An approximation to the exponential function

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Abstract

An implementation of the exponential function is introduced and compared to the standard exponential function from the math library.

1 Introduction

An exponential function is a type of function that is most often seen on the form,

$$f(x) = e^x \ . \tag{1}$$

This form is often seen since the derivative of $f(x) = f'(x)$ and this is a differential equation often needed to be solved in e.g. physics. One of its characteristics is that it has a well-defined half-life which can be used to describe e.g. radioactive decay. One of many simple approximation to the function (1) is a power series which is accurate for an infinite number of terms [1],

$$\exp x := \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \tag{2}$$

This can easily be seen as true by applying the power rule of differentiation.

2 The implementation

Our implementation reacts differently for x in different intervals. For negative x the reciprocal function is used which in general is possible for exponential functions since $a^{-b} = 1/a^b$. Furthermore, the implementation evaluates values for x in the interval $[0, 1/8]$ via the Tayler expansion to the 10th term. The trick in this implementation is that for larger values of x it is used that $e^x = (e^{x/2})^2$. This is used recursively to propagate the error of the Taylor function. Let us say we wish to evaluate $e^{1/2}$. This would be reinterpreted as $(e^{1/8})^4$ and from here evaluated via the Taylor expansion. This is a very clever way of propagating the error.

3 The convoluted Taylor series

For small values of x, the exponential function is computed via the following implementatin:

$$\begin{aligned} \exp x \approx 1 + x(1 + x/2(1 + x/3(1 + x/4(1 + x/5(1 \\ +x/6(1 + x/7(1 + x/8(1 + x/9(1 + x/10))))))))). \end{aligned} \tag{3}$$

This would work because every term in the series (3) can be made up of a product of the terms previous to it-only dividing by an integer equal to 1 added to the number of terms previous to the term. This is exactly why it is useful: it reuses the multiplication of the previous term instead of doing the factorial from the ground up for every new term.

4 Comparison

Here is an illustration of the implementation, compared with the built-in exponential function from the standard C-library. It is seen that the overlap of the graphs are so large that only one of them is in fact seen. This suggets a very large corelation between the two i.e. the approximation is good.

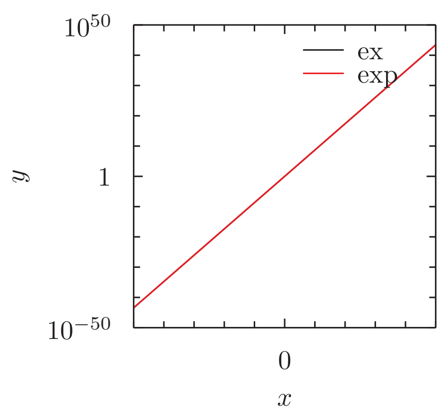

Figure 1: Our implementation vs. C's implementation of the exponential function

References

[1] Rudin, Walter (1987). Real and complex analysis (3rd ed.). New York: McGraw-Hill. p. 1. ISBN 978-0-07-054234-1,