The Disruption of Long Term Bank Credit

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Abstract

This paper studies the disruption of bank business credit during a financial crisis in a model with optimal long term contracting under agency frictions and a directed search market for bank funding. Banks commit to long term contracts with entrepreneurs but then face heterogeneous shocks to their cost of raising funds during a crisis. The optimal contract can be implemented using standard debt securities and a "covenant" that allows bankers with high funding costs to adjust debt terms once the entrepreneur has accumulated sufficiently many losses. This is consistent with empirical evidence from the recent financial crisis. In equilibrium, the contracting frictions amplify the crisis by increasing the project termination rate and decreasing the financing rate. The model is extended to incorporate working capital and project heterogeneity. The frictions then reduce project size and skew the economy towards lower volatility projects during the crisis.

1 Introduction

A large empirical literature has documented how problems in the banking sector were transmitted to the real economy during the 2007-9 financial crisis. This has generated a wave of macroeconomic models with financial sector frictions that amplify business cycle fluctuations through the expansion and contraction of credit.[1] One limitation of these models is that they typically abstract from the details of bank-firm contracting by imposing representative financial intermediaries and short-term bank-firm loans. This paper constructs a equilibrium model of the market for long-term bank credit that is consistent with stylised

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[1] For example, see Bernanke, Gertler, and Gilchrist [1999], Gertler and Kiyotaki [2010], He and Krishnamurthy [2013] and Brunnermeier and Sannikov [2014].

empirical facts on how banks reduced credit to firms during the financial crisis. This model is then used to explore the impact of bank credit disruption on aggregate output and the distribution of firms.

I focus on the following stylised facts from the empirical research on bank-firm lending. Firstly, the majority of loans take the form of credit lines or long-term debt contracts, both of which have covenants that give the banker additional control rights upon violation (Sufi [2009], Roberts and Sufi [2009] and Acharya, Almeida, Ippolito, and Perez [2014]). Secondly, bankers were heterogeneously exposed to adverse shocks during the financial crisis (Chodorow-Reich [2014] and Begenau, Bigio, and Majerovitz [2017]). Thirdly, bankers used loan covenants to condition contracts on their idiosyncratic crisis exposure. That is, "good" banks provided "waivers" for covenant violations during the crisis whereas "bad" banks increased interest rates, recalled debt and sometimes terminated loan contracts (Chodorow-Reich and Falato [2017]). Fourthly, bank-firm relationships are sticky, suggesting that it was costly for businesses to switch from "bad" banks to "good" banks during the crisis (Chodorow-Reich [2014] and Schwert [2018]).

I model these stylised facts by embedding a long-term bank-entrepreneur contracting problem into a directed search market that is subject to aggregate crisis shocks. The contracting problem extends DeMarzo and Sannikov [2006] to incorporate lender funding cost risk. Entrepreneurs have idiosyncratically risky long-term project opportunities but no resources to undertake the initial investment. Bankers are able to raise funds in order to provide financing to the entrepreneurs but their cost of funds is idiosyncratically risky. In the good aggregate state, all bankers can raise funds at low cost but, when the crisis occurs, some bankers "go bad" and can only raise funds at high cost. This generates heterogeneity amongst bankers during the crisis, in line with the second stylised fact. Given these shock processes, bankers contract optimally with entrepreneurs subject to three frictions: the entrepreneur can privately steal resources from the project (a moral hazard problem), the entrepreneur can walk away from the contract (a lack of entrepreneur commitment) and it is costly for the banker to condition the contract on their idiosyncratic cost of funds. In this case, the optimal contract has an implementation that is consistent with with first and third stylised facts: the banker provides financing to the entrepreneur by making a long term loan, setting up a credit line and imposing two thresholds based on the entrepreneur's draw on the credit line[2]. If the entrepreneur's draw exceeds the first threshold, then the project is flagged as having breached a covenant and the banker gains the right to adjust the debt contracts based on their funding cost. If the second threshold is breached, then the project is terminated.

Bankers and entrepreneurs match bilaterally in a credit market to sign contracts but the process has frictions, which I model using directed search, as in Moen [1997]. In this market, bankers post contracts, then entrepreneurs observe the contracts and choose where to target their search. The two parties

[2]For certain parameters, the entrepreneur will also issue outside equity.

then meet according to a constant returns to scale matching function. This makes it costly for entrepreneurs to leave a bank and search for a new contract, in line with stylised fact four. From the perspective of the contracting problem, the introduction of the credit market endogenises both the outside option of the entrepreneur and the initial contract terms. So, in equilibrium, it is the frictions in the contracting problem that determine the rate of financing in the search market. From the perspective of the macroeconomy, this generates a distribution of funded projects that differ by their current draw on their credit line (and so their likelihood of termination) and whether or not they have a covenant flag (and so the banker can easily adjust the contract when their cost of funds changes).

The contracting and matching frictions combine to create a credit disruption channel. When a crisis occurs, projects become less attractive to bankers, either because they have received an adverse cost of funds shock or because they believe they may receive one in the future. Bankers respond by recalling debt from entrepreneurs, potentially forcing costly early liquidation of long-term projects if the entrepreneur has already made sufficiently large draws on their credit line. As a result of these risks, bankers become more reluctant to fund projects leading to a lower rate of entrepreneur matching in the search market. So, in aggregate, the frictions distort the economy during a crisis by amplifying the rate of project termination and decreasing the rate of project financing, both of which lead to lower output.

The role of the covenant threshold in the credit disruption process is to characterise the extent to which good and bad banks cut credit differently during a crisis. If the optimal covenant threshold is low, then few projects are flagged and so good and bad banks are typically constrained to reduce credit by the same moderate amount. By contrast, if the optimal covenant threshold is high, then most projects have flags allowing bad banks to cut credit by relatively more than the good banks. This means that a higher covenant threshold leads to a greater dispersion in debt recalls during a crisis, which, in the macroeconomy, increases the aggregate rate of project termination and decreases the ratio of bad operating banks to good operating banks.

I calibrate the model to match Chodorow-Reich and Falato [2017], who estimate the frequency at which "covenants" bind on bank-firm contracts and the relative difference in credit reduction at good and bad banks when covenants bind. Although my model is highly stylised, I find suggestive evidence that both the credit disruption channel and the covenant flags have significant quantitative effects. Overall, the search and contracting frictions decrease output during a crisis by approximately 5 percentage points, while varying the covenant threshold from minimum to maximum potentially adjusts the output decrease by approximately ± 2 percentage points.

I consider two extensions to my baseline model. The first extension introduces project heterogeneity. In this case, entrepreneurs can have one of two project types: "safer" projects which have relatively low mean return and volatility, and "riskier" projects which have relatively high mean return and volatility. In equilibrium, bankers choose a higher covenant threshold for the higher volatility projects. This means that during a crisis the termination rate of risky projects increases and the financing rate decreases skewing the economy towards the lower volatility projects.

The second extension introduces working capital. In this case, bankers fund the initial investment to start the project and also provide working capital each period, which increases output according to a decreasing returns to scale production function. This means that the search and contracting frictions now reduce credit on both the extensive and intensive margins. The reduction on the intensive margin comes from two effects. Firstly, the search friction means that entrepreneurs may stay with bad banks who have high funding costs and so provide lower working capital. Secondly, bankers do not want to extend working capital to projects near their termination threshold because it will amplify project productivity shocks and further increase the likelihood of termination. This second effect has a complex impact on output decline during a crisis relative to the decline under first best contracting. On the one hand, the crisis shifts projects closer to their termination threshold which leads to a reduction in working capital. On the other hand, the agency frictions mean that many projects start a crisis with low working capital and so their capital reduction ends ups being small, even if their bank gets an adverse cost of funds shock.

Related Literature: My paper connects to the literature studying how bank-firm relationships evolve over the business cycle. The closest paper in that literature is Boualam [2018], which also combines optimal dynamic contracting with a directed search market. The contribution of my paper is the introduction of an alternative contracting model that features idiosyncratic project risk, heterogenous project types and endogenous early termination.[3] This means that my model differs by featuring both entrepreneur heterogeneity and banker heterogeneity, the interaction of which ends up creating the role for credit lines with covenant thresholds and an amplification of output decline during a crisis. Having idiosyncratic risk along both the lender and borrower dimensions is also a point of difference to the wider banking literature without search frictions. My baseline model complements the credit line literature (e.g. Holmström and Tirole [1998], Acharya et al. [2014]) by providing an additional reason why credit lines are hard to provide during a crisis and my working capital extension complements the relationship banking literature (e.g. Sharpe [1990], Rajan [1992], Bolton, Freixas, Gambacorta, and Mistrulli [2016]) by providing a different channel for how long term bank-entrepreneur attachments can be costly. Finally, there have also been recent money theory papers which focus on the transmission of monetary policy through financial frictions in a random search environment (e.g. Rocheteau, Wong, Zhang, et al. [2018a] and Rocheteau, Wright, and Zhang [2018b]). Although my paper does not study monetary policy, it could be seen as providing an alternative set of tools for connecting long term bank-firm contracting with search frictions.

[3] Boualam [2018] use a version of Albuquerque and Hopenhayn [2004] with risk-averse entrepreneurs and exogenous termination.

More broadly, my paper is part of a large macro-finance literature that attempts to connect frictions in the financial sector to behavior in the macroeconomy. Many papers in that literature have studied how frictions on bank fund raising can create a feedback effect that amplifies economic downturns (e.g. Gertler and Kiyotaki [2010] He and Krishnamurthy [2013], Brunnermeier and Sannikov [2014]). In these models, banks accumulate losses and then cut back on lending in order to maintain a low leverage ratio. In aggregate, this decreases the price of bank assets which causes further losses and deleveraging. By contrast, I focus on how frictions in the contracting between banks and firms can amplify crises when switching between banks is costly. This attempts to makes precise a "folk" story in the banking literature that there are real costs to credit disruption. In this sense, my paper can be seen as complementary to the financial accelerator literature by focusing on frictions in a different part of the bank's balance sheet.

From a technical perspective, I study a bank-entrepreneur contracting problem that uses the continuous time long-term contracting theory developed in DeMarzo and Sannikov [2006], Sannikov [2008] and Cvitanic and Zhang [2013], who show how to express a continuous time dynamic contracting problem recursively. In particular, the environment in DeMarzo and Sannikov [2006] is extended to incorporate Poisson shocks to the banker's cost of raising funds. This is similar to the methodology used in Piskorski and Tchistyi [2010] and DeMarzo, Fishman, He, and Wang [2012], who introduce Poisson shocks to the lender's discount rate and to aggregate TFP respectively. The main technical differences in my paper are the introduction of flagged and unflagged regimes and the embedding of the contracting problem into equilibrium. Mathematically, the flags turn the model into an optimal regime switching problem, which is solved using tools from Pham [2009]. There are also four smaller extensions in my contracting problem: the introduction of working capital, the coexistence of endogenous and exogenous project termination, the economic interpretation of the Poisson shocks as the bank's cost of raising deposits and the form of the implementation.

The directed search market is solved using the tools developed by Moen [1997], Shi [2002, 2009] and Menzio and Shi [2010, 2011]. The specific continuous time formulation uses the set up from Gilbukh and Roldan [2018].

Outline: The paper is structured in the following way. Section 2 describes the environment. Section 3 solves the bank-entrepreneur contracting problem. Section 4 puts the contracting problem into a directed search market for bank credit. Section 5 calibrates the model and then performs a series of crisis experiments. Section 6 extends the model to incorporate project heterogeneity and working capital. Section 7 concludes.

2 Environment

Time is continuous and the economy has an infinite horizon. There is a single, perishable consumption good. The economy is populated by a mass-one continuum of risk-neutral, infinitely-lived entrepreneurs and a continuum of bankers. There is free entry into the banking sector so the size of the banking sector is determined in equilibrium. Bankers raise funds from an exogenous deposit market and then provide those funds to entrepreneurs.

2.1 Aggregate State Process

Let $z \in \mathcal{Z} \equiv \{z_G, z_B\}$ denote the aggregate state of the economy. The state, z, follows a continuous time two-state Markov process with switching rate $\lambda_z(z_t)$. The aggregate state affects both the productivity in the economy and the rate at which banks receive adverse shocks. The "bad" state z_B will be interpreted as a "financial crisis", in which case productivity drops and banks get more adverse shocks.

2.2 Entrepreneurs's Project Technology

Entrepreneurs are risk neutral with discount rate ρ_e. Each entrepreneur has a project opportunity that requires an initial investment, I, and generates a project of unit size. While in operation, the project generates cumulative output, y_t, according to the process:

$$dy_t = zA(1 - \xi_t)dt + \sigma dB_t$$

where B_t is an idiosyncratic Brownian motion process, z is the aggregate productivity of the economy, A is the average productivity of the project and $\xi_t \in \{0, 1\}$ is a control variable chosen by the entrepreneur that gives them a benefit at rate $\beta z A \xi_t$ with $\beta \in (0, 1)$. Throughout this paper, the variable ξ_t will be referred to as the fraction of flow output that is "stolen" by the entrepreneur but it could also be interpreted as "shirking" by the entrepreneur. Observe that since $\beta \in (0, 1)$, stealing is never efficient.

A key friction in the economy is an information asymmetry between the entrepreneur and the banker. All variables about the entrepreneur's project are public information except for ξ_t and B_t, which are private information for the entrepreneur. Instead, the banker must rely upon the entrepreneur's reporting process, $\widetilde{B}_t$, where:

$$\sigma d\widetilde{B}_t = \sigma dB_t - A\xi_t dt$$

This means that the banker cannot tell whether flow output, dy_t, is low because there was a negative project productivity shock or because the entrepreneur has stolen output.

There are two ways that the project can be terminated. Firstly, the project terminates exogenously at rate λ_c in which case it can be liquidated for full value 1[4]. The project can also be endogenously terminated early and liquidated for $L < 1$.

There is heterogeneity within the population of entrepreneurs. Different entrepreneurs have project opportunities with different idiosyncratic project productivity, A, and project volatility, σ. The characteristics of the entrepreneur are denoted by $\mathbf{x} \equiv (A, \sigma)$ and the set of possible entrepreneur types in the economy is denoted by $\mathcal{X}$.

Finally, entrepreneurs cannot save or borrow directly from the deposit market. Instead, they must raise funds from bankers.[5].

2.3 Banker's Funding Technology

Bankers are risk neutral with relatively low discount rate $\rho_h < \rho_e$. Unlike the entrepreneur, the banker has access to a deposit market where they can raise funds. However, since the focus of this model is the bank-entrepreneur contracting problem, restrictions are placed on the bankers' balance sheets in order to simplify the model. Bankers do not hold equity. Instead, they raise debt, D, in order to fund the initial investment, I. The entrepreneur then keeps the debt level constant, potentially with negative consumption if the project flows are negative.[6]

The key feature of the bankers' simplified balance sheets is that they face individual funding cost risk. The interest rate they pay on their deposits follows an exogenous process:

$$r_t = \overline{r} + \epsilon_t$$

where $\epsilon_t \in \mathcal{E} \equiv \{\epsilon_G, \epsilon_B\}$ follows an idiosyncratic continuous time two-state Markov process with switching rate, denoted by $\lambda_\epsilon(\epsilon, z)$, that depends on both ϵ and the aggregate state z. It is imposed that $\epsilon_B > \epsilon_G$ so that the "G" subscript refers to the "good" idiosyncratic bank state, in which case the bank's cost of funds is low, and the "B" subscript refers to the "bad" idiosyncratic bank state, in which case the bank's cost of funds is high. In the good aggregate state, banks never get a shock that increases their cost of funds. That is, $\lambda_{\epsilon_G}(\epsilon_G, z) = 0$ and so all banks have ϵ_G. By contrast, in the bad aggregate state, the banks

[4]Exogenous termination is introduced in order to be able to match the average length of a debt contract.

[5]This is not an important constraint for the contracting problem (see DeMarzo and Sannikov [2006] and **?**, who study how private entrepreneur saving affects the problem with risk neutral and risk averse agents) but it is necessary to ensure that entrepreneurs cannot save their way out of needing to go to the search market.

[6]There are a number of possible interpretations of negative consumption in this environment. One interpretation is that the entrepreneur can raise equity in order to cover project losses but not to fund initial project investment. For example, the bank may have an equity buffer that is only used to cover losses. An alternative interpretation is that the entrepreneur is effectively "diversified" across projects even though there is bilateral matching. This follows because $\mathbb{E}[dB_t] = 0$ and so the positive or negative project shocks, dB_t, have no impact on the expected utility of the risk neutral banker. That is, the banker is contracting "as if" they are diversified across a continuum of projects and so the output volatility affects the moral hazard problem with the entrepreneur but not the balance sheet of the bank.

get shocks that increase their cost of funds with positive probability. That is, $\lambda_{\epsilon_G}(\epsilon_G, z) > 0$. I impose the parameter restriction that $z_B A - r(\epsilon_B)I > 0$.

The banker receives their initial cost of funds draw from the stationary distribution of ϵ after the contract terms have been set. The stationary distribution depends on the aggregate state and is denoted by $\overline{\pi}_\epsilon(\cdot|z)$. The bankers' cost of funds are reshuffled when the aggregate state changes. That is, when the bad state arrives, all the bankers get a new cost of funds draw from the stationary distribution of ϵ in the bad aggregate state. Likewise, when the good state arrives, all the banks revert to a low cost of funds.

The entrepreneur knows the aggregate state and the stationary distribution in each aggregate state, $\{\overline{\pi}_\epsilon(z) : z \in \mathcal{Z}\}$, but cannot observe the bankers cost of funds or the banker's consumption and has no independent way of learning about either of these variables. The banker can credibly reveal their cost of funds to the entrepreneur but the process is costly, as described in subsection 2.4.

2.4 Contracting Technology

Upon meeting, the banker and entrepreneur sign a contract that provides financing for the project. Under the contract, the banker gives the entrepreneur the funds required to start the project, I, and promises the entrepreneur a lifetime value, w_0. In exchange, the entrepreneur must report the project output shock process, $\widetilde{B}_t$, to the banker and the banker has the right to control the termination time of the project and the division of reported flows from the project. The banker can commit to the contract but the entrepreneur cannot. Instead, at any time, the entrepreneur may abandon the contract and return to the search market. The value to the entrepreneur of being in the search market is denoted by W_t^U and will be determined in equilibrium.

The banker has the following contracting technology for revealing information and conditioning the contract on stochastic processes. The banker can always costlessly condition the contract on the entrepreneur's project output reports, $\widetilde{B}$, and the aggregate state, z, but must pay to condition the contract on the idiosyncratic component of their funding cost, ϵ. Formally, contracts have a "covenant flag", denoted by $f \in \{0, 1\}$. A flag $f = 0$ indicates that the banker is not revealing their ϵ and cannot condition the contract on ϵ. By contrast, a flag $f = 1$ indicates that the banker is revealing ϵ and has the option to condition the contract on ϵ. All contracts start with a flag $f = 0$ at project initialisation. The banker then chooses a stopping time at which point the flag is set to $f = 1$ and the banker incurs a lump sum cost, $\Phi(z)$, which potentially depends upon the aggregate state of the economy.[7] The flag then stays at $f = 1$ until the contract is terminated. Conceptually, the flag changing to $f = 1$ will be interpreted as a "covenant" being triggered on the contract which gives the bank additional control rights over the project. Mathematically, the flags rep-

[7]This will be necessary in the calibration in order to match data on the frequency of covenants binding in each state.

resent different "regimes" under which the state variables of the problem evolve according to different stochastic processes.

Although the cost, Φ, is not being modeled explicitly in this paper, there are many possible interpretations. It could represent an administrative cost for triggering a covenant on a loan. Alternatively, it could be interpreted as a loss of market value that the bank experiences when it publicly reveals it has suffered an adverse shock. This could be due to government regulation or commercial pressures. A richer model might incorporate the work of **?**, who suggest that banks derive their value from being secret keepers and not revealing their idiosyncratic information to the market.

2.5 Bank Contract Market

Banks and entrepreneurs match bilaterally in a decentralised directed search market. Banks post contracts in order to attract entrepreneurs. Entrepreneurs observe the posted contracts and choose where to direct their search. Neither entrepreneurs nor bankers can coordinate in the search market.

In section 3, I will show that the expected lifetime value promised to the entrepreneur by the contract, W, is a sufficient statistic for the posted contracts. Thus, without loss of generality, a bank is referred to as posting a promised value in the credit market and the credit markets are organised into a continuum of submarkets, each consisting of a collection of bankers and entrepreneurs, and indexed by the promised value, W, implied by the contract. Within a submarket, contracts are allocated according to a matching function, $M(E, B)$, where E and B denote the measure of entrepreneurs and bankers in the search market respectively. M can be interpreted as a reduced form way of modeling the banker's screening technology and standards. It is imposed that the function M is continuous, concave, and homogeneous of degree one in both variables. The tightness of submarket W is defined to be the ratio of entrepreneurs to bankers, $\theta(W) \equiv E/B$. The rate at which entrepreneurs find bankers is then given by $\mu = M(E, B)/E = M(1, \theta) \equiv \mu(\theta)$ and the rate at which bankers find entrepreneurs is given by $\eta = M(E, B)/B = M(1/\theta, 1) \equiv \eta(\theta)$. This implies that $\eta(\theta) = \theta\mu(\theta)$. Banks can freely enter the credit market. However, they must pay a flow cost, ζ, while posting a contract.

2.6 Discussion of Model Environment and Relation to Stylised Facts

The model environment has been set up to qualitatively match the four stylised empirical facts I discussed in the introduction: 1) most bank loans are credit lines or long-term debt contracts with a covenant threshold, 2) bankers were heterogeneously exposed to the financial crisis, 3) bankers used covenants as an option to condition debt terms on their idiosyncratic crisis exposure and 4) it is costly for lenders to switch banks. Stylised facts 2) and 4) come directly from the specification of the banker's funding cost process and the introduction of the search friction in the market for bank contracts. As I will show in the next section, facts 1) and 3) come from the solution to the optimal contracting problem in this environment.

3 Contracting Problem

This section describes and solves the optimal contracting problem between a banker and an entrepreneur after they have matched in the search market. I present the problem heuristically here in the main text. A more technical description and the relevant proofs are provided in appendix A.

Methodologically, my approach builds on the contracting models of DeMarzo and Sannikov [2006], Piskorski and Tchistyi [2010] and DeMarzo et al. [2012] who show how to set up a continuous time bank-entrepreneur contracting problem recursively. I choose this style of contracting environment so that the optimal contract can be implemented with securities that are consistent with stylised fact 1). That is, with a combination of a credit line, a long-term debt contract, and collection of endogenous thresholds that give the banker additional control rights when the project accrues sufficiently many net accumulated losses. My main point of departure from their work is the introduction of regime shifting as the covenant flag is triggered. Conceptually, I do this to be able to match fact stylised 3. That is, bankers condition credit reduction on their individual crisis exposure once the project has breached financial covenants. From a technical perspective, solving the problem in regime 0 is both novel and difficult because the banker has to choose contract terms that cannot be conditioned on their idiosyncratic state. Other minor differences to the DeMarzo and Sannikov [2006] contracting problem include the coexistence of endogenous and exogenous project termination and the economic interpretation of the Poisson shock process. In the next section, I extend the DeMarzo and Sannikov [2006] environment further by putting the contracting problem into general equilibrium.

3.1 First Best Contract

In order to understand how the agency frictions distort the contracting problem, it is helpful to start by considering the contracting problem under full information. In this case, since $\rho_e > \rho_h$, the first best allocation can be implemented with 100% outside equity financing. That is, the banker purchases the project from the entrepreneur and then hires the entrepreneur back to run the project. This result is formalised in Lemma 1.

Lemma 1 (First Best Contract). *Suppose that all processes are publicly observable and the banker has promised W in the search market. Then, under the optimal contract, the banker pays W to the entrepreneur at contract initiation, receives all future resource flows from the project until exogeneous termination, and enforces entrepreneur action $\xi_t = 0$. This contract can be implemented by the banker purchasing the project from the entrepreneur for price W and then hiring the entrepreneur to run the project with no stealing, $\xi_t = 0$.*

Proof. See Appendix A.2. □

The key feature of the first best contract is that the banker's payout and termination decisions do not depend on the idiosyncratic project shocks. That is, the banker only makes payments to the entrepreneur at contract initialisation and never terminates early, regardless of how many negative shocks the project accumulates.[8] In this case, the banker's value function at contract initialisation is given by:

$$V(W, \epsilon, z) = S(\epsilon, z) - W$$

where W is the initial payment to the entrepreneur and $S(\epsilon, z)$ is the expected present discounted value of the future surplus generated by the project under the banker's discount rate. The surplus value function, $S(\epsilon, z)$, is characterised by the Hamilton-Jacobi Bellman equation:

$$\begin{aligned} \rho_h S(\epsilon, z) = zA - r(\epsilon)D + \lambda_c(1 - D - S(\epsilon, z)) - \lambda_\epsilon(\epsilon, z)(S(\epsilon^c, z) - S(\epsilon, z)) \\ + \lambda_z(z)\left(\sum_{\epsilon \in \mathcal{E}} \bar{\pi}_\epsilon(\epsilon'|z) S(\epsilon', z^c) - S(\epsilon, z)\right) \end{aligned} \tag{3.1}$$

where ϵ^c and z^c are the complement of ϵ and z respectively and $\bar{\pi}_\epsilon(\epsilon|z)$ denotes the stationary distribution of ϵ conditional on the aggregate state z.

The terms in equation (3.1) can be interpreted as follows. The term $zA - r(\epsilon)D$ denotes the flow of resources that the project generates minus the interest cost that the banker is paying to depositors. The term $1 - D - S(\epsilon, z)$ denotes the loss of value when the project terminates exogenously. This is multiplied by the rate of exogenous termination, λ_c. The term $S(\epsilon^c, z) - S(\epsilon^c, z)$ denotes the change in the total surplus when the bank's idiosyncratic funding cost state changes but the aggregate state remains the same. This term is multiplied by the rate at which ϵ changes, $\lambda_\epsilon(\epsilon, z)$. The final term, $\sum_{\epsilon \in \mathcal{E}} \bar{\pi}_\epsilon(\epsilon'|z) S(\epsilon', z^c) - S(\epsilon, z)$, denotes the expected change in surplus when the aggregate state changes and the banker gets a new cost of funds drawn from the new stationary distribution. For example, if the economy enters a crisis, then the term reflects the probability that the banker gets an adverse draw from the new stationary distribution $\bar{\pi}_\epsilon(\cdot|z_B)$. This final term is multiplied by the rate at which the aggregate state changes, $\lambda_z(z)$.

3.2 Contracting With Agency Frictions

Now, I consider the optimal contracting problem in the environment described in subsection 2. There are three frictions that prevent the first best contract from being implemented: (i) the entrepreneur's private information, (ii) entrepreneur's lack of commitment and (iii) the banker's cost of revealing their state. In the end, these frictions will mean that the optimal contract will be

[8] Because the Brownian innovations, dB_t, are i.i.d., it is always optimal for the banker to continue the project so long as the drift of the project is greater than their cost of funds.

implemented with a collection of "debt-like" securities, a termination threshold based on net accumulated losses and an "covenant" threshold at which the banker adjusts the terms of the debt securities based on their cost of funds. The intuition for this contract being optimal is the following. The information asymmetry means that that the banker must expose the entrepreneur to project risk in order to induce them to not steal output. This creates the "debt-like" nature of the optimal contract. However, the entrepreneur's lack of commitment means that they will walk away from the contract if they are punished for sufficiently many losses. This creates the endogeneous "termination" threshold. Finally, because it is costly for the banker to reveal their state, they will only adjust the terms of the debt securities when the value of adjustment is relatively high. This creates the "covenant" threshold.

3.2.1 Contract Form

At contract initialisation, denoted by time $t = 0$, the banker finances the investment required to start the project, I, and commits to providing the entrepreneur with value w_0. In exchange, they receive control rights over the project. Formally, this means that the banker chooses a contract $\mathcal{C} = \{\tau, \delta, \mathscr{T}\}$ where τ is the stochastic stopping time at which the project is terminated, δ is the stochastic payout process to the entrepreneur and $\mathscr{T}$ is the stochastic stopping time at which the banker pays the cost, Φ, and starts to condition the contract on their idiosyncratic state. This means that, before the cost has been paid (regime $f = 0$), the stochastic processes $\{\tau, \delta, \mathscr{T}\}$ can only be conditioned on the entrepreneur's reports, $\widetilde{B}$, and the aggregate state, z. I denote the contract terms in this regime by $\mathcal{C}^0 = \{\tau_0, \delta_0, \mathscr{T}_0\}$. Once the cost has been paid (regime $f = 1$), the stochastic processes can be conditioned on $\widetilde{B}$, z *and* the banker's cost of funds, ϵ. In this regime, the stochastic process $\mathscr{T}$ is redundant and so I denote the contract terms by $\mathcal{C}^1 = \{\tau_1, \delta_1\}$.[9]

3.2.2 Problem for Funded Entrepreneurs

Since the banker's cost of funds is not public information in regime 0, it is necessary to define the entrepreneur's beliefs before stating the entrepreneur's problem. Let $\pi_t^e \equiv [\pi_t^e(\epsilon_G|z_t), \pi_t^e(\epsilon_B|z_t)]$ denote the entrepreneur's belief about the bankers state at time t, conditional on the aggregate state, z_t. Let $\mathbb{E}_t^e[\cdot]$ denote the expectation conditional on the observable history up to time t and under the entrepreneur's belief, π_t^e.[10]

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[9]In the appendix, this is defined formally in the following way. Let $\mathcal{F}_t^{\tilde{B},z} \equiv \sigma(\{\tilde{B}_s, z_s : 0 \leq s \leq t\})$ denote the filtration generated by $\tilde{B}$ and z up to time t. Let $\mathcal{F}_t^{\tilde{B},z,\epsilon} \equiv \sigma(\{\tilde{B}_s, z_s : 0 \leq s \leq t\}, \{\epsilon_s : \mathscr{T}_0 \leq s \leq t\})$ denote the filtration generated by $\tilde{B}$, z from 0 to time t and by ϵ from $\mathscr{T}_0$ to time t. Then $\mathcal{C}^1$ is adapted to $\mathcal{F}_t^{\tilde{B},z,\epsilon}$ whereas $\mathcal{C}^0$ is only adapted to $\mathcal{F}_t^{\tilde{B},z}$. Observe that the banker is not able to condition on all past values of ϵ after paying the cost Φ. Instead, they are able to condition on current and future values of ϵ.

[10]Formally, in regime 0, the expectation under the entrepreneur's belief about the banker's state is given by $\mathbb{E}_t^e[\cdot] = \sum_{\epsilon \in \mathcal{E}} \pi_t^e(\epsilon|z)\mathbb{E}[\cdot|\mathcal{F}_t^{\tilde{B},z}, \epsilon]$, where $\mathcal{F}_t^{\tilde{B},z} \equiv \sigma(\{\tilde{B}_s, z_s : 0 \leq s \leq t\})$

Consider a type-$\mathbf{x}$ entrepreneur who has signed a contract, $\mathcal{C}$. Their problem is to choose the stealing process that solves:

$$W_0^{F,0} \equiv \sup_{\xi \in \mathcal{A}(\mathcal{C})} \left[W_0^{F,0,\xi} \right] \tag{3.2}$$

where $\mathcal{A}(\mathcal{C})$ denotes the set of processes satisfying[11]

$$\mathbb{E}_0 \left[\left(\int_0^{\tau_c} e^{-\rho_e s} (d\delta_s + \beta \xi_s A) ds \right)^2 \right] < \infty$$

and where $W_0^{F,0,\xi}$ denotes the initial value to an entrepreneur of being funded (F) by a contract, $\mathcal{C}$, in regime 0 under an arbitrary stealing process, ξ. $W_0^{F,0,\xi}$ is given by:

$$W_0^{F,0,\xi} \equiv \mathbb{E}_0^e \left[\int_0^{T_0} e^{-\rho_e s} (d\delta_{0,s} + \beta \xi_s A ds) + e^{-\rho_e T_0} \left(\mathbb{1}_{\{T_0 \in \{\tau_c, \tau_0\}\}} W_{T_0}^U + \mathbb{1}_{\{T_0 = \mathscr{T}_0\}} W_{\mathscr{T}_0}^{F,1,\xi} \right) \right]$$

where $T_0 \equiv \min\{\tau_c, \tau_0, \mathscr{T}_0\}$ is the first stopping time at which the project is terminated or the regime switches, W_t^U denotes the value to an entrepreneur of being in the search market at time t and $W_t^{F,1,\xi}$ denotes the value to an entrepreneur of being funded (F) by a contract, $\mathcal{C}^1$, in regime 1 under an arbitrary stealing process, ξ at time t. $W_t^{F,1,\xi}$ is given by:

$$W_t^{F,1,\xi} \equiv \mathbb{E}_t \left[\int_t^{T_1} e^{-\rho_e (s-t)} (d\delta_{1,s} + \beta \xi_s A ds) + e^{-\rho_e (T_1 - t)} W_{T_1}^U \right]$$

where $T_1 \equiv \min\{\tau_c, \tau_1\}$. Observe that the entrepreneur's value function at time 0 is calculated under their belief about the banker's state. However, once the entrepreneur arrives at regime 1, they learn the banker's state and so their beliefs become degenerate at the true value. For this reason, I denote the expectation in regime 1 without the e superscript.

