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# Basics of game theory
# The electricity market and MPEC

# Let's play: the prisoners dilema

| Suspect B / Suspect A | Remain silent | Blame |
|---|---|---|
| Remain silent | 1 / 1 | 5 / 0 |
| Blame | 0 / 5 | 3 / 3 |

https://corporatefinanceinstitute.com/resources/economics/prisoners-dilemma/

# What is a game?

- Set of players *J*
- Action space *A*
  - $a = (a_j, a_{-j})$ is a strategy profile
- Payoff function $P{:}A \rightarrow \mathbb{R}^j$
  - So, j's payoff is $P(a) \in \mathbb{R}$

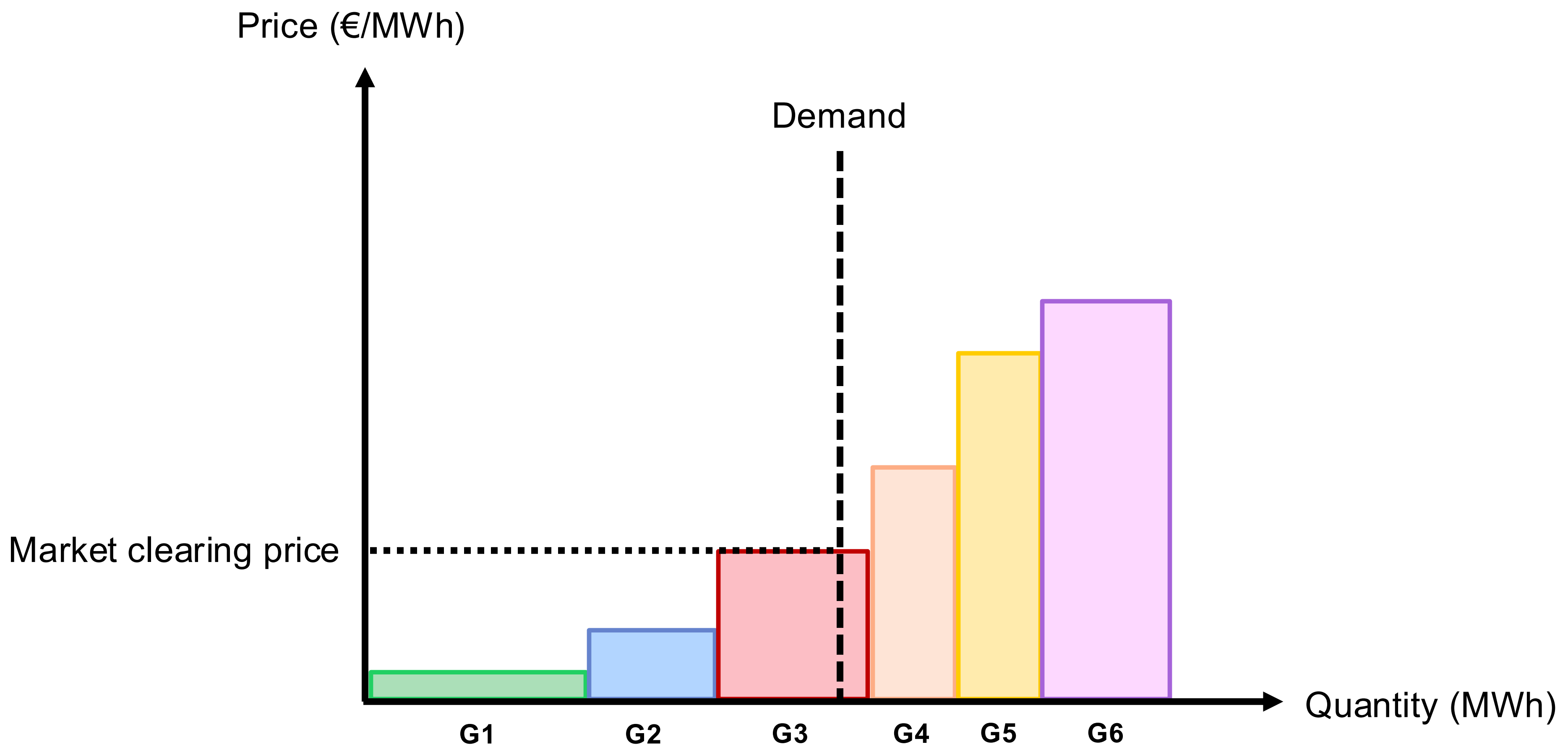
A strategy profile $a = (a_j, a_{-j})$ is a (pure) **Nash Equilibrium** if

