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# Final Report – Option Pricing

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# Contents

## Abstract

In this paper, we introduce the idea of option contracts and derive the fair value market price for American and European-style options. We use the Black Scholes pricing model as a benchmark, and Monte Carlo simulation methods to derive the fair value price of European Options. We provide a comprehensive review of the Least Squares Monte Carlo approach and its advantages over the existing Black Scholes closed form solution. Overall, this paper contributes to the understanding of Least Squares Monte Carlo simulation and its ability to price financial derivatives, primarily American Options with the European component.


## Introduction/Problem Statement

The volatile nature of financial markets imposes various sources of uncertainty on an investor's portfolio. The options market provides a way for investors to combat those uncertainties, by hedging their portfolio against the risk of adverse price movements. Thus, pricing option contracts are an important daily activity for banks and other financial institutions. It is also common for retail investors like us to study options for either short term gains or to be a beneficiary from a company as another retirement strategy. Therefore, for this paper we will compare the findings from analytical pricing methods and from simulation techniques, to approximate the fair value of the European and American option price today.

## Background and Description

### History Of Options

The history of option contracts[1] can be traced back to 332 BC, when a man named Thales bought the rights to buy olives prior to a harvest. Thales went on to amass a fortune from the purchase of these rights. In 1636, options were widely bought to speculate on the soaring prices of tulips in Europe. At the time, tulips became a sign of high status and beauty, which fueled a craze of speculation from all levels in the society. Tulip prices would skyrocket, increasing the number of growers and dealers looking to get into the trade. Dealers began selling the rights to own tulips in advance of the harvest to buyers looking to secure a definite buying price. A year later, the price of tulips had gotten so high that no buyers were willing to pay the inflated price, resulting in a selling frenzy. The price of tulips had fallen faster than it rose; and almost all the speculators were wiped out as the price declined lower than the originally agreed upon price to buy. The Dutch economy would collapse shortly after the selling frenzy, giving options a notorious reputation for being a dangerous speculative instrument. In the early 1700s options surfaced in London with financiers acknowledging its speculative power through its inherent leverage. Groups opposed the speculative nature leading to the outlaw of options trading in 1733 before making a return in 1860.

In 1872 a well-known American financier Russel Sage was the first to create call and put options for trading in the U.S. The options that Sage created were unstandardized and highly illiquid, resulting in an inefficient market for participants. However, Sage still managed to make millions within a few short years, before losing a fortune in the stock market crash of 1884. These unregulated options continued to trade until the creation of the U.S. Securities and Exchange Commission (SEC) after the stock market crash of the 1930s. In 1973, the Chicago Board of Exchange (CBOE) and Options Clearing Corporation (OCC) were formed to standardize and allow retail investors to participate in the trade under the performance guarantee of the OCC, and liquidity of the market maker system. By 1977 put options were introduced to the CBOE, which created the options market that know and trade today.

## Definitions and Literature Review

Option contracts refer to an agreement between two parties to facilitate a potential transaction on an underlying security at a predefined strike price (K) prior to, or on a specified expiration date (t). A single stock option contract gives the option holder the right to 100 shares of the underlying security.

### Vanilla Options

A Vanilla option[8] is a simple call or put with no special features or observation dates. Call options give the option holder the right to buy 100 shares of an asset at the predefined strike price, prior to or on a defined expiration date. Put options give the option holder the right to sell 100 shares of an underlying asset, prior to or on a defined expiration date. If an investor is unsure of the future performance of a specific stock, and they have a strong belief that the price will rise in the future, the investor may choose to buy a call option on the stock. This allows the investor to gain exposure from a future price increase, while not having to commit the capital to purchase the hundred shares on the specific asset. The European is the simplest form of a vanilla option, giving the investor the right to buy or sell shares of an asset at a specified expiration date. American options differ from the European counterpart, in the fact that they can be exercised at any point until or on the expiration date. Vanilla options are some of the most traded option contracts due to their simplistic nature and lack of exotic characteristics.

