Macrofinance, Segmentation, and Heterogeneity

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Introduction

- Historically, government policies have created very different financial sectors. E.g. [Payne et al., 2023b], [Payne et al., 2023a], [Lehner et al., 2024]
  - 1863-1933: Insurance companies faced few restrictions;
    Banks largely restricted to US debt and short-term commercial paper
  - 1934-2007: Banks allowed to hold long-term risky assets; insurance companies match duration
- Evidence these "institutional constraints" are important for explaining asset pricing. [Koijen and Yogo, 2019], [Koijen and Yogo, 2023], [Vayanos and Vila, 2021], [Payne and Szőke, 2024]
- Much interest in how these arrangements affect household welfare.
- But exploring this in a macro model has proven technically challenging.

Should all financial intermediaries be able to participate in all asset markets?

Today's Talk

- Model 1: Illustrative Heterogeneous Agent Macro-Finance (HAMF) Model
  - Environment with heterogeneous households, capital stock, and a financial expert.
    (Heterogeneous household version of the models you have in the Princeton Initiative.)
  - Show how to setup equilibrium and characterize using deep learning.
  - Study how asset pricing and participation constraints impact household inequality.
- Model 2: Heterogeneous Agent Institutional Asset Pricing (HAIAP) Model
  - Enrich the model to incorporate banks, insurers and multiple long-term assets.
  - Revisit historical questions about the optimal segmentation of financial markets.
  - Study how financial sector segmentation affects the allocation of risk and household welfare.

Literature Review: I Study the "Macro-Design" of the Financial Sector

- Asset pricing and inequality
  [Gomez, 2017], [Cioffi, 2021], [Gomez and Gouin-Bonenfant, 2024], [Fagereng et al., 2022], [Basak and Chabakauri, 2023], [Fernández-Villaverde and Levintal, 2024], [Irie, 2024]
  - *This talk:* endogenous capital market participation and price volatility.
- Historical asset pricing, market segmentation, and inelastic demand
  Krishnamurthy and Vissing-Jorgensen (2012), Daglish & Moore (2018), Choi et al. (2022), Payne et al. (2022), Jiang et al. (2022a), Chen et al. (2022), Jiang et al. (2022b), [Payne and Szőke, 2024], Koijen and Yogo (2019)
  - *This talk:* government strategically chooses market segmentation.
- Deep learning for macroeconomic models
  [Azinovic et al., 2022], [Han et al., 2021], [Maliar et al., 2021], [Kahou et al., 2021], [Bretscher et al., 2022], [Fernández-Villaverde et al., 2023], [Han et al., 2018], [Huang, 2022], [Duarte, 2018], [Gopalakrishna, 2021], [Fernandez-Villaverde et al., 2020], [Sauzet, 2021], [Gu et al., 2023], [Barnett et al., 2023], [Payne et al., 2024]
  - *This talk:* non-trivial agent optimization, distribution dynamics, and asset pricing.
- Deep learning and portfolio choice
  [Fernández-Villaverde et al., 2023], [Huang, 2023], [Azinovic and Žemlička, 2023], [Azinovic et al., 2023], [Kubler and Scheidegger, 2018]
  - *This talk:* enforces market clearing in neural network.

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Environment

- Continuous time. One good produced by technology $y_t = e^{z_t} k_t$, where:
  - Aggregate productivity follows: $dz_t = \alpha_z(\bar{z} - z_t)dt + \sigma_z dW_{z,t}$,
  - Capital stock follows $dk_t = (\phi(\iota_t)k_t - \delta k_t)dt$, where ι_t is the investment rate.
- Continuum of price taking OLG households ($i \in I$):
  - Idiosyncratic death shocks at rate λ_h; dying households replaced by new with wealth $\underline{a}_h = \varphi_h A$. (new wealth financed by transfer $\tau_{i,t}$ from surviving agents)
  - While alive households get flow utility $u(c_{i,t}) = c_{i,t}^{1-\gamma}/(1-\gamma)$ from consuming $c_{i,t}$.
  - *Friction:* cannot contract across generations (later, insurance sector does it).
  - *Friction:* penalty on holding capital $\psi_{h,t}(k_{i,t}, a_{i,t})$, $\uparrow$ in capital $k_{i,t}$ and $\downarrow$ in wealth $a_{i,t}$.
- Financial "experts" with death rate λ_e, log preferences, and no equity raising.
- Competitive markets for goods, risk-free bonds (at r_t), and capital (with price q_t, return $R_{k,t}$).

$$\frac{dq_t}{q_t} = \mu_{q,t}dt + \sigma_{q,t}dW_{z,t}, \qquad dR_{k,t}(\iota_t) := \frac{e^{z_t} - \iota_t k_t}{q_t k_t} + \frac{d(q_t k_t)}{q_t k_t} =: r_{k,t}dt + \sigma_{q,t}dW_{z,t}$$

Optimization and Equilibrium

- Given belief about price processes $(\hat{r}, \hat{q})$, household i with wealth $a_{i,t} = b_{i,t} + q_t k_{i,t}$ solves:

$$\max_{c_i, k_i, \iota_i} \left\{ \mathbb{E}_0 \left[\int_0^\infty e^{-\rho_i t} \left(u(c_{i,t}) - \Psi_t(k_{i,t}, a_{i,t}) \right) dt \right] \right\}$$

$$\begin{aligned} s.t. \quad da_{i,t} &= (a_{i,t} - k_{i,t})\hat{r}_{i,t} dt + k_{i,t} d\hat{R}_{k,t}(\iota_t) - c_{i,t} dt - \tau_{i,t} dt \\ &=: \mu_{a_i} a_{i,t} dt + \sigma_{a,i} a_{i,t} dW_{z,t} \end{aligned}$$

- Expert problem similar but without Ψ and with Epstein-Zin preferences More

- Equilibrium:
  1. Given $\hat{r}, \hat{q}$, households and expert optimize.
  2. Prices (q_t, r_t) solves market clearing:
     (i) Goods market $\sum_i c_{i,t} + \sum_i \Phi(\iota_{i,t}) k_{i,t} = y_t$,
     (ii) Capital market $\sum_i k_{i,t} = K_t$ and (iii) Bond market $\sum_i b_{i,t} = 0$.
  3. Agent beliefs are consistent with equilibrium $(\hat{r}, \hat{q}) = (r, q)$.

Recursive Characterization of Equilibrium (Three Blocks)

- ⋆ Aggregation within the expert sector but not within the household sector. Why?

- ⋆ Individual household state = $a_{i,t}$, Aggregate states = $(z_t, K_t, g_t) = \boldsymbol{s}_t$, where $g_t(a)$ is the household wealth measure (and $\int_a g_t(a)da$ is total household wealth share).
  $\Rightarrow$ Prices are a function g_t so beliefs about prices become beliefs about the evolution of g_t.

- ⋆ *Block 1:* Distribution evolution.

- ⋆ *Block 2:* Agent optimization.

- ⋆ *Block 3:* Equilibrium consistency.

Block 1: Distribution Evolution (the Kolmogorov Forward Equation)

- ⋆ Aggregation within the expert sector but not within the household sector.
- ⋆ Individual household state = $a_{i,t}$, Aggregate states = $(z_t, K_t, g_t) = \boldsymbol{s}_t$, where $g_t(a)$ is the household wealth measure (and $\int_a g_t(a)da$ is total household wealth share).
- ⋆ The household wealth measure $g_t(a)$ evolves according to:

$$dg_t(a) = \overbrace{\Big[\underbrace{\lambda_h \varphi_h A_t}_{\text{Birth}} - \underbrace{\lambda_h g_t(a)}_{\text{Death}} - \underbrace{\partial_a[\mu_a(a, \boldsymbol{s}_t, g_t) a g_t(a)]}_{\text{Wealth drift}} + \underbrace{\frac{1}{2}\partial_a \left[(\sigma_a^2(a, \boldsymbol{s}_t, g_t)) a^2 g_t(a)\right]}_{\text{Wealth volatility}}\Big]}^{=:\mu_{g,t}(a) = \text{distribution “drift”}} dt$$

$$- \underbrace{\partial_a[\sigma_a(a, \boldsymbol{s}_t, g_t) a g_t(a)]}_{=:\sigma_{g,t}(a) = \text{distribution “volatility”}} \quad dW_{z,t}$$

How would you derive this KFE?

KFE Proof Sketch: Setup for "Propagation of chaos" technique (1/2)

- ⋆ Idea: Study dynamics of finite agent population then take limit as number of agents $\to \infty$.

