents within a particular type make the same portfolio decisions. These models can generate heterogeneous portfolio

[4] In order to generate the full yield curve model in Vayanos and Vila (2021), we would also need to introduce variation in $\lambda_h$ across households and bonds with different average maturities.

choices across the different types in the population and so can generate the aggregate asset portfolio for the household sector. However, they cannot match any of the portfolio data at the micro level, which shows that portfolio decisions vary with household wealth.

## 2.2 Equilibrium

We let $a_{j,t}$ denote the wealth of a type $j \in \{h, b, f\}$ agent, where the indices $h$, $b$, $f$ refer to household, banker and fund manager respectively. So, $a_{h,t} = q_t^k k_{h,t} + q_t^n n_{h,t} + d_{h,t}$, $a_{b,t} = q_t^k k_{b,t} + q_t^m m_{b,t} + d_{h,t}$, and $a_{f,t} := q_t^k k_{f,t} + q_t^m m_{f,t} + q_t^n n_{f,t}$. We let $\mu_{a_j,t}$ and $\sigma_{a_j,t}$ denote the geometric drift and volatility for the wealth of a type $j \in \{h, b, f\}$ agent. We let $\theta_{j,t}^l = q_t^l l_{j,t}/a_{j,t}$ denote the share of wealth that an agent of type $j$ with wealth $a_{j,t}$ has in asset $l$ and let $\theta_{j,t}$ denote the vector of wealth shares chosen by an agent of type $j$ with wealth $a_{j,t}$ at time $t$. We use the notation $\mathsf{x} = (x_t)_{t\geq 0}$ to denote the stochastic process for variable $x_t$ and $\tilde{\mathsf{x}}$ to denote an agent belief about the process for $x_t$.

*Household problem:* Given their belief about price processes, $(\tilde{\mathsf{r}}, \tilde{\mathsf{q}})$, and initial wealth, $a_{h,0}$, a household chooses consumption, portfolio, and investment processes $(\mathsf{c}_h, \theta_h, \iota_h)$ to solve Problem (2.1) below:

$$
\begin{aligned}
&\max_{\mathsf{c}_h, \theta_h} \mathbb{E}\left[\int_0^T e^{-\rho_h t}\left(u(c_{h,t}) + \psi_{h,k}(\theta_{h,t}^k, \eta_{h,t})\Xi_{h,t} a_{h,t}\right) dt + e^{-\rho T}\mathcal{U}(\mathcal{C}_{h,T})\right] \\
&s.t.\ \ \frac{da_{h,t}}{a_{h,t}} = \theta_{h,t}^n d\tilde{R}_{h,t}^n + \theta_{h,t}^k d\tilde{R}_t^k + \left((1 - \theta_{h,t}^k - \theta_{h,t}^n)\tilde{r}_{d,t} - c_{h,t}/a_{h,t} - \tau_{h,t}\right) dt \qquad (2.1) \\
&\qquad \mathcal{C}_{h,T} \leq \left(1 - \theta_{h,t}^n + \frac{\theta_{h,t}^n}{q_t^n}\right) a_{h,t}
\end{aligned}
$$

where $\tau_{h,t}$ is the net tax or transfer (per unit of wealth) while agents are alive. We can use the return processes to express the drift and volatility of household wealth in terms of their choices:

$$
\begin{aligned}
\mu_{a_h,t} &= \tilde{r}_t^d + \sum_{l\in\{n,k\}} \theta_{h,t}^l(\tilde{r}_t^l - \tilde{r}_t^d) - c_{h,t}/a_{h,t} - \tau_{h,t} \\
\sigma_{a_h,t} &= \sum_{l\in\{n,k\}} \theta_{h,t}^l \tilde{\sigma}_{q^l,t}
\end{aligned}
$$

*Financial intermediary problems:* Given their belief about price processes, $(\tilde{\mathsf{r}}, \tilde{\mathsf{q}})$, and initial wealth, $a_{j,0}$, a financial intermediary of type $j \in \{b, f\}$ chooses their consumption, portfolio, and investment processes $(\mathsf{c}_j, \theta_j, \iota_j)$ to solve the Problem (2.2) below:

$$\begin{aligned}
&\max_{\mathsf{c}_j,\theta_j,\iota_j} \left\{ \int_0^\infty e^{-\rho_j t} \left( u(c_{j,t}) + \psi_{j,k}(\theta^k_{j,t})\Xi_{j,t}a_{j,t} + \psi_{j,m}(\theta^m_{j,t})\Xi_{j,t}a_{j,t} \right) dt \right\} \\
&s.t. \;\; \frac{da_{j,t}}{a_{j,t}} = \theta^k_{j,t}d\tilde{R}^k_t + \theta^m_{j,t}d\tilde{R}^m_t + (1-\theta^k_{j,t}-\theta^m_{j,t})d\tilde{R}^j_t - (c_{j,t}/a_{j,t} + \tau_{j,t})\, dt
\end{aligned} \tag{2.2}$$

where $d\tilde{R}^j_t$ is return on liabilities for type $j$ ($\tilde{r}^d_t dt$ for bankers $j = b$ and $d\tilde{R}^n_t$ for fund managers $j = f$). Like for the households, we can express the drift and volatility of financial intermediary wealth in terms of their choices:

$$\begin{aligned}
\mu_{a_h,t} &= \tilde{r}^j_t + \sum_{l\in\{m,k\}} \theta^l_{h,t}(\tilde{r}^l_t - \tilde{r}^j_t) - c_{h,t}/a_{h,t} - \tau_{h,t} \\
\sigma_{a_h,t} &= \theta^m_{h,t}\tilde{\sigma}_{q^m,t} + \theta^k_{h,t}\tilde{\sigma}_{q^k,t} + (1-\theta^m_{h,t}-\theta^k_{h,t})\tilde{\sigma}_{q^j,t}
\end{aligned}$$

where $\tilde{r}^j_t$ and $\tilde{\sigma}_{q^j,t}$ are the average return and volatility of return process for the liabilities of type $j$.

*Distribution:* Throughout this paper, we work with the distribution of wealth shares, rather than wealth levels, so that the state space is bounded. The bank and fund sectors aggregate so we will only need to track the aggregate states for each sector. We let $\eta_{b,t} := a_{b,t}/A_t$ and $\eta_{f,t} := a_{f,g}/A_t$ denote the share of aggregate wealth held by the banking and fund sectors. The uninsurable idiosyncratic shocks and wealth dependent differences in household portfolio constraints generate a non-degenerate cross-section distribution of household wealth across the economy. We let $g_{h,t} = \{\eta_{i,t} = a_{i,t}/A_t : i \in \mathcal{I}\}$ denote the measure of household wealth shares across the economy at time $t$ for a given filtration $\mathcal{F}_t$, where $\mathcal{F}_t$ is generated the by aggregate shock processes $\{W_t\}_{t\geq 0}$. With some abuse of notation, we let $G = (\eta_{b,t}, \eta_{f,t}, g_{h,t})$ denote the collection of "distribution" states in the economy.

**Definition 1** (Equilibrium)**.** For a given set of government taxation policies, an equilibrium is a collection of $\mathcal{F}_t$-adapted processes $(\mathsf{K}, \mathsf{r}, \mathsf{q}, \mathsf{G})$ and household decision processes $(\mathsf{c}_i, \iota_i, \theta_i)$ for $i \in I$ and financial intermediary decision processes $(\mathsf{c}_i, \iota_i, \theta_i)$ for $j \in \{b, f\}$ such that:

1. Given beliefs $(\tilde{\mathsf{r}}, \tilde{\mathsf{q}})$, households solve Problem 2.1 and financial intermediaries solve Problem 2.2.

2. The price processes $(\mathsf{r}, \mathsf{q})$ satisfies market clearing conditions at each $t$ (where the capital letter $Y_{j,t}$ refers to the aggregate quantity of variable $y$ for an agent of type $j \in \{h, b, f\}$ and capital letter $Y_t$ refers to the aggregate quantity of variable $y$ across the economy):

(a) Goods market clears: $\sum_{j\in\{h,b,n\}} C_{j,t} + \lambda_h \mathcal{C}_{h,t} = e^{z_t} K_t - \iota_t K_t$, where $\lambda_h \mathcal{C}_{h,t}$ is the aggregate household consumption upon retirement.

(b) Capital market clears: $\sum_{j\in\{h,b,n\}} K_{j,t} = K_t$

(c) Annuity market clears: $\sum_{j\in\{b,n\}} N_{j,t} = 0$

(d) Deposit market clears: $\sum_{j\in\{b,n\}} D_{j,t} = 0$

(e) Bond market clears: $\sum_{j\in\{b,n\}} M_{j,t} = M_t$

3. Agent beliefs are consistent with equilibrium $(\tilde{\mathsf{r}}, \tilde{\mathsf{q}}) = (\mathsf{r}, \mathsf{q})$.

## 2.3 Recursive Characterization of Equilibrium

We characterize the equilibrium recursively. We denote the finite dimensional components of the aggregate state vector by $\boldsymbol{s} := (z, K, \eta_b, \eta_f)$ and the full aggregate state vector incorporating the household distribution by $\boldsymbol{S} := (\boldsymbol{s}, g_h)$. Under the recursive formulation of the problem, agent beliefs about the price process become agent beliefs about the evolution of sector wealth, $(\tilde{\mu}_{\eta,j}, \tilde{\boldsymbol{\sigma}}_{\eta,j})_{j\in\{h,f\}}$, and the evolution of the household wealth share density characterized by $(\tilde{\mu}_g, \tilde{\sigma}_g)$ satisfying:

$$dg_{h,t}(a) = \tilde{\mu}_g(a, \boldsymbol{S})dt + \tilde{\boldsymbol{\sigma}}_g(a, \boldsymbol{S})^T d\boldsymbol{W}_t$$

Individual agents also have their wealth $a$ as an idiosyncratic state. For convenience, we define the state spaces for an individual agent by $\boldsymbol{x} := (a, z, K, \eta_b, \eta_f)$ and $\boldsymbol{X} := (\boldsymbol{x}, g_h)$. We let $V_j(a, \boldsymbol{S})$ and $\xi_j(a, \boldsymbol{S}) := \partial_a V_j(a, \boldsymbol{S})$ denote the value function and the derivative of the value function for an agent of type $j \in \{h, b, f\}$ with individual state $a$. We let $\mu_{\xi_h}(a, \boldsymbol{S})$ and $\sigma_{\xi_h}(a, \boldsymbol{S})$ denote the geometric drift and volatility of the process for $\xi_j$.

Theorems 1, 2, and 3 is the following subsections summarize the recursive characterization of equilibrium. We group the characterization into three blocks: (i) the optimization problems of the agents, (ii) the evolution of the distribution, and (iii) market clearing.

### 2.3.1 Block 1: Agent Optimization

**Theorem 1** (Agent Optimization)**.** *Given the price functions* $(r^d, (q^l, r^l, \sigma_{q^l})_{l\in\{k,n,m\}})$*, the household choices* $(c_h, \iota_h, \theta_h)$ *satisfy the first-order-conditions (FOCs):*

$$
\begin{aligned}
&[c_h]: && u'(c_h) = \xi_h(a, \boldsymbol{S}) \\
&[\iota_h]: && \Phi'(\iota_h) = (q^k(\boldsymbol{S}))^{-1} \\
&[\theta_h^k]: && r^k(\boldsymbol{S}) - r^d(\boldsymbol{S}) = -\partial_{\theta_h^k}\psi_{h,k}(\theta_h^k, \eta_h) - \sigma_{\xi_h}(a, \boldsymbol{S})\sigma_{q^k}(\boldsymbol{S}) \\
&[\theta_h^n]: && r_h^n(\boldsymbol{S}) - r^d(\boldsymbol{S}) = -\lambda_h\left(\frac{1}{q^n(\boldsymbol{S})} - 1\right)\frac{\mathcal{U}'(\mathcal{C})}{a\xi_h(a, \boldsymbol{S})} - \sigma_{\xi_h}(a, \boldsymbol{S})\sigma_{q^n}(\boldsymbol{S}),
\end{aligned}
$$

*and the household SDF* $\xi_h$ *satisfies the Euler equation:*

$$
\rho_h + \lambda_h = \mu_{\xi_h}(a, \boldsymbol{S}) + r^d(\boldsymbol{S}) - \tau_h + \psi_{h,k}(\theta_h^k, \eta_h) - \partial_{\theta_h^k}\psi_{h,k}(\theta_h^k, \eta_h)\theta_h^k
$$

*where the drift and volatility of* $\xi_h$ *are characterized by Itô's lemma:*

$$
\begin{aligned}
\mu_{\xi_h}(a, \boldsymbol{S})\xi_h(a, \boldsymbol{S}) &= (D_x\xi_h(a, \boldsymbol{S}))^T\boldsymbol{\mu}_x + \frac{1}{2}tr\Big\{(\boldsymbol{\sigma}_x \odot \boldsymbol{x})^T(\boldsymbol{\sigma}_x \odot \boldsymbol{x})D_x^2\xi_h(a, \boldsymbol{A})\Big\} + \mathcal{L}_g\xi_h(a, \boldsymbol{S}) \\
\sigma_{\xi_h}(a, \boldsymbol{S})\xi_h(a, \boldsymbol{S}) &= (\boldsymbol{\sigma}_x \odot \boldsymbol{x})^T D_x\xi_h(a, \boldsymbol{S})
\end{aligned}
$$

*and where* $\mathcal{L}_g\xi_h(\boldsymbol{X})$ *denotes the collection of terms with Frechet derivatives of* $\xi_h$ *w.r.t.* $g$ *specified by equation* (B.2) *in Appendix B.1. The financial intermediary variables* $(\xi_j, c_j, \theta_j)_{j\in\{b,f\}}$ *satisfy analogous conditions specified in Appendices B.2 and B.3.*

*Proof.* See Appendix B. □

The FOC for $c_h$ equates the household's marginal utility of consumption to their marginal value of wealth (i.e. their stochastic discount factor). The FOC for $\iota_h$ equates the marginal benefit of investment to its marginal cost. The remaining FOCs characterize the household portfolio choice for $\theta_h = (\theta_h^k, \theta_h^n)$. These equations capture the standard trade-off between earning an expected return, $r^l - r^d$ for asset $l \in \{k, n\}$, and facing the risk from the co-movement between the household's stochastic discount factor, $\sigma_{\xi_h}^T\sigma_{q^l}$ for asset $l \in \{k, n\}$. In addition, they also capture the distortions from household participation constraints and retirement consumption demand.

To understand the role of the capital market participation constraint, observe from the FOC for $\theta_h^k$ that the term $-\partial_{\theta_h^k}\psi_{h,k}(\theta_h^k, \eta_h)$ decreases household demand for capital by penalizing capital holding. The penalty is less pronounced as the household gets wealthier ($\eta_h$ increases) so, all else equal, wealthier households hold more capital. This increasing relationship between household wealth and capital holdings is what ultimately allows us to match the empirical data showing that poorer households predominately hold deposits while wealthier households predominately hold capital. It is also what prevents breaks aggregation results across the household sector and so necessitates keeping track of the household wealth distribution. By contrast, the portfolio choice problems for the financial intermediaries in Appendices B.2 and B.3 do not depend upon financial intermediary wealth and so the banker and fund manager sectors can be aggregated.

To understand the impact of household retirement consumption, we can rearrange the demand for pension contracts and use the exponential form of the Vayanos and Vila (2021) preferences for the retirement utility $\mathcal{U}(\mathcal{C}) = a\xi_h \exp(\mathcal{C})$ to get the following expression for the FOC for $\theta_h^n$:

$$\underbrace{\lambda_h \left(\frac{1}{q_t^n(\boldsymbol{S})} - 1\right) \exp\left(-\frac{q_t^k(\boldsymbol{S})}{q_t^n(\boldsymbol{S})}\theta_i^n \eta_i\right)}_{\substack{\text{“Preferred habitat”} \\ \text{component}}} = -\underbrace{\left(r_{h,t}^n(\boldsymbol{S}) - r_t^d(\boldsymbol{S})\right)}_{\substack{\text{Excess} \\ \text{return}}} - \underbrace{\sigma_{\xi_h}(a, \boldsymbol{S})\sigma_{q_t^n}(\boldsymbol{S})}_{\text{“risk shifter”}} \tag{2.3}$$

which can be interpreted as a generalization of the preferred habitat demand function from Vayanos and Vila (2021). Like Vayanos and Vila (2021), we have the “inelastic” preferred habitat term coming from the need for assets with duration $\lambda_h$ (the LHS in equation (2.3)). However, instead of exogenous demand shifters, we instead have that the household's risk-return tradeoff (the RHS in equation (2.3)) shifts household demand. We expand on this comparison quantitatively in our counterfactual experiments in Subsection 5.4.

**Corollary 1.** *The firm and banking sectors aggregate but the household sector does not.*

*Proof.* See Appendix B. □

### 2.3.2 Block 2: Distribution Evolution

Given the individual optimal decisions, we proceed to study how the distribution evolves over time. For the bank and fund sectors, we only need to keep track of the aggregate wealth share dynamics because we can aggregate within each of those sectors. However, for the household sector, the capital market participation constraint generates portfolio heterogeneity and so prevents aggregation. This means that we need to keep track of the full distribution of household wealth. We summarize this in Theorem 2 below.

**Theorem 2** (Distribution evolution)**.** *Given price functions* $(r^d, (q^l, r^l, \sigma_{q^l})_{l\in\{k,n,m\}})$ *and agent decisions* $(\xi_j, c_j, \theta_j, \iota)_{j\in\{h,b,f\}}$*, we can characterize the distribution evolution. At the* *sector level, the wealth share for financial intermediary $j \in \{b, f\}$ evolves according to:*

$$\frac{d\eta_{j,t}}{\eta_{j,t}} = \left(\mu_{A_j}(\boldsymbol{S}_t) - \mu_A(\boldsymbol{S}_t) + (\sigma_A(\boldsymbol{S}_t) - \sigma_{A_j}(\boldsymbol{S}_t))\sigma_A(\boldsymbol{S}_t)\right) dt + (\sigma_{A_j}(\boldsymbol{S}_t) - \sigma_A(\boldsymbol{S}_t))dW_t$$

*where $(\mu_{A_j}, \sigma_{A,j})$ are the drift and volatility of the aggregate wealth in sector $j$ and $(\mu_A, \sigma_A)$ are the drift and volatility of aggregate wealth in the economy:*

$$\begin{aligned}
\mu_{A_j}(\boldsymbol{S}) &= \mu_{a_j}(\boldsymbol{S}) + \lambda_j\left(\phi_j/\eta_{j,t} - 1\right) \\
\sigma_{A_j}(\boldsymbol{S}) &= \sigma_{a_j}(\boldsymbol{S}) \\
\mu_A(\boldsymbol{S}) &= \vartheta(\boldsymbol{S})(\mu_{q^k}(\boldsymbol{S}) + \Phi(\iota_j) - \delta) + (1 - \vartheta_t(\boldsymbol{S}))\mu_{q^m}(\boldsymbol{S}) \\
\sigma_A(\boldsymbol{S}) &= \vartheta(\boldsymbol{S})\sigma_{q^k}(\boldsymbol{S}) + (1 - \vartheta(\boldsymbol{S}))\sigma_{q^m}(\boldsymbol{S})
\end{aligned}$$

*where $r^j$ is the expected return on the liabilities of financial intermediary $j$ ($r^d$ for $j = b$ and $r^n$ for $j = f$), aggregate wealth is given by $A_t = q_t^k K_t + q_t^m M$, and the aggregate wealth in capital is $\vartheta_t := q_t^k K_t/(q^k K_t + q^m M)$. Within the household sector, the density of household wealth shares evolves according to:*

$$\begin{aligned}
dg_{h,t}(\eta) = &\left(\lambda_h\phi(\eta) - \lambda_h g_{h,t}(\eta) - \partial_\eta[\mu_\eta(\eta, \boldsymbol{S}_t)g_{h,t}(\eta)] + \frac{1}{2}\partial_\eta\left[\sigma_\eta^2(\eta, (\boldsymbol{S}_t))g_{h,t}(\eta)\right]\right) dt \\
&- \partial_\eta[\sigma_\eta(\eta, \boldsymbol{S}_t)g_{h,t}(\eta)]dW_t
\end{aligned} \tag{2.4}$$

*where:*

$$\begin{aligned}
\mu_\eta(\eta, (\boldsymbol{S})) &= \mu_a(a, \boldsymbol{S}) - \mu_A(\boldsymbol{S}) + (\sigma_A(\boldsymbol{S}) - \sigma_a(a, \boldsymbol{S}))\sigma_A, \\
\sigma_\eta(\eta, (\boldsymbol{S})) &= \sigma_a(a, \boldsymbol{S}) - \sigma_A(\boldsymbol{S})
\end{aligned}$$

*Proof.* See Appendix B. □

Theorem 2 shows that the distribution evolution equations are directly exposed to the aggregate shock process, $dW_t$, and so the distributional dynamics in the economy follow a *stochastic Kolmogorov Forward Equation (KFE)*. This is because all economic agents have a direct heterogeneous exposure to the fundamental Brownian shocks through the instantaneous asset price responses. By contrast, many classic models in the heterogeneous agent macroeconomics literature (e.g. the continuous time version Krusell and Smith (1998) described in Gu et al. (2023)) do not have stochastic KFEs because all assets offer short-term risk free returns and so the aggregate shocks do not directly impact household wealth.

### 2.3.3 Block 3: Market Clearing and Belief Consistency

Finally, we impose equilibrium by ensuring markets clear and agent beliefs are consistent with the emergent price processes. The following theorem restates the equilibrium conditions under belief consistency.

**Theorem 3** (Consistency)**.** *Under belief consistency:*

$$((\tilde{\mu}_{\eta,j}, \tilde{\boldsymbol{\sigma}}_{\eta,j})_{j\in\{h,f\}}, \tilde{\mu}_g, \tilde{\sigma}_g) = ((\mu_{\eta,j}, \sigma_{\eta,j})_{j\in\{h,f\}}, \mu_g, \sigma_g),$$

*the equilibrium prices satisfy:*

$$\begin{aligned}
\int \frac{c_h}{q^k(\boldsymbol{S})K + q^m(\boldsymbol{S})M} g_h(\eta)d\eta + \frac{C_b}{A_b}\eta_b + \frac{C_f}{A_f}\eta_f + \lambda_h \int \mathcal{C}_h g_h(\eta)d\eta &= \frac{(e^z - \iota)K}{q^k(\boldsymbol{S})K + q^m(\boldsymbol{S})M} \\
\int \theta_h g_h(\eta)d\eta + \theta_f \eta_f + \theta_b \eta_b &= \vartheta(\boldsymbol{S}) \\
\theta_f^n \eta_f + \int \theta_h^n g_h(\eta)d\eta &= 0 \\
\int \theta_h^d g_h(\eta)d\eta + \theta_b^d a_b &= 0 \\
\theta_b^m \eta_b + \theta_f^m \eta_f &= 1 - \vartheta(\boldsymbol{S})
\end{aligned}$$

*and the long term assets prices must satisfy consistency with Itô's Lemma for $l \in \{k, n, m\}$:*

$$\begin{aligned}
\mu_{q^l}(\boldsymbol{S})q^l(\boldsymbol{S}) &= (D_x q^l(\boldsymbol{S}))^T \boldsymbol{\mu}_s + \frac{1}{2} tr\left\{(\boldsymbol{\sigma}_s(\boldsymbol{S}) \odot \boldsymbol{S})^T (\boldsymbol{\sigma}_s(\boldsymbol{S}) \odot \boldsymbol{s}) D_s^2 q^l(\boldsymbol{S})\right\} + \mathcal{L}_g q^l(\boldsymbol{S}) \\
\boldsymbol{\sigma}_{q^l}(\boldsymbol{S})q^l(\boldsymbol{S}) &= (\boldsymbol{\sigma}_s(\boldsymbol{S}) \odot \boldsymbol{S})^T (D_s q^l(\boldsymbol{S})) + \langle \partial q^l/\partial g, \sigma_g \rangle
\end{aligned}$$

*where once again $\mathcal{L}_g$ is the collection of Frechet derivatives defined by equation* (B.2) *in the Appendix and $\partial q^l/\partial g$ is the Frechet derivative.*

*Proof.* See Appendix B. □

Theorem 3 characterizes the equilibrium conditions for our recursive setup. The first five equations restate the market clearing conditions. One of the difficulties for macro-finance is that these market clearing equations only implicitly pin down the pricing functions for the long-term assets $(q^k, q^n, q^m)$. Consequently, we also need to impose additional cross equations restricts to ensure that long-term asset prices processes are consistent with equilibrium through Itô's Lemma.

| Models | Non-Trivial Blocks | | | Method |
|---|---|---|---|---|
| | 1 (Opt.) | 2 (Dist.) | 3 (Asset q) | |
| Representative Agent (RA) (à la Lucas (1978)) | simple | NA | simple | Finite difference |
| Heterogeneous Agents (HA) (à la Krusell and Smith (1998)) | ✓ | ✓ | simple | Gu et al. (2023) |
| Long-lived assets (à la Brunnermeier and Sannikov (2014)) | closed-form | low-dim | ✓ | Gopalakrishna (2021) Duarte et al. (2024) |
| HA + Long-lived assets | ✓ | ✓ | ✓ | This paper |

**Table 1:** Summary of key models and solution methods.

## 2.4 Technical Comparison to Other Models

This system of equations is difficult to solve because, unlike in most models, all three blocks are non-trivial. To understand why this is the case, it is instructive to compare our model to other macro-finance models, as summarized in Table 1 and discussed below:

(i). For a representative agent model, block 2 is not applicable because there is no distribution and block 3 is less complicated because the goods market condition simply becomes $c + (\iota - \phi(\iota))K = y$, which can be substituted into equations in block 1. In this case, the set of equations can be simplified to a differential equation for $q$.