### Exotic Options

Exotic options[8] are options that differ from Vanilla options in their payment structures, expiration dates and predefined strike prices. The underlying asset can vary in an exotic option, thus giving investors alternative opportunities. Exotic options can be customized to meet the risk tolerance and investment objective of the investor. Asian Options are an exotic option in that they are dependent on the average price of the underlying asset over a specified time. Bermuda options are also an example of an exotic option as they give the investor the right to buy or sell an asset at many preset expiration dates. Exotic options provide investors more control over when the option is exercised, but they often come with a higher initial cost when compared to a vanilla option contract.

### Time Value of Money

The Discount factor incorporates "time value of money" to calculate the options price more accurately. Time value of money represents the idea that value of money changes with time driven by factors such as inflation, changes in interest rates (although the model assumes it is constant), company-specific & country-specific premiums or unforeseen events such as a pandemic. Theoretically, it can be explained by the concepts of present value (PV) and future value (FV) where it is believed that the sum of money is worth more now than the sum will be at a future date due to its earning potential in the interim[2]. It can be expressed as below where $n$ is the number of compounding periods and $t$ is the number of years.

$$\boldsymbol{PV = \frac{FV}{(1+r)^{nt}}}$$

## Brownian Motion

Brownian Motion[3] is the random chaotic movement observed in particles as the result of collisions with other surrounding particles. The Standard Brownian Motion generates random points and is defined as the following:

$$\mathcal{W}\left(\frac{i}{n}\right) = \mathcal{W}\left(\frac{i-1}{n}\right) + \frac{Y_i}{\sqrt{n}}, \quad i, = 1, 2, \ldots$$

Some important properties of Standard Brownian motion are that it has stationary and independent increments between points. The Stationarity property assumes the change in Brownian motion between two time periods is due to the length of the increment ($h$) only.

$$\mathcal{W}(t+h) - \mathcal{W}(t)$$

The independence property states that the change in motion between two pairs of time periods have no relationship with each other. Specifically, if $a < b < c < d$, the independence property states:

$$\mathcal{W}(d) - \mathcal{W}(c) \perp \mathcal{W}(b) - \mathcal{W}(a)$$

## Using Geometric Brownian Motion to Model a Stock Price:

Geometric Brownian[3] Motion is commonly used to model stock prices. Geometric Brownian process can be shown as below where standard Brownian motion is denoted $\mathcal{W}_t$, μ is the drift, $\boldsymbol{\sigma}$ is the volatility, and $S_0$ is the current stock price:

$$S_t = S_0\, e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma \mathcal{W}_t}, \qquad t \geq 0$$

## Input data

We obtained historical data from Yahoo Finance[7] for three major indices to track overall movements in the stock market. The data was obtained over the past year, beginning 2022 and collected twice a day, at the open and close. The stocks chosen are the SPDR S&P 500 ETF (SPY), Invesco QQQ Trust (QQQ), and the iShares Russel 2000 (IWM). The options on these stocks are traded heavily by financial institutions and retail investors alike. The tickers we chose are regular leaders in stock option volume traded each day, so there is a very large interest in trading options on these assets.

# Main findings

## European Options

### Defining Black-Scholes:

The Black Scholes model was developed in 1973 by Robert Merton and Myron Scholes. The equation estimates the theoretical value of an option based on the risk-free rate of return, and underlying stock price at the time of expiration. The Black Scholes model is still considered one of the best methods to price an option contract today[1]. However, there are some major assumptions proposed by the model. The model assumes that stock prices follow the Geometric Brownian Motion defined above, no dividends are paid, a constant risk-free rate of return for all maturities, and that there are no costs or restraints associated with trade transactions. Black-Scholes derives the price of a European call option[3]:

$$C(S,t) = e^{-rt}E[(S_t - K)^+]$$

$$= e^{-rt}E\left[S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t} - K)^+\right]$$

$$= e^{-rt}\int_{\infty}^{\infty}\left[S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}\,z} - K)^+\right]\phi(z)\,dz$$

$$= S_0\Phi(b + \sigma\sqrt{t}) - Ke^{-rt}\Phi(b)$$

$$\text{where } b \equiv \frac{rt - \frac{\sigma^2 t}{2} - \ln\left(\frac{K}{S(0)}\right)}{\sigma t}$$