- ⋆ Finite population approximation: $N < \infty$ agents with a_t^i that follow equation:
  (Where $\check{\mu}_{a_i}$ is the wealth drift without taxes and $\tau_{i,j,t}$ is tax when j reborn)

$$da_{i,t} = \check{\mu}_{a_i} a_{i,t} dt + \sigma_{a,i} a_{i,t} dW_{z,t} + (\varphi_h A_t - a_{i,t}) dN_t^i - \sum_j \tau_{i,j,t} dN_t^j.$$

- ⋆ Define the (empirical) density function for the finite population economy:

$$\hat{g}_t^N(a) := \frac{1}{N} \sum_{i=1}^{N} \delta_{a_t^i}(a), \quad \text{where } \delta_{a_t^i}(a) \text{ is the Dirac-delta measure.}$$

- ⋆ We would like to use Ito's Lemma to get evolution of $\hat{g}_t^N(a)$ and then take limit as $N \to \infty$
  …But Dirac-delta functions are too difficult to differentiate directly.
  …So instead we apply Ito's lemma to $\frac{1}{N}\sum_{i=1}^N \phi_t(a_{i,t})$, for arbitrary "test function" ϕ_t
  (where ϕ is smooth and has compact support)

KFE Proof Sketch: Apply Ito's Lemma and Take Limit (2/2)

- Applying Ito's Lemma to $\frac{1}{N}\sum_{i=1}^{N}\phi_t(a_{i,t})$ and rearranging gives:

$$\frac{1}{N}\sum_{i=1}^{N}\phi_t(a_t^i) - \frac{1}{N}\sum_{i=1}^{N}\phi_0(a_0^i) = \frac{1}{N}\sum_{i=1}^{N}\int_0^t \Big(\partial_s\phi_s(a_{i,s}) + \check{\mu}_a a_{i,s}\partial_a\phi_s(a_{s,i}) + \frac{1}{2}\sigma_{a,s}^2 a_{i,s}^2\partial_{aa}\phi_s(a_{i,s})\Big)ds$$
$$+ \frac{1}{N}\sum_{i=1}^{N}\int_0^t(\phi_s(\varphi_h A_s) - \phi_s(a_{i,s}))dN_s^i + \frac{1}{N}\sum_{i=1}^{N}\int_0^t \sigma_{a,s}a_{i,s}\partial_x\phi_s(a_{i,s})dW_{z,s} - \ (\tau_{i,t}\text{ terms})$$

- Take the limit as $N \to \infty$ so the idiosyncratic noise averages out
(With transfer $\tau_{i,t}$ terms implicitly moved to the drift $\mu_{a,s}$)

$$\int_{\mathcal{A}}(\phi_t(a)g_t(a) - \phi_0(a)g_0(a))\,da = \int_{\mathcal{A}}\int_0^t \Big(\partial_s\phi_s(a) + \mu_{a,s}a\partial_a\phi_s(a) + \frac{1}{2}\sigma_{a,s}^2 a^2\partial_{aa}\phi_s(a)\Big)g_s(a)dsda$$
$$+ \int_{\mathcal{A}}\int_0^t(\phi_s(\varphi_h A) - \phi_s(a))\lambda_h g_s(a)dsda + \int_{\mathcal{A}}\int_0^t \sigma_{a,s}a\partial_a\phi_s(a)g_s(a)dW_{z,s}da$$

- To finish the proof, use integration by parts to swap differentiation from ϕ to g. $\square$

Block 2: Agent Optimization: (Recursive in Wealth Levels)

- Individual household state $= a_{i,t}$, Aggregate states $= (z_t, K_t, g_t) = \boldsymbol{s}_t$.

- Given belief about evolution of the distribution, $(\tilde{\mu}_g(\boldsymbol{s}_t), \tilde{\sigma}_g(\boldsymbol{s}_t))$, household i chooses (c_i, ι_i) and capital wealth share $\theta_i^k := q^k k_i / a_i$ to solve:

$$\rho V_i(a_i, \boldsymbol{s}) = \max_{c_i, \theta_i, \iota_i} \Big\{ u(c_i) - \Psi(\theta_i^k, a_i, \boldsymbol{s}) + \frac{\partial V_i}{\partial a_i} \mu_{a_i}(a_i, c_i, \theta_i, \iota, \boldsymbol{s}) a_i + \frac{\partial V_i}{\partial z} \mu_z + \frac{\partial V_i}{\partial K} \tilde{\mu}_K(\boldsymbol{s})$$
$$+ \frac{1}{2} \frac{\partial^2 V_i}{\partial a_i^2} \sigma_{a_i}^2(\theta_i, \boldsymbol{s}) a_i^2 + \frac{1}{2} \frac{\partial^2 V_i}{\partial z^2} \sigma_z^2 + \frac{\partial^2 V_i}{\partial a_i \partial z} \sigma_{a_i}(\theta_i, \boldsymbol{s}) \sigma_z + \int_{\mathcal{A}} \frac{\partial V}{\partial g}(a, z, g(x)) \tilde{\mu}_g(x, z, g) dx$$
$$+ \int_{\mathcal{A}} \frac{\partial V_i}{\partial g \partial z}(a, z, g(x)) \tilde{\sigma}_g(x, z, g) \sigma_z dx + \int_{\mathcal{A}} \int_{\mathcal{A}} \frac{\partial V_i}{\partial g^2}(a, z, g(x, x')) \tilde{\sigma}_g(a, z, g(x)) \tilde{\sigma}_g(a, z, g(x')) dx dx' \Big\}$$

- Expert HJBE is similar but without $\Psi(k_i, a_i, \cdot)$ and with log utility.

- In equilibrium, beliefs are consistent: $(\mu_g(\boldsymbol{s}_t), \sigma_g(\boldsymbol{s}_t)) = (\tilde{\mu}_g(\boldsymbol{s}_t), \tilde{\sigma}_g(\boldsymbol{s}_t))$.

Block 3: Equilibrium Price Consistency

- Clearing conditions pin down the prices:

$$\sum_i c_{i,t} + \Phi(\iota_t)K_t = y_t \qquad \sum_i (1-\theta_{i,t})a_{i,t} = 0 \qquad \sum_i \theta_{i,t}a_{i,t} = q_t K_t$$

- But q process is implicit so we must impose consistency conditions on q to close the model:

$$\mu_{q,t}q_t dt + \sigma_{q,t}q_t dW_{z,t} = ITO(q(\boldsymbol{s}_t))$$

Comparison to Models With Existing Solution Techniques

Models	Non-Trivial Blocks			Method
	1 (Dist.)	2 (Opt.)	3 (Asset q)	
Representative Agent (à la [Lucas, 1978])	NA	simple	simple	Finite difference
Heterogeneous Agents (à la [Krusell and Smith, 1998])	✓	✓	simple	[Gu et al., 2023]
Long-lived assets (à la [Brunnermeier and Sannikov, 2014])	low-dim	closed-form	✓	[Gopalakrishna, 2021]
HA + Long-lived assets	✓	✓	✓	This talk

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Approach (“Projection” onto a Neural Network)

- High level idea:
  1. Replace agent continuum by high but finite dimensional approximation to the distribution.
  2. Represent equilibrium functions by neural networks with states as inputs.
  3. Train neural network parameters to minimize loss in equilibrium conditions on randomly sampled points from the state space.
- Easy to describe but tricky to implement in practice.
- One “art” of deep learning is resolving how to rewrite the problem to “help” the neural net.

How would you approximate the distribution?

1. Finite Dimensional “Distribution” Approximations [Gu et al., 2023]

	Finite Population	Discrete State	Projection
Dist. approx. (params $\hat{\varphi}_t$)	Agent states $\hat{\varphi}_t = \{a_t^i\}_{i \leq N}$	Masses on grid $\sum_{i=1}^{N} \hat{\varphi}_{i,t}\delta_{(a^i)}$	Basis coefficients $\sum_{i=0}^{N} \hat{\varphi}_{i,t} b_i(a)$
KFE approx. ($\mu^{\hat{\varphi}}$)	Evolution of other agents’ states with idio. noise averaged	Evolution of mass between grid points (e.g. finite diff.)	Evolution of projection coefficients (least squares)
Dimension (N)	$\approx 20 - 40$	≈ 200	≈ 5

We don’t need a very high dimensional finite population approximation. Why?

Finite Population KFE With Averaged Idiosyncratic Noise

- Applying Ito's Lemma to $\frac{1}{N}\sum_{i=1}^{N}\phi_t(a_{i,t})$ and rearranging gives:

$$\frac{1}{N}\sum_{i=1}^{N}\phi_t(a_t^i) - \frac{1}{N}\sum_{i=1}^{N}\phi_0(a_0^i) = \frac{1}{N}\sum_{i=1}^{N}\int_0^t \Big(\partial_s\phi_s(a_{i,s}) + \mu_a a_{i,s}\partial_a\phi_s(a_{s,i}) + \frac{1}{2}\sigma_{a,s}^2 a_{i,s}^2\partial_{aa}\phi_s(a_{i,s})\Big)ds$$

$$+\frac{1}{N}\sum_{i=1}^{N}\int_0^t(\phi_s(\varphi_h A_s) - \phi_s(a_{i,s}))dN_s^i + \frac{1}{N}\sum_{i=1}^{N}\int_0^t \sigma_{a,s}a_{i,s}\partial_x\phi_s(a_{i,s})dW_{z,s} + (\tau_{i,j}\ terms)$$

- Informally, take the limit as $N \to \infty$ selectively (so only the idiosyncratic noise averages out)

$$\frac{1}{N}\sum_{i=1}^{N}\phi_t(a_t^i) - \frac{1}{N}\sum_{i=1}^{N}\phi_0(a_0^i) = \frac{1}{N}\sum_{i=1}^{N}\int_0^t \Big(\partial_s\phi_s(a_{i,s}) + \mu_a a_{i,s}\partial_a\phi_s(a_{s,i}) + \frac{1}{2}\sigma_{a,s}^2 a_{i,s}^2\partial_{aa}\phi_s(a_{i,s})\Big)ds$$

$$+\int_{\mathcal{A}}\int_0^t(\phi_s(\varphi_h A) - \phi_s(a))\lambda_h g_s(a)dsda + \frac{1}{N}\sum_{i=1}^{N}\int_0^t \sigma_{a,s}a_{i,s}\partial_x\phi_s(a_{i,s})dW_{z,s}$$

Which variables would you represent by a Neural Network?

Practical Technical Decisions

1. How do we approximate the distribution? **A.** Finite population.

2. Which variables to represent by NNs? **A.** Consumption/wealth & price volatilities.
   - ⋆ We fit neural networks to the variables that are "easiest" to train.
   - ⋆ Better to represent $\xi = \partial_a V$ than V so we can easily impose V concavity.
   - ⋆ Better to represent $\omega = c/a$, then get $\xi = (\omega \eta q K)^{-\gamma}$ so extreme curvature is analytic.

3. Which equilibrium conditions go into loss function? **A.** Avoid market clearing.
   - ⋆ We work with wealth shares $\{\eta_i\}_{1 \leq i \leq I}$ rather than wealth levels $\{a_i\}_{1 \leq i \leq I}$
   - ⋆ We instead impose market clearing in the equations and the sampling
   - ⋆ Similar in spirit to [Azinovic and Žemlička, 2023].