(ii). For the continuous time version of Krusell and Smith (1998) discussed in Gu et al. (2023), there is a distribution of agents so block 2 is non-trivial. However, this model has no long-term assets and so we can derive closed form expressions for all prices in term of the distribution. This implies that block 3 can be trivially satisfied and we can combine all equilibrium conditions into one "master" partial differential equation. In addition, many researchers follow the original Krusell and Smith (1998) paper and approximate the distribution by the first moment and so eliminate the need to track the full distribution.

(iii). For models such as Basak and Cuoco (1998) and Brunnermeier and Sannikov (2014) discussed in Gopalakrishna (2021), the HJBE can be solved partially in closed form so that we can get an analytical expression for the dependence of the value function on idiosyncratic wealth. This means that block 1 can be reduced into scaled PDE and substituted into the block 3.

# 3 Computational Methodology and Algorithm

In this section, we outline our algorithm for characterizing solutions to our model. Conceptually, our approach can viewed as a type "projection" onto a neural network (similar to Duarte et al. (2024), Gopalakrishna (2020), Gu et al. (2023), and other papers that build on the PINN literature for using deep learning to solve differential equations). This involves:

(a) Replacing the agent continuum by a finite dimensional distribution approximation,

(b) Representing the equilibrium functions by neural networks with the states as inputs,

(c) Finding the neural network parameters that minimize the loss in the equilibrium conditions on randomly sampled points from the state space (often referred to as "training" the neural network), and

Once we have trained a neural network approximation to the equilibrium functions, we use it to simulate the equilibrium evolution of the state space and compute ergodic outcomes, impulse response functions, and other analysis.

Although this approach may appear straightforward to describe, implementing it successfully for macro-finance models has proven difficult and involves many non-trivial decisions. In particular, we need to decide which distribution approximation is most effective and find the optimal subset of equilibrium objects to approximate by neural networks. In this section we explain the tradeoffs associated with these decisions. At a high level, the main reason difficulties arise is because the endogenous volatility generates high curvature in the equilibrium functions and makes the long-term asset markets are hard to clear.

## 3.1 Part (a): Finite Dimensional Distribution Approximation

A fundamental challenge for heterogeneous agent macro-finance models is that the state space contains a density, $g_h$, which is an infinite-dimensional object. To make computational progress, we must adopt a finite dimensional approximation to the density. In this paper, we use a hybrid approach. When training the neural network to approximate the equilibrium functions, we approximate the distribution by a finite collection of $I < \infty$ price taking agents. So, the approximate aggregate state space becomes:

$$\hat{\boldsymbol{S}} := (z, K, \eta_1, \ldots, \eta_I, \eta_b, \eta_f) \in \mathcal{S}$$

where $\eta_i$ is the wealth share of agent $i$, $(\eta_b, \eta_f)$ are again the wealth shares of the financial intermediaries, and $\mathcal{S} = \mathbb{R} \times \mathbb{R}^+ \times [0,1]^{I+2}$ is space of possible aggregate states. When simulating the economy we use our neural network approximations to the equilibrium functions to construct a finite difference approximation to the equilibrium KFE (2.4). We do this following the approach developed in Gu et al. (2023) and summarized in Appendix D to this paper.

**Discussion:** Previous work has discussed the benefits and costs associated with different distribution approximations (e.g. see Gu et al. (2023)). We highlight key points here:

(i) A natural and frequently raised concern with the finite-agent approach is that simulations of a finite $I$ agent economy contain idiosyncratic noise that could make the training unstable and lead to an equilibrium that is inconsistent with the original model. However, we mitigate these concerns in a number of ways:

  (a) We solve for an equilibrium in which agents behave as price takers and forecast prices under the assumption that the idiosyncratic exit shocks have averaged out. This means that the perceived state space evolution does not contain idiosyncratic risk and so our equilibrium differential equations do not contain idiosyncratic noise. In addition, we do not need to use simulations to train the neural networks because we are working with a continuous time analytical formulation of the Euler equations that contain derivatives rather than expectations. So, concerns about simulation accuracy are unrelated to the accuracy of the neural network approximation.

  (b) When we do need to simulate the solved model to generate time paths or impulse responses, we use our neural network solution to approximate a finite difference approximation to the KFE and so are able to simulate the limiting economy with a continuum of agents rather than the finite agent economy. In this sense, we exploit the continuous-time analytical formulation to be able to maintain the convenience of a finite agent economy without having to deal with the finite sample noise problem that appears in discrete-time simulation based training methods. Figure 7 in the Appendix shows a histogram of the difference in the impulse responses computed using the finite difference approximation to the KFE and the N-agent simulation to illustrate the importance of working with the KFE for simulations.

(ii) Another concern that is sometimes raised about finite dimensional distributional approximations is that they lead to master equations with high dimensional Ito terms that are impossible to solve. Our hybrid approach of training on a finite agent economy and then reconstructing the KFE does not require such a large population to make the Ito terms intractable, even when calculating the Ito terms naively. Moreover, Duarte et al. (2024) shows that the Ito term in high-dimensional differential equations can be computed very efficiently by calculating second order directional derivatives. For all these reasons, we find that dimensionality is not what makes the problem very challenging. Instead, it is the high curvature and sensitivity of the market clearing conditions that make the problem challenging. In the next section, we discuss how we construct the loss function to overcome these issues.

## 3.2 Part (b): Neural Network Representation and Loss Function

We rewrite the equilibrium characterization to construct a loss function for the deep learning algorithm to minimize. First, we express the equilibrium objects as functions of the aggregate states without any explicit dependence on idiosyncratic states. To understand what this means, observe that in equilibrium all the price, policy, and value functions can be expressed directly in terms of the aggregate state $\hat{\boldsymbol{S}} := (z, K, \eta_1, \dots, \eta_I, \eta_b, \eta_f)$ for the finite agent approximation. For example, the household stochastic discount factor function is defined in the partial equilibrium household problem as a function of individual household wealth $\xi_h(a_i, \hat{\boldsymbol{S}})$ but also has an equilibrium representation that is only a function of the aggregate states:

$$\Xi_h(\hat{\boldsymbol{S}}) := \xi_h\left(\eta_i A\left(\hat{\boldsymbol{S}}\right), \hat{\boldsymbol{S}}\right)$$

where $A(\hat{\boldsymbol{S}}) = q^k(\hat{\boldsymbol{S}})K + q^m(\hat{\boldsymbol{S}})M$ is equilibrium aggregate wealth. We solve directly for the equilibrium functions (i.e. $\Xi_h$) rather than for the partial equilibrium functions (i.e. $\xi_h$).

Second, we parametrize the following variables by Neural Nets:

$$\begin{aligned}
\hat{\omega}_h &: \mathcal{S} \to \mathbb{R},\ (\hat{\boldsymbol{S}}, \Theta_{\omega_h}) \mapsto \hat{\omega}_h(\hat{\boldsymbol{S}}; \Theta_{\omega_h}), \\
\hat{\Omega}_h &: \mathcal{S} \to \mathbb{R},\ (\hat{\boldsymbol{S}}, \Theta_{\Omega_h}) \mapsto \hat{\Omega}_h(\hat{\boldsymbol{S}}; \Theta_{\Omega_h}), \\
\hat{\theta}^l_i &: \mathcal{S} \to \mathbb{R},\ (\hat{\boldsymbol{S}}, \Theta_{\theta_j}) \mapsto \hat{\boldsymbol{\theta}}_j(\hat{\boldsymbol{S}}; \Theta_{\theta_j}), \quad \forall l \in \{k, n, m\}, i \in \{h, f, b\} \\
\hat{q}^l &: \mathcal{S} \to \mathbb{R},\ (\hat{\boldsymbol{S}}, \Theta_{q^l}) \mapsto \hat{q}^l(\hat{\boldsymbol{S}}; \Theta_{q^l}), \quad \forall l \in \{n, m\} \\
\hat{\mu}_{q^l} &: \mathcal{S} \to \mathbb{R},\ (\hat{\boldsymbol{S}}, \Theta_{\mu,q^l}) \mapsto \hat{\mu}_{q^l}(\hat{\boldsymbol{S}}; \Theta_{\mu,q^l}), \quad \forall l \in \{k, n, m\} \\
\hat{\boldsymbol{\sigma}}_{q^l} &: \mathcal{S} \to \mathbb{R},\ (\hat{\boldsymbol{S}}, \Theta_{\sigma,q^l}) \mapsto \hat{\boldsymbol{\sigma}}_{q^l}(\hat{\boldsymbol{S}}; \Theta_{\sigma,q^l}), \quad l \in \{k, n, m\}
\end{aligned}$$

where $\omega := c/a$ denotes the household consumption-to-wealth ratio during their lifetime, $\Omega := \mathcal{C}/a$ denotes the household consumption-to-wealth ratio at retirement, $\Theta_\nu$ denotes the parameters for the Neural Net approximation of variable $\nu$, and we use $\Theta$ to refer the collection of neural network parameters across all approximations.

Finally, we construct the loss function. In doing so, where possible, we impose market clearing explicitly rather rather than including the market clearing conditions as part of the loss function. Given the neural network approximations ($\hat{\omega}_h$, $\hat{\Omega}_h$, $\hat{\boldsymbol{\theta}}_j$, $\hat{q}^l$, $\hat{\mu}_{q^k}$, $\boldsymbol{\sigma}_{q^l}$), we can solve for the other equilibrium variables explicitly using linear algebra. The neural network approximations then need to satisfy the following equations (after imposing market clearing and with $\hat{\Xi} = u'(\hat{\omega}(\hat{\boldsymbol{S}})\eta\hat{q}^k(\hat{\boldsymbol{S}})K)$):

$$
\begin{aligned}
\mathcal{L}_\omega(\hat{\boldsymbol{S}}) &= (r^d - \tau_h - \rho_h - \lambda_h)\hat{\Xi} + \mu_\Xi(\hat{\boldsymbol{S}}) + \psi_{h,k}(\theta_h^k(\hat{\boldsymbol{S}})) \\
&\quad - \partial_{\theta_h^k}\psi_{h,k}(\theta_h^k)\theta_h^k(\hat{\boldsymbol{S}}) + \psi_{h,n}(\theta_h^n(\hat{\boldsymbol{S}})) - \partial_{\theta_h^n}\psi_{n,k}(\theta_h^n)\theta_h^n(\hat{\boldsymbol{S}}) && \ldots\text{Euler eq.} \\
\mathcal{L}_\Omega(\hat{\boldsymbol{S}}) &= \hat{\Omega} - \eta A(\hat{\boldsymbol{S}})\mathcal{W}(\theta_h^k(\hat{\boldsymbol{S}}), \theta_h^n(\hat{\boldsymbol{S}})) && \ldots\text{Death cons.} \\
\mathcal{L}_{\theta_h^l}(\hat{\boldsymbol{S}}) &= r^l - r^d + \lambda\partial_{\theta_h^l}\mathcal{W}(\theta_h^k, \theta_h^n)\frac{\mathcal{U}'(\mathcal{C})}{a\xi_h} + \partial_{\theta_h^l}\psi_{h,l} + \sigma_{\xi_h}\sigma_{q^l}, \quad l \in \{k, n\} && \ldots\text{FOC hh.} \\
\mathcal{L}_{\theta_f^l}(\hat{\boldsymbol{S}}) &= r^l - r_f^n + \sigma_{\xi_f}(\sigma_{q^l} - \sigma_{q^n}), \quad l \in \{k, m\} && \ldots\text{FOC fund.} \\
\mathcal{L}_{\theta_b^l}(\hat{\boldsymbol{S}}) &= r^l - r_f^d + \sigma_{\xi_b}(\sigma_{q^l} - \sigma_{q^n}), \quad l \in \{k, m\} && \ldots\text{FOC bank.} \\
\mathcal{L}_{\mu_{q^l}}(\hat{\boldsymbol{S}}) &= (D_{\hat{\boldsymbol{S}}}q^l)^T\boldsymbol{\mu}_{\hat{\boldsymbol{S}}} + \frac{1}{2}\text{tr}\left\{(\boldsymbol{\sigma}_{\hat{\boldsymbol{S}}}(\hat{\boldsymbol{S}}, \boldsymbol{\theta}_h) \odot \hat{\boldsymbol{S}})^T(\boldsymbol{\sigma}_x(\hat{\boldsymbol{S}}, \boldsymbol{\theta}_h) \odot \hat{\boldsymbol{S}})D^2_{\hat{\boldsymbol{S}}}q^l\right\} && \ldots\text{Consistency} \\
\mathcal{L}_\sigma(\hat{\boldsymbol{S}}) &= \hat{\boldsymbol{\sigma}}_q(\hat{\boldsymbol{S}}) - (\boldsymbol{\sigma}_{\hat{\boldsymbol{S}}} \odot \hat{\boldsymbol{S}})^T(D_{\hat{\boldsymbol{S}}}q^l), \quad \forall l \in (k, n, m) && \ldots\text{Consistency}
\end{aligned}
\tag{3.1}
$$

so the loss function for the deep learning algorithm becomes:

$$
\hat{\mathcal{L}}(\hat{\boldsymbol{S}}; \boldsymbol{\Theta}) = (\mathcal{L}_\omega + \mathcal{L}_\Omega + \mathcal{L}_{\theta_h^l} + \mathcal{L}_{\theta_f^l} + \mathcal{L}_{\theta_b^l} + \mathcal{L}_{\mu_{q^k}} + \mathcal{L}_\sigma)(\hat{\boldsymbol{S}}; \boldsymbol{\Theta}) \tag{3.2}
$$

**Discussion** Our loss function attempts to balance a collection of difficulties that the macro-finance deep learning literature has faced. We describe the key ideas here.

(i) We express the state space in terms of wealth shares rather than wealth levels because wealth shares are bounded variables from which it is easier to sample.

(ii) We solve directly for the equilibrium functions (e.g. $\Xi_h(\mathbf{S})$) rather than for the partial equilibrium functions (e.g. $\xi_h(a, \mathbf{S})$) because it is a more computationally efficient way of imposing general equilibrium. If we want to impose general equilibrium on the partial equilibrium expressions when we sample, then we need to impose that $a_i = \eta_i q^k$. However, this then requires that we nest the neural network approximation for prices (i.e. $q^k$, $q^m$) inside the neural network approximation for other other variables, which we find to be computationally unstable.

(iii) We attempt to approximate variables that are easier for the deep learning algorithm to train, which typically means approximating variables that are smooth, bounded functions of the aggregate states. To understand this, observe that it is easier for the Neural network to approximate $\xi_h = \partial_a V_h$ than $V_h$ because it is easier to impose concavity on the derivative than the function. It is even easier to approximate the consumption-to-wealth ratio $\omega_h(\eta_i)$ and then reconstruct $\xi_h(\eta_i) = (\omega_h(\eta_i)\eta_i A)^{-\gamma}$ because $\omega_h(\eta_i)$ is typically bounded and so the explosive curvature in the SDF is encoded analytically.

(iv) We impose the market clearing conditions explicitly whenever possible because we find that allowing the deep learning algorithm to violate market clearing during the training process leads to instability. This is a similar observation to that made by Azinovic and Žemlička (2023) in their work on deep learning solutions to portfolio choice problems.

## 3.3 Part (c) Training Algorithm

We outline the steps for the training the neural networks in Algorithm 1 below. Given the current guesses of the neural network equilibrium function approximations, we sample states, calculate the equilibrium at those states, compute the loss at those points using equation (3.2), and then update the parameters to decrease the loss. We use Latin hypercube sampling and make sure that each agent's wealth share sample ranges from 0.0 to 0.5.

Each neural network is a fully-connected feed-forward type and has 2 hidden layers, with 256 neurons in each layer. We train using an ADAM optimizer with a learning rate scheduler (ReduceLROnPlateau) for a maximum of 200k iterations. Every 50 epochs, the algorithm evaluates the validation loss on a test dataset. If the validation loss improves, it saves a checkpoint and resets the patience counter. Otherwise, it increments the counter. Training terminates either when the patience counter exceeds 15,000 validation checks without improvement (early stopping) or when the variance of validation loss is lower than a threshold.

Figure 6 in the Appendix presents the L-2 loss from the quantitative model over iterations. The loss decreases over time, although not monotonically due to the stochastic nature of the learning process. After 60,000 iterations, the average Euler equation training loss (MSE) is $8.5 \times 10^{-5}$ in marginal utility units.[5] The right panel figure reveals that different components in the loss function converge at different speeds. The algorithm focuses on HJB equation first, followed by the consistency and portfolio choice conditions.

**Discussion:** The algorithm includes a few key features:

(i) We evaluate the progress of the algorithm on an "out-of-sample" validation dataset and save models that reduce loss on the validation set. We do this to try and limit

[5]For demonstration, Figure 6 displays losses until 200k epochs. The early stopping criterion gets triggered after 60k epochs since there is not much improvement in the validation loss when trained longer.

**Algorithm 1:** Neural Network Training with Validation and Adaptive Learning

Initialize neural network objects ($\hat{\omega}_h$, $\hat{\Omega}_h$, $\hat{\boldsymbol{\theta}}_j$, $\hat{q}^l$, $\hat{\mu}_{q^k}$, $\boldsymbol{\sigma}_{q^l}$) with parameters $\boldsymbol{\Theta}$

Initialize Adam optimizer and ReduceLROnPlateau scheduler

Generate validation dataset: $\left(\hat{\boldsymbol{S}}^{val} = \left(z^{val}, \zeta^{val}, K^{val}, (\eta_i)_{i\leq I}^{val}, \eta_b^{val}, \eta_f^{val}\right)\right)$

Set $patience \leftarrow 0$, $max\_patience \leftarrow 1000$, $max\_epochs \leftarrow 60,000$

**while** Epochs $< max\_epochs$ AND patience $< max\_patience$ **do**

  Compute adaptive learning rate lr using inbuilt scheduler

  Sample $N$ new training points: $\left(\hat{\boldsymbol{S}}^n = \left(z^n, \zeta^n, K^n, (\eta_i)_{i\leq I}^n, \eta_b^n, \eta_f^n\right)\right)_{n=1}^{N}$

  Calculate equilibrium at each training point $\hat{\boldsymbol{S}}^n$ given current neural network approximations

  Construct the loss as:

$$\hat{\mathcal{L}}(\hat{\boldsymbol{S}}^n) = (\mathcal{L}_\omega + \mathcal{L}_\Omega + \mathcal{L}_{\theta_h^l} + \mathcal{L}_{\theta_f^l} + \mathcal{L}_{\theta_b^l} + \mathcal{L}_{\mu_{q^k}} + \mathcal{L}_\sigma)(\hat{\boldsymbol{S}}^n; \boldsymbol{\Theta})$$

  where $\hat{\mathcal{L}}_v$ is defined by equation (3.1) for each variable v.

  Apply gradient clipping and update parameters: $\boldsymbol{\Theta} \leftarrow \boldsymbol{\Theta} - lr \cdot \nabla_{\boldsymbol{\Theta}} \hat{\mathcal{L}}_{total}$

  Update learning rate scheduler based on $\hat{\mathcal{L}}_{val}$

  **if** $epoch \bmod 50 = 0$ **then**

    Compute validation loss $\hat{\mathcal{L}}_{val}$ on validation set $\hat{\boldsymbol{S}}^{val}$

    **if** $\hat{\mathcal{L}}_{val} < min\_loss$ **then**

      Update the minimum loss and patience counter $min\_loss \leftarrow \hat{\mathcal{L}}_{val}$, $patience \leftarrow 0$

      Save all model states, optimizer, and scheduler

    **else**

      $patience \leftarrow patience + 1$

    **end if**

  **end if**

  IF validation loss variance (normalized by mean) $< 1e-4$ over window, BREAK

  $epoch \leftarrow epoch + 1$

**end while**

Save final checkpoint model.

the possibility of overfitting. This echoes standard practice for the machine learning literature.

(ii) To test the efficacy of our numerical algorithm in a controlled environment, in the Online Appendix E we use our algorithm to solve three canonical models from the macro-finance literature that can also be solved with traditional methods: a representative agent Lucas asset pricing model, Basak and Cuoco (1998), and Brunnermeier and Sannikov (2014). In each case our Neural Network solution lines up closely to the traditional solution approaches.

# 4 Calibration

For standard macro-finance parameters, we externally calibrate using accepted values in the literature. For the preference and regulatory parameters governing the household and financial intermediary portfolio problems, we internally calibrate using simulated method of moments to match target moments. Our data sample spans 2010Q1-2025Q2 so we interpret our model as indirectly estimating the Basel III regulatory parameters. Table 2 reports the complete set of parameters and the set of targeted moments. We classify the calibration targets into three groups — (i) macroeconomic, (ii) asset pricing, and (iii) cross-sectional moments. For each group, we discuss how successfully we fit the targeted moments and how robust our calibration is to matching "out-of-sample" untargeted moments.

*Macroeconomic parameters:* The mean-reversion of TFP ($\beta_z$) is set to 0.3, following Gertler et al. (2019), and the volatility of TFP ($\sigma_z$) is set to target a 3.5% output growth volatility. The depreciation rate ($\delta$) is chosen to match an annualized output growth rate of 2.6%, which is consistent with the real long-run output growth rate used in the literature. The investment friction parameter ($\kappa$) is calibrated to match the volatility of the private investment-to-capital ratio. The model also successfully generates an investment-to-capital ratio of 23.9%, close to the data, even though this is untargeted in the calibration.

*Asset pricing parameters:* The risk aversion parameters are internally calibrated to match spreads. We set household risk aversion to target an average equity risk premium ($r^k - r^d$ in the model) of 8.5%. We set bank risk aversion to target an average capital Sharpe ratio ($(r^k - r^d)/\sigma_{q^k}$ in the model) of 0.66, computed by estimating the risk premium using a factor model. Specifically, we regress equity market returns on dividend yield and estimate the Sharpe ratio from the fitted values. Finally, we set fund risk aversion to target an average pension return to the fund ($r^m - r^d$ in the model) of 3.5%.[6] For all agents, the internally calibrated risk aversion parameters are less than one, which is low relative to the frictionless consumption based asset pricing literature. This reflects that the participation and regulatory constraints generate significant curvature in the agent value function and so we do not need high risk aversion to match spreads. For all agents, the discount rate is set at 0.05, consistent with the literature (e.g., Gertler et al. (2019)).