Where $\Phi$ is the CDF of the Normal Distribution. The final equation above can also be written as a closed form solution[4] below to help us understand the formula more intuitively. We let $d_1, d_2$:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}, \quad d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} = d1 - \sigma\sqrt{t}$$

$$C(S,t) = S_0\Phi(d_1) - Ke^{-rt}\Phi(d_2)$$

Where $-Ke^{-rt}\Phi(d_2)$ is the present value of the expected payout for a given strike price. $S_0\Phi(d_1)$ is the conditional expectation of the stock given that the price is higher than the strike price multiplied by the probability that the strike price is higher than the stock price for a call option. Similarly, the European put option can be expressed as follows:

$$P(S,t) = Ke^{-rt} - S_t + C(S_t, t)$$

## Monte Carlo Simulation

We can estimate the fair value of the European Style options through the means of Monte Carlo simulation. Monte Carlo simulation methods are mathematical techniques that predict possible outcomes of an uncertain event. The more outcomes we can predict, the closer we get to the true value of the mathematical function. When applying this to the Black Scholes model, we begin with the initial stock price, $S_{0,}$ to simulate $N$ possible stock price paths using the Geometric Brownian motion. Depending on the option type, we find the average difference of the strike price $K$, and the stock price $S_t$. Finally, the discount factor $e^{-rt}$ is applied to find the fair value of the price today. The Monte Carlo method is shown below for European call and put options below[3].

$$\text{European Call Option Price: } C_0 = e^{-rT}\frac{1}{N}\sum_{i}^{N} max(S_T - K, 0)$$

$$\text{European Put Option Price: } P_0 = e^{-rT}\frac{1}{N}\sum_{i}^{N} max(K - S_T, 0)$$

Using Monte Carlo method is a very simple and elegant way to estimate the fair value of an option price today, if you are only able to exercise said option by the defined expiration date.

## American options

### Least Squares Monte Carlo Method (LSMC)

We begin with the same simulation of the stock price as in the European case. However, since we are pricing an American option, we are not limited to one but any possible exercise points before the expiration of the option contract. The option holder of an American style option needs to continuously compare the immediate payoff of the option with its expected future payoff to determine if they exercise or to continue to hold the option. This conditional expectation function can be approximated using cross-sectional information as an output of least squares regression.

The least squares Monte Carlo method begins with obtaining stock price paths from the Geometric Brownian Motion like the European case, however for the least squares approach, we only consider the paths where the option contract is in the money. The algorithm iterates over each exercise point recursively beginning with the time of expiration and determines the payoff for each path.

$$\text{Payoff for Call Option } = max(S_t - k, 0)$$

$$\text{Payoff for Put Option } = max(K - S_t, 0)$$

Obtaining the payoff for each in the money path helps improve the efficiency of the program and allows us to better estimate the region where the payoff is relevant. We proceed by applying functions of the given state variables to approximate the continuation value. The continuation is the value of an option contract at the next time step. State variables are the stock price at the current iteration for each of the simulated in the money price paths obtained in the first step.

The functions we apply to the given state variables are referred to as Basis functions. Basis functions are typically used in machine learning for approximating complex functions by generalizing a simple linear regression function. Schwartz and Longstaff convey that no more than 3 basis functions are needed to achieve the true value of the American option[5]. A common basis function that can be used is the simple polynomial function: let $X$ be the given state variable (stock price of the current iteration), and $Y$ be the continuation value we wish to predict. Then the Conditional Expectation function for the polynomial Basis function is:

$$E[Y|X] = \beta_0 + \beta_1(X) + \beta_2(X^2) + \beta_3(X^3)$$

where the coefficients $\beta_i$'s are calculated from traditional regression techniques

We repeat this process for each of the paths, discounting the expected cashflows one period at every exercise point until we get back to the current time. Finally, we discount the payoff one more time to get the net present value and obtain the average of all cashflows to estimate the current fair market value of the American option. We will be comparing the use of the polynomial function, the lognormal function, and the use of the control variate as basis functions in our simulation study. The control variate variance reduction for estimating the mean is as follows with an approximation for the parameter β.

$$C = \bar{X} - \beta(Y - E[Y])$$

$$\beta = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

The least squares method could easily be extended to price the exotic option Bermudan-style option as they have predefined exercise points. As the number of specified exercise points increase, we can get closer to the estimate of an American style option. However, this does mean that more computational power is used to complete the algorithm.