Near Solutions | Details on Imposing Market Clearing

Alternative Recursive Characterization for the Neural Network

- ⋆ Change variable to marginal value of wealth: $\xi_i := \partial V_i / \partial a_i$ in the optimization equations.

- ⋆ Change distribution to wealth shares $\{\eta_i\}_{1 \leq i \leq I}$, where $\eta_i := a_i / A$ is agent i's share.

- ⋆ At state $\boldsymbol{X} = (z, K, (\eta_i)_{i \leq I})$, the equilibrium objects $(\boldsymbol{\xi}, q, \boldsymbol{\omega}, \boldsymbol{\sigma_\eta}, s, \sigma_q, \boldsymbol{\theta}, \mu_\eta, \mu_q, r)$ must satisfy (where $\xi_i = u'(\omega_i \eta_i q K)$):

$$0 = (r - \rho_i)\xi_i + \frac{\partial \xi_i}{\partial z}\mu_z + \frac{\partial \xi_i}{\partial K}(\phi((\phi')^{-1}(q^{-1}))K_t - \delta K_t) + \sum_j \frac{\partial \xi_i}{\partial \eta_j}\eta_j \mu_{\eta_j,t}$$

$$+ \sum_j \frac{\partial^2 \xi_i}{\partial z \partial \eta_j}\eta_j \sigma_{\eta_j,t}\sigma_z + \frac{1}{2}\frac{\partial^2 \xi_i}{\partial z^2}\sigma_z^2 + \frac{1}{2}\sum_{j,j'} \frac{\partial^2 \xi_i^2}{\partial \eta_j \partial \eta_{j'}}\eta_j \eta_{j'} \sigma_{\eta_j,t}\sigma_{\eta_{j'},t}$$

$$0 = -q\sigma_q + \sum_j \frac{\partial q}{\partial \eta_j}\eta_j \sigma_{\eta_j} + \frac{\partial q}{\partial z}\sigma_z$$

and s.t. FOCs and wealth share evolution equations (with equilibrium imposed) All Equations

Neural Network Approximation

- Approximate ($\omega_h := c_h/a_h, \sigma_q$) by neural networks with parameters (Θ_{ω_h}, Θ_q):

$$\hat{\omega}_h(\boldsymbol{X}; \Theta_{\omega_h}), \qquad \hat{\sigma}_q(\boldsymbol{X}; \Theta_q)$$

- At state $\boldsymbol{X}$, the error (or "loss") in the Neural network approximations is given by: (with $\hat{\xi}_h = u'(\hat{\omega}_h(\boldsymbol{X}; \Theta_{\omega_h}))$ and $\hat{\sigma}_q = \hat{\sigma}_q(\boldsymbol{X}; \Theta_q)$)

$$\begin{aligned}
\mathcal{L}_{\omega_h}(\boldsymbol{X}) = {} & (r - \rho_h)\hat{\xi}_h + \frac{\partial \hat{\xi}_h}{\partial z}\mu_z + \frac{\partial \hat{\xi}_h}{\partial K}(\phi((\phi')^{-1}(q^{-1}))K_t - \delta K_t) + \sum_j \frac{\partial \hat{\xi}_h}{\partial \eta_j}\eta_j \mu_{\eta_j,t} \\
& + \sum_j \frac{\partial^2 \hat{\xi}_h}{\partial z \partial \eta_j}\eta_j \sigma_{\eta_j,t}\sigma_z + \frac{1}{2}\frac{\partial^2 \hat{\xi}_h}{\partial z^2}\sigma_z^2 + \frac{1}{2}\sum_{j,j'} \frac{\partial^2 \hat{\xi}_h^2}{\partial \eta_j \partial \eta_{j'}}\eta_j \eta_{j'}\sigma_{\eta_j,t}\sigma_{\eta_{j'},t} \\
\mathcal{L}_{\sigma}(\boldsymbol{X}) = {} & -q\hat{\sigma}_q + \sum_j \frac{\partial q}{\partial \eta_j}\eta_j \sigma_{\eta_j} + \frac{\partial q}{\partial z}\sigma_z
\end{aligned}$$

NN Structure | Algorithm Details | Sampling Approaches | Implementation Details | Test Models

Algorithm ("EMINN" or "Economic Deep Galerkin")

Initialize neural networks $\{\hat{\omega}_h, \hat{\sigma}_q\}$ with parameters $\{\Theta_{\omega_h}, \Theta_q\}$.

while Loss > tolerance **do**

  Sample N new training points: $\left(\boldsymbol{X}^n = \left(z^n, K^n, (\eta_i)^n_{i\leq I}\right)\right)_{n=1}^N$.

  Calculate equilibrium at each training point $\boldsymbol{X}^n$ given current $\{\hat{\omega}_h, \hat{\sigma}_q\}$:

  (a) Compute $(\hat{\omega}^n_i)_{i\leq I}$ using current approximation $\hat{\omega}_h$ evaluated at $\boldsymbol{X}^n$.

  (b) Compute q^n and $(\xi^n_i)_{i\leq I}$ using $(\hat{\omega}^n_i)_{i\leq I}$.

  (c) Solve for $(\boldsymbol{\theta}^n, \boldsymbol{\sigma}^n_{\boldsymbol{\eta}}, s^n)$ the current approximations for $\{\hat{\omega}_h, \hat{\omega}_e, \hat{\sigma}_q\}$.

  (d) Compute μ_η, μ_q, r.

  Construct loss as: $\hat{\mathcal{L}}(\boldsymbol{X}) = \frac{1}{N}\sum_n |\hat{\mathcal{L}}_{\omega_h}(\boldsymbol{X}^n)| + \frac{1}{N}\sum_n |\hat{\mathcal{L}}_\sigma(\boldsymbol{X}^n)|$

  Update $\{\Theta_{\omega_h}, \Theta_q\}$ using ADAM (extension of stochastic gradient descent that $\downarrow \hat{\mathcal{L}}$).

end while

NN Structure | Algorithm Details | Sampling Approaches | Implementation Details | Test Models

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Q. How Does Asset Pricing Impact Inequality?

- Difference between the drift of the wealth share of any two households i and j is given by:

$$\mu_{\eta_j,t} - \mu_{\eta_i,t} = (\theta_{j,t} - \theta_{i,t})(r_{k,t} - r_t - \sigma_{q,t}^2) - (\omega_j - \omega_i) + \frac{\tau\lambda}{I-1}\left(\frac{1}{\eta_{j,t}} - \frac{1}{\eta_{i,t}}\right)$$

1. Participation constraint: means low wealth agents hold less capital and earn less risk premium. E.g. for log utility and quadratic participation cost $(\psi_{i,t} = 0.5\bar{\psi}\sigma^2\theta_{i,t}^2/\eta_{i,t})$:

$$\theta_{i,t} = \frac{k_{i,t}}{a_{i,t}} \approx \frac{r_{k,t} - r_{f,t}}{\sigma_{q,t}^2 + \bar{\psi}\sigma^2/\eta_{i,t}}, \quad i \in \{1, \ldots, I-1\}$$

2. Differential consumption: low wealth agents consume less to escape participation constraint.

3. Redistribution: through death (and wealth taxes).

Equilibrium For Different Participation Constraints

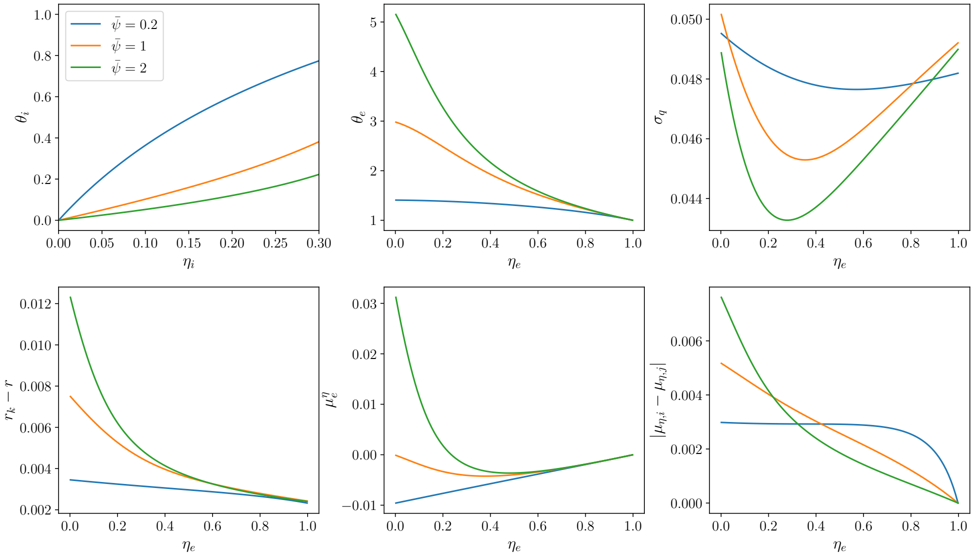

Figure: $\rho_e = 0.04, \rho_h = 0.03, \mu = 0.02, \sigma = 0.05$, Household i has 10 times wealth of household j.

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Environment: Setting, Production, and Households

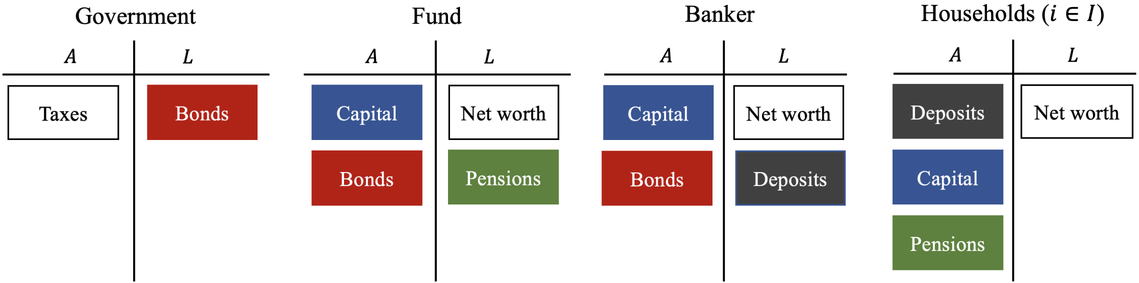
- Continuous time $t \in [0, \infty)$. One perishable consumption good, one capital stock.

