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# Abstract

Maybe use the first paragraph below and restructure to including findings. Then start background straight with History.

# Introduction/Problem Statement

The volatile nature of financial markets imposes various sources of uncertainty on an investor's portfolio. The options market provides a way for investors to combat those uncertainties, by hedging their portfolio against the risk of adverse price movements. Thus, pricing option contracts are an important daily activity for banks and other financial institutions. It is also common for retail investors like ourselves to study options for either short term short term gains or to be a beneficiary from a company as another retirement strategy. Therefore, for this paper we will compare the findings from analytical pricing methods and from simulation techniques, to approximate the fair value of the European and American option price today.

# Background and Description

## History Of Options

Before getting into terminologies and definitions, we want to start with a brief history of how options came to be.

The history of Option contracts can be traced back to 332 BC, when Thales bought the rights to buy olives prior to a harvest, who went on to amass a fortune from the purchase of these rights.

In 1636 options were widely bought to speculate on the soaring prices of tulips in Europe. At the time tulips became a sign of high status, and beauty, which fueled a craze of speculation from all levels in society. The wild speculation caused tulip prices to skyrocket, which also increased the number of growers and dealers looking to get into the trade. Dealers began selling the rights to own tulips in advance of the harvest, for buyers looking to secure a definite buying price. A year later, the price of tulips had gotten so high that no buyers were willing to pay the inflated price, resulting in a selling frenzy. The price of tulips had fallen faster than it rose; and almost all the speculators were wiped out as the price declined lower than the originally agreed upon price to buy. The Dutch economy would collapse after the sharp decline, leading to the notorious reputation for options being a dangerous speculative instrument.

Although options gained a bad reputation, they later surfaced in London in the early 1700’s with financiers acknowledging its speculative power through its inherent leverage. It was later declared illegal in 1733, as groups opposed the speculative nature of options trading. It was outlawed for about 100 years, making a return in 1860 almost 100 years later.

In 1872 Russel Sage, a well-known American financier from New York; was the first to create calls and puts for trading in the US. Sage created options that were unstandardized and highly illiquid, resulting in an inefficient market for participants. However, Sage still managed to make millions within a few short years, before losing a fortune in the stock market crash of 1884. These unregulated Options continued to trade until the creation of the SEC after the stock market crash of the 1930’s. In 1973 The Chicago Board of Exchange and Options Clearing Corporation were formed to standardize and allow retail investors to participate in the trade under the performance guarantee of the OCC, and liquidity of the market maker system. By 1977 put options were introduced to the CBOE, which created the options market that we can trade today.

## Definitions

Some definitions we will continuously use throughout the paper.

Option Contract: an agreement between two parties to facilitate a potential transaction involving an asset at a preset price and date.

Call:

Put:

Payoff: For a Put Option Stock - Strike, for a Call Option Strike - Stock; This is the value of the option contract.

Continuation Value: The value of the option contract at the next time step, ie. You choose to continue to hold the option.

In the money: For a Put Option, the stock has fallen below the chosen strike price. For a Call Option, the stock has risen above the chosen strike price.

Volatility tells us how likely the price deviates from true mean price in a given period. Stocks with higher volatility are more likely to have a very high or very low value at maturity which positively impacts both put and call options. Although stocks with higher volatility result in higher returns, it makes the predicting returns challenging which is not preferred by investors while exercising call options.

## Literature Review

Option contracts refer to an agreement between two parties to facilitate a potential transaction on an underlying security at a predefined strike price prior to, or on a specified expiration date. A single stock option contract gives the option holder the right to 100 shares of the underlying security. Call options give the option holder the right to buy 100 shares of an asset at the predefined strike price, prior to or on a defined expiration date. Put Options give the option holder the right to sell 100 shares of an underlying asset, prior to or on a defined expiration date.

### Vanilla Options

To extend our understanding of options, it is good to know that options fall into a couple of different types of classifications. Two basic classifications we can start with are Vanilla options and Exotic options. As we alluded to earlier, we are going to focus on the two most common Vanilla options, European and American. Both types of options factor in the stock price at the specified time of expiration of the contract. Even though the American option is priced at the point of expiration, it does provide the opportunity to sell earlier for an even larger potential payoff than what is projected at expiration unlike the European option that only allows to exercise at the expiration time.