$$P(a_j, a_{-j}) \geq P(a'_j, a_{-j}),$$
$$\forall\, a'_j \in A_j, \forall j \in J$$

$$\boldsymbol{G} = \{\boldsymbol{J}, \boldsymbol{A}, \boldsymbol{P}\}$$

More at: https://ocw.mit.edu/courses/14-126-game-theory-spring-2016/

# The electricity market



# The economic dispatch under perfect information

**We know the true cost of the generators,** $\mathrm{c}_i^{\mathrm{G}}$

$$\min_{X_{\mathrm{MO}}} \left\{ \sum_{i \in \mathcal{I}} \mathrm{c}_i^{\mathrm{G}} P_i^{\mathrm{Gopt}} \right\}$$

subject to:

$$\mathrm{D} - \sum_i P_i^{\mathrm{Gopt}} = 0 : \boxed{\lambda^{\mathrm{opt}}}$$

$$\mathrm{P}_i^{\mathrm{Gmin}} \leq P_i^{\mathrm{Gopt}} \leq \mathrm{P}_i^{\mathrm{Gmax}} : \mu_i^{\mathrm{G}_{\mathrm{min}}^{\mathrm{opt}}}, \mu_i^{\mathrm{G}_{\mathrm{max}}^{\mathrm{opt}}}, \qquad \forall i \in \mathcal{I}$$

Where,

$$X_{\mathrm{MO}} = \{P_{i \in \mathcal{I}}^{\mathrm{Gopt}}\}$$

- $\mathrm{c}_i^{\mathrm{G}}$ : cost of generator $i$
- $P_i^{\mathrm{Gopt}}$: power dispatch of generator $i$
- D: demand
- $\mathrm{P}_i^{\mathrm{Gmin}}, \mathrm{P}_i^{\mathrm{Gmax}}$: power limits of generator $i$
- $\lambda^{\mathrm{opt}}$: dual variable of the balance constraint
- $\mu_i^{\mathrm{G}_{\mathrm{min}}^{\mathrm{opt}}}, \mu_i^{\mathrm{G}_{\mathrm{max}}^{\mathrm{opt}}}$: dual variables of the power limits of gen $i$

**How can we know the market clearing price?**
**Is there a variable that represents it?**

# The market-based economic dispatch

The **true cost** of the generators ( $c_i^{\mathrm{G}}$) are **unknown**, market participants submit a **bid** ($\alpha_i^{\mathrm{G}}$),

$$\min_{X_{\mathrm{MO}}} \left\{ \sum_{i \in \mathcal{I}} \alpha_i^{\mathrm{G}} P_i^{\mathrm{G}} \right\}$$

subject to:

$$\mathrm{D} - \sum_i P_i^{\mathrm{G}} = 0 : \lambda,$$

$$\mathrm{P}_i^{\mathtt{Gmin}} \leq P_i^{\mathrm{G}} \leq \mathrm{P}_i^{\mathtt{Gmax}} : \mu_i^{\mathtt{Gmin}}, \mu_i^{\mathtt{Gmax}}, \qquad \forall i \in \mathcal{I}$$

Where,

$$X_{\mathrm{MO}} = \{P_{i \in \mathcal{I}}^{\mathrm{G}}\}$$

**How can we know the market clearing price?**
**Is there a variable that represents it?**

**Is the market clearing price going to be always the same regardless we optimize over costs or bids?**

$$\lambda \stackrel{?}{=} \lambda^{\mathtt{opt}}$$

**What does it mean if it is lower? And higher?**
**Which one is more likely to occur in reality?**

# Market vs perfect information

$$SC^{\mathrm{eq}} = \sum_{i \in \mathcal{I}} \mathrm{c}_i^{\mathrm{G}} P_i^{\mathrm{G}}$$

$$SC^{\mathrm{opt}} = \sum_{i \in \mathcal{I}} \mathrm{c}_i^{\mathrm{G}} P_i^{\mathrm{Gopt}}$$