- Goods production **technology** $y_t = e^{z_t} k_t$, where capital $dk_t = (\phi(\iota_t) - \delta) k_t dt$ and:
  - Aggregate productivity follows: $dz_t = \alpha_z(\bar{z} - z_t)dt + \sigma_z \sqrt{\zeta_t} dW_{z,t}$,
  - Stochastic volatility follows: $d\zeta_t = \alpha_\zeta \left(\bar{\zeta} - \zeta_t\right) dt + \sigma_\zeta \sqrt{\zeta_t} dW_{\zeta,t}$

- Continuum of price taking **households** (index by $i \in [0, 1]$)
  - Idiosyncratic death shocks at rate λ_h; dying households replaced by new with $\underline{a}_h = \varphi_h A$.
  - While alive: households get flow utility $\beta u(c_{i,t}) = \beta c_{i,t}^{1-\gamma}/(1-\gamma)$ from consuming $c_{i,t}$.
  - At death: get utility $(1-\beta)\mathcal{U}(\mathcal{C}_{i,t})$ from consuming $\mathcal{C}_{i,t}$.
  - *Friction:* cannot provide death insurance contracts to each other.
  - *Friction:* penalty on holding capital $\psi_{h,t}(k_{i,t}, a_{i,t})$, $\uparrow$ in capital $k_{i,t}$ and $\downarrow$ in wealth $a_{i,t}$.

  leads to non-degenerate density across household wealth, $g_h(a)$.

Environment: Financial Intermediaries and Balance Sheets



- **Bankers** (b): issue deposits (at r_t^d) and holds capital or government bonds.
- **Fund managers** (f): issue (pension/insurance) contracts and holds capital or gov bonds. (A contract pays 1 good to the household holding the contract when they die.)
- **Government**: issues fixed supply of zero coupon bonds B that mature at rate λ_B
- Asset prices for capital, contracts, bonds, $\boldsymbol{q}_t = (q_t^k, q_t^n, q_t^B)$

Portfolio Choice: Pension/Insurance Contracts

Individual state $= a_i$, Aggregate states $= (z, \zeta, K, g) =: \boldsymbol{S}$,
(where g is the wealth distribution across households and financial intermediaries)

Recursive characterization

Let $V_j(a_j, \boldsymbol{S})$ denote value function for type $j \in \{h, b, f\}$ and let $\xi_j = \partial_{a_j} V_j(a_j, \boldsymbol{S})$.

Then the FOCs in the pension/insurance contract market:

$$\underbrace{r^n - r^l}_{\substack{\text{Excess}\\\text{return}}} + \underbrace{\frac{\lambda_h}{q^n}\frac{\mathcal{U}'(\mathcal{C})}{\xi_i}}_{\substack{\text{“Inelastic demand”}\\\text{component}}} = - \underbrace{\boldsymbol{\sigma}_{\xi_i} \cdot \boldsymbol{\sigma}_{q^n}}_{\substack{\text{Comovement of}\\\text{SDF and price}}} \qquad \text{...Household FOC}$$

$$\underbrace{r^n - r^l}_{\substack{\text{Excess}\\\text{return}}} = - \underbrace{\boldsymbol{\sigma}_{\xi_f} \cdot \boldsymbol{\sigma}_{q^n}}_{\substack{\text{Comovement of}\\\text{SDF and price}}} \qquad \text{...Fund FOC}$$

Nesting Vayanos-Vila Preferences

Portfolio Choice: Capital

FOCs for Capital market:

$$\underbrace{r^k - r^l}_{\substack{\text{Excess}\\ \text{return}}} + \underbrace{\lambda_h(1-\tau_d)\frac{\mathcal{U}'(\mathcal{C})}{\xi_i}}_{\substack{\text{“Insurance demand”}\\ \text{component (after tax)}}} = -\boldsymbol{\sigma}_{\xi_i} \cdot \boldsymbol{\sigma}_{q^k} - \underbrace{\partial_k \psi_{i,k}}_{\substack{\text{“Participation”}\\ \text{constraint}}} \qquad \text{. . . Household FOC}$$

$$r^k - r^l = -\boldsymbol{\sigma}_{\xi_b} \cdot \boldsymbol{\sigma}_{q^k} \qquad \text{. . . Bank FOC}$$

$$r^k - r^l = -\boldsymbol{\sigma}_{\xi_f} \cdot \boldsymbol{\sigma}_{q^k} \qquad \text{. . . Fund FOC}$$

- ⋆ Bank and fund liabilities have different exposure:
  - ⋆ Bank short-term deposits are not exposed to TFP or volatility shocks,
  - ⋆ Pension annuities increase with TFP and decrease with TFP volatility

How are households affected when the government restricts which asset the funds and bankers can hold?

Q. How Does Asset Pricing Impact Inequality? Within Households

- Difference between the drift of the wealth share of any two households i and j is:

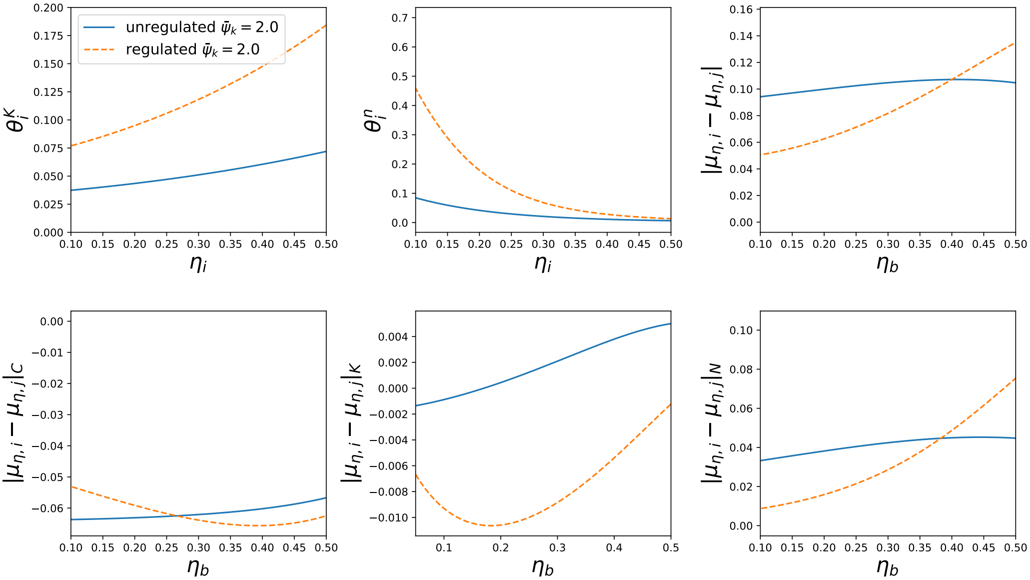
$$\mu_{\eta_j,t} - \mu_{\eta_i,t} = \underbrace{(\theta^k_{j,t} - \theta^k_{i,t})(r^k_t - r^l_t - \boldsymbol{\sigma}^k_{q,t} \cdot \boldsymbol{\sigma}^k_{q,t})}_{=:(\mu_{\eta_j,t}-\mu_{\eta_i,t})^K} + \underbrace{(\theta^n_{j,t} - \theta^n_{i,t})(r^n_t - r^l_t - \boldsymbol{\sigma}^k_{q,t} \cdot \boldsymbol{\sigma}^n_{q,t})}_{=:(\mu_{\eta_j,t}-\mu_{\eta_i,t})^N}$$
$$- (\omega_j - \omega_i) + \varphi_h \lambda \left(\eta^{-1}_{j,t} - \eta^{-1}_{i,t}\right)$$

1. Participation constraint: low wealth agents hold less capital and earn less risk premium. (θ^k_i is agent i's wealth share in capital)
2. Pension needs: low wealth agents save through low return pensions (θ^n_i is share in pensions).
3. Consumption: low wealth agents consume less to escape participation constraint.
4. Redistribution: through death (and wealth taxes).

- Compare economies with two different regulatory regimes:

1. Unregulated: allows funds to participate in all asset markets.
2. Regulated (Segmented): only allows funds to hold to LT government bonds

Inequality Decomposition: Segmentation Has Ambiguous Impact



Economic Questions

- ⋆ **Q.** How is risk allocated between households, bankers, and funds? Details
  - ⋆ **A.** In the unregulated economy, well capitalized funds absorb risk.
  - ⋆ So banks less exposed to TFP and households less exposed to volatility.
  - ⋆ However, distressed funds now charge much higher premia to rebuild wealth.
- ⋆ **Q.** How does segmentation impact household welfare? Details
  - ⋆ **A.** Restricting the fund from holding capital helps low wealth households who end up paying the high premiums to recapitalize the fund in bad times in the unregulated economy.
  - ⋆ Regulation also increases the price of government debt by creating a captive market.
- ⋆ **Q.** How does inequality impact asset pricing amongst households? Details
  - ⋆ **A.** Household inequality allows it to better act as a buffer and stabilize financial sector.

May make sense to restrict funds if fund participants will be forced to “recapitalize” it.

Table of Contents

Conclusion

- Economics: We should study how the government chooses asset market segmentation strategically!
- Technical: can train neural networks to characterize equilibria for macro-finance models with:
  - Large numbers of heterogeneous agents.
  - Financial frictions that prevent finding a closed form solution to the value function.
  - Multiple long-lived assets.
- We believe this offers a pathway to link institutional finance to macroeconomics.

