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# Abstract

In this paper we introduce option contracts and derive the fair value market price for American and European-style options. We use the Black Scholes pricing model as a benchmark, and Monte Carlo simulation methods to derive the fair value price of European Options. We provide a comprehensive review of the Least Squares Monte Carlo approach and its advantages over the existing Black Scholes closed form solution. Overall, this paper contributes to the understanding of Least Squares Monte Carlo simulation and its ability to price financial derivatives, primarily American Options with the European component.

# Introduction/Problem Statement

The volatile nature of financial markets imposes various sources of uncertainty on an investor's portfolio. The options market provides a way for investors to combat those uncertainties, by hedging their portfolio against the risk of adverse price movements. Thus, pricing option contracts are an important daily activity for banks and other financial institutions. It is also common for retail investors like us to study options for either short term gains or to be a beneficiary from a company as another retirement strategy. Therefore, for this paper we will compare the findings from analytical pricing methods and from simulation techniques, to approximate the fair value of the European and American option price today.

# Background and Description

## History Of Options

The history of Option contracts can be traced back to 332 BC, when Thales bought the rights to buy olives prior to a harvest, who went on to amass a fortune from the purchase of these rights. In 1636 options were widely bought to speculate on the soaring prices of tulips in Europe. At the time tulips became a sign of high status, and beauty, which fueled a craze of speculation from all levels in society. The wild speculation caused tulip prices to skyrocket, which also increased the number of growers and dealers looking to get into the trade. Dealers began selling the rights to own tulips in advance of the harvest, for buyers looking to secure a definite buying price. A year later, the price of tulips had gotten so high that no buyers were willing to pay the inflated price, resulting in a selling frenzy. The price of tulips had fallen faster than it rose; and almost all the speculators were wiped out as the price declined lower than the originally agreed upon price to buy. The Dutch economy would collapse after the sharp decline, leading to the notorious reputation for options being a dangerous speculative instrument. Although options gained a bad reputation, they later surfaced in London in the early 1700’s with financiers acknowledging its speculative power through its inherent leverage. It was later declared illegal in 1733, as groups opposed the speculative nature of options trading. It was outlawed for about 100 years, making a return in 1860 almost 100 years later.

In 1872 Russel Sage, a well-known American financier from New York; was the first to create calls and puts for trading in the US. Sage created options that were unstandardized and highly illiquid, resulting in an inefficient market for participants. However, Sage still managed to make millions within a few short years, before losing a fortune in the stock market crash of 1884. These unregulated Options continued to trade until the creation of the SEC after the stock market crash of the 1930’s. In 1973 The Chicago Board of Exchange and Options Clearing Corporation were formed to standardize and allow retail investors to participate in the trade under the performance guarantee of the OCC, and liquidity of the market maker system. By 1977 put options were introduced to the CBOE, which created the options market that we can trade today.

## Definitions and Literature Review

Option contracts refer to an agreement between two parties to facilitate a potential transaction on an underlying security at a predefined strike price prior to, or on a specified expiration date. A single stock option contract gives the option holder the right to 100 shares of the underlying security.

### Vanilla Options

A Vanilla Option is a simple call or put with no special features or observation dates. Call options give the option holder the right to buy 100 shares of an asset at the predefined strike price, prior to or on a defined expiration date. Put Options give the option holder the right to sell 100 shares of an underlying asset, prior to or on a defined expiration date. If an investor is unsure of the future performance of a specific stock, and they have a strong belief that the price will rise in the future, the investor may choose to buy a call option on the stock. This allows the investor to gain exposure from a future price increase, while not having to commit the capital to purchase the hundred shares on the specific asset. The European is the simplest form of a vanilla option, giving the investor the right to buy or sell shares of an asset a specified expiration date. American options differ from the European counterpart, in the fact that they can be exercised at any point until or on the expiration date. Vanilla options are some of the most traded option contacts due to their simplistic nature and lack of exotic characteristics.

### Exotic Options

Exotic Options are options that differ from Vanilla options in their payment structures, expiration dates and predefined strike prices. The underlying asset can vary in an exotic option, thus giving investors alternative opportunities. Exotic options can be customized to meet the risk tolerance and investment objective of the investor. Asian Options an exotic option in that they are dependent on the average price of the underlying asset over a specified time. Bermuda options are also an example of an exotic option, they give the investor the right to buy or sell an asset at many preset expiration dates. Exotic options provide investors more control over when the option is exercised, and they often come with a higher initial cost when compared to a vanilla option contract.

