# Counting, Discrete Probability, Conditional Probability, Monty Hall 6

Computer Science Mentors 70

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## 1 Intro to Counting

### 1.1 Introduction

**Rules of counting:**

1. If your event is composed of different independent events then you can multiply together the probabilities of the independent events.
2. If order does not matter then count with order and then divide by the number of orderings/sorted objects

### 1.2 When Order Matters

1. (a) You have 15 chairs in a room and there are 9 people. How many different ways can everyone sit down?

   (b) How many ways are there to fill 9 of the 15 chairs? (We dont care who sits in them)

2. **Identical Digits** The numbers 1447, 1005, and 1231 have something in common. Each of them is a four digit number that begins with 1 and has two identical digits. How many numbers like this are there?

## 1.3 More Practice

1. At Starbucks, you can choose either a Tall, a Grande, or a Venti drink. Further, you can choose whether you want an extra shot of espresso or not. Furthermore, you can choose whether you want a Latte, a Cappuccino, an Americano, or a Frappuccino.

   How many different drink combinations can you order?

2. We grab a deck of cards and its poker time. Remember, in poker, order doesnt matter.
   (a) How many ways can we have a hand with exactly one pair? This means a hand with ranks (a, a, b, c, d)
   (b) How many ways can we have a hand with four of a kind? This means a hand with ranks (a, a, a, a, b)
   (c) How many ways can we have a straight ? A straight is 5 consecutive cards, that dont all necessarily have the same suit. straight can be (2, 3, 4, 5, 6); (3, 4, 5, 6, 7); ; (10, J, Q, K, A) can start from 2 - 10, which is 9 possibilities each number in hand has 4 possibilities (suits)
   (d) How many ways can we have a hand of all of the same suit?
   (e) How many ways can we have a straight flush? This means we have a consecutive-rank hand of the same suit. For examples, (2, 3, 4, 5, 6), all of spades is a straight flush, while (2, 3, 5, 7, 8) of all spades is NOT, as the ranks are not consecutive.
3. How many solutions does $x+y+z = 10$ have, if all variables must be positive integers?

4. How many ways are there to arrange the letters of the word SUPERMAN
   (a) On a straight line?
   (b) On a straight line, such that SUPER occurs as a substring?

(c) On a straight line, such that SUPER occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?

(d) On a circle?

(e) On a circle, such that SUPER occurs as a substring?

(f) On a circle, such that SUPER occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?

# 2 Counting

## 2.1 Introduction

**Theorem 1** : Distributing $k$ distinguishable balls into $n$ distinguishable boxes, with exclusion, corresponds to forming a permutation of size $k$, taken from a set of size $n$. Therefore, there are $P(n, k) = n_k = n * (n-1) * (n-2) \ldots (n-k+1)$ different ways to distribute $k$ distinguishable balls into $n$ distinguishable boxes, with exclusion

**Theorem 2** : Distributing $k$ distinguishable balls into $n$ distinguishable boxes, without exclusion, corresponds to forming a permutation of size $k$, with unrestricted repetitions, taken from a set of size $n$. Therefore, there are $n^k$ different ways to distribute $k$ distinguishable balls into $n$ distinguishable boxes, without exclusion.

**Theorem 3** : Distributing $k$ indistinguishable balls into $n$ distinguishable boxes, with exclusion corresponds to forming a combination of size $k$, taken from a set of size $n$. Therefore, there are $C(n, k) = \binom{n}{k}$ different ways to distribute $k$ indistinguishable balls into $n$ distinguishable boxes, with exclusion.

## 2.2 Questions

1. How many 5-digit sequences have the digits in non-decreasing order?
2. How many ways can you deal 13 cards to each of 4 players so that each player gets one card of each of the 13 values (ace-2-3-. . . -king)?

3. How many ways can you give 10 cookies to 4 friends if each friend gets at least 1 cookie?

## 2.3 Extra Practice

1. In Jorge Luis Borges The Library of Babel, the narrator describes a massive library: Every book in the library has 410 pages, each page has 40 lines, and each line has 80 characters. Besides lowercase letters, the only characters appearing in the books are the period, the comma, and the space. In order to catch up with the 21st century, the Library got itself a Twitter account (@LibraryofBabel )! It appears to have been active for only a short time, but while it was running it diligently tweeted one 140-character message per day. Assume that it uses only the 26 letters of the English alphabet (plus the period, comma, and space) and that any 140-character combination is possible (e.g. asdfasdf. . . as and ,,,,,,. . . ,,,, are both perfectly valid).