## Fundamental results

We applied the Least Squares Monte Carlo approach for American-style options, and the Monte Carlo approach for European-style options beginning with 1000 independent random stock price paths. We then completed nine independent replication runs, each time with different initial seeds of the random number generator for variance reduction. We obtained the standard errors from the simulation output as shown here.

**Table 1: Standard Error from simulation output**

| Option Contract | LSMC Normal | LSMC Poly | LSMC CV | MC |
|---|---|---|---|---|
| IWM $185 Call | 0.06 | 0.08 | 0.09 | 0.08 |
| IWM $185 Put | 0.03 | 0.05 | 0.06 | 0.06 |
| QQQ $307 Call | 0.02 | 0.04 | 0.04 | 0.04 |
| QQQ $305 Put | 0.04 | 0.09 | 0.11 | 0.1 |
| SPY $398 Call | 0.07 | 0.13 | 0.15 | 0.14 |
| SPY $398 Put | 0.04 | 0.08 | 0.09 | 0.09 |

The standard errors for the simulated values range from 0.02 to 0.15, which are well within the bid-ask spread for these options in normal market conditions. Below we find the error of the fair value approximation done by our simulation methods, compared to the true observed option price for the two days before expiration. We note that the simulated errors seem to all be within the same general range. The true approximations can be found in the appendix. Overall, the simulation methods seemed to be well within the standard error reported above.

**Table 2: Error vs Actual American Option Price (2 days to expiration)**

| type | T | Ticker | K | S | σ | LSMC Normal | LSMC Poly | LSMC CV | MC | BS |
|---|---|---|---|---|---|---|---|---|---|---|
| Call | 2 | iwm | 185.0 | 173.06 | 0.39 | 0.0 | 0.01 | 0.01 | 0.01 | 0.0 |
| Call | 1 | iwm | 185.0 | 175.96 | 0.34 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 |
| Put | 2 | iwm | 185.0 | 173.06 | 0.0 | 0.32 | 0.32 | 0.29 | 0.29 | 0.29 |
| Put | 1 | iwm | 185.0 | 175.96 | 0.53 | 0.08 | 0.19 | 0.16 | 0.09 | 0.09 |
| Call | 2 | qqq | 304.0 | 297.37 | 0.28 | 0.17 | 0.06 | 0.05 | 0.02 | 0.03 |
| Call | 1 | qqq | 304.0 | 305.34 | 0.23 | 0.26 | 0.26 | 0.28 | 0.22 | 0.26 |
| Put | 2 | qqq | 305.0 | 297.37 | 0.0 | 0.51 | 0.51 | 0.47 | 0.47 | 0.47 |
| Put | 1 | qqq | 305.0 | 305.34 | 0.23 | 0.32 | 0.29 | 0.32 | 0.28 | 0.32 |
| Call | 2 | spy | 398.0 | 388.65 | 0.21 | 0.05 | 0.02 | 0.0 | 0.03 | 0.01 |
| Call | 1 | spy | 398.0 | 394.67 | 0.13 | 0.14 | 0.13 | 0.13 | 0.12 | 0.13 |
| Put | 2 | spy | 398.0 | 388.65 | 0.32 | 0.62 | 0.01 | 0.06 | 0.05 | 0.03 |
| Put | 1 | spy | 398.0 | 394.67 | 0.32 | 0.5 | 0.46 | 0.44 | 0.43 | 0.46 |

