Counting, Discrete Probability, Conditional Probability, Monty Hall

Computer Science Mentors 70

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1 Counting

1.1 Introduction

Theorem 1 : Distributing k distinguishable balls into n distinguishable boxes, with exclusion, corresponds to forming a permutation of size k, taken from a set of size n. Therefore, there are $P(n, k) = n_k = n * (n-1) * (n-2) \ldots (n-k+1)$ different ways to distribute k distinguishable balls into n distinguishable boxes, with exclusion

Theorem 2 : Distributing k distinguishable balls into n distinguishable boxes, without exclusion, corresponds to forming a permutation of size k, with unrestricted repetitions, taken from a set of size n. Therefore, there are n^k different ways to distribute k distinguishable balls into n distinguishable boxes, without exclusion.

Theorem 3 : Distributing k indistinguishable balls into n distinguishable boxes, with exclusion corresponds to forming a combination of size k, taken from a set of size n. Therefore, there are $C(n, k) = \binom{n}{k}$ different ways to distribute k indistinguishable balls into n distinguishable boxes, with exclusion.

1.2 Questions

1. How many 5-digit sequences have the digits in non-decreasing order?

Solution: This is a stars-and-bars problem with 5 slots and 9 dividers between digits, so the answer is $\binom{14}{9} = 2002$

2. How many ways can you deal 13 cards to each of 4 players so that each player gets one card of each of the 13 values (ace-2-3-. . . -king)?

Solution: Approach this problem value by value: there are 24 ways to distribute the aces to the 4 players $4!$, 24 ways to distribute the twos, and so on, so the number of ways to deal the cards in this manner is $4!^{13} = 24^{13}$.

3. How many ways can you give 10 cookies to 4 friends if each friend gets at least 1 cookie?

Solution: Give out 4 necessary cookies, then count the number of ways to give 6 cookies to 4 friends if some can get no cookies. 6 choices from 4 options with repetition, so the number of ways is $(6 + 4 - \binom{1}{4} - 1) = \binom{9}{3} = 84$.

1.3 Extra Practice

1. In Jorge Luis Borges The Library of Babel, the narrator describes a massive library: Every book in the library has 410 pages, each page has 40 lines, and each line has 80 characters. Besides lowercase letters, the only characters appearing in the books are the period, the comma, and the space. In order to catch up with the 21st century, the Library got itself a Twitter account (@LibraryofBabel)! It appears to have been active for only a short time, but while it was running it diligently tweeted one 140-character message per day. Assume that it uses only the 26 letters of the English alphabet (plus the period, comma, and space) and that any 140-character combination is possible (e.g. asdfasdf. . . as and ,,,,,,. . . ,,,, are both perfectly valid).

   (a) How many possible tweets are there?

   Solution: 29^{140}

   (b) How many tweets use no spaces?

   Solution: 28^{140}

   (c) How many tweets consist entirely of whitespace?

Solution: 1

(d) Let T be some particular tweet. How many tweets differ from T by exactly one character?

Solution: $140 * 28 = 3920$

(e) How many have exactly six spaces and five commas?

Solution: $\binom{140}{6} * \binom{134}{5} * 27^{129} = \frac{140!}{6!5!128!} * 27^{129}$

2 Combinatorial Proofs

2.1 Questions

1. $k\binom{n}{k} = n\binom{n-1}{k-1}$

Solution: Choose a team of k players where one of the players is the captain.
LHS: Pick a team with k players. This is $\binom{n}{k}$. Then make one of the players the captain. There are k options for the captain so we get $k \times \binom{n}{k}$.
RHS: Pick the captain. There are n choices for the captain. Now pick the last $k-1$ players on the team. There are now $n-1$ people to choose from. So we get $n \times \binom{n-1}{k-1}$.

2. $n! = \binom{n}{k}k!(n-k)!$

Solution: Arrange n items
LHS: number of ways to order n items
RHS: Choose k items without ordering. Order these k items. Order the remaining $n-k$ items.