3.2.3 Banker's Problem

Consider a banker who has promised an $\mathbf{x}$-entrepreneur that they will fund their project and deliver them value w_0. Let $V_0^0(W_0, z_0, \mathbf{x})$ denote the value to a banker of promising a value W_0 to the entrepreneur at contract initialisation when the contract starts in regime $f = 0$ and the aggregate state is z_0. Then,

denote the filtration generated by $\tilde{B}$ and z up to time t. Observe that in region 1, the entrepreneur's belief distribution is degenerate at the true state and $\mathbb{E}_t^e[\cdot] = \mathbb{E}_t[\cdot]$.

[11] As described in the appendix, this restriction is required to ensure that the martingale representation theorem can be used to represent the entrepreneur's contination value.

the banker's problem is to choose a contract, $\mathcal{C}$, that solves:

$$
\begin{aligned}
V_0^0(W_0, z_0, \mathbf{x}) = \sup_{\mathcal{C}} \Bigg\{ \sum_{\epsilon_0 \in \mathcal{E}} \overline{\pi}(\epsilon_0|z_0)\mathbb{E}\Bigg[\int_0^{T_0} & e^{-\rho_h s}\big((zA - r(\epsilon_s)D)ds - d\delta_{0,s}\big) \\
& + e^{-\rho_h T_0}\Big(\mathbb{1}_{\{T_0 = \mathscr{T}_0\}}(V^1_{\mathscr{T}_0} - \Phi(z_{\mathscr{T}_0})) \\
& + \mathbb{1}_{\{T_0=\tau_c\}}(1-D) + \mathbb{1}_{\{T_0=\tau_0\}}L_B\Big)\Big|\epsilon_0, z_0\Bigg]\Bigg\}
\end{aligned} \tag{3.3}
$$

where $T_0 \equiv \min\{\tau_c, \tau_0, \mathscr{T}_0\}$, $\Phi(z)$ is the cost of switching regime, $L_B \equiv \max\{L - D, 0\}$ is the liquidation value left for the banker after repaying debt holders, $V^1_{\mathscr{T}_0}$ denotes the value of the banker at time $\mathscr{T}_0$ when the covenant is triggered and the contract enters regime 1, and the expectation is taken over ϵ_0 because the banker gets a random draw of ϵ_0 from the stationary distribution at contract initialisation. The banker's choice of $\mathcal{C}$ must satisfy the promise keeping (PK), participation (PC) and incentive compatibility (IC) constraints:

(PK): $W_0^{F,0} = w_0$,

(PC): $W_t^{F,0} \geq W_t^U$, for all $0 \leq t \leq T_0$, and

(IC): $\xi_t = 0$ solves the entrepreneur's problem (3.2),

The banker's value $V^1_{\tau_0}$ is given by:

$$
\begin{aligned}
V^1_{\tau_0} = \sup_{\mathcal{C}^1} \Bigg\{ \mathbb{E}_{\tau_0}\Bigg[\int_{\tau_0}^{T_1} & e^{-\rho_h s}\big((zA - r(\epsilon_s)D)ds - d\delta_{1,s}\big) \\
& + e^{-\rho_h T_1}(\mathbb{1}_{\{T_1=\tau_c\}}(1-D) + \mathbb{1}_{\{T_1=\tau_1\}}L_B)\Bigg]\Bigg\}
\end{aligned} \tag{3.4}
$$

where $T_1 \equiv \min\{\tau_c, \tau_1\}$ and the choice of $\mathcal{C}^1$ must satisfy the constraints:

(PK): $\mathbb{E}^e_{\mathscr{T}_0}[W^{F,1}_{\mathscr{T}_0}] = W^{F,0}_{\mathscr{T}_0}$,

(PC): $W_t^{F,1} \geq W_t^U$, for all $\tau_0 \leq t \leq T_1$, and

(IC): $\xi_t = 0$ solves the entrepreneur's problem (3.2).

The three conditions in each regime make the restrictions on the contracting problem precise. The (PK) constraint requires that the contract actually delivers the promised value. In regime 0, this means that the contract delivers the value that was promised in the search market, w_0. In regime 1, this means that, under the entrepreneur's belief about the banker's state, the entrepreneur's value just before the covenant is triggered must be equal to the entrepreneur's expected value just after the covenant is triggered when the banker's state is revealed and the contract terms are adjusted. Conceptually, this constraint says that the banker can condition the entrepreneur's continuation value at $\mathscr{T}_0$ on their cost of funds so long as the entrepreneur's expected value just before they hit the boundary is the value that the entrepreneur has been promised.

The (PC) constraint comes from the entrepreneur's inability to commit to the contract. At any time, the entrepreneur can abandon the contract and return to the search market. This imposes a participation constraint that the entrepreneur's continuation value implied by the contract, $W_t^F(\mathcal{C})$, must always be greater than their value in the search market W_t^U at all times.

The (IC) constraint come from the moral hazard problem between the banker and the entrepreneur. Recall that the entrepreneur can steal $\xi_t > 0$ and report $d\widetilde{B}_t = dB_t - (zA/\sigma)\xi_t dt$ rather than dB_t to the banker. This means that the banker must provide incentives in order to get truthful reporting. A contract $\mathcal{C}$ is referred to as incentive compatible if truthful reporting, $\xi_t = 0$ and $d\widetilde{B}_t = dB_t$, is optimal for the entrepreneur under the contract at all times $t \leq \tau_1$. In this paper, attention is restricted to incentive compatible contracts. By the revelation principle, this is without loss of generality.[12]

In order to solve the problem, I first find a differential equation for the evolution of the entrepreneur's continuation value. This allows me to express the banker's problem recursively. I then solve the recursive problem by using backward induction across the two regimes. That is, I solve the problem in regime 1 when the covenant flag has been triggered and then, given the solution in regime 1, I solve the problem in regime 0 before the covenant flag has been triggered.

3.3 Evolution of Entrepreneur's Continuation Value

Let $\{N_{c,s}, N_{z,s}, N_{\epsilon,s} : s \geq 0\}$ describe the Poisson processes that, respectively, generate the exogeneous termination of the project and the state changes in the continuous Markov chains for z and ϵ.[13] This means that $dN_{c,t} = 1$ describes the time at which the project terminates while $dN_{z,t} = 1$ and $dN_{\epsilon,t} = 1$ describe the times at which the z and ϵ states change.

Lemma 2 shows that the evolution of entrepreneur's continuation value can be described by a stochastic differential equation specifying the entrepreneur's exposure to changes to the underlying stochastic processes in the economy: dB_t, $dN_{z,t}$, $dN_{\epsilon,t}$ and $dN_{c,t}$.

Lemma 2. *The entrepreneur's continuation value under truth-telling evolves according to following differential equations in each regime:*

(a) In regime 1*, there exist stochastic processes* Ψ_b^1*,* Ψ_ϵ^1 *and* $\{\Psi_{z,\epsilon}^1\}_{\forall \epsilon \in \mathcal{E}}$ *such*

[12]See **?** for a discussion of the revelation principle in this context.

[13]A formal definition is provided in the appendix.

that:

$$dW_t^{F,1} = \rho_e W_t^{F,1} dt - \underbrace{d\delta_{1,t}}_{\substack{\text{Payout from}\\ \text{banker}}} + \underbrace{\sigma \Psi_{b,t}^1 dB_t}_{\substack{\text{Exposure to}\\ \text{project shocks}}}$$
$$+ \underbrace{(W_t^U - W_t^{F,1})(dN_{c,t} - \lambda_c dt)}_{\substack{\text{Exposure to exogeneous}\\ \text{project completion shocks}}} + \underbrace{\Psi_{\epsilon,t}^1 (dN_{\epsilon,t} - \lambda_\epsilon(\epsilon_t) dt)}_{\substack{\text{Exposure to idiosyncratic}\\ \text{banker shocks}}}$$
$$+ \sum_{\epsilon' \in \mathcal{E}} \underbrace{\Psi_{z,\epsilon',t}^1 (dN_{z,t} \mathbb{1}_{\epsilon'} - \overline{\pi}(\epsilon' | z_t^c) \lambda_z(z_t) dt)}_{\text{Exposure to aggregate shocks}} \tag{3.5}$$

(b) In regime 0*, there exist stochastic processes* $\Psi_{b,t}^0$ *and* $\Psi_{z,t}^0$ *such that:*

$$dW_t^{F,0} = \rho_e W_t^{F,0} dt - \underbrace{d\delta_{0,t}}_{\substack{\text{Payout from}\\ \text{banker}}} + \underbrace{\sigma \Psi_{b,t}^0 dB_t}_{\substack{\text{Exposure to}\\ \text{project shocks}}}$$
$$+ \underbrace{(W_t^U - W_t^{F,1})(dN_{c,t} - \lambda_c dt)}_{\substack{\text{Exposure to exogeneous}\\ \text{project completion shocks}}} + \underbrace{\Psi_{z,t}^0 (dN_{z,t} - \lambda_z dt)}_{\substack{\text{Exposure to exogeneous}\\ \text{aggregate shocks}}} \tag{3.6}$$

(c) In regime 1, the contract satisfies the participation constraint if and only if, $W_t^{F,1} \geq W_t^U$ *and* $W_t^{F,1} + \Psi_{i,t}^1 \geq W_t^U$*, for all* $t \leq T_1$ *and for all* $i \in \{\epsilon, (z,\epsilon), (z,\epsilon^c)\}$*. In regime 0, the contract satisfies the participation constraint if and only if* $W_t^{F,0} \geq W_t^U$ *and* $W_t^{F,0} + \Psi_{z,t}^0 \geq W_t^U$ *for all* $t \leq T_0$*.*

(d) In both regimes, truth telling is optimal if and only if $\Psi_{b,t}^f \geq \beta$*.*

Proof. See appendix A.3. The proof is essentially a direct application of the martingale representation theorem. □

The stochastic processes, Ψ_b^f, Ψ_z^f and Ψ_ϵ^1 characterise how the entrepreneur's continuation value is exposed to the shocks in the economy under a particular contract, $\mathcal{C}$. First, consider the entrepreneur's continuation value in regime 1, which is governed by equation (3.5). In this equation, the process $\Psi_{b,t}^1$ characterises how the contract exposes the entrepreneur to shocks to project output, dB_t. The incentive compatibility condition, $\Psi_{b,t}^1 \geq \beta$, says that, in order to induce truth-telling, the banker must punish the entrepreneur for low output by a factor that is at least as large as the benefit that the entrepreneur gets from stealing. The process $\Psi_{\epsilon,t}^1$ characterises how a particular contract exposes the entrepreneur to shocks to the banker's cost of funds. This consists of two components: the jump in continuation value that occurs when the bank's funding cost shock arrives and a flow term that compensates the entrepreneur for that jump. Finally, the process $\Psi_{z,\epsilon,t}^1$ characterises how the contract exposes the entrepreneur to an aggregate shock in which the banker ends up with a new idiosyncratic cost of funds, ϵ.

Likewise, the entrepreneur's continuation value in regime 0 is governed by equation (3.6). The equation is similar but there is an important difference that the entrepreneur's continuation value is no longer affected by shocks to ϵ since $\mathcal{C}^0$ is not conditioned on the banker's cost of funds. Instead, the entrepreneur's continuation value is only exposed to shocks to the project, dB_t, and shocks to the aggregate state, $dN_{z,t}$.

I close this section by noting that the entrepreneur's belief in regime 0 is always the stationary distribution of ϵ. This result significantly simplifies the contracting problem because it means that I don't have to keep track of entrepreneur beliefs as a state variable.[14]

Claim 1. *The entrepreneur's belief in regime* 0 *under aggregate state z is* $\pi_t^e = \bar{\pi}(\cdot|z)$*, where $\bar{\pi}(\cdot|z)$ is the stationary distribution of ϵ in state z.*

Proof. This follows by construction since the banker's initial cost of funds is drawn randomly from the stationary distribution, bankers get a new draw of funds when the aggregate state changes and the $f = 1$ regime with triggered covenant is an absorbing regime. □

3.4 Optimal Contract

The optimal contract is solved by backward induction across the regimes. I start by solving the problem in regime 1, where the covenant binds and banker can freely adjust the contract with the cost of funds shocks. I then use that solution to solve the contract in regime 0.

3.4.1 Regime 1: Covenant Binding

The banker's problem in regime 1 can now be written recursively. The state variables for a banker who has contracted to fund an entrepreneur are (W, ϵ, z, $\mathbf{x}$), where W is the continuation value promised to the entrepreneur under the chosen contract, ϵ is the banker's funding cost state, z is the aggregate state and $\mathbf{x}$ is the entrepreneur's type. I use the notation $V^1(W, \epsilon, z, \mathbf{x})$ to denote the banker's value function in state (W, ϵ, z, $\mathbf{x}$) and regime 1. Since $\mathbf{x}$ does not change throughout the contract, I will sometimes drop $\mathbf{x}$ in order to streamline the notation. In section 4, I show that the entrepreneur's value of being unfunded in the search market is $W_t^U = W^U(z, \mathbf{x})$ so I use that notation in this section.

The banker's problem in regime 1 is to choose a contract $\mathcal{C}^1 = (\tau_1, \delta_1)$ that solves 3.4. Proposition 1 characterises the optimal contract.

[14] In the appendix, I investigate the more complex problem in which the entrepreneur's belief is moving over time. I don't discuss this in the main text because introducing the entrepreneur's belief as a state variable in this problem significantly increases the technical complexity of the problem without delivering significant economic insight because the entrepreneur doesn't have anything to learn about. The only additional dynamic that the richer model incorporates is that beliefs gradually trends to the stationary distribution after a change in the aggregate state. Instead, I set up the problem so that beliefs jump to the stationary distribution as soon as the aggregate state changes.

Proposition 1 (Optimal Contract: Regime 1)**.** *Under the optimal contract, $\mathcal{C}^{1*} = (\tau_1^*, \delta_1^*)$:*

(a) The banker terminates the project once W_t reaches $W^U(z, \mathbf{x})$. That is,

$$\tau_1^* = \inf\{t \geq 0 : W_t \leq W^U(z, \mathbf{x})\},$$

(b) The banker only makes payments to the entrepreneur when W_t is greater than a state contingent threshold $\overline{W}^{1*}(\epsilon, z, \mathbf{x})$ satisfying:

$$\partial_W V^1(\overline{W}^{1*}, \epsilon, z, \mathbf{x}) = -1$$

That is, $d\delta_{1,t}^* = 0$ when $W_t < \overline{W}^{1*}(\epsilon, z, \mathbf{x})$ and $d\delta_{1,t}^* \geq 0$ when $W_t \geq \overline{W}^{1*}(\epsilon, z, \mathbf{x})$.

This means that, under the optimal contract, the banker keeps the entrepreneur's continuation value inside the region $[W^U(z, \mathbf{x}), \overline{W}^{1*}(\epsilon, z, \mathbf{x})]$, where the lower boundary is a termination boundary and the upper boundary is a reflecting boundary.

Proof. See Appendix A.4. □

Given the form of the contract described in Proposition 1, the banker's value function in the region $[W^U(z, \mathbf{x}), \overline{W}^{1*}(\epsilon, z, \mathbf{x})]$ is the solution to the Hamilton-Jacobi-Bellman equation:

$$\begin{aligned}
&\rho_h V^1(W, \epsilon, z, \mathbf{x}) \\
&\quad = \sup_{\Psi_b^1, \Psi_\epsilon^1, \Psi_{z,\cdot}^1} \Big\{ zA - r(\epsilon)D \\
&\qquad\qquad + \Big(\rho_e W - \lambda_\epsilon(\epsilon)\Psi_\epsilon - \sum_{\epsilon' \in \mathcal{E}} \overline{\pi}(\epsilon'|z)\lambda_z(z)\Psi_{z,\epsilon'} - \lambda_c(W^U(z, \mathbf{x}) - W)\Big)\partial_W V^1(W, \epsilon, z, \mathbf{x}) \\
&\qquad\qquad + (\Psi_{b,t}^1)^2\sigma^2\partial_{WW} V^1(W, \epsilon, z, \mathbf{x}) \\
&\qquad\qquad + \lambda_\epsilon(\epsilon|z)(V^1(W + \Psi_\epsilon^1, \epsilon^c, z, \mathbf{x}) - V^1(W, \epsilon, z, \mathbf{x})) \\
&\qquad\qquad + \lambda_z(z)\Big(\sum_{\epsilon'} \overline{\pi}(\epsilon', z)V^1(W + \Psi_{z,\epsilon'}^1, \epsilon', z^c, \mathbf{x}) - V^1(W, \epsilon, z, \mathbf{x})\Big) \\
&\qquad\qquad + \lambda_c(L_B - V^1(W, \epsilon, z, \mathbf{x}))\Big\}
\end{aligned}$$

subject to the constraints:

$$\begin{aligned}
&(IC) : \Psi_{b,t}^1 \geq \beta \\
&(PC) : \Psi_i^1 \geq W^U(z, \mathbf{x}) - W, \quad \forall i \in \{\epsilon, (z, \epsilon), (z, \epsilon^c)\}
\end{aligned}$$

and with the boundary conditions:

$$\begin{aligned}
(BC_L) &: V^1(W, \epsilon, z, \mathbf{x}) = L_B \\
(BC_{U,1}) &: \partial_W V^1(\overline{W}^{1*}, \epsilon, z, \mathbf{x}) = -1 \\
(BC_{U,2}) &: \partial_{WW} V^1(\overline{W}^{1*}, \epsilon, z, \mathbf{x}) = 0
\end{aligned}$$

The lower boundary condition is the absorbing boundary condition that says the banker gets the liquidation value when the project is terminated. The first upper boundary condition says that the banker is paying resources at rate 1 with respect to W in order to reflect the W_t away from the upper boundary. The third boundary condition says that the free boundary is chosen optimally.

The key point to note about the banker's HJB equation is that the banker's optimal control problem has been reduced to choosing how to expose the entrepreneur's continuation value to the underlying stochastic processes, as characterised by the choice of $\Psi_b^1, \Psi_\epsilon^1, \Psi_{(z,\cdot)}^1$. In the appendix, I show that the function V^1 is concave in W and so the optimal choices satisfy:

(Ψ_b^1) : $\Psi_b^1 = \beta$

(Ψ_ϵ^1) : $\Psi_\epsilon^1 = \max\{\widetilde{\Psi}_\epsilon, W^U - W\}$, where $\widetilde{\Psi}_\epsilon$ solves:

$$\partial_W V^1(W + \widetilde{\Psi}_\epsilon, \epsilon^c, z, \mathbf{x}) = \partial_W V^1(W, \epsilon, z, \mathbf{x})$$

$(\Psi_{z,\epsilon'}^1)$: $\Psi_{z,\epsilon'}^1 = \max\{\widetilde{\Psi}_{z,\epsilon'}, W^U - W\}$, where, for $\epsilon' \in \mathcal{E}$, $\widetilde{\Psi}_{z,\epsilon'}$ solves:

$$\partial_W V^1(W + \widetilde{\Psi}_{z,\epsilon'}, \epsilon', z^c, \mathbf{x}) = \partial_W V^1(W, \epsilon, z, \mathbf{x})$$

A number of features of this contract are standard in the literature. However, the intuition is reviewed since understanding the contracting problem is central to this paper. First consider the banker's value function in regime 1, when the aggregate state is z_B and there is a possibility of an adverse cost of funds shock. For a particular (ϵ, z_B), this is depicted graphically in figure 1. There are three "forces" driving the shape of the banker's value function. The first force is that $\rho_e > \rho_h$ which makes the banker want to decrease the entrepreneur's continuation value (promise less in the future) and pay the entrepreneur more now. The second force is the (IC) constraint, which requires the bank to decrease the entrepreneur's continuation value after a negative project shock in order to incentivise truthful reporting. The difficulty for the banker is that there is a limit on how much they can decrease the entrepreneur's continuation value before the (PC) constraint binds and entrepreneur abandons the contract causing the project to be liquidated. The desire to avoid costly liquidation gives the banker a precautionary motive in the problem and introduces curvature into their value function.

The terms Ψ_ϵ^1, $\Psi_{z,\epsilon'}^1$ and $\Psi_{z,\epsilon}^1$ capture how the banker conditions the contract on their idiosyncratic cost of funds and the aggregate state, given the three forces described in the previous paragraph. All three terms have first-order conditions of the same form: the banker adjusts the entrepreneur's continuation value in order to keep the marginal value of W the same after the shock. By way of an example, the choice of Ψ_ϵ^1 is depicted graphically in figure 1.[15] The intuition

[15] The diagram would look very similar for a shock to the aggregate state. The one important difference is that a change to aggregate state also changes the equilibrium in the search market and so the entrepreneur's outside option. This shifts the value functions. I will return to a discussion of the interaction between aggregate shocks, the search market, and the banker's value functions in section 4 once the search market has been defined.

for all these first-order conditions can be seen in the banker's HJB equation. For example, again consider the choice of Ψ^1_ϵ, which solves the maximization problem:

$$\sup_{\Psi^1_\epsilon} \Big\{ -\lambda_\epsilon(\epsilon|z)\partial_W V^1(W,\epsilon,z,\mathbf{x}) + \lambda_\epsilon(\epsilon|z)(V^1(W+\Psi^1_\epsilon,\epsilon^c,z,\mathbf{x}) - V^1(W,\epsilon,z,\mathbf{x}))\Big\} \tag{3.7}$$

The second term in (3.7) captures the net benefit the banker gets from being able to shift the entrepreneur's continuation value when their cost of funds shock occurs. The first term in (3.7) captures the flow change in continuation value that the banker has to provide to the entrepreneur in order to compensate them for the jump when the cost of funds shock arrives. The first order condition then equates the marginal benefit and the marginal cost of shifting the entrepreneur's continuation value at a cost of funds shock. The intuition for $\Psi^1_{z,\epsilon'}$ and $\Psi^1_{z,\epsilon}$ follows the same line of argument.

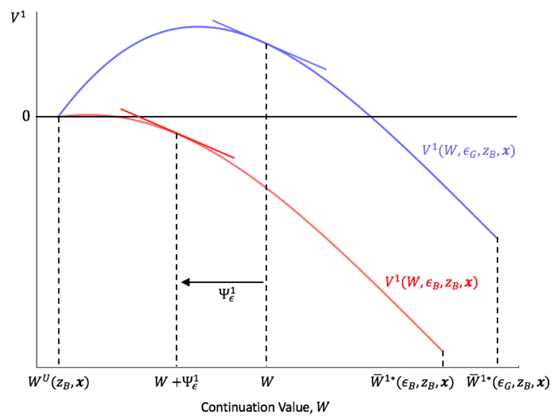

Figure 1: *Crisis State (z_B): Regime 1: Banker's Value Function.* The figure displays the value function for the banker in each cost of funding state, ϵ, when the aggregate state is z_B. It also displays the banker's choice of Ψ^1_ϵ when the their cost of funds increases (ϵ changes from ϵ_G to ϵ_B) and the economy remains in the crisis state, z_B.

In order to extend the intuition for the example of Ψ^1_ϵ, the change in the entrepreneur's value after an idiosyncratic bank shock, Ψ^1_ϵ, and the corresponding change in the banker's value, $V^1(W+\Psi^1_\epsilon,\epsilon^c,z,\mathbf{x}) - V^1(W,\epsilon,z,\mathbf{x})$, are depicted in figure 2 for different values of W before the shock. The first point to note is that if W is sufficiently close to the entrepreneur's outside option, W^U, before the shock, then the banker will terminate the project after a funding cost increase. The second point to note is that, outside of the termination region, the banker adjusts the entrepreneur's continuation value by less when the entrepreneur's continuation value is higher. The intuition for this can be thought about in the following way. The banker only wants to offer the entrepreneur a continuation value above their outside option because liquidation is costly (the banker takes a loss on liquidation and no longer gets surplus from the project). If the banker's cost of funds increases, then the surplus from the project decreases and so their precautionary motive to keep W away from the W^U also decreases. This is what causes the banker to want to decrease the entrepreneur's continuation value. Since the banker's precautionary motive is stronger when W is closer to W^U, this effect is larger when W is smaller and so the magnitude of Ψ^1_ϵ is decreasing in W. Once again, the story is similar for $\Psi^1_{z,\epsilon'}$ and $\Psi^1_{z,\epsilon}$.

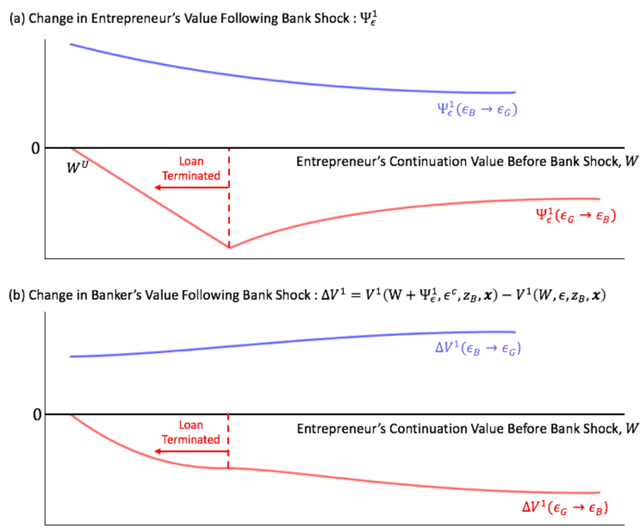

Figure 2: *Crisis State (z_B): Regime 1: Change in value functions following bank funding cost shocks.* The upper graph shows the change in the entrepreneur's value following a bank shock, Ψ^1_ϵ, while the economy is in the crisis aggregate state. The red line shows the change when the bank's cost of funds increases. The blue line shows the change when the bank's cost of funds decreases. The lower graph shows the corresponding change in the banker's value when the entrepreneur's continuation value is changed.

Finally, figure 3 displays the evolution of the entrepreneur's continuation value under different histories of the project shocks and bank funding cost shocks when the economy is in the crisis state. As can be seen on the blue line, if the project gets positive shocks ($dB_t > 0$), then the banker increases the entrepreneur's continuation value until the payout boundary is reached, at which point the banker gives output to the entrepreneur to consume. The orange line depicts the time path in which the project gets negative shocks ($dB_t < 0$). In order to satisfy the incentive compatibility constraint, the banker responds by decreasing the entrepreneur's continuation value until it reaches the lower boundary and the project is terminated. The green line depicts the time path in which the banker gets a bad cost of funds shock. In this case, the banker shifts the entrepreneur's continuation value down by Ψ^1_ϵ, which leads to earlier project termination.

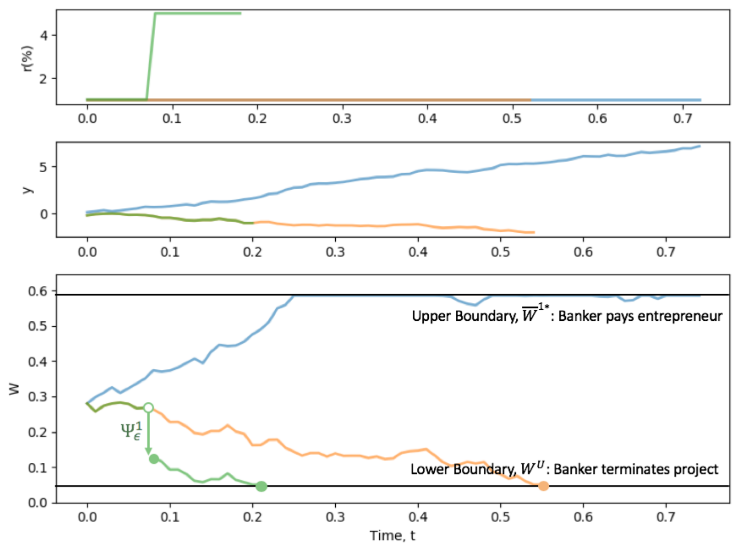

Figure 3: *Crisis State (z_B): Regime 1: Time paths under optimal contracting.* The plots show the evolution of key variables while the economy is in the crisis state. The upper plot shows the evolution of the bank's interest rate. The middle plot shows the evolution of the cumulative output process. The lower plot shows the evolution of the entrepreneur's continuation value. The blue line shows the evolution of the key variables when the project receives positive shocks and the entrepreneur's continuation value hits the upper payout boundary. The orange line shows the evolution of the key variables when the project receives negative shocks and the entrepreneur's continuation value hits the termination boundary. The green line shows the evolution of the key variables when the project receives negative shocks and the bank's cost of funds increases.

3.4.2 Regime 0: Covenant Not Binding

The previous subsection describes the banker's problem once the project has entered regime 1 and the contract can be conditioned on their cost of funds. This subsection describes the problem for a banker in regime 0 where the entrepreneurs don't know the idiosyncratic component of the banker's cost of funds and the banker cannot condition the contract on the cost of funds. In this case, the optimal contract is characterised by Proposition 2.

Proposition 2 (Optimal Contract: Regime 0)**.** *Under the optimal contract,* $C^{0*} = (\tau_0^*, \delta_0^*, \mathscr{T}_0)$*:*

(a) The banker terminates the project once W_t *reaches* $W^U(z, \mathbf{x})$*. That is,*

$$\tau_0^* = \inf\{t \geq 0 : W_t \leq W^U(z, \mathbf{x})\},$$

(b) There exists a threshold, $\overline{W}^{0*}(z, \mathbf{x})$*, such that the banker only makes payments to the entrepreneur when* W_t *is greater* $\overline{W}^{0*}(z, \mathbf{x})$*. That is,* $d\delta_{0,t}^* = 0$ *when* $W_t < \overline{W}^{0*}(z, \mathbf{x})$ *and* $d\delta_{0,t}^* \geq 0$ *when* $W_t \geq \overline{W}^{0*}(z, \mathbf{x})$*.*

(c) For sufficiently small Φ*, the optimal choice of* $\mathscr{T}_0^*$ *satisfies* $\mathscr{T}_0^* < \tau_0^*$ *which means that the covenant binds before the project is terminated. At* $\mathscr{T}_0^*$*, the banker adjusts the entrepreneur's continuation value to:*

$$W_{\mathscr{T}_0^*}^{F,1} = W_{\mathscr{T}_0^*}^{F,0} + \Psi_\Phi(\epsilon, z, \mathbf{x})$$

where:

$$\Psi_\Phi(\epsilon, z, \mathbf{x}) = \max\{\widetilde{\Psi}_\Phi, W^U(z, \mathbf{x}) - W_{\mathscr{T}_0^*}^{F,0}\}$$

and where $\widetilde{\Psi}_\Phi$ *solves:*

$$\partial_w V^1(W_{\mathscr{T}_0^*}^{F,0} + \widetilde{\Psi}_\Phi(\epsilon_G, z, \mathbf{x}), \epsilon_G, z, \mathbf{x}) = \partial_w V^1(W_{\mathscr{T}_0^*}^{F,0} + \widetilde{\Psi}_\Phi(\epsilon_B, z, \mathbf{x}), \epsilon_B, z, \mathbf{x})$$
$$\sum_{\epsilon \in \mathcal{E}} \overline{\pi}(\epsilon|z)\widetilde{\Psi}_\Phi(\epsilon, z, \mathbf{x}) = 0$$

Proof. See Appendix A.4.[16] □

Part (a) specifies that the optimal termination time has the same form as in regime 1. That is, the banker does not terminate the project until the participation constraint binds. Part (b) specifies that, once again, the banker only makes payments to the entrepreneur once the entrepreneur's continuation value crosses an upper threshold. The difference is that now the threshold no longer

[16] The optimal choice of τ_0, $W^{0*}(z, \mathbf{x})$ and $\mathscr{T}_0$ are complex to characterise because the banker has to solve an optimisation problem in which they cannot condition on all the state variables. The details are not conceptually edifying but they are essential for all the numerical algorithms developed to solve the problems in this paper.

depends on ϵ. This means that the banker must choose an "average" payout boundary, $\overline{W}^{0*}(z, \mathbf{x})$, satisfying:

$$\overline{W}^{1*}(\epsilon_B, z, \mathbf{x}) \leq \overline{W}^{0*}(z, \mathbf{x}) \leq \overline{W}^{1*}(\epsilon_G, z, \mathbf{x})$$

This demonstrates the ex-post inefficiency from not being able to condition the contract on the entrepreneur's cost of funds. When the bank is in state ϵ_B, the payout boundary is too high ex-post and the banker ends up promising the entrepreneur too much given the low ex-post surplus. Likewise, when the banker is in state ϵ_G, the payout boundary is too low ex-post and the banker ends up promising the entrepreneur too little given how valuable it is to keep the entrepreneur away from the lower boundary when the surplus is high.