**Table 3: Early Exercise component**

| Ticker | type | T | K | S | σ | LSMC Normal | LSMC Poly | LSMC CV | MC |
|---|---|---|---|---|---|---|---|---|---|
| iwm | Call | 2 | 185.0 | 173.06 | 0.39 | 0.0 | 0.01 | 0.0 | 0.01 |
| iwm | Call | 1 | 185.0 | 175.96 | 0.34 | 0.0 | 0.0 | 0.0 | 0.0 |
| iwm | Put | 2 | 185.0 | 173.06 | 0.0 | 0.03 | 0.03 | 0.0 | 0.0 |
| iwm | Put | 1 | 185.0 | 175.96 | 0.53 | 0.0 | 0.11 | 0.07 | 0.01 |
| qqq | Call | 2 | 304.0 | 297.37 | 0.28 | 0.15 | 0.03 | 0.03 | 0.0 |
| qqq | Call | 1 | 304.0 | 305.34 | 0.23 | 0.01 | 0.01 | 0.03 | 0.04 |
| qqq | Put | 2 | 305.0 | 297.37 | 0.0 | 0.04 | 0.04 | 0.0 | 0.0 |
| qqq | Put | 1 | 305.0 | 305.34 | 0.23 | 0.01 | 0.02 | 0.01 | 0.03 |
| spy | Call | 2 | 398.0 | 388.65 | 0.21 | 0.04 | 0.01 | 0.01 | 0.02 |
| spy | Call | 1 | 398.0 | 394.67 | 0.13 | 0.01 | 0.01 | 0.01 | 0.01 |
| spy | Put | 2 | 398.0 | 388.65 | 0.32 | 0.65 | 0.04 | 0.03 | 0.02 |
| spy | Put | 1 | 398.0 | 394.67 | 0.32 | 0.04 | 0.0 | 0.02 | 0.03 |

The European component of an option price comes from the Black-Scholes Model and is the price of the option at expiration. The early exercise component of an American option is the difference between the

European price and the simulated American option price we have obtained from the methods above. The early exercise value is the current value of the option compared to if you held it to expiration. Table 3 shows the early exercise prices across for short-term contracts.

## Conclusion & Further Research Opportunities

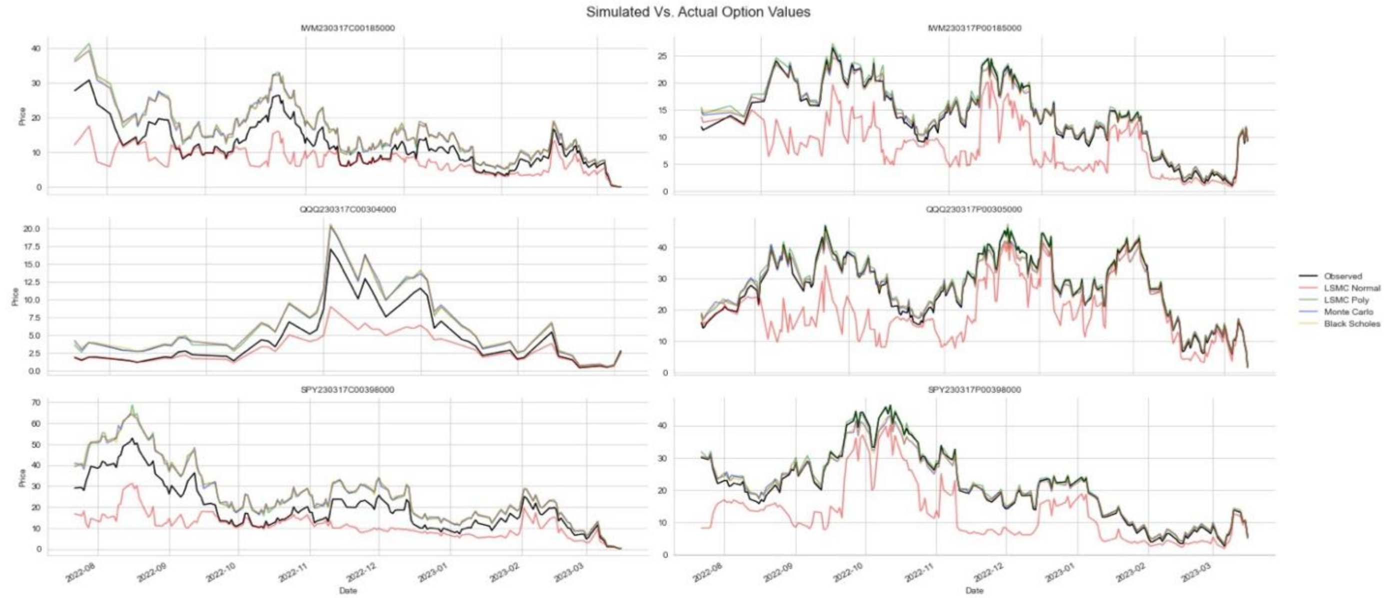
In conclusion, we can see that the LSMC method is able to estimate the fair value of an American option reasonably well. The LSMC method can also be used with a wide variety of Basis functions making it versatile and easy to test. However, one should choose a basis function that does not allow high correlations as this can lead to convergence failures. Further, the LSMC method can be extended to various types of options such as equity, commodities, mortgage, and swaps. We have found these methods to show better results with a shorter time until expiration. The simulation for this paper took about 30 minutes to run. In the future, one could attempt to vectorize the LSMC process above to increase the computation speed of the algorithm.