Thank you

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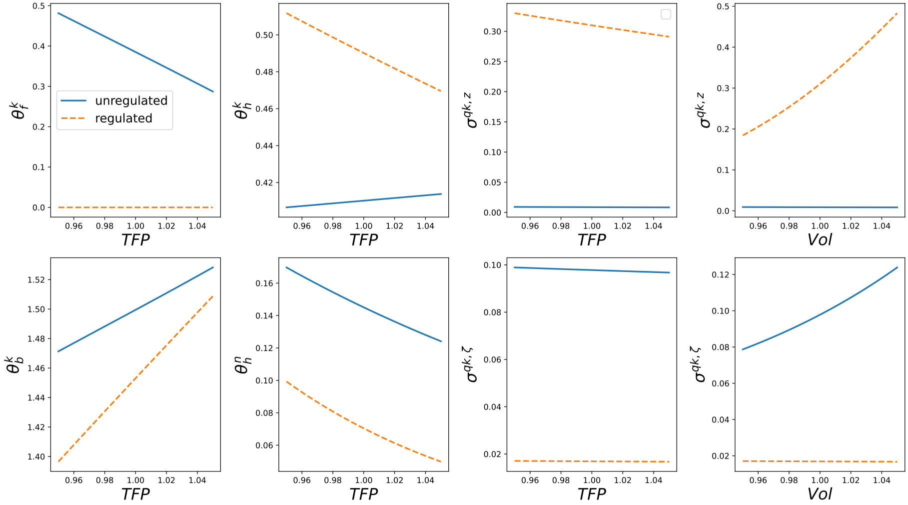
Sector Level Intermediary Asset Pricing

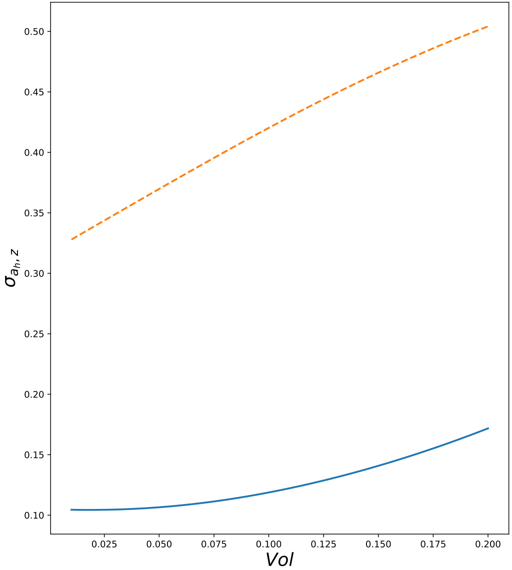
- Banks and funds issue liabilities with different exposure to aggregate shocks:
  - Bank short-term risk-free deposits are not exposed to TFP or volatility shocks,
  - Price of fund pension/insurance contracts:
    - Decreases in recessions (when TFP is low and goods are scarce)
    - Increases or decreases when TFP volatility is high (and households move to safer assets)
- Implies banks and funds face different net-worth shocks in recessions:
  - Bank net-worth $\downarrow$ in recessions $\Rightarrow$ sell capital (generating capital price volatility).
  - Fund net-worth $\uparrow$ in recession $\Rightarrow$ natural "backstop" in recessions.
- Compare economies with two different regulatory regimes:
  1. Unregulated: allows funds to participate in all asset markets.
  2. Regulated: only allows funds to hold to LT government bonds

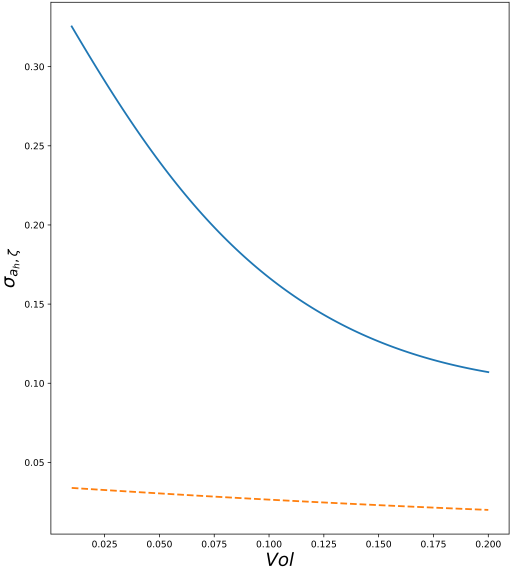
Fund Regulation Determines Where Endogenous Volatility Appears



Funds Insure Household TFP-risk But Not Volatility-risk

Prices Spreads





Expert Problem

Given their belief about price processes $(\hat{r}, \hat{q})$, expert i solves:

$$V_{e,t} = \max_{c_i, k_i, \iota_i} \left\{ \mathbb{E}_0 \left[\int_0^\infty e^{-\rho_e t} f(c_t, V_{e,t}) dt \right] \right\}$$
$$s.t. \quad da_{i,t} = (a_{i,t} - k_{i,t})\hat{r}_{i,t} dt + k_{i,t} d\hat{R}_{k,t} - c_{i,t} dt + \tau_t dt$$

where the "felicity" function is:

$$f(c, V) = \frac{1-\gamma}{1-\frac{1}{\varrho}} V \left[\left(\frac{c}{((1-\gamma)V)^{1/(1-\gamma)}} \right)^{1-\frac{1}{\varrho}} - \rho \right]$$

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Expert Euler Equations

- Given price processes $(r, r_k, q, \mu_q, \sigma_q)$, expert optimization implies:

$$
\begin{aligned}
\textit{Euler equation:} \quad & \frac{\partial f(c_e, V_e)}{\partial V_e}\xi_h = r\xi_e + \frac{\partial \xi_e}{\partial z}\mu_z + \frac{\partial \xi_e}{\partial K}\mu_K + \frac{1}{2}\frac{\partial^2 \xi_e}{\partial z^2}\sigma_z^2 + \sum_j \frac{\partial \xi_e}{\partial \eta_j}\eta_j\mu_{\eta_j,t} \\
& \quad + \sum_j \frac{\partial^2 \xi_e}{\partial z \partial \eta_j}\eta_j\sigma_{\eta_j,t}\sigma_z + \frac{1}{2}\sum_{j,j'} \frac{\partial^2 \xi_e^2}{\partial \eta_j \partial \eta_{j'}}\eta_j\eta_{j'}\sigma_{\eta_j,t}\sigma_{\eta_{j'},t} \\
\textit{Consumption FOC:} \quad & \xi_e = \frac{\partial f(c_e, V_e)}{\partial c_e} \\
\textit{Portfolio FOC:} \quad & \xi_e(r_k - r) = -\left(\frac{\partial \xi_e}{\partial z}\sigma_z + \sum_j \frac{\partial \xi_e}{\partial \eta_j}\sigma_{j,\eta}\right)\sigma_q
\end{aligned}
$$

- where the "felicity" function is:

$$
f(c, V) = \frac{1-\gamma}{1-\frac{1}{\varrho}}V\left[\left(\frac{c}{((1-\gamma)V)^{1/(1-\gamma)}}\right)^{1-\frac{1}{\varrho}} - \rho\right]
$$

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Sampling Approaches

- Sampling (a, z, ζ, K): draw from uniform distribution, then add draws where error high.

- Sampling the parameters in the distribution approximation $(\hat{\varphi}^i)_{i \leq N}$:
  - *Moment sampling:*
    1. Draw samples for selected moments of the distribution (that are important for $\hat{Q}(z, \hat{\varphi})$).
    2. Sample $\hat{\varphi}$ from a distribution that satisfies the moments drawn in the first step.
  - *Mixed steady state sampling:*
    1. Solve for the steady state for a collection of fixed aggregate states z.
    2. Draw random, perturbed mixtures of this collection of steady state distributions.
  - *Ergodic sampling:*
    1. Simulate economy using current value function approximation.
    2. Use simulated distributions as training points.

Need to choose economically relevant subspace on which to sample. Back

Approximate ω by Neural Network (Feed Forward, Fully Connected)

- Let $\boldsymbol{X} = (z, K, \{\eta_i\}_{i \leq I})$. We approximate surplus $\omega(\boldsymbol{X})$ by neural network with form:

$$
\begin{aligned}
\boldsymbol{h}^{(1)} &= \phi^{(1)}(W^{(1)}\boldsymbol{X} + \boldsymbol{b}^{(1)}) && \text{...Hidden layer 1} \\
\boldsymbol{h}^{(2)} &= \phi^{(2)}(W^{(2)}\boldsymbol{h}^{(1)} + \boldsymbol{b}^{(2)}) && \text{...Hidden layer 2} \\
&\vdots \\
\boldsymbol{h}^{(H)} &= \phi^{(H)}(W^{(H)}\boldsymbol{h}^{(H-1)} + \boldsymbol{b}^{(H)}) && \text{...Hidden layer H} \\
\omega &= \sigma(\boldsymbol{h}^{(H)}) && \text{...Variable}
\end{aligned}
$$

- Terminology (our parameter choices are in blue):
  - H: is the number of *hidden layers*, $(H = 4)$
  - Length of vector $\boldsymbol{h}^{(i)}$: number of *neurons* in hidden layer i, $(Length = 32)$
  - $\phi^{(i)}$: is the *activation function* for hidden layer i, $(\phi^i = tanh)$
  - σ: is the *activation function* for the final layer, $(\sigma = tanh)$
  - $\boldsymbol{\Theta} = (W^1, \dots W^{(H)}, b^{(1)}, \dots, b^{(H)})$ are the *parameters*,

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Additional Implementation Details

	Description
Neural Network	
(i) Structure	Feed-forward with 4 hidden layers and 80 neurons each layer
(ii) Initialization	Random
Sampling	
(i) (z, ζ, K)	Uniform sampling
(ii) $(\eta_i)_{i \leq N}$	Moment sampling: sample moments of the distribution and then population distributions satisfying moments
Loss Function	
(i) Learning rate	0.0005

