Parameters related to the minimum rank problem

Jephian C.-H. Lin

Department of Mathematics, Iowa State University

Dec 2, 2014
Preliminary Exam

Minimum rank problem

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

Minimum rank problem

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

Real, Symmetric

What is the smallest possible rank?

Minimum rank problem

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

$$\Big\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Real, Symmetric

What is the smallest possible rank?

Minimum rank problem

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

Real, Symmetric

What is the smallest possible rank?

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{rank}} 3$$

Minimum rank problem

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{spectrum}} 1,\ 1 \pm \sqrt{2}$$

Real, Symmetric

What is the smallest possible rank?

Minimum rank problem

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Real, Symmetric

What is the smallest possible rank?

Minimum rank problem

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

Real, Symmetric

What is the smallest possible rank?

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\text{rank}} 2$$

Minimum rank problem

- Let G be a simple graph.
- Denote $\mathcal{S}^F(G)$ as the family of symmetric matrices over the field F whose i,j-entry, $i \neq j$, is nonzero if $i \sim j$ and zero otherwise. (Diagonal entries are free.)
- The minimum rank of G is defined as

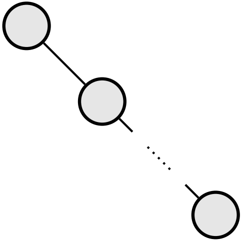
$$\mathrm{mr}^F(G) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}^F(G)\}.$$

The maximum nullity is

$$M^F(G) = \max\{\mathrm{null}(A) : A \in \mathcal{S}^F(G)\}.$$

- $M^F(G) + \mathrm{mr}^F(G) = |V(G)|$ for any G and F.

Example: Paths P_n



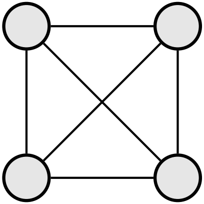
$$\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & & \ddots & -2 & 1 \\ 0 & \cdots & 0 & 1 & -1 \end{bmatrix}$$

$M(G) \neq 0$ for all G.

$M(G) = 1$ iff G is a path.

[Fiedler (1969), Bento and Leal Duarte (2005)]

Example: Complete Graphs K_n



$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$M(G) = n \text{ iff } G = \overline{K_n}.$$
$$M(G) = n - 1 \text{ iff } G = K_n \dot{\bigcup} \overline{K_m},\ n \geq 2.$$

Inverse eigenvalue problem

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

Real, Symmetric

What is the possible spectrum?

- We know $\mathrm{mr}(G) = 2$ and $M(G) = 1$ and Spec= $\{1, 1 \pm \sqrt{2}\}$ is possible.
- Can Spec= $\{1, 5, 5\}$?
- No, for otherwise $\mathrm{null}(A - 5I) = 2 > M(G)$.
- Largest possible multiplicity = $M(G)$.

Inverse eigenvalue problem

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

Real, Symmetric

What is the possible spectrum?

- We know $\mathrm{mr}(G) = 2$ and $M(G) = 1$ and Spec= $\{1, 1 \pm \sqrt{2}\}$ is possible.
- Can Spec= $\{1, 5, 5\}$?
- No, for otherwise $\mathrm{null}(A - 5I) = 2 > M(G)$.
- Largest possible multiplicity = $M(G)$.

Inverse eigenvalue problem

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

Real, Symmetric

What is the possible spectrum?

- We know $\mathrm{mr}(G) = 2$ and $M(G) = 1$ and Spec= $\{1, 1 \pm \sqrt{2}\}$ is possible.
- Can Spec= $\{1, 5, 5\}$?
- No, for otherwise $\mathrm{null}(A - 5I) = 2 > M(G)$.
- Largest possible multiplicity = $M(G)$.

Inverse eigenvalue problem

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

Real, Symmetric

What is the possible spectrum?

- We know $\text{mr}(G) = 2$ and $M(G) = 1$ and Spec= $\{1, 1 \pm \sqrt{2}\}$ is possible.
- Can Spec= $\{1, 5, 5\}$?
- No, for otherwise $\text{null}(A - 5I) = 2 > M(G)$.
- Largest possible multiplicity = $M(G)$.

Inverse eigenvalue problem

Theorem (K. H. Monfared, B. L. Shader 2013)

For a graph G and distinct real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, there is a matrix $A \in \mathcal{S}^{\mathbb{R}}(G)$ such that the spectrum of A is $\lambda_1, \lambda_2, \ldots, \lambda_n$.

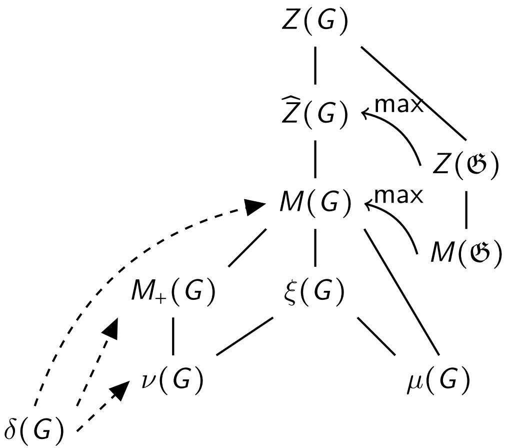
Inverse eigenvalue problem

Theorem (K. H. Monfared, B. L. Shader 2013)

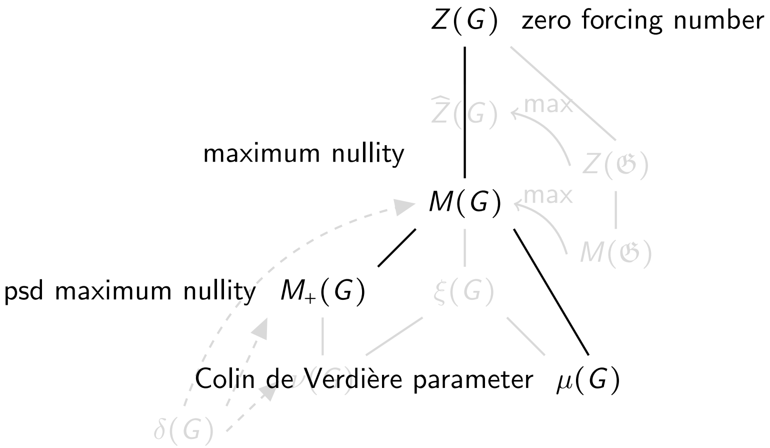
For a graph G and distinct real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, there is a matrix $A \in \mathcal{S}^{\mathbb{R}}(G)$ such that the spectrum of A is $\lambda_1, \lambda_2, \ldots, \lambda_n$.

For the case multiplicity $\neq 1$, it is still unknown, but the minimum rank problem provides a restriction.

The landscape of minimum rank problems

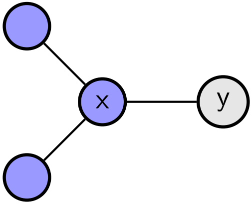


The landscape of minimum rank problems



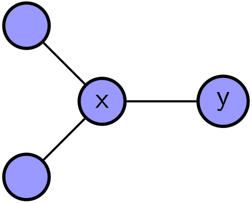
Zero forcing number

- A zero forcing game on a simple graph G starts by setting a set $B \subseteq V(G)$ of vertices blue and the others white, and then repeatedly applies the color-change rule (CCR):
  - if $y \in V(G)$ is the only white neighbor of $x \in V(G)$ and x is blue, then y turns blue.



Zero forcing number

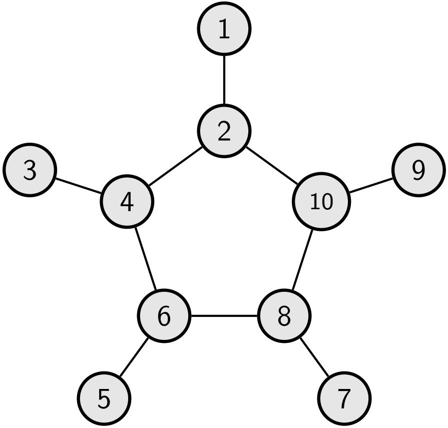
- A zero forcing game on a simple graph G starts by setting a set $B \subseteq V(G)$ of vertices blue and the others white, and then repeatedly applies the color-change rule (CCR):
  - if $y \in V(G)$ is the only white neighbor of $x \in V(G)$ and x is blue, then y turns blue.



Zero forcing number

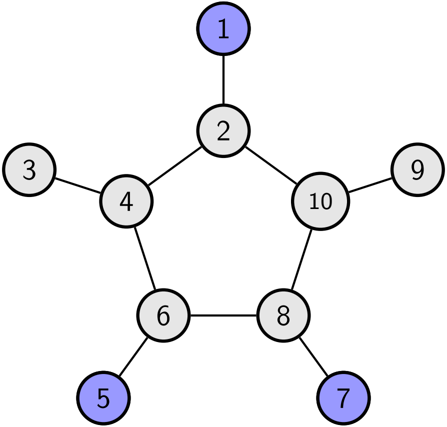
- A zero forcing game on a simple graph G starts by setting a set $B \subseteq V(G)$ of vertices blue and the others white, and then repeatedly applies the color-change rule (CCR):
  - if $y \in V(G)$ is the only white neighbor of $x \in V(G)$ and x is blue, then y turns blue.
- The final coloring is the set of blue vertices when no more CCR applies.
- The initial set B is called a zero forcing set if its final coloring is $V(G)$.
- The zero forcing number of G, denoted as $Z(G)$, is the minimum cardinality of a zero forcing set on G.