   (a) How many possible tweets are there?

   (b) How many tweets use no spaces?

   (c) How many tweets consist entirely of whitespace?

   (d) Let $T$ be some particular tweet. How many tweets differ from $T$ by exactly one character?

   (e) How many have exactly six spaces and five commas?

# 3 Combinatorial Proofs

## 3.1 Questions

1. $n! = \binom{n}{k} k!(n-k)!$

2. $\sum_{k=0}^{n} k^2 = \binom{n+1}{2} + 2\binom{n+1}{3}$

3. Prove $a(n-a)\binom{n}{a} = n(n-1)\binom{n-2}{a-1}$ by a combinatorial proof.

## 3.2 Challenge

1. Prove the Hockey Stick Theorem:

$$\sum_{t=k}^{n} \binom{t}{k} = \binom{n+1}{k+1}$$

where $n, t$ are natural numbers and $n > t$

# 4 Discrete Probability

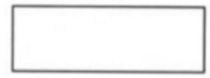
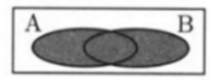
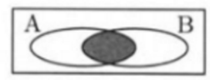
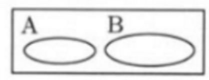
## 4.1 Introduction

1. What is a sample (event, outcome) space?
2. What is an event?
3. Given a uniform probability space $\Omega$ such that $|\Omega| = N$, how many events are possible?

## 4.2 Questions

1. **Probably Poker**
   (a) What is the probability of drawing a hand with a pair?

Figure 1: From Pitman

| Event language | Set language | Set notation | Venn diagram |
|---|---|---|---|
| outcome space | universal set | $\Omega$ |  |
| event | subset of $\Omega$ | $A$, $B$, $C$, etc. |  |
| impossible event | empty set | $\emptyset$ |  |
| not $A$, opposite of $A$ | complement of $A$ | $A^c$ |   |
| either $A$ or $B$ or both | union of $A$ and $B$ | $A \cup B$ |   |
| both $A$ and $B$ | intersection of $A$ and $B$ | $AB$, $A \cap B$ |   |
| $A$ and $B$ are mutually exclusive | $A$ and $B$ are disjoint | $AB = \emptyset$ |   |
| if $A$ then $B$ | $A$ is a subset of $B$ | $A \subseteq B$ |   |

(b) What is the probability of drawing a hand with four of a kind?

(c) What is the probability of drawing a straight?

(d) What is the probability of drawing a hand of all of the same suit?

(e) What is the probability of drawing a straight house?

2. Suppose you arrange 12 different cars in a parking lot, uniformly at random. Three of the cars are Priuses, four of the cars are Teslas, and the other five are Nissan Leaves. What is the probability that the three Prius's are all together?

# 5 Conditional Probability

## 5.1 Introduction

**Bayes' Rule**

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

**Total Probability Rule**

$$P[B] = P[A \cap B] + P[\bar{A} \cap B] = P[B|A] * P[A] + P[B|\bar{A}] * (1 - P[A])$$

**Independence**
Two events $A$, $B$ in the same probability space are independent if

$$P[A \cap B] = P[A] * P[B]$$

## 5.2 Questions

1. You have a deck of 52 cards. What is the probability of:

(a) Drawing 2 Kings with replacement?

(b) Drawing 2 Kings without replacement?

(c) The second card is a King without replacement?

(d) **[EXTRA]** The nth card is a King without replacement ($n < 52$)?

2. Find an example of 3 events A, B, and C such that each pair of them are independent, but they are not mutually independent.

3. A lie detector is known to be 80% reliable when the person is guilty and 95% reliable when the person is innocent. If a suspect is chosen from a group of suspects where only 1% have ever committed a crime, and the test indicates that the person is guilty, what is the probability they are innocent?