Because the European option is simpler, there is a closed form analytical solution to price those options. This model is called Black-Scholes and is very common for market makers. However, one can also use Monte Carlo simulation can be chosen to price out contracts. Because American options are more complex, there is no closed form analytical solution, they must be derived using many different methodologies. A couple of examples are using a partial differential equation (Black-Sholes) or simulation has to be chosen, like Monte Carlo (1). If the Black-Scholes PDE is chosen, it has to be combined with a finite difference model to. Define or give example of finite d model

### Exotic Options

One of the most common option falling into the exotic realm are Asian Options. Fundamentally they are different than the Vanilla options because it depends on the average price of the stock over a certain time period whereas European and American options are a specific point in time. How do they get priced ie simulation or closed form?

### The Greeks

Regardless of the options investors choose, traders typically desire KPIs to evaluate or predict how the option might change as the market adjusts day to day. The KPIs or risk measurements are known as The Greeks. The Delta is commonly referred to as the most important Greek. This helps gauge the likelihood of the options expiring in the money (profitable) or out of the money.

* + Measure and monitoring items
    - The Greeks ([URL to bullets below](https://www.simplertrading.com/blog/getting-started/greek-alphabet-in-trading?utm_source=gapm-ads&utm_medium=paid_dynamic&utm_campaign=st_homepage-rt_traffic_gapm&coupon-code=gapm-ads-hp&=u_ag=adgroup&utm_content=&utm_term=&u_cid=18635413390&u_agid=&u_loc=9005495&u_mt=&u_net=x&u_plmt=&gclid=Cj0KCQjw27mhBhC9ARIsAIFsETHfhwtO-B1xLjpncKZ-dswwuiwzH-pqM8SixZwX-43KaYgdERbH_dgaAiETEALw_wcB)) are risk measurements that trader use to evaluate what the option might do if the market adjusts
      * Delta – Considered the most important Greek measurement, can help gauge the likelihood an option will expire in the money (ITM) or out of the money (OTM). This means the strike price is below (for calls) or above (for puts) the underlying stock’s market price.
      * Theta – measures how much value an option contract will lose each day the contract approaches its expiration date.
      * Vega – measures how sensitive an option is in relation to large price swings in the underlying stock or asset.
      * Gamma – measures the rate of change in delta in relation to movement in the underlying stock or asset.
      * Rho – While not a primary formula, Rho measures the effect of interest rate changes on an option. As Rho does not significantly impact options trades, we don’t use this one often, but it still deserves mention.

## Input data

We obtained historical data from Yahoo Finance for three major indices to track overall movements in the stock market. The data was obtained over the past year, beginning 2022 and collected twice a day, at the open and close. The stocks chosen are the SPDR S&P 500 ETF (SPY), Invesco QQQ Trust (QQQ), and the iShares Russel 2000 (IWM). The options on these stocks are traded heavily by financial institutions and retail investors alike. The tickers we chose are regular leaders in stock option volume traded each day, so there is a very large interest in trading options on these assets.

# Main findings

|  |  |
| --- | --- |
| ***S*** | *current stock price* |
| ***K*** | *strike price* |
| ***r*** | *risk free rate* |
| ***σ*** | *volatility* |
| ***t*** | *time to expiration* |
| ***µ*** | *expected return* |
| ***N*** | *number of simulations* |

## Brownian Motion

Standard Brownian motion W(t) developed by Brown and modified by Einstein and Winer, is a stochastic process which starts at time 0. It is a continuous process with normal distribution with mean 0 at any particular time (t). W(t) has stationary independent increments which assumes that the change in Brownian motion between two time periods is due to the length of the increment only and the change in motion between two pair of time periods have no relationship with each other (i.e., independent).

Graphical user interface, chart

Description automatically generated

Brownian motion starts at 0 and can be negative as well.