[6] Pension returns are difficult to compute and lack consensus. We rely on Mitchell, Poterba, Warshawsky and Brown (1999) to obtain a conservative estimate of 3.5% net of fees.

| *Targeted moments* | Parameter | Value | Target (Data) | Target (Model) |
|---|---|---|---|---|
| TFP mean reversion | $\beta_z$ | 0.30 | Literature | - |
| TFP volatility | $\sigma_z$ | 0.03 | Output growth vol = 0.034 | 0.032 |
| Depreciation | $\delta$ | 0.01 | Output growth =0.025 | 0.0263 |
| Investment friction | $\kappa$ | 40 | Investment vol =0.076 | 0.086 |
| Household risk aversion | $\gamma, \Gamma$ | 0.3 | Risk premium = 0.085 | 0.071 |
| Bank risk aversion | $\gamma_b$ | 0.85 | Capital Sharpe Ratio = 0.66 | 0.46 |
| Fund risk aversion | $\gamma_f$ | 0.10 | Annuity return = 0.035 | 0.033 |
| Discount rate (hh.) | $\rho_h$ | 0.05 | Literature | - |
| Discount rate (fund) | $\rho_e$ | 0.05 | Literature | - |
| Discount rate (bank) | $\rho_b$ | 0.05 | Literature | - |
| Capital constraint | $\bar{\psi}_k$ | 1e-5 | Median hh. capital share =0.40 | 0.51 |
| Death shock intensity (hh.) | $\lambda_h$ | 0.10 | Average life =35 | 10 |
| Death shock intensity (fund) | $\lambda_f$ | 0.20 | Equity recapitalization | 5yrs |
| Death shock intensity (bank) | $\lambda_b$ | 0.20 | Equity recapitalization | 5yrs |
| Transfer weight (hh.) | $\phi_h$ | 0.03 | Median hh. annuity share = 0.37 | 0.40 |
| Transfer weight (fund and bank) | $\phi_f = \phi_b$ | 0.10 | Fin. wealth/Total wealth = 0.33 | 0.35 |
| Bond maturity | $\lambda_m$ | 0.10 | Avg. Maturity of LT bonds | 10 yrs |
| *Untargeted moments* | | | Data | Model |
| Investment/capital rate (%) | | | 24.0 | 23.9 |
| 25th pctl. capital share (hh.) | | | 0.33 | 0.46 |
| 75th pctl. capital share (hh.) | | | 0.58 | 0.59 |
| 25th pctl. deposit share (hh.) | | | 0.35 | 0.24 |
| 75th pctl. deposit share (hh.) | | | 0.19 | 0.10 |
| 25th pctl. pension share (hh.) | | | 0.32 | 0.30 |
| 75th pctl. pension share (hh.) | | | 0.35 | 0.29 |

**Table 2:** List of model parameters and calibration targets. All values are annualized. The time period is from 2010 Q1 to 2025Q2.

The transfer weight of the fund targets the equity share of financial institutions as the fraction of U.S. corporate equities held by pension funds, insurance companies, and US-chartered banks using the Federal Reserve's Z.1. Table L.223. Over 2010-2024, this share averages about one-third of the total market value, consistent with our target of 33%. We assume the same transfer weight for the bank to reduce degrees of freedom. The household transfer weight is calibrated to match the ratio of risky assets to total wealth for the median household in the Survey of Consumer Finance (SCF) data. We set bond maturity to 30 years to represent a long-duration bond.

The exit rates are externally calibrated. The household retirement intensity reflects the average retirement period in the population and is set to 0.1, taking into account the exponentially distributed death shock. The exit intensity of intermediaries is set to 0.20, motivated by their average equity recapitalization duration. Black, Floros and Sengupta (2016) reports that between 2001 and 2014, around 30% of the publicly listed banks (BHCs) in the US raised equity. We are not aware of comparable statistics for the fund sector and so assume the same number.

The household participation constraint and regulatory parameters are set to target agent portfolios. We set the household capital constraint parameter $\psi_h$ to match the risky asset holdings of the median household in the SCF data, and then evaluate how well the model replicates moments across the distribution.[7] The intermediary portfolio penalty parameters $\psi_b^k, \psi_f^k, \bar{\theta}_b^k, \bar{\theta}_f^k$ are set according to Table 5 in the baseline model and will be varied in counter-factual experiments. Specifically, the thresholds $\bar{\theta}_b^k, \bar{\theta}_f^k$ are chosen to target the leverage ratios of the bank and the fund, respectively. Bank leverage in the data is around 5 (Krishnamurthy and Li (2020), while firms' leverage is much lower (around 1.3 from COMPUSTAT). Since our "bank" combines both financial and non-financial institutions that hold risky capital, we target a conservative leverage ratio of 1.5.[8] For the pension fund sector, the leverage ratio is around 1.4, closer to the data.[9]

We set $\beta = 0.5$ so that flow and terminal utility carry the same weight. We pick the smallest value of penalty on bond holdings for intermediaries to prevent shorting (i.e., $\psi_b^m = \psi_f^m = 5$), and keep this parameter fixed in all counter-factual experiments.

As an out-of-sample test, we compare study the risk exposure of the financial sector. The model also directionally matches the beta of the fund's equity exposure to capital return,

[7] We remove households who have short portfolio positions to be consistent with the SCF data.

[8] Since the year 2010, non-financial corporate sector assets have been around \$45 trillion USD, of which \$30 trillion are equity. Banking sector assets have been around \$30 trillion (net of reserves and Treasuries), of which \$2.57 trillion are equity. The combined sector implies a leverage ratio of 1.77 which is closer to our target.

[9] Source: FRED database. We use 'BOGZ1FL594090005Q' for total pension fund assets, and 'BOGZ1FL574190005Q' for liabilities.

even though the betas are not directly targeted. Table (3) reports the factor regression results where the fund equity returns are proxied by changes in the fund wealth share in the model. The results are consistent with Koijen and Yogo (2022), who show that the variable annuity insurers' stock returns have a positive beta with respect to stock and a negative beta with respect to 10-year Treasury bond returns. While the interest rate is endogenous in our model, the quantitative exercise indicates that an unexpected increase in the rate does decrease the fund's wealth, in line with Koijen and Yogo (2022).

| Factor | Data (1999-2017) | Data (2010-2017) | Model |
|---|---|---|---|
| Stock market return | 1.36 | 1.11 | 1.84 |
| | (0.19) | (0.08) | (0.00) |
| Long term bond return | -0.01 | -1.28 | -1.67 |
| | (0.32) | (0.43) | (0.00) |
| Observations | 228 | 96 | 1499 |

**Table 3:** Risk exposure of fund sector. The table reports betas from a factor regression of fund equity returns on stock returns and long-term bond returns. Data values are taken from Koijen and Yogo (2022), and corresponds to the period 2010-2017. Heteroskedasticity adjusted standard errors are given in parenthesis.

*Cross-sectional moments:* Our model generates rich cross-sectional moments that are broadly consistent with the data, even though they are not explicitly targeted. Figure 1 shows households' risky capital, annuity, and risk-free asset shares as a fraction of total wealth in both the data and the model. We can see that the model does a reasonable job of matching the large blocks in the distribution at the 25th, 50th, and 95th percentiles, particularly since these moments are untargeted. We also assess the model's performance in matching the response of household wealth to negative TFP shocks. Empirically, we measure changes in the wealth distribution following shocks to the net worth of U.S. insurance funds. Shocks are estimated as innovations in the autoregression $\hat{a}_t = \beta_0 + \beta_1 \hat{a}_{t-1} + u_t$, normalized by lagged equity value $\hat{f} := \frac{u_t}{\hat{a}_{t-1}}$. We then apply local projections in the spirit of Jordà (2005), regressing shocks to fund net worth on the wealth share of households at different percentiles. Specifically, we run the following regression

$$\log\left(\frac{W_{p,t+h}}{W_{p,t}}\right) = \alpha_{p,h} + \beta_{p,h}\hat{f}_t + \epsilon_{p,t+h}$$

for horizons $h = 1$ to 25 quarters where $w_{p,t+h}$ denotes household wealth at percentile $p$ at horizon $h$, where $p$ denotes the middle 40 percent wealth. The household wealth distribution

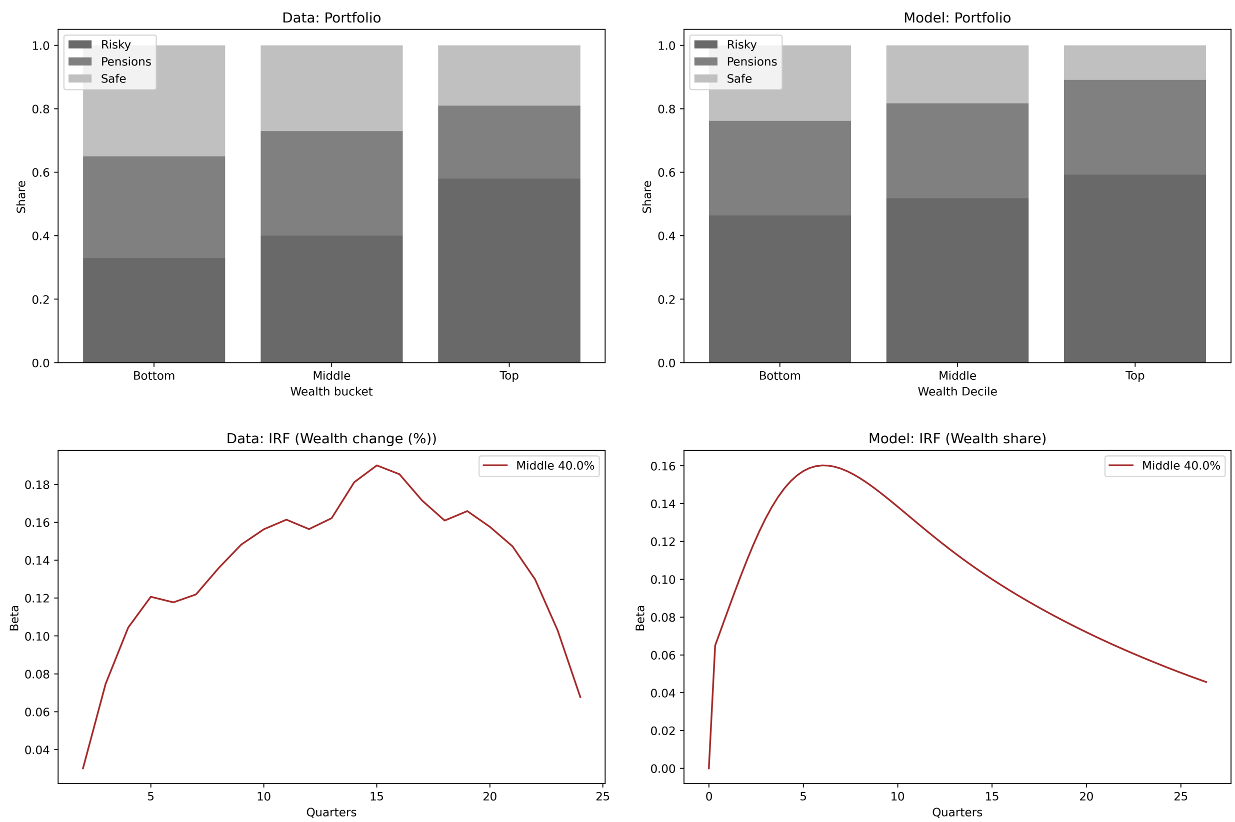


**Figure 1:** The figures present the portfolio holdings of households across 25th (Bottom), median (Middle), and 95th (Top) percentiles from the SCF data and model, respectively. The bottom left figure plots the impulse response of wealth distribution to risk premium ($\beta_{p,h}$) obtained from the regression $\log{(W_{p,t+h}/W_{p,t})} = \alpha_{p,h} + \beta_{p,h}\hat{f}_t + \epsilon_{p,t+h}$, where $p$ denotes middle 40 percent. The data for wealth percentiles come from Saez and Zucman (2016), and risk premium is estimated using a factor model. The bottom right figure plots the impulse response of wealth share of households at different percentiles to a 2 std. deviation negative TFP shock.

is taken from Saez and Zucman (2016), and the fund wealth share series is constructed from the Financial Accounts of the United States (FRB) following Koijen and Yogo (2021). The empirical evidence shows that disruptions to the insurance sector's wealth raise the wealth share of middle 40 percent household wealth, which our model quantitatively matches.

# 5 Counterfactual: Revisiting Basel III and IV

In this section, we use our calibrated model to study the macroeconomic impact of changes to the current Basel III regulatory system. Previous macro-finance work has focused on the high level regulatory tradeoff between growth and stability. Our model allows us to extend this analysis to consider the heterogeneous impact of macro-prudential regulation across different financial intermediaries and households.

In Section 4, we calibrated our baseline economy using data from 2010Q1-2024Q4, a period during which the Basel III rules and the Dodd-Frank Act introduced tighter regulatory oversight over financial intermediaries. The regulatory parameters from this calibration are summarized in the first column of Table 5, which shows tighter leverage requirements on the banking sector than on the fund sector. This is consistent with the commonly held view that bank risk taking has been particularly restricted since the Global Financial Crisis and so non-bank financial intermediaries–including pension and insurance funds–have absorbed a larger share of long-duration risk assets (e.g. Koijen and Yogo (2022), Begenau et al. (2024)).

We use our calibrated environment to study two counterfactual regulatory regimes. In the first, we tighten the leverage requirement on banks by increasing the parameter $\psi_b$, which captures the strength of the leverage constraint penalty. This "bank restricted" economy reflects proposals from critics concerned that banks have been able to circumvent Basel III regulatory requirements and stay excessively leveraged (Admati and Hellwig (2013), Acharya, Engle and Pierret (2014)). In the second, we impose the current bank leverage restrictions on the fund sector by decreasing the parameter $\bar{\theta}^k_f$ that captures the maximum capital holdings the fund can have without incurring a penalty. This "fund restricted" economy eliminates any regulatory incentives for non-banks to take on risk. Together, these counterfactuals allow us to quantify how extending or strengthening leverage constraints across institutions would affect asset prices and the distribution of household wealth. We include the full set of regulatory parameters for the counterfactual experiments in Table 5 and summarize the key macroeconomic statistics from the different counterfactual policy regimes in Table 5.[10]

[10] We also modify the transfer parameters $\phi_b$ and $\phi_f$ from 0.1 to 0.06 in the counterfactual economies to target the same mean financial-sector wealth-share across the regimes. This enables us to focus on the regulatory impact within the household sector.

| | Baseline | Target | Counterfactual: Bank-restricted | Target | Counterfactual: Fund-restricted | Target |
|---|---|---|---|---|---|---|
| *Bank regulation parameters* | | | | | | |
| $\bar{\theta}_b^k$ | 0.9 | Bank A/E = 1.10 | 0.7 | Bank A/E = 0.96 | 0.9 | |
| $\phi_f$ | 0.10 | $\eta_f + \eta_b$ = 0.33 | 0.06 | $\eta_f + \eta_b$ = 0.33 | 0.06 | $\eta_f + \eta_b$ = 0.33 |
| *Fund regulation parameters* | | | | | | |
| $\bar{\theta}_f^k$ | 1.0 | Fund A/E = 1.80 | 1.0 | | 0.5 | Fund A/E = 1.60 |
| $\phi_b$ | 0.10 | $\eta_f + \eta_b$ = 0.33 | 0.06 | $\eta_f + \eta_b$ = 0.33 | 0.06 | $\eta_f + \eta_b$ = 0.33 |

**Table 4:** Regulatory parameters across different regimes. Parameters $\bar{\theta}_b^k$ and $\bar{\theta}_f^k$ denotes the upper limit of capital holdings for the bank and the fund, respectively. Parameters $\phi_b$ and $\phi_f$ denotes the transfer weights. The thresholds in counterfactual experiments are chosen to target a 5% drop in leverage ratios from the baseline case.

## 5.1 A Growth-Stability-Inequality Tradeoff

Many researchers have emphasized that macro-prudential policy makers face a trade-off between ensuring stability and generating growth. The first two rows of Table 5 show that our model also delivers such a tradeoff. Increasing restrictions on the bank and the fund both lead to lower household risk, as measured by the volatility of aggregate household wealth, while also lowering the investment rate. However, our model also suggests an additional dimension to the macro-prudential tradeoff: inequality. The economies with greater restrictions on the bank and fund both lead to higher inequality, as measured by the Gini coefficient for the economy (the third row of Table 5), with the effects more pronounced when additional restrictions are placed on the funds.

There are two key mechanisms that are important for understanding the growth-stability-inequality tradeoff in Table 5: (a) funds are the "natural" backstop for the economy in bad times, and (b) dispersion in the wealth distribution is driven by the spread between capital returns and the returns on intermediary products (deposits and pensions). We discuss the intuition behind these mechanisms in the subsequent subsections.

## 5.2 Mechanism (a): Funds Are a Natural Backstop in Bad Times

Absent regulation, the key difference between banks and funds is that they have different liability structures: banks issue short-term deposits while funds issue tradable long-term pensions. This means that, all else equal, banks and funds experience different liability

| | Baseline | Counterfactual: Bank-restricted | Counterfactual: Fund-restricted |
|---|---|---|---|
| *Growth-Stability-Inequality Tradeoff* | | | |
| Output Growth (%) | 2.446 | 2.245 | 2.303 |
| Wealth share. Risk (hh.) | 0.281 | 0.222 | 0.242 |
| Gini Coefficient | 0.093 | 0.101 | 0.097 |
| *Price results* | | | |
| Sharpe Ratio $(r^k - r)/\sigma_{q^k}$ | 0.463 | 0.567 | 0.738 |
| Govt. bond price $q^m$ | 0.135 | 0.241 | 0.221 |
| Pension spread $r^n - r^d$ (%) | -2.833 | -3.492 | -3.433 |
| Pension price $q^n$ | 0.771 | 0.753 | 0.761 |
| *Aggregate results* | | | |
| Investment (%) | 23.900 | 19.982 | 21.054 |
| Total consumption (hh.) | 0.651 | 0.580 | 0.592 |
| *Sector level results* | | | |
| Fund Risky A/E | 1.686 | 1.907 | 1.655 |
| Bank Risky A/E | 1.076 | 0.869 | 1.084 |
| Wealth (hh.) | 1.212 | 1.099 | 1.116 |
| *Household distributional results (hh.)* | | | |
| C/W (Top 1%/Median) | 1.226 | 1.171 | 1.187 |
| Wealth (Top 1%/Median) | 1.832 | 1.970 | 1.928 |
| Wealth share Risk (Top 1%/Median) | -0.019 | -0.018 | 0.0055 |

**Table 5:** Equilibrium across different regulatory regimes. Wealth share risk (hh.) represents the average wealth share volatility ($\sigma_{\eta_h,z}$) across households. Total consumption (hh.) refers to the sum of flow and terminal consumption, averaged across houeholds. Fund Risky A/E and Risky Bank A/E represent the respective asset to equity ratios with zero weights on bond holdings. C/W is the flow consumption to wealth ratio of households. Wealth share risk denotes $\sigma_{\eta_h,z}$, the loading on the TFP shock to the household wealth-share.

wealth effects during a recession. Banks owe short-term deposits so the market value of their liabilities is unaffected by productivity decreases. They have to repay the full amount next period regardless of the state of the economy. By contrast, funds owe long-term tradable pensions that decrease in price during recessions (as do the prices of all long-term assets because goods are scarce and the marginal value of consumption is high) so the market value of their liabilities decreases in recessions. This means that, all else equal, funds gain wealth relative to banks during recessions. In this sense, funds are a natural "backstop" for the banking sector: they have the balance sheet space to purchase risky assets from banks during bad times. This is illustrated in Tables 6 and 7, which uses arrows to show how asset price changes during a recession affect the financial intermediary balance sheets.

**Table 6:** Bank Balance Sheet

| **Assets** | | **Liabilities** | |
|---|---|---|---|
| Capital | $\downarrow q_t^k k_t^b$ | Deposits | $d_t^b$ |
| Govt. bonds | $\downarrow q_t^m m_t^b$ | Equity | $\downarrow a_t^b$ |

**Table 7:** Fund Balance Sheet

| **Assets** | | **Liabilities** | |
|---|---|---|---|
| Capital | $\downarrow q_t^k k_t^f$ | Pensions | $\downarrow q_t^n n_t$ |
| Govt. bonds | $\downarrow q_t^m m_t^f$ | Equity | (?) $a_t^f$ |

Figure 2 illustrates the sector level risk sharing in our quantitative model by showing risk premia and and sector level dynamics in the baseline and counterfactual economies. The first two subplots in the top row show financial intermediary capital holdings while the final subplot shows the three main spreads in the economy: the capital risk premium ($r^k - r^d$), the term premium ($r^m - r^d$), and the annuity premium ($r^n - r^d$). The bottom row shows the impulse responses for financial intermediary wealth shares and fund capital holdings functions during a recession. These impulse response functions illustrate how funds act as shock absorbers in the baseline economy: during downturns their relative wealth share and capital holdings increase relative to banks. That is, they indirectly "insure" the banking sector.

The extent to which funds tax households through annuity premiums to absorb risk changes with regulation. In the fund-restricted economy, funds contract their balance sheet by holding less capital and supplying less annuity. The aggregate capital holding from the financial sector falls, pushing up the capital return spread. The household sector responds by substituting away from annuities and into risky capital. Lower capital prices depress aggregate output and investments but also lower consumption and wealth share risk of households since low financial sector leverage leads to low endogenous price volatility. This highlights a central trade-off: tighter regulation stabilizes household risk exposures but at the expense of macroeconomic outcomes. In the bank-regulated economy, banks shrink their balance sheets by reducing deposits and capital holdings. Funds expand instead, issuing more annuities at attractive terms and thereby increasing their absorbing capacity. This shift raises fund capital holdings and wealth shares in response to shocks. The macroeconomic and distributional trade-offs are similar to those in the fund-regulated economy, although quantitatively less pronounced.

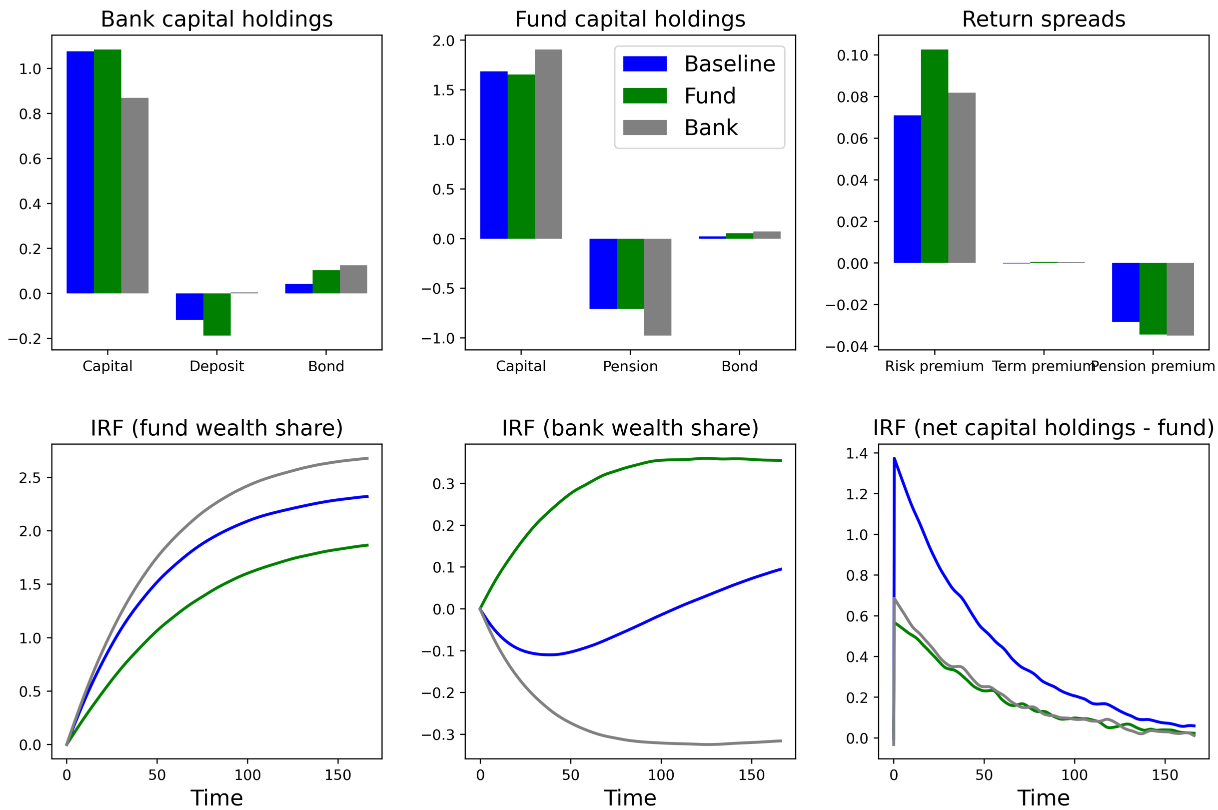


**Figure 2:** Sector level: The top left and center panel presents the bank and fund capital holdings across three economies, respectively. The top right figure presents the spreads. The bottom left and center panel figures present the impulse response of wealth share of fund, bank, and median household to a 2 std. deviation negative TFP shock. The bottom right panel figures trace out the capital holdings of the fund net of bank in response to the shock.