Part (c) of proposition 2 characterises how the banker chooses to shift the entrepreneur's continuation value at $\mathscr{T}_0$ when the covenant binds and the banker is able to start conditioning the contract on their cost of funds. In this case, the banker can adjust the entrepreneur's continuation value so long as they maintain the promise-keeping constraint that, under the entrepreneur's beliefs, the contract actually delivers $W^0_{\mathscr{T}_0}$ in expectation at the boundary. That is, it must hold that:

$$\mathbb{E}^e_{\mathscr{T}_0}[W^1_{\mathscr{T}_0}(\epsilon, z, \mathbf{x})] = \mathbb{E}^e_{\mathscr{T}_0}[W^0_{\mathscr{T}_0} + \Psi^1_\Phi(\epsilon, z, \mathbf{x})] = W^0_{\mathscr{T}_0}$$

3.4.3 Simulations in Regime 0.

The key feature of the regime 0 contracting problem is that the optimal contract now potentially has additional thresholds beyond the termination threshold, $W^U(z, \mathbf{x})$, and the payout threshold, $\overline{W}^{0*}(z, \mathbf{x})$. For sufficiently small Φ, there will also be at least one threshold at which the banker pays to condition the contract on their cost of funds. In the quantitative examples in the paper, I choose Φ so that there is only a lower threshold, denoted by $\underline{W}^{\Phi}(z, \mathbf{x}) = W^0_{\mathscr{T}_0} \leq W_0$, at which the covenant flag is triggered and the regime switches to $f = 1$.[17] The termination threshold, covenant flag threshold, and payout threshold are depicted graphically in figure 4.

In order to illustrate how the multiple thresholds operate, two simulations of the economy during a crisis state are included in figures 5 and 6. The first of the two figures depicts the evolution of the contract when a bank gets an increase in their cost of funds and then the project breaches the covenant threshold and gets flagged (denoted in the diagram as by f going from 0 to 1). As can be seen, the contracts evolve the same way until the covenant threshold is reached.

[17] In appendix A.4, I discuss the possibility of an upper threshold, $\overline{W}^{\Phi}(z, \mathbf{x}) > W_0$, at which the regime changes. Conceptually, the banker is more willing to pay to start conditioning the contract on their cost of funds in a region of the state space where the contractual flexibility is more valuable. This occurs when W_t is low because, in that region of the state space, "bad" banks want to terminate projects whereas "good" banks want to shift projects away from the termination boundary. However, the flexibility is also potentially valuable when W_t is high because then the banker can avoid paying out too soon or too late. In numerical exercises, I verify that the termination effect dominates and set Φ so that there is only a lower boundary.

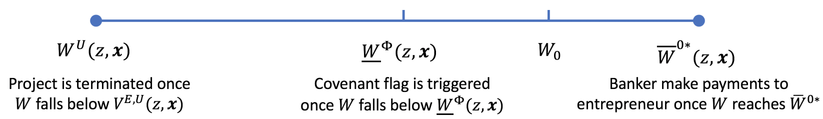

Figure 4: Thresholds in the Optimal Contract.

At that point, the bad bank chooses to terminate the project whereas the good bank chooses to recapitalise the project. The second of the two figures depicts the evolution of the contract when the covenant threshold is breached first and then later the banker gets an increase in their cost of funds. As can be seen, in this case, the bad banker chooses to decrease the continuation value of the entrepreneur even though their continuation value is well above the covenant threshold. This illustrates the option value of having a flagged project in which the covenant is binding and the banker can adjust the contract terms.

3.5 Implementation with Standard Securities

The 2007-09 financial crisis has provided extensive data on bank-firm contracting during times of bank distress. Papers such as Chodorow-Reich [2014], Chodorow-Reich and Falato [2017], **?** and **?** study large collections of individual bank business contracts and find evidence that borrowers from relatively "bad" banks ended up with significantly worse outcomes during the crisis that borrowers from relatively "good" banks. In particular, Chodorow-Reich and Falato [2017] study how relatively unhealthy banks cut back long-term credit. They create an index of bank health during the crisis[18] and record breaches in financial loan covenants that would give the bank additional control rights over the project[19]. They find that good banks typically provided waivers when covenants were violated whereas bad banks increased interest rates, recalled debt, reduced credit lines, and (sometimes) terminated projects when covenants were violated. In this sense, they argue that the banks appeared to use the financial covenants not only to resolve agency frictions with the entrepreneur but also as an option to adjust debt terms when they got into trouble.

In this section, I show that the abstract optimal contract described in the previous subsection can be implemented (non uniquely) using standard debt-like securities, a debt recall process, and covenants that, once breached, give bankers additional flexibility in how they can recall debt. This will mean that the implementation generates similar behaviour to that described by Chodorow-Reich and Falato [2017]. Technically, the form of the implementation extends the implementation in DeMarzo and Sannikov [2006] to include bank shocks and regime shifting.[20]

[18] The index is based on the bank's exposure to Lehman Brothers, their exposure to mortgage-backed securities and their ratio of trading revenue to assets

[19] Their focus is on two financial covenants: a lower bound on the EBITDA to debt ratio and a lower bound on the EBITDA to interest expense ratio.

[20] Many researchers have proposed implementations for the moral hazard contracting prob-

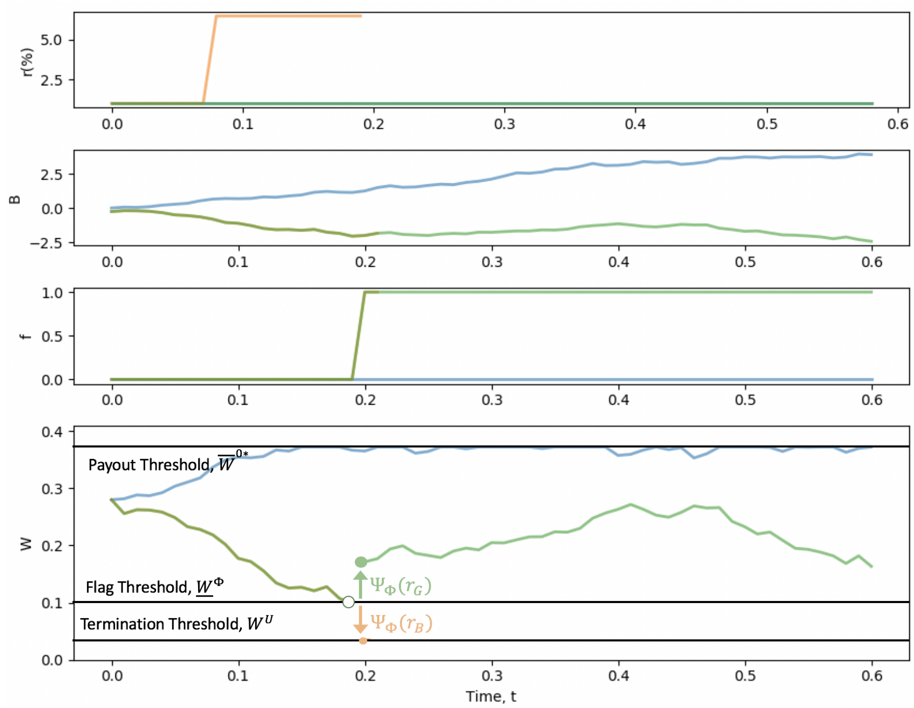

Figure 5: *Crisis State (z_B): Simulation of the contract: Example 1.* The blue line depicts the evolution of the contract when the project receives positive shocks and the entrepreneur's continuation value hits the upper payout boundary. In this case, the covenant flag threshold is never breached and the entrepreneur is unaffected by idiosyncratic bank shocks. The green line depicts the evolution of the contract for a project that receives negative shocks but is attached to a good bank when the covenant binds. The orange line depicts the evolution of the contract for a project that receives negative shocks and is attached to a bad bank when the covenant binds.

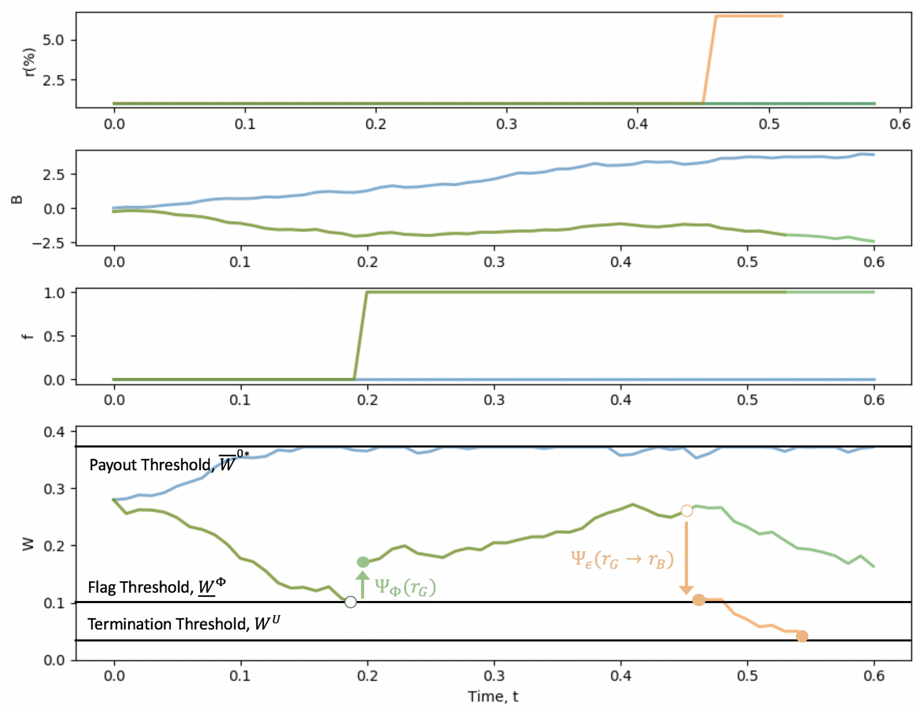

Figure 6: *Crisis State (z_B): Simulation of the contract: Example 2.* The blue line depicts the evolution of the contract when the project receives positive shocks and the entrepreneur's continuation value hits the upper payout boundary. In this case, the covenant flag threshold is never breached and the entrepreneur is unaffected by idiosyncratic bank shocks. The green line depicts the evolution of the contract for a project that receives negative shocks but is always attached to a good bank. The orange line depicts the evolution of the contract for a project that receives negative shocks and ends up attached to a bad bank after the covenant binds.

3.5.1 Implemenation

Consider an implementation in which the banker provides the entrepreneur with funds, I, in exchange for three types of securities:

> *Equity:* which gives the holder the right to dividend flows paid out by the entrepreneur.[21]
>
> *Long-term Contingent Debt Consol:* which gives the holder the right to continuous coupons at rate x_t into perpetuity. The banker recalls debt according to a state-contingent process γ_t.[22]
>
> *Credit Line:* which commits the holder to providing funds to the entrepreneur, at the discretion of the entrepreneur. The balance of the credit line, denoted by M_t, has an interest rate of r_c. If the draw on the credit line is greater than the threshold $\overline{M}_t^L$, then the banker terminates the project.

The debt contract and credit line are accompanied by a covenant threshold, $\overline{M}_t^c \leq \overline{M}_t^L$. The contracts start without a covenant flag ($f = 0$). If the balance on the credit line exceeds the threshold $\overline{M}_t^c \leq \overline{M}_t^L$, then the banker gives the contracts a covenant flag ($f = 1$):

(i) When the covenant is not binding ($f = 0$), the coupon process x_t, the debt recall process γ_t and the limit on the credit line $\overline{M}_t^L$ can only depend upon z_t and M_t, and

(ii) When the covenant is binding ($f = 1$), the coupon process x_t, the debt recall process γ_t and the limit on the credit line $\overline{M}_t^L$ can depend on z_t, M_t *and* ϵ_t.

Proposition 3 characterises how these securities can implement the optimal contract from the previous subsection. I drop $\mathbf{x}$ from the notation in order to save space (and to avoid confusion with the coupon process, x_t).

---

lem with only project risk (e.g. see DeMarzo and Sannikov [2006] and **?**). However, they have had more difficulty trying to implement a contracting problem which also includes non-project risk. In this paper, that means capturing the impact of the bank-specific shocks and the aggregate shocks using standard securities. My approach can be seen as complementary to the implementation in Piskorski and Tchistyi [2010] but with regime shifting and debt recalls.

[21] In this paper, I will consider the bank holding the outside equity issued by the entrepreneur. Alternatively, the problem could be set up so that the banker acts as an intermediary and sells the equity on to households.

[22] I choose to implement the contract using debt recalls because debt recalls were observed during the crisis. An alternative would be to implement the contract by using option adjustable-rate contracts, as in Piskorski and Tchistyi [2010]. In this implementation, the banker commits to a debt recall process in advance. The reason this is necessary is that the optimal contract is not renegotiation proof and a bank with discretion over debt recalls would choose to forgive debt when their value function is upward sloping in the entrepreneur's continuation value. If I also allowed renegotiation, then I could implement the optimal contract with a full commitment to the credit line and discretionary debt recalls. However, renegotiation also introduces other counterfactual behaviour so I restrict attention in the main text to the case of full bank commitment.

Proposition 3. *Suppose that the entrepreneur takes on the following capital structure in exchange for receiving I funds to start the project:*

- *The entrepreneur gives the banker outside equity for a fraction $1-\beta$ of the project and retains inside equity for a fraction β of the project,*
- *The entrepreneur takes out a long-term debt consol, with starting amount denoted by D_0:*
  - *(i) Before the covenant is breached ($f=0$), the flow coupon payment and debt recall processes are:*

$$dx_t^0 = \Big(zA + \frac{1}{\beta}\Big(-r_c\overline{W}^{0*}(z_t) + \Psi_z^0(\overline{W}^{0*}(z_t) - \beta M_t, z_t)\lambda_z(z_t) + W^U(z_t)\lambda_c\Big)\Big)dt$$
$$d\gamma_t^0 = \Big(-\frac{1}{\beta}\Psi_z^0(\overline{W}^{0*}(z_t) - \beta M_t, z_t) + \overline{M}_t^{L0}(z_t^c) - \overline{M}_t^{L0}(z_t) - (W^U(z_t^c) - W^U(z_t))\Big)dN_{z,t}$$

  - *(ii) After the covenant is breached ($f=1$), the flow coupon payment and debt recall processes are:*

$$\begin{aligned} dx_t^1 = \Big(zA + \frac{1}{\beta}\Big(&-r_c\overline{W}^{0*}(z_t) + \Psi_\epsilon^0(\overline{W}^{1*}(\epsilon_t, z_t) - \beta M_t, \epsilon_t, z_t)\lambda_\epsilon(\epsilon|z_t) \\ &+ \sum_{\epsilon'\in\mathcal{E}} \Psi_{z,\epsilon'}^1(\overline{W}^{1*}(\epsilon_t, z_t) - \beta M_t, \epsilon_t, z_t)\mathbb{1}_{\epsilon'}(z_t^c)\lambda_z(z_t) + W^U(z_t)\lambda_c\Big)\Big)dt \\ d\gamma_t^1 = \Big(&-\frac{1}{\beta}\Psi_\epsilon^1(\overline{W}^{1*}(\epsilon_t, z_t) - \beta M_t, \epsilon_t, z_t) + \overline{M}^{L1}(\epsilon_t^c, z_t) - \overline{M}^{L1}(\epsilon_t, z_t)\Big)dN_{\epsilon,t} \\ &+ \sum_{\epsilon'\in\mathcal{E}} \Big(-\frac{1}{\beta}\Psi_{z,\epsilon'}^1(\overline{W}^{1*}(\epsilon_t, z_t) - \beta M_t, \epsilon_t, z_t) + \overline{M}^{L1}(\epsilon', z_t) - \overline{M}^{L1}(\epsilon_t, z_t^c) \\ &\qquad - (W^U(z_t^c) - W^U(z_t))\Big)\mathbb{1}_{\epsilon'}(z_t^c)dN_{z,t} \end{aligned}$$

- *The entrepreneur has starting balance on the credit line equal:*

$$M_0 = I - D_0 - (1-\beta)E_0$$

  where E_0 is the value of the equity claim. The interest rate on the credit line is $r^c = \rho_e + \lambda_c$. The limit on the credit line in regimes 0 and 1 are:

$$\overline{M}_t^{L0}(z_t) = \frac{1}{\beta}\Big(\overline{W}^{0*}(z_t) - W^U(z_t)\Big)$$
$$\overline{M}_t^{L1}(\epsilon_t, z_t) = \frac{1}{\beta}\Big(\overline{W}^{1*}(\epsilon_t, z_t) - W^U(z_t)\Big)$$

- *The covenant threshold is:*

$$\overline{M}^c(z_t) = \frac{1}{\beta}\Big(\overline{W}^{0*}(z_t) - \underline{W}^\Phi(z_t)\Big)$$

This capital structure implementes the optimal contract, $\mathcal{C}^*$. In particular, it is incentive compatible for the entrepreneur to report project flows truthfully. It is also weakly optimal for the entrepreneur to pay no dividends (or consumption) until the credit line is paid off and then, once $M_t = 0$, pay out all excess project flows as dividends (and consumption).

In this case, the continuation value of the entrepreneur in regimes 0 *and* 1 *can be expressed as:*

$$\begin{aligned} W_t^0 &= \underline{W}^{\Phi}(z_t) + \beta(\overline{M}^c(z_t) - M_t) \\ &= W^U(z_t) + \beta(\overline{M}^{L0}(z_t) - M_t) \\ W_t^1 &= W^U(z_t) + \beta(\overline{M}^{L1}(\epsilon_t, z_t) - M_t) \end{aligned}$$

Proof. See Appendix A.5. □

There are two important points to note about this implementation. Firstly, observe that there is a one-to-one mapping between the continuation value of the entrepreneur and remaining balance on the credit line, $\overline{M}_t^{Lf} - M_t$. So, we can interpret the remaining balance on the credit line as the "economic" state variable for the contracting problem.

Secondly, we can see how the bank's debt recall process works. Consider the case in which the banker gets an adverse cost of funds shock but the aggregate state does not change (i.e. $dN_{\epsilon,t} = 1$). If the project does not have a covenant flag, then the terms of the debt securities do not change. However, if the project does have a covenant flag, then the banker responds by:

(i) *Decreasing* the credit limit:

$$\overline{M}^{L1}(\epsilon_B, z_B) - \overline{M}^{L1}(\epsilon_G, z_B) = \frac{1}{\beta}(W^{1*}(\epsilon_B, z_B) - W^{1*}(\epsilon_G, z_B)) < 0$$

(ii) *Recalling* debt in the amount of:

$$-\frac{1}{\beta}\Psi_\epsilon^1(\overline{W}^{1*}(\epsilon_G, z_B) - \beta M_t, \epsilon_G, z_B) + \overline{M}^{L1}(\epsilon_B, z_G) - \overline{M}^{L1}(\epsilon_G, z_B) > 0$$

where the sign comes from $\Psi_\epsilon^1(\cdot, \epsilon_G, z_B) < 0$ (since the banker is going from the good state ϵ_G to the bad state ϵ_B).

(iii) *Increasing* the coupon rate by:

$$\frac{1}{\beta}\left(\Psi_\epsilon^1(\overline{W}^{1*}(\epsilon_B, z_B) - \beta M_t, \epsilon_B, z_B) - \Psi_\epsilon^1(\overline{W}^{1*}(\epsilon_G, z_B) - \beta M_t, \epsilon_G, z_B)\right) > 0$$

where the sign comes from $\Psi_\epsilon^1(\cdot, \epsilon_B, z_B) > 0$ and $\Psi_\epsilon^1(\cdot, \epsilon_G, z_B) < 0$

Chodorow-Reich and Falato [2017] show empirically that these were the adjustments to debt terms that bad banks made during the recent financial crisis to firms that breached covenants.

In order to repay the debt recall, the entrepreneur must draw down on their credit line. If the draw is sufficiently large, then the project is terminated. This means that the entrepreneur's continuation value changes by:

$$\Delta W_t^1 = \Delta W^U(z_t) + \beta(\Delta \overline{M}^{L1}(\epsilon_t, z_t) - \Delta M_t) = +\Psi_\epsilon^1(\overline{W}^{1*}(\epsilon_G, z_B), \epsilon_G, z_B) < 0$$

which illustrates why the optimal contract is implemented by securities in Proposition 3.

4 Credit Market Equilibrium

This section puts the contracting problem studied is section 3 into general equilibrium by introducing a bilateral directed search market for bank contracts. This endogenises both the outside option of the entrepreneur and the initial value promised by the banker at the start of the contract. So, in equilibrium, it is now the frictions in the contracting problem that determine the rate of financing in the search market and amplify the impact of a financial crisis. In the normal aggregate state, all banks have the same cost of funds but then, when the crisis occurs, banks get hit by idiosyncratic shocks to their cost of funds. In a frictionless economy, entrepreneurs would leave the "bad" banks and move to the "good" banks but because of the search friction this is costly and so, instead, the long-term bank-entrepreneur contracts are conditioned on the bank's cost of funds (in the threshold manner described in section 3). This means that, during a crisis, there is a distribution of funded projects in the economy that differs by the continuation value promised to the entrepreneur (or, in the implementation, the remaining draw on the entrepreneur's credit line) and whether or not they have a covenant flag.

4.1 Agent Problems

This subsection sets up the problems for bankers and entrepreneurs in the directed search market.

4.1.1 Unattached Entrepreneurs

Let $W^U(\mathbf{x}, z)$ denote the value function for an unmatched (U) $\mathbf{x}$-type entrepreneur when the aggregate state is z. Then,

$$W^U(\mathbf{x}, z) = \max_{w \in \mathcal{W}} \left\{ W^{U,w}(\mathbf{x}, z) \right\} \tag{4.1}$$

where $W^{U,w}(\mathbf{x}, z)$ denotes the value function for an entrepreneur searching in an arbitrary submarket w and $W^{U,w}(\mathbf{x}, z)$ solves the Hamilton-Jacobi Bellman equation:

$$\rho_e W^{U,w}(\mathbf{x}, z) = \mu(\theta(w))(w - W^{U,w}(\mathbf{x}, z)) + \lambda_z(z)(W^{U,w}(\mathbf{x}, z^c) - W^{U,w}(\mathbf{x}, z))$$

Since entrepreneurs only choose to apply to submarkets with the highest expected valuation offers, markets that are active in equilibrium must be solutions to the entrerpeneur's problem. That is, for all $(w, z) \in \mathcal{W} \times \mathcal{Z}$

$$W^{U,w}(\mathbf{x}, z) \leq W^{U}(\mathbf{x}, z), \quad \text{with equality if and only if } \theta(w; z) > 0$$

Let $\mathcal{W}^*(z) \equiv \{w \in \mathcal{W} : \theta(w; z) > 0\} \subseteq \mathcal{W}$ denote the set of active markets in state z. Then, for all $w \in \mathcal{W}^*(z)$, it holds that:

$$\mu(\theta(w))(w - W^{U,w}(\mathbf{x}, z)) = \Gamma^B(z)$$

where the left hand side is the expected value of a match to the entrepreneur and $\Gamma^B(z)$ is the opportunity cost of searching:

$$\Gamma^B(z) \equiv \rho_e W^{U,w}(\mathbf{x}, z) - \lambda_z(z)(W^{U,w}(\mathbf{x}, z^c) - W^{U,w}(\mathbf{x}, z))$$

4.1.2 Unattached Bankers

Let $V^U(z)$ denote the value function for an unmatched banker (U) when the aggregate state is z. Then, $V^U(z)$ solves

$$\rho_h V^U(z) = \max_{w \in \mathcal{W}, \mathbf{x} \in \mathcal{X}} \{V^{U,w}(z)\}$$

where $V^{U,w}(z)$ denotes the value for a banker posting in an arbitrary submarket w. This solves the Hamilton-Jacobi-Bellman equation:

$$\rho V^U(z) = \max_{w \in \mathcal{W}, \mathbf{x} \in \mathcal{X}} \Big\{ -\zeta + \eta(\theta(w))(V^{F,0}(w, \mathbf{x}, z) + (I - D) - V^U(z)) \Big\} + \lambda_z(z)(V^U(z^c) - V^U(z))$$

where $V^{F,0}(w, \epsilon, \mathbf{x}, z)$ denotes the banker's value from promising w to an $\mathbf{x}$-type entrepreneur when the aggregate state is z and contract starts in regime 0. In equilibrium, free entry implies that

$$V^U(z) = 0 \tag{4.2}$$

and $I = D$. This means that the U and F notation is redundant for the banker and $V^{F,0}(w, \epsilon, \mathbf{x}, z)$ will be denoted by $V^0(w, \epsilon, \mathbf{x}, z)$. It follows that, in equilibrium, for active markets, it must hold that:

$$\zeta = \eta(\theta(w))V^0(w, \epsilon, \mathbf{x}, z) \tag{4.3}$$

That is, in active markets, the banker equates the expected value from matching with the cost of posting.

4.2 Evolution of Project Distribution

Given the optimal contract derived in the previous section, the evolution of the distribution of projects can be expressed with a Kolmogorov Forward equation. For each type, $\mathbf{x}$, the population of entrepreneurs with that particular type has its own Kolmogorov Forward equation so the variable $\mathbf{x}$ is dropped to streamline the notation.

Let $m_t^0(w,\epsilon)$ denote the distribution of funded entrepreneurs with projects that have not breached a covenant. There is a mass point at $w_0(z)$, the equilibrium value promised $\mathbf{x}$-type entrepreneurs in the search market. For $w \in [\underline{W}^{\Phi}(z), W^{0*}(z)]/\{w_0(z)\}$, the Kolmogorov equation follows:

$$
\begin{aligned}
&dm_t^0(w,\epsilon)\\
&= \underbrace{-\lambda_c m_t^0(w,\epsilon)}_{\substack{\text{Exit from exogeneous}\\ \text{project termination}}} \quad - \underbrace{\partial_w\left[(\rho_e w - \lambda_c(W^u(z)-w) - \lambda_z(z)\Psi_z(w,z))m_t^0(w,\epsilon)\right]dt}_{\substack{\text{Change in density from the drift in}\\ \text{entrepreneur continuation value}}}\\
&\quad + \frac{1}{2}\underbrace{\partial_{WW}\left[\sigma^2\beta^2 m_t^0(w,\epsilon)\right]}_{\substack{\text{Change in density from}\\ \text{project volatility}}} + \lambda_\epsilon(\epsilon)\underbrace{\left(m_t^0(w,\epsilon^c) - m_t^0(w,\epsilon)\right)}_{\substack{\text{Change in density}\\ \text{when } \epsilon \text{ changes}}}\\
&\quad + \underbrace{\left[(1-\partial_w\gamma_z(w,z))\sum_{\epsilon'\in\mathcal{E}}\overline{\pi}(\epsilon'|z)m_t^0(w-\gamma_z(w,z),\epsilon') - m_t^0(w,\epsilon)\right]}_{\substack{\text{Change in density when } z \text{ changes and the entrepreneur's}\\ \text{continuation value is shifted by } \Psi_{z,t}^0(W_t,z)}} dN_{z,t}
\end{aligned}
\tag{4.4}
$$

where $\gamma_z(w,z)$ is defined by the equivalence:

$$
W_t + \Psi_{z,t}^0(W_t,z) = w \quad \Leftrightarrow \quad W_t = w - \gamma_z(w,z)
$$

and there is an absorbing lower boundary condition and a reflecting upper boundary condition:

$$
\begin{aligned}
m_t^0(\underline{W}^{\Phi}(z),\epsilon) &= 0, \quad \forall \epsilon \in \mathcal{E}, z \in \mathcal{Z}\\
\partial_W m_t^0(\overline{W}^{0*}(z),\epsilon) &= 0, \quad \forall \epsilon \in \mathcal{E}, z \in \mathcal{Z}
\end{aligned}
$$

Likewise, let $m_t^1(w,\epsilon)$ denote the distribution of funded entrepreneurs with projects that have breached a covenant. In this regime, there are mass points at $\{\underline{W}^{\Phi} + \Psi_{\Phi}(\epsilon,z)\}_{\epsilon\in\mathcal{E}}$. For $w \in [W^U(z), \overline{W}^{1*}(\epsilon,z)]/\{\underline{W}^{\Phi} + \Psi_{\Phi}(\epsilon,z)\}_{\epsilon\in\mathcal{E}}$, the

Kolmogorov forward equation is:

$$
\begin{aligned}
&dm_t^1(w,\epsilon)\\
&= \underbrace{-\lambda_c m_t^1(w,\epsilon)}_{\substack{\text{Exit from exogeneous}\\ \text{project termination}}} + \frac{1}{2}\underbrace{\partial_{WW}\left[\sigma^2\beta^2 m_t^1(w,\epsilon)\right]}_{\substack{\text{Change in density from}\\ \text{project volatility}}}\\
&\quad \underbrace{-\,\partial_w\left[\Big(\rho_e w - \lambda_c(W^u(z)-w) - \lambda_\epsilon(\epsilon,z)\Psi_\epsilon(w,\epsilon,z) - \lambda_z(z)\sum_{\epsilon\in\mathcal{E}}\overline{\pi}(\epsilon|z)\Psi^1_{z,\epsilon}(w,\epsilon,z)\Big)m_t^1(w,\epsilon)\right]}_{\text{Change in density from the drift in entrepreneur continuation value}} dt\\
&\quad + \lambda_\epsilon(\epsilon)\underbrace{\Big((1-\partial_w\gamma_\epsilon^1(w,\epsilon,z))m_t^1(w-\partial_w\gamma_\epsilon^1(w,\epsilon,z),\epsilon^c) - m_t^1(w,\epsilon)\Big)}_{\text{Change in density when } \epsilon \text{ changes}}\\
&\quad + \Big[\underbrace{\sum_{\epsilon'\in\mathcal{E}}(1-\partial_w\gamma^1_{z,\epsilon'}(w,\epsilon,z))\overline{\pi}(\epsilon'|z)m_t^1(w-\gamma^1_{z,\epsilon'}(w,\epsilon,z),\epsilon') - m_t^1(w,\epsilon)}_{\substack{\text{Change in density when } z \text{ changes and the entrepreneur's}\\ \text{continuation value is shifted by } \Psi^1_{z,\epsilon,t}(W_t,\epsilon,z)}}\Big]dN_{z,t}
\end{aligned}
\tag{4.5}
$$

where $\gamma_\epsilon^1(w,\epsilon,z)$ and $\gamma^1_{z,\epsilon}(w,\epsilon,z)$ are defined by the respective equivalences:

$$
\begin{aligned}
W_t + \Psi_\epsilon^1(W_t,\epsilon,z) = w \quad &\Leftrightarrow \quad W_t = w - \gamma_z^1(w,\epsilon,z)\\
W_t + \Psi_{z,\epsilon}^1(W_t,\epsilon,z) = w \quad &\Leftrightarrow \quad W_t = w - \gamma_{z,\epsilon}^1(w,\epsilon,z)
\end{aligned}
$$

and there is an absorbing lower boundary condition and a reflecting upper boundary condition:

$$
\begin{aligned}
m_t^0(W^U(z),\epsilon) = 0, \quad &\forall \epsilon\in\mathcal{E}, z\in\mathcal{Z}\\
\partial_W m_t^0(\overline{W}^{1*}(\epsilon,z),\epsilon) = 0, \quad &\forall \epsilon\in\mathcal{E}, z\in\mathcal{Z}
\end{aligned}
$$

Given the evolution of the Kolmogorov Forward equation, we can find expressions for the transition rates from regimes. This is done in corollary 1. Conceptually, there are two ways that the project can hit the lower boundary: it can drift towards the boundary or a shock can cause the project to hit the boundary.

Corollary 1. *The exit rates are:*

(a) In regime 0, the rate of outflow at the lower boundary, $w = \underline{W}^\Phi$, is:

$$
\begin{aligned}
J_t^0(w,\epsilon) &= (\rho_e w - \lambda_c(W^u(z)-w) - \lambda_z(z)\Psi_z(w,z))m_t^0(w,\epsilon)\\
&\quad + \partial_w\left[\frac{1}{2}\sigma^2\beta^2 m_t^0(w,\epsilon)\right] + \lambda_z(z)\int_{\underline{W}^\Phi}^{W^{0,c}} m_t^0(w,\epsilon)dW
\end{aligned}
\tag{4.6}
$$

where $W^{0,c}$ is the cutoff at which the shock to z causes the project to hit the lower boundary. That is, $W^{0,c}$ satisfies $W^{0,c} + \Psi_z^0(W^{0,c},z) = W^U(z)$.