### Time Value of Money

Discount factor incorporates “time value of money” to calculate the options price more accurately. Time value of money represents the idea that value of money changes with time driven by factors such as inflation, changes in interest rates (although the model assumes it is constant), company-specific & country-specific premiums, unforeseen events such as pandemic, among others. Theoretically, it can be explained by the concepts of present value (PV) and future value (FV) where it is believed that the sum of money is worth more now than the sum will be at a future date due to its earning potential in the interim:

## Brownian Motion

Brownian Motion is the random chaotic movement observed in particles as the result of collisions with other surrounding particles. The Standard Brownian Motion generates random points and is defined as the following:

Some important properties of Standard Brownian motion are that it has stationary and independent increments between points. The Stationarity property assumes the change in Brownian motion between two time periods is due to the length of the increment ( only.

The independence property states that the change in motion between two pairs of time periods have no relationship with each other. Specifically, if , the independence property states:

### Using Geometric Brownian Motion to Model a Stock Price:

Geometric Brownian Motion is commonly used to model stock prices. Where μ is the drift, 𝝈 is the volatility, and is the current stock price. The standard Brownian motion is denoted , thus the Geometric Brownian process can be shown as follows:

## Input data

We obtained historical data from Yahoo Finance for three major indices to track overall movements in the stock market. The data was obtained over the past year, beginning 2022 and collected twice a day, at the open and close. The stocks chosen are the SPDR S&P 500 ETF (SPY), Invesco QQQ Trust (QQQ), and the iShares Russel 2000 (IWM). The options on these stocks are traded heavily by financial institutions and retail investors alike. The tickers we chose are regular leaders in stock option volume traded each day, so there is a very large interest in trading options on these assets.

# Main findings

### European Options

### Defining Black-Scholes:

The Black Scholes model was developed in 1973 by Robert Merton and Myron Scholes. The equation estimates the theoretical value of an option based on the risk-free rate of return, and the underlying stock price at the time of expiration. The Black Scholes model is still considered one of the best methods to price an option contract today. However, there are some major assumptions proposed by the model. First the model assumes that stock prices follow the Geometric Brownian Motion defined above, no dividends are paid, a constant risk-free rate of return for all maturities, and that there are no costs or restraints associated with trade transaction. Black-Scholes derives the price of a European call option:

The final equation above can also be written as a closed form solution below to help us understand the formula more intuitively. We let :

,

Where is the present value of the expected payout for a given strike price. is the conditional expectation of the stock given that the price is higher than the strike price multiplied by the probability that the strike price is higher than the stock price for a call option. Similarly, the European put option can be expressed as follows:

## Monte Carlo Simulation

We can estimate the fair value of the European Style options through the means of Monte Carlo simulation. Monte Carlo simulation methods are mathematical techniques that predict possible outcomes of an uncertain event. The more outcomes we can predict, the closer we get to the true value of the mathematical function. When applying this to the Black Scholes model, we begin with the initial stock price S0 to simulate possible stock price paths using the Geometric Brownian motion. Depending on the option type, we find the average difference of the strike price , and the stock price . Finally, the discount factor e-rt is applied to find the fair value of the price today. The Monte Carlo method is shown below for European call and put options below.

Using Monte Carlo method is a very simple and elegant way to the fair value of an option price today, if you are only able to exercise said option by the defined expiration date.

### American options

### Least Squares Monte Carlo Method

We begin with the same simulation of the stock price as in the European Case. However, since we are pricing an American Option, we are not limited to one but any possible exercise point before the expiration of the option contract. The option holder of an American style option needs to continuously compare the immediate payoff of the option with its expected future payoff. At each exercise point, the option holder needs to determine if it is optimal to exercise the option or to continue to hold the option. This conditional expectation function can be approximated using cross-sectional information as an output of least squares regression.

The least squares Monte Carlo Method begins with obtaining stock price paths from the Geometric Brownian Motion similarly to the European case, however for the least squares approach, we only consider the paths where the option contract is in the money. The algorithm iterates over each exercise point recursively beginning with the time of expiration and determines the payoff for each path.

After we obtain the payoff for each path, we proceed by applying functions of the given state variables to approximate the continuation value. The continuation is the value of an option contract at the next time step. State variables are the stock price at the current iteration for each of the simulated in the money price paths obtained in the first step. Using in the money paths help improve the efficiency of the program allow us to better estimate the region where the payoff is relevant.

The functions we apply to the given state variables are referred to as Basis functions. Basis functions are typically used in machine learning for approximating complex functions by generalizing a simple linear regression function. Schwartz and Longstaff convey that no more than 3 basis functions are needed to achieve the true value of the American option. A common basis function that can be used is the simple polynomial function. Let be the given state variable (Stock price of the current iteration), and be the continuation value we wish to predict.