- Average training error $\approx 10^{-5}$
- Training time:
  - 10 agents: $<$ 10 minutes on laptop.
  - 25 agents: $<$ 20 minutes on laptop.
  - 50 agents: $\approx$ 1 hour on cluster
- Test on [Lucas, 1978], [Basak and Cuoco, 1998], [Brunnermeier and Sannikov, 2014] More

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Summary (Part 1)

At state $\boldsymbol{X} = (z, K, (\eta_i)_{i\leq I})$, the equilibrium objects $(\boldsymbol{\xi}, q, \boldsymbol{\omega}, \boldsymbol{\sigma_\eta}, s, \sigma_q, \boldsymbol{\theta}, \mu_\eta, \mu_q, r)$ must satisfy:

$$0 = (r-\rho_i)\xi_i + \frac{\partial \xi_i}{\partial z}\mu_z + \frac{\partial \xi_i}{\partial K}(\phi((\phi')^{-1}(q^{-1}))K_t - \delta K_t) + \sum_j \frac{\partial \xi_i}{\partial \eta_j}\eta_j \mu_{\eta_j,t} \tag{1}$$

$$+ \sum_j \frac{\partial^2 \xi_i}{\partial z \partial \eta_j}\eta_j \sigma_{\eta_j,t}\sigma_z + \frac{1}{2}\frac{\partial^2 \xi_i}{\partial z^2}\sigma_z^2 + \frac{1}{2}\sum_{j,j'} \frac{\partial^2 \xi_i^2}{\partial \eta_j \partial \eta_{j'}}\eta_j \eta_{j'}\sigma_{\eta_j,t}\sigma_{\eta_{j'},t} \tag{2}$$

$$q = \frac{e^z K + \Phi((\phi')^{-1}(q^{-1}))}{\sum_{i=1}^I \omega_i \eta_i}, \tag{3}$$

$$\xi_i = u'\left(\omega_i \eta_i q K\right), \text{ for } i \in \{1, ..., I\}, \tag{4}$$

$$0 = -\left[\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{\eta}} \odot \begin{bmatrix} \boldsymbol{\eta} \\ | \end{bmatrix} \quad \boldsymbol{\xi}\right]\begin{bmatrix}\boldsymbol{\sigma_\eta} \\ s\end{bmatrix} - \sigma_z \frac{\partial \boldsymbol{\xi}}{\partial z} - \frac{1}{\sigma_q}\mathrm{diag}\left(\frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{b}}\right) \tag{5}$$

$$q\sigma_q = \sum_j \frac{\partial q}{\partial \eta_j}\eta_j \sigma_{\eta_j} + \frac{\partial q}{\partial z}\sigma_z \tag{6}$$

$$1 - \theta_i = -\frac{\eta_j \sigma_{\eta_j}}{\sigma_q}, \text{ for } i \in \{1, ..., I\}, \tag{7}$$

Summary (Part 2)

…continuation of the conditions:

$$K\mu_K = \left(\phi\left((\phi')^{-1}\left(q^{-1}\right)\right)K - \delta K_t\right) \tag{8}$$

$$r_k - \mu_q = \frac{e^z}{q} - \frac{(\phi')^{-1}\left(q^{-1}\right)}{q} + (\phi(\iota_t) - \delta) \tag{9}$$

$$\eta_i\mu_{\eta_i} = r_k - \mu_q + \theta_i\sigma_q s - \mu_K - \omega_i + \theta_i\sigma_q^2, \text{ for } i \in \{1, ..., I\} \tag{10}$$

$$q\mu_q = \sum_j \frac{\partial q}{\partial \eta_j}\eta_j\mu_{\eta_j} + \frac{\partial q}{\partial z}\mu_z + \frac{\partial q}{\partial K}\mu_K + \sum_j \frac{\partial^2 \xi_i}{\partial z \partial \eta_j}\eta_j\sigma_{\eta_j}\sigma_z \tag{11}$$

$$r = \sigma_q s + \frac{e^z}{q} - \frac{(\phi')^{-1}(q^{-1})}{q} + (\phi((\phi')^{-1}(q^{-1})) - \delta) + \mu_q \tag{12}$$

Market for Pension Shares (Vayanos-Vila Preferences)

Individual state $= a_i$, Aggregate states $= (z, \zeta, K, \{a_j\}_{j \neq i}) = (\cdot)$ Recursive characterization

Let $V_j(a_j, \cdot)$ denote value function for type $j \in \{h, b, f\}$ and let $\xi_j = \partial_{a_j} V_j(a_j, \cdot)$.

Then the FOCs in the pension share market:

$$\underbrace{r^n - r^l}_{\text{Excess return}} + \underbrace{\frac{\lambda}{q^n} \exp\left(-\alpha \frac{q^k}{q^n} \theta_i^n \eta_i\right)}_{\text{“Preferred habitat” component}} = -\underbrace{\boldsymbol{\sigma}_{\xi_i} \cdot \boldsymbol{\sigma}_{q^n}}_{\text{“}shifter''} \qquad \ldots \text{Household FOC}$$

$$\underbrace{r^n - r^l}_{\text{Excess return}} = -\underbrace{\boldsymbol{\sigma}_{\xi_f} \cdot \boldsymbol{\sigma}_{q^n}}_{\text{Comovement of SDF and price}} \qquad \ldots \text{Fund FOC}$$

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Roadmap

Solutions to example models

Macroprudential Policy and Inequality

Recursive Equilibrium

Three Testable Models

Model	Layers	Neurons	Learning Rate	Error
"As-if" Complete Model	4	64	0.001	1.0×10^{-5}
[Basak and Cuoco, 1998]	5	64	0.001	4.9×10^{-4}
[Brunnermeier and Sannikov, 2014]	5	32	0.001	7.0×10^{-5}

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Representative Agent Model ([Lucas, 1978])

Suppose there are no financial frictions: $\Psi(a_{i,t}, b_{i,t}) = 0$ and no investment.

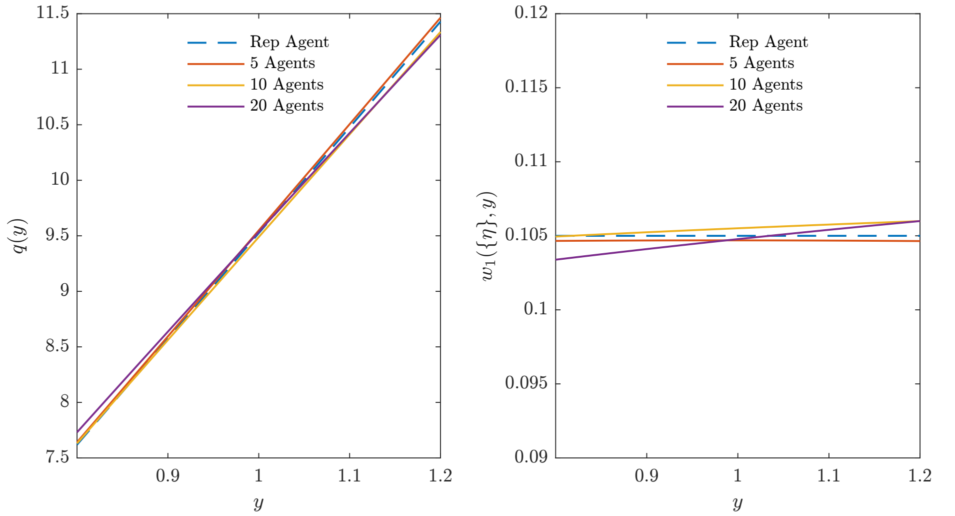
In this case, households are identical so there is a "representative agent".

The model has closed form solution:

$$q(y) = \frac{y}{\rho + (\gamma - 1)\mu - \frac{1}{2}\gamma(\gamma - 1)\sigma^2}$$

$$\omega(y) = \left[\rho + (\gamma - 1)\mu - \frac{1}{2}\gamma(\gamma - 1)\sigma^2\right]$$

[Lucas, 1978] Model solution. MSE: $< 10^{-4}$



As-if Complete Market Model, $\gamma = 5$, $\mu = 2\%$, $\sigma = 5\%$, $\rho = 5\%$.

Limited Participation Model ([Basak and Cuoco, 1998])

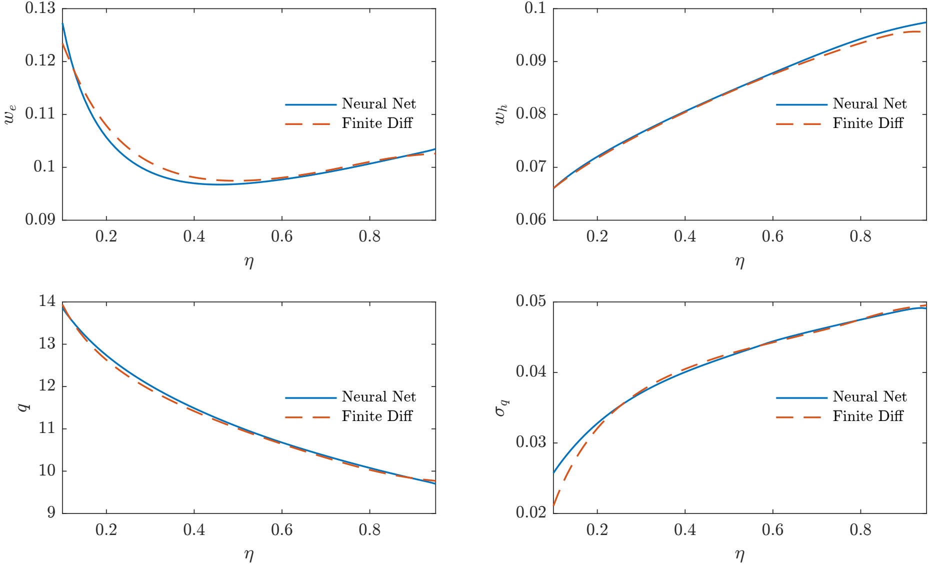
Two types of agents: experts (e) and households (h).