(b) *In regime 1, the rate of outflow at the lower boundary, $w = W^U(z)$, is:*

$$J_t^1(w,\epsilon) = (\rho_e w - \lambda_c(W^u(z) - w) - \lambda_\epsilon(\epsilon,z)\Psi_\epsilon(w,\epsilon) - \lambda_z(z)\sum_{\epsilon\in\mathcal{E}}\overline{\pi}(\epsilon|z)\Psi^1_{z,\epsilon}(\epsilon,z))m_t^1(w,\epsilon)$$
$$+ \partial_w\left[\frac{1}{2}\sigma^2\beta^2 m_t^1(w,\epsilon)\right] + \lambda_\epsilon(\epsilon|z)\int_{W^U(z)}^{W_\epsilon^{1,c}(\epsilon)} m_t^1(w,\epsilon)dw$$
$$+ \lambda_z(z)\sum_{\epsilon'\in\mathcal{E}}\overline{\pi}(\epsilon',z)\int_{W^U(z)}^{W^{1,c}_{z,\epsilon}(\epsilon,z)} m_t^1(w,\epsilon)dw \tag{4.7}$$

where $W_\epsilon^{1,c}(\epsilon)$ is the cutoff at which a shock to ϵ causes the project to hit the boundary and $W^{1,c}_{z,\epsilon}(\epsilon,z)$ is the cutoff at which a shock to z causes the project to hit the boundary. That is, $W_\epsilon^{1,c}$ satisfies $W_\epsilon^{1,c} + \Psi_\epsilon^0(W_\epsilon^{1,c},\epsilon,z) = W^U(z)$ and $W^{1,c}_{\epsilon,z}$ satisfies $W^{1,c}_{\epsilon,z} + \Psi^0_{z,\epsilon}(W^{1,c}_{\epsilon,z},\epsilon,z) = W^U(z)$

4.3 Equilibrium

The directed search equilibrium is defined following the work of Moen [1997].

Definition 1 (Equilibrium)**.** A directed search equilibrium is a collection of a market tightness function $\theta(\cdot,z) : \mathcal{X} \to \mathbb{R}$, a value function of an unattached entrepreneur, $W^U(z,\mathbf{x}) : \mathcal{Z}\times\mathcal{X} \to \mathbb{R}$, a value function for a funded entrepreneur: $W^{F,f}(\mathcal{C})$, for each regime $f \in \{0,1\}$, an initial promised continuation value function, $w_0(z,\mathbf{x}) : \mathcal{Z}\times\mathcal{X} \to \mathbb{R}$, a value function for an unattached banker, $V^U(z) : \mathcal{Z} \to \mathbb{R}$, a value function for a funding banker $V^f(W,\epsilon,z,\mathbf{x}) : \mathcal{W}\times\mathcal{E}\times\mathcal{Z}\times\mathcal{X} \to \mathbb{R}$ for each regime $f \in \{0,1\}$, a distribution of contracts: $m_t^f(W,\epsilon;z,\mathbf{x})$ for $f \in \{0,1\}$, an aggregate measure of unfunded entrepreneurs, $m_t^U(\mathbf{x},z)$ such that:

1. Given the market tightness function, θ, the unattached entrepreneur's value function, W^U and choice of market, w_0, satisfy (4.1),

2. The funded entrepreneur's value functions, $\{W^{F,f} : f \in \{0,1\}\}$ solves (3.2),

3. Given the market tightness function, θ, the unattached banker's value function, V^U and value choice to post in market w_0 satisfy (4.2) and the free entry condition (4.3),

4. The funding banker's value function, $\{V^f : f \in \{0,1\}\}$, solves (3.3),

5. For each type, $\mathbf{x}$, the distribution of contracts $\{m_t^f : f \in \{0,1\}\}$, satisfies (4.4) and (4.5), and

6. The aggregate measure of unfunded entrepreneurs satisfies $dm_t^U = (-\mu(\theta) + J_t^1(w,\epsilon))dt$, where $J_t^1(w,\epsilon)$ satisfies (4.7).

4.4 Endogenising the Outside Option and Initial Terms

We can now return to the banker's value function, which is plotted in all states and regimes in figure 7. The shape of the value function follows from the description in section 3. However, the interaction between the contracting problem and the search market can now be seen. First, consider the value functions for the initial regime 0 (depicted on the left of the figure). As can be seen, the initial value posted by the bankers, W_0, is between the peak of the value function and the horizontal intercept. This illustrates how the directed search market compares to other market structures. If the banker was a monopolist, then they would set the value at the peak of their value function. At the other extreme, if the banker was in a competitive market, then they would end up with zero expected value (given the probability of getting an adverse cost of funds shocks). It can also be observed that because the banker cannot condition the contract on their cost of funds, they must set the same payout boundary, W^{0*} and covenant threshold boundary, $\overline{W}$, regardless of their cost of funds. Finally, the difference in height between the value functions in regime 0 and regime 1 at the lower boundary is the cost, Φ, of switching to regime 1.

Now, consider the banker's value function in regime 1 (depicted on the right of the figure). It can be observed that the crisis state decreases the outside option of the entrepreneur (as will be seen, this is because they match more slowly and receive a lower initial value). This is depicted graphically by the decrease in $V^U(z)$. Decreasing the outside option actually partially offsets the loss of surplus during a crisis because it makes termination less likely and so decreases the impact of the agency friction.

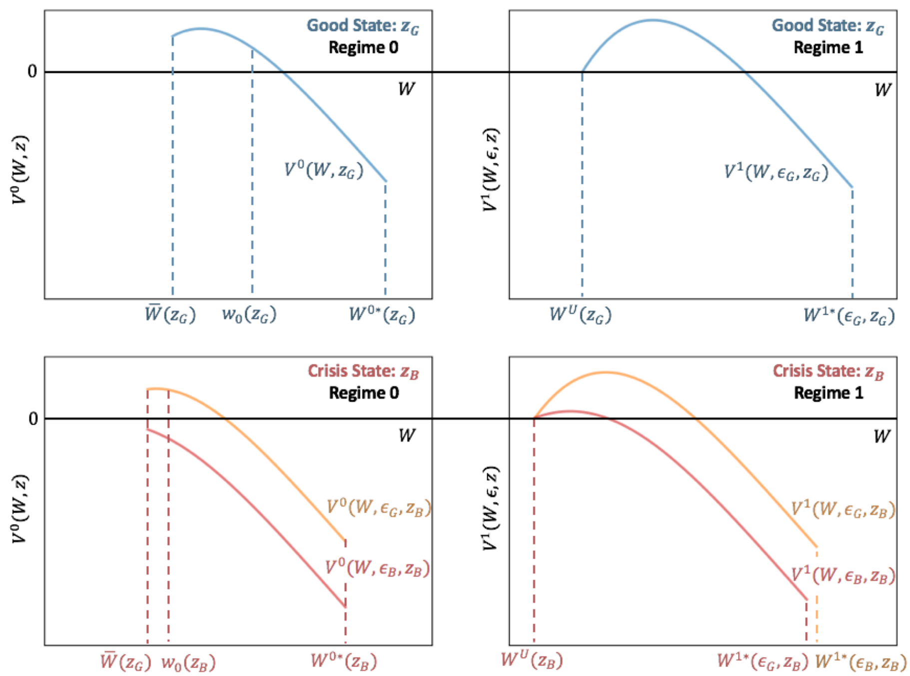

Figure 7: *Regime 1: Banker's Value Function* The left upper plot depicts the banker's value function in regime 0 (before the covenant binds) and the good aggregate state. The right upper plot depicts the banker's value function in regime 1 (after the covenant binds) and the good aggregate state. The left lower plot depicts the banker's value function in regime 0 and the bad aggregate state. The right lower plot depicts the banker's value function in regime 1 and the bad aggregate state.

5 Numerical Exercise

In this section, I undertake a numerical exercise in order to understand and quantify the impact of the contracting frictions on the aggregate economy during a financial crisis. I start by calibrating a baseline model to the work of Chodorow-Reich and Falato [2017]. I then use the calibrated model to investigate the channels by which the contracting frictions lead to credit disruption and output decline during the crisis. The primary channel is that project surplus decreases and so bankers want to recall debt. This leads to an increase in the aggregate termination rate and a decrease in the aggregate financing rate. A secondary effect is that banker heterogeneity potentially amplifies this channel. When the crisis arrives, "bad" bankers want to recall more debt than "good" bankers which increases the dispersion of debt recall size and, ultimately, further increases the aggregate termination rate. The size of this secondary effect is governed by the level of the covenant threshold, which determines the equilibrium fraction of projects that have breached their covenant and can be conditioned on the banker's idiosyncratic cost of funds.

5.1 Calibration Strategy

In this subsection, I discuss how I calibrate a baseline model of the economy with one project type. My approach is to choose parameters for the project technology and shock processes that are consistent with the broader banking literature. I then use the work of Chodorow-Reich and Falato [2017] to discipline my choice of the novel parameters from my contracting problem: the bank's cost of changing regime, Φ, the size of the adverse idiosyncratic shock during a crisis, ϵ_B, and the agency friction, β.

The only additional functional form that needs to be specified is the matching function in the search market. For the rest of this paper, I impose a Cobb-Douglas matching function:

$$M(E, B) = E^{\gamma} B^{1-\gamma}$$

It follows that the matching rates for entrepreneurs and bankers are:

$$\mu(\theta) = \frac{M(E, B)}{E} = \theta^{\gamma - 1}$$
$$\eta(\theta) = \frac{M(E, B)}{B} = \theta^{\gamma}$$

5.1.1 Discussion of Parameters

The full set of parameters for my baseline model is shown in table 1. I choose most of the project technology parameters to be consistent with DeMarzo et al. [2012] and Piskorski and Tchistyi [2010]. The extra parameter is the exogenous termination rate, which I set to target an average loan maturity of 3.3 years, as reported by Chodorow-Reich and Falato [2017].

The switching rates for the continuous time Markov processes for ϵ and z are taken from the banking literature. I choose the rate at which banking crises occur to match **?**, who estimates that, on average, a banking crisis occurs every 33 years. In the crisis aggregate state, the rate at which banks switch between idiosyncratic states is set to reflect the rate at which bank balance sheets recover following adverse shocks. In doing this, I reference the work of Begenau et al. [2017] who estimate that 80% of the impact of an adverse bank shock dissipates within five years.[23] This is not an exact match to my model because I have discrete changes whereas the data shows banks recovering gradually. However, using this as a guide, I choose $\lambda_\epsilon(\epsilon_B, z_B) = 0.05$. This implies that the rate at which a bank recovers from an adverse state (either because the idiosyncratic state of the bank changes or because the crisis ends) is approximately given by $\lambda_z(z_B) + \lambda_\epsilon(\epsilon_B, z_B) = 0.25$ and so, on average, it takes a bank approximately four years to completely recover.

In order to calibrate the remaining parameters, I need a simple approximation for the interest rate implied by the contract. I use the constant interest rate implied by the contract (which I refer to as the annuity rate and denote by R) as a proxy for the interest rate the banker charges the entrepreneur.[24] Using this proxy, I then choose the mean productivity, A, and the common deposit rate, $\bar{r}$, so that the model approximation for the Net Interest Margin (NIM) in the non-crisis state, $R(z_G) - r(\epsilon_G)$, is approximately 5%, as estimated by **??**. Under this calibration, the banker's discount rate is higher than the common deposit rate $\bar{r}$. This is consistent with a model where bankers raise equity and deposits from risk neutral households who have discount rate ρ_h and receive a liquidity benefit from holding deposits (see **?** for an example model).

I choose the search parameters to be approximately consistent with **?**. They refer to Eurostat data for the UK that estimates that the annual success rate for firms seeking credit fell by approximately 30% during the crisis. I choose my banker posting cost to be approximately consistent with this number. I set the elasticity of the matching function to be 0.5. **?** sets this parameter to get that a 1% increase in the interest rate on bank loans leads to a 15% increase in the entrepreneur's matching rate. I don't have a direct comparison for this in my model but I use the responsiveness of μ to the annuity rate, R, as a sense check comparison.

Finally, the level of the adverse bank shock, ϵ_B, the agency friction, β, and

[23] In particular, they find that, after a 10% return shock, the bank's leverage ratio increases and, after five years, has only returned to 80% of its initial level.

[24] Suppose that the contract, $\mathcal{C}$, gives the entrepreneur a value of w_0. Then the constant interest rate implied by the contract (referred to as the annuity rate) is the constant flow, R, that satisfies:

$$\frac{w_0(z) - W^u(z)}{I} = \mathbb{E}\left[\int_0^{\tau_c} e^{-\rho_e s}(dy_s - Rds)\right]$$

In other words, the implied annuity rate is the constant interest rate that delivers the gain $w_0 - W^u(z)$ to the entrepreneur. As was shown earlier, the actual implementation of the optimal contract is significantly more complicated than a constant rate debt contract but the annuity rate can serve as a sense check.

the cost of changing regimes, Φ, are calibrated to match the work Chodorow-Reich and Falato [2017]. The cost in each aggregate state is chosen to match the frequency with which covenants bind in that state. The other two parameters are chosen to match the coefficients in their regression of credit reduction on covenants binding and banks getting adverse shocks. I focus on the results from this calibration exercise in the next subsection.

Externally Calibrated Parameters		
Parameter	Value	Comments
z	$(1.0, 1.02)$	Recession TFP drop
λ_z	(0.2, 0.03)	Crisis frequency from **?**
(ρ_h, ρ_e)	(0.05, 0.06)	From DeMarzo et al. [2012] and Piskorski and Tchistyi [2010]
σ_κ	0.25	From DeMarzo et al. [2012]
L	0.95	From DeMarzo et al. [2012]
I	0.95	Bank issued deposits are risk free
λ_c	0.3	Average loan maturity of 3.3 years (Chodorow-Reich and Falato [2017])
$\bar{r}$	0.01	Risk free rate
ϵ_G	0.0	Normalisation
$\lambda_\epsilon(z_B)$	(0.01, 0.05)	Reflects frequency of bank losses and recapitalisation
γ	0.5	Approximately consistent with **?**

Internally Calibrated Parameters		
ζ	(0.05, 0.05)	Targets $\approx 30\%$ decline in μ during crisis
A	0.09	Targets average Bank NIM from **??**
Φ	$(0.10, 0.06)$	Targets $\mathbb{P}$[Covenent Binds] from Chodorow-Reich and Falato [2017]
(ϵ_B, β)	$(0.08, 0.45)$	Targets $\mathbb{E}$[Credit Reduction|Bad Bank, Covenant] from Chodorow-Reich and Falato [2017]

Table 1: Calibration for Baseline Model: This table contains the baseline parameters that are chosen for the baseline model.

5.1.2 Replicating Chodorow-Reich (2017)

The work of Chodorow-Reich and Falato [2017] provides a natural empirical counterpart to test the model. In that paper, the authors use matched bank-entrepreneur contract data to estimate how firms deleveraged long-term loans during the 2007-09 financial crisis. Two estimates are particularly relevant: (i) the probability that a contract ends up with a covenant flag and (ii) the expected credit reduction conditional a contract receiving a covenant flag and being attached to a "bad" bank.

The model and empirical counterparts for statistic (i) are included in table 2. The model counterpart is calculated by starting the economy in the steady state for z_G, then running the economy with one particular aggregate state for two years and calculating the fraction of projects that start without a covenant flag but end up w ith a covenant flag during the two years. The results from this exercise are shown in table 2.

Chodorow-Reich and Falato [2017] estimate statistic (ii) by running the regression:

$$\%\Delta\text{Credit} = \beta_0 + \beta_1[\text{Bad Lender}] + \beta_2[\text{Binds}] + \beta_3[\text{Bad Lender} \times \text{Bind}]$$

where $\%\Delta$Credit is the percentage change in credit during a year of the 2007-09 financial crisis, [Bad Lender] is an index of lender quality[25] and [Binds] is an indicator variable for whether a covenant bound over the year. The regression is repeated using the model in this paper and the results are shown in table 3. As can be seen, the model is able to replicate the results of the regression closely. One caveat to note here is that the numbers from Chodorow-Reich and Falato [2017] represent the total decline in credit along both the intensive and extensive margins whereas the model in this section only has credit reduction along the extensive margin.

	CRF (2017)	Model
$\mathbb{P}(\text{Covenant binds}; z_G)$	0.28	0.20
$\mathbb{P}(\text{Covenant binds}; z_B)$	0.37	0.35

Table 2: *Probability of Covenant Binding: Comparison to Chodorow-Reich and Falato [2017].* In Chodorow-Reich and Falato [2017], $\mathbb{P}(\text{Covenant binds}; z)$ is calculated as the fraction of loans that breached a covenant over the period 2006-7 (for z_G) and 2008-9 (for z_B). The model counterpart is calculated in the following way. The economy is started in the steady state for z_G. $\mathbb{P}(\text{Covenant binds}; z)$ is then calculated as the fraction of projects that start without a covenant flag but end up with a covenant flag after two years if the aggregate state is z over the two year period.

5.1.3 Other Key Moments

I show a selection of other moments from the calibration in table 4. The top part of the table shows the targetted moments that are unrelated to Chodorow-Reich and Falato [2017]. The lower part of the table shows moments that are not targetted but are of general interest. We can first observe that the entrepreneur's matching rates in the good and bad states are 1.66 and 1.14 respectively. These imply that the entrepreneur's expected search durations in the good and bad states are approximately 7 months and 10.5 months, which are similar in magnitude to the estimates in **?**. Like the authors of that paper, I interpret the duration as representing not only the time spent filling in bank

[25] THe bad lender index is based on the bank's exposure to Lehman Brother's, their holdings of mortgage-backed securities and the fraction of their revenue that comes from trading.

Regression		
Dependent Variable: %ΔCredit		
Independent Variables:	CRF (2017)	Model
Bad Bank (β_1)	0.2	0.0
Binds (β_2)	-3.2	-5.0
Bad Bank $\times$ Binds (β_3)	-22.9	-23.1

Table 3: Replication of Table 10 in Chodorow-Reich and Falato [2017]. In the model, the data for the regression is generated in the following way. The economy starts in the steady state for z_G. Then, the economy switches to the crisis state z_B and is run for one year. %ΔCredit is an indicator of whether the project is terminated over the year. [Bad Bank] is an indicator for whether the bank starts and finishes the year with a high cost of funds. [Binds] is an indicator of whether the project has a binding covenant at some time during the year.

applications but also the time the entrepreneur spends preparing to raise debt, the time the bank spends evaluating loan quality and the likelihood that a loan is deemed not creditworthy.

We can also compare the fraction of projects with funding and the fraction of funded projects with covenant flags in the crisis and non-crisis steady states. As can be seen, when the crisis occurs, the fraction projects with funding falls by 9 percentage points and the fraction of projects with covenant flags increases to almost 90%. However, it should be noted that even in the non-crisis steady state there are still 61% of contracts with binding covenant flags so many bankers have the option to adjust terms straight away when a crisis occurs.

5.2 Steady State Distribution of Project Credit Balances

A feature of this model is that, in equilibrium, the idiosyncratic project shocks generate a distribution of entrepreneur continuation values. In the implementation, this leads to a distribution of entrepreneur draws on their credit lines. Figure 8 depicts the simulated steady-state distribution of credit line balances for each aggregate state and in each regime, $f \in \{0, 1\}$.

We can see a number of key features of the model by comparing the crisis and non-crisis steady-state distributions. First, we can see that the support of the distribution expands during the crisis. This occurs because the entrepreneur's outside option, W^U, decreases and so entrepreneurs are less likely to default on their bank contracts and return to the search market. Ultimately, this decreases the impact of the agency friction in the contracting problem, which allows the banker to extend more credit. This effect is the general equilibrium mitigation of the crisis impact. Second, we can see that the proportion of banks at the upper boundary in the $f = 1$ regime decreases when the economy moves to the crisis steady-state. This is because banks start to recall debt, which causes

Targetted Moments		
Moment	Model	Data
$\%\Delta\mu(\theta, z_G \to z_B)$	-38.8%	$\approx -30\%$ during crisis
Net Interest Margin	5.2%	4.6%

Non-targetted Moments		
Moment	Model	Comment / Interpretation
$\mu(\theta, z_G)$	1.66	Expected duration of loan search is $\approx$ 7 months
$\mu(\theta, z_B)$	1.14	Expected duration of loan search is $\approx$ 10.5 months
Steady state fraction funded (z_G)	0.83	17% of entrepreneurs are searching
Steady state fraction funded (z_B)	0.74	26% of entrepreneurs are searching
Steady state fraction $f = 1$ (z_G)	0.61	61% of projects have binding covenants
Steady state fraction $f = 1$ (z_B)	0.88	88% of projects have binding covenants
$\partial\mu/\partial R$	49%	Higher than **?**

Table 4: Key moments from the calibrated model.

entrepreneurs to draw down on their credit lines and so move away from the upper boundary. Together the first two observations tell part of the story about how the heterogeneity between good and bad banks affects credit lines balances during the crisis. Conditional on being with a good bank, entrepreneurs find that they have a larger credit line limit during the crisis because the banker knows that it is a tough time for the entrepreneur to default on their contract. However, conditional on being with a bad bank, the entrepreneur finds that the banker starts to recall debt and so they end up closer to their termination limit even if the support of the credit line distribution has increased.

We can also observe how the covenant threshold choice and the initial contract terms change as the economy switches to the crisis state. The covenant threshold is the lower boundary on the support of the distribution in the $f = 0$ plot. As can be seen, the covenant threshold is higher during a crisis. This reflects the greater value of being able to adjust credit terms during the crisis. The initial contract values are the spikes in the distribution away from the upper boundary on the $f = 0$ plot. As can be seen, the banker offers an initial value that is very close to the boundary during a crisis. This is why the steady-state fraction of projects with covenants in the crisis state is so high in table 4.

5.3 Crisis Shocks

Now, I simulate a crisis in the economy and study how long term bank credit is disrupted. Figure 9 depicts the impulse response functions for a one year crisis shock under first best contracting (denoted in the figure as 1B for first best) and optimal contracting with agency frictions (denoted in the figure as 2B for

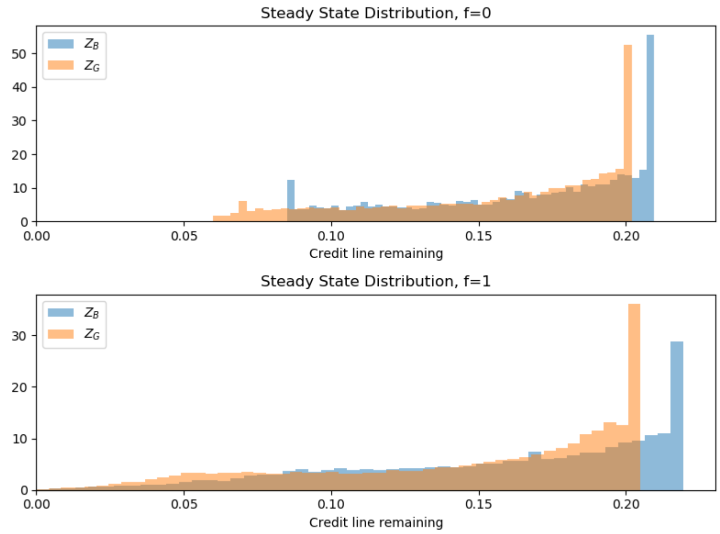

Figure 8: *Steady State Distribution of Credit Line Balances.* The figures show the frequency plots for the remaining draw entrepreneur's can make on their credit line in the good aggregate state (in blue) and the crisis aggregate state (in orange). Observe that the histograms are renormalised within each regime so they do not depict the relative distribution between the two regimes.

second best). As can be seen in the plot of normalised output, the aggregate impact of the contracting frictions is to amplify the peak decrease in output during the financial crisis by approximately 5 percentage points.

The remaining plots illustrate how the contracting frictions affect the economy. When the crisis state occurs, expected project surplus decreases for two reasons: productivity is lower and banks have either received an adverse funding cost shock or believe that they may receive one in the future. Under first best contracting, this leads to a lower entry rate and, consequently, a lower fraction funded but no impact on the exit rate which stays permanently at the exogenous exit rate of 0.3. The main differences when contracting under agency frictions are that there is endogenous early termination in equilibrium and that the rate of early termination is amplified during the crisis. The amplification occurs firstly as a spike in the exit rate and then secondly as an increase in the average exit rate during the crisis. Conceptually, this occurs because the bankers respond to the lower project surplus during the crisis by decreasing the entrepreneur's continuation value (or, in the implementation, recalling long-term debt). This can cause the participation constraint to bind forcing termination (or, in the implementation, cause borrowers to hit the termination limit on the credit line). The amplification of the termination rate and the decrease in project surplus then also have the combined impact of amplifying the decrease in the entry rate. Ultimately, all these effects amplify the decrease in the fraction of funded projects and, so, aggregate output.

Most of the plots depict the aggregate impact but there is significant heterogeneity in bank responses. This can be seen in the lower left plot which depicts the fraction of bad bankers to good bankers. As can be seen, when the crisis occurs, the fraction of bad bankers jumps up. In the case of first best contracting, it jumps up to the new steady state distribution of ϵ and stays there for the duration of the crisis. By contrast, in the case of contracting under agency frictions, it jumps up to less than the new steady state and then declines over time. The reason for this is that bad bankers who have projects with covenant violations recall more debt than other bankers and so end up exiting at a faster rate. Ultimately, the fraction of bad bankers does not impact aggregate output in this model because the projects produce the same output no matter what type of bank they are attached to. However, once working capital is introduced in section 6, this distributional impact starts to matter.

5.4 Role of Covenants

One of the novel features of this model is the introduction of covenants into the bank-firm contracting problem. This was done to match one of the stylised facts about contracting during the crisis and allowed me to calibrate the model to match the work of Chodorow-Reich and Falato [2017]. We can now explore the role that covenants play in the general equilibrium economy. Conceptually, the level of the covenant threshold (which is characterised by the size of the switching cost, Φ), governs the extent to which good and bad banks can act differently. For example, as Φ decreases, bankers raise the covenant threshold and so projects are more likely to end up with binding covenants that give bankers the right to condition debt recalls on their idiosyncratic cost of funds. This means that, in general equilibrium, the level of the covenant threshold has two effects in my model. The first effect increases the dispersion in the magnitude of the shocks to entrepreneur continuation values (or, in the implementation, to debt recalls). If $\Phi = \infty$ and so the covenant threshold never binds, then good and bad banks are forced to respond to an adverse ϵ shock with the same moderate decrease in the entrepreneur's continuation value. However, if $\Phi = 0$ and so the covenant threshold always binds, then bad banks decrease the entrepreneur's continuation value by relatively more and good banks by relatively less. Ultimately, this means that lowering Φ effectively has a similar impact on the contracting problem as increasing project volatility – it increases the probability that the project hits the lower boundary and is terminated. The second, related, effect is that the level of the covenant threshold also governs the rate at which the fraction of bad banks declines during the crisis. If $\Phi = 0$, then the fraction remains the same. If $\Phi > 0$, then the fraction starts to decline over time, as was observed in the previous subsection.

Figure 10 shows a quantitive estimate of the role that covenants play in amplifying the output decline. In order to generate the figure, the crisis experiment from subsection 5.3 was rerun but with two additional cases. In the first case, I set $\Phi = \infty$ so that the covenant threshold is never breached leaving the project

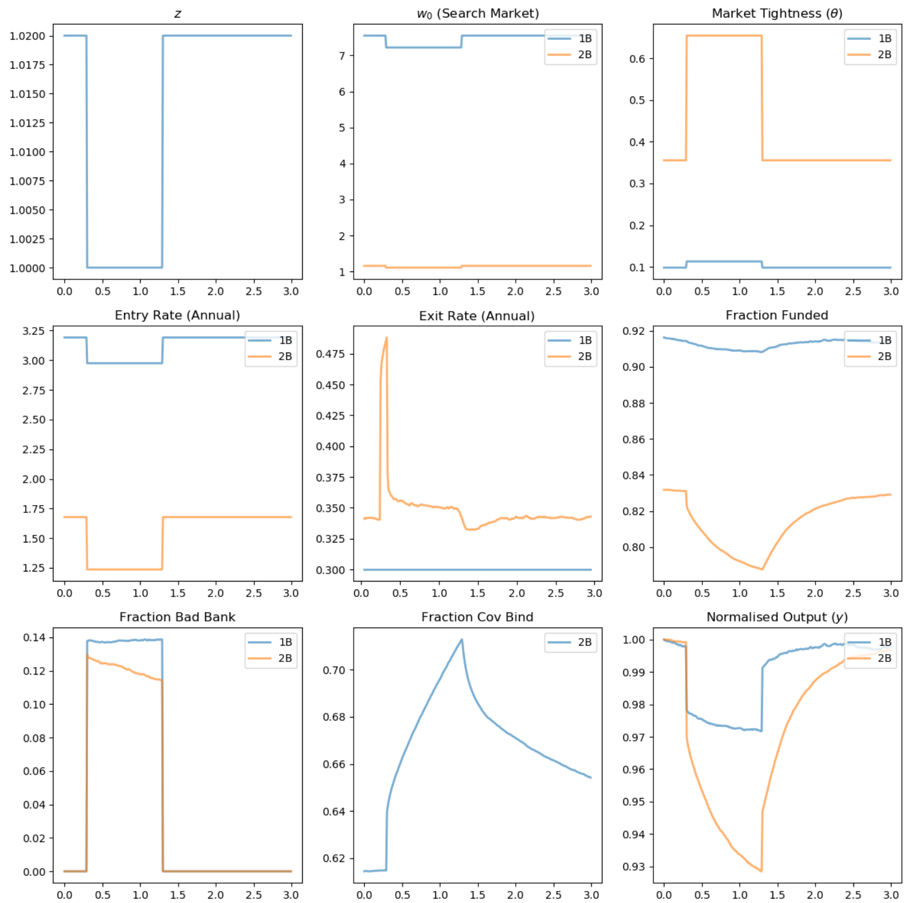

Figure 9: *Impulse Response Functions: First Best and Baseline Model.* The figures show the time paths for the economy following a year long crisis shock. The $1B$ lines depict the impulse response functions under first best contracting. The $2B$ lines depict the impulse response functions under optimal contracting with frictions (second best contracting).

with flag $f = 0$ for the contract duration. The path for normalised output in this case is denoted by $2B(\Phi = \infty)$ in the diagram. In the second case, I set $\Phi = 0$ so the covenant threshold is breached at contract initialisation leaving the project with flag $f = 1$ for the contract duration. The path for normalised output in this case is denoted by $2B(\Phi = 0)$ in the diagram. As can be seen, the level of Φ appears to play a significant role in amplifying output decline. In a world where covenants are never breached, the peak fall in output would be approximately 2 percentage points lower and, in a world where covenants are always binding, the peak fall in output would be approximately 4 percentage points larger.

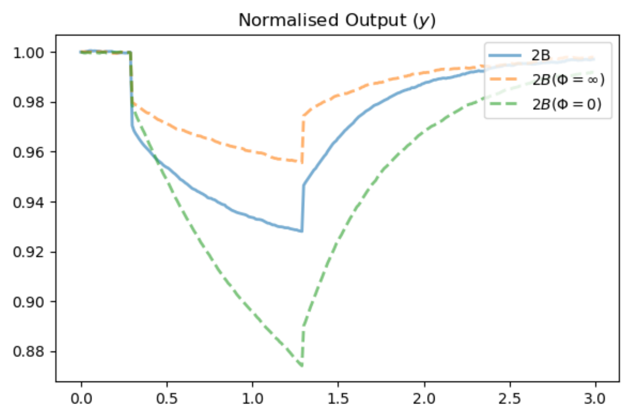

Figure 10: *Output Loss: Without and Without Covenant Flags:* The figures show the time path for normalised output during a year long crisis shock. The $2B$ line depicts the impulse response function under optimal contracting with frictions (second best contracting). The $2B(\Phi = \infty)$ line depicts the impulse response function under second best contracting when $\Phi = \infty$ and so the covenant flag is never triggered (the project stays in $f = 0$ for the duration of the contract). Finally, the $2B(\Phi = 0)$ line depicts the impulse response functions under second best contracting when $\Phi = 0$ and so the covenant flag is triggered immediately after project initialisation (the project stays in $f = 1$ for the duration of the contract.)

It is important to note that a larger output decline in figure 9 or figure 10 does not necessarily imply a larger decline in welfare. The bankers choose to cut more credit during the crisis when the adjustment cost is lower so, from the perspective of the banker, greater deleveraging is optimal given the constraints that they face. This could also be optimal for the aggregate economy if it leads to an increase in the average rate of financing.

5.5 Lessons

I learn two main lessons from this counterfactual exercise. Firstly, under optimal long-term contracting in an environment with agency frictions and search frictions, bankers significantly amplify output decline during a crisis by recalling long-term debt contracts. This provides a response to Chodorow-Reich and Falato [2017] who ask whether the scale of deleveraging we saw during the crisis could have come from optimal contracting under rational expectations (as opposed to an environment in which entrepreneurs underestimated the probability of a crisis). I find an answer in the affirmative. In my model, banks and entrepreneurs sign optimal contracts with full knowledge about the probability distributions in the economy and, quantitatively, these contracts are able to match key estimates from the work of Chodorow-Reich and Falato [2017] and generate significant aggregate credit reduction and output decline.

A second lesson is that banker heterogeneity can further amplify output decline if bankers can condition the magnitude of their debt recalls on their idiosyncratic crisis exposure. I interpret this to mean that greater banker heterogeneity has a similar aggregate effect to higher project return volatility. As bankers get shocks to their funding costs, they recall debt, which, just like a project loss, pushes the entrepreneur towards the termination limit on their credit line. So, an increase in the dispersion of banker shocks leads to an increase in the dispersion of entrepreneur credit line draws and, ultimately, higher credit termination rate.