$$\boxed{PoA = \frac{SC^{\mathrm{eq}}}{SC^{\mathrm{opt}}} \geq 1}$$

# Producers bids

**How do producers determine their bid ($\alpha_i^G$)?**

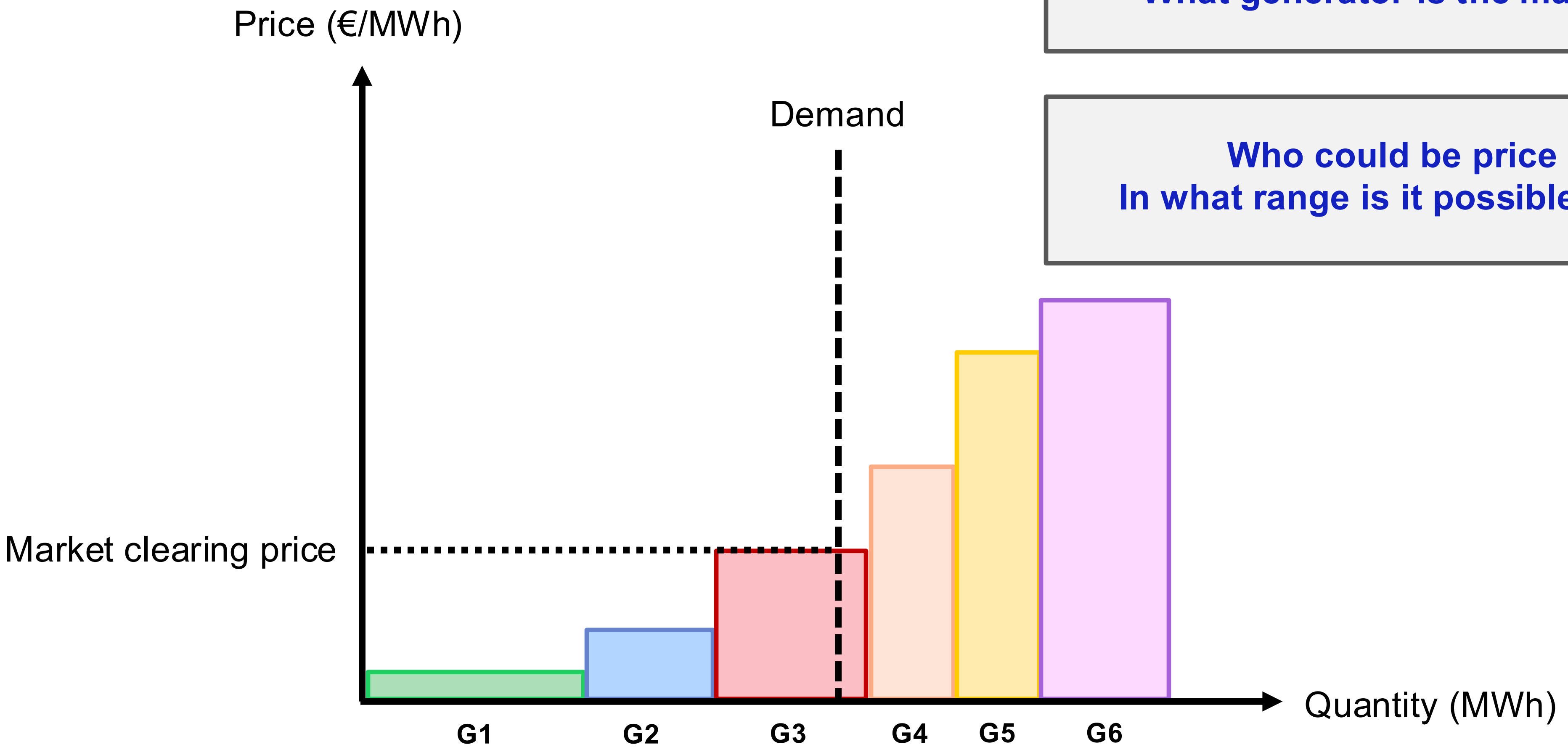
- Price takers
- Price makers

A **price taker** is a market participant (often a small generator) whose individual bids or offers are too small to influence the market-clearing price. They accept the market price as given and decide only on the quantity they are willing to produce or consume at that price.

A **price maker** is typically a large generator or dominant player whose bidding strategy can influence the market-clearing price. Because of their significant market share or strategic importance (e.g., being needed to meet demand in certain hours), their actions can shift supply-demand balance and thereby set or strongly affect the price.

