Conditioning on X_0, the log-likelihood of MTS data is then given by

$$l_{MTS}(\phi, \rho) = n_{01} \log \phi + n_{10} \log (1-\phi) + (n_{01} + n_{10}) \log \left(1 - e^{-\rho}\right) \\ + n_{00} \log \left(1 - \phi + \phi e^{-\rho}\right) + n_{11} \log \left[\phi + (1-\phi) e^{-\rho}\right]. \tag{1}$$

The score function is given by

$$\frac{\partial l_{MTS}}{\partial \phi} = \frac{n_{01}}{\phi} - \frac{n_{10}}{1-\phi} - \frac{n_{00}}{\frac{1}{1-e^{-\rho}} - \phi} + \frac{n_{11}}{\frac{e^{-\rho}}{1-e^{-\rho}} + \phi}$$

$$\frac{\partial l_{MTS}}{\partial \rho} = e^{-\rho} \left[\frac{n_{01} + n_{10}}{1 - e^{-\rho}} - \frac{n_{00}}{\frac{1-\phi}{\phi} + e^{-\rho}} - \frac{n_{11}}{\frac{\phi}{1-\phi} + e^{-\rho}}\right]$$

and the elements of the Hessian are

$$\frac{\partial^2 l_{MTS}}{\partial \phi^2} = -\frac{n_{01}}{\phi^2} - \frac{n_{10}}{(1-\phi)^2} - \frac{n_{00}}{\left(\frac{1}{1-e^{-\rho}} - \phi\right)^2} - \frac{n_{11}}{\left(\frac{e^{-\rho}}{1-e^{-\rho}} + \phi\right)^2}$$

$$\frac{\partial^2 l_{MTS}}{\partial \rho^2} = -e^{-\rho} \left[\frac{n_{01} + n_{10}}{(1 - e^{-\rho})^2} - \frac{\frac{1-\phi}{\phi} n_{00}}{\left(\frac{1-\phi}{\phi} + e^{-\rho}\right)^2} - \frac{\frac{\phi}{1-\phi} n_{11}}{\left(\frac{\phi}{1-\phi} + e^{-\rho}\right)^2}\right]$$

$$\frac{\partial^2 l_{MTS}}{\partial \phi \partial \rho} = -e^{-\rho} \left[\frac{n_{00}}{(1 - \phi + \phi e^{-\rho})^2} - \frac{n_{11}}{(\phi + (1-\phi) e^{-\rho})^2}\right].$$