6 Extensions

In this section, I consider two extensions to the baseline model from section 5. The first extension introduces project type heterogeneity. The second extension introduces working capital.

6.1 Project Heterogeneity

In section 5, all entrepreneurs had the same project type, as characterised by the mean and volatility parameters generating the idiosyncratic project shocks. However, there is evidence that different types of borrowers were affected very differently during the 2007-09 financial crisis and subsequent recovery (e.g. see **?**, **?**, and **?**). In particular, the empirical studies suggest that projects with higher return and volatility experienced comparatively greater credit reduction. In this subsection, I ask whether the contracting frictions in my model can generate the result that funding is disproportionately cut to high-risk projects, even when the underlying return and risk characteristics of the projects do not change during the crisis.

Consider the environment from section 2 but with the specification that entrepreneurs now permanently have one of two types of projects: a unit measure have L-type projects with characteristics $\mathbf{x} = (A_L, \sigma_L)$ and a unit measure have H-type project with characteristics $\mathbf{x} = (A_H, \sigma_H)$, where $A_H > A_L$ and $\sigma_H > \sigma_L$ so that the H-type project has both higher mean return and higher volatility. My choices of numerical values for the replacement parameters are given in table 5.

Additional Parameters				
Parameter	A_L	A_H	σ_L	σ_H
Value	0.08	0.12	0.15	0.3

Table 5: Additional parameters for the project heterogeneity extension.

The crisis period is then simulated using the approach taken in section 5 and the impulse response curves are shown in figure 11. Firstly, the L-type projects do not have binding covenants in equilibrium. This reflects that, as project volatility decreases, the bankers eventually stop paying the regime switching cost and set the covenant and termination thresholds at the same level. Secondly, in all aggregate states, the H-type projects have both the highest exit rate and the lowest entry rate since bankers are reluctant to finance projects that are more likely to terminate early. The three graphs in the bottom row describe the impact on the aggregate economy. Reading from right to left, the right plot shows the decline in total output across all project types under first-best and second-best contracting. The middle plot decomposes the aggregate decline into the decline amongst L-type and H-type projects. Finally, the left plot shows the ratio of H-type to L-type projects, which falls by 4 percentage points generating the outcome observed in the empirical work.

This behaviour reflects an important feature of how the project parameters affect the value of a bank-firm match. The mean productivity, A, affects the value by changing the joint surplus whereas the volatility, σ, affects the value by changing the likelihood of termination. In this sense, σ acts to amplify the impact of the agency friction in the model. What the impulse response functions show is that, when a crisis state occurs, the value distortion caused by the high σ becomes relatively more significant. This means that projects with a high σ are terminated at a relatively higher rate and financed at a relatively lower rate leading to a decrease in the ratio of H-type to L-type projects.[26]

6.2 Working Capital

The previous sections focused on modeling bank credit reduction along the extensive margin. However, both empirical and theoretical research suggests that there is also significant credit reduction along the intensive margin (e.g. **?**, **?**, Chodorow-Reich and Falato [2017]). In order to investigate this possibility in my model, I introduce working capital so that bankers are able to vary project size as well as terminate projects. This introduces a new dynamic into the model that maintaining matches between bad banks and/or projects with high net accumulated losses imposes a cost on the aggregate economy because the bankers allocate lower levels of working capital to thoses matches.

[26]Although it is beyond the scope of this paper, it is worth noting that a model of this kind could be extended to generate stochastic volatility over the business cycle by introducing correlation across the entrepreneur project shocks.

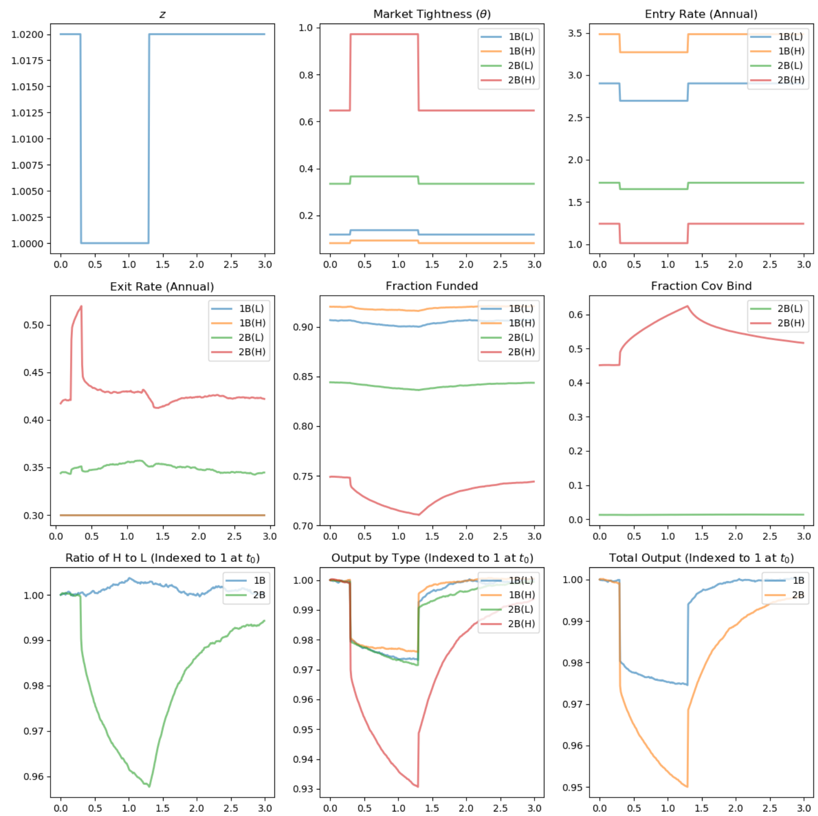

Figure 11: *High Risk and Low Risk Projects During a Crisis:* The figures show selected time paths following a year long crisis shock. The $1B(L)$ and $1B(H)$ lines depict the response curves for low risk and high risk projects under first best contracting. The $2B(L)$ and $2B(H)$ lines depict the response curves for a low risk and high risk project when contracting under agency frictions.

Given these new channels introduced by interacting working capital with the agency and search frictions, I return to the question I asked in section 5 – to what extent do the contracting frictions amplify credit decline (now along both the intensive and extensive margins) during a crisis?

6.2.1 Environment Changes

The environment is the same as in section 2 except that now working capital can be added to projects in order to increase output according to a decreasing returns to scale production function. Formally, all entrepreneurs have projects that require an initial investment, I, in order to generate non-adjustable capital, κ (this was normalised to unit size in section 2). Once the project has been created, additional working capital, k, can be allocated.[27] The project then generates cumulative output, y_t, according to the process:

$$dy_t = zA(\kappa + f(k))((1 - \xi_t)dt + \sigma dB_t)$$

where the production function $f(k)$ satisfies

$$f(k) = k^\alpha$$

and where $\alpha \in (0, 1)$. Once again, z is the aggregate productivity, A is the project specific productivity and ξ_t is the fraction of output stolen by the entrepreneur. The entrepreneur now gets benefit $\beta zA(\kappa+f(k))\xi_t$ from stealing ξ_t. The level of working capital can be adjusted without cost, subject to the constraint that the level of working capital cannot be conditioned on the banker's idiosyncratic cost of funds while the contract is in regime $f = 0$ and the covenant has not been breached.

The banker has the same access to the deposit market as in subsection 2.3. However, in addition to raising I deposits for the initial investment, they must now also raise k short-term deposits each period in order to fund the provision of working capital to the project. That is, total debt is now $D = I + k$. As before, they pay an exogenously fluctuating interest rate on the deposits they raise

$$r_t = \bar{r} + \epsilon_t$$

where $\epsilon_t \in \mathcal{E} \equiv \{\epsilon_G, \epsilon_B\}$ follows the Poisson process described in subsection 2.3.

6.2.2 Optimal Contract

The banker chooses a contract $\mathcal{C} = (\mathcal{C}^0, \mathcal{C}^1)$, where $\mathcal{C}^f = \{\tau_f, \delta_f, k_f\}$ now also specifies a process for working capital, k_f. Most features of the optimal contract are very similar to the one discussed in section 3 so the details are relegated to

[27]The reason that I have non-adjustable capital, κ, as well as adjustable capital, k, rather than just k is to prevent the banker from eliminating the possibility of early termination by decreasing k to 0 as the project accumulates negative shocks and the entrepreneur's continuation value falls.

the appendix and, instead, I focus on the new features of the problem. Once again, the contracting problem has two regimes: regime 1 in which the covenant is binding and contract can be exposed to the banker's idiosyncratic cost of funds and regime 0 in which the contract can only be exposed to entrepreneur reports and aggregate shocks. Although the HJB equations are different across the two regimes, the optimal choice of working capital is similar so, for ease of exposition, only the HJB equation in regime 1 is discussed here. As before, in this regime, the banker keeps the entrepreneur's continuation value inside a region $[W^U(z,\mathbf{x}), W^{1*}(\epsilon, z, \mathbf{x})]$. Inside that region, their value function then evolves according to:

$$
\begin{aligned}
&\rho_h V^1(W,\epsilon,z,\mathbf{x}) \\
&\quad = \sup_{k,\Psi_b,\Psi_\epsilon,\Psi_{z,\cdot}} \Big\{ z(\kappa + f(k)) - r(\epsilon)k \\
&\qquad\qquad + \Big(\rho_e W - \lambda_\epsilon(\epsilon)\Psi_\epsilon - \sum_{\epsilon'\in\mathcal{E}} \overline{\pi}(\epsilon', z)\lambda_z(z)\Psi_{z,\epsilon} \\
&\qquad\qquad\qquad - \lambda_c(W^U(z,\mathbf{x}) - W)\Big)\partial_W V^1(W,\epsilon,z,\mathbf{x}) \\
&\qquad\qquad + \lambda_\epsilon(\epsilon)(V^1(W+\Psi_\epsilon, \epsilon^c, z, \mathbf{x}) - V^1(W,\epsilon,z,\mathbf{x})) \\
&\qquad\qquad + \lambda_z(z)\Big(\sum_{\epsilon'} \overline{\pi}(\epsilon', z)V^1(W+\Psi_{z,\epsilon'}, \epsilon', z^c, \mathbf{x}) - V^1(W,\epsilon,z,\mathbf{x})\Big) \\
&\qquad\qquad + \lambda_c(L_B - V^1(W,\epsilon,z,\mathbf{x})) \\
&\qquad\qquad + \Psi_{b,t}^2 z^2 A^2(\kappa + f(k))^2\sigma^2\partial_{WW}V^1(W,\epsilon,z,\mathbf{x})\Big\}
\end{aligned}
$$

Most terms in the HJB equation are familiar from section 3 and the first order conditions for Ψ_b, Ψ_ϵ and $\Psi_{z,\cdot}$ are the same as before. The key difference is in the last term where we can see that the banker's choice of working capital, k, now amplifies the impact of the project productivity shocks. Since there is no cost of adjusting capital, choosing k is an intratemporal problem characterised by the first order condition:

$$
(k) : zAf'(k) = r(\epsilon) - 2\Psi_b^2 z^2(\kappa + f(k))\sigma^2\partial_{WW}V^1(W,\epsilon,z,\mathbf{x})f'(k)
$$

This equation equates the marginal benefit from extending more working capital (LHS) with the marginal cost (RHS). The marginal cost has two components. The first term, $r(\epsilon)$, is the interest rate that the bank pays to depositors. The second term, $-2\Psi_b^2 z^2 A^2(\kappa + f(k))f'(k)\sigma^2\partial_{WW}V^1$, is positive since $f'(k) > 0$ and $\partial_{WW}V^1 < 0$. It reflects the marginal cost to the banker of extending more working capital and making project output more volatile.

6.2.3 Capital Allocation Distortion

The economy now has distortions on both the extensive and intensive margins. The extensive margin distortion is the same distortion that was discussed at length in section 3: bankers terminate projects early and, as a consequence, provide financing at lower rates. The new distortion is on the intensive margin. If the project accumulates negative shocks or the banker's cost of funds increases, then the banker chooses to decrease the working capital available to the project.

Figure 12 compares the banker's optimal choice of working capital, k, to the first best choice of working capital, k_{FB}, conditional on the banker's idiosyncratic cost of funds. As can be seen, both the agency friction and the search friction distort the level of working capital. The distortion from the agency friction is that the banker only chooses the first best level of working capital when the entrepreneur's continuation value is high and so costly termination is essentially no longer an issue. The distortion from the search friction is that banks with high funding costs offer less working capital than banks with low funding costs. In a competitive market, this would not occur because entrepreneurs would move to the good banks that offer high working capital.

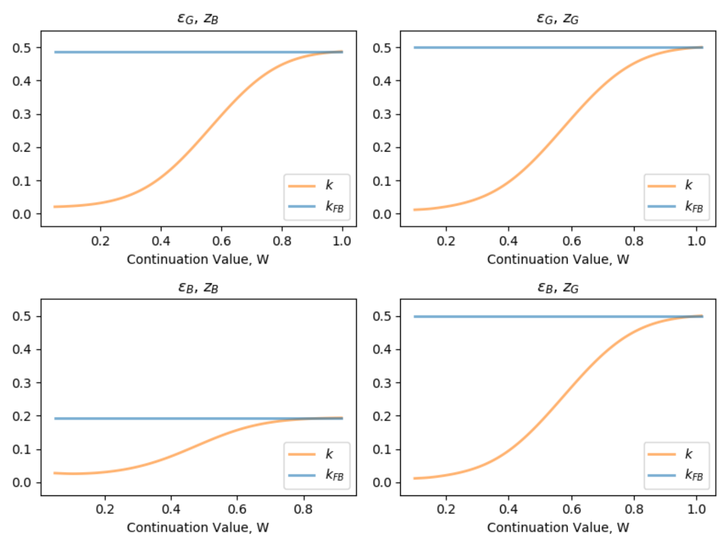

Figure 12: *Working Capital Distortion in Regime* $f = 1$. These plots show the banker's optimal choice of working capital, k, and the first best choice of working capital, k_{FB}, conditional on the banker's state when the covenant is binding.

6.2.4 Financial Crisis

The crisis simulation from the previous chapter is repeated for the economy with working capital. The additional parameter, α, is set to 0.25 in order to be consistent with **?**. The results are shown in figure 13. In order to interpret the output, I first consider the new working capital related plots. As can be seen, the average working capital per entrepreneur (which can be interpreted as the average project size) decreases during the crisis under both first best contracting and second best contracting. This occurs because both the decrease in productivity and the increase in funding costs make extending working capital less profitable. We can also observe that the decrease in average working capital per entrepreneur is larger under first best contracting than under second best contracting. The reason for this can be seen in figure 12. Under second best contracting, many projects have low continuation value and so already have low working capital before their banker get hit with an adverse cost of funds shock. In other words, the intensive margin distortion due to the agency friction ends up dampening the intensive margin distorion due to banker heterogeneity and search frictions.

Now, consider the gap between output decline under first best contracting and the output decline under optimal contracting with agency frictions. In theory, there are forces at play that could widen or tighten that gap. On the one hand, working capital per capita decreases by more under first best contracting, which brings the output decline under the first and second best closer together. On the other hand, the crisis shifts projects closer to their termination threshold which amplifies the intensive margin distortion from the agency frictions. As can be seen in the figure, it is the first effect that dominates and so the gap between the output decline under first best and second best contracting narrows.

7 Conclusion

This paper studies how long-term bank credit is disrupted during a financial crisis when bankers and entrepreneurs contract optimally under agency frictions and a directed search market for bank funding. I show that the optimal contract can be implemented using a long-term debt security, a credit line, a termination threshold and a covenant threshold that give the banker the right to condition debt terms on their idiosyncratic state. This allows me to calibrate the model to recent research on cross-sectional bank-firm matching and run counter-factual experiments in general equilibrium. I learn a number of lessons from these exercises. Firstly, under optimal long-term contracting, bankers significantly amplify output decline during a crisis by recalling long-term debt contracts. Secondly, banker heterogeneity can further amplify output decline if the covenant threshold is high and so bankers can typically condition the magnitude of their debt recalls on their idiosyncratic crisis exposure. Thirdly, the contracting frictions skew the economy away from the high return, high-risk projects during a crisis. Finally, the search friction amplifies the decrease in average project size during a crisis while the agency friction moderates the decrease in size.

There are a number of possible extensions that could be considered in future work. One natural extension would be to investigate the constrained planner problem so I can be explicit about the welfare implications of credit reduction. Another would be to allow entrepreneurs to search for new banks when they are attached to a "bad" bank. This would restrict the bank's capacity to share an adverse funding cost shock with an entrepreneur and so may moderate the amplification effects discussed in this paper. A further alternative would be to introduce stochastic project quality so that banks can end up committed to

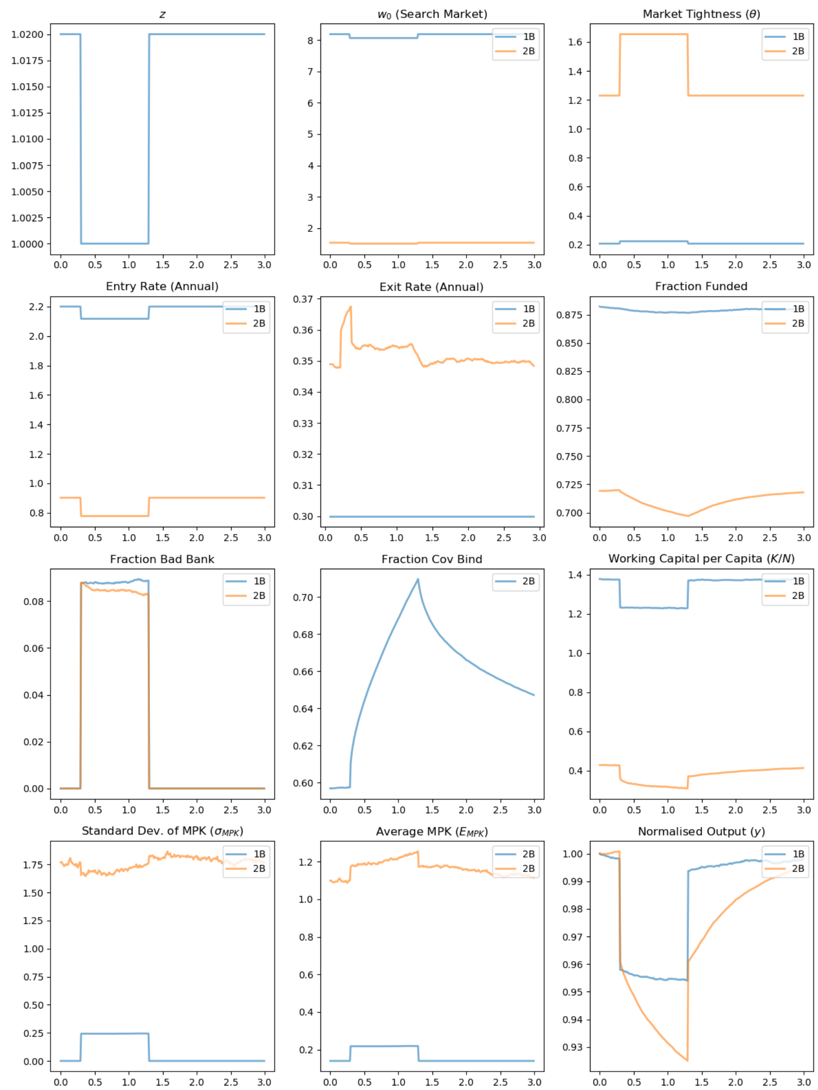

Figure 13: *Impulse Response Functions.* The figures show the time paths for the economy following a year long crisis shock. The $1B$ lines depict response curves under first best contracting. The $2B$ lines depict the response curves under contracting with agency frictions (second best contracting).

low-quality projects. This would introduce a novel approach for studying "bad debt" in the financial system. Finally, introducing a richer balance sheet for the bank would allow explicit bank regulation to be investigated. This final extension would enable the large literature of bank balance sheet models to be connected to the optimal dynamic contracting literature.

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A Supplementary Proofs to Section 3 (Contracting Problem)

A.1 Additional Definitions

I start the appendix by making some additional definitions to set up the problem formally. These definitions will be used throughout the appendix.

A.1.1 Definitions of Stochastic Processes

Let $(\Omega, \mathcal{C}, \mathbb{P})$ denote a probability space. The stochastic process for the aggregate state, z, is defined in the following way. As in the main text, let $\mathcal{Z} \equiv \{z_G, z_B\}$ denote the support for the aggregate state. Then, the aggregate state process, z_t, is defined by the differential equation and initial value:

$$
\begin{aligned}
dz_t &= (z_t^c - z_t)dN_{z,t} \\
z_0 &= z_G
\end{aligned}
$$

where z^c denotes the complement of z and $N_z = \{N_{z,s}, s \geq 0\}$ is a compound Poisson process defined on $(\Omega, \mathcal{C}, \mathbb{P})$ with intensity $\lambda_z(z_t)$.

Now, suppose that bank j matches with entrepreneur i at time t_0 and starts a project. Once the project begins, there are two idiosyncratic stochastic processes: the entrepreneur's idiosyncratic project productivity process and the banker's idiosyncratic funding cost process. Both of these processes continue until the project is terminated. As described in the main text, entrepreneur i's productivity process, denoted by B, is given by an idiosyncratic Brownian motion process defined on $(\Omega, \mathcal{C}, \mathbb{P})$.[28]

The construction of the process for bank j's cost of funds, denoted by ϵ, is more complicated because it is shifted by both the bank's idiosyncratic process and the aggregate shock process. As in the main text, let $\mathcal{E} \equiv \{\epsilon_G, \epsilon_B\}$ denote the support of ϵ. Now, for a given z, define the auxiliary stochastic process by:

$$
\begin{aligned}
d\tilde{\epsilon}_t &= (\tilde{\epsilon}_t^c - \tilde{\epsilon}_t)dN_{\epsilon,t} \\
\tilde{\epsilon}_0 &= \tilde{\epsilon}_G
\end{aligned}
$$

where, again, $\tilde{\epsilon}^c$ denotes the complement of $\tilde{\epsilon}$ and $N_\epsilon = \{N_{\epsilon,s}, s \geq 0\}$ is an idiosyncratic compound Poisson process defined on $(\Omega, \mathcal{C}, \mathbb{P})$ with intensity $\lambda_\epsilon(\epsilon, z)$. Observe that λ_ϵ depends on both the bank's current idiosyncratic state, ϵ, and the aggregate state, z. Let $\overline{\pi}_\epsilon(\cdot|z)$ denote the stationary distribution for $\tilde{\epsilon}$ when the aggregate state stays at z. Then, the stochastic process for ϵ is defined by:

$$
\begin{aligned}
d\epsilon_t &= (\epsilon_t^c - \epsilon_t)dN_{\epsilon,t} + (\mathbb{1}_{\epsilon_G}(z_t^c)\epsilon_G + \mathbb{1}_{\epsilon_B}(z_t^c)\epsilon_B - \epsilon_t)dN_{z,t} \\
\epsilon_{t_0} &= \mathbb{1}_{\epsilon_G}(z_{t_0})\epsilon_G + \mathbb{1}_{\epsilon_B}(z_{t_0})\epsilon_B
\end{aligned}
$$

[28] Observe that technically the idiosyncratic processes could be indexed by i and j but I drop the indices in order to keep the notation light.

where $\mathbb{1}_{\epsilon'}(z)$ denotes an indicator random variable satisfying:

$$\mathbb{1}_{\epsilon'}(z) = \begin{cases} 1, & \text{w.p. } \overline{\pi}_{\epsilon}(\epsilon'|z) \\ 0, & \text{w.p. } 1 - \overline{\pi}_{\epsilon}(\epsilon'|z) \end{cases}$$

That is, $\mathbb{1}_{\epsilon'}(z)$ is random variable with value 1 if ϵ' is drawn from the stationary distribution, $\overline{\pi}$, when the aggregate state is z. The notation for the ϵ process is complicated but in words it simply says that, at contract initialisation, the initial ϵ is drawn from the current stationary distribution for $\tilde{\epsilon}$. Then, over time, the value of ϵ can change in two possible ways. Firstly, a $dN_{\epsilon,t}$ shock could occur, in which case ϵ switches. Alternatively, a $dN_{z,t}$ shock could occur, in which case the aggregate state changes and the banker gets a new draw of ϵ from the new stationary distribution.

Finally, let $N_c = \{N_{c,s}, s \geq 0\}$ denote a compound Poisson process defined on $(\Omega, \mathcal{C}, \mathbb{P})$ with intensity λ_c. The project completes exogenously the first time that $N_{c,t} = 1$. That is, the exogenous completion time is $\tau_c \equiv \{t \geq t_0 : N_{c,t} = 1\}$.

The relevant filtrations can now be defined. Let $\mathcal{F}_t^z \equiv \sigma(\{z_s : 0 \leq s \leq t\})$ denote the filtration generated by z up to time t. Let $\mathcal{F}_t^{\tilde{B},z} \equiv \sigma(\{z_s : 0 \leq s \leq t\}, \{\widetilde{B}_s, N_{c,s} : t_0 \leq s \leq t\})$ denote the filtration generated by z up to time t and by $\widetilde{B}$ & N_c from contract initialisation up to time t. Finally, let $\mathcal{F}_{t^{\tilde{B},z,\epsilon}} \equiv \sigma(\{z_s : 0 \leq s \leq t\}, \{\widetilde{B}_s, z_s, N_{c,s} : t_0 \leq s \leq t\}, \{\epsilon_s : \mathscr{T}_0 \leq s \leq t\})$ denote the filtration generated by z up to time t, by $\widetilde{B}$ & N_c from contract initialisation up to time t, and by ϵ from time $\mathscr{T}_0$ to t.

A.1.2 Definition of Contract Terms

The banker chooses a contract $\mathcal{C} = \{\tau, \delta, \mathscr{T}\}$ where τ is the stochastic stopping time at which the project is terminated, δ is the stochastic payout process to the entrepreneur and $\mathscr{T}$ is the stochastic stopping time at which the banker pays the cost, Φ, and starts to condition the contract on their idiosyncratic state. I use the notation $f \in \{0, 1\}$ to indicate whether the cost, Φ, has been paid and, following the mathematics literature, I refer to $f = 0$ and $f = 1$ as different regimes. The key difference between the regimes is in how the contract terms can be conditioned. Before the cost has been paid (in regime $f = 0$), the stochastic processes $\{\tau, \delta, \mathscr{T}\}$ are only adapted to $\mathcal{F}_t^{\tilde{B},z}$. I denote the contract terms in this regime by $\mathcal{C}^0 = \{\tau_0, \delta_0, \mathscr{T}_0\}$. Once the cost has been paid (in regime $f = 1$), the stochastic processes $\{\tau, \delta\}$ are adapted to $\mathcal{F}_t^{\tilde{B},z,\epsilon}$ and the stochastic process $\mathscr{T}$ is redundant. I denote the contract terms in this regime by $\mathcal{C}^1 = \{\tau_1, \delta_1\}$.

A.1.3 Definition of Beliefs

In regime 0, the banker's cost of funds is not known to the entrepreneur. Instead, the entrepreneur forms a belief based on their knowledge of the stochastic processes in the economy. Let $\pi_t^e \equiv [\pi_t^e(\epsilon_G|\mathcal{F}_t^z), \pi_t^e(\epsilon_B|\mathcal{F}_t^z)]$ denote the entrepreneur's belief about their bankers state at time t, conditional on the history $\mathcal{F}_t^z$. In general, the evolution of the entrepreneur's belief in regime 0 can depend upon z_t, which is observable. Let the general stochastic process for π_t^e be denoted by:

$$d\pi_t^e = \mu_{\pi,t}dt + \gamma_{\pi,t}dN_{z,t}$$

In the main text, I imposed a number of assumptions in order to simplify the evolution of the entrepreneur's beliefs. These are summarised in Assumption 1 below.

Assumption 1. Suppose that the following hold:

1. At contract initialisation, the banker's initial cost of funds is drawn from the stationary distribution of $\widetilde{\epsilon}(\cdot|z_t)$,
2. When the aggregate state changes from z_t to z_t^c, the bankers receive a new cost of funds drawn from the new stationary distribution, $\widetilde{\epsilon}(\cdot|z_t^c)$, and
3. The entrepreneur has full knowledge of the stochastic processes in the economy. However, the entrepreneur does not observe the realisation of ϵ process and has no independent way of learning about the ϵ process.

From Claim 1 in the main text, we have that, under Assumption 1, $\pi_t^e = \overline{\pi}(\cdot|z)$ and so the stochastic process for the evolution of ϵ is satisfies $\mu_{\pi,t} = 0$ and $\gamma_{\pi,t} = \overline{\pi}(z_t^c) - \overline{\pi}(z_t)$.

Finally, I define the expectations. Let $\mathbb{E}_t[\cdot]$ denote the expectation conditional on the observable history up to time t. In regime 0, this is $\mathbb{E}_t[\cdot] \equiv \mathbb{E}[\cdot|\mathcal{F}_t^{\tilde{B},z}]$. In regime 1, this is $\mathbb{E}_t[\cdot] \equiv \mathbb{E}[\cdot|\mathcal{F}_t^{\tilde{B},z,\epsilon}]$. Let $\mathbb{E}_t^e[\cdot] \equiv \sum_{\epsilon\in\mathcal{E}} \pi_t^e(\epsilon|\mathcal{F}_t^z)\mathbb{E}[\cdot|\mathcal{F}_t^{\tilde{B}}, \epsilon]$ denote the expectation conditional on the observable history up to time t and under the entrepreneur's belief, π_t^e. Observe that, in region 1, the entrepreneur's belief distribution is degenerate at the true state and so $\mathbb{E}_t^e[\cdot] = \mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_t]$.

A.2 Supplementary Proofs for Subsection 3.1 (The First Best Contract)

Proof of Lemma 1: This result follows immediately from $\rho_e > \rho_h$ and risk neutrality. Given these preferences, if the banker has promised an agent a value W and there is no agency friction, then it is optimal for the banker to pay the agent W in goods immediately. □

A.3 Supplementary Proofs for Subsection 3.3 (The Evolution of the Entrepreneur's Continuation Value)

This section contains the proofs for the derivation of the entrepreneur's continuation value. I first prove some preliminary results that are not stated in the main text. Then, I prove the main results from subsection 3.3.

A.3.1 Preliminary Proofs

Before proceeding to a proof of lemma 2 from the main text, I first prove a series of preliminary results. The first result is Lemma 3, which uses the Martingale Representation Theorem to re-express the funded entrepreneur's continuation value in terms of the underlying stochastic processes relevant to the contract: B, N_ϵ, N_z and N_c.