Taken together, these observations show that regulation determines (i) which agent, and to what extent the agent acts as a "back-stop" during bad times, and (ii) the magnitude of return spreads in the risky assets.

## 5.3 Mechanism (b): Return Spreads Generate Inequality

An important feature of the model is the ability to characterize the general equilibrium relationship between participation constraints, inequality, and asset price dynamics. Formally, the difference between the drift of the wealth share of any two households $i$ and $j$ can be expressed as:

$$\mu_{\eta_j,t} - \mu_{\eta_i,t} = \underbrace{(\theta^k_{j,t} - \theta^k_{i,t})(r^k_t - r^d_t - \bar{\boldsymbol{\sigma}}^k_{q,t} \cdot \boldsymbol{\sigma}^k_{q,t})}_{=:(\mu_{\eta_j,t}-\mu_{\eta_i,t})^K} + \underbrace{(\theta^n_{j,t} - \theta^n_{i,t})(r^n_t - r^d_t - \bar{\boldsymbol{\sigma}}^k_{q,t} \cdot \boldsymbol{\sigma}^n_{q,t})}_{=:(\mu_{\eta_j,t}-\mu_{\eta_i,t})^N}$$
$$- (\omega_j - \omega_i) + \lambda_h \phi_h \left( \frac{1}{\eta_{j,t}} - \frac{1}{\eta_{i,t}} \right) \tag{5.1}$$

The first term in equation (5.1) captures how participation constraints and risk aversion impact the excess return that different agents can earn. If agent $j$ is wealthier, $\eta_{j,t} > \eta_{i,t}$, then agent $j$ holds more wealth in capital and so gains wealth share compared to the poorer agents who are unwilling to pay the cost to participate in the capital market. The literature has sometimes been referred to this as a "scaling" effect—wealthier agents have access to better investment opportunities and so gain wealth more quickly. Our model endogenizes how strong this scaling effect is in general equilibrium by determining $r^k_t - r^d_t > 0$. The second term in equation (5.1) captures how pension costs impact inequality. Poorer households hold more pensions and so are more exposed the cost of the pension system, $r^n_t - r^d_t < 0$. The third term in (5.1) captures how a lower marginal propensity to consume out of wealth, $\omega_j < \omega_i$, leads to greater wealth accumulation. The final term captures redistribution from the exit and transfer system. This is the main force that stabilizes the wealth distribution in the economy. Other models have attributed this to many possible features (e.g. new entrants with better skills, idiosyncratic risk, etc.). We have little to say about it in our model and so simply allocate it to an exit rate.

Figure 3 reports household capital allocation and the decomposition of wealth share drift across percentiles. In the baseline economy, substantial heterogeneity in capital holdings emerges from participation constraints: wealthier households hold more capital due to easy access to capital markets, while annuity holdings remain similar across the distribution due to the "preferred-habitat" nature of annuity demand. The decomposition result in the bottom panel reveals that portfolio return differences contribute significantly to the difference in the wealth share drift between the top 1% and top 10% households. This "scaling effect" reinforces tail inequality. Because capital return spikes during recessions, households with greater risk exposure eventually gain wealth share, generating a fat-tailed distribution and procyclical inequality dynamics.

Regulation amplifies these effects through its effect on asset prices. Relative to the baseline, both the fund and the bank regulated economies exhibit a larger difference in capital holdings and wealth share drift between the rich and poor households. This is because of the endogenous increase in capital return spread, pushing up the first component of equation

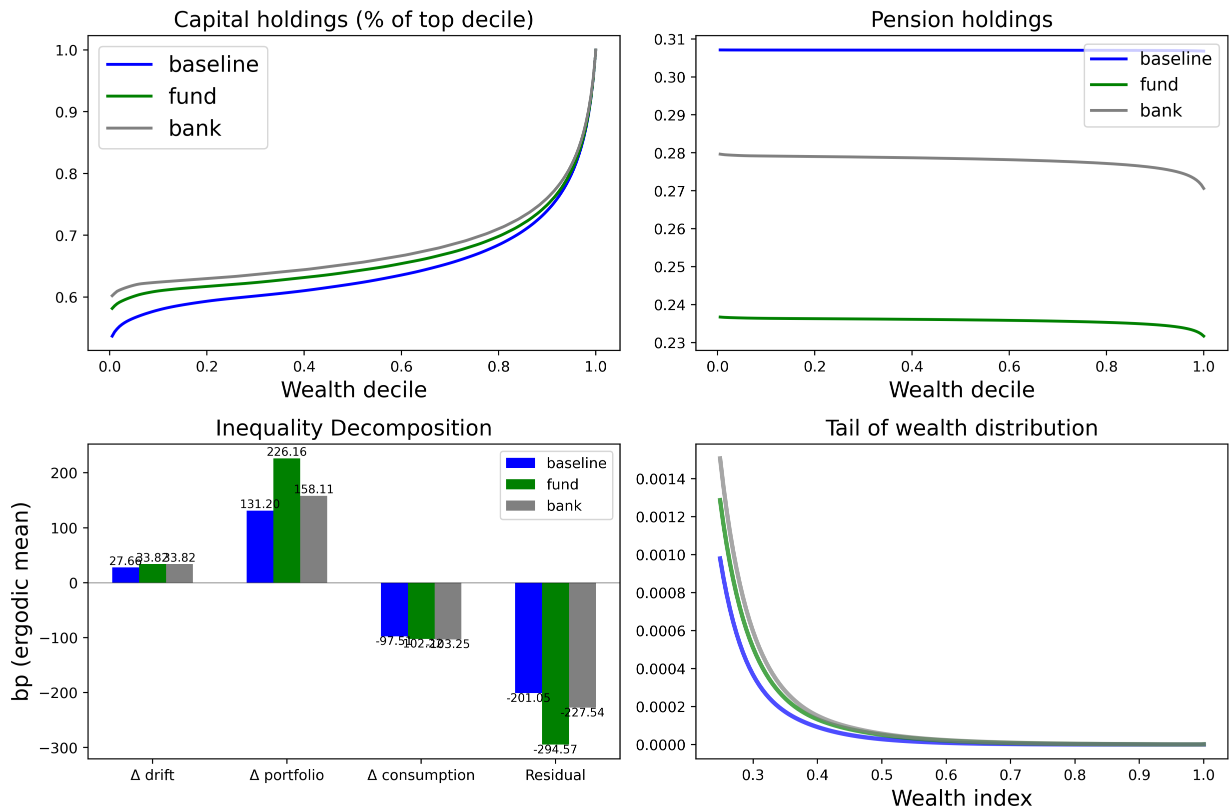


**Figure 3:** Distributional outcomes: The top left panel presents the capital holdings of households relative to the top wealth decile in the baseline, fund-regulated, and bank-regulated economy. The top right panel presents the level of annuity holdings. The bottom left panel presents the decomposition of difference in wealth share drift between 99-th pctl. and 90-th pctl. household. The bottom right panel presents the tail of wealth distribution. In all plots, the blue line corresponds to baseline economy, green line corresponds to fund restricted economy, and gray line corresponds to bank restricted economy.

5.1. In turn, higher return spread intensifies inequality dynamics: in response to a shock, the Gini coefficient raises more in regulated economies due to a combination of high risk exposure among the wealthy and high expected capital returns. The effect is especially more pronounced in the fund economy, where the capital return spread is the largest. The basic intuition is that regulations make financial intermediation difficult, which opens up spreads. Wealthy agents can better take advantage of these spreads because they can better circumvent the regulations and invest directly into capital markets to earn high returns.

## 5.4 Policy Variant Preferred Habitat

Environments with preferred habitat preferences and inelastic demand have been much studied and estimated over the past decade. In this section, we map our model into the Vayanos and Vila (2021) preferred habitat environment and show that our FOCs can be interpreted as endogenizing the policy dependence of the inelastic demand "shifters" in Vayanos and Vila (2021). This is what enables us to study the general equilibrium impact of macro prudential policies.

To understand these connections, we first express our pension demand in an analogous form to the literature. The original Vayanos and Vila (2021) paper has a bond demand function:

$$\mathcal{U}'(C_t) = q_t^n \beta_t, \quad \beta_t = \theta_0 + \sum_{k=1}^{K} \theta_k \beta_{k,t} \tag{5.2}$$

where the $\{\beta_{k,t}\}$ are exogenous, mean reverting factors and the $\{\theta_k\}$ are exogenous factor loadings. Our asset pricing condition for pensions can be rearranged to get:

$$\mathcal{U}'\left(C\right) = \frac{1}{\lambda_h} \frac{q^n}{1-q^n} \xi_h \left(r^d - r^n - \sigma_{\xi_h} \sigma_{q^n}\right) \tag{5.3}$$

where the RHS can also be expressed as:

$$\frac{1}{\lambda_h} \frac{q^n}{1-q^n} E\left[\int_0^\infty e^{-\rho t} \xi_{h,t} dc_{h,t} dt\right]$$

Formula (5.3) implicitly characterizes our equilibrium demand function for pensions. The demand function shares the same basic trade-off as the micro-foundation in the appendix of Vayanos and Vila (2021): households can invest an extra unit in the pension for marginal benefit $\mathcal{U}'(C)$ (the LHS) at the cost of shifting down their consumption path (the RHS). However, our formula differs in four ways: (i) we model a random horizon while Vayanos and Vila (2021) has fixed horizon[11]; (ii) we have one good so the gain from purchasing one share of pension right before retirement shock hits is $1/q_n - 1$, while in Vayanos and Vila (2021), there are two goods; (iii) in Vayanos and Vila (2021), they assume risk-neutral flow utility function $\xi_{h,t} = 1$; and most importantly (iv) in Vayanos and Vila (2021) the RHS "shifter" $\beta_h = \xi_h \left(r^d - r^n - \sigma_{\xi_h}\sigma_{q^n}\right)/\lambda_h$ is treated as exogenous whereas in our model it comes from the portfolio choice problem of a risk averse agent.

To make these connections precise, we take the linear approximation for our endogenous RHS shifter $\beta_h = \xi_h \left(r^d - r^n - \sigma_{\xi_h}\sigma_{q^n}\right)/\lambda_h$ and compare it to the exogenous linear factor model for $\beta_h$ in Vayanos and Vila (2021). We do this by taking a first-order Taylor expansion of $\beta_h(\boldsymbol{\eta}, \eta_f, z)$ around the stochastic steady state (SSS) point $(\eta_{h,ss}, \ldots, \eta_{h,ss}, \eta_{f,ss}, z_{ss})$:

$$\beta_h(\boldsymbol{\eta}, \eta_f, z) \approx \beta_{h,0} + \sum_{i=1}^{I} \left.\frac{\partial \beta_h}{\partial \eta_i}\right|_{\text{SSS}} (\eta_i - \eta_{h,ss}) + \left.\frac{\partial \beta_h}{\partial \eta_f}\right|_{\text{SSS}} (\eta_f - \eta_{f,ss}) + \left.\frac{\partial \beta_h}{\partial z}\right|_{\text{SSS}} (z - z_{ss}).$$

Define the centered factors $\beta_{k,t} = \eta_{k,t} - \eta_{h,ss}$ for $k = 1, \ldots, I$, $\beta_{I+1,t} = \eta_{f,t} - \eta_{f,ss}$, $\beta_{I+2,t} = z_t - z_{ss}$. Then $K = I + 2$, and the loadings in Vayanos and Vila (2021) formulation in equation (5.2) map to the gradient at SSS:

$$\theta_0 = \beta_{h,0}, \qquad \theta_k = \left.\frac{\partial \beta_h}{\partial \eta_k}\right|_{\text{SSS}} \ (k = 1, \ldots, I), \qquad \theta_{I+1} = \left.\frac{\partial \beta_h}{\partial \eta_f}\right|_{\text{SSS}}, \qquad \theta_{I+2} = \left.\frac{\partial \beta_h}{\partial z}\right|_{\text{SSS}}.$$

Thus, the VV factor representation nests the linear response of $\beta_h$ to distributional and output shocks in equilibrium.

We plot our "factors" in Figure 4. The left plot shows the drift of the preferred habitat demand shifter projected onto $z$ in our model under the different policy regimes. The right plot presents the conditional CDF of $\beta_h$. In both cases, we can see that regulatory changes move around the factor loading, indicating that without the additional structure in our model it would not be possible to identify the general equilibrium impact of policy change. This illustrates the importance of generalizing the preferred habitat style environments for doing policy analysis.

# 6 Comparative Statics: Demographic change

Pension and insurance funds are inherently exposed to demographic risks because they are short mortality risk through the sale of annuities. Capital regulation further distorts their ability to absorb these risks: when longevity increases, annuity liabilities rise, but insurers

[11]To see this more clearly, we could further rewrite $1/\lambda_h$ as: $\int_0^\infty e^{-\lambda_h s} ds$.

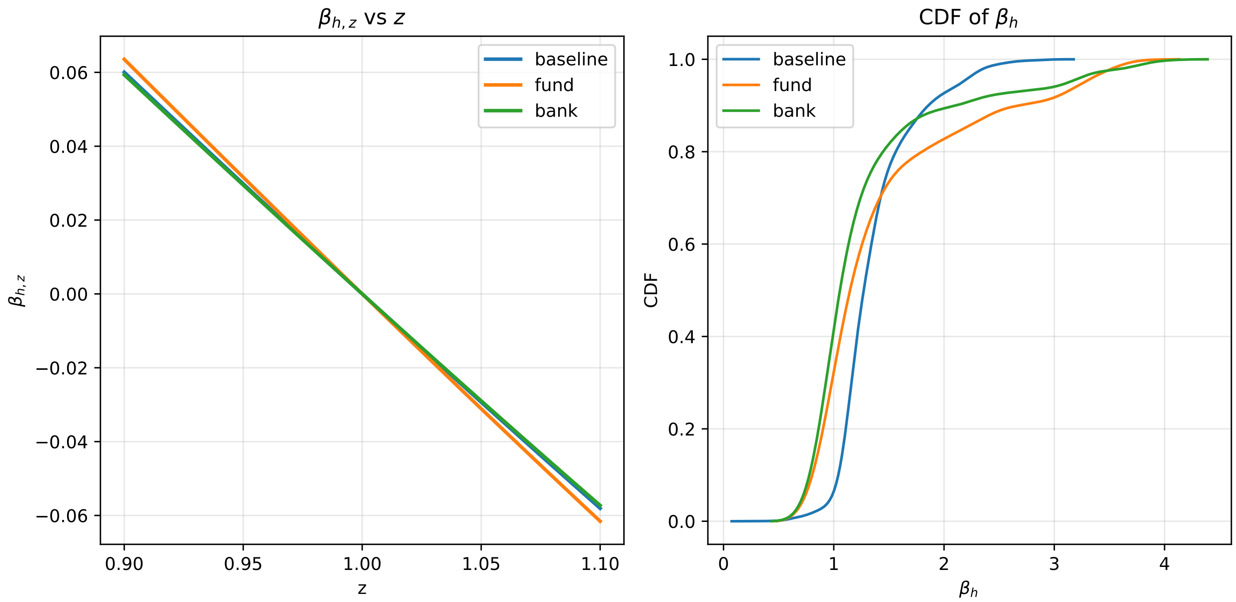


**Figure 4:** Preferred-Habitat demand: The left panel presents the household preferred habitat demand $\beta_h$'s drift projected onto productivity level across three regulatory regimes, with the household distribution and intermediary wealth share at the stochastic steady state in baseline economy; The right panel presents the conditional CDF of $\beta_h$ with regulatory implied stationary household distribution, and intermediaries' wealth level set to be at the stochastic steady state.

often retrench rather than expand supply since regulatory capital charges make equity financing costly (Koijen and Yogo, 2016). On the demand side, demographic pressures also shape pension fund behavior. Rising longevity expands liabilities and underfunding, creating strong incentives to seek higher returns by tilting toward risky assets (Novy-Marx and Rauh, 2011). In this way, demographic-driven liability growth directly influences balance sheets and portfolio choices. To capture this mechanism, we study a counterfactual economy with longer household life span that increase longevity-driven liability pressure, modeled as an decrease in the parameter $\lambda_h$. This allows us to quantify how demographic risks affect asset prices and household wealth distribution.

Figure 5 illustrates how demographic shifts sectoral balance sheets, portfolio allocations, and the distribution of household wealth, while Table 8 summarizes the corresponding moments. The counterfactual economy with a lower death rate produces notable changes in who acts as shock absorbers in downturns, alongside distributional changes across households. The most immediate effect of a lower retirement rate is the increase of the household wealth share to the other agents in the economy. Because of lower retirement rate, the demand for annuity and capital goes up, increasing their prices. In equilibrium, this translates to smaller spreads. From the fund perspective, low prices reduce the attractiveness of capital. Therefore, they shrink their balance sheet by scaling down on capital, and retreating from

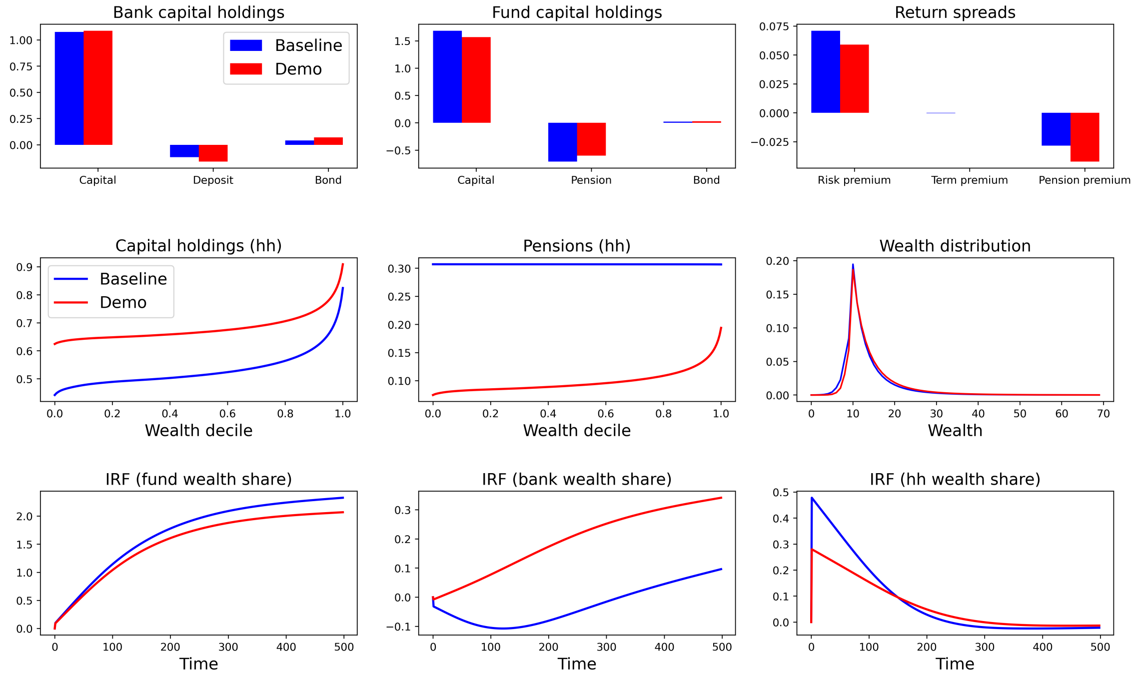


**Figure 5:** Demographic change: The top panel presents the financial sector portfolio holdings and return spreads. The second row presents household portfolio holdings and wealth distribution. The last row presents impulse response of sectoral wealth shares to a 2 s.d. negative TFP shock. In all plots, the blue represents baseline economy with $\lambda_h = 0.10$, and red represents a counterfactual economy with $\lambda_h = 0.07$.

| | Baseline | Demographic |
|---|---|---|
| *Growth-Stability-Inequality Tradeoff* | | |
| Output Growth (%) | 2.446 | 2.585 |
| Cons. Risk (hh.) (%) | 3.743 | 4.080 |
| Gini Coefficient | 0.093 | 0.101 |
| *Price results* | | |
| Sharpe Ratio $(r^k - r)/\sigma_{q^k}$ | 0.463 | 0.347 |
| Govt. bond price $q^b$ | 0.135 | 0.176 |
| Pension spread $r^n - r^d$ (%) | -2.833 | -4.209 |
| *Aggregate results* | | |
| Investment (%) | 23.900 | 27.006 |
| Total consumption (hh.) | 0.651 | 0.740 |

**Table 8:** Baseline ($\lambda_h = 0.10$) vs Demographic economy ($\lambda_h = 0.07$). Cons. Risk (hh.) represents the average consumption volatility across households. Total consumption (hh.) refers to the sum of flow and terminal consumption, averaged across houeholds. Fund A/E and Bank A/E represent the respective asset to equity ratios. C/W is the flow consumption to wealth ratio of households. Wealth share risk denotes $\sigma_{\eta_h,z}$, the loading on the TFP shock to the household wealth-share.

their role as marginal risk absorbers. The vacuum left by the fund sector is taken by the banking sector. They expand their balance sheet by taking on more leverage, and in doing so, assume the stabilizing role during downturns that the funds played in the baseline economy. The larger banking sector improves their aggregate capacity to absorb shocks, which stabilizes capital prices more directly. The result is higher output and capital investment relative to the baseline economy.

Taken together, our counterfactual results demonstrate that financial sector segmentation is not only an asset pricing phenomenon but also a central determinant of household welfare and inequality. Who absorbs risk in downturns, whether banks, funds, or households, critically shapes the trade-offs between growth, stability, and inequality. By embedding intermediary frictions into a heterogeneous agent framework, we show how regulation and demographics alter these channels in ways that standard representative agent or pricing-only models cannot capture.

# 7 Conclusion

This paper has examined how segmentation within the financial sector interacts with household heterogeneity to shape asset prices, stability, and inequality. We developed a new deep learning methodology that enables global solutions in environments with long-term assets, stochastic volatility, and binding portfolio constraints. This approach bridges advances in intermediary asset pricing and heterogeneous agent macroeconomics, providing a flexible framework for studying how regulation and demographics influence the joint distribution of wealth and asset market dynamics. Quantitatively, we calibrated the model to the post financial crisis United States economy from 2010 to 2024 and showed that it matches key macroeconomic, asset pricing, and cross sectional portfolio moments. The model reproduces realistic heterogeneity in capital holdings across households, delivers endogenous volatility in asset returns, and generates a fat tailed wealth distribution consistent with survey and administrative data. The interaction between intermediaries is central. Banks issue short term liabilities that remain stable in downturns, while funds issue long term annuities whose value falls in recessions, enabling them to expand balance sheets and absorb risk. This mechanism explains why funds act as stabilizers in the baseline economy but also why their behavior imposes costs on households through higher annuity premia.

Counterfactual experiments highlight the tradeoffs associated with alternative regulatory and demographic regimes. Fund regulation, which imposes bank style leverage limits, reduces volatility but raises inequality by shifting risk to households with greater capital exposure. Bank regulation, in the form of tighter capital requirements, instead amplifies volatility while also increasing inequality, as wider return spreads disproportionately benefit wealthier households. Demographic change operates through a different channel. Longer household working lives shifts risk absorption away from funds and toward banks. This resulting increase in inequality coincides with more household risk and higher investment, underscoring the complex role that demographics play in shaping both macroeconomic outcomes and distributional patterns.