3. $\sum_{k=0}^{n} k^2 = \binom{n+1}{2} + 2\binom{n+1}{3}$

Solution: Number of ordered triplets of the form (i, j, k) where i and j are less than or equal to k for every k from 0 to n
LHS: For each k there are k options for i and k options for j so k^2 options for all.
RHS: Consider the case where $i = j$. Then we must choose two numbers from $\{0, \dots, n\}$ which amounts to $\binom{n+1}{2}$. If $i \neq j$ then we choose 3 numbers from $n + 1$. But i can be less than j or greater than j so we must multiply by 2.

4. Prove $a(n-a)\binom{n}{a} = n(n-1)\binom{n-2}{a-1}$ by a combinatorial proof.

Solution: Suppose that you have a group of n players. The left-hand side is the number of ways to pick a team of a of these players, designate one member of the team as captain, and then pick one reserve player from the remaining $n - a$ people. The right-hand side is the number of ways to pick the captain, then the reserve player, and then the other $a - 1$ members of the team.

2.2 Challenge

1. Prove the Hockey Stick Theorem:

$$\sum_{t=k}^{n} \binom{t}{k} = \binom{n+1}{k+1}$$

where n, t are natural numbers and $n > t$

Solution: $\sum\limits_{t=k}^{n} \binom{t}{k} = \binom{n+1}{k+1}$
where n, t are natural numbers and $n > t$
In the homework you were asked to show the following:
$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
Plug this in for:
$\binom{t+1}{k+1}$
$\binom{t+1}{k+1} = \binom{t}{k} + \binom{t}{k+1} \implies \binom{t}{k} = \binom{t+1}{k+1} - \binom{t}{k+1}$
Now do the summation:
$\sum\limits_{t=k}^{n} \binom{t}{k} = \sum\limits_{t=k}^{n} \binom{t+1}{k+1} - \sum\limits_{t=k}^{n} \binom{t}{k+1}$
Split apart the two summations as follows: take out the last term in the first summation and the last term in the second summation
$\sum\limits_{t=k}^{n} \binom{t}{k} = \left(\sum\limits_{t=k}^{n-1} \binom{t+1}{k+1} + \binom{n+1}{k+1}\right) - \left(\sum\limits_{t=k+1}^{n} \binom{t}{k+1} + \binom{k}{k+1}\right)$

Look at all subsets of $\{1, 2, 3, \ldots, 2015\}$ that have 1000 elements. Choose the least element from each subset. Find the average of all least elements. Since $\binom{k}{k+1}$ is 0 we can just remove this term:

$\sum\limits_{t=k}^{n} \binom{t}{k} = \binom{n+1}{k+1} + \sum\limits_{t=k}^{n-1} \binom{t+1}{k+1} - \sum\limits_{t=k+1}^{n} \binom{t}{k+1}$

We want the summations to match. So change t such that the new t, t, goes from $k+1$ to $n-1$. So let t = t - 1. Then $t = k+1 \rightarrow t = k$ and $t = n \rightarrow t = n-1$.

$\sum\limits_{t=k}^{n} \binom{t}{k} = \binom{n+1}{k+1} + \sum\limits_{t=k}^{n-1} \binom{t+1}{k+1} - \sum\limits_{t'=k}^{n-1} \binom{t'+1}{k+1}$

Notice that the summations cancel out. We are left with the statement we were trying to prove.

3 Discrete Probability

3.1 Introduction

1. What is a sample (event, outcome) space?

   Solution: The set of all possible outcomes

2. What is an event?

   Solution: A partition of the sample space.

3. Given a uniform probability space Ω such that $|\Omega| = N$, how many events are possible?

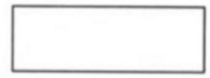
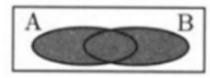
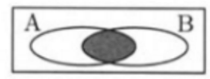
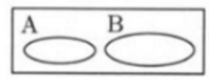
   Solution: 2^N. Each point is either included or excluded from any particular subset, and the number of subsets is the number of events.

