Laboratory 1

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1 Simulating Random Permutations

1.1

The following algorithm will simulate a random permutation of values $\{1, ..., n\}$:

STEP 1: Simulate and store n random variables $U \sim Uniform(0,1)$

STEP 2: Define some f that maps each $Uniform\,(0,1)$ random variable to the index at which it was created

STEP 3: Order the $Uniform\,(0,1)$ random variables in decreasing order

STEP 4: Apply f to each U to produce the permutation of integers

1.2

In this situation, X has a *Binomial* distribution because it is looking at a fixed number of independent trials, each with possible outcome "success" or "failure". Additionally, the probability of success is fixed and the same for each trial. Additionally, because we are examining a permutation σ, the probability of success is $p = \frac{1}{7!}$, and the probability of failure is therefore $(1-p) = (1 - \frac{1}{7!})$.

Now, given that X has a *Binomial* distribution, we know that $E[X] = np$, but we can also prove this fact using the definition of expected value:

$$E[X] = \sum_{k=0}^{n} kP(X = k) = \sum_{k=0}^{n} k\binom{n}{k}p^k(1-p)^{n-k} = \sum_{k=0}^{n} k\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k} = \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!}p^k(1-p)^n$$

because we can remove the term corresponding to $k = 0$. Now, if we introduce two new variables, j and m such that $j = k - 1$ and $m = n - 1$ then we have:

$$E[X] = \sum_{j=0}^{m} \frac{(m+1)!}{j!(m-j)!}p^{j+1}(1-p)^{m-j} = (m+1)p\sum_{j=0}^{m} \frac{m!}{j!(m-j)!}p^j(1-p)^{m-j}$$

Now, remembering the Binomial Theorem:

$$(a+b)^m = \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} a^y b^{m-y}$$

we get the following result when we let $a = p$ and $b = 1 - p$:

$$\sum_{j=0}^{m} \frac{m!}{y!(m-y)!} p^j (1-p)^{m-j} = \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} a^y b^{m-y} = (a+b)^m = (p-(1-p))^m = 1^m = 1$$

and therefore we can conclude that

$$E[X] = (m+1)p \cdot 1 = np$$

1.3

1.4

1.5

2 Computing Performance of Retrieving Algorithms

2.1

2.2

2.3

2.4

2.5