### Algorithm of Brownian Motion:

Assume Y1, Y2…Yi’s are normal random variables with mean µ (0 in this example), variance *σ2*(1 in this example), t is points of time, and n is sample size. Using Donsker’s Central Limit Theorem, Brownian motion W(t) can be defined as the sum of random variables divided by square root of n. The number of random variables to add is dictated by n times t which changes for various values of t. As t changes, it converges to a whole stochastic process “Brownian motion” as n gets big.

Using the CLT theorem above, constructing a simple Brownian motion random walk with mean 0 and variance 1 using steps below:

1. Take a large value of n (at least 100)
2. Take random variables Yi each with probability ½
3. Time (t) = 1/n, 2/n, 3/n,…n/n
4. Starting with W(0)=0, calculate W(1/n), W(2/n),….,W(n/n) by summing each prior Brownian motion with the next Yi divided by square root of n
5. Where W(i-1/n) is the prior Brownian motion

### Using Geometric Brownian Motion to estimate the fair value of the option:

***Simple definition of expected profit or fair value:***

*E[Profit] =**cash inflow of selling – cash outflow at exercising option – price of options contract*

Geometric Brownian Motion, specifically, is commonly used in the world of Finance to estimate option prices. The equation of Brownian Motion is used to price stock at time (t).

* S(0) means that it is estimating the price to buy at time 0 (now or today).
* (µ – *σ2*)/2 represents the drift factor with µ is the drift rate & σ2/2 is the volatility penalty (positive trend or increased drift is preferred)
* W(t) is the Brownian motion (source of randomness)

Example of Geometric Brownian Motion to estimate option price using equation of European call option:

where S(t) – K is the expected value after exercising the option at time (t) (should be positive to break-even or make profit). exp-rt is the discount factor or time value of money element that is present-valuing the expected profit. In other words, it is risk-adjusting the future returns to make it worth the present time value.

## Simulation Methods

### To estimate the fair values, run multiple simulations:

Take average of the differences between all stock price paths (St) from the simulation and the strike price (K) which is the predetermined price of the stock. It is then multiplied with the discount factor (e-rt) to find the fair value of the price today. The fair value of the price is the minimum acceptable stock price for the contract to break even or make profit when exercised. Stock price - Strike price is also known as option premium, which is the price that you pay for the option. If r is our continuously compounded interest rate for a period of t years, then the discount factor will be e-rt. Discount factor incorporates “time value of money” to calculate the options price more accurately. Time value of money represents the idea that value of money changes with time driven by factors such as inflation, changes in interest rates (although the model assumes it is constant), company-specific & country-specific premiums, unforeseen events such as pandemic, among others. Theoretically, it can be explained by the concepts of present value (PV) and future value (FV) where it is believed that the sum of money is worth more now than the sum will be at a future date due to its earning potential in the interim (Investopedia):

Compare the simulation results with the actual observed prices and results from Black-Scholes which is the analytical closed form of pricing option.

### Defining Black-Scholes European Call Option Value:

Let (.) denote the usual Nor (0,1) p.d.f and c.d.f.

Constant b is defined as = (interest rate times T minus sigma squares over 2 times t minus log of strike price over stock price at time 0)/sigma times square root of T

Black-Scholes starts with replacing the value of S(t) from Geometric Brownian motion equation from above in the expected value section of the equation of European call option below. It is then multiplied with phi of z and integrated over . The Brownian motion becomes as it is Nor (0,1).

(after lots of algebra)

The final equation above can also be written as a ‘closed form’ solution below to help us understand the formula more intuitively (<https://richnewman.wordpress.com/2007/07/01/a-beginner%e2%80%99s-guide-to-the-black-scholes-option-pricing-formula-part-3/> ). We will show it for a European call option, c, as an example:

We pay strike price (K) if the underlying stock price (S) is above the strike price at maturity, thus the probability of that can be defined as KN(d2) where N is the cumulative normal function (a statistical operator) and d1 and d2 are expressed as below. To get the value of this cash flow today, we discount it by factor exp-rt as explained above. Thus, the value of cash to buy option today is KN(d2) *exp-rt.*

By the same logic, SN(d1) is the value of the stock received once the option is bought whose expected value is proportional to S, stock price today, and can be written as SN(d1).