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# A Additional Model Features

## A.1 Modeling Retirement

In this subsection, we offer a microfoundation for the terminal utility $\mathcal{U}(\mathcal{C})$. Consider the setup from Section 2 but with agent retirement. Formally, agents are now born as active participants in the economy, then transition to retirement at rate $\lambda_h$, and then finally die at rate $\lambda_d$. When agents retire they stop participating in economic production and exchange their assets for resources to consume during their retirement. They receive flow utility $u(c_{h,t}) = (1-\alpha)c_{h,t}^{1-\gamma}/(1-\gamma)$ from consuming during retirement. The mathematically simplest way to model this is for the households to purchase a stock of goods at retirement, which they progressive eat. We focus on this case in this Appendix since it leads to exactly the same market clearing conditions as the main text. An alternative model would be for households to purchase an annuity that pays goods throughout their retirement, which would lead to a similar household problem but potentially different market clearing.

As in the main text, let $V_h(a, \boldsymbol{S})$ denote the value function of household with wealth $a$ when the aggregate state is $\boldsymbol{S}$. Now, let $W_h(A, \boldsymbol{S})$ denote the value function of a retired household with remaining wealth $A$. Proposition

**Proposition 1.** *The household's value function at retirement is:*

$$W_h(a, \boldsymbol{S}) = \frac{(1-\alpha)\varrho A^{1-\gamma}}{(1-\gamma)(\rho+\lambda_d)}$$

*where $\varrho$ satisfies:*

$$(\rho+\lambda_d)\frac{\varrho}{(\rho+\lambda_d)(1-\gamma)} = \frac{1}{1-\gamma}\left(\frac{\rho+\lambda_d}{\varrho}\right)^{(1-\gamma)/\gamma} + \frac{\varrho}{\rho+\lambda_d}\left(1-\left(\frac{\rho+\lambda_d}{\varrho}\right)^{1/\gamma}\right)$$

*Proof.* The optimization problem of a household that retires at time $T_r$ is:

$$\begin{aligned}
W_h(A_{T_r}, \boldsymbol{S}_{T_r}) &= \max_c \int_{T_r}^{T_d} e^{-\rho(t-T_r)} u(c_{T+t})dt \\
dA_t &= (A_t - c_t)dt \\
A_{T_r} &= (1-\theta^n_{h,t} + \theta^n_{h,t}/q^n_t)a_{T_r}
\end{aligned}$$

where $A_{T_r}$ is the total household wealth after redeeming pension contracts with the fund managers. From the setup we can see that $W$ does not depend upon $\mathbf{S_{T_r}}$ and so we drop it from function. It follows that $W_h(A)$ satisfies the HJBE:

$$(\rho + \lambda_d)W_h(A) = \max_c\{u(c) + \partial W(A)(A - c)\}$$

for which the FOC is:

$$u'(c) = \partial_A W(A).$$

We guess (and verify) that:

$$W(a) = \frac{(1-\alpha)\varrho A^{1-\gamma}}{(1-\gamma)(\rho + \lambda_d)}$$

and impose the function form $u(c_{h,t}) = (1-\alpha)c_{h,t}^{1-\gamma}/(1-\gamma)$. Then the FOC becomes:

$$c = \left(\frac{\rho + \lambda_d}{\varrho}\right)^{1/\gamma} A$$

and the HJBE becomes:

$$(\rho + \lambda_d)\frac{\varrho A^{1-\gamma}}{(\rho + \lambda_d)(1-\gamma)} = \frac{1}{1-\gamma}\left(\left(\frac{\rho + \lambda_d}{\varrho}\right)^{1/\gamma} A\right)^{1-\gamma} + \frac{\varrho}{\rho + \lambda_d} A^{-\gamma}\left(A - \left(\frac{\rho + \lambda_d}{\varrho}\right)^{1/\gamma} A\right)$$

so that $\varrho$ implicitly satisfies:

$$(\rho + \lambda_d)\frac{\varrho}{(\rho + \lambda_d)(1-\gamma)} = \frac{1}{1-\gamma}\left(\frac{\rho + \lambda_d}{\varrho}\right)^{(1-\gamma)/\gamma} + \frac{\varrho}{\rho + \lambda_d}\left(1 - \left(\frac{\rho + \lambda_d}{\varrho}\right)^{1/\gamma}\right)$$

and our guess is verified. □

*Extension:* If the household invested in an annuity with coupon rate $\zeta$ instead of storing consumption goods, then the household problem is the same except that now $da_t = (\zeta a_t - c_t)dt$ and so the implicit expression for $\varrho$ is adjusted by $\zeta$ to become:

$$(\rho + \lambda_d)\frac{\varrho}{(\rho + \lambda_d)(1-\gamma)} = \frac{1}{1-\gamma}\left(\frac{\rho + \lambda_d}{\varrho}\right)^{(1-\gamma)/\gamma} + \frac{\varrho}{\rho + \lambda_d}\left(\zeta - \left(\frac{\rho + \lambda_d}{\varrho}\right)^{1/\gamma}\right).$$

*Micro-foundation for main text:* Returning to the formulation in the main text, we can micro-found consumption at death with this retirement model by setting $\mathcal{U}(\cdot) = W(\cdot)$ and noting that at retirement:

$$C_{T_r} = (1 - \theta^n_{h,t} + \theta^n_{h,t}/q^n_t) a_{T_r} = A_{T_r}$$

and so we have:

$$U(C_{T_r}) = V(A_{T_r}).$$

as required. Under this formulation, we have the parameter mapping:

$$1 - \beta = \frac{1 - \alpha}{\rho + \lambda_d}.$$

## A.2 Government Budget Constraint Accounting

Let $q^m_t$ denote the price of a government bond at time $t$. Then the flow rate of bond maturity is $\lambda_m M$ and the proceeds from the re-issuance of the bonds is $q^m_t \lambda_m M$. The government raises a flow wealth tax, $\tau_j$, on agent of type $j \in \{h, b, f\}$. This implies that the government budget constraint is given by:

$$\sum_j \tau_{j,t} A_{j,t} + \lambda_b A_{b,t} + \lambda_f A_{f,t} + q^m_t \lambda_m M = \lambda_h \underline{a}_{h,t} + \lambda_b \underline{a}_{b,t} + \lambda_f \underline{a}_{f,t}$$

where $A_j$ is the wealth of agents in sector $j \in \{h, b, f\}$, $K_h$ is capital owned by the household, and $D_h$ is deposits owned by the household. If all agents have the same proportional flow wealth tax, $\tau_j = \tau_{a,t}$ for all $j \in \{h, b, f\}$, then:

$$\tau_a = \frac{\sum_j \lambda_j \phi_j A_t - \lambda_b A_{b,t} - \lambda_f A_{f,t} + (1 - q^m_t)\,\lambda_m M}{A_t}$$

# B Recursive Characterization of Equilibrium

In this section, we derive the recursive representation of equilibrium described in Theorems 1, 2, and 3 in the main text. Recall the notation for the finite dimensional aggregate states, $\boldsymbol{s}$, the total aggregate states, $\boldsymbol{S}$, the finite dimensional individual and aggregate state states, $\boldsymbol{x}$, and the total individual and aggregate states, $\boldsymbol{X}$:

$$\boldsymbol{s} := (z, K, \eta_b, \eta_f), \qquad \boldsymbol{S} := (\boldsymbol{s}, g_h)$$

$$\boldsymbol{x} := (a, z, K, \eta_b, \eta_f), \qquad \boldsymbol{X} := (\boldsymbol{x}, g_h)$$

We denote the law of motion for the finite dimensional states as:

$$\begin{aligned}
d\boldsymbol{s}_t &= (\boldsymbol{\mu}_s(\boldsymbol{s}_t) \odot \boldsymbol{s}_t)dt + (\boldsymbol{\sigma}_s(\boldsymbol{s}_t) \odot \boldsymbol{s}_t)^T dW_t \\
d\boldsymbol{x}_t &= (\boldsymbol{\mu}_x(\boldsymbol{x}_t, c_h, \boldsymbol{\theta}_h) \odot \boldsymbol{x}_t)dt + (\boldsymbol{\sigma}_x(\boldsymbol{x}_t, \boldsymbol{\theta}_h) \odot \boldsymbol{x}_t)^T dW_t
\end{aligned}$$

where $\boldsymbol{\mu}_s$, $\boldsymbol{\mu}_x$, $\boldsymbol{\sigma}_s$ and $\boldsymbol{\sigma}_x$ are the geometric drift and volatility for $\boldsymbol{s}_t$ and $\boldsymbol{x}_t$. We denote the law of motion for household wealth distribution by:

$$dg_{h,t}(a) = \mu_g(a, \boldsymbol{S})dt + \sigma_g(a, \boldsymbol{S})^T dW_t$$

## B.1 Household Optimization

In this subsection, we solve the household optimization problem. We start by setting up the beliefs of the agents. We then construct the HJB equation for the households and take the first order conditions. Finally, we derive the law of motion for the household SDF and construct their Euler equation.

*Beliefs:* Let $V_h(\boldsymbol{X})$ denote the value function for a household with state variable $\boldsymbol{X}$. Let $(\tilde{\boldsymbol{\mu}}_s, \tilde{\boldsymbol{\sigma}}_s, \tilde{\mu}_g, \tilde{\sigma}_g)$ denote the household's belief about the evolution of the aggregate state space[12]. So, under their beliefs, the geometric drift and volatility of the household's total finite state vector $\boldsymbol{x}_t$ are:

$$\boldsymbol{\mu}_x(\boldsymbol{x}; c_h, \boldsymbol{\theta}_h, \iota_h) = \begin{bmatrix} \mu_a(x; c_h, \boldsymbol{\theta}_h, \iota_h) \\ \mu_z \\ \tilde{\mu}_K \\ \tilde{\mu}_{\eta_b} \\ \tilde{\mu}_{\eta_f} \end{bmatrix}, \qquad \boldsymbol{\sigma}_x(\boldsymbol{x}; \boldsymbol{\theta}_h)^T = \begin{bmatrix} \sigma_a(\boldsymbol{x}; \boldsymbol{\theta}_h) \\ \sigma_z \\ 0 \\ \tilde{\sigma}_{\eta_b} \\ \tilde{\sigma}_{\eta_f} \end{bmatrix}$$

and their belief about the law of motion of $g_{h,t}$ satisfies the equation:

$$dg_{h,t}(a) = \tilde{\mu}_g(a, \boldsymbol{S})dt + \tilde{\sigma}_g(a, \boldsymbol{S})^T dW_t$$

*HBJE:* Given their beliefs about the evolution of the aggregate states, the value function

[12] We assume that all agents have the same beliefs

$V_h(\boldsymbol{X})$ for a household solves the HJBE (B.1) below (written in matrix form):

$$\begin{aligned}\rho_h V_h(\boldsymbol{X}) = \max_{c_h,\boldsymbol{\theta}_h,\iota_h} \Big\{ & u(c_h) + \psi_{h,k}(\theta_h^k, \eta_h)\Xi_h a + \lambda\left(\mathcal{U}\left(1-\theta_h^n + \theta_h^n/q^n\right) - V_h(\boldsymbol{X})\right) \\ & + (\boldsymbol{\mu}_x(\boldsymbol{x}, c_h, \boldsymbol{\theta}_h, \iota_h)\odot \boldsymbol{x})^T D_x V_h(\boldsymbol{X}) + \langle \partial V_h/\partial g(\cdot, \boldsymbol{S}),\ \tilde{\mu}_g(\cdot,\boldsymbol{S})\rangle \\ & + \frac{1}{2}\mathrm{tr}\Big\{(\boldsymbol{\sigma}_x(\boldsymbol{x},\boldsymbol{\theta}_h)\odot\boldsymbol{x})^T(\boldsymbol{\sigma}_x(\boldsymbol{x},\boldsymbol{\theta}_h)\odot\boldsymbol{x})D_x^2 V_h(\boldsymbol{X})\Big\} \\ & + \langle((\boldsymbol{\sigma}_x(\boldsymbol{x},\boldsymbol{\theta}_h)\odot\boldsymbol{x})^T D_{x,g}V_h)(\cdot,\boldsymbol{S}), \tilde{\sigma}_g(\cdot,\boldsymbol{S})\rangle \\ & + \frac{1}{2}\langle\langle D_{gg}V(\cdot,\boldsymbol{S}), \tilde{\sigma}_g(\cdot,\boldsymbol{S})\otimes\tilde{\sigma}_g(\cdot,\boldsymbol{S})\rangle\rangle\Big\},\end{aligned} \tag{B.1}$$

where the gradient vector and Hessian are given by:

$$D_x V_h(\boldsymbol{x}) = \begin{bmatrix} \partial_a V_h(\boldsymbol{x}) \\ \partial_z V_h(\boldsymbol{x}) \\ \partial_K V_h(\boldsymbol{x}) \\ \partial_{\eta_b} V_h(\boldsymbol{x}) \\ \partial_{\eta_f} V_h(\boldsymbol{x}) \end{bmatrix}, \qquad D_x^2 V_h(\boldsymbol{x}) = \begin{bmatrix} \partial^2_{aa}V_h & \dots & \partial^2_{aA_f}V_h \\ \partial^2_{za}V_h & \dots & \partial^2_{zA_f}V_h \\ \partial^2_{Ka}V_h & \dots & \partial^2_{KA_f}V_h \\ \partial^2_{\eta_b a}V_h & \dots & \partial^2_{A_bA_f}V_h \\ \partial^2_{\eta_f a}V_h & \dots & \partial^2_{A_fA_f}V_h \end{bmatrix}.$$

and where $\frac{\partial V_h}{\partial g}(a')$ is the first order Frechet derivative kernel of $V_h$ with respect to $g$, $D_{xg}V(a') := \frac{\partial}{\partial x}\left(\frac{\partial V_h}{\partial g}(a')\right)$ is the cross partial Frechet derivative, $D_{gg}V(a', a'') := \frac{\partial^2 V_h}{\partial g(a')\partial g(a'')}$ is the second order Frechet derivative kernel, and the inner products are $\langle f, h\rangle := \int f(a')h(a')da'$ and $\langle\langle K, u\otimes v\rangle\rangle := \int\int K(a', a'')u(a')v(a'')da'da''$. For notational convenience, we define the Frechet derivative operator by:

$$\begin{aligned}\mathcal{L}_g V_h := & \langle \partial V_h/\partial g(\cdot,\boldsymbol{S}),\ \tilde{\mu}_g(\cdot,\boldsymbol{S})\rangle \\ & + \langle((\boldsymbol{\sigma}_x(\boldsymbol{x},\boldsymbol{\theta}_h)\odot\boldsymbol{x})^T D_{x,g}V_h)(\cdot,\boldsymbol{S}), \tilde{\sigma}_g(\cdot,\boldsymbol{S})\rangle \\ & + \frac{1}{2}\langle\langle D_{gg}V(\cdot,\boldsymbol{S}), \tilde{\sigma}_g(\cdot,\boldsymbol{S})\otimes\tilde{\sigma}_g(\cdot,\boldsymbol{S})\rangle\rangle\end{aligned} \tag{B.2}$$

*HBJE (partial expansion):* To see the optimization more clearly, we can rewrite the HJBE with the controlled variables taken out of the matrices. In this case, the HJBE is given by:

$$\begin{aligned}\rho_h V_h(a,\boldsymbol{s},g_h) = \max_{c_h,\boldsymbol{\theta}_h,\iota_h} \Big\{ & u(c_h) + \psi_{h,k}(\theta_h^k)\Xi_h a + \lambda\left(\mathcal{U}\left(1-\theta_h^n + \theta_h^n/q^n\right) - V_h(a,\boldsymbol{s},g_h)\right) \\ & + \mu_a(a,\boldsymbol{s},g_h; c_h, \boldsymbol{\theta}_h, \iota_h)a + (\boldsymbol{\mu}_s(\boldsymbol{s},g_h)\odot\boldsymbol{s})^T D_s V_h(a,\boldsymbol{s},g_h) \\ & + \frac{1}{2}\partial^2_{aa}V_h(a,\boldsymbol{s},g_h)\sigma_a^2(\boldsymbol{s},g_h;\boldsymbol{\theta}_h)a^2 + \sum_{l\le|\boldsymbol{s}|}\partial_{as_l}V_h(a,\boldsymbol{s},g_h)\sigma_a(\boldsymbol{s},g_h;\boldsymbol{\theta}_h)\sigma_{s_j}as_j \\ & + \frac{1}{2}\mathrm{tr}\Big\{(\boldsymbol{\sigma}_s(\boldsymbol{s},g_h)\odot\boldsymbol{s})^T(\boldsymbol{\sigma}_s(\boldsymbol{s},g_h)\odot\boldsymbol{s})D_s^2V_h(a,\boldsymbol{s},g_h)\Big\} + \mathcal{L}_g V_h(a,\boldsymbol{s},g_h)\Big\}\end{aligned}$$

where:

$$
\begin{aligned}
\psi_h(\theta^k_{h,t}) &= \frac{\bar{\psi}_k}{2}\left(\theta^k_{h,t}\right)^2 \\
\mu_a(a, \boldsymbol{s}, g_h; c_h, \boldsymbol{\theta}_h, \iota_h) &= \left(\tilde{r}^d_t + \theta^n_{h,t}(\tilde{r}^n_t - \tilde{r}^d_t) + \theta^k_{h,t}(\tilde{r}^k_t - \tilde{r}^d_t) - c_{h,t}/a_{h,t} - \tau_{h,t}\right) \\
\sigma_a(\boldsymbol{s}, g_h; \boldsymbol{\theta}_h) &= \boldsymbol{\theta}^T_h \tilde{\sigma}_q = \theta^n_h \tilde{\sigma}_{q^n,z} + \theta^k_h \tilde{\sigma}_{q^k,z} \\
\sigma^2_a(\boldsymbol{s}, g_h; \boldsymbol{\theta}_h) &= (\theta^n_h \tilde{\sigma}_{q^n,z} + \theta^k_{h,t} \tilde{\sigma}_{q^k,z})^2 \\
\sigma_a(\boldsymbol{s}, g_h; \boldsymbol{\theta}_h)\tilde{\sigma}_{\eta_j} &= (\theta^n_{h,t} \tilde{\sigma}_{q^n,z} + \theta^k_h \tilde{\sigma}_{q^k,z})\tilde{\sigma}_{\eta_j,z} \\
\sum_{l \leq |\boldsymbol{s}|} \partial_{as_l} V_h(a, \boldsymbol{s}, g_h)\boldsymbol{\sigma}^T_a(\boldsymbol{s}, g_h; \boldsymbol{\theta}_h)\boldsymbol{\sigma}_{s_l} &= \partial^2_{az} V_h(a, \boldsymbol{s}, g_h)\sigma_a(\boldsymbol{s}, g_h; \boldsymbol{\theta}_h)\sigma_z a z \\
&\quad + \sum_{j \in \{b,f\}} \partial^2_{a\eta_j} V_h(a, \boldsymbol{s}, g_h)\boldsymbol{\sigma}^T_a(\boldsymbol{s}, g_h; \boldsymbol{\theta}_h)\tilde{\boldsymbol{\sigma}}_{\eta_j}(\boldsymbol{s}) a \eta_j
\end{aligned}
$$

The HJBE then becomes:

$$
\begin{aligned}
\rho_h V_h(a, \boldsymbol{s}, g_h) = \max_{c_h, \boldsymbol{\theta}_h, \iota_h} &\Big\{ u(c_h) + \psi_{h,k}(\theta^k_h)\Xi_h a + \lambda\left(\mathcal{U}\left(1 - \theta^n_h + \theta^n_h/q^n\right) - V_h(a, \boldsymbol{s}, g_h)\right) \\
&+ \mu_a(a, \boldsymbol{s}, g_h; c_h, \boldsymbol{\theta}_h, \iota_h) a \ \partial_a V_h(a, \boldsymbol{s}, g_h) + (\boldsymbol{\mu}_s(\boldsymbol{s}) \odot \boldsymbol{s})^T D_s V_h(a, \boldsymbol{s}, g_h) \\
&+ \frac{1}{2}\partial^2_{aa} V_h(a, \boldsymbol{s}, g_h)(\boldsymbol{\theta}^T_h \tilde{\boldsymbol{\sigma}}_q)^2 a^2 + \sum_j \partial_{as_j} V_h(a, \boldsymbol{s}, g_h)\boldsymbol{\theta}^T_h \tilde{\boldsymbol{\sigma}}_q \tilde{\sigma}_{s_j} a s_j \\
&+ \frac{1}{2}\mathrm{tr}\left\{(\boldsymbol{\sigma}_s(\boldsymbol{s}) \odot \boldsymbol{s})^T (\boldsymbol{\sigma}_s(\boldsymbol{s}) \odot \boldsymbol{s}) D^2_s V_h(a, \boldsymbol{s}, g_h)\right\} + \mathcal{L}_g V_h(a, \boldsymbol{s}, g_h) \Big\}
\end{aligned}
$$

*FOCs:* The first order conditions are given by the following equations:

$$
\begin{aligned}
[c_h]: \qquad & 0 = u'(c_h) - \partial_a V_h(a, \boldsymbol{s}, g_h) \\
[\iota_h]: \qquad & 0 = \Phi'(\iota) - \frac{1}{q^k(\boldsymbol{s}, g_h)} \\
[\theta_h^k]: \qquad & 0 = (\tilde{r}^k(\boldsymbol{s}, g_h) - \tilde{r}^d(\boldsymbol{s}, g_h))a\partial_a V_h(a, \boldsymbol{s}, g_h) + \partial_{\theta_h^k}\psi_{h,k}(\theta_h^k)\Xi_h(\boldsymbol{s}, g_h)a \\
& \qquad + \partial_{aa}V_h(a, \boldsymbol{s}, g_h)\tilde{\sigma}_{q^k}(\boldsymbol{s}, g_h)\sigma_a(\boldsymbol{s}, g_h)a^2 \\
& \qquad + \sum_j \partial_{as_j}V_h(a, \boldsymbol{s}, g_h)\tilde{\sigma}_{q^k}(\boldsymbol{s}, g_h)\tilde{\sigma}_{s_j}(\boldsymbol{s}, g_h)as_j \\
& \quad = (\tilde{r}^k(\boldsymbol{s}, g_h) - \tilde{r}^d(\boldsymbol{s}, g_h))a\partial_a V_h(a, \boldsymbol{s}, g_h) + \partial_{\theta_h^k}\psi_{h,k}\Xi_h a \\
& \qquad + (D_x(\partial_a V_h(a, \boldsymbol{s}, g_h))^T(\boldsymbol{\sigma}_x \odot \boldsymbol{x}))^T\tilde{\sigma}_{q^k}(\boldsymbol{s}, g_h))a \\
[\theta_h^n]: \qquad & 0 = (\tilde{r}^n(\boldsymbol{s}, g_h)) - \tilde{r}^d(\boldsymbol{s}, g_h))a\partial_a V_h(a, \boldsymbol{s}, g_h) + \lambda(1/q^n(\boldsymbol{s}, g_h) - 1)\mathcal{U}'(\mathcal{C}) \\
& \qquad + \partial_{\theta_h^n}\psi_{h,n}(\theta_h^k)\Xi_h(\boldsymbol{s}, g_h)a + \partial_{aa}V_h(a, \boldsymbol{s}, g_h)\tilde{\sigma}_{q^n}(\boldsymbol{s}, g_h)\sigma_a(a, \boldsymbol{s}, g_h)a^2 \\
& \qquad + \sum_j \partial_{as_j}V_h(a, \boldsymbol{s}, g_h)\tilde{\sigma}_{q^n}(\boldsymbol{s}, g_h))\sigma_{s_j}(\boldsymbol{s}, g_h)as_j \\
& \quad = (\tilde{r}^n(\boldsymbol{s}, g_h) - \tilde{r}^d(\boldsymbol{s}, g_h))a\partial_a V_h(a, \boldsymbol{s}, g_h) + \lambda(1/q^n(\boldsymbol{s}, g_h) - 1)\mathcal{U}'(\mathcal{C}) \\
& \qquad + \partial_{\theta_h^n}\psi_{h,k}(\theta_h^n)\Xi_h(\boldsymbol{s}, g_h)a + (D_x(\partial_a V_h(a, \boldsymbol{s}, g_h))^T(\boldsymbol{\sigma}_x \odot \boldsymbol{x}))^T\tilde{\sigma}_{q^n}(\boldsymbol{s}, g_h)a
\end{aligned}
$$