The **marginal generator** is the unit whose offer is the last accepted in the supply stack (merit order) to meet demand in each settlement period. Its bid sets the market-clearing price for all dispatched generators, regardless of their own costs.

# The electricity market




**What generator is the marginal generator?**

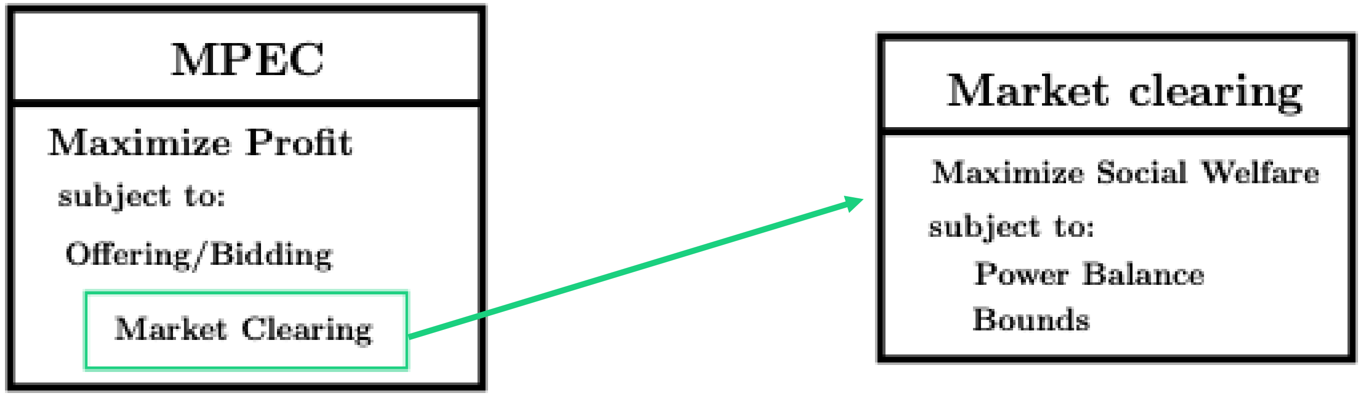
**Who could be price maker here?**
**In what range is it possible to shift the price?**

# Strategic producers – price makers

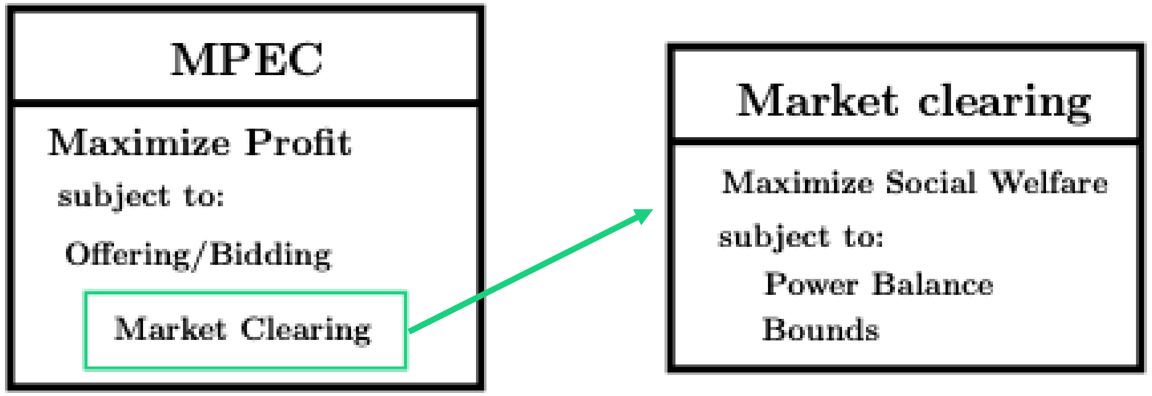
**How do strategic producers determine their bid ( $\alpha_i^G$ )?**