3.2 Questions

1. **Probably Poker**

   (a) What is the probability of drawing a hand with a pair?

Figure 1: From Pitman

Event language	Set language	Set notation	Venn diagram
outcome space	universal set	Ω	
event	subset of Ω	A, B, C, etc.	
impossible event	empty set	$\emptyset$	
not A, opposite of A	complement of A	A^c	
either A or B or both	union of A and B	$A \cup B$	
both A and B	intersection of A and B	AB, $A \cap B$	
A and B are mutually exclusive	A and B are disjoint	$AB = \emptyset$	
if A then B	A is a subset of B	$A \subseteq B$	

Solution: $\frac{13 * \binom{4}{2} * \binom{12}{3} * 4^3}{\binom{52}{5}}$

(b) What is the probability of drawing a hand with four of a kind?

Solution: $\frac{13 * 12 * 4}{\binom{52}{5}}$

(c) What is the probability of drawing a straight?

Solution: $\frac{9 * 4^5}{\binom{52}{5}}$

(d) What is the probability of drawing a hand of all of the same suit?

Solution: $\frac{4 * \binom{13}{5}}{\binom{52}{5}}$

(e) What is the probability of drawing a straight house?

Solution: $\frac{4*9}{\binom{52}{5}}$

2. Suppose you arrange 12 different cars in a parking lot, uniformly at random. Three of the cars are Priuses, four of the cars are Teslas, and the other five are Nissan Leaves. What is the probability that the three Prius's are all together?

Solution: There are $12!$ possible ways to arrange the 12 cars. Now, there are $12 - 3 + 1 = 10$ different places the Priuss could go (positions 1,2,3, positions 2,3,4, all the way until positions 10,11,12). For each of those 10 places, there are 3! ways to arrange the Priuss and 9! ways to arrange the other 9 cars in the 9 remaining spots. So, in total, there are $10 * 3! * 9!$ ways of arranging the cars so that the 3 Priuss are together. So the probability we get what we want is $\frac{10*3!*9!}{12!}.0455$

4 Conditional Probability

4.1 Introduction

Bayes' Rule

$$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$$

Total Probability Rule

$$Pr[B] = Pr[A \cap B] + Pr[\bar{A} \cap B] = Pr[B|A] * Pr[A] + Pr[B|A] * (1 - Pr[A])$$

Independence

Two events A, B in the same probability space are independent if

$$Pr[A \cap B] = Pr[A] * Pr[B]$$

4.2 Questions

1. You have a deck of 52 cards. What is the probability of:

(a) Drawing 2 Kings with replacement?

Solution: Uniform Probability space enumerates the $52 * 52$ combinations of two cards. There are $4 * 4$ that are 2 Kings, so the probability is $\frac{1}{13^2}$

(b) Drawing 2 Kings without replacement?

Solution: We need to change our uniform probability space to the $\binom{52}{2} = 52 * 51$ pairs of cards possible without replacement (with order mattering, but this is not important in this case, as long as you keep it consistent). Of these $4 * 3$ represent pairs of kings. Therefore the probability is $\frac{4*3}{52*51}$. We can also use conditional probability (which we will have to use in the next part):

$$P(K \text{ on 2nd and } K \text{ on 1st}) = P(\text{K on 2nd}|\text{K on 1st})P(\text{K on 1st}) = \frac{3}{51} * \frac{4}{52}$$

(c) The second card is a King without replacement?

Solution:

$$\begin{aligned} P(Kon2nd) &= P(K \text{ on 2nd } |K \text{ on 1st})P(K \text{ on 1st}) \\ &+ P(K \text{ on 2nd } | \text{ no } K \text{ on 1st})P(\text{no } K \text{ on 1st}) \\ &= (3/51)(4/52) + (4/51)(48/52) \\ &= (4/52) * ((48 + 3)/51) = 4/52 \end{aligned} \tag{1}$$

Note that this is the same as the $P(K \text{ on 1st})$, because a K is equally likely to be anywhere in the deck.