Expert sector can hold stocks and bonds.

Household sector can only hold bonds: $\Psi_h(a_{h,t}, b_{h,t}) = a_{h,t} - b_{h,t} = 0$.

State space is (y, η), where η is expert's wealth share.

[Basak and Cuoco, 1998] Model solution. L2 Loss: $< 10^{-5}$



2 Agents Limited Participation Model, $\gamma = 5, \rho_e = \rho_h = 5\%$, $\mu = 2\%$, $\sigma = 5\%$.

Productivity Gap Model ([Brunnermeier and Sannikov, 2014])

Two types of agents: experts (e) and households (h).

We allow households to hold capital but in a less productive way. The productivity of experts and households is z_h, z_e $(z_h < z_e)$ respectively. Their relative risk-aversion are both γ.

Output grows at exogenous drift $\mu_y = y\mu$, volatilty $y\sigma$, and experts cannot issse outside equities.

State space is (y, η), where η is expert's wealth share.

[Brunnermeier and Sannikov, 2014] Model solution. L2 Loss: $< 10^{-5}$

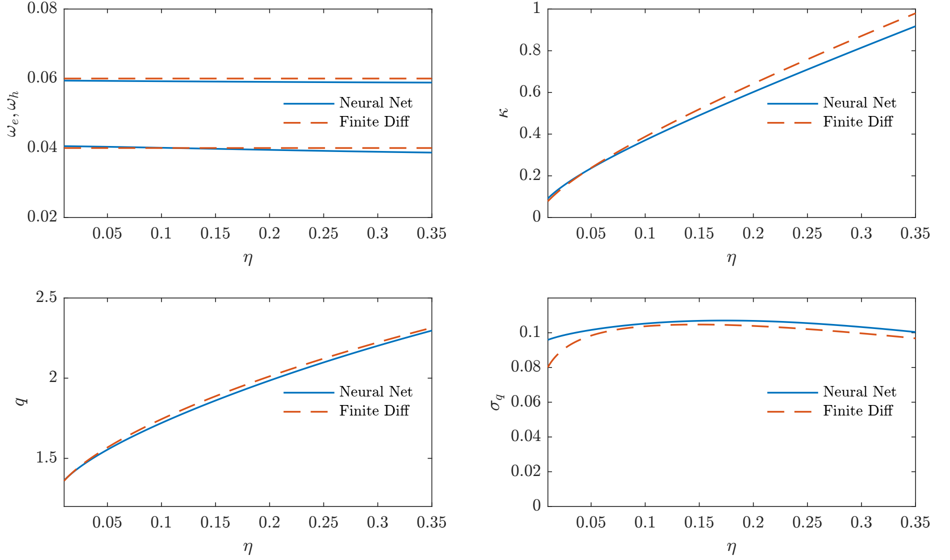
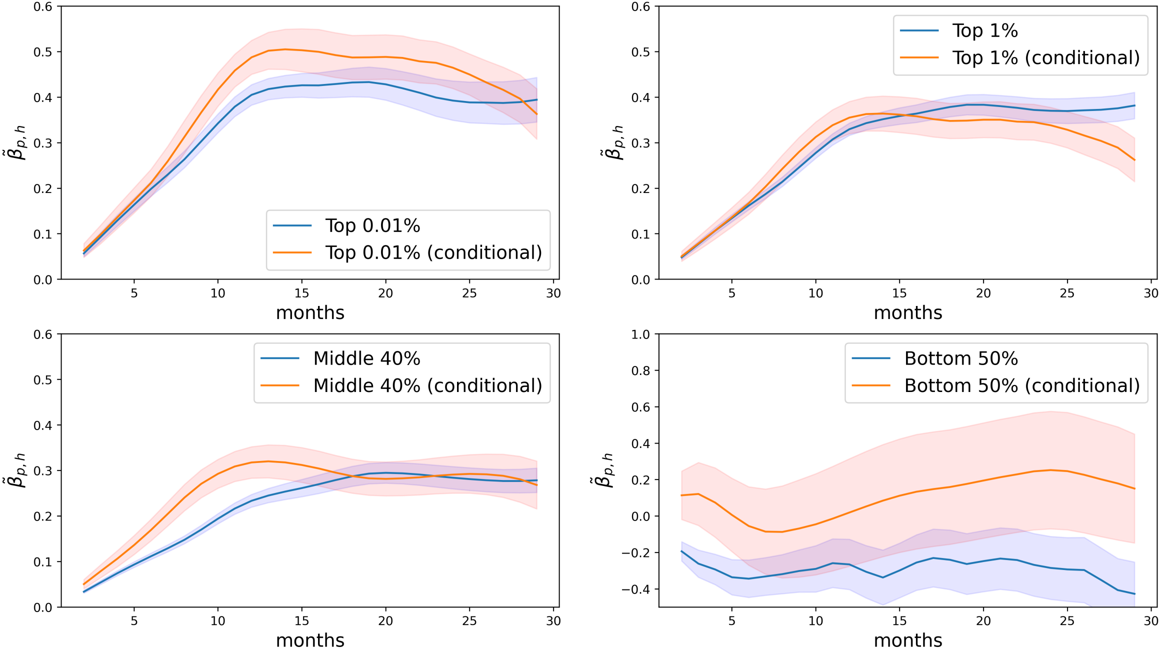


Figure: Solution to the model with productivity gap.



Top 0.01%
Top 0.01% (conditional)
Top 1%
Top 1% (conditional)
Middle 40%
Middle 40% (conditional)
Bottom 50%
Bottom 50% (conditional)
$\tilde{\beta}_{p,h}$
months

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Distributional Impact of Macroprudential Policy

- ⋆ We introduce an exogenous leverage constraint on financial expert: $\theta_e \leq \bar{\ell}$.
- ⋆ Then simulate a collection of recession paths and track the distribution evolution.

Optimization

- ⋆ *Banker problem:* Given their belief about price processes, $(\tilde{\mathsf{r}}, \tilde{\mathsf{q}})$, and initial wealth, $a_{b,0}$, a banker chooses processes $(\mathsf{c}_b, \boldsymbol{\theta}_b, \iota_b)$ to solve the Problem (13) below:

$$\begin{aligned} &\max_{\mathsf{c}_b, \boldsymbol{\theta}_b, \iota_b} \left\{ \int_0^\infty e^{-\rho_b t} u(c_{b,t}) dt \right\} \quad s.t. \\ &\frac{da_{b,t}}{a_{b,t}} = \theta^k_{b,t} d\tilde{R}^k_t + \theta^m_{b,t} d\tilde{R}^m_t + \left((1 - \theta^k_{b,t} - \theta^m_{b,t})\tilde{r}^d_t - c_{b,t}/a_{b,t} - \tau_{b,t}\right) dt \end{aligned} \tag{13}$$

- ⋆ *Fund problem:* Given their belief about price processes, $(\tilde{\mathsf{r}}, \tilde{\mathsf{q}})$, and initial wealth, $a_{f,0}$, a fund manager chooses processes $(\mathsf{c}_f, \boldsymbol{\theta}_f, \iota_f)$ to solve the Problem (14) below:

$$\begin{aligned} &\max_{\mathsf{c}_f, \boldsymbol{\theta}_f, \iota_f} \left\{ \int_0^\infty e^{-\rho_f t} u(c_{f,t}) dt \right\} \quad s.t. \\ &\frac{da_{f,t}}{a_{f,t}} = \theta^k_{f,t} d\tilde{R}^k_t + \theta^m_{f,t} d\tilde{R}^m_t + (1 - \theta^k_{f,t} - \theta^m_{f,t}) d\tilde{R}^n_t + \left(-c_{f,t}/a_{f,t} - \tau_{f,t}\right) dt \end{aligned} \tag{14}$$

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Equilibrium

An equilibrium is a collection of aggregate processes $(\mathbf{K}, \mathbf{r}, \mathbf{q}, G)$ and agent decision processes $(\mathbf{c}_i, \boldsymbol{\iota}_i, \mathbf{k}_i, \mathbf{n}_i, \mathbf{d}_i, \mathbf{l}_i)$:

1. Given beliefs $(\tilde{\mathbf{r}}, \tilde{\mathbf{q}})$, households, bankers, and fund managers optimize.

2. The price processes $(\mathbf{r}, \mathbf{q})$ satisfies market clearing conditions at each time t: (Where capital letters refer to sector aggregates.)
   (i) Goods market: $C_{h,t} + C_{b,t} + C_{f,t} + \lambda \mathcal{C}_{h,t} = e^{z_t} K_t - \iota_t K_t$,
   (ii) Pension share, deposit, and loan markets: $\sum_i N_{i,t} = \sum_i D_{i,t} = 0$,
   (iii) Capital market: $\sum_i K_{i,t} = K_t$,
   (iv) Bond market clears: $B_{b,t} + B_{f,t} = B$

3. Agent beliefs are consistent with equilibrium $(\tilde{\mathbf{r}}, \tilde{\mathbf{q}}) = (\mathbf{r}, \mathbf{q})$.

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Recursive Characterization of Equilibrium: (in Wealth Levels)