*SDF Evolution:* Let $\xi_h(\boldsymbol{X}) := \partial_a V_h(a, \boldsymbol{s}, g_h)$, where we have reverted to the condensed notation $\boldsymbol{X} = (a, \boldsymbol{s}, g_h)$ now that the FOCs have been taken. From Ito's Lemma, we have that the drift and volatility of $\xi_h$ are given by:

$$
\begin{aligned}
\mu_{\xi_h}(\boldsymbol{X})\xi_h(\boldsymbol{X}) &= (D_x\xi_h(\boldsymbol{X}))^T\boldsymbol{\mu}_x(\boldsymbol{X}) \\
&\qquad + \frac{1}{2}\mathrm{tr}\left\{(\boldsymbol{\sigma}_x(\boldsymbol{X}) \odot \boldsymbol{x})^T(\boldsymbol{\sigma}_x(\boldsymbol{X}) \odot \boldsymbol{x})D_x^2\xi_h(\boldsymbol{X})\right\} + \mathcal{L}_g\xi_h(\boldsymbol{X}) \\
\sigma_{\xi_h}(\boldsymbol{X})\xi_h(\boldsymbol{X}) &= (\boldsymbol{\sigma}_x(\boldsymbol{X}) \odot \boldsymbol{x})^T D_x\xi_h(\boldsymbol{X}) + \langle \partial\xi_h/\partial g, \sigma_g \rangle
\end{aligned}
$$

Thus, we can rewrite the FOCs as:

$$
\begin{aligned}
[\theta_h^k]: \qquad & 0 = \xi_h(\boldsymbol{X})(\tilde{r}^n(\boldsymbol{X}) - \tilde{r}^d(\boldsymbol{X})) + \lambda(1/q^n(\boldsymbol{X}) - 1) \\
& \qquad + \partial_{\theta_h^k}\psi_{h,k}(\theta_h^k)\Xi_h(\boldsymbol{X}) + (\sigma_{\xi_h}(\boldsymbol{X})\xi_h(\boldsymbol{X}))^T\sigma_{q^k}(\boldsymbol{X}) \\
[\theta_h^n]: \qquad & 0 = \xi_h(\tilde{r}^n - \tilde{r}^d) + \lambda\partial_{\theta_h^n}\mathcal{W}(\theta_h^k, \theta_h^n)\frac{\mathcal{U}'(\mathcal{C})}{a} + \partial_{\theta_h^n}\psi_{h,n}\Xi_h + (\sigma_{\xi_h}\xi_h)^T\sigma_{q^n}
\end{aligned}
$$

Imposing belief consistency and using the equilibrium result that $\Xi_h = \xi_h$, we get the simplified FOCs:

$$[\theta_h^k]: \qquad r^k - r^d = -\lambda \partial_{\theta_h^k} \mathcal{W}(\theta_h^k, \theta_h^n) \frac{\mathcal{U}'(\mathcal{C})}{a\xi_h} - \partial_{\theta_h^k} \psi_{h,k} - \sigma_{\xi_h} \sigma_{q^k}$$

$$[\theta_h^n]: \qquad r^n - r^d = -\lambda \partial_{\theta_h^n} \mathcal{W}(\theta_h^k, \theta_h^n) \frac{\mathcal{U}'(\mathcal{C})}{a\xi_h} - \partial_{\theta_h^n} \psi_{h,n} - \sigma_{\xi_h} \sigma_{q^n}$$

*Euler equation:* We close this section by using the Envelope theorem to get the Euler equation (using the partial matrix representation). To do this, we treat $\theta a$ as the choice rather than $\theta$ when taking the envelope theorem. This gives:

$$\begin{aligned}
&\rho_h \xi_h(a, \boldsymbol{s}) \\
&= (\psi_{h,k}(\theta_h^k a/a) + \psi_{h,n}(\theta_h^n a/a))\Xi_h \\
&\quad - \left( \partial_{\theta_h^k} \psi_{h,k}(\theta_h^k a/a) \frac{\theta_h^k}{a} + \partial_{\theta_h^k} \psi_{n,k}(\theta_h^n a/a) \frac{\theta_h^n}{a} \right) \Xi_h a \\
&\quad - \lambda \xi_h(a, \boldsymbol{s}) + \partial_a \xi_h(a, \boldsymbol{s}) \mu_a(a, c_h, \boldsymbol{\theta}_h, \cdot) a + \xi_h(a, \boldsymbol{s})(r^d + \tau_h) \\
&\quad + (D_s \xi_h(a, \boldsymbol{s}))^T (\boldsymbol{\mu}_s(\boldsymbol{s}) \odot \boldsymbol{s}) \\
&\quad + \frac{1}{2} \partial_{aa}^2 \xi_h(a, \boldsymbol{s}) \sigma_a^2(\boldsymbol{\theta}_h, \boldsymbol{s}) a^2 \\
&\quad + \sum_j \partial_{as_j} \xi_h(a, \boldsymbol{s}) \sigma_a(\boldsymbol{\theta}_h, \boldsymbol{s}) \sigma_{s_j} a s_j \\
&\quad + \frac{1}{2} \mathrm{tr} \left\{ (\boldsymbol{\sigma}_s(\boldsymbol{s}) \odot \boldsymbol{s})^T (\boldsymbol{\sigma}_s(\boldsymbol{s}) \odot \boldsymbol{s}) D_s^2 \xi_h(a, \boldsymbol{s}) \right\} + \mathcal{L}_g \xi_h \\
&= (\psi_{h,k}(\theta_h^k) - \partial_{\theta_h^k} \psi_{h,k}(\theta_h^k a/a) \theta_h^k + \psi_{h,n}(\theta_h^n) - \partial_{\theta_h^k} \psi_{n,k}(\theta_h^n a/a) \theta_h^n) \Xi_h \\
&\quad - \lambda \xi_h(a, \boldsymbol{s}) + (D_s \xi_h(\boldsymbol{x}))^T (\boldsymbol{\mu}_x(\boldsymbol{x}) \odot \boldsymbol{x}) + \xi_h(a, \boldsymbol{s}) \left( r^d + \tau_h \right) \\
&\quad + \frac{1}{2} \mathrm{tr} \left\{ (\boldsymbol{\sigma}_x(\boldsymbol{x}, \boldsymbol{\theta}_h) \odot \boldsymbol{x})^T (\boldsymbol{\sigma}_x(\boldsymbol{x}, \boldsymbol{\theta}_h) \odot \boldsymbol{x}) D_x^2 \xi_h(\boldsymbol{x}) \right\} + \mathcal{L}_g \xi_h \\
&= (\psi_{h,k}(\theta_h^k) - \partial_{\theta_h^k} \psi_{h,k}(\theta_h^k) \theta_h^k + \psi_{h,n}(\theta_h^n) - \partial_{\theta_h^n} \psi_{n,k}(\theta_h^n) \theta_h^n) \Xi_h \\
&\quad - \lambda_h \xi_h(a, \boldsymbol{s}) + \mu_{\xi_h} \xi_h(a, \boldsymbol{s}) + \xi_h(a, \boldsymbol{s}) \left( r^d - \tau_h \right)
\end{aligned}$$

So, imposing belief consistency and we get that:

$$\rho_h + \lambda_h = \mu_{\xi_h} + r^d - \tau_h + \psi_{h,k}(\theta_h^k) - \partial_{\theta_h^k} \psi_{h,k}(\theta_h^k) \theta_h^k + \psi_{h,n}(\theta_h^n) - \partial_{\theta_h^n} \psi_{n,k}(\theta_h^n) \theta_h^n$$

## B.2 Banker Optimization

The banker HJBE is given by:

$$\rho_b V_b(a, \boldsymbol{s}, g_h) = \max_{c_b, \boldsymbol{\theta}_b, \iota_b} \Big\{ u(c_b) + \mu_a(a, \boldsymbol{s}, c_b, \boldsymbol{\theta}_b, \iota_b)a + (\boldsymbol{\mu}_s(\boldsymbol{s}) \odot \boldsymbol{s})^T D_s V_b(a, \boldsymbol{s}, g_h)$$
$$+ \frac{1}{2}\partial^2_{aa} V_b(a, \boldsymbol{s}, g_h)\sigma_a^2(\boldsymbol{\theta}_b, \boldsymbol{s})a^2 + \sum_j \partial_{as_j} V_b(a, \boldsymbol{s}, g_h)\sigma_a(\boldsymbol{\theta}_b, \boldsymbol{s})\sigma_{s_j} a s_j$$
$$+ \frac{1}{2}\mathrm{tr}\left\{(\boldsymbol{\sigma}_s(\boldsymbol{s}) \odot \boldsymbol{s})^T(\boldsymbol{\sigma}_s(\boldsymbol{s}) \odot \boldsymbol{s})D_s^2 V_b(a, \boldsymbol{s})\right\} + \mathcal{L}_g V_b(a, \boldsymbol{s}, g_h) + \Big[\psi_{b,k}(\theta_b^k) + \psi_{b,m}(\theta_b^m)\Big]\xi_b a_b \Big\}$$

where:

$$\mu_a(a, c_b, \theta_b, \cdot) = \left(\tilde{r}^d + \theta_h^k(\tilde{r}^k - \tilde{r}^d) - c_{b,t}/a_b - \tau_b\right)$$
$$\sigma_a(\theta_b, \cdot) = \theta_b^k \tilde{\sigma}_{q^k,z} + \theta_b^m \tilde{\sigma}_{q^m,z}$$

Following the same steps, the equilibrium FOCs for the banker are given by:

$$[c_b]: \qquad 0 = u'(c_b) - \partial_a V_b(a, \boldsymbol{s}, g_h)$$
$$[\iota_b]: \qquad 0 = \Phi'(\iota) - \frac{1}{q_t^k}$$
$$[\theta_b^k]: \qquad 0 = r^k - r^d + \sigma_{\xi_b}\sigma_{q^k,z} + \partial_{\theta_b^k}\psi_{b,k}$$
$$[\theta_b^m]: \qquad 0 = r^m - r^d + \sigma_{\xi_b}\sigma_{q^m,z} + \partial_{\theta_b^m}\psi_{b,m}$$

and the Euler equation is:

$$\rho_b = \mu_{\xi_b} + r^d - \tau_b + \Big[\psi_{b,k} + \psi_{b,m} - \partial_{\theta_b^k}\psi_{b,k}\theta_b^k - \partial_{\theta_b^m}\psi_{b,m}\theta_b^m\Big]$$

## B.3 Fund Manager Optimization

The banker HJBE is given by:

$$\rho_f V_f(a, \boldsymbol{s}, g_h) = \max_{c_f, \boldsymbol{\theta}_f, \iota_f} \Big\{ u(c_f) + \mu_a(a, \boldsymbol{s}, c_f, \boldsymbol{\theta}_f, \iota_f)a + (\boldsymbol{\mu}_s(\boldsymbol{s}) \odot \boldsymbol{s})^T D_s V_f(a, \boldsymbol{s}, g_h)$$
$$+ \frac{1}{2}\partial^2_{aa} V_f(a, \boldsymbol{s})\sigma_a^2(\boldsymbol{\theta}_f, \boldsymbol{s}, g_h)a^2 + \sum_j \partial_{as_j} V_f(a, \boldsymbol{s}, g_h)\sigma_a(\boldsymbol{\theta}_f, \boldsymbol{s})\sigma_{s_j} a s_j$$
$$+ \frac{1}{2}\mathrm{tr}\left\{(\boldsymbol{\sigma}_s(\boldsymbol{s}) \odot \boldsymbol{s})^T(\boldsymbol{\sigma}_s(\boldsymbol{s}) \odot \boldsymbol{s})D_s^2 V_f(a, \boldsymbol{s}, g_h)\right\} + \mathcal{L}_g V_f(a, \boldsymbol{s}, g_h) + \Big[\psi_{f,k}(\theta_f^k) + \psi_{f,m}(\theta_f^m)\Big]\xi_f a_f \Big\}$$

where:

$$\mu_a(a, c_f, \boldsymbol{\theta}_f, \boldsymbol{s}, g_h) = \tilde{r}_f^n + \theta_f^k(\tilde{r}^k - \tilde{r}_f^n) + \theta_f^m(\tilde{r}_t^m - \tilde{r}_f^n) - c_{f,t}/a_f - \tau_f$$
$$\sigma_a(\boldsymbol{\theta}_f, \boldsymbol{s}, g_h) = \theta_f^k \tilde{\sigma}_{q^k,z} + \theta_f^m \tilde{\sigma}_{q^m,z} + (1 - \theta_f^k - \theta_f^m)\tilde{\sigma}_{q^n,z} = \boldsymbol{\theta}_f^T \tilde{\boldsymbol{\sigma}}_{q,t}$$

Following the same steps, the equilibrium FOCs for the fund are given by:

$$\begin{aligned}
[c_f] :& \qquad 0 = u'(c_f) - \partial_a V_f(a, \cdot) \\
[\iota_f] :& \qquad 0 = \Phi'(\iota) - \frac{1}{q^k} \\
[\theta_f^k] :& \qquad 0 = r^k - r_f^n + \sigma_{\xi_f}(\sigma_{q^k,z} - \sigma_{q^n,z}) + \partial_{\theta_f^k}\psi_{f,k} \\
[\theta_f^m] :& \qquad 0 = r^m - r_f^n + \sigma_{\xi_f}(\sigma_{q^m,z} - \sigma_{q^n,z}) + \partial_{\theta_f^k}\psi_{f,m}
\end{aligned}$$

and the Euler equation is:

$$\rho_f = \mu_{\xi_f} + r_f^n - \tau_h + \left[\psi_{f,k} + \psi_{f,m} - \partial_{\theta_f^k}\psi_{f,k}\theta_f^k - \partial_{\theta_f^m}\psi_{f,m}\theta_f^m\right] + \sigma_{q^n,z}\sigma_{\xi_f}$$

## B.4 Equilibrium Functions

The agent optimization problem has the terms:

$$\xi_h(a, z, K, g), \quad D_x\xi_h(a, z, K, g) = \begin{bmatrix} \partial_a\xi_h(a, z, K, g) \\ \partial_z\xi_h(a, z, K, g) \\ \partial_K\xi_h(a, z, K, g) \\ \partial_{\eta_h}\xi_h(a, z, K, g) \\ \partial_{\eta_f}\xi_h(a, z, K, g) \end{bmatrix}$$

In equilibrium we have that that $a = \eta_h A(\boldsymbol{s})$ where $A(\boldsymbol{s}) = q^k(\boldsymbol{s})K + q^m(\boldsymbol{s})M$:

$$\Xi_h(z, g, K) = \xi_h(a, z, K, g)|_{a=\eta_h A(\boldsymbol{s})}$$
$$\Xi_h'(z, g, K) = D_x\xi_h(a, z, K, g)|_{a=\eta_h A(\boldsymbol{s})}$$

The term $\Xi_i'(z, g, K)$ appears throughout the FOCs equations so we need approximate it. However, we have:

$$\Xi_h'(z, g, K) \neq D_s\Xi_h(z, g, K)$$

for the obvious reason that the dimension is different. Instead, we have that:

$$D_s\Xi_h(z,g,K) = \begin{bmatrix} \partial_z\xi_h(\eta_h A(s),z,K,g) \\ \partial_\zeta\xi_h(\eta_h A(s)z,K,g) \\ \partial_K\xi_h(\eta_h A(s),z,K,g) \\ \partial_a\xi_h(a,z,K,g)|_{a=\eta_h A(s)}A(s) + \partial_{\eta_h}\xi_h(a,z,K,g)|_{a=\eta_h A(s)} \\ \partial_{\eta_f}\xi_h(\eta_h A(s),z,K,g) \end{bmatrix}$$

**Proposition 2.** *In equilibrium, we have that for $j \in \{h,b,f\}$:*

$$\mu_{\xi_h}\xi_j(a,\boldsymbol{s})|_{a=\eta_j A(\boldsymbol{s})} = \mu_{\Xi_j}\Xi_j(\boldsymbol{s})$$
$$\sigma_{\xi_j}\xi_j(a,\boldsymbol{s})|_{a=\eta_j A(\boldsymbol{s})} = \sigma_{\Xi_j}\Xi_j(\boldsymbol{s})$$

*Proof.* For clarity, I show this in the non-matrix form for the household rather than using the matrix chain rule. For the volatility, we have that:

$$\begin{aligned}\sigma_{\xi_h}\xi_h &= (\boldsymbol{\sigma}_x \odot \boldsymbol{x})^T(D_x\xi_h) \\ &= \partial_a\xi_h\sigma_{a,z}a + \partial_z\xi_h\sigma_z z + \sum_j \partial_{\eta_j}\xi_h\sigma_{\eta_j,z}\eta_j\end{aligned}$$

After imposing equilibrium $a = \eta_1 A(\boldsymbol{s})$, where $A(\boldsymbol{s}) = q^k K + q^m M$, we have that the RHS is:

$$\begin{aligned} RHS &= \partial_a\xi_h\sigma_{a,z}\eta_1 A(\boldsymbol{s}) + \partial_z\xi_h\sigma_z z + \sum_j \partial_{\eta_j}\xi_h\sigma_{\eta_j,z}\eta_j \\ &= \begin{bmatrix} \sigma_{a,z} \\ \sigma_z \\ 0 \\ 0 \\ \sigma_{\eta_h,z} \\ \sigma_{\eta_f,z} \end{bmatrix}^T \begin{bmatrix} \partial_z\xi_h(\eta_h A(s),z,K,g) \\ \partial_\zeta\xi_h(\eta_h A(s)z,K,g) \\ \partial_K\xi_h(\eta_h A(s),z,K,g) \\ \partial_a\xi_h(\eta_h A(s),z,K,g)A(s) + \partial_{\eta_h}\xi_h(\eta_h A(s),z,K,g) \\ \partial_{\eta_f}\xi_h(\eta_h A(s),z,K,g) \end{bmatrix} \\ &= (\boldsymbol{\sigma}_s \odot \boldsymbol{s})^T(D_s\Xi_h) \\ &= \sigma_{\Xi_h}\Xi_h \end{aligned}$$

□

*Equilibrium Portfolio Choice:* Imposing Proposition 2 we have that the portfolio choices satisfy:

$$
\begin{aligned}
&[\theta_h^k]: && r^k - r^d = -\lambda \partial_{\theta_h^k} \mathcal{W}(\theta_h^k, \theta_h^n) \frac{\mathcal{U}'(\mathcal{C})}{a\Xi_h} - \partial_{\theta_h^k} \psi_{h,k} - \sigma_{\Xi_h} \sigma_{q^k} \\
&[\theta_h^n]: && r_h^n - r^d = -\lambda \partial_{\theta_h^n} \mathcal{W}(\theta_h^k, \theta_h^n) \frac{\mathcal{U}'(\mathcal{C})}{a\Xi_h} - \partial_{\theta_h^n} \psi_{h,n} - \sigma_{\Xi_h} \sigma_{q^n} \\
&[\theta_b^k]: && r^k - r^d = -\sigma_{\Xi_b} \sigma_{q^k} - \partial_{\theta_b^k} \psi_{b,k} \\
&[\theta_b^m]: && r^m - r^d = -\sigma_{\xi_b} \sigma_{q^m,z} - \partial_{\theta_b^m} \psi_{b,m} \\
&[\theta_f^k]: && r^k - r_f^n = -\sigma_{\Xi_f} (\sigma_{q^k} - \sigma_{q^n}) \\
&[\theta_f^m]: && r^m - r_f^n = -\sigma_{\Xi_f} (\sigma_{q^m} - \sigma_{q^n})
\end{aligned}
$$

where:

$$
r_f^n = r_h^n + \left( \frac{1}{q_t^n} - 1 \right) \lambda_h
$$

*Equilibrium Euler Equations:* Imposing Proposition 2 we have that the Euler equations satisfy:

$$
\begin{aligned}
\rho_h + \lambda_h &= \mu_{\Xi_h} + r^d - \tau_h + \psi_{h,k}(\theta_h^k) - \partial_{\theta_h^k} \psi_{h,k}(\theta_h^k)\theta_h^k + \psi_{h,n}(\theta_h^n) - \partial_{\theta_h^n} \psi_{n,k}(\theta_h^n)\theta_h^n \\
\rho_b &= \mu_{\Xi_b} + r^d - \tau_b + \left[ \psi_{b,k} + \psi_{b,m} - \partial_{\theta_b^k} \psi_{b,k} \theta_b^k - \partial_{\theta_b^m} \psi_{b,m} \theta_b^m \right] \\
\rho_f &= \mu_{\Xi_f} + r_f^n - \tau_f + \left[ \psi_{f,k} + \psi_{f,m} - \partial_{\theta_f^k} \psi_{f,k} \theta_f^k - \partial_{\theta_f^m} \psi_{f,m} \theta_f^m \right] + \sigma_{q^n,z} \sigma_{\Xi_f}
\end{aligned}
$$

## B.5 Equilibrium Block 1: Summary of Optimization

Given equilibrium prices and price processes:

$$
(r^d, q^k, r^k, \sigma_{q^k}, q^n, r^n, \sigma_{q^n}, q^m, r^m, \sigma_{q^m})
$$

the household, banker, and fund optimization variables (14 variables):

$$
(\Xi_h, \Xi_b, \Xi_f, c_h, \mathcal{C}_h, c_b, c_f, \theta_h^k, \theta_h^n, \theta_b^k, \theta_b^m, \theta_f^k, \theta_f^m, \iota)
$$

satisfy the optimization equations (14 equations):

$$
\begin{aligned}
0 =& -(\rho_h + \lambda_h) + \mu_{\Xi_h} + r^d - \tau_h + \psi_{h,k}(\theta_h^k) - \partial_{\theta_h^k}\psi_{h,k}(\theta_h^k)\theta_h^k \\
& + \psi_{h,n}(\theta_h^n) - \partial_{\theta_h^n}\psi_{n,k}(\theta_h^n)\theta_h^n \\
0 =& -\rho_b + \mu_{\Xi_b} + r^d - \tau_b + \left[\psi_{b,k} + \psi_{b,m} - \partial_{\theta_b^k}\psi_{b,k}\theta_b^k - \partial_{\theta_b^m}\psi_{b,m}\theta_b^m\right] \\
0 =& -\rho_f + \mu_{\Xi_f} + r_f^n - \tau_h + \left[\psi_{f,k} + \psi_{f,m} - \partial_{\theta_f^k}\psi_{f,k}\theta_f^k - \partial_{\theta_f^m}\psi_{f,m}\theta_f^m\right] + \sigma_{q^n,z}\sigma_{\Xi_f} \\
0 =& \, u'(c_h) - \Xi_h \\
0 =& \, u'(c_b) - \Xi_b \\
0 =& \, u'(c_f) - \Xi_f \\
0 =& -\mathcal{C}_h + a(\theta_h^k(1-\tau))^\alpha((q^n)^{-1}\theta_h^n)^{1-\alpha} \\
0 =& \, r^k - r^d + \lambda_h \partial_{\theta_h^k}\mathcal{W}(\theta_h^k, \theta_h^n)\frac{\mathcal{U}'(\mathcal{C})}{a\Xi_h} + \partial_{\theta_h^k}\psi_{h,k} + \sigma_{\Xi_h}\sigma_{q^k} \\
0 =& \, r^n - r^d + \lambda_h \partial_{\theta_h^n}\mathcal{W}(\theta_h^k, \theta_h^n)\frac{\mathcal{U}'(\mathcal{C})}{a\Xi_h} + \partial_{\theta_h^n}\psi_{h,n} + \sigma_{\Xi_h}\sigma_{q^n} \\
0 =& \, r^k - r^d + \sigma_{\Xi_b}\sigma_{q^k} + \partial_{\theta_b^k}\psi_{b,k} \\
0 =& \, r^m - r^d + \sigma_{\Xi_b}\sigma_{q^m} + \partial_{\theta_b^m}\psi_{b,m} \\
0 =& \, r^k - r_f^n + \sigma_{\Xi_f}(\sigma_{q^k} - \sigma_{q^n}) + \partial_{\theta_f^k}\psi_{f,k} \\
0 =& \, r^m - r_f^n + \sigma_{\Xi_f}(\sigma_{q^m} - \sigma_{q^n}) + \partial_{\theta_f^m}\psi_{f,m} \\
0 =& \, \Phi'(\iota) - \frac{1}{q^k}
\end{aligned}
$$

## B.6 Equilibrium Block 2: Distribution Evolution

*Kolmogorov Forward Equation (KFE): Financial Sector:* We consider two levels of the distribution evolution. Let $A_{h,t}$, $A_{b,t}$, and $A_{f,t}$ denote the aggregate wealth in the household, banking, and fund sectors. Let $g_{j,t}$ denote the measure function of wealth for type $j \in \{h, b, f\}$.