And

### European Options:

* + - European – black-Scholes
      * Can also use the same Monte Carlo as American options, but use different T values. I found it to be more clearly explained in (<https://www.codearmo.com/blog/pricing-options-monte-carlo-simulation-python>)

### American optIons

* + - American – find a simplified version
      * Pose a question about accounting for the ability of sell early. Then paraphrase the paper that Jerald sent. https://people.math.ethz.ch/~hjfurrer/teaching/LongstaffSchwartzAmericanOptionsLeastSquareMonteCarlo.pdf

One of the methodologies we have employed for simulating American options is the concept of Least Squared Monte Carlo (LSM). The major factor for consideration of the American option is that the contract holder has the option to sell said contract early if they believe it to be more profitable than at expiration. This is important to understand and what the LSM takes into account. In this methodology, we are starting with basic simulation and use Brownian motion to derive all the different paths a stock can take in the future. There in then an iterative process of keeping track of decisions on whether to sell at the current time, or to keep until expiration. The is done by taking only the paths in the money at time t, and using a regression model to calculate the projected payoff would be at the next discounted time increment t+1 and compare to the current payoff at time t. This loops backwards in time and tracks when it would have been earliest optimal to sell for each path. Then based on those optimal time, the paths are discounted to Net Present Value and averaged to get the average option price for the specific contract. For the regression models, we have used a normal distribution model as well as a polynomial model to evaluate the paths future payoff.

### Volatility

* Variance Reduction from latest lesson by using same randomly variables, etc*.*

## Fundamental results

* *Describe some of the* ***fundamental results*** *(e.g., this paper gives an explicit expression for an estimator’s expected value in terms of the covariance function).*
  + *Compare analytical way vs. simulation*

The simulation methods seemed to perform better for the Put contracts than the calls with the data we have provided. The Black Scholes model seemed to give a close approximate to the observed price of the option; however as seen below, the LSMC method seems to be more accurate. The errors for the Put contracts (Right) are much closer to 0 than compared to the Calls (Left). We can note that the LSMC with a Normal Basis function performed the worst for all contracts, having a very high variability in its fair value approximation. We note that the LSMC method can estimate the value of the option contract very well especially with a shorter time until expiration. Simulations methods proved to be more accurate with less days until the expiration, with the LSMC Normal Basis Function performing the worst across all time horizons. We can see that the Simulation methods give a noticeably lower standard error for short term contracts than longer term, with simulated results differing within 1 cent of the average simulated fair value.

Since we can get the European component of an option price from the Black-Scholes Model, we already know the price of the option at expiration. The early exercise component of an American option is the difference between the European price and the simulated American option price we have obtained from the methods above. The early exercise value can be thought of the excess value of the option compared to if you held it to expiration. So, if we notice that the early exercise is high, we might want to exercise the option early, to get the most value out of the option, and vice versa if the early exercise value is negative. Thus, the table above shows the difference between the simulated methods and the Black Scholes method, and it clearly shows that the LSMC Normal Basis Function is the worst at estimating the early exercise value. The Polynomial Basis Function performed the best, and the finite difference simulation had similar results to the polynomial basis function. The difference between the Black Scholes method and the simulated methods is well within the bid-ask spread for these option contracts, which suggest that the simulated methods can approximate the early exercise value of the option contracts very well.

## Opportunities

* *Formulate (but you don’t have to solve)* ***a couple of research problems*** *that might be derived from this work.  (Jerald added some in the conclusion*

# Conclusion

* *What did you find/learn from the project?* 
  + *Challenges encountered*
* *Any ideas for future work, if appropriate*
* Determining accuracy of the simulation by using Stand Error

In conclusion, we can see that the LSMC method is able to estimate the fair value of an American option reasonably well and can be extended to various types of options such as equity, commodities, mortgage, and swaps. We have found these methods to show better results with a shorter time until expiration, with the polynomial basis function preforming the best. Choosing a strong basis function should be a priority when using the LSMC method. In future research one should give care in choosing a basis function, to ensure that the choice of basis functions is not highly correlated with one another. Another interesting direction to explore would be to implement the LSMC method with an underlying asset that follows a jump diffusion process.