Where the coefficients ’s found from traditional regression techniques. We repeat this process for each of the in the money paths, discounting the expected cashflows one period at every exercise point until we get back to the current time. Finally, we discount the payoff one more time to get the net present value and obtain the average of all cashflows to estimate the current fair market value of the American option. We will be comparing the use of the polynomial function, the lognormal function, and the use of the control variate as Basis functions in our simulation study. The control variate variance reduction for estimating the mean is as follows with an approximation for the parameter .

The least squares method could easily be extended to price a Bermudan-style option as they have predefined exercise points. As the number of specified exercise points increase, we can get closer to the estimate of an American style option, however this does entail that more compute power is used to complete the algorithm.

## Fundamental results

We apply the Least Squares Monte Carlo approach for American-style options, and the Monte Carlo approach for European-style options beginning with 1000 independent random stock price paths then complete nine independent replication runs, each time with different initial seeds of the random number generator. We obtain the standard error from the simulation output here.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Standard Error from simulation output | | | | |
| Option Contract | LSMC Normal | LSMC Poly | LSMC CV | MC |
| IWM $185 Call | 0.06 | 0.08 | 0.09 | 0.08 |
| IWM $185 Put | 0.03 | 0.05 | 0.06 | 0.06 |
| QQQ $307 Call | 0.02 | 0.04 | 0.04 | 0.04 |
| QQQ $305 Put | 0.04 | 0.09 | 0.11 | 0.1 |
| SPY $398 Call | 0.07 | 0.13 | 0.15 | 0.14 |
| SPY $398 Put | 0.04 | 0.08 | 0.09 | 0.09 |

The standard errors for the simulated values range from 0.02 to 0.15 which is well within the market bid-ask spread for these options in normal market conditions. Below we find the error of the fair value approximation done by our simulation methods, as well as the error from the observed option price for the two days before expiration. We note that the simulated errors seem to all be within the same general range, the true approximations can be found in the appendix. Overall, the simulation methods seemed to be well within the standard error reported above.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Error vs Actual American Option Price (2 days to expiration ) | | | | | | | | | | |
| type | T | Ticker | K | S | 𝛔 | LSMC Normal | LSMC Poly | LSMC CV | MC | BS |
| Call | 2 | iwm | 185.0 | 173.06 | 0.39 | 0.0 | 0.01 | 0.01 | 0.01 | 0.0 |
| Call | 1 | iwm | 185.0 | 175.96 | 0.34 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 |
| Put | 2 | iwm | 185.0 | 173.06 | 0.0 | 0.32 | 0.32 | 0.29 | 0.29 | 0.29 |
| Put | 1 | iwm | 185.0 | 175.96 | 0.53 | 0.08 | 0.19 | 0.16 | 0.09 | 0.09 |
| Call | 2 | qqq | 304.0 | 297.37 | 0.28 | 0.17 | 0.06 | 0.05 | 0.02 | 0.03 |
| Call | 1 | qqq | 304.0 | 305.34 | 0.23 | 0.26 | 0.26 | 0.28 | 0.22 | 0.26 |
| Put | 2 | qqq | 305.0 | 297.37 | 0.0 | 0.51 | 0.51 | 0.47 | 0.47 | 0.47 |
| Put | 1 | qqq | 305.0 | 305.34 | 0.23 | 0.32 | 0.29 | 0.32 | 0.28 | 0.32 |
| Call | 2 | spy | 398.0 | 388.65 | 0.21 | 0.05 | 0.02 | 0.0 | 0.03 | 0.01 |
| Call | 1 | spy | 398.0 | 394.67 | 0.13 | 0.14 | 0.13 | 0.13 | 0.12 | 0.13 |
| Put | 2 | spy | 398.0 | 388.65 | 0.32 | 0.62 | 0.01 | 0.06 | 0.05 | 0.03 |
| Put | 1 | spy | 398.0 | 394.67 | 0.32 | 0.5 | 0.46 | 0.44 | 0.43 | 0.46 |

Since the European component of an option price from the Black-Scholes Model is the price of the option at expiration. The early exercise component of an American option is the difference between the European price and the simulated American option price we have obtained from the methods above. The early exercise value is the current value of the option compared to if you held it to expiration. Table 3 shows the early exercise prices across for short term contracts, we can see that the