Lemma 3. *Let $W_t^{F,f,\xi}$ denote a type-$\mathbf{x}$ entrepreneur's continuation value at time t in regime f under stealing process ξ and contract $\mathcal{C}^1$. Then:*

(a) In regime 1, there exist stochastic processes Ψ_b^1, Ψ_ϵ^1 and $\{\Psi_{z,\epsilon}^1\}_{\forall\epsilon\in\mathcal{E}}$ such that:

$$\begin{aligned}
W_t^{F,1,\xi} = &\int_t^{T_1} e^{-\rho_e(s-t)}(d\delta_{1,s} + zA\xi_s\beta ds) \\
&+ e^{-\rho_e T_1} W^U(z_{T_1}, \mathbf{x}) \\
&- \int_t^{T_1} e^{-\rho_e(s-t)} \left(W_t^U - W_t^{F,1,\xi}\right)(dN_{c,s} - \lambda_c ds) \\
&- \int_t^{T_1} e^{-\rho_e(s-t)}\sigma\Psi_{b,s}^1 dB_s - \int_t^{T_1} e^{-\rho_e(s-t)}\Psi_{\epsilon,s}^1(dN_{\epsilon,s} - \lambda_\epsilon(\epsilon_s)ds) \\
&- \int_t^{T_1} e^{-\rho_e(s-t)} \sum_{\epsilon'\in\mathcal{E}} \Psi_{z,\epsilon',s}^1(\mathbb{1}_{\epsilon'}(z_s^c)dN_{z,s} - \overline{\pi}_\epsilon(\epsilon'|z_s^c)\lambda_z(z_s)ds)
\end{aligned} \tag{A.1}$$

where $T_1 \equiv \min\{\tau_c, \tau_1\}$. It follows that $dW_t^{F,1,\xi}$ evolves according to:

$$\begin{aligned}
dW_t^{F,1,\xi} = &\rho_e W_t^{F,1,\xi}dt - d\delta_t^1 - \beta\xi_t zAdt + \sigma\Psi_{b,t}^1 dB_t \\
&+ (W_t^U - W_t^{F,1,\xi})(dN_{c,t} - \lambda_c dt) + \Psi_{\epsilon,t}^1(dN_{\epsilon,t} - \lambda_\epsilon(\epsilon_t, z_t)dt) \\
&+ \sum_{\epsilon'\in\mathcal{E}} \Psi_{z,\epsilon',t}^1(\mathbb{1}_{\epsilon'}(z_t^c)dN_{z,t} - \overline{\pi}_\epsilon(\epsilon'|z_t^c)\lambda_z(z_t)dt)
\end{aligned} \tag{A.2}$$

(b) In regime 0, there exist stochastic processes Ψ_b^0 and Ψ_z^0 such that:

$$\begin{aligned}
W_t^{F,0,\xi} = &\int_t^{T_0} e^{-\rho_e(s-t)}(d\delta_{0,s} + zA\xi_s\beta ds) \\
&+ e^{-\rho_e(T_0-t)}(\mathbb{1}_{\{T_0\in\{\tau_c,\tau_0\}}W_t^U + \mathbb{1}_{\{T_0=\mathscr{T}_0\}}\widehat{W}_t^{e,\Phi}) \\
&- \int_t^{T_0} e^{-\rho_e(s-t)} \left(W_t^U - W_t^{F,0,\xi}\right)(dN_{c,s} - \lambda_c ds) \\
&- \int_t^{T_0} e^{-\rho_e(s-t)}\sigma\Psi_{b,s}^0 dB_s - \int_t^{T_1} e^{-\rho_e(s-t)}\Psi_{z,s}^0(dN_{z,s} - \lambda_z(z_s)ds)
\end{aligned} \tag{A.3}$$

where $T_0 \equiv \min\{\tau_c, \tau_0, \mathscr{T}_0\}$ *and* $\widehat{W}_t^{e,\Phi}$ *is the entrepreneur's expected continuation value immediately after the regime change given their beliefs at time* t*. It follows that* $dW_t^{F,0,\xi}$ *evolves according to:*

$$\begin{aligned} dW_t^{F,0,\xi} &= \rho_e W_t^{F,0,\xi} dt - d\delta_{0,t} - \beta \xi_t z A dt + \sigma \Psi_{b,t}^0 dB_t \\ &\quad + (W_t^U - W_t^{F,0,\xi})(dN_{c,t} - \lambda_c dt) + \Psi_{z,t}^0 (dN_{z,t} - \lambda_z(z_t) dt) \end{aligned} \tag{A.4}$$

Proof. (a) Consider the regime 1 case. First define the discounted continuation value, $U_t^{F,1,\xi}$, and the continuation value, $W_t^{F,1,\xi}$ by:

$$\begin{aligned} U_t^{F,1,\xi}(\mathbf{x}) &\equiv \mathbb{E}_t \left[\int_t^{T_1} e^{-\rho_e s} (d\delta_{1,s} + \beta \xi_s z A ds) + e^{-\rho_e T_1} W^U(z_{T_1}, \mathbf{x}) \right] \\ W_t^{F,1,\xi}(\mathbf{x}) &\equiv e^{\rho_e t} U_t^{F,1,\xi}(\mathbf{x}) \end{aligned}$$

Since $\xi \in \mathcal{A}(\mathcal{C}, \mathbf{x})$, we have that:

$$\mathbb{E}_t \left[\left(\int_t^{T_1} e^{-\rho_e s} (d\delta_s + z A \xi_s \beta ds) \right)^2 \right] < \infty$$

and so the process $U_t^{F,1,\xi}(\mathbf{x})$ is square integrable. Using the the martingale representation theorem (for Poisson processes (**?** Theorem 5.3.5.) and for controlled processes (Cvitanic and Zhang [2013] Lemma 10.4.6), this implies that there exists Ψ_c^1, Ψ_b^1, Ψ_ϵ^1 and $\{\Psi_{z,\epsilon}^1\}_{\forall \epsilon \in \mathcal{E}}$ such that:

$$\begin{aligned} &\int_t^{T_1} e^{-\rho_e s} (d\delta_{1,s} + \beta \xi_s z A ds) + e^{-\rho_e T_1} W^U(z_{T_1}, \mathbf{x}) \\ &\quad = \mathbb{E}_t \left[\int_t^{T_1} e^{-\rho_e s} (d\delta_{1,s} + \beta \xi_s z A ds) + e^{-\rho_e T_1} W^U(z_{T_1}, \mathbf{x}) \right] \\ &\qquad + \int_t^{T_1} e^{-\rho_e s} \Psi_{c,s}^1 (dN_{c,s} - \lambda_c ds) \\ &\qquad + \int_t^{T_1} e^{-\rho_e s} \sigma \Psi_{b,s} dB_s + \int_t^{T_1} e^{-\rho_e s} \Psi_{\epsilon,s} (dN_{\epsilon,s} - \lambda_\epsilon(\epsilon_s) ds) \\ &\qquad + \int_t^{T_1} e^{-\rho_e s} \sum_{\epsilon' \in \mathcal{E}} \Psi_{z,\epsilon',s}^1 (dN_{z,s} \mathbb{1}_{\epsilon'}(z_s^c) - \overline{\pi}(\epsilon' | z_s^c) \lambda_z(z_s) ds) \end{aligned}$$

It follows that:

$$\begin{aligned}
U_t^{F,1,\xi}(\mathcal{C}^1,\mathbf{x}) = & \int_t^{T_1} e^{-\rho_e s}(d\delta_{1,s} + \beta\xi_s zAds) + e^{-\rho_e T_1} W^U(z_{T_1},\mathbf{x}) \\
& - \int_t^{T_1} e^{-\rho_e s}\Psi_{c,s}^1(dN_{c,s} - \lambda_c ds) \\
& - \int_t^{T_1} e^{-\rho_e s}\sigma\Psi_{b,s}dB_s + \int_t^{T_1} e^{-\rho_e s}\Psi_{\epsilon,s}(dN_{\epsilon,s} - \lambda_\epsilon(\epsilon_s)ds) \\
& - \int_t^{T_1} e^{-\rho_e s}\sum_{\epsilon'\in\mathcal{E}}\Psi_{z,\epsilon',s}^1(dN_{z,s}\mathbb{1}_{\epsilon'}(z_s^c) - \overline{\pi}(\epsilon'|z_s^c)\lambda_z(z_s)ds)
\end{aligned}$$

From the construction of the problem, we know that $\Psi_{c,t} = W_t^U - W_t^{F,1,\xi}$. Substituting $\Psi_{c,t}$ into the equation and multiplying by $e^{\rho_e t}$ gives equation (A.1). Converting the equation to differential notation then gives equation (A.2).

(b) Now consider regime 0. For this case, it is necessary to incorporate the additional choice of the regime-switching time $\mathscr{T}_0$. The undiscounted continuation value, $U_t^{F,0,\xi}$, and the discounted continuation value, $W_t^{F,0,\xi}$, are then now defined by the following:

$$\begin{aligned}
U_t^{F,0,\xi} &\equiv \mathbb{E}_t^e\left[\int_t^{T_0} e^{-\rho_e s}(d\delta_{0,s} + \beta\xi_s zAds) + e^{-\rho_e T_0}(\mathbb{1}_{\{T_0\in\{\tau_c,\tau_0\}\}}W_{T_0}^U + \mathbb{1}_{\{T_0=\mathscr{T}_0\}}W_{\mathscr{T}_0}^{F,1,\xi})\right] \\
W_t^{F,0,\xi} &\equiv e^{\rho_e t}U_t^{F,0,\xi}
\end{aligned}$$

As will be discussed further when I solve the banker's problem, for a given current continuation value W_t, there could be two potential regime switching times: a regime switching time at which the entrepreneur's continuation value hits a lower boundary and a regime switching time at which it hits an upper boundary. In addition, at $\mathscr{T}_0$, the banker is constrained to condition $W_{\mathscr{T}_0}^{F,1,\xi}$ on the state $\epsilon_{\mathscr{T}_0}$, rather than the history of ϵ. This means that the value at the lower and upper boundaries only depends upon $(\epsilon_{\mathscr{T}_0}, \mathcal{F}_{\mathscr{T}_0}^{\tilde{B},z})$. Let $\underline{\mathscr{T}}_0$ denote the stopping time at which W_t hits the lower regime switching boundary and let $\underline{W}^\Phi(\epsilon_{\mathscr{T}_0}) \in [-\infty, w_0]$ denote the $(\epsilon_{\mathscr{T}_0}, \mathcal{F}_{\mathscr{T}_0}^{\tilde{B},z})$-adapted value at the lower boundary. Observe that I allow $\underline{W}^\Phi(\epsilon_{\mathscr{T}_0})$ to attain the value $-\infty$ in order to include the case in which $\underline{\mathscr{T}}_0 = \infty$ (the stopping time is never reached), Likewise, let $\overline{\mathscr{T}}_0$ denote the stopping time at which W_t hits the upper regime switching boundary and let $\overline{W}^\Phi(\epsilon_{\mathscr{T}_0}) \in [w_0, \infty]$ denote the $(\epsilon_{\mathscr{T}_0}, \mathcal{F}_{\mathscr{T}_0}^{\tilde{B},z})$-adapted value at the upper boundary. Then, $\mathscr{T}_0 = \inf\{\underline{\mathscr{T}}_0, \overline{\mathscr{T}}_0\}$ is the first time that one of the two boundaries is hit and the regime changes.

Now, observe that the continuation value can be written as:

$$
\begin{aligned}
U_t^{F,0,\xi} &\equiv \mathbb{E}_t^e\Big[\int_t^{\tau_0} e^{-\rho_e s}(d\delta_{0,s} + A\xi_s\beta ds) \\
&\qquad + e^{-\rho_e T_0}(\mathbb{1}_{\{T_0\in\{\tau_c,\tau_0\}\}}W_{T_0}^U + \mathbb{1}_{\{T_0=\underline{\mathscr{T}}_0\}}\underline{W}^\Phi(\epsilon_{\mathscr{T}_0}) + \mathbb{1}_{\{T_0=\overline{\mathscr{T}}_0\}}\overline{W}^\Phi(\epsilon_{\mathscr{T}_0}))\Big] \\
&= \sum_{\epsilon_t\in\mathcal{E}} \pi_t(\epsilon_t|\mathcal{F}_t^z)\mathbb{E}\Bigg[\int_t^{\tau_0} e^{-\rho_e s}(d\delta_{0,s} + A\xi_s\beta ds) \\
&\qquad + e^{-\rho_e T_0}(\mathbb{1}_{\{T_0\in\{\tau_c,\tau_0\}\}}W_{T_0}^U + \mathbb{1}_{\{T_0=\underline{\mathscr{T}}_0\}}\underline{W}^\Phi(\epsilon_{\mathscr{T}_0}) + \mathbb{1}_{\{T_0=\overline{\mathscr{T}}_0\}}\overline{W}^\Phi(\epsilon_{\mathscr{T}_0}))\Big|\mathcal{F}_t^{\widetilde{B},z}, \epsilon_t\Bigg] \\
&= \mathbb{E}\Big[\int_t^{\tau_0} e^{-\rho_e s}(d\delta_{0,s} + A\xi_s\beta ds) + e^{-\rho_e T_0}\mathbb{1}_{\{T_0\in\{\tau_c,\tau_0\}\}}W_{T_0}^U\Big|\mathcal{F}_t^{\widetilde{B},z}\Big] \\
&\qquad + \sum_{\epsilon_t\in\mathcal{E}} \pi_t(\epsilon_t|\mathcal{F}_t^z)\mathbb{E}\Big[e^{-\rho_e T_0}\mathbb{1}_{\{T_0=\underline{\mathscr{T}}_0\}}\underline{W}^\Phi(\epsilon_{\mathscr{T}_0})\Big|\mathcal{F}_t^{\widetilde{B},z}, \epsilon_t\Big] \\
&\qquad + \sum_{\epsilon_t\in\mathcal{E}} \pi_t(\epsilon_t|\mathcal{F}_t^z)\mathbb{E}\Big[e^{-\rho_e T_0}\mathbb{1}_{\{T_0=\overline{\mathscr{T}}_0\}}\overline{W}^\Phi(\epsilon_{\mathscr{T}_0})\Big|\mathcal{F}_t^{\widetilde{B},z}, \epsilon_t\Big] \qquad \text{(A.5)}
\end{aligned}
$$

where I have used that $(\tau_0, \delta_0, \mathscr{T}_0)$ are not conditioned on the bank's idiosyncratic process, ϵ. The value at the lower regime switching boundary can be expressed as:

$$
\begin{aligned}
&\sum_{\epsilon_t\in\mathcal{E}} \pi_t(\epsilon_t|\mathcal{F}_t^z)\mathbb{E}\Big[e^{-\rho_e T_0}\mathbb{1}_{\{T_0=\underline{\mathscr{T}}_0\}}\underline{W}^\Phi(\epsilon_{\underline{\mathscr{T}}_0})\Big|\mathcal{F}_t^{\widetilde{B},z}, \epsilon_t\Big] \\
&\quad = \sum_{\epsilon_t\in\mathcal{E}} \pi_t(\epsilon_t|\mathcal{F}_t^z)\mathbb{E}\Big[\mathbb{E}\Big[e^{-\rho_e T_0}\mathbb{1}_{\{T_0=\underline{\mathscr{T}}_0\}}\underline{W}^\Phi(\epsilon_{\underline{\mathscr{T}}_0})\Big|\mathcal{F}_{\underline{\mathscr{T}}_0}^{\widetilde{B},z}, \mathcal{F}_t^{\widetilde{B},z}, \epsilon_t\Big]\Big|\mathcal{F}_t^{\widetilde{B},z}, \epsilon_t\Big] \\
&\quad = \sum_{\epsilon_t\in\mathcal{E}} \pi_t(\epsilon_t|\mathcal{F}_t^z)\mathbb{E}\Big[e^{-\rho_e T_0}\mathbb{1}_{\{T_0=\underline{\mathscr{T}}_0\}}\mathbb{E}\Big[\underline{W}^\Phi(\epsilon_{\underline{\mathscr{T}}_0})\Big|\mathcal{F}_{\underline{\mathscr{T}}_0}^{\widetilde{B},z}, \epsilon_t\Big]\Big|\mathcal{F}_t^{\widetilde{B},z}, \epsilon_t\Big] \\
&\quad = \mathbb{E}\Big[e^{-\rho_e T_0}\mathbb{1}_{\{T_0=\underline{\mathscr{T}}_0\}}\sum_{\epsilon_t\in\mathcal{E}} \pi_t(\epsilon_t|\mathcal{F}_t^z)\mathbb{E}\Big[\underline{W}^\Phi(\epsilon_{\underline{\mathscr{T}}_0})\Big|\mathcal{F}_{\underline{\mathscr{T}}_0}^{\widetilde{B},z}, \epsilon_t\Big]\Big|\mathcal{F}_t^{\widetilde{B},z}\Big] \\
&\quad = \mathbb{E}\Big[e^{-\rho_e T_0}\mathbb{1}_{\{T_0=\underline{\mathscr{T}}_0\}}\sum_{\epsilon\in\mathcal{E}} \overline{\pi}(\epsilon|z_t)\mathbb{E}\Big[\underline{W}^\Phi(\epsilon_{\underline{\mathscr{T}}_0})\Big|\mathcal{F}_{\underline{\mathscr{T}}_0}^{\widetilde{B},z}, \epsilon\Big]\Big|\mathcal{F}_t^{\widetilde{B},z}\Big] \\
&\quad = \mathbb{E}_t\Big[e^{-\rho_e T_0}\mathbb{1}_{\{T_0=\underline{\mathscr{T}}_0\}}\mathbb{E}_{t,\underline{\mathscr{T}}_0}^e\Big[\underline{W}^\Phi(\epsilon_{\underline{\mathscr{T}}_0})\Big]\Big]
\end{aligned}
$$

where

$$
\mathbb{E}_{t,\underline{\mathscr{T}}_0}^e\big[\underline{W}^\Phi(\epsilon_{\underline{\mathscr{T}}_0})\big] \equiv \sum_{\epsilon\in\mathcal{E}} \overline{\pi}(\epsilon|z_t)\mathbb{E}\left[\underline{W}^\Phi(\epsilon_{\underline{\mathscr{T}}_0})|\mathcal{F}_{\underline{\mathscr{T}}_0}^{\widetilde{B},z}, \epsilon\right]
$$

since, by proposition 1, the belief process for the entrepreneur only depends upon the aggregate state and so $\pi_t(\epsilon_t|\mathcal{F}_t^z) = \bar{\pi}(\epsilon_t|z_t)$. Observe that this means that $\mathbb{E}_{t,\underline{\mathscr{T}}_0}^e\big[\underline{W}^\Phi(\epsilon_{\underline{\mathscr{T}}_0})\big]$ depends upon z_t but *not* upon ϵ_t since the entrepreneur does not know ϵ_t and changes in ϵ_t do not change the entrepreneur's belief process. The value at the upper boundary can be expressed in similar way and so equation (A.5) becomes:

$$\begin{aligned}
U_t^{F,0,\xi} &= \mathbb{E}_t\Big[\int_t^{\tau_0} e^{-\rho_e s}(d\delta_{0,s} + A\xi_s\beta ds) + e^{-\rho_e T_0}\mathbb{1}_{\{T_0\in\{\tau_c,\tau_0\}\}}W_{T_0}^U\Big] \\
&\qquad + \mathbb{E}_t\Big[e^{-\rho_e T_0}\Big(\mathbb{1}_{\{T_0=\underline{\mathscr{T}}_0\}}\mathbb{E}^e_{t,\underline{\mathscr{T}}_0}\big[\underline{W}^\Phi(\epsilon_{\underline{\mathscr{T}}_0})\big] + \mathbb{1}_{\{T_0=\overline{\mathscr{T}}_0\}}\mathbb{E}^e_{t,\overline{\mathscr{T}}_0}\big[\overline{W}^\Phi(\epsilon_{\overline{\mathscr{T}}_0})\big]\Big)\Big] \\
&\equiv \mathbb{E}_t\Big[\int_t^{\tau_0} e^{-\rho_e s}(d\delta_{0,s} + A\xi_s\beta ds) + e^{-\rho_e T_0}\mathbb{1}_{\{T_0\in\{\tau_c,\tau_0\}\}}W_{T_0}^U + e^{-\rho_e T_0}\mathbb{1}_{\{T_0=\mathscr{T}_0\}}\widehat{W}_t^{e,\Phi}\Big]
\end{aligned} \tag{A.6}$$

where

$$\begin{aligned}
W^\Phi(\epsilon_{\mathscr{T}_0}) &\equiv \mathbb{1}_{\{T_0=\underline{\mathscr{T}}_0\}}\underline{W}^\Phi(\epsilon_{\underline{\mathscr{T}}_0}) + \mathbb{1}_{\{T_0=\overline{\mathscr{T}}_0\}}\overline{W}^\Phi(\epsilon_{\overline{\mathscr{T}}_0}) \\
\widehat{W}_t^{e,\Phi} &\equiv \mathbb{E}^e_{t,\overline{\mathscr{T}}_0}\Big[W^\Phi(\epsilon_{\mathscr{T}_0})\Big] \\
&= \mathbb{1}_{\{T_0=\underline{\mathscr{T}}_0\}}\mathbb{E}^e_{t,\underline{\mathscr{T}}_0}\big[\underline{W}^\Phi(\epsilon_{\underline{\mathscr{T}}_0})\big] + \mathbb{1}_{\{T_0=\overline{\mathscr{T}}_0\}}\mathbb{E}^e_{t,\overline{\mathscr{T}}_0}\big[\overline{W}^\Phi(\epsilon_{\overline{\mathscr{T}}_0})\big]
\end{aligned}$$

Once again, the process $U_t^{F,0,\xi}(\mathcal{C}^1, \mathbf{x})$ is square integrable since $\xi \in \mathcal{A}(\mathcal{C}, \mathbf{x})$ and so, by the martingale representation theorem, there exists stochastic processes Ψ_c^0, Ψ_b^0 and Ψ_z^0 such that:

$$\begin{aligned}
U_t^{F,0,\xi} = &\int_t^{\tau_0} e^{-\rho_e s}(d\delta_{0,s} + zA\xi_s\beta ds) + e^{-\rho_e T_0}(\mathbb{1}_{\{T_0\in\{\tau_c,\tau_0\}\}}W_{T_0}^U + \mathbb{1}_{\{T_0=\mathscr{T}_0\}}\widehat{W}_t^{e,\Phi}) \\
&+ \int_t^{T_0} e^{-\rho_e s}\Psi_{c,t}^0(dN_{c,s} - \lambda_c ds) \\
&+ \int_t^{T_0} e^{-\rho_e s}\sigma\Psi_{b,s}dB_s + \int_t^{T_1} e^{-\rho_e s}\Psi_{z,s}(dN_{z,s} - \lambda_z(z_s)ds)
\end{aligned}$$

where here I have used that the terms in equation (A.6) are adapted to $\mathcal{F}_t^{\widetilde{B},z}$ not $\mathcal{F}_t$. Substituting $\Psi_{c,t}^0 = W_t^U - W_t^{F,1,\xi}$ into the equation and multiplying by $e^{\rho_e t}$ gives equation (A.3). Converting the equation to differential notation then gives equation (A.4). □

Now, we can characterise the participation constraint and the incentive compatibility constraint.

Lemma 4. *The participation and incentive compatibility constraints in each regime are characterised by:*

(a) Regime 1:

(i) The contract satisfies the participation constraint if and only if, for all $t \leq T_1$, $W_t^{F,1,\xi} \geq W_t^U$ and $W_t^{F,1,\xi} + \Psi_{i,t}^1 \geq W_t^U$, for all $i \in \{\epsilon, (z, \epsilon)\}$.

(ii) The contract is incentive compatible ($\xi_t = 0$ is optimal) if and only if $\Psi^1_{b,t} \geq \beta$.

(b) Regime 0:

(i) The contract satisfies the participation constraint if and only if, for all $t \leq T_0$, $W^{F,0,\xi}_t \geq W^U_t$ and $W^{F,0,\xi}_t + \Psi^0_{z,t} \geq W^U_t$.

(ii) The contract is incentive compatible ($\xi_t = 0$ is optimal) if and only if $\Psi^0_{b,t} \geq \beta$.

Proof. The characterisation of the participation constraint follows immediately from the definition. It remains to show the characterisation of the incentive compatibility constraint. I will only show it for regime 0. The proof for regime 1 is analogous. From lemma 3 we have:

$$
\begin{aligned}
W^{F,0,\xi}_t(\mathcal{C}) = & \int_t^{T_0} e^{-\rho_e(s-t)}(d\delta_{0,s} + zA\xi_s\beta ds) \\
& + e^{-\rho_e(T_0-t)}(\mathbb{1}_{\{T_0\in\{\tau_c,\tau_0\}\}}W^U_{T_0} + \mathbb{1}_{\{T_0=\mathscr{T}_0\}}\widehat{W}^{e,\Phi}_t)) \\
& - \int_t^{T_0} e^{-\rho_e(s-t)}\left(W^U_t - W^{F,0,\xi}_t\right)(dN_{c,s} - \lambda_c ds) \\
& - \int_t^{T_0} e^{-\rho_e(s-t)}\sigma\Psi^0_{b,s}dB_s - \int_t^{T_1} e^{-\rho_e(s-t)}\Psi^0_{z,s}(dN_{z,s} - \lambda_z(z_s)ds) \\
= & \int_t^{T_0} e^{-\rho_e(s-t)}d\delta_{0,s} + \int_t^{T_0} e^{-\rho_e(s-t)}zA\xi_s(\beta - \Psi_{b,s})ds) \\
& + e^{-\rho_e(T_0-t)}(\mathbb{1}_{\{T_0\in\{\tau_c,\tau_0\}\}}W^U_{T_0} + \mathbb{1}_{\{T_0=\mathscr{T}_0\}}\widehat{W}^{e,\Phi}_t)) \\
& - \int_t^{T_0} e^{-\rho_e(s-t)}\left(W^U_t - W^{F,0,\xi}_t\right)(dN_{c,s} - \lambda_c ds) \\
& - \int_t^{T_0} e^{-\rho_e(s-t)}\sigma\Psi^0_{b,s}d\widetilde{B}_s - \int_t^{T_1} e^{-\rho_e(s-t)}\Psi^0_{z,s}(dN_{z,s} - \lambda_z(z_s)ds)
\end{aligned}
$$

where I have used that $\sigma d\widetilde{B}_t \equiv \sigma dB_t - \beta A\xi_t$. The entrepreneur chooses the process ξ in order to maximise the second term. It follows that the entrepreneur chooses $\xi_t = 0$ at time t if and only if $\beta \leq \Psi^0_{b,t}$. □

A.3.2 Proving Results from the Main Text

We can now return to the proof of lemma 2 from the main text.

Proof of Lemma 2: Part (a) follows from substituting $\xi = 0$ into equation (A.2). Part (b) follows from substituting $\xi = 0$ into equation (A.4). Parts (c) and (d) follow immediately from lemma 4. □

A.4 Supplementary Proofs for Subsection 3.4 (The Optimal Contract)

In this subsection of the appendix, I derive the Hamilton Jacobi Bellman equation for the banker's value function and the optimal contract in each regime. In some parts, the approach is similar to that used in Cvitanic and Zhang [2013] and Piskorski and Tchistyi [2010]. I leave details that replicate their work as references and include only the working that is new or will be helpful in later proofs. In particular, I do not provide a verification theorem since the approach essentially repeats Cvitanic and Zhang [2013] section 7.3.3.

A.4.1 Optimal Contract in Regime 1

I start by deriving the optimal contract for times $t > \mathscr{T}_0$ (that is strictly after the banker has paid the cost and started conditioning the contract on their idiosyncratic state). In this case, the banker solves the continuation value problem:

$$V_t^1(w_t, \epsilon_t, z_t, \mathbf{x}) \equiv \sup_{\mathcal{C}^1} \Bigg\{ \mathbb{E}_t \Bigg[\int_t^{T_1} e^{-\rho_h s}\big((zA - r(\epsilon_s)D)ds - d\delta_{1,s}\big) + e^{-\rho_h T_1}(\mathbb{1}_{\{T_1=\tau_1\}} L_B + \mathbb{1}_{\{T_1=\tau_c\}}(1-D)) \Bigg] \Bigg\}$$

where $T_1 \equiv \min\{\tau_c, \tau_1\}$ and $L_B \equiv \max\{L - D, 0\}$ is the liquidation value left for the banker after repaying debt holders and the choice of $\mathcal{C}^1$ must satisfy the constraints:

(PK): $W_t^{F,1} = w_t$,

(PC): $W_s^{F,1} \geq W_s^U$, for all $t \leq s \leq T_1$, and

(IC): $\xi_t = 0$ solves the entrepreneur's problem (3.2).

I solve this problem in Proposition 4.

Proposition 4 (Optimal Contract in Regime 1). *Under the optimal contract:*

(a) If $zA - r(\epsilon_B)I > 0$, then the banker chooses to terminate the project the first time that W falls below $W^U(z, \mathbf{x})$. This means that $W^U(z, \mathbf{x})$ is a lower absorbing boundary for the entrepreneur's continuation value W_t and so the optimal termination time is:

$$\tau_1 = \inf\{t \geq 0 : W_t \leq W^U(z, \mathbf{x})\}$$

(b) For each state $(\epsilon, z, \mathbf{x})$, let $\overline{W}^{1*}(\epsilon, z, \mathbf{x})$ denote the continuation value that satisfies:

$$\partial_W V^1(\overline{W}^{1*}, \epsilon, z, \mathbf{x}) = -1$$

Then the banker chooses a payout process, δ_1, such that $\overline{W}^{1*}(\epsilon, z, \mathbf{x})$ is an upper reflecting boundary for the entrepreneur's continuation value W_t:

- *When $W_t < W^{1*}(\epsilon_t, z_t, \mathbf{x})$, the banker makes no payments to the entrepreneur ($d\delta_{1,t} = 0$), and*
- *When $W_t \geq W^{1*}(\epsilon_t, z_t, \mathbf{x})$, the banker makes a sufficiently large payment ($d\delta_{1,t} \geq 0$) to keep W_t weakly below $W^{1*}(\epsilon_t, z_t, \mathbf{x})$.*

(c) Inside the region $[W^U(z, \mathbf{x}),\ \overline{W}^{1*}(\epsilon, z, \mathbf{x})]$, the banker's value function solves the Hamilton Jacobi Bellman equation:

$$
\begin{aligned}
&\rho_h V^1(W, \epsilon, z, \mathbf{x}) \\
&\quad = \sup_{\Psi_b, \Psi_\epsilon, \Psi_{z,\cdot}} \Big\{ zA - r(\epsilon)D \\
&\qquad\qquad + \Big(\rho_e W - \lambda_\epsilon(\epsilon)\Psi_\epsilon - \sum_{\epsilon' \in \mathcal{E}} \overline{\pi}(\epsilon'|z^c)\lambda_z(z)\Psi^1_{z,\epsilon'} - \lambda_c(W^U(z, \mathbf{x}) - W)\Big)\partial_W V^1(W, \epsilon, z, \mathbf{x}) \\
&\qquad\qquad + \Psi^2_{b,t}\sigma^2 \partial_{WW} V^1(W, \epsilon, z, \mathbf{x}) \\
&\qquad\qquad + \lambda_\epsilon(\epsilon|z)(V^1(W + \Psi_\epsilon, \epsilon^c, z, \mathbf{x}) - V^1(W, \epsilon, z, \mathbf{x})) \\
&\qquad\qquad + \lambda_z(z)\Big(\sum_{\epsilon'} \overline{\pi}(\epsilon'|z^c)V^1(W + \Psi^1_{z,\epsilon'}, \epsilon', z^c, \mathbf{x}) - V^1(W, \epsilon, z, \mathbf{x})\Big) \\
&\qquad\qquad + \lambda_c(L_B - V^1(W, \epsilon, z, \mathbf{x}))\Big\}
\end{aligned}
$$

subject to the constraints:

$$
\begin{aligned}
&(IC): \Psi^1_{b,t} \geq \beta \\
&(PC): \Psi^1_i \geq W^U(z, \mathbf{x}) - W, \quad \forall i \in \{\epsilon, (z, \epsilon), (z, \epsilon^c)\}
\end{aligned}
$$

and with the boundary conditions:

$$
\begin{aligned}
(BC_L) &: V^1(W^U(z, \mathbf{x}), \epsilon, z, \mathbf{x}) = L_B \\
(BC_{U,1}) &: \partial_W V^1(\overline{W}^{1*}, \epsilon, z, \mathbf{x}) = -1 \\
(BC_{U,2}) &: \partial_{WW} V^1(\overline{W}^{1*}, \epsilon, z, \mathbf{x}) = 0
\end{aligned}
$$

and where ϵ^c is the complement of ϵ.

(d) Suppose that $V^1(W, \epsilon, z, \mathbf{x}) \in C^2$. Then, the value function $V^1(W, \epsilon, z, \mathbf{x})$ is concave in W. It follows that the optimal choices of Ψ^1_b, Ψ^1_ϵ and $\Psi^1_{(z,\cdot)}$ satisfy:

(Ψ^1_b) : $\Psi^1_b = \beta$

(Ψ^1_ϵ) : $\Psi^1_\epsilon = \max\{\widetilde{\Psi}_\epsilon, W^U - W\}$, where $\widetilde{\Psi}_\epsilon$ solves:

$$\partial_W V^1(W + \widetilde{\Psi}_\epsilon, \epsilon^c, z, \mathbf{x}) = \partial_W V^1(W, \epsilon, z, \mathbf{x})$$

($\Psi^1_{z,\epsilon'}$) : $\Psi^1_{z,\epsilon'} = \max\{\widetilde{\Psi}_{z,\epsilon'}, W^U - W\}$, *where, for* $\epsilon' \in \mathcal{E}$, $\widetilde{\Psi}_{z,\epsilon'}$ *solves:*

$$\partial_W V^1(W + \widetilde{\Psi}_{z,\epsilon'}, \epsilon', z^c, \mathbf{x}) = \partial_W V^1(W, \epsilon, z, \mathbf{x})$$

Proof. In order to economise on space, throughout this proof, I drop the $\mathbf{x}$ state variable from the notation since it is not changing.

(a): This follows directly from Cvitanic and Zhang [2013] Lemma 7.3.2. Essentially the banker never wants to terminate the contract before the participation constraint binds because the project generates a positive surplus, the project has mean zero Brownian motion increments and the banker is committed to paying the entrepreneur their promised continuation value if the contract is terminated early.