Example: 5-sun H_5



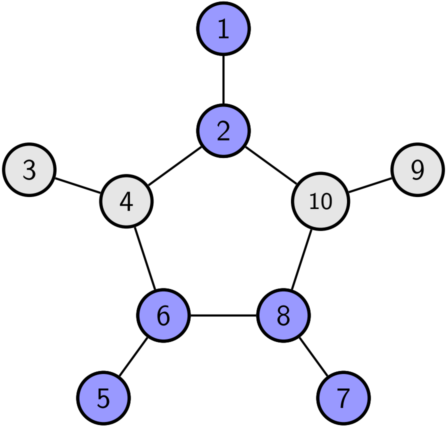
$$Z(H_5) = 3.$$

Example: 5-sun H_5



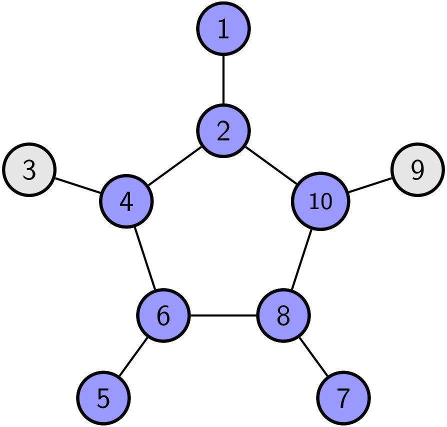
$$Z(H_5) = 3.$$

Example: 5-sun H_5



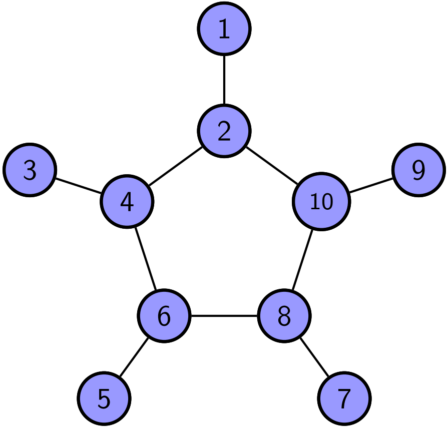
$$Z(H_5) = 3.$$

Example: 5-sun H_5



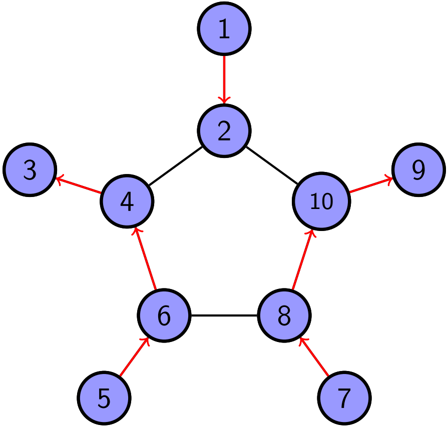
$$Z(H_5) = 3.$$

Example: 5-sun H_5



$$Z(H_5) = 3.$$

Example: 5-sun H_5



$$Z(H_5) = 3.$$

Triangle number

- Let Q be a pattern (a matrix with entries $\in \{0, *, ?\}$).
- An upper triangular subpattern is a square submatrix of Q such that the lower part is all 0, diagonals are $*$.
- The triangle number of Q, denoted as $\mathrm{tri}(Q)$, is the largest size of an upper triangular subpattern that can be found in Q through row/column permutations.

$$\begin{bmatrix} * & 0 & 0 \\ ? & * & ? \\ 0 & 0 & * \end{bmatrix} \longrightarrow \begin{bmatrix} ? & * & ? \\ * & 0 & 0 \\ 0 & 0 & * \end{bmatrix} \longrightarrow \begin{bmatrix} * & ? & ? \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

- If Q is the pattern of a graph G, then $\mathrm{mr}(G) \geq \mathrm{tri}(Q)$ and $M(G) \leq n - \mathrm{tri}(Q)$.

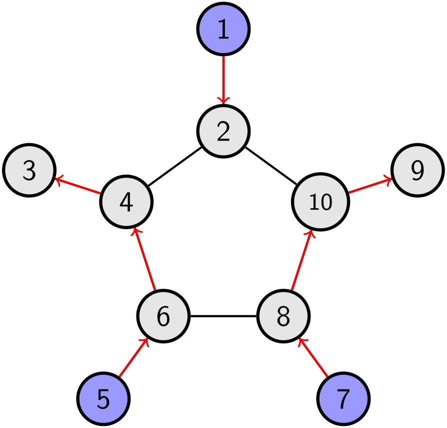
Triangle number of H_5?

The pattern Q below is the pattern for H_5. What is $\mathrm{tri}(Q)$?

$$\begin{bmatrix} ? & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & ? & 0 & * & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & ? & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & ? & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ? & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & ? & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ? & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & ? & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ? & * \\ 0 & * & 0 & 0 & 0 & 0 & 0 & * & * & ? \end{bmatrix}$$

Triangle number of H_5?

$1 \to 2$
$5 \to 6$
$7 \to 8$
$6 \to 4$
$8 \to 10$
$4 \to 3$
$10 \to 9$



Triangle number of H_5?

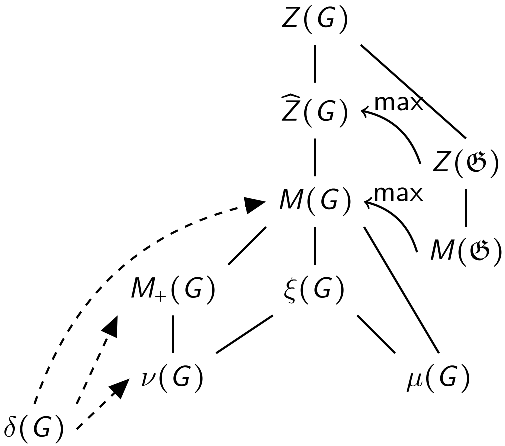
$\text{tri}(Q) = 7$ and $Z(H_5) = 3$.

$$
\begin{array}{c c}
 & \begin{array}{cccccccccc} 1 & 5 & 7 & 6 & 8 & 4 & 10 & 2 & 3 & 9 \end{array} \\
\begin{array}{c} 2 \\ 6 \\ 8 \\ 4 \\ 10 \\ 3 \\ 9 \\ 1 \\ 5 \\ 7 \end{array} &
\begin{bmatrix}
\color{red}{*} & 0 & 0 & 0 & 0 & * & * & ? & 0 & 0 \\
0 & \color{red}{*} & 0 & ? & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & \color{red}{*} & * & ? & 0 & * & 0 & 0 & 0 \\
0 & 0 & 0 & \color{red}{*} & 0 & ? & 0 & * & * & 0 \\
0 & 0 & 0 & 0 & \color{red}{*} & 0 & ? & * & 0 & * \\
0 & 0 & 0 & 0 & 0 & \color{red}{*} & 0 & 0 & ? & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \color{red}{*} & 0 & 0 & ? \\
? & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\
0 & ? & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & ? & 0 & * & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

Zero forcing vs Triangle

- Number of forces $x_i \to y_i$ = size of triangle.
- $Z(G) = n - \text{tri}(Q)$, where Q is the pattern of G.
- $M^F(G) \leq Z(G)$, for any simple graph G, any field F [AIM Group 2007].
- It doesn't matter if $\mathcal{S}^F(G)$ is defined to be symmetric or not.
- $M(G) = Z(G)$ when $|V(G)| \leq 7$ or G is a tree, a cycle, a complete bipartite graph, ...
- $Z(H_5) = 3$ but $M(H_5) = 2$.

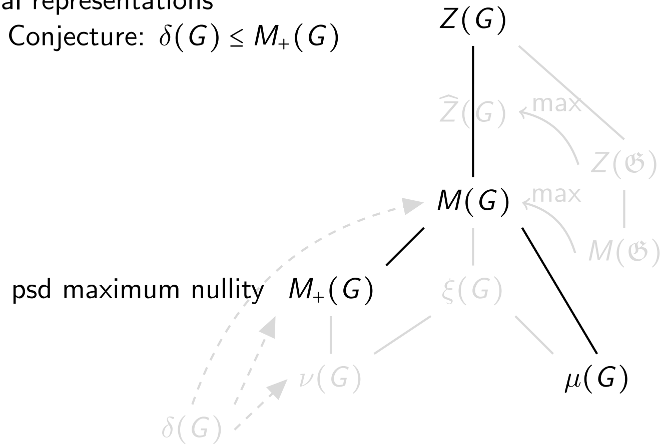
The landscape of minimum rank problems



The landscape of minimum rank problems

orthogonal representations

Maehara Conjecture: $\delta(G) \leq M_+(G)$



PSD maximum nullity

- Denote $\mathcal{S}^{\mathbb{F}}(G)$ as the family of symmetric matrices over $\mathbb{F}$ whose i,j-entry, $i \neq j$, is nonzero if $i \sim j$ and zero otherwise. (Diagonal entries are free.)
- $\mathbb{F} = \mathbb{R}$, or $\mathbb{C}$.
- $\mathrm{mr}_+^{\mathbb{F}}(G) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}^{\mathbb{F}}(G),\ A \text{ is psd}\}$.
- $M_+^{\mathbb{F}}(G) = \max\{\mathrm{null}(A) : A \in \mathcal{S}^{\mathbb{F}}(G),\ A \text{ is psd}\}$.