Other topics could include implementing the LSMC method with an underlying asset that follows a jump diffusion process or using out of sample regression coefficients for the conditional expectation of the continuation value. We could also consider simulating risk measurements such as The Greeks. As the name alludes, there are 5 common KPIs used to monitor option movements as market conditions adjust. There are 4 first order derivatives Delta, Vega, Theta, and Rho second order derivative Gamma. Considered the most important Greek, Delta helps gauge the likelihood of the options expiring in the money (profitable) or out of the money. Vega measures how sensitive an option is to large swings of the stock price. Theta is the measure of the value lost each day as the contract approaches T. Rho may be considered the least used Greek, but it is used to measure the effect of interest rates on your options. Lastly, Gamma as a second order derivative measures the change in Delta[6]. In practice these KPIs can be simulated to be used in more complex models for pricing options.

Ultimately, options are a very complex idea and there are many ways we could have enhanced our simulation to make more diverse conclusions. The least squares Monte Carlo method has proven to be intuitive and very flexible with its applications.

## Appendix

| Observed and Simulated Fair Value Option Price. (2 Days to expiration) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| type | T | Ticker | K | S | σ | Observed | BS | LSMC Normal | LSMC Poly | LSMC CV | MC |
| Call | 2 | iwm | 185.0 | 173.06 | 0.39 | 0.06 | 0.06 | 0.06 | 0.05 | 0.06 | 0.06 |
| Call | 1 | iwm | 185.0 | 175.96 | 0.34 | 0.04 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| Put | 2 | iwm | 185.0 | 173.06 | 0.0 | 11.6 | 11.89 | 11.92 | 11.92 | 11.89 | 11.89 |
| Put | 1 | iwm | 185.0 | 175.96 | 0.53 | 9.28 | 9.19 | 9.2 | 9.09 | 9.12 | 9.19 |
| Call | 2 | qqq | 304.0 | 297.37 | 0.28 | 0.79 | 0.76 | 0.62 | 0.73 | 0.74 | 0.77 |
| Call | 1 | qqq | 304.0 | 305.34 | 0.23 | 2.82 | 2.56 | 2.56 | 2.56 | 2.54 | 2.6 |
| Put | 2 | qqq | 305.0 | 297.37 | 0.0 | 7.08 | 7.55 | 7.59 | 7.59 | 7.55 | 7.55 |
| Put | 1 | qqq | 305.0 | 305.34 | 0.23 | 1.9 | 1.59 | 1.58 | 1.61 | 1.58 | 1.62 |
| Call | 2 | spy | 398.0 | 388.65 | 0.21 | 0.36 | 0.36 | 0.32 | 0.35 | 0.36 | 0.34 |
| Call | 1 | spy | 398.0 | 394.67 | 0.13 | 0.4 | 0.27 | 0.26 | 0.28 | 0.28 | 0.28 |
| Put | 2 | spy | 398.0 | 388.65 | 0.32 | 10.53 | 10.56 | 9.91 | 10.52 | 10.59 | 10.58 |
| Put | 1 | spy | 398.0 | 394.67 | 0.32 | 5.54 | 5.08 | 5.04 | 5.08 | 5.1 | 5.11 |


