We obtained historical data from Yahoo Finance for three major indices to track overall movements in the stock market. The data was obtained over the past year, beginning 2022 and collected twice a day, at the open and close. The stocks chosen are the SPDR S&P 500 ETF (SPY), Invesco QQQ Trust (QQQ), and the iShares Russel 2000 (IWM). The options on these stocks are traded heavily by financial institutions and retail investors alike. The tickers we chose are regular leaders in stock option volume traded each day, so there is a very large interest in trading options on these assets. The research we are doing will be highly valued if seen by the public to get a better gauge on how options price behave in the real world setting.

**Payoff**: For a Put Option Stock - Strike, for a Call Option Strike - Stock; This is the value of the option contract.

**Continuation** **Value**: The value of the option contract at the next time step, ie. You choose to continue to hold the option.

**In the money**: For a Put Option, the stock has fallen below the chosen strike price. For a Call Option, the stock has risen above the chosen strike price.

**Exercising an option:** Exercising the right to buy or sell shares of an option contract.

The LMSC Methods attempts to approximate the immediate payoff of an option given the continuation value.

For American Options on the day of expiration, it is optimal to exercise the contract if you are in the money. However, prior to the expiration date the option holder needs to compare the current payoff with the expected payoff if they decide to hold the contract. If the immediate value of the option is higher than the expected continuation value; the investor will choose to exercise the contract early to avoid future losses. If the immediate value of the option is lower than the expected continuation value, the investor will choose to hold onto the option contract, in hopes of realizing future gains on the option contract. Thus, it is beneficial to the option holder to be able to approximate the expected cash flows from holding on to the option rather than exercising early.

The Least Squares Monte Carlo approach uses cross-sectional information from simulated stock price paths using Geometric Brownian Motion to identify the conditional expectation function using least squares regression. We regress the payoffs from continuation at times

$t+1, t+2, ..$ on basis functions of the state variable at time $t$. The fitted value from the regression is an unbiased estimate of the conditional expectation function and allows us to accurately estimate the optimal stopping rule for that option for each simulated path.

The algorithm is recursive and works backwards from the expiration date to get the current value given the continuation value of the option.

The Least squares Monte Carlo approach works backwards from the time of

For example, say we have put option contract that we can exercise at any time $t = \{1, 2, 3\}$

The payoff at time $t=3$ is $max(0, K - S\_{t=3})$

The payoff at time $t=2$ is $max(0, K - S\_{t=2}) | max(0, K - S\_{t=3})$

Let $X$ denote the stock prices at time $t=2$ for the paths that we have simulated and decided to exercise on day $t=3$.

Let $Y$ denote the corresponding dicounted payoff recieved at time $t=3$ if the put option is not exercised at time $t=2$.

We only use in the money paths since it allows us to better estimate the conditional expectation function in the region where exercise is relevant.

To get the immediate value of the option at time $t=2$, we regress $Y$ on a constant $X$ and $X^2$ using least squares regression, to obtain the conditional expectation function $E(Y|X) = \beta\_0 + \beta\_1 X + \beta\_2 X^2$. Where $\beta\_0, \beta\_1, \beta\_2$ are the coefficients of the regression model. We compare the fitted value from the function to the payoff at time $t = 3$.

If the immediate payoff is greater than the continuation value, we exercise the option. If the continuation value is greater than the immediate payoff, we continue to hold the option. If stock price is situated so that the option is out of the money, the option is worthless and thus the payoff is zero. If we choose to exercise before the expiration date, then all future payoffs for that path are also zero. So the realized payoff will be a matrix of zeros and discounted payoffs.

Proceeding backwards we examine if the option should be exercised at time $t=1$. Let $Y$ denote the dicounted value of future realized payoff along each path. We use the actual realized payoff along each path at time $t=2$ to regress on the fitted value of the conditional expectation function at time $t=1$, not the the conditional expected value of $Y$ at time $t=2$. As stated in the paper, this leads to an upward bias in the value of the option.

Finally after obtaining the cashflows from the regression for each path, we discount the cashflows to the time $t = 0$ and taking the average of all paths, to obtain the present value of the option contract. This is the estimated value of the option contract, and can be thought of as the sample mean of the simulation study.

So in total there are $K = 3$ times where the option can be exercised. If we increase the number of time steps $K$ to be sufficiently large we can estiamte the value of the option using many exercise points, which resemeble the continuous exercise feature of the american option. $0 < t\_1 ≤ t\_2 ≤ t\_3 ≤ ... ≤ t\_K ≤ T$.

Since we use the sample mean of the cashflows as the estimator of the expected value of the option. We run $r$ independent simulations to get the sample mean $Z\_i$ of the cashflows. The sample mean is an unbiased estimator of the conditional expectation of the option price. Thus, we can estimate the variance of $Z\_i$ by taking the sample variance of all $Z\_i$'s

$$\hat{\sigma}^2 = \frac{1}{r}\sum\_{i=1}^r (Z\_i - \bar{Z})^2$$

where $\bar{Z}$ is the sample mean of all $Z\_i$'s.

The nice part about this method is that it is easy to implement, and it is very flexible. We can use various kinds of basis functions to approximate the conditional expectation function. Another nice feature is that it is easy to extend to other types of options, such as Asian options, barrier options, and lookback options.

If we use the method of independent replications we obtain the following results: