Conditioning on X_0, the log-likelihood of MTS data is then given by

$$\begin{aligned} l_{MTS}(\phi,\rho) = n_{01}\log\phi + n_{10}\log(1-\phi) + (n_{01}+n_{10})\log\left(1-e^{-\rho}\right) \\ + n_{00}\log\left(1-\phi+\phi e^{-\rho}\right) + n_{11}\log\left[\phi+(1-\phi)e^{-\rho}\right]. \end{aligned} \tag{1}$$

The score function is given by

$$\begin{aligned} \frac{\partial l_{MTS}}{\partial\phi} &= \frac{n_{01}}{\phi} - \frac{n_{10}}{1-\phi} - \frac{n_{00}}{\frac{1}{1-e^{-\rho}}-\phi} + \frac{n_{11}}{\frac{e^{-\rho}}{1-e^{-\rho}}+\phi} \\ \frac{\partial l_{MTS}}{\partial\rho} &= e^{-\rho}\left[\frac{n_{01}+n_{10}}{1-e^{-\rho}} - \frac{n_{00}}{\frac{1-\phi}{\phi}+e^{-\rho}} - \frac{n_{11}}{\frac{\phi}{1-\phi}+e^{-\rho}}\right] \end{aligned}$$

and the elements of the Hessian are

$$\begin{aligned} \frac{\partial^2 l_{MTS}}{\partial\phi^2} &= -\frac{n_{01}}{\phi^2} - \frac{n_{10}}{(1-\phi)^2} - \frac{n_{00}}{\left(\frac{1}{1-e^{-\rho}}-\phi\right)^2} - \frac{n_{11}}{\left(\frac{e^{-\rho}}{1-e^{-\rho}}+\phi\right)^2} \\ \frac{\partial^2 l_{MTS}}{\partial\rho^2} &= -e^{-\rho}\left[\frac{n_{01}+n_{10}}{(1-e^{-\rho})^2} - \frac{\frac{1-\phi}{\phi}n_{00}}{\left(\frac{1-\phi}{\phi}+e^{-\rho}\right)^2} - \frac{\frac{\phi}{1-\phi}n_{11}}{\left(\frac{\phi}{1-\phi}+e^{-\rho}\right)^2}\right] \\ \frac{\partial^2 l_{MTS}}{\partial\phi\partial\rho} &= -e^{-\rho}\left[\frac{n_{00}}{(1-\phi+\phi e^{-\rho})^2} - \frac{n_{11}}{(\phi+(1-\phi)e^{-\rho})^2}\right]. \end{aligned}$$