**Payoff: For a Put Option Stock - Strike, for a Call Option Strike - Stock; This is the value of the option contract.**

**Continuation Value: The value of the option contract at the next time step, ie. You choose to continue to hold the option.**

**In the money: For a Put Option, the stock has fallen below the chosen strike price. For a Call Option, the stock has risen above the chosen strike price.**

**\*\*State Variable\*\***

The LMSC Methods attempts to approximate the conditional expectation of the payoff now given the continuation value. On the day of expiration, it is optimal to exercise the contract if you are in the money. Before the expiration, we need to compare the current value with the expected value if we continue to hold the contract. Therefore, with American Options, we are interested in identifying the conditional expected value of continuation given the current value. The LSMC method uses cross-sectional information from the simulated paths of the geometric Brownian motion to identify the conditional expectation function using least squares regression. Particularly we regress the payoffs from continuation at times on basis functions of the state variable at time. The fitted value from the regression is an unbiased estimate of the conditional expectation function and allows us to accurately estimate the optimal stopping rule for that option for each simulated path. The algorithm is recursive and works backwards from the expiration date to get the current value given the continuation value of the option.

For example, say we have put option contract that we can exercise at any time $t = \{1, 2, 3\}$

The payoff at time $t=3$ is $max(0, K - S\_{t=3})$

The payoff at time $t=2$ is $max(0, K - S\_{t=2}) | max(0, K - S\_{t=3})$

Let $X$ denote the stock prices at time $t=2$ for the paths that we have simulated and decided to exercise on day $t=3$.

Let $Y$ denote the corresponding dicounted payoff recieved at time $t=3$ if the put option is not exercised at time $t=2$.

We only use in the money paths since it allows us to better estimate the conditional expectation function in the region where exercise is relevant.

To get the immediate value of the option at time $t=2$, we regress $Y$ on a constant $X$ and $X^2$ using least squares regression, to obtain the conditional expectation function $E(Y|X) = \beta\_0 + \beta\_1 X + \beta\_2 X^2$. Where $\beta\_0, \beta\_1, \beta\_2$ are the coefficients of the regression model. We compare the fitted value from the function to the payoff at time $t = 3$.

If the immediate payoff is greater than the continuation value, we exercise the option. If the continuation value is greater than the immediate payoff, we continue to hold the option. If stock price is situated so that the option is out of the money, the option is worthless and thus the payoff is zero. If we choose to exercise before the expiration date, then all future payoffs for that path are also zero. So the realized payoff will be a matrix of zeros and discounted payoffs.

Proceeding backwards we examine if the option should be exercised at time $t=1$. Let $Y$ denote the dicounted value of future realized payoff along each path. We use the actual realized payoff along each path at time $t=2$ to regress on the fitted value of the conditional expectation function at time $t=1$, not the the conditional expected value of $Y$ at time $t=2$. As stated in the paper, this leads to an upward bias in the value of the option.

Finally after obtaining the cashflows from the regression for each path, we discount the cashflows to the time $t = 0$ and taking the average of all paths, to obtain the present value of the option contract. This is the estimated value of the option contract, and can be thought of as the sample mean of the simulation study.

So in total there are $K = 3$ times where the option can be exercised. If we increase the number of time steps $K$ to be sufficiently large we can estiamte the value of the option using many exercise points, which resemeble the continuous exercise feature of the american option. $0 < t\_1 ≤ t\_2 ≤ t\_3 ≤ ... ≤ t\_K ≤ T$.

Since we use the sample mean of the cashflows as the estimator of the expected value of the option. We run $r$ independent simulations to get the sample mean $Z\_i$ of the cashflows. The sample mean is an unbiased estimator of the conditional expectation of the option price. Thus, we can estimate the variance of $Z\_i$ by taking the sample variance of all $Z\_i$'s

$$\hat{\sigma}^2 = \frac{1}{r}\sum\_{i=1}^r (Z\_i - \bar{Z})^2$$

where $\bar{Z}$ is the sample mean of all $Z\_i$'s.

The nice part about this method is that it is easy to implement, and it is very flexible. We can use various kinds of basis functions to approximate the conditional expectation function. Another nice feature is that it is easy to extend to other types of options, such as Asian options, barrier options, and lookback options.

If we use the method of independent replications we obtain the following results:

**Fundamental Results**

The simulation methods seemed to perform better for the Put contracts than the calls with the data we have provided. The Black Scholes model seemed to give a close approximate to the observed price of the option; however as seen below, the LSMC method seems to be more accurate. The errors for the Put contracts (Right) are much closer to 0 than compared to the Calls (Left). We can note that the LSMC with a Normal Basis function performed the worst for all contracts, having a very high variability in its fair value approximation. We note that the LSMC method can estimate the value of the option contract very well especially with a shorter time until expiration. Simulations methods proved to be more accurate with less days until the expiration, with the LSMC Normal Basis Function performing the worst across all time horizons. We can see that the Simulation methods give a noticeably lower standard error for short term contracts than longer term, with simulated results differing within 1 cent of the average simulated fair value.

**Early Exercise:**

Since we can get the European component of an option price from the Black-Scholes Model, we already know the price of the option at expiration. The early exercise component of an American option is the difference between the European price and the simulated American option price we have obtained from the methods above. The early exercise value can be thought of the excess value of the option compared to if you held it to expiration. So, if we notice that the early exercise is high, we might want to exercise the option early, to get the most value out of the option, and vice versa if the early exercise value is negative. Thus, the table above shows the difference between the simulated methods and the Black Scholes method, and it clearly shows that the LSMC Normal Basis Function is the worst at estimating the early exercise value. The Polynomial Basis Function performed the best, and the finite difference simulation had similar results to the polynomial basis function. The difference between the Black Scholes method and the simulated methods is well within the bid-ask spread for these option contracts, which suggest that the simulated methods can approximate the early exercise value of the option contracts very well.

**Conclusion:**

In conclusion, we can see that the LSMC method is able to estimate the fair value of an American option reasonably well and can be extended to various types of options such as equity, commodities, mortgage, and swaps. We have found these methods to show better results with a shorter time until expiration, with the polynomial basis function preforming the best. Choosing a strong basis function should be a priority when using the LSMC method. In future research one should give care in choosing a basis function, to ensure that the choice of basis functions is not highly correlated with one another. Another interesting direction to explore would be to implement the LSMC method with an underlying asset that follows a jump diffusion process.