# Appendix

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|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Actual Vs. Simulated Option Price Errors | | | | | | | | | |
| Term | 90 Days | | | 60 Days | | | 10 Days | | |
|  | BS error | LSMC Poly error | MC error | BS error | LSMC Poly error | MC error | BS error | LSMC Poly error | MC error |
| Option |  |  |  |  |  |  |  |  |  |
| IWM $185 Call | 1.615 | 1.642 | 1.572 | 0.813 | 0.847 | 0.829 | 0.211 | 0.214 | 0.207 |
| IWM $185 Put | 0.430 | 0.444 | 0.427 | 0.379 | 0.423 | 0.361 | 0.231 | 0.337 | 0.199 |
| QQQ $307 Call | 1.794 | 1.685 | 1.758 | 1.047 | 1.000 | 1.036 | 0.312 | 0.301 | 0.295 |
| QQQ $305 Put | 1.048 | 0.638 | 1.078 | 0.335 | 0.600 | 0.385 | 0.378 | 0.522 | 0.447 |
| SPY $398 Call | 3.155 | 3.150 | 3.155 | 1.455 | 1.467 | 1.473 | 0.425 | 0.422 | 0.396 |
| SPY $398 Put | 0.644 | 0.680 | 0.674 | 0.573 | 0.687 | 0.553 | 0.304 | 0.421 | 0.265 |

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Standard Errors From Independent Replications | | | | | | | | | |
| Term | 90 Days | | | 60 Days | | | 10 Days | | |
|  | LSMC Normal SE | LSMC Poly SE | MC SE | LSMC Normal SE | LSMC Poly SE | MC SE | LSMC Normal SE | LSMC Poly SE | MC SE |
| Option |  |  |  |  |  |  |  |  |  |
| IWM $185 Call | 0.0686 | 0.1333 | 0.1543 | 0.0355 | 0.0895 | 0.0979 | 0.0147 | 0.0257 | 0.0218 |
| IWM $185 Put | 0.0373 | 0.0909 | 0.1085 | 0.0343 | 0.0535 | 0.0669 | 0.0416 | 0.0495 | 0.0574 |
| QQQ $307 Call | 0.0806 | 0.1421 | 0.1643 | 0.0627 | 0.0900 | 0.1061 | 0.0359 | 0.0401 | 0.0480 |
| QQQ $305 Put | 0.0495 | 0.1369 | 0.1934 | 0.0479 | 0.1008 | 0.1166 | 0.0312 | 0.0486 | 0.0662 |
| SPY $398 Call | 0.1066 | 0.2541 | 0.2719 | 0.0587 | 0.1483 | 0.1553 | 0.0342 | 0.0613 | 0.0571 |
| SPY $398 Put | 0.0696 | 0.1791 | 0.2114 | 0.0763 | 0.1262 | 0.1397 | 0.1036 | 0.1192 | 0.1265 |

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Early Exercise Compare | | | | | | | | | |
| Term | 90 Days | | | 60 Days | | | 10 Days | | |
|  | LSMC Normal | LSMC Poly | MC | LSMC Normal | LSMC Poly | MC | LSMC Normal | LSMC Poly | MC |
| Option |  |  |  |  |  |  |  |  |  |
| IWM $185 Call | 3.708 | 0.194 | 0.176 | 2.298 | 0.100 | 0.080 | 0.376 | 0.015 | 0.029 |
| IWM $185 Put | 1.936 | 0.182 | 0.113 | 0.966 | 0.069 | 0.056 | 0.407 | 0.121 | 0.063 |
| QQQ $307 Call | 3.267 | 0.164 | 0.160 | 1.742 | 0.085 | 0.100 | 0.316 | 0.038 | 0.038 |
| QQQ $305 Put | 2.534 | 1.129 | 0.183 | 1.937 | 0.267 | 0.166 | 0.499 | 0.149 | 0.085 |
| SPY $398 Call | 7.176 | 0.268 | 0.201 | 3.915 | 0.120 | 0.115 | 0.832 | 0.042 | 0.079 |
| SPY $398 Put | 4.375 | 0.349 | 0.188 | 2.820 | 0.161 | 0.137 | 1.435 | 0.185 | 0.153 |

Chart

Description automatically generated

Chart

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