PSD Decomposition

- Let A be an $n \times n$ (symmetric) psd matrix with $\text{rank}(A) = r$.
- Then

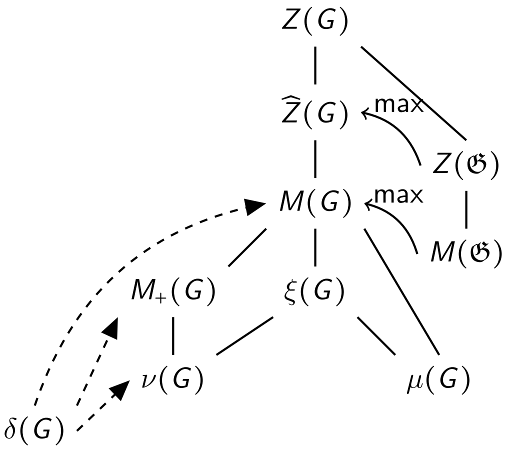
$$S^* S = \begin{bmatrix} - & v_1^* & - \\ - & v_2^* & - \\ & \vdots & \\ - & v_n^* & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = [\langle v_i, v_j \rangle],$$

where $v_i \in \mathbb{F}^r$.

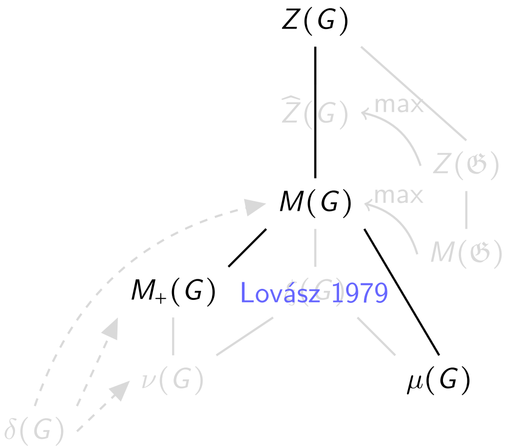
Orthogonal representation (faithful)

- $$S^*S = \begin{bmatrix} - & v_1^* & - \\ - & v_2^* & - \\ & \vdots & \\ - & v_n^* & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = [\langle v_i, v_j \rangle],$$
  where $v_i \in \mathbb{F}^r$.
- A (faithful) orthogonal representation is a function:
  $$\begin{array}{ccc} V(G) & \longrightarrow & \mathbb{F}^d \\ i & \longmapsto & v_i \end{array} \text{ such that } \langle v_i, v_j \rangle \begin{cases} \neq 0 & \text{if } i \sim j \\ = 0 & \text{if } i \nsim j. \end{cases}$$
- For a given graph G, $\min r = \min d$, so $M_+(G) = n - \min d$.

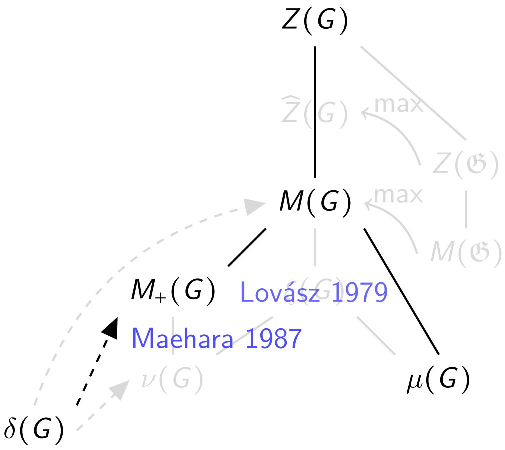
delta conjecture



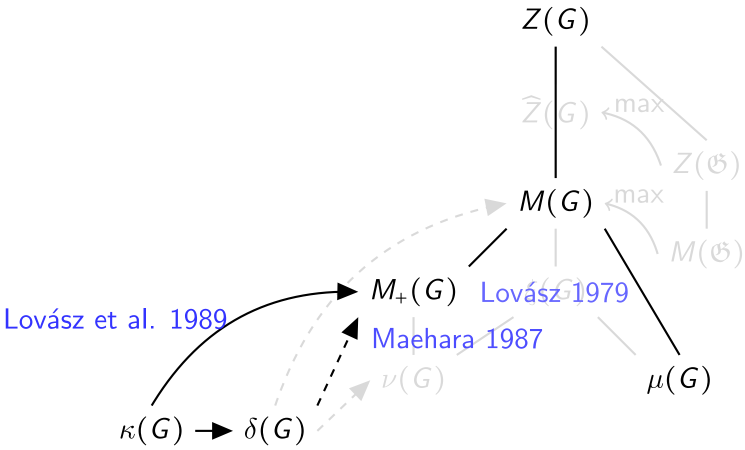
delta conjecture



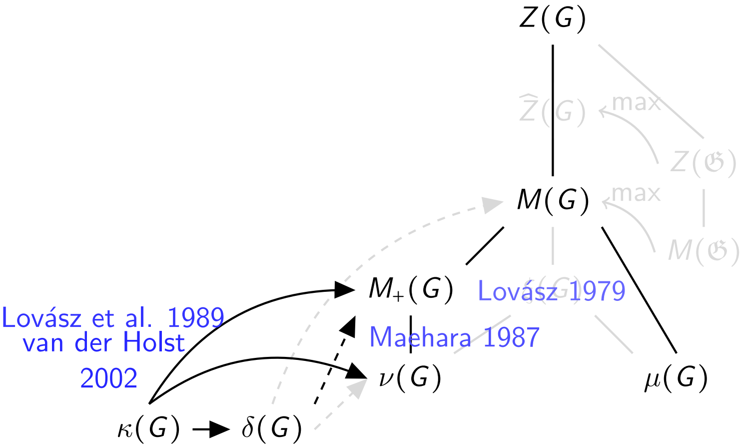
delta conjecture



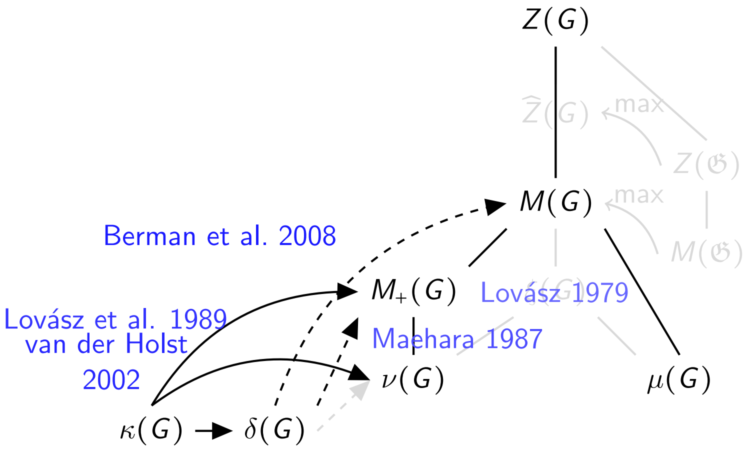
delta conjecture



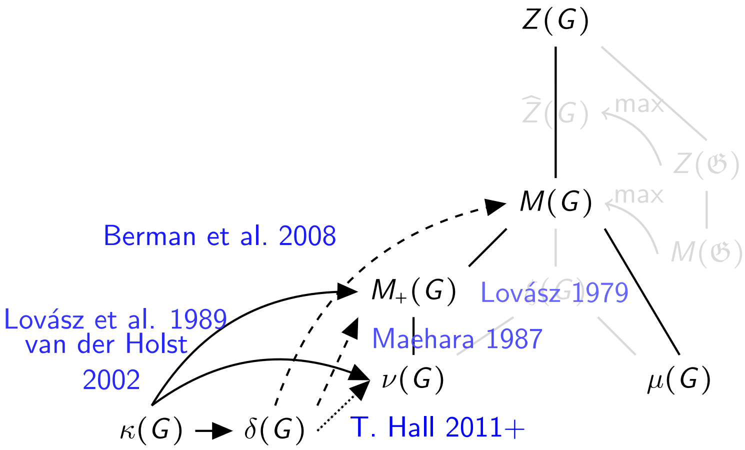
delta conjecture



delta conjecture



delta conjecture



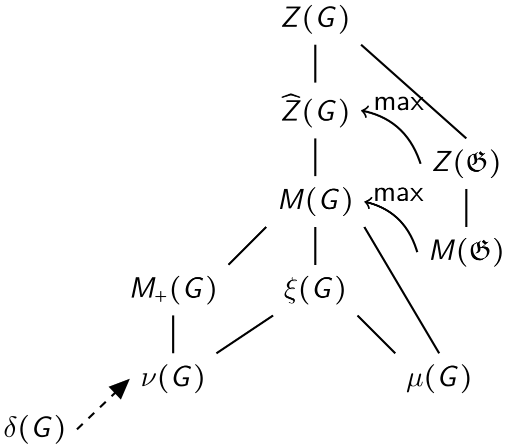
What is ν?

- We say a matrix A satisfies strong Arnol'd Hypothesis (SAH) if there is no nonzero symmetric matrix X satisfying

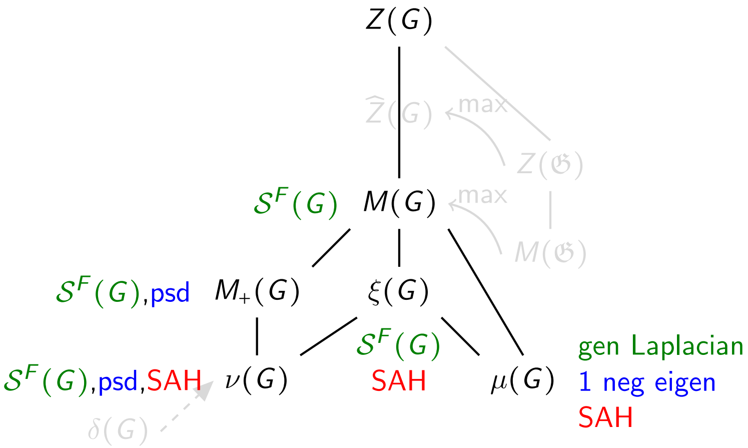
$$\begin{cases} I \circ X = O \\ A \circ X = O \\ AX = O \end{cases},$$

  where $\circ$ is the Hadamard (entrywise) product.
- $\nu(G) = \max\{\text{null}(A) : A \in \mathcal{S}^{\mathbb{R}}(G),\ A \text{ is psd},\ \text{SAH}\}$
- Colin de Verdière (1998) proved that if H is a minor of G, then $\nu(H) \leq \nu(G)$.

Colin de Verdière type parameters



Colin de Verdière type parameters



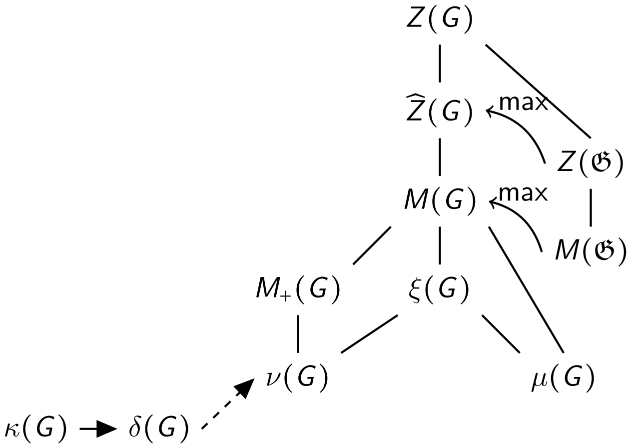
Colin de Verdière parameter μ

- $\mu(G)$ is defined as the maximum nullity among matrices M with the following properties:
  - Generalized Laplacian: $M_{ij} \begin{cases} < 0 & \text{if} \quad i \sim j, i \neq j \\ = 0 & \text{if} \quad i \nsim j, i \neq j \\ \text{free} & \text{if} \quad i = j \end{cases}$.
  - M has exactly one negative eigenvalue.
  - M satisfies SAH.
- $\mu(G)$ bridges algebraic and topological properties of a graph [Colin de Verdière, Robertson et al., Lovász et al.]:
  - $\mu(G) \leq 1$ iff G is a disjoint union of paths;
  - $\mu(G) \leq 2$ iff G is outerplanar;
  - $\mu(G) \leq 3$ iff G is planar;
  - $\mu(G) \leq 4$ iff G is linklessly embedable.
- Colin de Verdière conjectured $\chi(G) \leq \mu(G) + 1$.

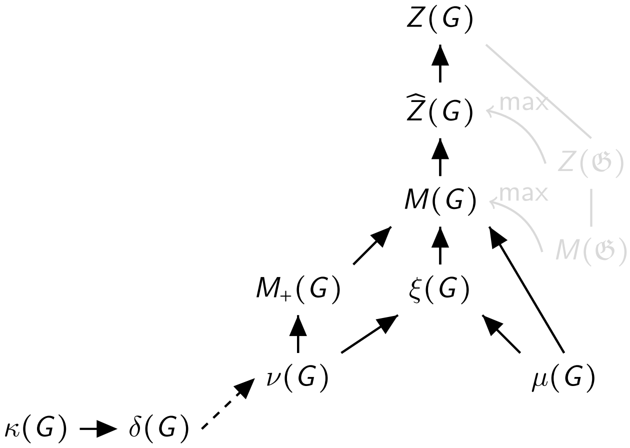
Graph Complement Conjecture (GCC)

- Let G be a simple graph and $\beta(G)$ a parameter of G. Then GCC-β states that $\beta(G) + \beta(\overline{G}) \geq n - 2$.
  - Kotlov (1997) conjectured GCC-μ.
  - Brualdi et al. (2007) conjectured GCC-M.
  - Barioli et al. (2012) conjectured GCC-M_+ and GCC-ν.
  - ISU EGR group (2011) proved GCC-Z, GCC-Z_+, and GCC-tw.

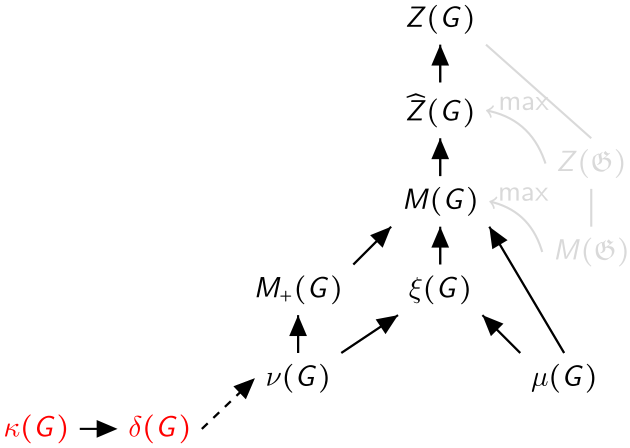
Graph Complement Conjecture



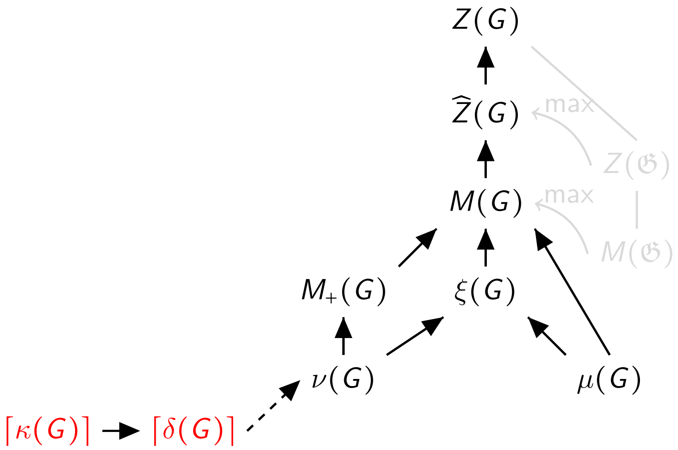
Graph Complement Conjecture



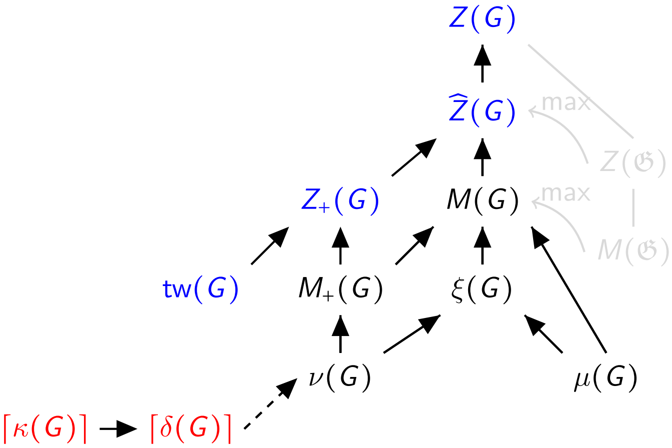
Graph Complement Conjecture



Graph Complement Conjecture



Graph Complement Conjecture

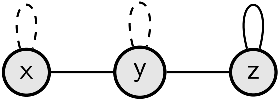


Loop graphs

- A loop graph $\mathfrak{G}$ is a graph where loops are allowed. (Each vertex has at most one loop.)
- A loop configuration $\mathfrak{G}$ of a simple graph G is a loop graph obtained from G by designating each vertex as having no loop or one loop. (There are 2^n possibilities.)
- $M^F(\mathfrak{G}) =$
$$\max\left\{\text{null}(A) : A \in \mathcal{S}^F(G), A_{i,i}\begin{cases} \neq 0 & \text{if} \quad i \text{ has a loop;} \\ = 0 & \text{if} \quad i \text{ has no loop.} \end{cases}\right\}.$$
- $M^F(G) = \max_{\mathfrak{G}} M^F(\mathfrak{G})$, where $\mathfrak{G}$ runs over all loop configurations of $\mathfrak{G}$.

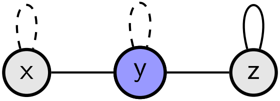
$Z(\mathfrak{G})$ for loop graphs / $\widehat{Z}(G)$ for simple graphs

- The color-change rule for loop graphs is:
  - if $y \in V(\mathfrak{G})$ is the only white neighbor of $x \in V(\mathfrak{G})$ ~~and x is blue~~, then y turns blue. ($x = y$ is possible.)



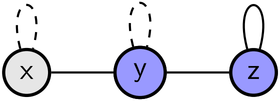
$Z(\mathfrak{G})$ for loop graphs / $\widehat{Z}(G)$ for simple graphs

- The color-change rule for loop graphs is:
  - if $y \in V(\mathfrak{G})$ is the only white neighbor of $x \in V(\mathfrak{G})$ ~~and x is blue~~, then y turns blue. ($x = y$ is possible.)



