Let $y_0, y_1, y_2, y_3, ..., y_K$ be a sample from an equilibrium DTMC on an $(R+1)$-dimensional space with observed frequency counts $k_{qr} = \sum_{i=1}^{K} I(y_{i-1} = q, y_i = r)$ and transition probabilities $\pi_{qr}(\boldsymbol{\theta}) = \Pr(y_1 = r | y_0 = q)$ that are functions of a $p \times 1$ parameter vector $\boldsymbol{\theta}$. The equilibrium distribution of the DTMC has the properties that

$$\Pr(y_0 = r) = \tilde{\pi}_r = \sum_{q=0}^{R} \tilde{\pi}_q \pi_{qr} \qquad \text{and} \qquad \sum_{r=0}^{R} \tilde{\pi}_r = 1. \tag{1}$$

The log-likelihood is given by

$$l(\boldsymbol{\theta} | y_0, ..., y_K) = \sum_{q=0}^{R} I(y_0 = q) \log \tilde{\pi}_q + \sum_{q=0}^{R} \sum_{r=0}^{R} k_{qr} \log \pi_{qr}. \tag{2}$$

The i^{th} entry of the score function is therefore given by

$$\frac{\partial l}{\partial \theta_i} = \sum_{q=0}^{R} \frac{I(y_0 = q)}{\tilde{\pi}_q} \frac{\partial \tilde{\pi}_q}{\partial \theta_i} + \sum_{q=0}^{R} \sum_{r=0}^{R} \frac{k_{qr}}{\pi_{qr}} \frac{\partial \pi_{qr}}{\partial \theta_i} \tag{3}$$

and the $(i,j)^{th}$ entry in the Hessian of the likelihood by

$$\frac{\partial^2 l}{\partial \theta_i \partial \theta_j} = \sum_{q=0}^{R} \left(\frac{I(y_0 = q)}{\tilde{\pi}_q} \frac{\partial^2 \tilde{\pi}_q}{\partial \theta_i \partial \theta_j} - \frac{I(y_0 = q)}{\tilde{\pi}_q^2} \frac{\partial \tilde{\pi}_q}{\partial \theta_i} \frac{\partial \tilde{\pi}_q}{\partial \theta_j} \right) + \sum_{q=0}^{R} \sum_{r=0}^{R} \left(\frac{k_{qr}}{\pi_{qr}} \frac{\partial^2 \pi_{qr}}{\partial \theta_i \partial \theta_j} - \frac{k_{qr}}{\pi_{qr}^2} \frac{\partial \pi_{qr}}{\partial \theta_i} \frac{\partial \pi_{qr}}{\partial \theta_j} \right). \tag{4}$$

Because the DTMC is in equilibrium, $E(I(y_0 = q)) = \Pr(y_0 = q) = \tilde{\pi}_q$ and

$$E(k_{qr}) = K \Pr(y_0 = q, y_1 = r) = K \Pr(y_0 = q) \Pr(y_1 = r | y_0 = q) = K \tilde{\pi}_q \pi_{qr}.$$

Furthermore, from the property of transition matrices that $\sum_{r=1}^{R} \pi_{qr} = 1$, it follows that

$$\sum_{r=0}^{R} \frac{\partial \pi_{qr}}{\partial \theta_i} = 0 \qquad \text{and} \qquad \sum_{r=0}^{R} \frac{\partial^2 \pi_{qr}}{\partial \theta_i \partial \theta_j} = 0;$$

similarly,

$$\sum_{r=0}^{R} \frac{\partial \tilde{\pi}_r}{\partial \theta_i} = 0 \qquad \text{and} \qquad \sum_{r=0}^{R} \frac{\partial^2 \tilde{\pi}_r}{\partial \theta_i \partial \theta_j} = 0.$$

The expected information matrix therefore has $(i,j)^{th}$ entry

$$-\mathrm{E} \left(\frac{\partial^2 l}{\partial \theta_i \partial \theta_j} \right) = \sum_{q=0}^{R} \frac{1}{\tilde{\pi}_q} \frac{\partial \tilde{\pi}_q}{\partial \theta_i} \frac{\partial \tilde{\pi}_q}{\partial \theta_j} + K \sum_{q=0}^{R} \sum_{r=0}^{R} \frac{\tilde{\pi}_q}{\pi_{qr}} \frac{\partial \pi_{qr}}{\partial \theta_i} \frac{\partial \pi_{qr}}{\partial \theta_j}. \tag{5}$$

For purposes of comparing the efficiency of different recording procedures, the first term in (5) can be ignored as K grows large. Thus I use

$$\mathcal{I}_{ij}^{E} = K \sum_{q=0}^{R} \sum_{r=0}^{R} \frac{\tilde{\pi}_q}{\pi_{qr}} \frac{\partial \pi_{qr}}{\partial \theta_i} \frac{\partial \pi_{qr}}{\partial \theta_j}. \tag{6}$$

1 Momentary time sampling

The transition matrix corresponding to momentary time sampling with intervals of length L is given by

$$\mathbf{\Pi}^M = \left[\begin{array}{cc} 1 - p_0(L) & p_0(L) \\ 1 - p_1(L) & p_1(L) \end{array} \right], \tag{7}$$

where $p_0(t) = \phi \left(1 - e^{-\rho l}\right)$ and $p_1(t) = (1-\phi)e^{-\rho t} + \phi$. The equilibrium distribution of this DTMC is given by $\tilde{\pi}_0 = 1 - \phi$, $\tilde{\pi}_1 = \phi$. The first derivatives of the transition probabilities with respect to ϕ and ρ are given by

$$\begin{aligned} \frac{\partial \mathbf{\Pi}^M}{\partial \phi} &= \left[\begin{array}{cc} -(1 - e^{-\rho L}) & (1 - e^{-\rho L}) \\ -(1 - e^{-\rho L}) & (1 - e^{-\rho L}) \end{array} \right], \\ \frac{\partial \mathbf{\Pi}^M}{\partial \rho} &= \left[\begin{array}{cc} -\phi L e^{-\rho L} & \phi L e^{-\rho L} \\ (1-\phi) L e^{-\rho L} & -(1-\phi) L e^{-\rho L} \end{array} \right]. \end{aligned}$$

The expected information matrix corresponding to the momentary time sampling procedure can then be evaluated directly from (6).