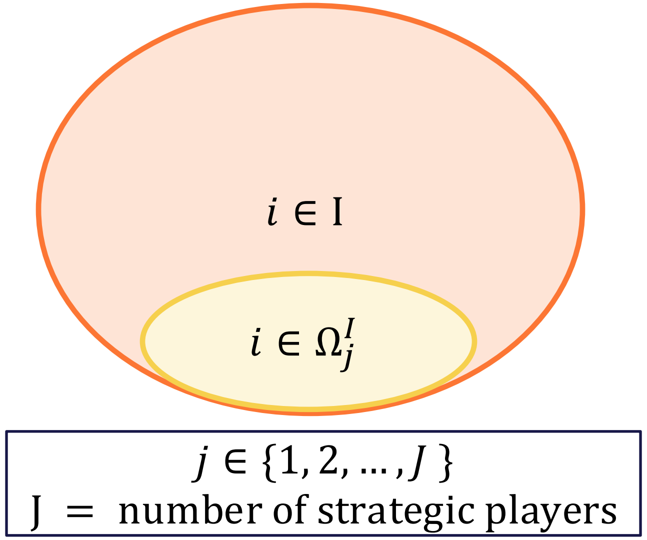
A strategic producer can anticipate other players (producers) bids. Based on this, they can determine their optimal bid.

- What does a strategic producer solve for? What is the objective function in their problem?
- What constraints does an strategic producer have?
- How does the information about other players enter the problem?



# Strategic producers – price makers




It is clear that a strategic producer needs to solve the MPEC problem (more on this name later). But let's take a closer look.

**How many problems do we need to solve?** Hint: how many objective functions are there?

Do all constraints need to be satisfied?

What kind of mathematical program characterizes this kind of problems?

# Bilevel programing

$$\min_{X_{\mathrm{UL}},\Xi_{\mathrm{UL}}} \left\{ \sum_{\boxed{i\in\Omega_j^{\mathrm{I}}}} \left( -\lambda P_i^{\mathrm{G}} + c_i^{\mathrm{G}} P_i^{\mathrm{G}} \right) \right\}$$

**Upper level**

subject to:

$$\alpha^{\min} \leq \alpha_i^{\mathrm{G}} \leq \alpha^{\max},$$

We can only control **our** generators!

$$\min_{X_{\mathrm{LL}}} \left\{ \sum_{i\in\mathcal{I}} \alpha_i^{\mathrm{G}} P_i^{\mathrm{G}} \right\}$$

**Lower level**

subject to:

$$\mathrm{D} - \sum_i P_i^{\mathrm{G}} = 0 : \lambda,$$

$$\mathrm{P}_i^{\mathrm{Gmin}} \leq P_i^{\mathrm{G}} \leq \mathrm{P}_i^{\mathrm{Gmax}} : \mu_i^{\mathrm{Gmin}}, \mu_i^{\mathrm{Gmax}}, \qquad \forall i \in \mathcal{I}$$

Where:

$$X_{\mathrm{UL}} = \left\{ \alpha^{\mathrm{G}}_{i\in\Omega_j^{\mathrm{I}}}, P^{\mathrm{G}}_{i\in\mathcal{I}} \right\}$$ → Upper level primal variables

$$\Xi_{\mathrm{UL}} = \left\{ \lambda_{n,t} \mu_i^{\mathrm{Gmin}}, \mu_i^{\mathrm{Gmax}} \right\}$$ → Upper level dual variables

$$X_{\mathrm{LL}} = \left\{ P^{\mathrm{G}}_{i\in\mathcal{I}} \right\}$$ → Lower level primal variables

| MPEC |
| --- |
| Maximize Profit<br>subject to:<br>Offering/Bidding<br>Market Clearing |

# How to solve a bilevel program?

We replace the Lower Level problem with its **KKT conditions**

The **Karush–Kuhn–Tucker (KKT) conditions** are a set of **first-order necessary conditions** for a solution to be optimal in a constrained optimization problem.

They extend the method of Lagrange multipliers to handle **inequality constraints** as well as equalities.

•They state that at an optimum, the **gradient of the objective** can be expressed as a combination of the gradients of the active constraints.

•They include **primal feasibility** (constraints are satisfied), **dual feasibility** (multipliers for inequalities are nonnegative), and **complementary slackness** (a multiplier is zero unless its constraint is binding).

So, the KKT conditions provide the mathematical link between the objective, constraints, and multipliers, characterizing candidate optimal solutions.