- ⋆ Individual state $= a_i$, Aggregate states $= (z, \zeta, K, \{a_j\}_{j\neq i}) = (\cdot)$.
- ⋆ Given belief about evolution of other agents, $(\tilde{\mu}_{a_j}(\cdot), \tilde{\sigma}_{a_j}(\cdot))$, household i chooses (c_i, ι_i) and asset wealth shares $\theta_i^k := q^k k_i / a_i$, $\theta_i^n := q^n n_i / a_i$, $\theta_i^d := d_i / a_i$ to solve:

$$
\begin{aligned}
(\rho+\lambda)V(a_i,\cdot) = \max_{c_i,\boldsymbol{\theta}_i,\iota_i} \Big\{ & u(c_i) + \nu(\theta_i^d, a_i) - \psi(\theta_i^k, a_i) + \lambda \left(\mathcal{U}(\mathcal{C}_i; \theta_i^n, \theta_i^k) - V(a_i,\cdot)\right) \\
& + \partial_{a_i} V(a_i,\cdot)\mu_{a_i}(a_i, c_i, \boldsymbol{\theta_i}, \cdot) + \partial_z V(a_i,\cdot)\mu_z + \partial_\zeta V(a_i,\cdot)\mu_\zeta + \textstyle\sum_{j\neq i} \partial_{a_j} V(a_i,\cdot)\tilde{\mu}_{a_j}(\cdot) \\
& + 0.5\left(\partial^2_{a_i^2} V(a_i,\cdot)\boldsymbol{\sigma}_{a_i}^T \boldsymbol{\sigma}_{a_i}(a_i,\boldsymbol{\theta}_i,\cdot) + \partial^2_{z^2} V(a_i,\cdot)\sigma_z^2 + \partial^2_{\zeta^2} V_i(a_i,\cdot)\sigma_\zeta^2\right) \\
& + \partial^2_{a_i z} V(a_i,\cdot)\sigma_{a_i,z}(a_i,\boldsymbol{\theta}_i,\cdot)\sigma_z + \partial^2_{a_i\zeta} V(a_i,\cdot)\sigma_{a_i,\zeta}(a_i,\boldsymbol{\theta}_i,\cdot)\sigma_\zeta + \partial^2_{z\zeta} V(a_i,\cdot)\sigma_z\sigma_\zeta + \sigma_\lambda\sigma_V \\
& + \sum_{j\neq i} \partial^2_{a_j z} V(a_i,\cdot)\tilde{\sigma}_{a_j,z}(\cdot)\sigma_z + \sum_{j\neq i} \partial^2_{a_j\zeta} V(a_i,\cdot)\tilde{\sigma}_{a_j,z}(\cdot)\sigma_z + 0.5 \sum_{j\neq i, j'\neq i} \partial^2_{a_j,a_{j'}} V_i \tilde{\boldsymbol{\sigma}}_{a_j}^T \tilde{\boldsymbol{\sigma}}_{a'_j}(\cdot) \Big\}
\end{aligned}
$$

- ⋆ Banker and fund HJBEs are similar but without ν or ψ_i terms.
- ⋆ Equilibrium belief consistency becomes: $(\hat{\mu}_{a_j}(\cdot), \hat{\sigma}_{a_j}(\cdot)) = (\mu_{a_j}(\cdot), \sigma_{a_j}(\cdot))$.

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Recursive Characterization of Equilibrium: (in Wealth Levels)

- ⋆ Individual state $= a_i$, Aggregate states $= (z, \zeta, K, \{a_j\}_{j \neq i}) = (\cdot)$.
- ⋆ Given belief about evolution of other agents, $(\tilde{\mu}_{a_j}(\cdot), \tilde{\sigma}_{a_j}(\cdot))$, household i chooses (c_i, ι_i) and asset wealth shares $\theta_i^k := q^k k_i / a_i$, $\theta_i^n := q^n n_i / a_i$, $\theta_i^d := d_i / a_i$ to solve:

$$
\begin{aligned}
\rho_h V(a_i, \cdot) = \max_{c_i, \boldsymbol{\theta}_i, \iota_i} \Big\{ & u(c_i) + \nu(\theta_{i,t}^d, a_i) - \psi(\theta_i^k, a_i) + \lambda \left(\mathcal{U}(\mathcal{C}_i; \theta_i^n, \theta_i^k) - V(a_i, \cdot) \right) \\
& + (\mathcal{L}_h V)(a_i, \cdot, c_i, \boldsymbol{\theta}_i) + \textstyle\sum_{j \neq i} \partial_{a_j} V(a_i, \cdot) \tilde{\mu}_{a_j}(\cdot) \\
& + \sum_{j \neq i} \partial^2_{a_j z} V(a_i, \cdot) \tilde{\sigma}_{a_j, z}(\cdot) \sigma_z + \sum_{j \neq i} \partial^2_{a_j \zeta} V(a_i, \cdot) \tilde{\sigma}_{a_j, z}(\cdot) \sigma_z + 0.5 \sum_{j \neq i, j' \neq i} \partial^2_{a_j, a_{j'}} V_i \tilde{\boldsymbol{\sigma}}_{a_j}^T \tilde{\boldsymbol{\sigma}}_{a_j'}(\cdot) \Big\}
\end{aligned}
$$

- ⋆ Banker and fund HJBEs are similar but without ν or ψ terms and with log utility and closed form solution.
- ⋆ Equilibrium belief consistency becomes: $(\tilde{\mu}_{a_j}(\cdot), \tilde{\sigma}_{a_j}(\cdot)) = (\mu_{a_j}(\cdot), \sigma_{a_j}(\cdot))$.

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Market for Pension Shares (Vayanos-Vila Preferences)

Individual state $= a_i$, Aggregate states $= (z, \zeta, K, \{a_j\}_{j\neq i}) = (\cdot)$ Recursive characterization

Let $V_j(a_j, \cdot)$ denote value function for type $j \in \{h, b, f\}$ and let $\xi_j = \partial_{a_j} V_j(a_j, \cdot)$.

Then the FOCs in the pension share market:

$$\underbrace{r^n - r^l}_{\text{Excess return}} + \underbrace{\frac{\lambda}{q^n} \exp\left(-\alpha \frac{q^k}{q^n} \theta_i^n \eta_i\right)}_{\text{"Preferred habitat" component}} = -\underbrace{\boldsymbol{\sigma}_{\xi_i} \cdot \boldsymbol{\sigma}_{q^n}}_{\text{"shifter"}} \qquad \ldots \text{Household FOC}$$

$$\underbrace{r^n - r^l}_{\text{Excess return}} = -\underbrace{\boldsymbol{\sigma}_{\xi_f} \cdot \boldsymbol{\sigma}_{q^n}}_{\text{Comovement of SDF and price}} \qquad \ldots \text{Fund FOC}$$

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Roadmap

Solutions to example models

Macroprudential Policy and Inequality

Recursive Equilibrium

Block 1: Optimization

- Given price processes $(r, r_k, q, \mu_q, \sigma_q)$, household optimization implies:

$$\textit{Euler equation:} \quad \rho_h \xi_h = r\xi_h + \frac{\partial \xi_h}{\partial z}\mu_z + \frac{\partial \xi_i}{\partial K}\mu_K + \frac{1}{2}\frac{\partial^2 \xi_h}{\partial z^2}\sigma_z^2 + \sum_j \frac{\partial \xi_h}{\partial \eta_j}\eta_j \mu_{\eta_j,t} + \sum_j \frac{\partial^2 \xi_h}{\partial z \partial \eta_j}\eta_j \sigma_{\eta_j,t}\sigma_z + \frac{1}{2}\sum_{j,j'} \frac{\partial^2 \xi_h^2}{\partial \eta_j \partial \eta_{j'}}\eta_j \eta_{j'}\sigma_{\eta_j,t}\sigma_{\eta_{j'},t}$$

$$\textit{Consumption FOC:} \quad \xi_h = u'(c_h)$$

$$\textit{Portfolio FOC:} \quad \xi_h(r_k - r) = -\left(\frac{\partial \xi_h}{\partial z}\sigma_z + \sum_j \frac{\partial \xi_h}{\partial \eta_j}\sigma_{j,\eta}\right)\sigma_q - \frac{\partial \Psi_h}{\partial k_i}$$

- Expert optimization is similar but adjusted for Epstein-Zin More

Block 2: Distribution Evolution

- Given the prices $(r, r_k, q, \mu_q, \sigma_q)$ and (c, ξ, k), the law of motion of wealth shares is given as

$$\frac{d\eta_{j,t}}{\eta_{j,t}} = \mu_{\eta_j,t}dt + \sigma_{\eta_j,t}dW_t$$

where:

$$\mu_{\eta_j,t} = r_t + \theta_{j,t}(r_{k,t} - r_t) - \omega_{j,t} - \mu_{q,t} - \mu_{K,t} + (1-\theta_{j,t})\sigma_{q,t}^2 + \lambda\tau\left(\frac{\frac{1}{I-1}(1-\eta_{j,t})}{\eta_{j,t}} - 1\right)$$

$$\sigma_{\eta_j,t} = -(1-\theta_{j,t})\sigma_{q,t}$$

where

- $\theta_{k,t} := k_{j,t}/(\eta_{j,t}q_tK_t)$ is agent j's share of wealth in capital,
- $\omega_{j,t} := c_{j,t}/(\eta_{j,t}q_tK_t)$ is agent j's consumption-to-wealth ratio.

Block 3: Equilibrium Consistency

- Clearing conditions pin down the prices:

$$\sum_i c_{i,t} + \Phi(\iota_t)K_t = y_t \qquad \sum_i (1-\theta_{i,t})a_{i,t} = 0 \qquad \sum_i \theta_{i,t}a_{i,t} = q_t K_t$$

- But q process is implicit so we must impose consistency conditions on q to close the model:

$$q\mu_{q,t} = \sum_j \frac{\partial q}{\partial \eta_j}\eta_j \mu_{\eta_j,t} + \frac{\partial q}{\partial z}\mu_{z,t} + \frac{\partial q}{\partial K}\mu_{K,t} + \sum_j \frac{\partial^2 \xi_i}{\partial z \partial \eta_j}\eta_j \sigma_{\eta_j,t}\sigma_z$$
$$+ \frac{1}{2}\sum_{j,j'} \frac{\partial^2 q}{\partial \eta_j \partial \eta_{j'}}\eta_j \eta_{j'}\sigma_{\eta_j,t}\sigma_{\eta_{j'},t} + \frac{1}{2}\frac{\partial^2 q}{\partial z^2}\sigma_z^2$$
$$q\sigma_{q,t} = \sum_j \frac{\partial q}{\partial \eta_j}\eta_j \sigma_{\eta_j,t} + \frac{\partial q}{\partial z}\sigma_{z,t}$$