(b) & (c): I prove parts (b) and (c) together. I start by re-expressing the banker's contract choice in terms of the shock exposures. From part (a), we can see there is an equivalence between the choice of τ_1 and the choice of $\{\Psi^1_b, \Psi^1_\epsilon, \Psi^1_{(z,\cdot)}\}$. Thus, we can write the continuation value and the discounted continuation value of the banker as:

$$\begin{aligned}
V^1_t(W_t, \epsilon_t, z_t) = \sup_{\Psi_b, \Psi_\epsilon, \Psi_{(z,\cdot)}, \delta_1} \mathbb{E}_t\Bigg[\int_t^{T_1} & e^{-\rho_h(s-t)}\big((A - r(\epsilon_s)D)ds - d\delta_{1,s}\big) \\
& + e^{-\rho_h(T_1-t)}(\mathbb{1}_{\{T_1=\tau_1\}}L_B + \mathbb{1}_{\{T_1=\tau_c\}}(1-D))\Bigg] \\
U^1_t(W_t, \epsilon_t, z_t) \equiv e^{-\rho_h t}V^1_t(W_t, \epsilon_t, z_t) &
\end{aligned}$$

I now derive the Hamilton Jacobi Bellman equation. By definition, we have that:

$$\begin{aligned}
&U^1_t(W_t, \epsilon_t, z_t) \\
&\qquad \geq \mathbb{E}_t\left[\int_t^{T_1} e^{-\rho_h s}\big((A - r(\epsilon_s)D)ds - d\delta_{1,s}\big) + e^{-\rho_h T_1}(\mathbb{1}_{\{T_1=\tau_1\}}L_B + \mathbb{1}_{\{T_1=\tau_c\}}(1-D))\right] \\
&\qquad = \mathbb{E}_t\left[\int_t^{t+\hat{h}} e^{-\rho_h s}\big((A - r(\epsilon_s)D)ds - d\delta_{1,s}\big) + U^1_{t+\hat{h}}(W_{t+\hat{h}}, \epsilon_{t+\hat{h}}, z_{t+\hat{h}})\right] \\
&\quad \Rightarrow 0 \geq \mathbb{E}_t\left[\int_t^{t+\hat{h}} e^{-\rho_h s}\big((A - r(\epsilon_s)D)ds - d\delta_{1,s}\big) + U^1_{t+\hat{h}}(W_{t+\hat{h}}, \epsilon_{t+\hat{h}}, z_{t+\hat{h}}) - U^1_t(W_t, \epsilon_t, z_t)\right]
\end{aligned}$$

where $t+\hat{h} \equiv (t+h) \wedge T_1$. Then, by Ito's Lemma we have that:

$$
\begin{aligned}
0 \geq \mathbb{E}_t \Bigg[& \int_t^{t+\hat{h}} e^{-\rho_h s}\big((A - r(\epsilon_s)D)ds - d\delta_{1,s}\big) + U^1_{t+\hat{h}}(W_{t+\hat{h}}, \epsilon_{t+\hat{h}}, z_{t+\hat{h}}) - U^1_t(W_t, \epsilon_t, z_t) \Bigg] \\
\geq \mathbb{E}_t \Bigg[& \int_t^{t+\hat{h}} e^{-\rho_h s}\big((A - r(\epsilon_s)D)ds - d\delta_{1,s}\big) \\
& + \int_t^{t+\hat{h}} \Big(\rho_e W_s - \lambda_\epsilon(\epsilon_s|z_s)\Psi^1_{\epsilon,s} - \sum_{\epsilon' \in \mathcal{E}} \overline{\pi}(\epsilon'|z_s)\lambda_z(z_s)\Psi^1_{z,\epsilon',s} - \lambda_c(W^U_s - W_s) - d\delta_{1,s}\Big)\partial_W U^1(W_s, \epsilon_s, z_s)ds \\
& + \int_t^{t+\hat{h}} (\Psi^1_{b,s})^2\sigma^2 \partial_{WW} U^1(W_s, \epsilon_s, z_s)ds \\
& + \int_t^{t+\hat{h}} \lambda_\epsilon(\epsilon_s|z_s)(U^1(W + \Psi^1_{\epsilon,s}, \epsilon^c_s, z_s) - U^1(W_s, \epsilon_s, z_s))ds \\
& + \int_t^{t+\hat{h}} \lambda_z(z_s)\Big(\sum_{\epsilon'} \overline{\pi}(\epsilon', z_s)U^1(W_s + \Psi^1_{z_s,\epsilon'}, \epsilon', z^c_s) - U^1(W_s, \epsilon_s, z_s)\Big)ds \Bigg]
\end{aligned}
$$

Dividing by h and then taking the limit as $h \downarrow 0$ gives the HJB equation:

$$
\begin{aligned}
0 = \sup_{\Psi^1_b, \Psi^1_\epsilon, \Psi^1_{(z,\cdot)}, \delta_1} \Big\{ & zA - r(\epsilon)D \\
& + \Big(\rho_e W - \lambda_\epsilon(\epsilon)\Psi^1_\epsilon - \sum_{\epsilon' \in \mathcal{E}} \overline{\pi}(\epsilon'|z)\lambda_z(z)\Psi^1_{z,\epsilon'} - \lambda_c(W^U(z) - W)\Big)\partial_W U^1(W, \epsilon, z) \\
& + (\Psi^1_{b,t})^2\sigma^2\partial_{WW}U^1(W, \epsilon, z) - (1 + \partial_W U^1(W, \epsilon, z))d\delta_1 \\
& + \lambda_\epsilon(\epsilon|z)(U^1(W + \Psi^1_\epsilon, \epsilon^c, z) - U^1(W, \epsilon, z)) \\
& + \lambda_z(z)\Big(\sum_{\epsilon'} \overline{\pi}(\epsilon'|z)U^1(W + \Psi^1_{z,\epsilon'}, \epsilon', z^c) - U^1(W, \epsilon, z)\Big) \\
& + \lambda_c(L_B - U^1(W, \epsilon, z))\Big\}
\end{aligned}
$$

From this equation, we can see that the optimal choice of δ_1 must satisfy:

$$
\sup_{d\delta_{1,t}} \{-(1 + \partial_W U^1(W_t, \epsilon_t, z_t))d\delta_{1,t}\}
$$

and so it follows that $d\delta_{1,t} = 0$ for $\partial_W U^1(W_t, \epsilon_t, z_t) > -1$ and $d\delta_{1,t} \geq 0$ for $\partial_W U^1(W_t, \epsilon_t, z_t) \leq -1$. When $\partial_W U^1(W_t, \epsilon_t, z_t) < -1$, the banker sets $d\delta_{1,t} > 0$ to immediately reduce W_t to a value at which $\partial_W U^1(W_t, \epsilon_t, z_t) \geq -1$. In other words, it describes a reflecting boundary. This gives part (b) of the proposition.

Finally, substituting the optimal δ_1 policy into the differential equation and using Ito's lemma gives that $V^1_t(W_t, \epsilon_t, z_t) = e^{\rho_h t}U_t(W_t, \epsilon_t, z_t)$ solves the differential equation in part (c) of the proposition.

(d): Concavity follows from Piskorski and Tchistyi [2010] Proposition 2. A version of their proof tailored specifically to this problem is available upon request to the interested reader. The steps are the same although there are additional Poisson jumps to handle. The first order conditions come directly from the HJB equation. □

A.4.2 Optimal Contract on the Boundary

At the regime switching time, $\mathscr{T}_0$, the participation constraint for the banker is more complicated because they have committed to delivering a continuation value to the entrepreneur in expectation under the entrepreneur's belief about the ϵ process. Conceptually, this means that the banker can shift the banker's continuation value at the boundary based on their idiosyncratic state so long as the entrepreneur believes they will get their promised value, in expectation. Formally, suppose that the promised continuation value is $W^{F,0}_{\mathscr{T}_0}$ at stopping time $\mathscr{T}_0$. Observe that this could be the value at the upper or lower boundary. That is $\mathscr{T}_0$ could be $\underline{\mathscr{T}}_0$ or $\overline{\mathscr{T}}_0$. At time 0, the problem for the banker at the boundary is:

$$\max_{W^{F,1}_{\mathscr{T}_0}} \left\{ \sum_{\epsilon_0 \in \mathcal{E}} \overline{\pi}(\epsilon_0|z_0) \mathbb{E}\left[e^{-\rho_h \mathscr{T}_0} V^1(W^{F,1}_{\mathscr{T}_0}, \epsilon_{\mathscr{T}_0}, z_{\mathscr{T}_0}, \mathbf{x}) \middle| \epsilon_0, z_0 \right] \right\}$$

subject to the promise keeping constraint that:

$$\mathbb{E}^e_{\mathscr{T}_0}[W^{F,1}_{\mathscr{T}_0}] = W^{F,0}_{\mathscr{T}_0}$$

Lemma 5. *Suppose that Assumption 1. Then, the banker chooses* $\Psi_\Phi(\epsilon, z, \mathbf{x})$ *such that:*

$$W^{F,1}_{\mathscr{T}_0} = W^{F,0}_{\mathscr{T}_0} + \Psi_\Phi(\epsilon_{\mathscr{T}_0}, z_{\mathscr{T}_0}, \mathbf{x}) \equiv W^\Phi(\epsilon_{\mathscr{T}_0}, z_{\mathscr{T}_0}, \mathbf{x})$$

where:

$$\Psi_\Phi(\epsilon, z, \mathbf{x}) \equiv \max\{\widetilde{\Psi}_\Phi, W^U(z, \mathbf{x}) - W^{F,0}_{\mathscr{T}_0}\}$$

and where $\widetilde{\Psi}_\Phi$ *solves:*

$$\partial_w V^1(W^{F,0}_{\mathscr{T}_0} + \widetilde{\Psi}_\Phi(\epsilon_G, z, \mathbf{x}), \epsilon_G, z, \mathbf{x}) = \partial_w V^1(W^{F,0}_{\mathscr{T}_0} + \widetilde{\Psi}_\Phi(\epsilon_B, z, \mathbf{x}), \epsilon_B, z, \mathbf{x})$$
$$\sum_{\epsilon \in \mathcal{E}} \overline{\pi}(\epsilon|z) \widetilde{\Psi}_\Phi(\epsilon, z, \mathbf{x}) = 0$$

Proof. Let W^Φ denote the value at the boundary, $W^{F,1}_{\mathscr{T}_0}$. The objective function becomes:

$$
\begin{aligned}
&\sum_{\epsilon_0\in\mathcal{E}}\overline{\pi}(\epsilon_0|z_0)\mathbb{E}\left[e^{-\rho_h\mathscr{T}_0}V^1(W^\Phi,\epsilon_{\mathscr{T}_0},z_{\mathscr{T}_0},\mathbf{x})\middle|\epsilon_0,z_0\right]\\
&\qquad=\sum_{\epsilon_0\in\mathcal{E}}\overline{\pi}(\epsilon_0|z_0)\mathbb{E}\left[\mathbb{E}\left[e^{-\rho_h\mathscr{T}_0}V^1(W^\Phi,\epsilon_{\mathscr{T}_0},z_{\mathscr{T}_0},\mathbf{x})\middle|\mathcal{F}^{\tilde{B},z}_{\mathscr{T}_0},\epsilon_0,z_0\right]\middle|\epsilon_0,z_0\right]\\
&\qquad=\mathbb{E}\left[e^{-\rho_h\mathscr{T}_0}\sum_{\epsilon_0\in\mathcal{E}}\overline{\pi}(\epsilon_0|z_0)\mathbb{E}\left[V^1(W^\Phi,\epsilon_{\mathscr{T}_0},z_{\mathscr{T}_0},\mathbf{x})\middle|z_{\mathscr{T}_0},\epsilon_0,z_0\right]\middle|\epsilon_0,z_0\right]\\
&\qquad=\mathbb{E}\left[e^{-\rho_h\mathscr{T}_0}\sum_{\epsilon_0\in\mathcal{E}}\overline{\pi}(\epsilon_0|z_0)\sum_{\epsilon_{\mathscr{T}_0}}V^1(W^\Phi,\epsilon_{\mathscr{T}_0},z_{\mathscr{T}_0},\mathbf{x})\mathbb{P}(\epsilon_{\mathscr{T}_0}|z_{\mathscr{T}_0},\epsilon_0,z_0)\middle|\epsilon_0,z_0\right]\\
&\qquad=\mathbb{E}\left[e^{-\rho_h\mathscr{T}_0}\mathbb{P}(z_{\mathscr{T}_0}|z_0)\sum_{\epsilon_{\mathscr{T}_0}}V^1(W^\Phi,\epsilon_{\mathscr{T}_0},z_{\mathscr{T}_0},\mathbf{x})\overline{\pi}(\epsilon_{\mathscr{T}_0}|z_{\mathscr{T}_0})\middle|z_0\right]
\end{aligned}
$$

The promise keeping constraint becomes:

$$
\begin{aligned}
\mathbb{E}^e_{\mathscr{T}_0}[W^\Phi]&=W^{F,0}_{\mathscr{T}_0}\\
\Rightarrow\mathbb{E}^e_{\mathscr{T}_0}[W^{F,0}_{\mathscr{T}_0}+\Psi_\Phi(\epsilon,z,\mathbf{x})]&=W^{F,0}_{\mathscr{T}_0}\\
\Rightarrow\mathbb{E}^e_{\mathscr{T}_0}[\Psi_\Phi(\epsilon,z,\mathbf{x})]&=0\\
\Rightarrow\sum_{\epsilon\in\mathcal{E}}\overline{\pi}(\epsilon|z_{\mathscr{T}_0})\Psi_\Phi(\epsilon,z_{\mathscr{T}_0},\mathbf{x})&=0
\end{aligned}
$$

Taking the first order condition completes the proof. □

A.4.3 Optimal Contract in Regime 0

The problem in contract regime 0 is less standard. To my knowledge, this is the first paper to consider regime switching and incomplete conditioning in a continuous time long-term contracting problem.

The reason that it is difficult to solve for the optimal contract in regime 0 is that the banker cannot condition the project on ϵ even though ϵ affects their output. This makes it difficult to set up the problem recursively. I resolve this issue by setting up an "auxiliary" problem in which the banker's forecast of ϵ given time 0 information is a state variable. This could be interpreted as solving the problem for a banker who does not know their own cost of funds but knows the stochastic process that governs the evolution of ϵ. The full list of state variables in the auxiliary problem is the entrepreneur's continuation value, W, the aggregate state, z, the banker's forecast, π^b_t, the entrepreneur's belief, π^e_t and the project type, $\mathbf{x}$. However, under assumption 1, these beliefs collapse to $\pi^b_t=\pi^b_t=\overline{\pi}$. Thus, the banker effectively only has the dynamic state variables: W and z. This means that I can use standard techniques to solve for the banker's value function in the auxiliary problem and get the banker's choice of contract terms. I can then go back and compute the banker's ex-post ϵ-dependent value under their chosen policy.

Banker's Problem

The banker's problem in regime 0 is to choose a contract, $\mathcal{C}^0 = \{\tau_0, \delta_0, \mathscr{T}_0\}$, that solves:

$$V_0^0(W_0, z_0, \mathbf{x}) = \sup_{\mathcal{C}^0} \Bigg\{ \sum_{\epsilon_0 \in \mathcal{E}} \overline{\pi}(\epsilon_0|z_0) \mathbb{E}\Bigg[\int_{t_0}^{T_0} e^{-\rho_h s}\big((zA - r(\epsilon_s)D)ds - d\delta_{0,s}\big)\big) \\ + e^{-\rho_h T_0}\Big(\mathbb{1}_{\{T_0 = \mathscr{T}_0\}} (V^1_{\mathscr{T}_0}(W^\Phi(\epsilon_{\mathscr{T}_0}, z_{\mathscr{T}_0}, \mathbf{x}), \epsilon_{\mathscr{T}_0}, z_{\mathscr{T}_0}, \mathbf{x}) - \Phi(z_{\mathscr{T}_0})) \\ + \mathbb{1}_{\{T_0 = \tau_c\}}(1-D) + \mathbb{1}_{\{T_0=\tau_0\}} L_B \Big) \Big| \epsilon_0, z_0 \Bigg] \Bigg\}$$

where the expectation over ϵ_0 reflects that the bankers gets a random draw of ϵ from the stationary distribution at contract initialisation, $T_0 \equiv \min\{\tau_c, \tau_0\}$, $L_B = \max\{L - D, 0\}$, $\Phi(z)$ is the cost of triggering the covenant to switch regime, and the banker's choice of $\mathcal{C}^0$ must satisfy the promise keeping (PK), participation (PC) and incentive compatibility (IC) constraints:

(PK): $W_0^{F,0} = w_0$,

(PC): $W_t^{F,0} \geq W_t^U$, for all $t_0 \leq t \leq T_0$, and

(IC): $\xi_t = 0$ solves the entrepreneur's problem (3.2),

and where $V^1_{\tau_0}(W^\Phi(\epsilon_{\mathscr{T}_0}, z_{\mathscr{T}_0}, \mathbf{x}), \epsilon_{\mathscr{T}_0}, z_{\mathscr{T}_0}, \mathbf{x})$ denotes the value function of the banker when they trigger the covenant to enter regime 1 at time $\mathscr{T}_0$, as characterised in lemma 5.

Recursive Formation of the Auxilliary Problem

Before setting up the auxilliary continuation value problem, I introduce some additional notation. Denote the probability of ϵ_t given contractible information by $\pi_t^b(\epsilon)$. In regime 0, this is given by $\pi_t^b(\epsilon) \equiv \mathbb{P}[\epsilon_t = \epsilon | \mathcal{F}_t^{\tilde{B},z}]$ and, in regime 1, this is degenerate at the true value. Denote the vector of probabilities by:

$$\pi_t^b \equiv [\pi_t^b(\epsilon_L), \pi_t^b(\epsilon_H)]^T$$

The process, π_t^b is adapted to $\mathcal{F}_t^z$ and so, in general, can evolve according to:

$$d\pi_t^b = \mu_\pi^b(\pi_t^b, z_t)dt + \gamma_\pi^b(\pi_t^b, z_t)dN_{z,t}$$

The auxiliary continuation value (with full potential state space) is then defined by:

$$\widetilde{V}_t^0(W_t, \pi_t^b, \pi_t^e, z_t, \mathbf{x}) = \sup_{\mathcal{C}^0} \Bigg\{ \sum_{\epsilon_t \in \mathcal{E}} \pi_t^b(\epsilon_t) \mathbb{E}\Bigg[\int_t^{T_0} e^{-\rho_h(s-t)} \big((zA - r(\epsilon_s)D)ds - d\delta_{0,s}\big) + e^{-\rho_h(T_0-t)} \Big(\mathbb{1}_{\{T_0 = \mathscr{T}_0\}} (V^1_{\mathscr{T}_0}(W^\Phi, \epsilon_{\mathscr{T}_0}, z_{\mathscr{T}_0}, \mathbf{x}) - \Phi(z_{\mathscr{T}_0})) + \mathbb{1}_{\{T_0 = \tau_c\}}(1-D) + \mathbb{1}_{\{T_0=\tau_0\}} L_B \Big) \Big| \mathcal{F}_t^{\widetilde{B},z}, \epsilon_t \Bigg] \Bigg\} \tag{A.7}$$

Suppose that assumption 1 holds. Then, $\pi_t^b = \pi_t^e = \overline{\pi}(z_t)$ and so:

$$d\pi_t^b = d\pi_t^e = (\overline{\pi}(z_t^c) - \overline{\pi}(z_t))dN_{z,t}$$

which means that the auxilliary continuation value can be expressed with a smaller state space:

$$\widetilde{V}_t^0(W_t, z_t, \mathbf{x}) = \sup_{\mathcal{C}^0} \Bigg\{ \sum_{\epsilon_t \in \mathcal{E}} \overline{\pi}(\epsilon_t | z_t) \mathbb{E}\Bigg[\int_t^{T_0} e^{-\rho_h(s-t)} \big((zA - r(\epsilon_s)D)ds - d\delta_{0,s}\big) + e^{-\rho_h(T_0-t)} \Big(\mathbb{1}_{\{T_0 = \mathscr{T}_0\}} (V^1_{\mathscr{T}_0}(W^\Phi, \epsilon_{\mathscr{T}_0}, z_{\mathscr{T}_0}, \mathbf{x}) - \Phi(z_{\mathscr{T}_0})) + \mathbb{1}_{\{T_0 = \tau_c\}}(1-D) + \mathbb{1}_{\{T_0=\tau_0\}} L_B \Big) \Big| \mathcal{F}_t^{\widetilde{B},z}, \epsilon_t \Bigg] \Bigg\} \tag{A.8}$$

Optimal Contract

I now consider the optimal choice of the contracting terms in regime 0. The banker must choose the termination time, τ_0, the regime switching time, $\mathscr{T}_0$, and the payout process δ_0. As in regime 1, there will be a lower termination boundary and an upper payout boundary. However, it could be that the banker chooses to pay the cost Φ and switch regimes before the termination or payout boundary is reached. I lay out the possibilities in proposition 5.

Proposition 5. *Suppose that assumption 1 holds. Suppose that both $\widetilde{V}^0$ and V^1 are $\mathscr{C}^2$. Then:*

(a) If $zA - \sum_{\epsilon \in \mathcal{E}} \overline{\pi}(\epsilon | z_B) r(\epsilon) I > 0$, then the banker chooses a termination time

$$\tau_0^* = \inf\{t \geq 0 : W_t^0 \leq W^U(z, \mathbf{x})\}$$

(b) For each state $(z, \pi^b, \pi^e, \mathbf{x})$, let $\overline{W}^{\delta *}(z, \pi^b, \pi^e, \mathbf{x})$ denote the continuation value that satisfies:

$$\partial_W \widetilde{V}^0(\overline{W}^{\delta *}, z, \pi^b, \pi^e, \mathbf{x}) = -1$$

Then the banker chooses an optimal payout process, δ_0^*, such that:

- *When* $W_t < \overline{W}^{\delta*}(z, \pi^b, \pi^e, \mathbf{x})$*, the banker makes no payments to the entrepreneur* ($d\delta_{0,t} = 0$)*, and*
- *When* $W_t \geq \overline{W}^{\delta*}(z, \pi^b, \pi^e, \mathbf{x})$*, the banker makes a sufficiently large payment* ($d\delta_{0,t} \geq 0$) *to keep* W_t *weakly below* $\overline{W}^{\delta*}(z, \pi^b, \pi^e, \mathbf{x})$.

(c) *Define the regime switching set by:*

$$\mathcal{S}_0(z, \pi^b, \pi^e, \mathbf{x}) \equiv \left\{ w \in \mathbb{R} : \mathbb{E}^b \left[V^1(w + \Psi_\Phi(\epsilon, z, \mathbf{x}; w), \epsilon, z, \mathbf{x}) \right] - \Phi(z) \geq \widetilde{V}^0(z, \pi^b, \pi^e, \mathbf{x}) \right\}$$

where $\mathbb{E}^b[\cdot]$ *is the expectation under belief* π^b*. The regime switching time,* $\mathscr{T}_0$*, is then given by* $\mathscr{T}_0 = \inf\{t \geq 0 : W_t^0 \in \mathcal{S}_0(z_t, \mathbf{x})\}$*. For* $\Phi(z)$ *sufficiently small,* $\mathcal{S}_0$ *is non-empty. In this case, for each* z*, at* $\mathscr{T}_0 < \infty$*, the value matching and smooth pasting conditions are satisfied:*

$$\begin{aligned}
\widetilde{V}^0(W^0_{\mathscr{T}_0}, z, \pi^b, \pi^e, \mathbf{x}) &= \mathbb{E}^b \left[V^1(W^0_{\mathscr{T}_0} + \Psi_\Phi(\epsilon, z, \mathbf{x}; W^0_{\mathscr{T}_0}), \epsilon, z, \mathbf{x}) \right] - \Phi(z) \\
\partial_W \widetilde{V}^0(W^0_{\mathscr{T}_0}, z, \pi^b, \pi^e, \mathbf{x}) &= \partial_W \mathbb{E}^b \left[V^1(W^0_{\mathscr{T}_0} + \Psi_\Phi(\epsilon, z, \mathbf{x}; W^0_{\mathscr{T}_0}), \epsilon, z, \mathbf{x}) \right]
\end{aligned}$$

(d) *Define the following thresholds*

$$\underline{W}^\Phi(z, \pi^b, \pi^e, \mathbf{x}) \equiv \begin{cases} \sup\{w \in \mathbb{R} : w \leq W_0, w \in \mathcal{S}_0\} & \text{if } \{w \leq W_0\} \cup \mathcal{S}_0 \neq \emptyset \\ -\infty & \text{otherwise} \end{cases}$$

$$\overline{W}^\Phi(z, \pi^b, \pi^e, \mathbf{x}) \equiv \begin{cases} \inf\{w \in \mathbb{R} : w \geq W_0, w \in \mathcal{S}_0\} & \text{if } \{w \geq W_0\} \cup \mathcal{S}_0 \neq \emptyset \\ +\infty & \text{otherwise} \end{cases}$$

Then the entrepreneur's continuation value in regime 0 *evolves in the region* $[\underline{W}^{0,*}, \overline{W}^{0,*}]$*, where*

$$\begin{aligned}
\underline{W}^{0,*} &= \max \left\{ \underline{W}^\Phi(z, \pi^b, \pi^e, \mathbf{x}), W^U(z, \mathbf{x}) \right\} \\
\overline{W}^{0,*} &= \min \left\{ \overline{W}^\Phi(z, \pi^b, \pi^e, \mathbf{x}), \overline{W}^{\delta 0}(z, \pi^b, \pi^e, \mathbf{x}) \right\}
\end{aligned}$$

with the following behaviour at the boundaries:

(i) *If* $\underline{W}^\Phi(z, \pi^b, \pi^e, \mathbf{x}) > \underline{W}^{0*}(z, \pi^b, \pi^e, \mathbf{x})$*, then* $\tau_0^* = \infty$ *and the project is never terminated in regime* 0*. In this case,* $\underline{W}^\Phi(z, \pi^b, \pi^e, \mathbf{x})$*, is a lower regime switching threshold. If* $\underline{W}^\Phi(z, \pi^b, \pi^e, \mathbf{x}) \leq \underline{W}^{0*}(z, \pi^b, \pi^e, \mathbf{x})$*, then* $\mathscr{T}_0^* = \infty$ *and there is no lower regime switching boundary. In this case,* $\underline{W}^{0*}(z, \pi^b, \pi^e, \mathbf{x})$ *is an absorbing lower boundary in regime* 0.

(ii) *If* $\overline{W}^\Phi(z, \pi^b, \pi^e, \mathbf{x}) < \overline{W}^{0*}(z, \pi^b, \pi^e, \mathbf{x})$*, then* $d\delta_{0,t} = 0$ *for all* $t < \mathscr{T}_0$*. In this case,* $\overline{W}^\Phi(z, \pi^b, \pi^e, \mathbf{x})$ *is an upper regime switching threshold. If* $\overline{W}^\Phi(z, \pi^b, \pi^e, \mathbf{x}) \geq \overline{W}^{0*}(z, \pi^b, \pi^e, \mathbf{x})$*, then there is no upper* *regime switching boundary. In this case, $\overline{W}^{0*}(z, \pi^b, \pi^e, \mathbf{x})$ is an upper reflecting boundary.*

Proof. I prove each part in order.
Parts (a) & (b): Follow from the same reasoning as the proof of parts (a) and (b) in Proposition 4.
Part (c): The characterisation of the regime-switching decision largely follows from Section 5.3 in Pham [2009]. However, I sketch the key details that are particular to this problem. The novel feature of this environment is that the banker's choice of regime switching time, $\mathscr{T}_0$, cannot depend upon the history of $\mathcal{F}_t^\epsilon$. This means that the banker chooses to switch regimes when the expected value of paying $\Phi(z)$ and switching to $V^1(w + \Psi_\Phi, \epsilon, z, \mathbf{x})$ is higher than remaining at $\widetilde{V}^0(w, z, \pi^b, \pi^e, \mathbf{x})$, where the expectation is taken under π_t^b – the banker's forecast of ϵ at $\mathscr{T}_0$ given public time 0 information.

Let $\mathcal{S}_0$ and $\mathscr{T}_0$ be defined as in the proof statement. Let $\mathcal{A}^f$ denote the set of contracts available to the banker in regime f. Since $\mathcal{A}^0 \subset \mathcal{A}^1$, it follows that:

$$\mathbb{E}_t^b\left[V^1(W + \Psi_\Phi(\epsilon, z, \mathbf{x}; W), \epsilon, z, \mathbf{x})\right] \geq \widetilde{V}^0(W, z, \pi^b, \pi^e, \mathbf{x}), \quad \forall W \in \mathbb{R}$$

and so, for sufficiently small $\Phi(z)$, $\mathcal{S}_0(z)$ is non-empty. In this case, by the continuity of V^1 and $\widetilde{V}^0$, we have the value matching condition at the regime switching time, $\mathscr{T}_0$:

$$\mathbb{E}^b\left[V^1(W^0_{\mathscr{T}_0} + \Psi_\Phi(\epsilon, z, \mathbf{x}; W^0_{\mathscr{T}_0}), \epsilon, z, \mathbf{x})\right] - \Phi(z_{\mathscr{T}_0}) = \widetilde{V}^0(W^0_{\mathscr{T}_0}, z_{\mathscr{T}_0}, \pi^b_{\mathscr{T}_0}, \pi^e_{\mathscr{T}_0}, \mathbf{x})$$

And, by standard arguments (e.g. Proposition 5.3.2. in Pham [2009]), we also have a smooth pasting condition at the regime switching time, $\mathscr{T}_0$:

$$\partial_W \mathbb{E}^b\left[V^1(W^0_{\mathscr{T}_0} + \Psi_\Phi(\epsilon, z, \mathbf{x}; W^0_{\mathscr{T}_0}), \epsilon, z, \mathbf{x})\right] = \partial_W \widetilde{V}^0(W^0_{\mathscr{T}_0}, z_{\mathscr{T}_0}, \pi^b_{\mathscr{T}_0}, \pi^e_{\mathscr{T}_0}, \mathbf{x})$$

Let $\underline{W}^\Phi$ and $\overline{W}^\Phi$ be defined as in the proof statement. If $\max\left\{\left|\underline{W}^\Phi(z)\right|, \left|\overline{W}^\Phi(z)\right|\right\} < \infty$, then the value matching and smooth pasting conditions can be expressed more explicitly as:

$$\begin{aligned}
\widetilde{V}^0(\underline{W}^\Phi, z) &= \mathbb{E}_t^e\left[V^1(\underline{W}^\Phi + \Psi_\Phi(\epsilon, z, \mathbf{x}; \underline{W}^\Phi), \epsilon, z)\right] - \Phi(z) \\
\widetilde{V}^0(\overline{W}^\Phi, z) &= \mathbb{E}_t^e\left[V^1(\overline{W}^\Phi + \Psi_\Phi(\epsilon, z, \mathbf{x}; \overline{W}^\Phi), \epsilon, z)\right] - \Phi(z) \\
\partial_W \widetilde{V}^0(\underline{W}^\Phi, z) &= \partial_W \mathbb{E}_t^e\left[V^1(\underline{W}^\Phi + \Psi_\Phi(\epsilon, z, \mathbf{x}; \underline{W}^\Phi, \epsilon, z)\right] \\
\partial_W \widetilde{V}^0(\overline{W}^\Phi, z) &= \partial_W \mathbb{E}_t^e\left[V^1(\overline{W}^\Phi + \Psi_\Phi(\epsilon, z, \mathbf{x}; \overline{W}^\Phi, \epsilon, z)\right]
\end{aligned}$$

(d): Follows immediately from (a), (b) and (c).

□

HAMILTON JACOBI BELLMAN EQUATION

I now derive the Hamilton-Jacobi-Bellman (HJB) equation for the banker's value function in regime 0. First, observe that, by Proposition 5, there is an equivalence between choosing the stopping times, $(\tau_0,\ \mathscr{T}_0)$, and choosing the entrepreneur's exposure to the stochastic processes in the economy, $(\Psi_b,\ \Psi_z)$. Thus, we can rewrite the banker's optimisation problem as:

$$\begin{aligned}\widetilde{V}_t^0(W_t, \pi_t^b, \pi_t^e, z_t, \mathbf{x}) = \sup_{\Psi_b^0, \Psi_z^0, \delta_0} \Bigg\{ \sum_{\epsilon_t \in \mathcal{E}} \pi_t^b(\epsilon_t) \mathbb{E}\Bigg[\int_t^{T_0} e^{-\rho_h(s-t)} \big((zA - r(\epsilon_s)D)ds - d\delta_{0,s}\big) \\ + e^{-\rho_h(T_0-t)} \Big(\mathbb{1}_{\{T_0=\mathscr{T}_0\}} (V_{\mathscr{T}_0}^1(W^\Phi, \epsilon_{\mathscr{T}_0}, z_{\mathscr{T}_0}, \mathbf{x}) - \Phi(z_{\mathscr{T}_0})) \\ + \mathbb{1}_{\{T_0=\tau_c\}}(1-D) + \mathbb{1}_{\{T_0=\tau_0\}} L_B \Big) \Big| \mathcal{F}_t^{\widetilde{B},z}, \epsilon_t \Bigg] \Bigg\}\end{aligned} \tag{A.9}$$

Lemma 6 then derives the HJB equation with the full state space.