Simulated Vs. Actual Option Values
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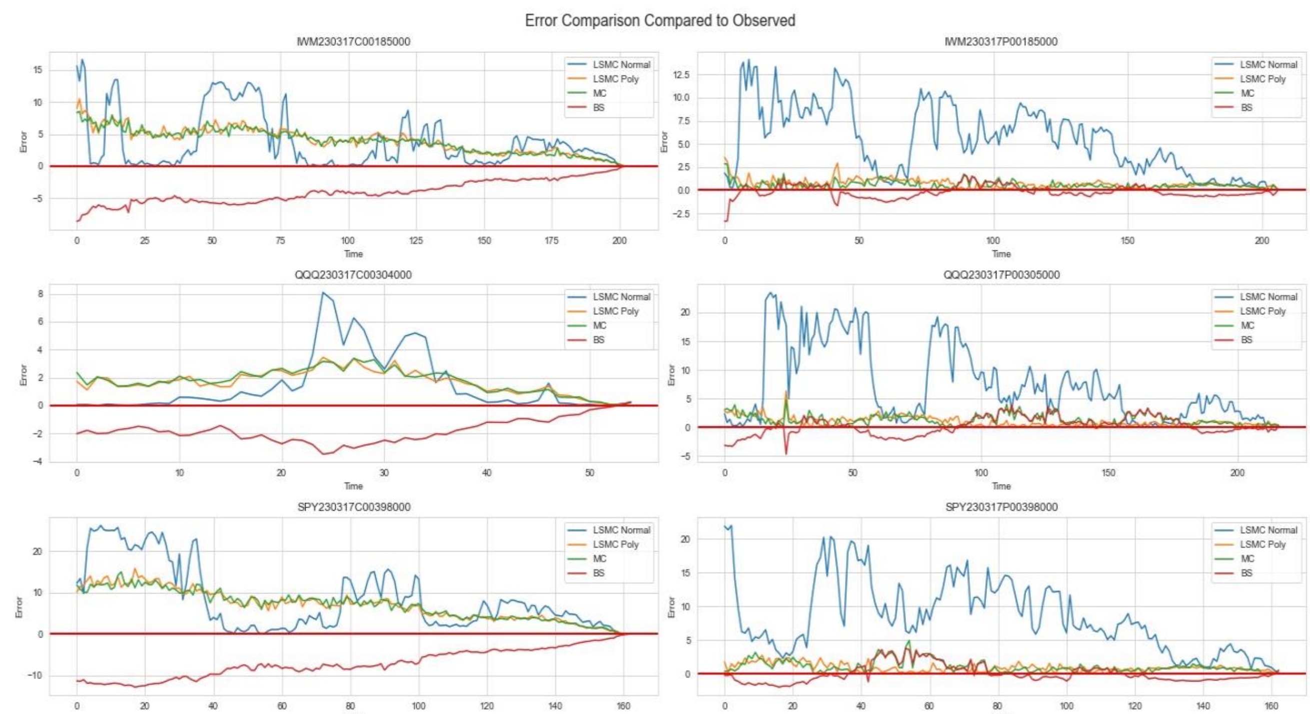
IWM230317P00185000
QQQ230317C00304000
QQQ230317P00305000
SPY230317C00398000
SPY230317P00398000
Price
Date
Observed
LSMC Normal
LSMC Poly
Monte Carlo
Black Scholes


Error Comparison Compared to Observed
IWM230317C00185000
IWM230317P00185000
QQQ230317C00304000
QQQ230317P00305000
SPY230317C00398000
SPY230317P00398000
Error
Time
LSMC Normal
LSMC Poly
MC
BS

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