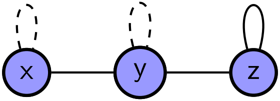
$Z(\mathfrak{G})$ for loop graphs / $\widehat{Z}(G)$ for simple graphs

- The color-change rule for loop graphs is:
  - if $y \in V(\mathfrak{G})$ is the only white neighbor of $x \in V(\mathfrak{G})$ ~~and x is blue~~, then y turns blue. ($x = y$ is possible.)



$Z(\mathfrak{G})$ for loop graphs / $\widehat{Z}(G)$ for simple graphs

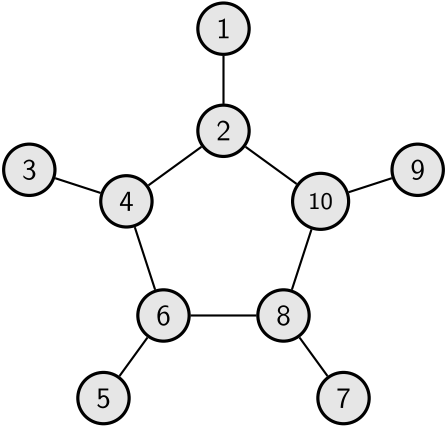
- The color-change rule for loop graphs is:
  - if $y \in V(\mathfrak{G})$ is the only white neighbor of $x \in V(\mathfrak{G})$ ~~and x is blue~~, then y turns blue. ($x = y$ is possible.)



$Z(\mathfrak{G})$ for loop graphs / $\widehat{Z}(G)$ for simple graphs

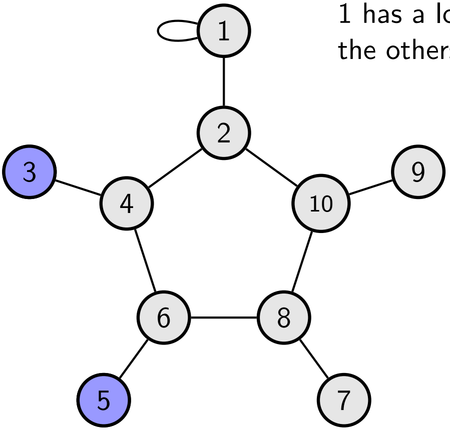
- The color-change rule for loop graphs is:
  - if $y \in V(\mathfrak{G})$ is the only white neighbor of $x \in V(\mathfrak{G})$ ~~and x is blue~~, then y turns blue. ($x = y$ is possible.)
- $Z(\mathfrak{G})$ is the smallest cardinality of a zero forcing set on $\mathfrak{G}$ using CCR for loop graphs.
- $M^F(\mathfrak{G}) \le Z(\mathfrak{G})$ for all loop graphs $\mathfrak{G}$ and fields F [Hogben (2010)].
- If $\mathfrak{G}$ is a loop configuration of G, then $Z(\mathfrak{G}) \le Z(G)$.
- The enhanced zero forcing number is defined as $\widehat{Z}(G) = \max_{\mathfrak{G}} Z(\mathfrak{G})$, where $\mathfrak{G}$ runs over all loop configurations of G.

H_5 revisited



$\widehat{Z}(H_5) = 2$ and $Z(H_5) = 3$.

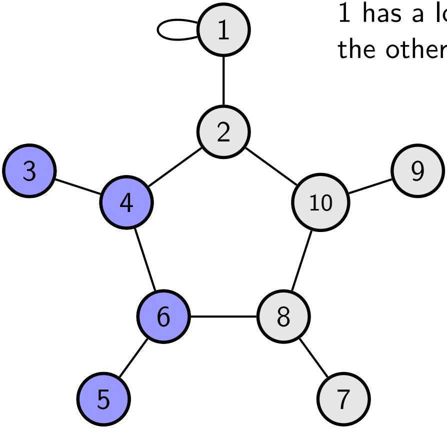
H_5 revisited



1 has a loop and
the others are unknown.

$\widehat{Z}(H_5) = 2$ and $Z(H_5) = 3$.

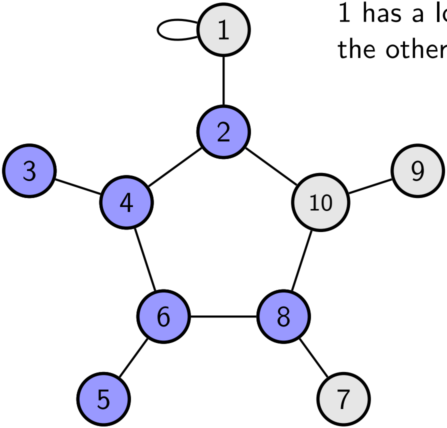
H_5 revisited



1 has a loop and
the others are unknown.

$\widehat{Z}(H_5) = 2$ and $Z(H_5) = 3$.

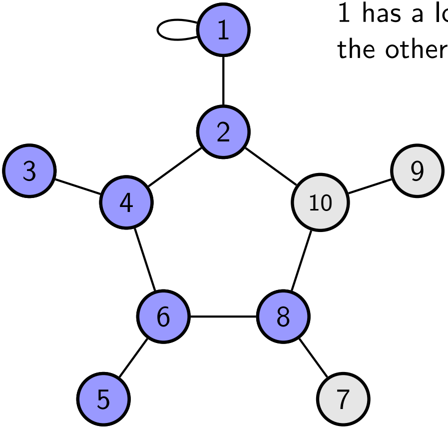
H_5 revisited



1 has a loop and
the others are unknown.

$\widehat{Z}(H_5) = 2$ and $Z(H_5) = 3$.

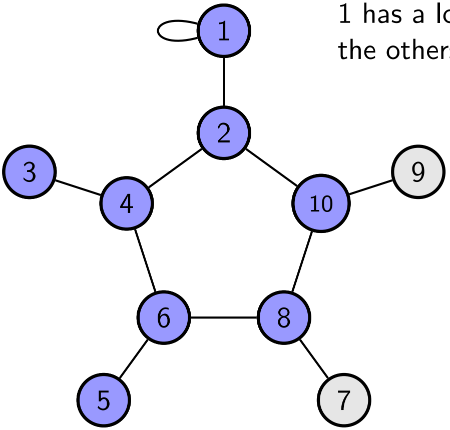
H_5 revisited



1 has a loop and the others are unknown.

$$\widehat{Z}(H_5) = 2 \text{ and } Z(H_5) = 3.$$

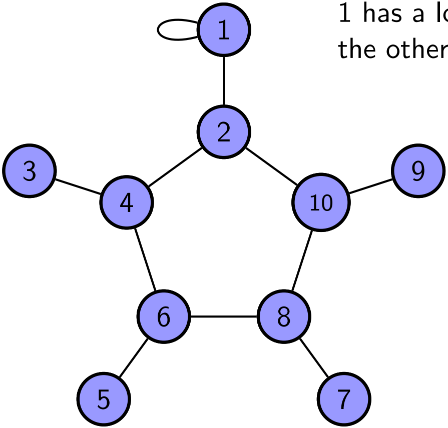
H_5 revisited



1 has a loop and
the others are unknown.

$\widehat{Z}(H_5) = 2$ and $Z(H_5) = 3$.

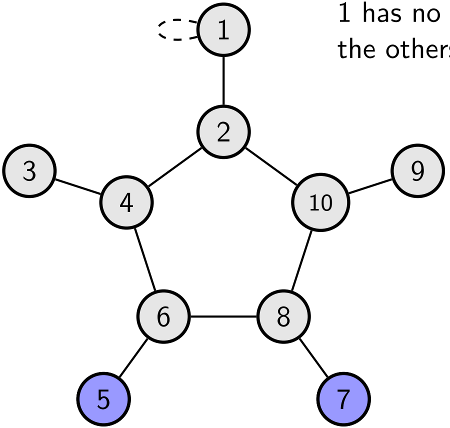
H_5 revisited



1 has a loop and
the others are unknown.

$$\widehat{Z}(H_5) = 2 \text{ and } Z(H_5) = 3.$$

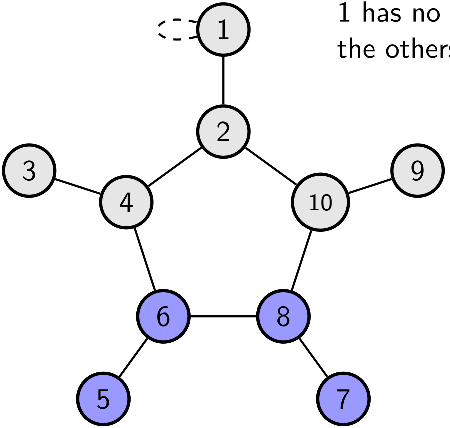
H_5 revisited



1 has no loop and
the others are unknown.

$$\widehat{Z}(H_5) = 2 \text{ and } Z(H_5) = 3.$$

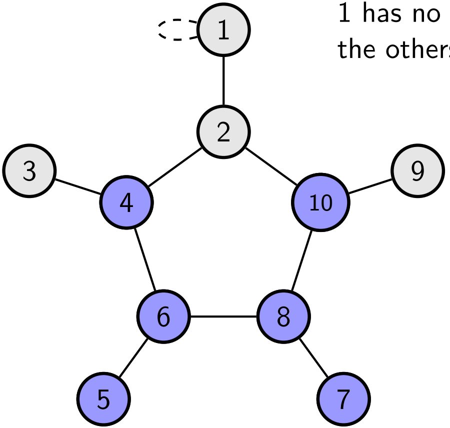
H_5 revisited



1 has no loop and
the others are unknown.