Standard format of optimization problem

$$\begin{aligned} &\underset{x}{\text{Minimize}} \quad f(x) \\ &\text{subject to:} \\ &h(x) = 0 \quad : \quad \lambda \\ &g(x) \leq 0 \quad : \quad \mu \end{aligned}$$

Lagrangian function

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda^\top h(x) + \mu^\top g(x)$$

KKT conditions

$$\begin{aligned} &\frac{\partial \mathcal{L}(x, \lambda, \mu)}{\partial x} = 0 \\ &h(x) = 0 \\ &0 \leq -g(x) \perp \mu \geq 0 \\ &\lambda \in \text{free} \end{aligned}$$

# How to solve a bilevel program?

A complementary constraint is:

$$
\begin{aligned}
&0 \leq x \\
&0 \leq y \\
&x \cdot y = 0,
\end{aligned}
$$

usually represented as,

$$0 \leq x \perp y \geq 0$$

This is why we call it **M**athematical **P**rogram with **E**quilibrium **C**onstraints (**MPEC**)

It can be linearized with an auxiliary binary

$$
\begin{aligned}
&0 \leq x \leq \mathrm{M} \cdot z \\
&0 \leq y \leq \mathrm{M} \cdot (1 - z), \\
&z \in \{0, 1\}
\end{aligned}
$$

# Exercises

# Exercise 1

Write the bilevel model of a strategic producer as a single level problem.

1. Write the LL problem in standard form
2. Construct the Lagrangian
3. Find the KKT conditions
4. Linearize the KKT conditions using the bigM method.
5. Write the equivalent single level problem.
6. Solve the MPEC using Pyomo, analyse the results.

Where is the LL problem objective function represented?
How many objective functions do we end up with?

$$\min_{X_{\mathrm{UL}},\Xi_{\mathrm{UL}}} \left\{ \sum_{i\in\Omega_j^{\mathrm{I}}} \left( -\lambda P_i^{\mathrm{G}} + \mathrm{c}_i^{\mathrm{G}} P_i^{\mathrm{G}} \right) \right\}$$

**subject to:**

$$\alpha^{\mathtt{min}} \leq \alpha_i^{\mathrm{G}} \leq \alpha^{\mathtt{max}},$$

$$\min_{X_{\mathrm{LL}}} \left\{ \sum_{i\in\mathcal{I}} \alpha_i^{\mathrm{G}} P_i^{\mathrm{G}} \right\}$$

**subject to:**

$$\mathrm{D} - \sum_i P_i^{\mathrm{G}} = 0 : \lambda,$$

$$\mathrm{P}_i^{\mathrm{G}\mathtt{min}} \leq P_i^{\mathrm{G}} \leq \mathrm{P}_i^{\mathrm{G}\mathtt{max}} : \mu_i^{\mathrm{G}\mathtt{min}}, \mu_i^{\mathrm{G}\mathtt{max}}, \qquad \forall i \in \mathcal{I}$$

**Where:**

$$X_{\mathrm{UL}} = \left\{ \alpha^{\mathrm{G}}_{i\in\Omega_j^{\mathrm{I}}}, P^{\mathrm{G}}_{i\in\mathcal{I}} \right\}$$

$$\Xi_{\mathrm{UL}} = \left\{ \lambda_{n,t} \mu_i^{\mathrm{G}\mathtt{min}}, \mu_i^{\mathrm{G}\mathtt{max}} \right\}$$

$$X_{\mathrm{LL}} = \left\{ P^{\mathrm{G}}_{i\in\mathcal{I}} \right\}$$

# Exercise 2

Write the bilevel model of a strategic producer as a single level problem.

1. Write the LL problem in standard form
2. Construct the Lagrangian
3. Find the KKT conditions
4. Use SOS1 constraints to encode complementary
5. Solve and analyze the results using Pyomo.

$$\min_{X_{\mathrm{UL}},\Xi_{\mathrm{UL}}} \left\{ \sum_{i\in\Omega_j^{\mathrm{I}}} \left( -\lambda P_i^{\mathrm{G}} + \mathrm{c}_i^{\mathrm{G}} P_i^{\mathrm{G}} \right) \right\}$$

**subject to:**

$$\alpha^{\mathtt{min}} \leq \alpha_i^{\mathrm{G}} \leq \alpha^{\mathtt{max}},$$

$$\min_{X_{\mathrm{LL}}} \left\{ \sum_{i\in\mathcal{I}} \alpha_i^{\mathrm{G}} P_i^{\mathrm{G}} \right\}$$

**subject to:**

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**Where:**

$$X_{\mathrm{UL}} = \left\{ \alpha^{\mathrm{G}}_{i\in\Omega_j^{\mathrm{I}}}, P^{\mathrm{G}}_{i\in\mathcal{I}} \right\}$$

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