2 Partial interval recording

The transition probabilities corresponding to partial interval recording with intervals of length L are given by

$$\pi_{j,j+1} = 1 - e^{-\phi \rho L} \left[1 - \overset{(j)}{\circ} f(0) \right], \qquad \pi_{j,0} = 1 - \pi_{j,j+1} \tag{8}$$

for $j = 0, 1, 2, 3, ...$, where

$$f(q) = \frac{\phi - (\phi - q)\, e^{-\rho L}}{1 - (1-q) e^{-\phi \rho L}}$$

and $\overset{(j)}{\circ} f$ denotes j-fold recursion of f:

$$\overset{(0)}{\circ} f(q) = q, \qquad \overset{(j)}{\circ} f(q) = f\left(\overset{(j-1)}{\circ} f(q) \right).$$

The equilibrium distribution of this DTMC is given by

$$\tilde{\pi}_0 = 1 - E(U_k) = (1-\phi) e^{-\phi \rho L},$$

and from (1),

$$\tilde{\pi}_j = \tilde{\pi}_{j-1} \pi_{j-1,j} = \tilde{\pi}_0 \prod_{i=1}^{j} \pi_{i-1,i}$$

for $j = 1, 2, 3, ...$ The derivatives of $\pi_{j,j+1}$ are given by

$$\begin{aligned} \frac{\partial \pi_{j,j+1}}{\partial \phi} &= \rho L \left(1 - \pi_{j,j+1}\right) + e^{-\phi \rho L} \left(\overset{(j)}{\circ} g_\phi(0,0) \right), \\ \frac{\partial \pi_{j,j+1}}{\partial \rho} &= \phi L \left(1 - \pi_{j,j+1}\right) + e^{-\phi \rho L} \left(\overset{(j)}{\circ} g_\rho(0,0) \right), \end{aligned}$$

for $j = 0, 1, 2, ...$ Here, g_ϕ and g_ρ are two-dimensional functions defined by $g_\theta(q, r) = (f(q), f'_\theta(q, r))$ with

$$f'_\phi(q,r) = \frac{\partial f(q)}{\partial \phi} = \frac{1 - e^{-\rho L}(1-r) - f(q)e^{-\phi\rho L}[\rho L(1-q)+r]}{1-(1-q)e^{-\phi\rho L}}$$
$$f'_\rho(q,r) = \frac{\partial f(q)}{\partial \rho} = \frac{e^{-\rho L}[L(\phi - q)+r] - f(q)e^{-\phi\rho L}[\phi L(1-q)+r]}{1-(1-q)e^{-\phi\rho L}},$$

where I write r for $\frac{\partial q}{\partial \theta}$.

The expected information matrix for this DTMC can then be evaluated as

$$\mathcal{I}^P_E = K \sum_{q=0}^{\infty} \frac{\tilde{\pi}_q}{\pi_{q,q+1}(1-\pi_{q,q+1})} \frac{\partial \pi_{q,q+1}}{\partial \theta_i} \frac{\partial \pi_{q,q+1}}{\partial \theta_j}. \tag{9}$$

3 Augmented interval recording

The transition matrix corresponding to augmented interval recording with intervals of length L is given by

$$\mathbf{\Pi}^A = \begin{bmatrix} e^{-\phi\rho L} & 1 - e^{-\phi\rho L} - p_0(L) & p_0(L) & 0 \\ 0 & \frac{1-\phi}{\phi} p_0(L) & 1 - e^{-(1-\phi)\rho L} - \frac{1-\phi}{\phi} p_0(L) & e^{-(1-\phi)\rho L} \end{bmatrix}. \tag{10}$$

The equilibrium distribution of this DTMC is given by $\tilde{\pi}_0 = 1 - \phi$, $\tilde{\pi}_1 = \phi$. The first derivatives of the transition probabilities with respect to ϕ and ρ are given by

$$\frac{\partial \mathbf{\Pi}^A}{\partial \phi} = \begin{bmatrix} -\rho L e^{-\phi\rho L} & \rho L e^{-\phi\rho L} - (1 - e^{-\rho L}) & (1 - e^{-\rho L}) & 0 \\ 0 & -(1 - e^{-\rho L}) & 1 - e^{-\rho L} - \rho L e^{-(1-\phi)\rho L} & \rho L e^{-(1-\phi)\rho L} \end{bmatrix},$$
$$\frac{\partial \mathbf{\Pi}^A}{\partial \rho} = \begin{bmatrix} -\phi L e^{-\phi\rho L} & \phi L\left(e^{-\phi\rho L} - e^{-\rho L}\right) & \phi L e^{-\rho L} & 0 \\ 0 & (1-\phi) L e^{-\rho L} & (1-\phi) L\left(e^{-(1-\phi)\rho L} - e^{-\rho L}\right) & -(1-\phi) L e^{-(1-\phi)\rho L} \end{bmatrix}.$$

The expected information matrix corresponding to the momentary time sampling procedure can then be evaluated directly from (6).