$\widehat{Z}(H_5) = 2$ and $Z(H_5) = 3$.

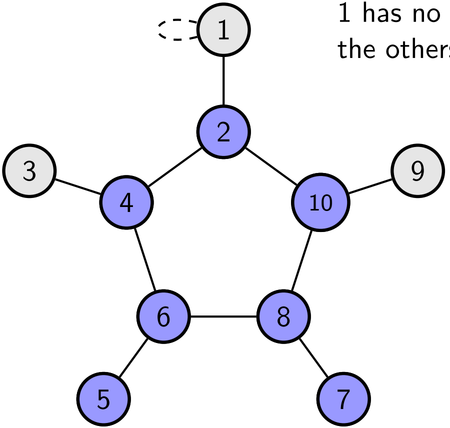
H_5 revisited



1 has no loop and
the others are unknown.

$\widehat{Z}(H_5) = 2$ and $Z(H_5) = 3$.

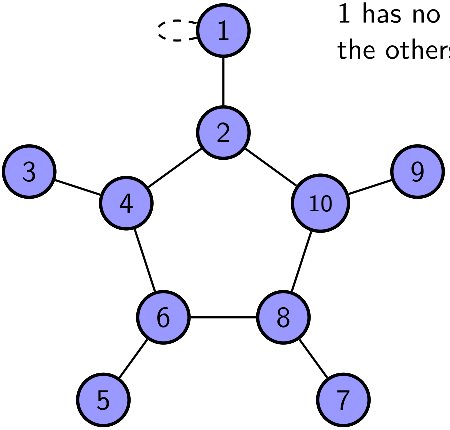
H_5 revisited



1 has no loop and
the others are unknown.

$\widehat{Z}(H_5) = 2$ and $Z(H_5) = 3$.

H_5 revisited



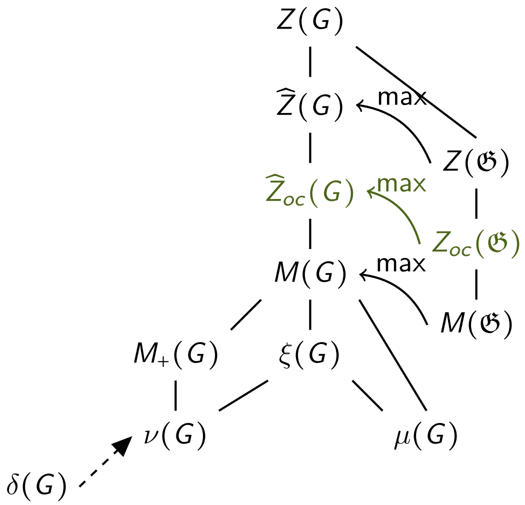
1 has no loop and
the others are unknown.

$\widehat{Z}(H_5) = 2$ and $Z(H_5) = 3$.

Sage Data

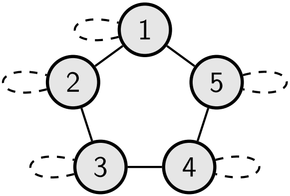
- $M(G) = \widehat{Z}(G) = Z(G)$ if $|V(G)| \leq 7$.
- For $n = 8$, there are 7 graphs with $\widehat{Z}(G) < Z(G)$.
- For $n = 9$, there are 412 graphs with $\widehat{Z}(G) < Z(G)$.
- For $n = 10$, there are 18700+ graphs with $\widehat{Z}(G) < Z(G)$.
- But $M(K_{3,3,3}) = 6$ and $Z(G) = \widehat{Z}(G) = 7$.

New parameters $\widehat{Z}_{oc}(G)$ and $Z_{oc}(\mathfrak{G})$



Odd cycles

- $\widehat{Z}(G)$ shows a bound for $M(\mathfrak{G})$ leads to a bound for $M(G)$; an improvement of bounds for loop graphs leads to an improvement for simple graphs.
- Let $\mathfrak{C}^0_{2k+1}$ be a loopless odd cycle, as a loop graph. Then $M(\mathfrak{C}^0_{2k+1}) = 0$ but $Z(\mathfrak{C}^0_{2k+1}) = 1$.



$$\det \begin{bmatrix} 0 & e_1 & & & e_{2k+1} \\ e_1 & 0 & e_2 & & \\ & e_2 & \ddots & \ddots & \\ & & \ddots & & e_{2k} \\ e_{2k+1} & & & e_{2k} & 0 \end{bmatrix} = 2\prod_{i=1}^{2k+1} e_i$$

Try to generalize triangle number

$$\operatorname{rank}\begin{bmatrix} a_{1,1} & ? & ? & ? & ? \\ 0 & a_{2,2} & ? & ? & ? \\ 0 & 0 & a_{3,3} & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix} \geq 3$$

Try to generalize triangle number

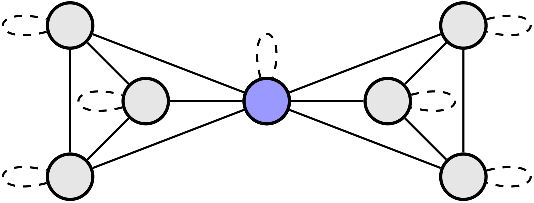
$$\operatorname{rank}\begin{bmatrix} A_{1,1} & ? & ? & ? & ? \\ O & A_{2,2} & ? & ? & ? \\ O & O & A_{3,3} & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix} \geq \sum_{i=1}^{3} \operatorname{rank}(A_{i,i})$$

Try to generalize triangle number

$$\operatorname{rank}\begin{bmatrix} A(\mathfrak{C}_5^0) & ? & ? & ? & ? \\ O & A(\mathfrak{C}_7^0) & ? & ? & ? \\ O & O & A(\mathfrak{C}_3^0) & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix} \geq 5 + 7 + 3 = 15$$

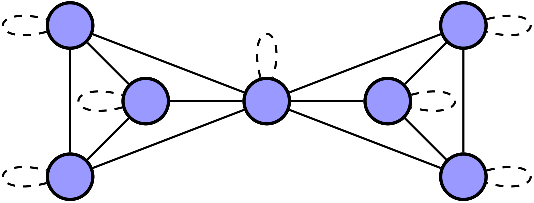
Odd cycle zero forcing number

- The color-change rule CCR-Z_{oc} for loop graphs is:
  - if $y \in V(\mathfrak{G})$ is the only white neighbor of $x \in V(\mathfrak{G})$ ~~and x is blue~~, then y turns blue ($x = y$ is possible);
  - if W is the set of white vertices, and $\mathfrak{G}[W]$ has a connected component $\mathfrak{C}$ such that $\mathfrak{C} \cong \mathfrak{C}^0_{2k+1}$, then all vertices in $V(\mathfrak{C})$ turn blue.



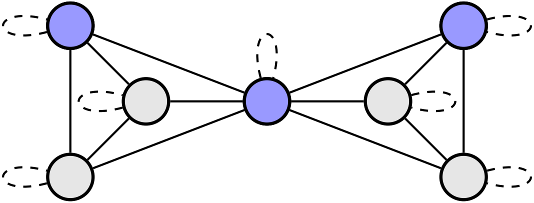
Odd cycle zero forcing number

- The color-change rule CCR-Z_{oc} for loop graphs is:
  - if $y \in V(\mathfrak{G})$ is the only white neighbor of $x \in V(\mathfrak{G})$ ~~and x is blue~~, then y turns blue ($x = y$ is possible);
  - if W is the set of white vertices, and $\mathfrak{G}[W]$ has a connected component $\mathfrak{C}$ such that $\mathfrak{C} \cong \mathfrak{C}^0_{2k+1}$, then all vertices in $V(\mathfrak{C})$ turn blue.



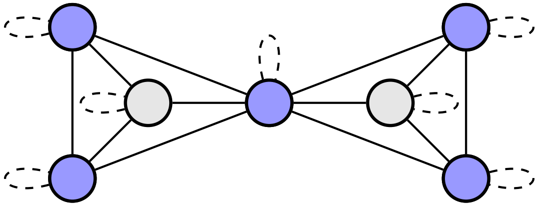
Odd cycle zero forcing number

- The color-change rule CCR-Z_{oc} for loop graphs is:
  - if $y \in V(\mathfrak{G})$ is the only white neighbor of $x \in V(\mathfrak{G})$ ~~and x is blue~~, then y turns blue ($x = y$ is possible);
  - if W is the set of white vertices, and $\mathfrak{G}[W]$ has a connected component $\mathfrak{C}$ such that $\mathfrak{C} \cong \mathfrak{C}^0_{2k+1}$, then all vertices in $V(\mathfrak{C})$ turn blue.



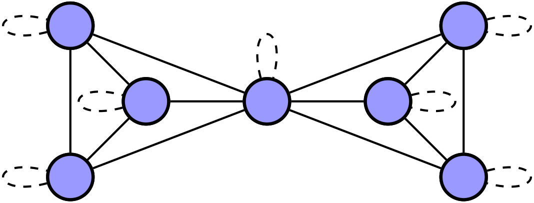
Odd cycle zero forcing number

- The color-change rule CCR-Z_{oc} for loop graphs is:
  - if $y \in V(\mathfrak{G})$ is the only white neighbor of $x \in V(\mathfrak{G})$ ~~and x is blue~~, then y turns blue ($x = y$ is possible);
  - if W is the set of white vertices, and $\mathfrak{G}[W]$ has a connected component $\mathfrak{C}$ such that $\mathfrak{C} \cong \mathfrak{C}^0_{2k+1}$, then all vertices in $V(\mathfrak{C})$ turn blue.