(d) **[EXTRA]** The nth card is a King without replacement ($n < 52$)?

Solution: The last problem is a hint that we can argue this by symmetry. Since we have no information about what any of the preceding cards were before it, it is equally likely that the nth card is any of the 52 possible cards, so the probability that it is a King is $\frac{4}{52}$.

2. Find an example of 3 events A, B, and C such that each pair of them are independent, but they are not mutually independent.

Solution: Consider a fair 4-sided die. Let A be the event that 1 or 2 appears in a die roll, B be the event that 1 or 3 appears, and C be the event that 1 or 4 appears.Then,

$$Pr(A) = Pr(B) = Pr(C) = \frac{1}{2}$$

Furthermore,

$$Pr(A \cap B) = Pr(1 \text{ appears}) = \frac{1}{4} = Pr(A)Pr(B).$$

So A and B are pairwise independent. Similarly (A, C) and (B, C) are pairwise independent. However,

$$Pr(A \cap B \cap C) = Pr(1 \text{ appears}) = \frac{1}{4} = Pr(A)Pr(B)Pr(C) = \frac{1}{8}$$

So these 3 events are not mutually independent.The answer is not unique; any other valid answer is acceptable.

3. A lie detector is known to be 80% reliable when the person is guilty and 95% reliable when the person is innocent. If a suspect is chosen from a group of suspects where only 1% have ever committed a crime, and the test indicates that the person is guilty, what is the probability they are innocent?

Solution: Let I and G be the events that the person is innocent and guilty respectively, and let L_I and L_G be the events that the test says innocent or guilty.

$$P(I|L_G) = \frac{P(L_G|I) * P(I)}{P(L_G|I) * P(I) + P(L_G|G) * P(G)} = \frac{0.05 * 0.99}{0.05 * 0.99 + 0.8 * 0.01} = 0.86$$

4. Jim and George are setting up venture capital portfolios. Suppose that Jim picks $n+1$ startups to fund and George picks n startups to fund. Suppose that the probability of any startup succeeding is $\frac{1}{2}$ and all of the startups succeed or fail independently. What is the probability that Jim picks more successful startups than George?

Solution: Since Jim picks one more startup than George, it is impossible that they pick both the same number of successful startups and the same number of unsuccessful startups. So Jim picks either more successful startups than George or more unsuccessful startups than George (but not both). Since the probability of succeeding is $\frac{1}{2}$, these events are equally likely by symmetry, so both events have probability $\frac{1}{2}$.

5 Monty Hall

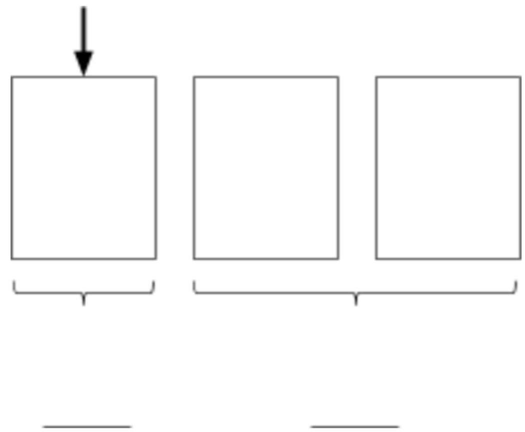
5.1 Introduction

The Problem :
Suppose a contestant is shown 3 doors. There is a car behind one of them and goats behind the rest. Then they do the following:

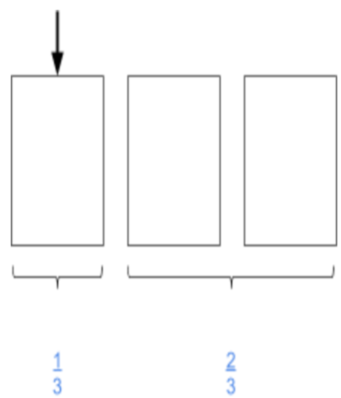
1. Contestant chooses a door.
2. Host opens a door with a goat behind it.
3. Contestant can choose to switch or stick to original choice

Is the contestant more likely to win if they switch?