We start with the evolution of aggregate wealth in the banking sector:

$$\begin{aligned}
\frac{dA_{b,t}}{A_{b,t}} &= \frac{1}{A_{b,t}}\int_0^\infty \mu_{a_b}(c_h(a),\boldsymbol{\theta}_b,\boldsymbol{S})ag_{b,t}(a)dadt + \frac{1}{A_{b,t}}\lambda_b(\phi_b A_t - A_{b,t})dt \\
&\quad + \frac{1}{A_{b,t}}\int_0^\infty \sigma_{b,t}(\boldsymbol{\theta}_b,\boldsymbol{S})adW_t a g_{b,t}(a)da \\
&= \left(\mu_{a_b}(c_h(a),\boldsymbol{\theta}_b,\boldsymbol{S}) + \lambda_b\left(\frac{\phi_b}{\eta_{b,t}} - 1\right)\right)dt + \sigma_{b,t}(\boldsymbol{\theta}_b,\boldsymbol{S})dW_t \\
&= \left(r^d + \theta_h^k(r^k - r^d) - (\rho_b + \lambda_b) - \tau_b + \lambda_b\left(\frac{\phi_b}{\eta_{b,t}} - 1\right)\right)dt \\
&\quad + \sigma_{b,t}(\boldsymbol{\theta}_b,\boldsymbol{S})dW_t
\end{aligned}$$

Likewise, the evolution of aggregate wealth in the fund section is:

$$\begin{aligned}
\frac{dA_{f,t}}{A_{f,t}} = &\left(r^b + \theta_f^k(r^k - r_f^b) + \theta_f^m(r^m - r_f^n) - (\rho_f + \lambda_f) - \tau_f + \lambda_f\left(\frac{\phi_f}{\eta_{f,t}} - 1\right)\right)dt \\
&+ \sigma_{f,t}(\boldsymbol{\theta}_f,\boldsymbol{S})dW_t
\end{aligned}$$

Aggregate wealth is given by $A_t = q_t^k K_t + q_t^m M$. Let $\vartheta_t = q_t^k K_t/(q^k K_t + q^m M)$. The evolution of aggregate wealth follows:

$$\begin{aligned}
\frac{dA_t}{A_t} &= \vartheta\left(\frac{dq^k}{q^k} + \frac{dK}{K}\right) + (1-\vartheta)\frac{dq^m}{q^m} \\
&= \underbrace{\vartheta(\mu_{q^k} + \Phi(\iota) - \delta) + (1-\vartheta)\mu_{q^m}}_{\mu_A} + \vartheta\sigma_{q^k,z}dW \\
&\quad + (1-\vartheta)\sigma_{q^m,z}dW_t
\end{aligned}$$

So, the evolution of $\eta_{b,t} = A_{b,t}/A_t$ is given by:

$$\begin{aligned}
\frac{d\eta_{b,t}}{\eta_{b,t}} &= \frac{dA_{b,t}}{A_{b,t}} - \frac{dA_t}{A_t} - \frac{dA_{b,t}}{A_{b,t}}\frac{dA_t}{A_t} + \left(\frac{dA_t}{A_t}\right)^2 \\
&= \left(\mu_{A_b,t} - \mu_{A,t} - \sigma_{A_b,t}\sigma_{A,t} + \sigma_{A,t}\sigma_{A,t}\right)dt + (\sigma_{A_b,t} - \sigma_{A,t})dW_t \\
&= \left(\mu_{A_b,t} - \mu_{A,t} + (\sigma_{A,t} - \sigma_{A_b,t})\sigma_{A,t}\right)dt + (\sigma_{A_b,t} - \sigma_{A,t})dW_t
\end{aligned}$$

and the evolution of $\eta_{f,t} = A_{f,t}/A_t$ is given by:

$$\frac{d\eta_{f,t}}{\eta_{f,t}} = \left(\mu_{A_f,t} - \mu_{A,t} + (\sigma_{A,t} - \sigma_{A_f,t})\sigma_{A,t}\right)dt + (\sigma_{A_f,t} - \sigma_{A,t})dW_t$$

*KFE: Within Households:* The KFE for the household distribution in levels, $a$, is given by:

$$\begin{aligned}
dg_{h,t}(a) = &+ \lambda_h \phi(a) A_t - \lambda_h g_{h,t}(a) - \partial_a[\mu_a(a, \boldsymbol{s}_t, g_{h,t}) g_{h,t}(a)] \\
&- \partial_a[\sigma_a(a, \boldsymbol{s}_t, g_{h,t}) dW_t g_{h,t}(a)] + \frac{1}{2}\partial_a \left[\boldsymbol{\sigma}_a^T \boldsymbol{\sigma}_a(a, \boldsymbol{s}_t, g_{h,t}) g_{h,t}(a)\right] dt \\
= &+ \lambda_h \phi(a) A_t - \lambda_h g_{h,t}(a) - \partial_a[\mu_a(a, \boldsymbol{s}_t, g_{h,t}) g_{h,t}(a)] \\
&- \partial_a[\sigma_{a,z}(a, \boldsymbol{s}_t, g_{h,t}) g_{h,t}(a)] dW_{z,t} \\
&+ \frac{1}{2}\partial_a \left[(\sigma_{a,z}^2(a, \boldsymbol{s}_t, g_{h,t})) g_{h,t}(a)\right] dt
\end{aligned}$$

The evolution $\eta_{i,t} := a_{i,t}/A_t$ is given by:

$$\begin{aligned}
\frac{d\eta_{i,t}}{\eta_{i,t}} &= \left(\mu_{a_i,t} - \mu_{A,t} + (\sigma_{A,t} - \sigma_{a_i,t})\boldsymbol{\sigma}_{A,t}\right) dt + (\sigma_{a_i,t} - \sigma_{A,t}) dW_t \\
&=: \mu_{\eta_i,t} dt + \sigma_{\eta_i,t} dW_t
\end{aligned}$$

where we have softened the entry function from $\phi_h$ to $\phi(a)$, where $\phi(a)$ is a function with mean $\phi_h$. For $a$, a natural candidate would be $\phi(a) = \text{LogNormal}(\phi_h, \sigma)$. Likewise, the KFE for the distribution in shares is:

$$\begin{aligned}
dg_{h,t}(\eta) = &+ \lambda_h \phi(\eta) - \lambda_h g_{h,t}(\eta) - \partial_\eta[\mu_\eta(\eta, \boldsymbol{s}_t, g_{h,t}) g_{h,t}(\eta)] \\
&- \partial_\eta[\sigma_\eta(\eta, \boldsymbol{s}_t, g_{h,t}) dW_t g_{h,t}(\eta)] + \frac{1}{2}\partial_\eta \left[\boldsymbol{\sigma}_\eta^T \boldsymbol{\sigma}_\eta(\eta, \boldsymbol{s}_t, g_{h,t}) g_{h,t}(\eta)\right] dt \\
= &+ \lambda_h \phi(\eta) - \lambda_h g_{h,t}(\eta) - \partial_\eta[\mu_\eta(\eta, \boldsymbol{s}_t, g_{h,t}) g_{h,t}(\eta)] \\
&- \partial_\eta[\sigma_{\eta,z}(\eta, \boldsymbol{s}_t, g_{h,t}) g_{h,t}(\eta)] dW_{z,t} \\
&+ \frac{1}{2}\partial_\eta \left[(\sigma_{\eta,z}^2(\eta, \boldsymbol{s}_t, g_{h,t}) + \sigma_{\eta,\zeta}^2(\eta, \boldsymbol{s}_t, g_{h,t})) g_{h,t}(\eta)\right] dt
\end{aligned}$$

where again we have softened the entry function $\phi_h$ to $\phi(\eta)$, where $\phi(\eta)$ is a function with mean $\phi_h$. For $\eta$, a natural candidate would be $\phi(\eta) \sim$ Beta with mean $\phi_h$.

# C Additional Details on Training

Figure 6 presents the Euler equation training MSE in marginal utility units over epochs in the top left panel, along with the validation loss in the top right panel. The algorithm converges successfully with the validation loss in the order $10^-4$. The error saturates after 60k epochs, where the gradient norm of all neural network parameters stabilizes.

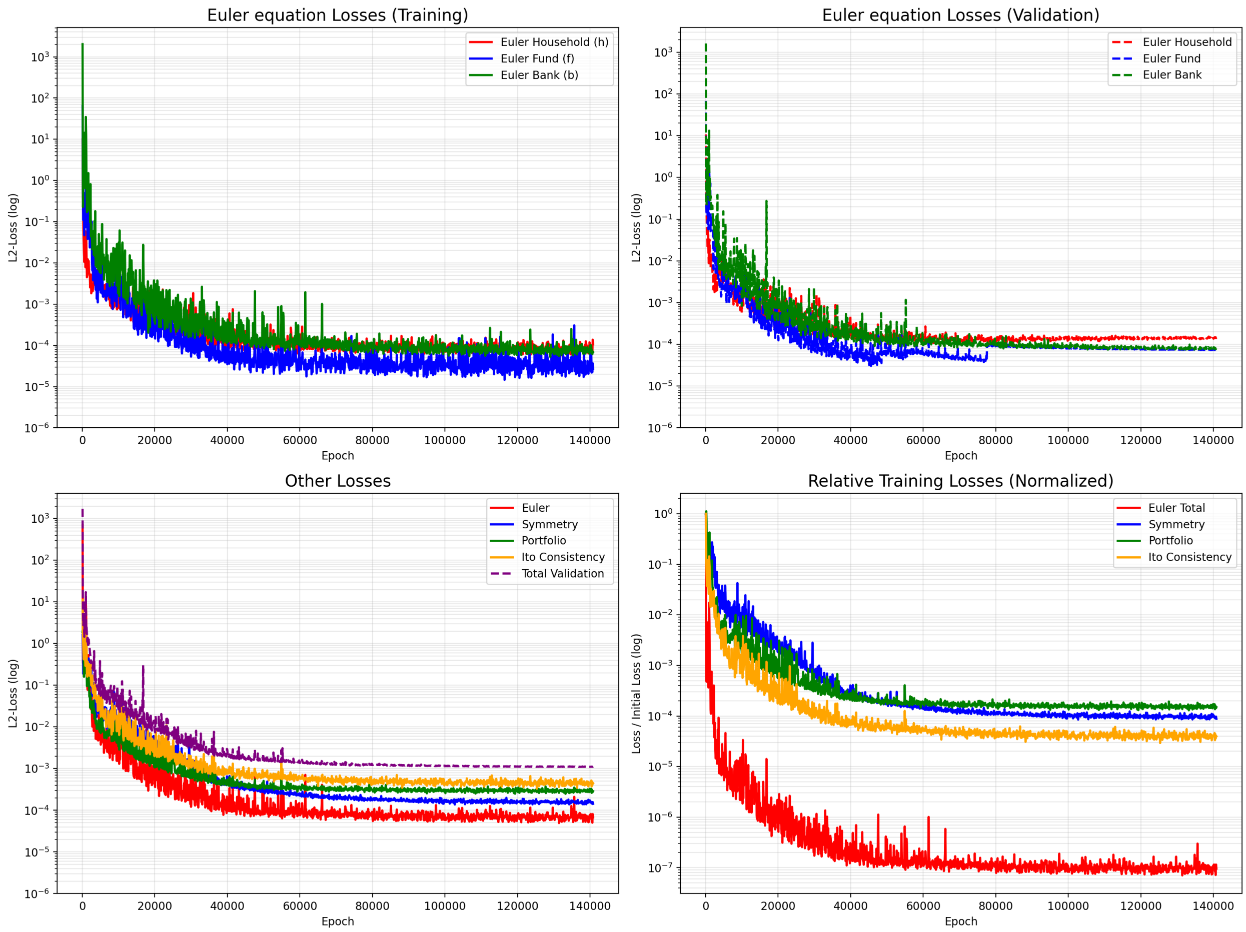


**Figure 6:** The top left panel plots the Euler equation L-2 training loss from the baseline model over iterations on a logarithmic scale. The neural network architecture is 4 hidden layers with 256 neurons in each layer trained using an ADAM optimizer with a learning rate scheduler. The top right panel produces the Euler equation validation loss. The bottom left panel produces other losses and the bottom right panel produces the normalized losses.

# D Additional Details on Simulating

In order to simulate the economy we need to compute the evolution of the household wealth distribution. This is complicated for the finite agent approximation method because the neural network policy rules are functions of the positions of the $N$ other agents rather than a continuous density. To overcome this difficulty, we deploy the "hybrid" approach described in Algorithm 2 that uses the neural network solution to approximate a finite difference approximation to the KFE. Let $\underline{\boldsymbol{a}} = (a_m : m \leq M)$ denote the grid in the $a$-dimension. Let $\underline{\boldsymbol{g}}_t = (g_{m,t} : m \leq M)$ denote the marginal density on the $a$-grid. At each time step, our method draws $N_{sim} \times N_{fit}$ different samples of $N$ agents from from the current density $g_t$. Since we only have finite agent representation for all equilibrium functions, we exploit law of iterated expectation that:

$$\mathbb{E}_{\hat{g}}[\mu_a(a_i|\hat{g}, z)|g, z] = \mu_a(a_i|g, z).$$

to reconstruct the drift given households' wealth share distribution, intermediary wealth level and aggregate state realization. In implementation, we divide the sample into $N_{sim}$ subsamples with sample size $N_{fit}$, then approximate $\mathbb{E}_{\hat{g}}[\mu_a(\underline{\boldsymbol{a}}|\hat{g}, z)|g, z]$ by the follow steps:

- For $z$, given $g$, we draw household distribution $\hat{g}$ for $N_{fit}$ times, we calculate $\mu_a(a_i|\hat{g}, z, \zeta)$.
- Run cubic spline fitting for equilibrium drift vs. $a_i$. Denote the fitted function $\hat{f}(\cdot|g, z)$.
- Fit the drifts at grid points: $\mu_{g,m} = \hat{f}(a_m|g, z), m \leq M$.

Similarly, we construct the volatility part $\boldsymbol{\sigma}^T_{g,m}$.

For each draw $k \leq N_{sim}$, denoted by $\hat{\varphi}^k_t = (a_i : 1 \leq i \leq N)$, the KFE is replaced by the following finite difference equation:

$$dg_{m,t} = \mu_{g,m}(\hat{\varphi}^k_t)dt + \boldsymbol{\sigma}^T_{g,m}(\hat{\varphi}^k_t)d\boldsymbol{W}_t, \qquad m \leq M \tag{D.1}$$

where the drift at point $(m)$ is defined by the finite difference approximation for the KFE using the policy rules from our finite population neural network solution. From this approximation we can calculate the transition matrix $\mathcal{A}_{t,k}$ for the finite difference approximation at the draw $\varphi^k$. We repeat this procedure many times then compute an average transition matrix, which we use for simulation. We summarize the steps in Algorithm 2.

For comparison, we also simulate the baseline economy using the N-agent equilibrium functions (referred to henceforth as 'N-agent simulation'). Specifically, we simulate using an Euler scheme for 500 years with a monthly frequency and compute the ergodic mean wealth shares of all N-agents. We then compute impulse responses from these ergodic states to a negative TFP shock and compare them with the responses from the ergodic states obtained using the KFE method. Figure 7 presents the difference between the two.

## D.1 Multiple stochastic steady states

Our model admits multiple stochastic steady states. Figure 8 displays the drift of the bank's wealth share as a function of its own wealth share. We sample 2000 points from the state space and compute the equilibrium drift using the evolution equation in Theorem 2 in two economies: the baseline economy and an alternate economy where households' demand for deposits dominates other assets. For illustration, we fit a cubic spline curve

**Algorithm 2:** Finding Transition Paths In Finite Agent Approximation

**Input :** Initial distribution, neural network approximations to the policy and price functions, number of agents $N$, time step size $\Delta t$, number of time steps $N_T$, number of simulations $N_{sim}$, grid $\underline{\boldsymbol{a}} = \{a_m : m \leq M\}$ for the finite difference approximation.

**Output:** A transition path $g = \{g_t : t = 0, \Delta t, \ldots, N_T \Delta t\}$

**for** $n = 0, \ldots, N_T - 1$ **do**
- **for** $k = 1, \ldots, N_{sim}$ **do**
  - Sample $\Delta W_{t,z}$ from the normal distribution $N(0, \Delta t)$, construct TFP shock paths by: $z_{t+\Delta t} = z_t + \eta(\bar{z} - z_t) + \sigma \Delta B_t^0$. Do likewise to construct the volatility shock path.
  - Draw states for $N$ agents $\{\varphi_i^k : i = 1, \ldots, N\}$ from density $g_t$ at $t = n\Delta t$.
  - Given state $(z_{t+\Delta t}, \varphi_t^k)$, compute equilibrium prices and returns.
  - At each grid point $a_m \in \underline{\boldsymbol{a}}$, calculate the consumption and portfolio choices.
  - Construct the transition matrix $\mathcal{A}_{t,k}$ using finite difference on the grid $\underline{\boldsymbol{a}}$, as described by (D.1).
- **end**
- Take the average: $\bar{\mathcal{A}}_t = \frac{1}{N_{sim}} \sum_{k=1}^{N_{sim}} \mathcal{A}_{t,k}$
- Update $g_t$ by implicit method: $g_{t+\Delta t} = (I - \bar{\mathcal{A}}_t^{\top} \Delta t)^{-1} g_t + \boldsymbol{\sigma}^T d\boldsymbol{W}_{t,z}$

**end**

to the scatter points. In the baseline economy, there is a non-degenerate stationary density around $\eta_b = 0.23$, as evidenced by the drift curve crossing zero at that point. In the alternate economy, the drift is increasing in the bank's wealth share, implying that the bank takes over the economy in the long run.

## D.2 Details on The Vayanos Vila (2021) Sampling

*Details in the constrained sampling.* Let $\eta_1, ..., \eta_I$ denote non-negative quantities satisfying:

$$\sum_{i=1}^{I} \eta_i = S := 1 - \eta_{fss} - \eta_{bss}, \qquad \eta_i \geq 0,$$

and suppose each $\eta_i$ is drawn "from" a baseline distribution $g(\cdot)$ supported on $[0, \infty)$. We require a sampling scheme for $\boldsymbol{\eta}$ conditional on the sum constraint. The conditional density is

$$\pi(\boldsymbol{\eta} \mid \sum_i \eta_i = S) \propto \prod_{i=1}^{I} g(\eta_i) \quad \text{on } \mathcal{S}_S := \{\boldsymbol{\eta} \geq 0 : \sum_i \eta_i = S\}.$$

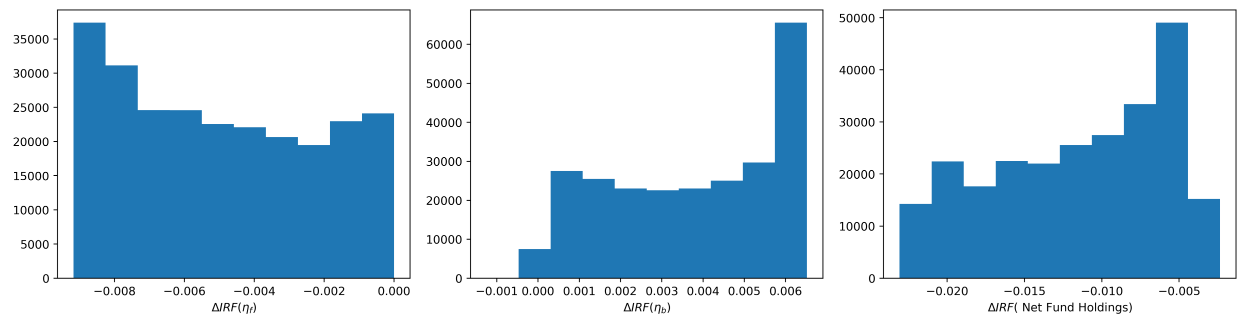


**Figure 7:** This figure presents the histogram of difference in the impulse responses computed using the KFE and the N-agent simulation. The left and right panel plots the level differences in the wealth share of fund and the wealth share of bank, respectively. The right panel presents the level differences in net asset holdings of fund (net of bank asset holdings).

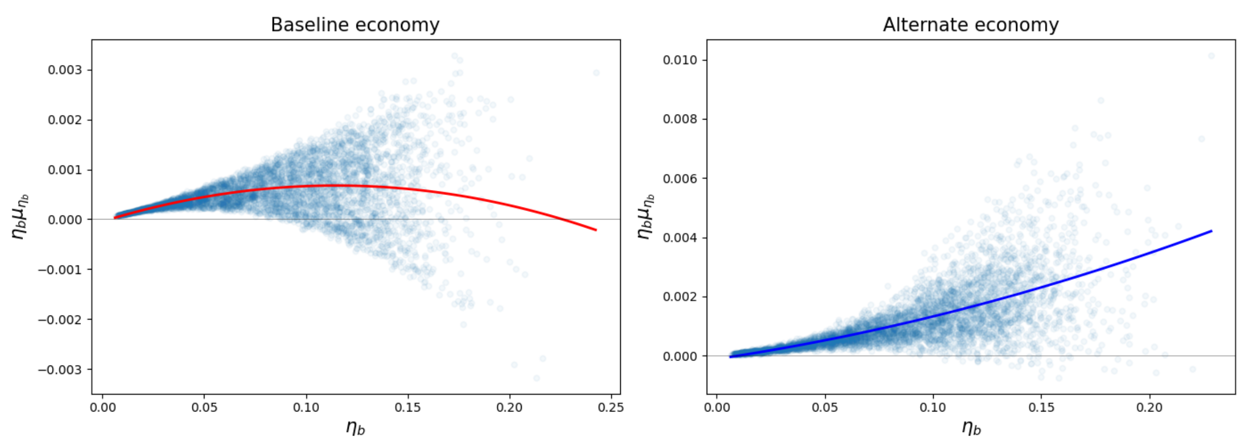


**Figure 8:** The left panel plots the drift of bank's wealth share as a function of $\eta_b$ in the baseline economy. The right panel plot the same in an alternative economy with high household deposit demand. In both panels, scatter points represent the equilibrium drift from a sample of 2000 randomly drawn points from the state space. The red and blue line represent a cubic spline fit to the points.

Since there is no closed-form sampler[13] in general, we draw from $\pi(\boldsymbol{\eta})$ via Metropolis algorithm:

1. **Initialization**. Start from any interior point, e.g. $\eta_i^{(0)} = S/I$.
2. **Random direction**. Draw a mean-zero direction $\boldsymbol{d}$ satisfying $\sum_i d_i = 0$.
3. **Feasible step range**. Compute bounds $[\ell, u]$ such that $\boldsymbol{\eta}^{(t)} + \delta \boldsymbol{d} \geq 0$ for all $\delta \in [\ell, u]$.
4. **Proposal and acceptance.** Draw $\delta \sim \text{Uniform}([\ell, u])$; set $\boldsymbol{\eta}' = \boldsymbol{\eta}^{(t)} + \delta \boldsymbol{d}$. Accept with probability $\alpha = \min\left\{1, \prod_{i=1}^{I} \frac{g(\eta_i')}{g(\eta_i^{(t)})}\right\}$.
5. **Iteration**. Repeat after burn-in, thinning if desired.

For graphical comparison across regulatory regimes, we estimate the cross-sectional distribution of $\beta_h$ using a Gaussian kernel density estimator with bandwidth chosen by Scott's rule. Kernel-smoothed cumulative distribution functions (CDFs) are obtained by numerically integrating the estimated PDFs and normalizing to one.