4. Jim and George are setting up venture capital portfolios. Suppose that Jim picks $n + 1$ startups to fund and George picks $n$ startups to fund. Suppose that the probability

of any startup succeeding is $\frac{1}{2}$ and all of the startups succeed or fail independently. What is the probability that Jim picks more successful startups than George?

5. Oski the bear has lost his dog in either forest $A$ (with a priori probability 0.4) or in forest $B$ (with a priori probability 0.6).
   On any given day, if the dog is in $A$ and Oski spends a day searching for it in $A$, the conditional probability that he will find the dog that day is 0.25. Similarly, if the dog is in $B$ and Oski spends a day looking for it there, the conditional probability that he will find the dog that day is 0.15.
   The dog cannot go from one forest to the other. Oski can search only in the daytime, and he can travel from one forest to the other only at night.
   (a) In which forest should Oski look to maximize the probability he finds his dog on the first day of the search?
   (b) Given that Oski looked in $A$ on the first day but didnt find his dog, what is the probability that the dog is in $A$?
   (c) If Oski flips a fair coin to determine where to look on the first day and finds the dog on the first day, what is the probability that he looked in $A$?

(d) If the dog is alive and not found by the $N$th day of the search, it will die that evening with probability $\frac{N}{N+2}$. Oski has decided to look in $A$ for the first two days. What is the probability that he will find a live dog for the first time on the second day?

# 6 Monty Hall

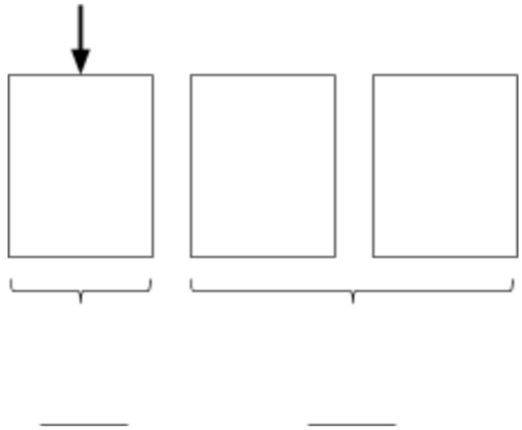
## 6.1 Introduction

**The Problem** :
Suppose a contestant is shown 3 doors. There is a car behind one of them and goats behind the rest. Then they do the following:

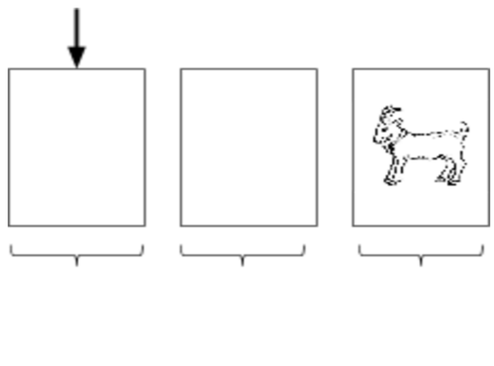
1. Contestant chooses a door.
2. Host opens a door with a goat behind it.
3. Contestant can choose to switch or stick to original choice

Is the contestant more likely to win if they switch?

At step 1, what is the probability that the car is behind the door the contestant chose? What is the probability that the car is behind the other two doors?



After the host opens a door with a goat, what are the probabilities of the car being behind each door?



## 6.2 Questions

1. **Grouping Doors**
   Now we have 6 doors. You pick 1 and the other 5 doors are divided into two groups: one with 2 doors and the other with 3 doors. He removes doors until each group has 1 door left. Do you switch? What do you switch to?

2. **Macs and Monty**
   Suppose instead of the normal Monty Hall scenario in which we have two empty doors and a car residing behind the third we have a car behind one door, a Mac behind

another, and nothing behind the third.
Let us assume that the contestant makes an initial pick at his/her discretion (random) and the host proceeds to ALWAYS open the empty door. When the contestant's initial choice corresponds to the empty door, the host will say so and the contestant must switch.
Does the typical Monty Hall paradox of $\frac{2}{3}$ chance of obtaining the car by switching versus a $\frac{1}{3}$ chance of obtaining the car by staying apply in this particular case?

3. **Generalizing Monty**
Now say we have $n$ doors and there is a car behind one of them. Monty opens $k$ doors, where $0 \leq k \leq n - 2$. Should you switch?