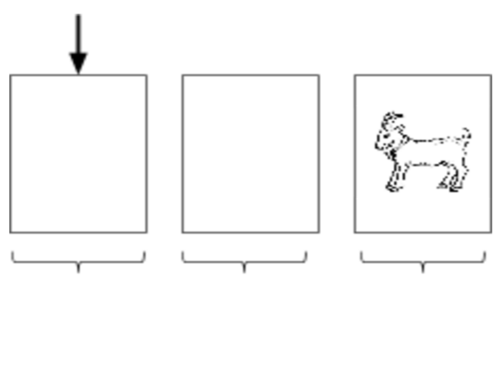
At step 1, what is the probability that the car is behind the door the contestant chose? What is the probability that the car is behind the other two doors?



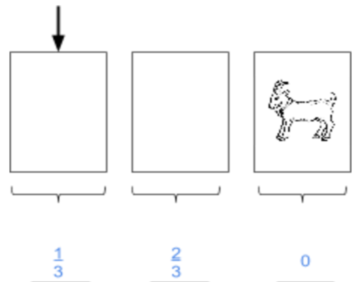
Solution:



After the host opens a door with a goat, what are the probabilities of the car being behind each door?



Solution:



5.2 Questions

1. **Grouping Doors**
   Now we have 6 doors. You pick 1 and the other 5 doors are divided into two groups: one with 2 doors and the other with 3 doors. He removes doors until each group has 1 door left. Do you switch? What do you switch to?

   Solution: Choose the door that was part of the group of three doors. When the doors were split up into three groups, those groups each had a probability of $\frac{1}{6}$, $\frac{2}{6}$, and $\frac{3}{6}$. These probabilities do not change when doors are removed, so three remaining doors each have probability $\frac{1}{6}$, $\frac{2}{6}$, and $\frac{3}{6}$.

2. **Macs and Monty**
   Suppose instead of the normal Monty Hall scenario in which we have two empty doors and a car residing behind the third we have a car behind one door, a Mac behind another, and nothing behind the third.
   Let us assume that the contestant makes an initial pick at his/her discretion (random) and the host proceeds to ALWAYS open the empty door. When the contestant's initial choice corresponds to the empty door, the host will say so and the contestant must switch.
   Does the typical Monty Hall paradox of $\frac{2}{3}$ chance of obtaining the car by switching versus a $\frac{1}{3}$ chance of obtaining the car by staying apply in this particular case?

   Solution: No. The normal Monty Hall paradox holds because when another door is opened, you learn nothing about your door, so the chance that you picked the right door from the beginning remains $\frac{1}{n}$ (which means the chance of the other door must be $\frac{n-1}{n}$). However, in this case, you may learn something about your door–you may be told that it must be wrong for example–thus shattering the paradox. These are all of the possible scenarios:

Initial Pick	Switch?	Win?
Car	Yes	Lose
Car	No	Win
Mac	Yes	Win
Mac	No	Lose
Empty	Yes (to car)	Win
Empty	Yes (to car)	Lose

Switching results in a win 50% of the time, so there is a 50% chance of winning regardless of strategy.

3. **Generalizing Monty**
Now say we have n doors and there is a car behind one of them. Monty opens k doors, where $0 \leq k \leq n-2$. Should you switch?

Solution: Yes.

$$\frac{n-1}{n} * \frac{1}{n-k-1} = \frac{1}{n} * \frac{n-1}{n-k-1} \geq \frac{1}{n}$$