Odd cycle zero forcing number

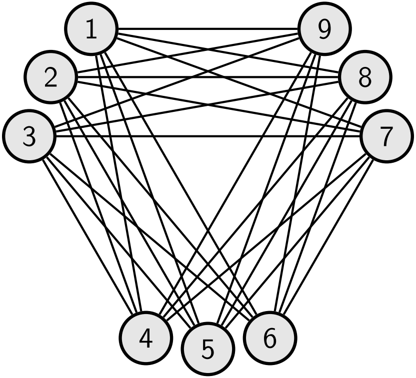
- The color-change rule CCR-Z_{oc} for loop graphs is:
  - if $y \in V(\mathfrak{G})$ is the only white neighbor of $x \in V(\mathfrak{G})$ ~~and x is blue~~, then y turns blue ($x = y$ is possible);
  - if W is the set of white vertices, and $\mathfrak{G}[W]$ has a connected component $\mathfrak{C}$ such that $\mathfrak{C} \cong \mathfrak{C}^0_{2k+1}$, then all vertices in $V(\mathfrak{C})$ turn blue.



Odd cycle zero forcing number

- The color-change rule CCR-Z_{oc} for loop graphs is:
  - if $y \in V(\mathfrak{G})$ is the only white neighbor of $x \in V(\mathfrak{G})$ ~~and x is blue~~, then y turns blue ($x = y$ is possible);
  - if W is the set of white vertices, and $\mathfrak{G}[W]$ has a connected component $\mathfrak{C}$ such that $\mathfrak{C} \cong \mathfrak{C}^0_{2k+1}$, then all vertices in $V(\mathfrak{C})$ turn blue.
- $Z_{oc}(\mathfrak{G})$ is the smallest cardinality of a zero forcing set on $\mathfrak{G}$ using CCR-Z_{oc} for loop graphs.
- $M^F(\mathfrak{G}) \leq Z_{oc}(\mathfrak{G})$ whenever char $F \neq 2$ and matrices are symmetric.
- The enhanced odd cycle zero forcing number is defined as $\widehat{Z}_{oc}(G) = \max_{\mathfrak{G}} Z_{oc}(\mathfrak{G})$, where $\mathfrak{G}$ runs over all loop configurations of G.

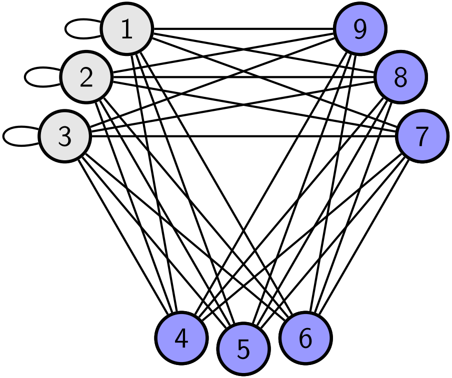
Example: $K_{3,3,3}$



$$\widehat{Z}_{oc}(K_{3,3,3}) = 6 \text{ and } \widehat{Z}(K_{3,3,3}) = Z(K_{3,3,3}) = 7.$$

Example: $K_{3,3,3}$

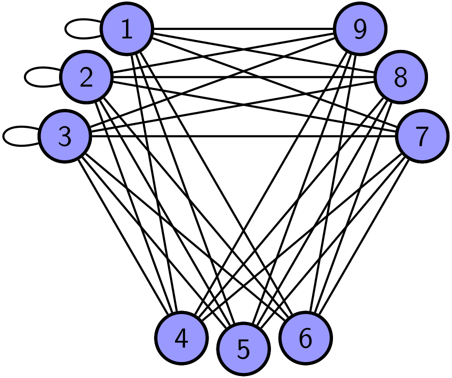
1,2,3 have loops
others are unknown



$$\widehat{Z}_{oc}(K_{3,3,3}) = 6 \text{ and } \widehat{Z}(K_{3,3,3}) = Z(K_{3,3,3}) = 7.$$

Example: $K_{3,3,3}$

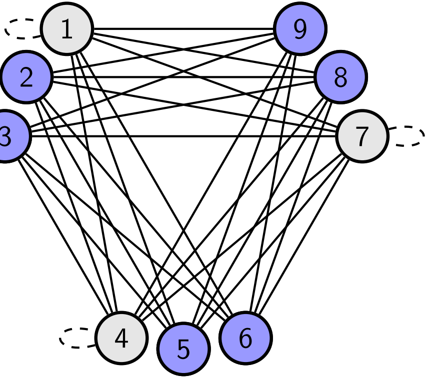
1,2,3 have loops
others are unknown



$$\widehat{Z}_{oc}(K_{3,3,3}) = 6 \text{ and } \widehat{Z}(K_{3,3,3}) = Z(K_{3,3,3}) = 7.$$

Example: $K_{3,3,3}$

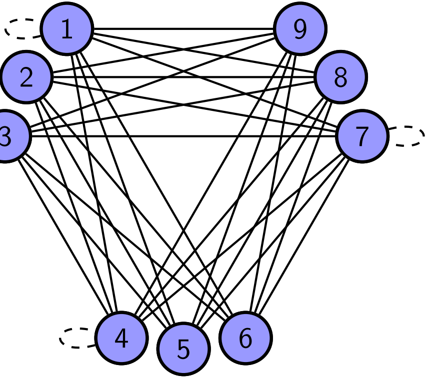
1,4,7 have no loops
others are unknown



$$\widehat{Z}_{oc}(K_{3,3,3}) = 6 \text{ and } \widehat{Z}(K_{3,3,3}) = Z(K_{3,3,3}) = 7.$$

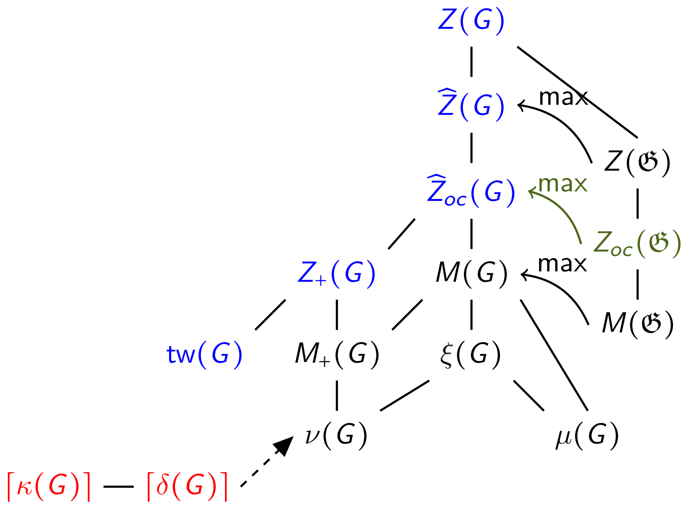
Example: $K_{3,3,3}$

1,4,7 have no loops
others are unknown

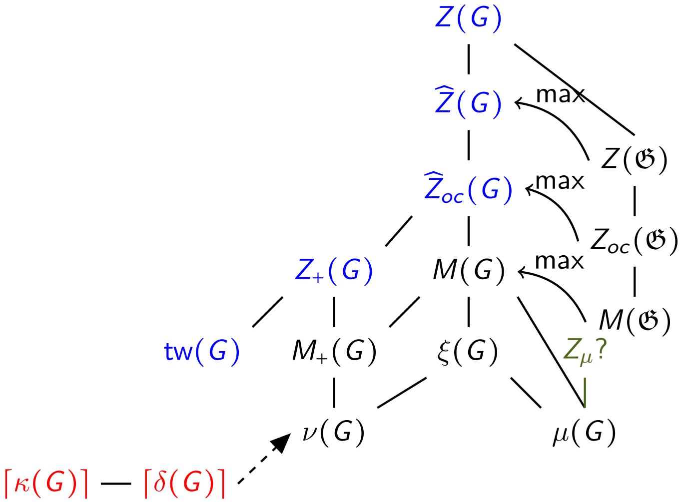


$$\widehat{Z}_{oc}(K_{3,3,3}) = 6 \text{ and } \widehat{Z}(K_{3,3,3}) = Z(K_{3,3,3}) = 7.$$

GCC-$\widehat{Z}_{oc}(G)$



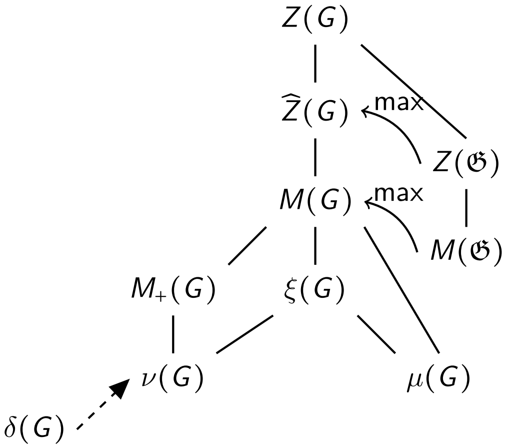
GCC-$\widehat{Z}_{oc}(G)$



Bound μ above

- Recall that $\mu(G)$ is the maximum nullity among matrices M with the following properties:
  - Generalized Laplacian: $M_{ij}\begin{cases} <0 & \text{if} \quad i \sim j, i \neq j \\ =0 & \text{if} \quad i \nsim j, i \neq j \\ \text{free} & \text{if} \quad i = j \end{cases}.$
  - M has exactly one negative eigenvalue.
  - M satisfies SAH.
- Goldberg and Berman (2014) found $Z_\pm$ to bound $M(Q_\pm)$.
- Butler et al. (2014) found Z_q to bound $M_q(G)$.
- So $\mu(G) \leq \min\{Z_\pm(G), Z_1(G)\}$, but can we do better?
- If such Z_μ exists, is GCC-Z_μ true or not?