# E Test Models (Online Appendix)

We "test" our approach by using our algorithm to characterize the solution to three macro-finance models that can be solved using conventional methods: a complete markets model, Basak and Cuoco (1998), and Brunnermeier and Sannikov (2014). First we summarize the key results, and then present the model details. For all models, we use simple feed-forward neural networks and an ADAM optimizer. The details of the neural network parameters for each model are shown in Table 9.

| Model | Num of Layers | Num of Neurons | Learning Rate |
|---|---|---|---|
| "As-if" Complete Model | 4 | 64 | 0.001 |
| Limited Participation Model | 5 | 64 | 0.001 |
| BruSan Model | 5 | 32 | 0.001 |

**Table 9:** Neural network parameters for the three testible models

Table 10 summarizes the mean squared error between the conventional solution and the neural network solution. Evidently, the neural network and conventional methods converge

[13]Exception is when $g(\cdot)$ follows Gamma distribution, then the draw is constructed directly via: $y_i \sim_{i.i.d.} g$, $\eta_i = S \cdot y_i / (\sum_i y_i)$.

to very similar characterizations of equilibrium. Each following subsection describes how the model in that section can be nested with the main model along with technical details.

| Method | L1-Error |
|---|---|
| Complete markets | $1.0 \times 10^{-5}$ |
| Basak and Cuoco (1998) | $4.9 \times 10^{-4}$ |
| Brunnermeier and Sannikov (2014) | $7.0 \times 10^{-5}$ |

**Table 10:** Summary of the algorithm performance and computational speed. Error calculates the difference between solution by neural network and finite difference. All errors are in absolute value (L1).

## E.1 Complete Market Model

We make the following modifications to map our $N$ agent segmentation model from the main text to a Lucas Tree model. We set both the capital depreciation rate $\delta$ and the capital participation constraint function to zero. We fix the capital level $K_t$ to be one and remove all portfolio penalty functions. To further simplify our notations, we introduce the output level $y_t = e^{z_t}$. Without financial frictions, there is simple aggregation of individuals' Euler equations, which coincides with the representative agent's pricing equation. Let us assume that $y_t$ follows a geometric Brownian motion process.

$$dy_t = \mu y_t dt + \sigma y_t dW_t^0.$$

**Analytical Solution.** In a representative agent's world, by the standard Lucas tree pricing formula, the asset price is determined by the discounted flow of dividends:

$$q(y_0) = \mathbb{E}\left[\int_0^\infty e^{-\rho t} \frac{u'(c_t)}{u'(c_0)} y_t dt\right] = y_0 \mathbb{E}\left[\int_0^\infty e^{-\rho t} (y_t/y_0)^{1-\gamma} dt\right]$$

Note that for geometric Brownian motion, the distribution of output is given by:

$$\ln(y_t/y_0) \sim \mathcal{N}\left((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t\right)$$

which means (the integral and expectation operator are interchangeable):

$$\begin{aligned}\mathbb{E}(y_t/y_0)^{1-\gamma} &= (1-\gamma)(\mu - \frac{1}{2}\sigma^2)t + \frac{1}{2}(1-\gamma)^2\sigma^2 t \\ &= (1-\gamma)\mu t + \frac{1}{2}(\gamma-1)\gamma\sigma^2 t \\ &\equiv -\check{g}t\end{aligned}$$

Therefore, asset prices are given by:

$$q(y_0) = y_0 \int_0^\infty e^{-\rho t} e^{-\check{g}t} dt = \frac{y_0}{\rho + \check{g}} = \frac{y_0}{\rho + (\gamma-1)\mu - \frac{1}{2}\gamma(\gamma-1)\sigma^2}$$

By goods market clearing condition, we know that $c_t = y_t$, which means that the consumption policy is:

$$c = \left[\rho + (\gamma-1)\mu - \frac{1}{2}\gamma(\gamma-1)\sigma^2\right] q$$

For $\gamma = 5, \mu = 0.02, \sigma = 0.05, \rho = 0.05$ in the numerical example, $c/q = 10.5\%$, which means: $q(1) = 1/10.5\% \approx 9.5$.

**Neural Network Solution.** Though aggregation results hold, we still incorporate the wealth heterogeneity in our solution algorithm, i.e., we have $N$ asset pricing conditions and $N$ Euler equations. We compare the equilibrium asset price $q(\cdot, y)$ and consumption to wealth ratio $\omega_i(\cdot, y)$ with the "as-if" representative agent economy. The estimated time cost for the model with 5, 10, and 20 agents is about 2 mins, 10 mins, and 20 mins, respectively. The difference between the consumption rule solved using our neural network method and the analytical solution is less than 0.1% for 5 and 10 agents, and 0.5% for 20 agents, respectively. The parameters for the complete market model are provided in Table 12.

| Num of Agents | Euler Eq Error | Diff | Time Cost |
|---|---|---|---|
| 5 | <1e-4 | <0.1% | 2 mins |
| 10 | <1e-4 | <0.5% | 10 mins |
| 20 | <1e-3 | <0.5% | 20 mins |

**Table 11:** Summary of the algorithm performance and computational speed. "Diff" means the difference between representative agent case's solution and brute-force. All errors are in absolute value (L1 loss).

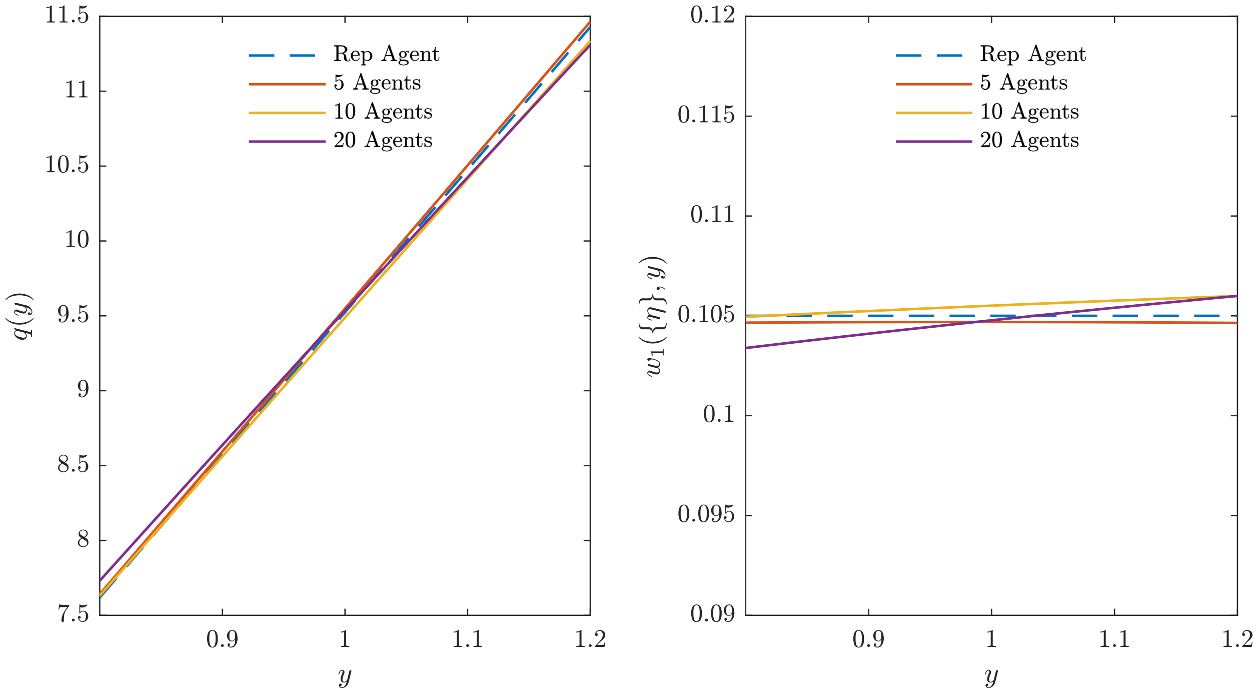


**Figure 9:** Solution to As-if representative agent model. Right panel: consumption-wealth ratio of agent 1.

| Parameter | Symbol | Value |
|---|---|---|
| Risk aversion | $\gamma$ | 5.0 |
| Agents' Discount rate | $\rho$ | 0.05 |
| Output Growth Rate | $\mu$ | 2% |
| Volatility of Growth | $\sigma$ | 5% |

**Table 12:** Model parameters.

## E.2 Asset Pricing with Restricted Participation

We carry forward modifications to the main model from the previous section to mimic the endowment economy. Consider an infinite-horizon economy with two types of price-taking agents: expert (indexed by $e$) and household (indexed by $h$). The financial friction is that households cannot participate in the capital market. Experts do not face this constraint. Mathematically, it is stated as

$$\Psi_i(a_i, b_i) = -\frac{\bar{\psi}_i}{2}(a_i - b_i)^2, \bar{\psi}_h = \infty, \bar{\psi}_e = 0.$$

As before, the output $y_t$ follows a geometric Brownian motion process.

$$dy_t = \mu y_t dt + \sigma y_t dZ_t.$$

**Finite Difference Solution.** We exploit the scalability for geometric Brownian motion's case to get a precise solution by focusing on one-dimensional differential equation. For a scalable income process, we postulate the price function as $q = f(\eta)y$, where $\eta$ is the expert's wealth share with no loss of generality, i.e., $\eta = \eta_e$ (and $1-\eta = \eta_h$). The value function can be written as:

$$V_i = \frac{1}{\rho_i}\frac{(\omega_i\eta_i q)^{1-\gamma}}{1-\gamma} = \frac{(\omega_i\eta_i f(\eta))^{1-\gamma}}{\rho_i}\frac{y^{1-\gamma}}{1-\gamma} \equiv v_i\frac{y^{1-\gamma}}{1-\gamma}, \quad i = e, h$$

where $v_i$ is the scaled value function. From the first-order condition, we get [14]

$$c_i^{-\gamma} = \frac{1}{\rho_i}\frac{(\omega_i\eta_i q)^{1-\gamma}}{\eta_i q} \Rightarrow \left(\frac{c_i}{y}\right)^{\gamma} = \frac{\eta_i f(\eta)}{v_i}, \omega_i = [\eta_i f(\eta)]^{\frac{1}{\gamma}-1} v_i^{-\frac{1}{\gamma}}$$

From the goods market clearing condition, we have:

$$1 = \frac{\sum_i c_i}{y} = \sum_i \left(\frac{\eta_i f(\eta)}{v_i}\right)^{\frac{1}{\gamma}} = y \Rightarrow f(\eta) = \frac{1}{\left[\sum_i \left(\frac{\eta_i}{v_i}\right)^{\frac{1}{\gamma}}\right]^{\gamma}} \tag{E.1}$$

The *HJB equation* for the scaled value function $v_i$ is given by

$$[\rho_i - (1-\gamma)\mu + \frac{\gamma}{2}(1-\gamma)\sigma^2 - \omega_i]v_i = [\mu_\eta + (1-\gamma)\sigma\sigma_\eta]\eta\frac{\partial v_i}{\partial \eta} + \frac{1}{2}\frac{\partial^2 v_i}{\partial \eta^2}\eta^2\sigma_\eta^2 \tag{E.2}$$

where $\mu_\eta, \sigma_\eta$'s expressions are as follows.

$$\begin{aligned}\mu_\eta =& (1-\eta)(\omega_h - \omega_e) + \left(-\frac{1-\eta}{\eta}\right)(r_f - r_q + (\sigma_q)^2)\\ \sigma_\eta =& \frac{1-\eta}{\eta}\sigma_q, \text{where } r_f - r_q = \sigma_\xi\sigma_q, \quad \sigma_q = \frac{\sigma}{1 - \frac{f'(\eta)}{f(\eta)}(1-\eta)}.\end{aligned}$$

The price of risk which appears in the asset pricing condition is determined by Itô's Lemma as follows.

$$\xi_i = \frac{v_i}{\eta_i f(\eta)}y^{-\gamma} \Rightarrow \sigma_\xi = \sigma_v - \sigma_f - \sigma_\eta - \gamma\sigma = \frac{v_i'(\eta)\eta\sigma_\eta}{v_i} - \frac{f'(\eta)\eta\sigma_\eta}{f} - \sigma_\eta - \gamma\sigma.$$

[14] This expression leads to the boundary condition at $\eta = 1$: $\frac{f(1)}{v_e} = 1$

In finite difference solution approach, we introduce a pseudo time-derivative. (E.2):

$$[\rho_i - (1-\gamma)\mu + \frac{\gamma}{2}(1-\gamma)\sigma^2 - \omega_i]v_i = [\mu_\eta + (1-\gamma)\sigma\sigma_\eta]\eta\frac{\partial v_i}{\partial \eta} + \frac{1}{2}\frac{\partial^2 v_i}{\partial \eta^2}\eta^2\sigma_\eta^2 + \frac{\partial v_i}{\partial t}$$

We then update the value function in an implicit scheme to solve the following equation.

$$\check{\boldsymbol{\rho}}\mathbf{I}\mathbf{v}_{t+dt} = \mathbf{M}\mathbf{v}_{t+dt} + \frac{\mathbf{v}_{t+dt} - \mathbf{v}_t}{dt},$$

where $\mathbf{M}$ is the differential matrix by upwind scheme, and $\mathbf{I}$ is the identity matrix.

*Boundary Conditions.* We focus on the case that $\eta \in (0,1]$, as the economy is ill-defined when experts are wiped out from the economy, i.e., there will be nobody left in the economy to hold the tree in equilibrium. To get the right boundary, we use the asset prices and consumption policy $\omega_e$ from the representative agent's solution:

$$\omega_e(1,y) = \rho_e + (\gamma-1)\mu - \frac{1}{2}\gamma(\gamma-1)\sigma^2, q(1,y) = \frac{y}{\omega_e(1,y)},$$

which implies the boundary condition: $v_e(1) = \frac{1}{\rho_e + (\gamma-1)\mu - \frac{1}{2}\gamma(\gamma-1)\sigma^2}$.

The estimated time to solve the limited participation problem by neural network is about 5 minutes. We compare the finite difference solution with the neural network solution on $\eta$'s dimension in figure 10 for $y = 1$. We can see that our method well captures the high non-linearity (left-upper panel) and amplification (right-lower panel). The parameters are provided in Table 13.

| Parameter | Symbol | Value |
|---|---|---|
| Risk aversion | $\gamma$ | 5.0 |
| Households' Discount rate | $\rho_h$ | 0.05 |
| Experts' Discount rate | $\rho_h$ | 0.05 |
| Output Growth Rate | $\mu$ | 2% |
| Volatility of Growth | $\sigma$ | 5% |

**Table 13:** Parameters for the restricted participation model.

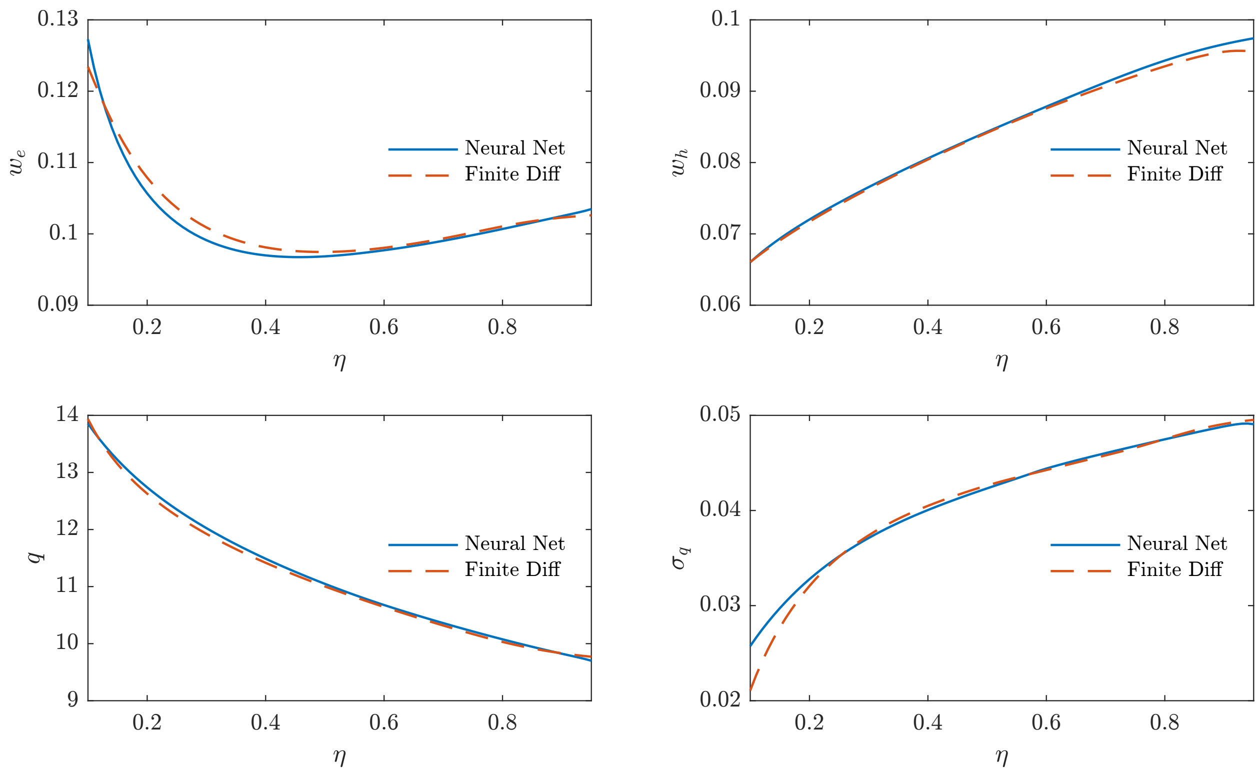


**Figure 10:** Solution to restricted stock market participation model.

## E.3 A Macroeconomic Model with Productivity Gap

The setup in this example follows Brunnermeier and Sannikov (2016). There are two types of agents in this infinite horizon economy: experts and households. Both types can hold capital, but experts have a higher productivity rate compared to households. The productivity rates are given by $z_h, z_e$ ($z_h < z_e$), respectively. Their relative risk aversions are the same, denoted by $\gamma$. Output grows exogenously by $\mu_y = y\mu$, with volatility $y\sigma$, and experts cannot issue outside equities. In addition, we assume that households cannot short capital, which can be formally written as:

$$\begin{cases} \Psi_h(a_h, b_h) = -\dfrac{\bar{\psi}_h}{2}(\min\{a_h - b_h, 0\})^2, & \bar{\psi}_h = \infty \\ \Psi_e(a_e, b_e) = -\dfrac{\bar{\psi}_e}{2}(a_e - b_e)^2, & \bar{\psi}_e = 0. \end{cases}$$

The output flow of households and experts follow

$$d_{e,t} = z_e y_t, d_{h,t} = z_h y_t, dy_t = y_t \mu dt + y_t \sigma dZ_t$$

The expected capital return is

$$r_{q,e,t} = \frac{d_{e,t}}{q_t} + \mu_{q,t}, r_{q,h,t} = \frac{d_{h,t}}{q_t} + \mu_{q,t}.$$

We rewrite the financial friction as the difference : $\frac{y_e - y_h}{q\sigma_q}$. For the first two equations, we have:

$$\begin{cases} -\frac{1}{\xi_e}\frac{\partial \xi_e}{\partial y}\sigma_y = \frac{1}{\xi_e}\frac{\partial \xi_e}{\partial \eta}\sigma_\eta - \frac{r_f - r_{q,h}}{\sigma_q} + \frac{y_e - y_h}{q\sigma_q} \\ -\frac{1}{\xi_h}\frac{\partial \xi_h}{\partial y}\sigma_y = \frac{1}{\xi_h}\frac{\partial \xi_h}{\partial \eta}\sigma_\eta - \frac{r_f - r_{q,h}}{\sigma_q} \end{cases}$$

Unlike the fully restricted participation's case where the experts hold all capital, we have to keep track of the capital allocation ratio of experts $\kappa$, which is parameterized as $\kappa = \eta + \lambda = \eta + \mathcal{N}_\lambda \eta^\beta$, where $\mathcal{N}_\lambda$ is a trainable neural network, and $\beta = \frac{1}{2}$ captures the power law for $\eta \to 0$. Given the expert's capital share holding $\kappa$, the volatility of wealth share $\sigma_\eta$ is $(\kappa - \eta)\sigma_q$. The goods market clearing condition (E.1) is replaced by

$$f(\eta) = \frac{\kappa\eta z_e + (1-\kappa)(1-\eta)z_h}{\left[\sum_i \left(\frac{\eta_i}{v_i}\right)^{\frac{1}{\gamma}}\right]^\gamma}$$

and the price volatility is revised as

$$\sigma_q = \frac{\sigma}{1 - \frac{f'(\eta)}{f(\eta)}(\kappa - \eta)}$$

At the left boundary, $f(\cdot)$ is determined by $f(0) = \frac{y_h}{\omega_h(0)}, f(1) = \frac{y_e}{\omega_e(1)}$.

The estimated time to solve the model by our neural network method is about 5 minutes. Again, we compare the finite difference solution with the neural network solution in figure 11 for $y = 1$. We restrict the range of $\eta$ to be the crisis region in Brunnermeier and Sannikov (2016), which is defined by the region where the capital share of experts $\kappa < 1$. This region captures the fire-sale region, where amplification takes place. We can see that the neural network solution well captures most of the amplification in that crisis region. The parameters are provided in Table 14.

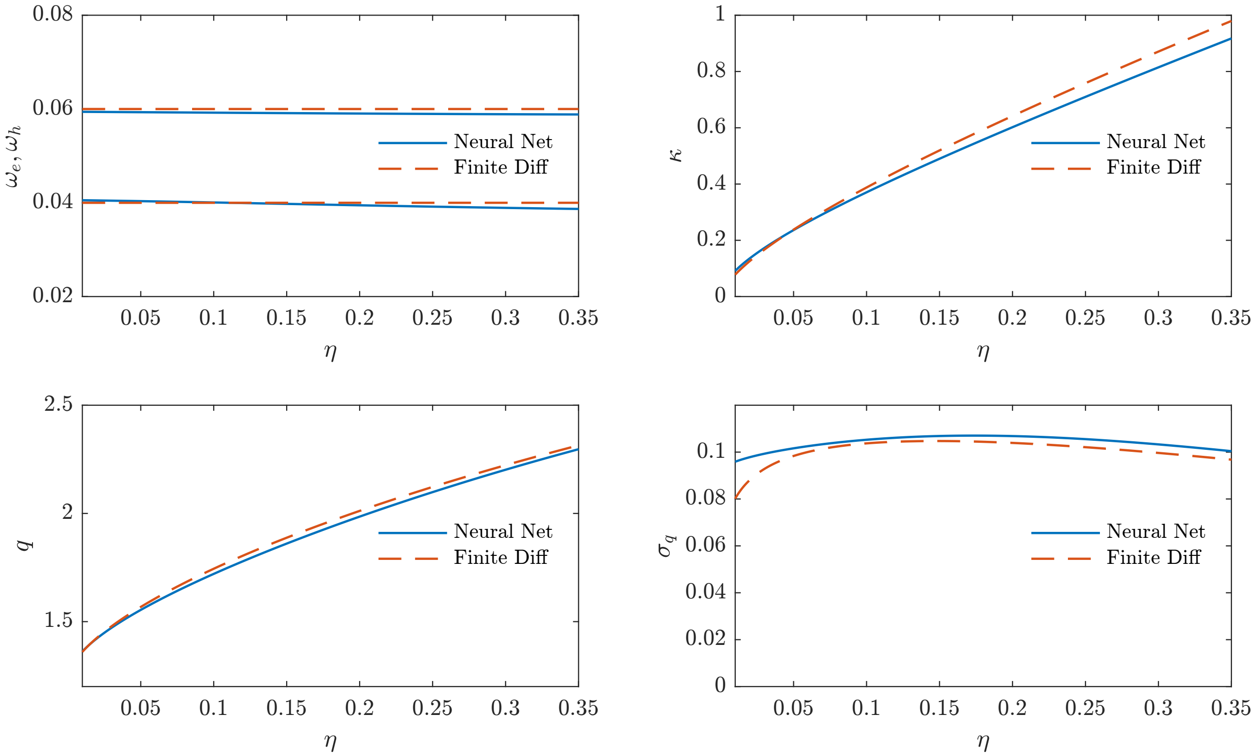


**Figure 11:** Solution to the model with productivity gap.

| Parameter | Symbol | Value |
|---|---|---|
| Risk aversion | $\gamma$ | 1.0 |
| Households' Discount rate | $\rho_h$ | 0.04 |
| Experts' Discount rate | $\rho_e$ | 0.06 |
| Households' Productivity | $z_e$ | 0.11 |
| Experts' Productivity | $z_h$ | 0.05 |
| Output Growth Rate | $\mu$ | 2% |
| Volatility of Growth | $\sigma$ | 5% |

**Table 14:** Parameters for the macroeconomic model.