Lemma 6. *The Hamilton Jacobi Bellman equation for* $\widetilde{V}^0(W, \pi^b, \pi^e, z, \mathbf{x})$ *is given by:*

$$\begin{aligned}\rho_h \widetilde{V}^0(W, z, \pi^b, \pi^e, \mathbf{x}) = \sup_{\delta_0, \Psi_b, \Psi_z} \Bigg\{ \sum_{\epsilon \in \mathcal{E}} \pi^b(\epsilon)(zA - r(\epsilon)D) \\ + (\rho_e W - \lambda_z(z)\Psi_z(z) - \lambda_c(W^U(z, \mathbf{x}) - W))\partial_W \widetilde{V}^0(W, \pi^b, \pi^e, z, \mathbf{x}) \\ - (1 + \partial_W \widetilde{V}^0(W, \pi^b, \pi^e, z, \mathbf{x}))d\delta_0 \\ + \mu_\pi^b \partial_{\pi^b} \widetilde{V}^0(W, \pi^b, \pi^e, z, \mathbf{x}) + \mu_\pi^e \partial_{\pi^e} \widetilde{V}^0(W, \pi^b, \pi^e, z, \mathbf{x}) \\ + (\Psi_{b,t}^0)^2 \sigma^2 \partial_{WW} \widetilde{V}^0(W, \pi^b, \pi^e, z, \mathbf{x}) \\ + \lambda_z(z)(\widetilde{V}^0(W + \Psi_z^0, \pi^b + \gamma^b, \pi^e + \gamma^e, z^c, \mathbf{x}) - \widetilde{V}^0(W, \pi^b, \pi^e, z, \mathbf{x})) \\ + \lambda_c(L_B - \widetilde{V}^0(W, \pi^b, \pi^e, z, \mathbf{x})) \Bigg\}\end{aligned} \tag{A.10}$$

Proof. I drop the $\mathbf{x}$ from the notation in order to save space. Let

$$\begin{aligned}G(W_s, \epsilon_s, z_s) &\equiv zA - r(\epsilon_s)D - d\delta_s \\ V_{T_0} &\equiv \mathbb{1}_{\{T_0=\mathscr{T}_0\}}(\mathbb{1}_{\{\mathscr{T}_0=\underline{\mathscr{T}}_0\}} V_{\underline{\mathscr{T}}_0}^1(W_{\underline{\mathscr{T}}_0}, \epsilon_{\underline{\mathscr{T}}_0}, z_{\underline{\mathscr{T}}_0}, \mathbf{x}) \\ &\quad + \mathbb{1}_{\{\mathscr{T}_0=\overline{\mathscr{T}}_0\}} V_{\overline{\mathscr{T}}_0}^1(W_{\overline{\mathscr{T}}_0}, \epsilon_{\overline{\mathscr{T}}_0}, z_{\overline{\mathscr{T}}_0}, \mathbf{x}) - \Phi(z_{\mathscr{T}_0})) \\ &\quad + \mathbb{1}_{\{T_0=\tau_c\}}(1-D) + \mathbb{1}_{\{T_0=\tau_0\}} L_B\end{aligned}$$

Then, we have:

$$
\begin{aligned}
&\widetilde{U}^0(W_t, \pi_t^b, \pi_t^e, z_t, \mathbf{x}) \\
&\geq \sum_{\epsilon_t \in \mathcal{E}} \pi_t(\epsilon_t | z_t) \mathbb{E}\Big[\int_t^{T_0} e^{-\rho_h s} G(W_s, \epsilon_s, z_s) ds + e^{-\rho_h T_0} V_{T_0} \Big| \mathcal{F}_t^{\widetilde{B}, z}, \epsilon_t \Big] \\
&= \sum_{\epsilon_t \in \mathcal{E}} \pi_t(\epsilon_t | z_t) \mathbb{E}\Big[\int_t^{t+h} e^{-\rho_h s} G(W_s, \epsilon_s, z_s) ds \Big| W_t, \epsilon_t, z_t \Big] \\
&\qquad + \sum_{\epsilon_t \in \mathcal{E}} \pi_t(\epsilon_t | z_t) \mathbb{E}\Big[\int_{t+h}^{T_0} e^{-\rho_h s} G(W_s, \epsilon_s) ds + e^{-\rho_h T_0} V_{T_0} \Big| W_t, \epsilon_t, z_t \Big] \\
&= \sum_{\epsilon_t \in \mathcal{E}} \pi_t(\epsilon_t | z_t) \mathbb{E}\Big[\int_t^{t+h} e^{-\rho_h s} G(W_s, \epsilon_s, z_s) ds \Big| W_t, \epsilon_t, z_t \Big] \\
&\qquad + \sum_{\epsilon_t \in \mathcal{E}} \pi_t(\epsilon_t | z_t) \mathbb{E}\Big[\sum_{\epsilon_{t+h} \in \mathcal{E}} \pi_{t+h,t}(\epsilon_{t+h} | z_{t+h}, \epsilon_t) \mathbb{E}\Big[\int_{t+h}^{T_0} e^{-\rho_h s} G(W_s, \epsilon_s) ds \\
&\qquad\qquad + e^{-\rho_h T_0} V_{T_0} \Big| W_{t+h}, \epsilon_{t+h}, z_{t+h} \Big] \Big| W_t, \epsilon_t, z_t \Big] \\
&= \sum_{\epsilon_t \in \mathcal{E}} \pi_t(\epsilon_t | z_t) \mathbb{E}\Big[\int_t^{t+h} e^{-\rho_h s} G(W_s, \epsilon_s, z_s) ds \Big| W_t, \epsilon_t, z_t \Big] \\
&\qquad + \mathbb{E}\Big[\sum_{\epsilon_{t+h} \in \mathcal{E}} \pi_{t+h,t}(\epsilon_{t+h} | z_{t+h}) \mathbb{E}\Big[\int_{t+h}^{T_0} e^{-\rho_h s} G(W_s, \epsilon_s) ds \\
&\qquad\qquad + e^{-\rho_h T_0} V_{T_0} \Big| W_{t+h}, \epsilon_{t+h}, z_{t+h} \Big] \Big| W_t, \epsilon_t, z_t \Big] \\
&= \sum_{\epsilon_t \in \mathcal{E}} \pi_t(\epsilon_t | z_t) \mathbb{E}\Big[\int_t^{t+h} e^{-\rho_h s} G(W_s, \epsilon_s, z_s) ds \Big| W_t, \epsilon_t, z_t \Big] + \mathbb{E}\Big[\widetilde{U}^0(W_{t+h}, \pi_{t+h}^b, \pi_{t+h}^e, z_t, \mathbf{x}) | W_t, \epsilon_t, z_t \Big]
\end{aligned}
$$

Rearranging gives:

$$
\begin{aligned}
0 \geq &\sum_{\epsilon_t \in \mathcal{E}} \pi_t(\epsilon_t | z_t) \mathbb{E}\Big[\int_t^{t+h} e^{-\rho_h s} G(W_s, \epsilon_s, z_s) ds \Big| W_t, \epsilon_t, z_t \Big] \\
&+ \mathbb{E}\Big[\widetilde{U}^0(W_{t+h}, \pi_{t+h}^b, \pi_{t+h}^e, z_{t+h}, \mathbf{x}) | W_t, \epsilon_t, z_t \Big] - \widetilde{U}^0(W_t, \pi_t^b, \pi_t^e, z_t, \mathbf{x})
\end{aligned}
$$

The standard procedure can now be applied in order to derive the HJB equation. That is, applying Ito's formula, dividing by h and then taking the limit as $h \to 0$

gives the HJB equation:

$$
\begin{aligned}
0 = \sup_{\delta_{0,t},\Psi^0_{b,t},\Psi^0_{z,t}} \Big\{ & \sum_{\epsilon_t\in\mathcal{E}} \pi_t(\epsilon_t|z_t)(zA - r(\epsilon_t)D) \\
& + (\rho_e W_t - \lambda_z(z_t)\Psi_z(z_t) - \lambda_c(W_t^U - W))\partial_W \widetilde{U}^0(W_t, \pi_t^b, \pi_t^e, z_t, \mathbf{x}) \\
& + \mu_\pi^b \partial_{\pi^b} \widetilde{U}^0(W_t, \pi_t^b, \pi_t^e, z_t, \mathbf{x}) + \mu_\pi^e \partial_{\pi^e} \widetilde{U}^0(W_t, \pi_t^b, \pi_t^e, z_t, \mathbf{x}) \\
& + \Psi_{b,t}^2 \sigma^2 \partial_{WW} \widetilde{U}^0(W_t, \pi_t^b, \pi_t^e, z_t, \mathbf{x}) \\
& - (1 + \partial_W \widetilde{U}^0(W_t, \pi_t^b, \pi_t^e, z_t, \mathbf{x}))d\delta_{0,t} \\
& + \lambda_z(z_t)(\widetilde{U}^0(W_t + \Psi^0_{z,t}, \pi_t^b + \gamma_t^b, \pi_t^e + \gamma_t^e, z_t^c, \mathbf{x}) - \widetilde{U}^0(W_t, \pi_t^b, \pi_t^e, z_t, \mathbf{x})) \\
& + \lambda_c(L_B - \widetilde{U}^0(W_t, \pi_t^b, \pi_t^e, z_t, \mathbf{x}))\Big\}
\end{aligned} \tag{A.11}
$$

Finally, again using Ito's formula, we get that the HJB equation for $\widetilde{V}_t^0(W_t, \pi_t, z_t) = e^{\rho_h t}\widetilde{U}_t^0(W_t, \pi_t, z_t)$ is given by equation (A.10). □

Corollary 2. *Suppose that assumption 1 holds. Then, $\pi_t^b = \pi_t^e = \overline{\pi}(z_t)$ and so:*

$$d\pi_t^b = d\pi_t^e = (\overline{\pi}(z_t^c) - \overline{\pi}(z_t))dN_{z,t}$$

In this case, π^b and π^e can be dropped from state space and the HJB equation becomes:

$$
\begin{aligned}
\rho_h \widetilde{V}^0(W, z, \mathbf{x}) = \sup_{\delta_0,\Psi^0_b,\Psi^0_z} \Big\{ & \sum_{\epsilon\in\mathcal{E}} \overline{\pi}(\epsilon|z)(zA - r(\epsilon)D) \\
& + (\rho_e W - \lambda_z(z)\Psi_z(z) - \lambda_c(W^U - W))\partial_W \widetilde{V}^0(W, z, \mathbf{x}) \\
& - (1 + \partial_W \widetilde{V}^0(W, z, \mathbf{x}))d\delta_0 \\
& + (\Psi_b^0)^2 \sigma^2 \partial_{WW} \widetilde{V}^0(W, z, \mathbf{x}) \\
& + \lambda_z(z)(\widetilde{V}^0(W + \Psi_z^0, z^c, \mathbf{x}) - \widetilde{V}^0(W, z, \mathbf{x})) \\
& + \lambda_c(L_B - \widetilde{V}^0(W, z, \mathbf{x}))\Big\}
\end{aligned} \tag{A.12}
$$

Proof. Follows immediately from subsituting $\pi_t^b = \pi_t^e = \overline{\pi}(z_t)$ into equation (A.11). □

Comment: Comparing equations (A.10) and (A.12) shows the role played by assumption 1. In equation (A.10), the evolution of π_t^b and π_t^e are state variables, which drift towards $\overline{\pi}$ until z changes. By contrast, in equation (A.12), π_t^b and π_t^e immediately jump to the new $\overline{\pi}$ when z changes. So, by imposing assumption 1 greatly simplifies the dynamics of problem but means we lose the dynamic that it takes some time for the new steady state distribution of bad banks to emerge.

Solving the Auxilliary Problem.

For the quantitative section of the paper, I choose $\Phi(z)$ such there is only a lower regime switching threshold $\underline{W}^{\Phi}$. I do this by numerical verification. For this reason, in this subsection, I describe the differential equation that governs the behaviour of the auxiliary problem when there is only a lower regime switching threshold. This is done in corollary 3.

Corollary 3. *Suppose that Assumption 1 holds. Suppose that* $\underline{W}^{0,*} = \underline{W}^{\Phi}(z, \mathbf{x})$ *and* $\overline{W}^{0,*} = \overline{W}^{\delta 0}(z, \mathbf{x})$. *Then:*

(a) Inside the region $[\underline{W}^{\Phi}(z, \mathbf{x}), \overline{W}^{\delta 0}(z, \mathbf{x})]$, *the banker's auxillary value function solves the Hamilton-Jacobi Bellman equation* (A.12) *subject to the constraints that:*

$$[IC] : \Psi_b^0 \geq \beta$$
$$[PC] : \Psi_z^0 \geq W^U(z, \mathbf{x}) - W$$

and with the boundary conditions:

$$\begin{aligned}
\widetilde{V}^0(\underline{W}^{\Phi}, z, \mathbf{x}) &= \sum_{\epsilon \in \mathcal{E}} \overline{\pi}(\epsilon|z) V^1(\underline{W}^{\Phi} + \Psi_{\Phi}(\epsilon, z, \mathbf{x}), \epsilon, z, \mathbf{x}) - \Phi(z) \\
\partial_W \widetilde{V}^0(\underline{W}^{\Phi}, z, \mathbf{x}) &= \sum_{\epsilon \in \mathcal{E}} \overline{\pi}(\epsilon|z) \partial_W V^1(\underline{W}^{\Phi} + \Psi_{\Phi}(\epsilon, z, \mathbf{x}), \epsilon, z, \mathbf{x}), \quad \forall \epsilon \in \mathcal{E} \\
\partial_W \widetilde{V}^0(\overline{W}^0, z, \mathbf{x}) &= -1 \\
\partial_{WW} \widetilde{V}^0(\overline{W}^0, z, \mathbf{x}) &= 0
\end{aligned}$$

(b) The auxilliary value function, $\widetilde{V}^0$, *is concave and so the banker chooses* $\Psi_b = \beta$.

Proof. Follows immediately from substituting the contract from Proposition 5 into the differential equation (A.12). □

Numerical Solution to the Banker's Ex-Ante Value Function

Although the banker cannot condition the contract on ϵ, the value of ϵ still affects the banker's value. For this reason, I also describe the differential equations for calculating the banker's ex-ante value function under the optimal contract. This is done in corollary 4.

Corollary 4. *Suppose that Assumption 1 holds. Suppose that* $\underline{W}^{0,*} = \underline{W}^{\Phi}(z, \mathbf{x})$ *and* $\overline{W}^{0,*} = \overline{W}^{\delta 0}(z, \mathbf{x})$. *Then inside the region* $[\underline{W}^{\Phi}(z, \mathbf{x}), \overline{W}^{\delta 0}(z, \mathbf{x})]$, *the banker's value function solves the Hamilton-Jacobi Bellman equation* (A.12) *subject to the constraints that:*

$$[IC] : \Psi_b \geq \beta$$
$$[PC] : \Psi_z \geq W^U(z, \mathbf{x}) - W$$

and with the boundary conditions:

$$\widetilde{V}^0(\underline{W}^\Phi, \epsilon, z, \mathbf{x}) = V^1(\underline{W}^\Phi + \Psi_\Phi(\epsilon, z, \mathbf{x}), \epsilon, z, \mathbf{x}) - \Phi(z)$$
$$\partial_W \widetilde{V}^0(\overline{W}^0, \epsilon, z, \mathbf{x}) = -1$$

COMMENT: The difference between differential equations in corollary 3 and corollary 4 illustrates the sub-optimality of not being able to condition the contract on the banker's cost of funds, ϵ. In 4, the upper boundary condition only satisfies the reflecting boundary condition $\partial_W \widetilde{V}^0(\overline{W}^0, \epsilon, z, \mathbf{x}) = -1$. It does not also satisfy the optimality condition $\partial_{WW} \widetilde{V}^0(\overline{W}^0, \epsilon, z, \mathbf{x}) = 0$. Instead, the suboptimal condition $\partial_{WW} \widetilde{V}^0(\overline{W}^0, z, \mathbf{x}) = 0$ is satisfied.

A.4.4 Proving Results from the Main Text

We can now finally return to the proofs of propositions 1 and 2 from the main text.

PROOF OF PROPOSITION 1: Follows immediately from parts (a) and (b) in Proposition 4. □

PROOF OF PROPOSITION 2: Follows immediately from the Lemma 5 and Proposition 5. □

A.5 Supplementary Proofs for Subsection 3.5 (Implementation with Standard Securities)

PROOF OF LEMMA 3: I use a guess and verify approach to prove the result. I do with the following steps. Firstly, I show that, after the covenant binds ($f = 1$), it is weakly optimal for the entrepreneur to choose no stealing, $\xi = 0$, and the consumption process δ_1^*. Secondly, I show that, before the covenant binds ($f = 0$), it is weakly optimal for the entrepreneur to choose no stealing, $\xi = 0$, and the consumption process δ_0^*. Finally, I show that the optimal stopping times are implemented.

Step 1: Entrepreneur's Problem when Covenant Flag is $f = 1$: Suppose that the entrepreneur has the loan contracts specified in lemma 3. That is, the entrepreneur raises initial investment by: (i) issuing outside equity for $1 - \alpha$ of the project, (ii) setting up a credit line with starting balance M_0 and interest rate $r_c = \rho_e + \lambda_c$, and (iii) taking out a long term loan requiring coupon payments:

$$dx_t = \mu_x(\epsilon_t, z_t)dt$$
$$\mu_x(M_t, \epsilon_t, z_t) \equiv zA + \frac{1}{\beta}\Big(- r_c\overline{W}^{0*}(z_t) + \Psi_\epsilon^0(\overline{W}^{1*}(\epsilon_t, z_t) - \beta M_t, \epsilon_t, z_t)\lambda_\epsilon(\epsilon|z_t)$$
$$+ \sum_{\epsilon' \in \mathcal{E}} \Psi_{z,\epsilon'}^1(\overline{W}^{1*}(\epsilon_t, z_t) - \beta M_t, \epsilon_t, z_t)\mathbb{1}_{\epsilon'}(z_t^c)\lambda_z(z_t) + W^U(z_t)\lambda_c\Big)$$

The banker's debt recall process follows:

$$d\gamma_t^1 = \gamma_\epsilon^1(M_t, \epsilon_t, z_t)dN_{\epsilon,t} + \sum_{\epsilon' \in \mathcal{E}} \gamma_{z,\epsilon'}^1(M_t, \epsilon_t, z_t)\mathbb{1}_{\epsilon'}(z_t^c)dN_{z,t}$$

where

$$\begin{aligned}
\gamma_\epsilon^1(M_t, \epsilon_t, z_t) &= -\frac{1}{\alpha}\Psi_\epsilon^1(\overline{W}^{1*}(\epsilon_t, z_t) - \beta M_t, \epsilon_t, z_t) + \overline{M}^{L1}(\epsilon_t^c, z_t) - \overline{M}^{L1}(\epsilon_t, z_t) \\
&= \frac{1}{\alpha}\Big(W^{1*}(\epsilon_t^c, z_t) - W^{1*}(\epsilon_t, z_t) - \Psi_\epsilon^1(\overline{W}^{1*}(\epsilon_t, z_t) - \alpha M_t, \epsilon_t, z_t)\Big) \\
\gamma_{z,\epsilon}^1(M_t, \epsilon_t, z_t) &= -\frac{1}{\alpha}\Psi_{z,\epsilon'}^1(\overline{W}^{1*}(\epsilon_t, z_t) - \alpha M_t, \epsilon_t, z_t) + \overline{M}^{L1}(\epsilon', z_t) - \overline{M}^{L1}(\epsilon_t, z_t^c) - (W^U(z_t^c) - W^U(z_t)) \\
&= \frac{1}{\alpha}\Big(W^{1*}(\epsilon, z_t^c) - W^{1*}(\epsilon_t, z_t) - \Psi_{z,\epsilon}^1(\overline{W}^{1*}(\epsilon_t, z_t) - \alpha M_t, \epsilon_t, z_t)\Big)
\end{aligned}$$

Let $d\nu_t$ denote the entrepreneur's flow draw on their credit line for paying dividends. Let $d\widetilde{y}_t = zA - \sigma dB_t$ denote the output the entrepreneur reports to the bank in order to pay back the credit line. Under this contract, the entrepreneur's draw on the credit line, M_t, evolves according to the following (where I simplify the notation for μ_x and γ by dropping the explicit dependence on (M_t, ϵ_t, z_t)):

$$\begin{aligned}
dM_t &= r_c M_t dt + dx_t + d\gamma_t^1 + d\nu_t - d\widetilde{y}_t \\
&= (r_c M_t + \mu_{x,t})dt + \gamma_{\epsilon,t}^1 dN_{\epsilon,t} + \sum_{\epsilon' \in \mathcal{E}} \gamma_{z,\epsilon',t}^1 \mathbb{1}_{\epsilon'}(z_t^c)dN_{z,t} + d\nu_t - d\widetilde{y}_t
\end{aligned}$$

Now, define the candidate entrepreneur continuation value by:

$$\begin{aligned}
\widetilde{W}^1 &\equiv W_t^U(z_t) + \alpha(\overline{M}^{L1}(\epsilon_t, z_t) - M_t) \\
&= W^{1*}(\epsilon_t, z_t) - \alpha M_t
\end{aligned}$$

which evolves according to:

$$
\begin{aligned}
d\widetilde{W}_t^1 &= dW^{1*}(\epsilon_t, z_t) - \alpha dM_t - \alpha(\overline{M}^{L1}(\epsilon_t, z_t) - M_t)dN_{c,t} \\
&= (W^{1*}(\epsilon_t^c, z_t) - W^{1*}(\epsilon_t, z_t))dN_{\epsilon,t} + \sum_{\epsilon' \in \mathcal{E}} (W^{1*}(\epsilon', z_t)\mathbb{1}_{\epsilon'}(z_t^c) - W^{1*}(\epsilon_t, z_t))dN_{z,t} \\
&\quad - \alpha\Bigg(\left(r_c\left(\frac{W^U(z_t) - \widetilde{W}_t^1 + \alpha\overline{M}^{L1}(\epsilon_t, z_t)}{\alpha}\right) + \mu_{x,t}\right)dt + \gamma_{\epsilon,t}^1 dN_{\epsilon,t} \\
&\qquad + \sum_{\epsilon' \in \mathcal{E}} \gamma_{z,\epsilon',t}^1 \mathbb{1}_{\epsilon'}(z_t^c)dN_{z,t} + d\nu_t - d\widetilde{y}_t\Bigg) + (W^U(\epsilon_t, z_t) - \widetilde{W}_t^1)dN_{c,t} \\
&= \left(\alpha z A - r_c\big(W^U(z_t) - W_t^1 + \alpha\overline{M}^L(\epsilon_t, z_t)\big) - \alpha\mu_{x,t}\right)dt \\
&\quad + \left(W^{1*}(\epsilon_t^c, z_t) - W^{1*}(\epsilon_t, z_t) - \alpha\gamma_{\epsilon,t}^1\right)dN_{\epsilon,t} \\
&\quad + \sum_{\epsilon' \in \mathcal{E}} \left(W^{1*}(\epsilon', z_t) - W^{1*}(\epsilon_t, z_t) - \alpha\gamma_{z,\epsilon',t}^1\right)\mathbb{1}_{\epsilon'}(z_t^c)dN_{z,t} \\
&\quad + (W^U(\epsilon_t, z_t) - \widetilde{W}_t^1)dN_{c,t} + \alpha\sigma d\widetilde{B}_t - \alpha d\nu_t
\end{aligned}
$$

Substituting in the values for $\mu_{x,t}$, $\gamma_{\epsilon,t}^1$, $\gamma_{z,\epsilon,t}$ and r_c gives:

$$
\begin{aligned}
d\widetilde{W}_t^1 &= \rho_e\widetilde{W}_t^1 dt - \alpha d\nu_t + \alpha\sigma d\widetilde{B}_t + (W^U(\epsilon_t, z_t) - \widetilde{W}_t^1)(dN_{c,t} - \lambda_c dt) \\
&\quad + \Psi_\epsilon^1(\widetilde{W}_t^1, \epsilon_t, z_t)(dN_{\epsilon,t} - \lambda_\epsilon(\epsilon_t|z_t)dt) + \sum_{\epsilon' \in \mathcal{E}} \Psi_{z,\epsilon'}^1(\widetilde{W}_t^1, \epsilon_t, z_t)\mathbb{1}_{\epsilon'}(z_t^c)(dN_{z,t} - \lambda_z(z_t)dt)
\end{aligned}
$$

Now, for a generic household policy (ν, ξ), define the present discounted utility by:

$$
\widehat{U}_t \equiv \int_0^t e^{-\rho_e s}(\alpha d\nu_s + \beta z A\xi_s ds) + e^{-\rho_e t}\widetilde{W}_t^1
$$

Converting to differential notation gives:

$$
\begin{aligned}
d\widehat{U}_t &= e^{-\rho_e t}(\alpha dc_t + \beta z A\xi_t dt) + e^{-\rho_e t}d\widetilde{W}_t^1 - \rho_e e^{-\rho_e t}\widetilde{W}_t^1 dt \\
\Rightarrow e^{-\rho_e t}d\widehat{U}_t &= \alpha dc_t + \beta z A\xi_t dt - \rho_e\widetilde{W}_t^1 dt \\
&\qquad \rho_e\widetilde{W}_t^1 dt - \alpha dc_t + \alpha\sigma d\widetilde{B}_t + (W^U(\epsilon_t, z_t) - \widetilde{W}_t^1)(dN_{c,t} - \lambda_c dt) \\
&\qquad + \Psi_\epsilon^1(\widetilde{W}_t^1, \epsilon_t, z_t)(dN_{\epsilon,t} - \lambda_\epsilon(\epsilon_t)dt) + \sum_{\epsilon' \in \mathcal{E}} \Psi_{z,\epsilon'}^1(\widetilde{W}_t^1, \epsilon_t, z_t)\mathbb{1}_{\epsilon'}(z_t^c)(dN_{z,t} - \lambda_z(z_t)dt) \\
&= zA\xi_t(\beta - \alpha)dt + \alpha\sigma dB_t + (W^U(\epsilon_t, z_t) - \widetilde{W}_t^1)(dN_{c,t} - \lambda_c dt) \\
&\qquad + \Psi_\epsilon^1(\widetilde{W}_t^1, \epsilon_t, z_t)(dN_{\epsilon,t} - \lambda_\epsilon(\epsilon_t)dt) + \sum_{\epsilon' \in \mathcal{E}} \Psi_{z,\epsilon'}^1(\widetilde{W}_t^1, \epsilon_t, z_t)\mathbb{1}_{\epsilon'}(z_t^c)(dN_{z,t} - \lambda_z(z_t)dt)
\end{aligned}
$$

For $\alpha > \beta$, this is a supermartingale. Thus, we have:

$$
\widehat{U}_0 = \widetilde{W}_0^1 \geq \mathbb{E}\left[\widetilde{W}_{T_1(c,\xi)}^1\right] = \mathbb{E}\left[\int_0^{T_1(c,\xi)} e^{-\rho_e s}(\alpha dc_s + \beta z A\xi_s ds) + e^{-\rho_e T_1(c,\xi)}\widetilde{W}_{T_1(c,\xi)}^1\right] \tag{A.13}
$$

For $\xi = 0$, $\alpha = \beta$ and $\nu = \delta_1^*/\beta$, the process $e^{-\rho_e t} d\widehat{U}_t$ is a martingale and equation (A.13) holds with equality. This shows that it is weakly optimal for the entrepreneur to choose no stealing and the consumption process δ_1^*.

Step 2: Entrepreneur's Problem when Covenant Flag is $f = 0$: Before the covenant is breached, the long-term loan has coupon payments that follow:

$$
\begin{aligned}
dx_t &= \mu_{0,x}(M_t, z_t)dt \\
\mu_{0,x}(M_t, z_t) &\equiv zA + \frac{1}{\beta}\Big(-r_c\overline{W}^{0*}(z_t) + \Psi_z^0(\overline{W}^{0*}(z_t) - \alpha M_t, z_t)\lambda_z(z_t) + W^U(z_t)\lambda_c\Big)\Big)
\end{aligned}
$$

and the banker's debt recall process follows:

$$
\begin{aligned}
d\gamma_t &= \gamma_z(M_t, z_t)dN_{z,t} \\
\gamma_\epsilon(M_t, z_t) &\equiv -\frac{1}{\alpha}\Psi_z^0(\overline{W}^{0*}(z_t) - \alpha M_t, z_t) + \overline{M}_t^{L0}(z_t^c) - \overline{M}_t^{L0}(z_t) - (W^U(z_t^c) - W^U(z_t)) \\
&= \frac{1}{\alpha}\Big(W^{0*}(\epsilon_t^c, z_t) - W^{0*}(\epsilon_t, z_t) - \Psi_z^0(\overline{W}^{0*}(z_t) - \alpha M_t, \epsilon_t, z_t)\Big)
\end{aligned}
$$

Under these contract terms, the entrepreneur's draw on the credit line, M_t, evolves according to:

$$
\begin{aligned}
dM_t &= r_c M_t dt + dx_t + d\gamma_t + d\nu_t - d\widetilde{y}_t \\
&= (r_c M_t + \mu_{x,t})dt + \gamma_{z,t}dN_{z,t} + d\nu_t - d\widetilde{y}_t
\end{aligned}
$$

Now define the candidate continuation value:

$$
\begin{aligned}
\widetilde{W}_t^0 &\equiv \underline{W}^\Phi(z_t) + \alpha(\overline{M}^c(z_t) - M_t) \\
&= W^{0*}(z_t) - \alpha M_t
\end{aligned}
$$

which evolves according to:

$$
\begin{aligned}
d\widetilde{W}_t^0 &= dW^{0*}(z_t) - \alpha dM_t - (W^U(z_t) - \widetilde{W}_t^0)dN_{c,t} \\
&= (W^{0*}(z_t^c) - W^{0*}(z_t))dN_{z,t} - (W^U(z_t) - \widetilde{W}_t^0)dN_{c,t} \\
&\quad - \alpha\Big((r_c M_t + \mu_x^0(z_t))dt + \gamma_z^0(z_t)dN_{z,t} + d\nu_t - d\widetilde{y}_t\Big) \\
&= (W^{0*}(z_t^c) - W^{0*}(z_t))dN_{z,t} - (W^U(z_t) - \widetilde{W}_t^0)dN_{c,t} \\
&\quad - \Big(r_c\Big(W^{0*}(z_t) - \widetilde{W}_t^0\Big) + \alpha\mu_x^0(z_t)\Big)dt - \alpha\gamma_z^0(z_t)dN_{z,t} - \alpha(d\nu_t - d\widetilde{y}_t) \\
&= \Big(zA - r_c\Big(W^{0*}(z_t) - \widetilde{W}_t^0\Big) - \alpha\mu_x^0(z_t)\Big)dt - \alpha d\nu_t + \alpha\sigma d\widetilde{B}_t \\
&\quad + \big(W^{0*}(z_t^c) - W^{0*}(z_t) - \alpha\gamma_z^0(z_t)\big)dN_{z,t} - (W^U(z_t) - \widetilde{W}_t^0)dN_{c,t}
\end{aligned}
$$

Substituting in μ_x^0, γ_z^0 and r_c gives:

$$
\begin{aligned}
d\widetilde{W}_t^0 &= \rho_e\widetilde{W}_t^0 dt - \alpha d\nu_t + \alpha\sigma d\widetilde{B}_t \\
&\quad + \Psi_z^0(z_t)(dN_{z,t} - \lambda_z(z_t)dt) + (W^U(z_t) - \widetilde{W}_t^0)(dN_{c,t} - \lambda_c dt)
\end{aligned}
$$

Now, for a generic houshold policy, (ν, ξ), define the present discounted value by:

$$\widehat{U}_t^0 = \int_0^t e^{-\rho_e s}(\alpha d\nu_s + \beta z A \xi_s ds) + e^{-\rho_e t}\widetilde{W}_t^0$$

Using the same working as before, we can show that $\widehat{U}_t^0$ is a supermartingale and so:

$$\widehat{U}_0^0 = \widetilde{W}_0^0 \geq \mathbb{E}\left[\widetilde{W}_{T_0(c,\xi)}^0\right] = \mathbb{E}\left[\int_0^{T_0(\nu,\xi)} e^{-\rho_e s}(\alpha dc_s + \beta z A \xi_s ds) + e^{-\rho_e \mathscr{T}_0(c,\xi)}\widetilde{W}_{T_0(c,\xi)}^0\right] \tag{A.14}$$

Once again, for $\xi = 0$, $\alpha = \beta$ and $\nu = \delta_0^*/\beta$, the process $e^{-\rho_e t} d\widehat{U}_t^0$ is a martingale and equation (A.14) holds with equality. This shows that it is weakly optimal for the entrepreneur to choose no stealing and the consumption process δ_0^*.

Step 3: The previous steps have shown that, under the terms of the debt securities, it is weakly optimal for the entrepreneur to choose $\xi = 0$ and $\delta = \delta^*$. By construction of the covenant threshold and credit limit, we have that $\tau \equiv \inf\{t : M_t \geq \overline{M}_t^L\} = \tau^*$ and $\mathscr{T} \equiv \inf\{t : M_t^0 \geq \overline{M}_t^c\} = \mathscr{T}^*$. Thus, the optimal contract is implemented. □

B Supplementary Proofs to Section 6 (Extensions)

A preliminary set of proofs for the working capital model are available upon request. The proofs follow a very similar to format to the baseline case.