Transferring from ν to μ



GCC-ν

- A *k-tree* is formed by starting from K_{k+1} and repeatedly adding one vertex joined to an existing k-clique.
- Sinkovic and van der Holst (2011) showed that if G is a k-tree, then $\nu(\overline{G}) \geq n - 2 - k$.
- So if G is a subgraph of a k-tree T_k and $\nu(G) \geq k$, then GCC-ν holds.

$$\nu(G) + \nu(\overline{G}) \geq k + n - 2 - k = n - 2,$$

since $\overline{T_k}$ is a subgraph of $\overline{G}$.

- Can we replace ν by μ?

GCC-ν

- Barioli et al. (2012) showed that if either
  - G and H each have an edge, or
  - G has an edge and $H = \overline{K_r}$ with $\nu(G) \leq |V(G)| - r$,

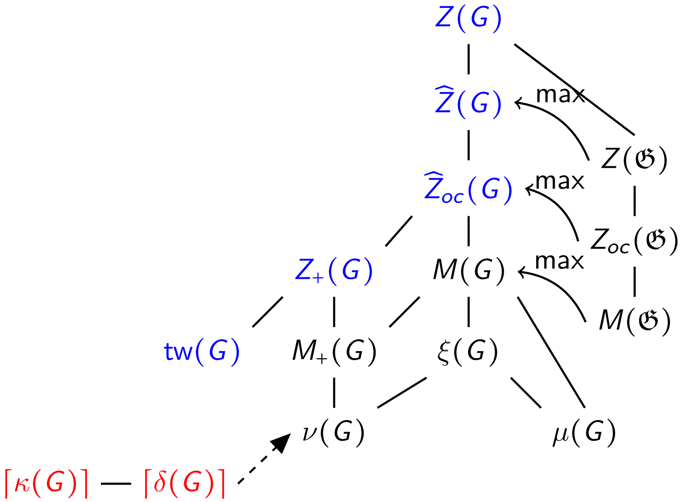
  then $\nu(G \vee H) = \min\{|V(G)| + \nu(H), \nu(G) + |V(H)|\}$;

  Otherwise,
  $\nu(G \vee H) = \min\{|V(G)| + \nu(H), \nu(G) + |V(H)|\} - 1.$
- Can we replace ν by μ?

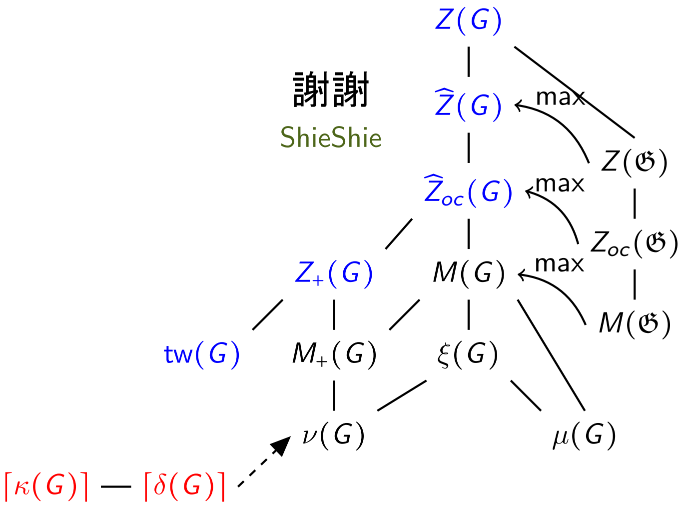
Partial answers

- $\mu(G) \leq \mu(G - v) + 1$.
- $\mu(G \vee H) \leq \min\{|V(G)| + \mu(H), \mu(G) + |V(H)|\}$.
- $\min\{|V(G)| + \mu(H), \mu(G) + |V(H)|\} - 1 \overset{?}{\leq} \mu(G \vee H)$.
- Up to $n \leq 7$, $\mu(G)$ can be determined.
  - $\mu(G) \leq 1$ iff G is a disjoint union of paths;
  - $\mu(G) \leq 2$ iff G is outerplanar;
  - $\mu(G) \leq 3$ iff G is planar;
  - $\mu(G) \leq 4$ iff G is linklessly embedable.
  - $\mu(G) \leq n - 1$, with the equality holds when G is $\overline{K_2}$ or K_n.
- The inequality holds for graphs with $n \leq 8$.

Keep going


$Z(G)$
$\widehat{Z}(G)$
max
$Z(\mathfrak{G})$
$\widehat{Z}_{oc}(G)$
max
$Z_{oc}(\mathfrak{G})$
$Z_+(G)$
$M(G)$
max
$M(\mathfrak{G})$
$\mathrm{tw}(G)$
$M_+(G)$
$\xi(G)$
$\nu(G)$
$\mu(G)$
$\lceil \kappa(G) \rceil$ — $\lceil \delta(G) \rceil$

Keep going



謝謝

ShieShie

References I

AIM Minimum Rank – Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, and A. Wangsness).
Zero forcing sets and the minimum rank of graphs.
Linear Algebra Appl., 428:1628–1648, 2008.

F. Barioli, W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben, and H. van der Holst.
On the graph complement conjecture for minimum rank.
Linear Algebra Appl., 436:4373–4391, 2012.

References II

A. Bento and A. Leal Duarte.
On Fiedler's characterization of tridiagonal matrices over arbitrary fields.
Linear Algebra Appl., 401:467–481, 2005.

A. Berman, S. Friedland, L. Hogben, U. G. Rothblum, and B. Shader.
An upper bound for the minimum rank of a graph.
Linear Algebra Appl., 429:1629–1638, 2008.

References III

R. A. Brualdi, L. Hogben, and B. Shader.
AIM Workshop on Spectra of Families of Matrices Described by Graphs, Digraphs and Sign patterns, Final report: Mathematical Results.
http://aimath.org/pastworkshops/matrixspectrumrep.pdf, 2007.

S. Butler, J. Grout, and H. T. Hall.
Using variants of zero forcing to bound the inertia set of a graph.
Electron. J. Linear Algebra, 2014.
(accepted).

References IV

Y. Colin de Verdière.
On a new graph invariant and a criterion for planarity.
In *Graph Structure Theory*, pp. 137–147, American Mathematical Society, Providence, RI, 1993.

Y. Colin de Verdière.
Multiplicities of eigenvalues and tree-width graphs.
J. Combin. Theory Ser. B, 74:121–146, 1998.

J. Ekstrand, C. Erickson, H. T. Hall, D. Hay, L. Hogben, R. Johnson, N. Kingsley, S. Osborne, T. Peters, J. Roat, A. Ross, D. D. Row, N. Warnberg, and M. Young.
Positive semidefinite zero forcing.
Linear Algebra Appl., 439:1862–1874, 2013.

References V

M. Fiedler.
A characterization of tridiagonal matrices.
Linear Algebra Appl., 2:191–197, 1969.

F. Goldberg and A. Berman.
Zero forcing for sign patterns.
Linear Algebra Appl., 447:56–67, 2014.

H. T. Hall.
http://www.math.iastate.edu/news/fp8.html, 2011.

L. Hogben.
Minimum rank problems.
Linear Algebra Appl., 432:1961–1974, 2010.

References VI

L. Hogben.
Survey of Nordhaus-Gaddum problems for Colin de Verdière type parameters, variants of tree-width, and related parameters, 2014.
(in preparation).

H. van der Holst.
Graphs with magnetic schrödinger operators of low corank.
J. Combin. Theory Ser. B, 84:311–339, 2002.

H. van der Holst, L. Lovász, and A. Schrijver.
The Colin de Verdière graph parameter.
In *Graph Theory and Computational Biology (Balatonlelle, 1996)*, pp. 29–85, Janos Bolyai Math. Soc., Budapest, 1999.

References VII

A. Kotlov, L. Lovász, and S. Vempala.
The Colin de Verdère number and sphere representations of a graph.
Combinatorica, 17:483–521, 1997.

L. Lovász, M. Saks, and A. Schrijver.
Orthogonal representations and connectivity of graphs.
Linear Algebra Appl., 114/115:439–454, 1989.

L. Lovász, M. Saks, and A. Schrijver.
A correction: Orthogonal representations and connectivity of graphs.
Linear Algebra Appl., 313:101–105, 2000.

References VIII

L. Lovász and A. Schrijver.
A Borsuk theorem for antipodal links and a spectral characterization of linklessly embeddable graphs.
Proc. Amer. Math. Soc., 126:1275–1285, 1998.

K. H. Monfared and B. Shader.
Construction of matrices with a given graph and prescribed interlaced spectral data.
Linear Algebra Appl., 438:4348 – 4358, 2013.

N. Robertson, P. Seymour, and R. Thomas.
A survey of linkless embeddings.
In *Graph Structure Theory*, pp. 125–136, American Mathematical Society, Providence, RI, 1993.

References IX

J. Sinkovic and H. van der Holst.
The minimum semidefinite rank of the complement of partial k-trees.
Linear Algebra Appl., 434:1468–1474, 2011.