Quantifiers, Methods of Proof

Computer Science Mentors 70

Independent review

1 Induction

1.1 Questions

1. The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$ is defined by:
   $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-2} + F_{n-1}$

   Using induction, prove that $F_1 + \ldots + F_n = F_{n+2} - 1$

2. Prove that a chocolate bar of n squares requires a minimum of $n - 1$ cuts to separate into single squares.

2 Stable Marriage

2.1 Introduction

The Algorithm:

1. **Every Morning**: Each man proposes to the most preferred woman on his list who has not yet rejected him.
2. **Every Afternoon**: Each woman collects all the proposals she received in the morning; to the man she likes best, she responds maybe, come back tomorrow (she now has him on a string), and to the others, she says never.
3. **Every Evening**: Each rejected man crosses off the woman who rejected him from his list. The above loop is repeated each successive day until each woman has a man on a string; on this day, each woman marries the man she has on a string.

Definitions:

M and W are a rogue couple if they prefer to be with each other as opposed to the people they are paired with

A pairing is stable if there are no rogue couples

Lemmas:

The algorithm halts.

If man M proposes to woman W on the kth day, then on every subsequent day W has someone on a string whom she likes at least as much as M.

2.2 Questions

1. Lemma: Algorithm terminates with a pairing.

2. Lemma: The pairing is stable.

3. A person x is said to prefer a matching A to a matching A' if x strictly prefers her/his partner in A to her/his partner in A' . Given two stable matchings A and A' , a person may prefer one to the other or be indifferent if she/he is matched with the same person in both. Suppose now that A and A' are stable matchings, and suppose that m and w are partners in A but not in A' . Prove that one of m and w prefers A to A' , and the other prefers A' to A.

3 Well Ordering Principle

3.1 Questions

1. Every non empty set of natural numbers contains a smallest element.
   In this question, we will go over how the well-ordering principle can be derived from (strong) induction. Remember the well-ordering principle states the following: For every non-empty subset S of the set of natural numbers N, there is a smallest element $x \in S$; i.e. $\exists x : \forall y \in S : x \leq y$

   1. What is the significance of S being non-empty? Does WOP hold without it? Assuming that S is not empty is equivalent to saying that there exists some number z in it.

   2. Induction is always stated in terms of a property that can only be a natural number. What should the induction be based on?

   3. Now that the induction variable is clear, formally state the induction hypothesis.

   4. Verify the base case.

   5. Now prove that the induction works, by writing the inductive step.

   6. What should you change so that the proof works by simple induction (as opposed to strong induction)?

4 Optimal, Pessimal

4.1 Introduction

A persons optimal partner is their most preferred partner among possible partners in stable pairings.
A male optimal pairing is a pairing in which all males are paired with his optimal woman.
Lemma: If a pairing is male optimal, then it is also female pessimal

4.2 Questions

1. Theorem: The pairing produced by the stable marriage algorithm is male optimal

5 More Practice

5.1 Questions

1. Imagine that in the context of stable marriage all men have the same preference list. That is to say there is a global ranking of women, and men's preferences are directly determined by that ranking. Use any method of proof to answer the following questions.

    1. Prove that the first woman in the ranking has to be paired with her first choice in any stable pairing.

    2. Prove that the second woman has to be paired with her first choice if that choice is not the same as the first woman's first choice. Otherwise she has to be paired with her second choice.

    3. Continuing this way, assume that we have determined the pairs for the first $k-1$ women in the ranking. Who should the k-th woman be paired with?

    4. Prove that there is a unique stable pairing.