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Early Exercise component | | | | | | | | | |
| Ticker | type | T | K | S | 𝛔 | LSMC Normal | LSMC Poly | LSMC CV | MC |
| iwm | Call | 2 | 185.0 | 173.06 | 0.39 | 0.0 | 0.01 | 0.0 | 0.01 |
| iwm | Call | 1 | 185.0 | 175.96 | 0.34 | 0.0 | 0.0 | 0.0 | 0.0 |
| iwm | Put | 2 | 185.0 | 173.06 | 0.0 | 0.03 | 0.03 | 0.0 | 0.0 |
| iwm | Put | 1 | 185.0 | 175.96 | 0.53 | 0.0 | 0.11 | 0.07 | 0.01 |
| qqq | Call | 2 | 304.0 | 297.37 | 0.28 | 0.15 | 0.03 | 0.03 | 0.0 |
| qqq | Call | 1 | 304.0 | 305.34 | 0.23 | 0.01 | 0.01 | 0.03 | 0.04 |
| qqq | Put | 2 | 305.0 | 297.37 | 0.0 | 0.04 | 0.04 | 0.0 | 0.0 |
| qqq | Put | 1 | 305.0 | 305.34 | 0.23 | 0.01 | 0.02 | 0.01 | 0.03 |
| spy | Call | 2 | 398.0 | 388.65 | 0.21 | 0.04 | 0.01 | 0.01 | 0.02 |
| spy | Call | 1 | 398.0 | 394.67 | 0.13 | 0.01 | 0.01 | 0.01 | 0.01 |
| spy | Put | 2 | 398.0 | 388.65 | 0.32 | 0.65 | 0.04 | 0.03 | 0.02 |
| spy | Put | 1 | 398.0 | 394.67 | 0.32 | 0.04 | 0.0 | 0.02 | 0.03 |

# Conclusion & Further research Opportunities

In conclusion, we can see that the LSMC method is able to estimate the fair value of an American option reasonably well. The LSMC method can also be used with a wide variety of Basis functions making it versatile and easy to test. However, one should choose a basis function that does not allow high correlations as this can lead to convergence failures. Further, the LSMC method and can be extended to various types of options such as equity, commodities, mortgage, and swaps. We have found these methods to show better results with a shorter time until expiration. The simulation for this paper took about 30 minutes to run. In the future, one could attempt to vectorize the LSMC process above to increase the compute speed of the algorithm. Other topics could include implementing the LSMC method with an underlying asset that follows a jump diffusion process, or using out of sample regression coefficients for the conditional expectation of the continuation value.

# Appendix

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Observed and Simulated fair value Option price. (2 Days to expiration) | | | | | | | | | | | |
| type | T | Ticker | K | S | 𝛔 | Observed | BS | LSMC Normal | LSMC Poly | LSMC CV | MC |
| Call | 2 | iwm | 185.0 | 173.06 | 0.39 | 0.06 | 0.06 | 0.06 | 0.05 | 0.06 | 0.06 |
| Call | 1 | iwm | 185.0 | 175.96 | 0.34 | 0.04 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 |
| Put | 2 | iwm | 185.0 | 173.06 | 0.0 | 11.6 | 11.89 | 11.92 | 11.92 | 11.89 | 11.89 |
| Put | 1 | iwm | 185.0 | 175.96 | 0.53 | 9.28 | 9.19 | 9.2 | 9.09 | 9.12 | 9.19 |
| Call | 2 | qqq | 304.0 | 297.37 | 0.28 | 0.79 | 0.76 | 0.62 | 0.73 | 0.74 | 0.77 |
| Call | 1 | qqq | 304.0 | 305.34 | 0.23 | 2.82 | 2.56 | 2.56 | 2.56 | 2.54 | 2.6 |
| Put | 2 | qqq | 305.0 | 297.37 | 0.0 | 7.08 | 7.55 | 7.59 | 7.59 | 7.55 | 7.55 |
| Put | 1 | qqq | 305.0 | 305.34 | 0.23 | 1.9 | 1.59 | 1.58 | 1.61 | 1.58 | 1.62 |
| Call | 2 | spy | 398.0 | 388.65 | 0.21 | 0.36 | 0.36 | 0.32 | 0.35 | 0.36 | 0.34 |
| Call | 1 | spy | 398.0 | 394.67 | 0.13 | 0.4 | 0.27 | 0.26 | 0.28 | 0.28 | 0.28 |
| Put | 2 | spy | 398.0 | 388.65 | 0.32 | 10.53 | 10.56 | 9.91 | 10.52 | 10.59 | 10.58 |
| Put | 1 | spy | 398.0 | 394.67 | 0.32 | 5.54 | 5.08 | 5.04 | 5.08 | 5.1 | 5.11 |

Chart

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Chart

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