INEQUALITIES, DISTRIBUTIONS, CONTINUOUS PROBABILITY, CONDITIONAL EXPECTATION 9

COMPUTER SCIENCE MENTORS 70

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1 Markov and Chebyshev Inequalities

1.1 Introduction

Markov's Inequality
For a non-negative random variable X with expectation $\mathrm{E}(X) = \mu$, and any $\alpha > 0$:

$$\mathrm{P}[X \geq \alpha] \leq \frac{\mathrm{E}(X)}{\alpha}$$

Chebyshev's Inequality
For a random variable X with expectation $\mathrm{E}(X) = \mu$, and any $\alpha > 0$:

$$\mathrm{P}[|X - \mu| \geq \alpha] \leq \frac{\mathrm{Var}(X)}{\alpha^2}$$

1.2 Questions

1. Use Markov's to prove Chebyshev's Inequality:

2. **Squirrel Standard Deviation**
   As we all know, Berkeley squirrels are extremely fat and cute. The average squirrel is 40% body fat. The standard deviation of body fat is 5%. Provide an upper bound on the probability that a randomly trapped squirrel is either too skinny or too fat? A skinny squirrel has less than 27.5% body fat, and a fat squirrel has more than 52.5% body fat?

3. **Bound It**
   A random variable X is always strictly larger than -100. You know that $\mathrm{E}(X) = 60$. Give the best upper bound you can on $\mathrm{P}[X \geq 20]$.

4. Give a distribution for a random variable where the expectation is 1,000,000 and the probability that the random variable is zero is 99%.

5. Consider a random variable Y with expectation μ whose maximum value is $\frac{3\mu}{2}$, prove

that the probability that Y is 0 is at most $\frac{1}{3}$.

6. Let X be the sum of 20 i.i.d. Poisson random variables $X_1, \ldots, X_{20}$ with $\mathrm{E}(X_i) = 1$. Find an upper bound of $\mathrm{P}[X \geq 26]$ using,

   (a) Markov's inequality:

   (b) Chebyshev's inequality:

   (c) Chernoff Bound:

7. The citizens of the country USD (the United States of Drumpf) vote in the following manner for their presidential election: if the country is liberal, then each citizen votes for a liberal candidate with probability p and a conservative candidate with probability $1p$, while if the country is conservative, then each citizen votes for a conservative candidate with probability p and a liberal candidate with probability $1p$. After the election, the country is declared to be of the party with the majority of the votes.

(a) Assume that $p = \frac{3}{4}$ and suppose that 100 citizens of USD vote in the election and that USD is known to be conservative. Provide a tight bound on the probability that it is declared to be a Liberal country.

(b) Now let p be unknown; we wish to estimate it. Using the CLT, determine the number of voters necessary to determine p within an error of 0.01, with probability at least 0.95.

2 Confidence Intervals

2.1 Questions

1. Define i. i. d. variables $A_k \sim \text{Bern}(p)$ where $k \in [1, n]$. Assume we can declare that $\text{P}[|\frac{1}{n}\sum_k A_k - p| > 0.25] = 0.01$.

   (a) Please give a 99% confidence interval for p if given A_k.

   (b) We know that the variables X_i, for i from 1 to n, are i.i.d. random variables and have variance. We also have a value (an observation) of $A_n = \frac{X_1 + ... + X_n}{n}$. We want to guess the mean, μ, of each X_i.

   Prove that we have 95% confidence μ lies in the interval $\left[A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}\right]$

   That is, $\text{P}\left[\mu \in \left[A_n - 4.5\frac{\sigma}{\sqrt{n}}, A_n + 4.5\frac{\sigma}{\sqrt{n}}\right]\right] \geq 95\%$

(c) Give the 99% confidence interval for μ:

2. We have a die whose 6 faces are values of consecutive integers, but we dont know where it starts (it is shifted over by some value k; for example, if $k = 6$, the die faces would take on the values $7, 8, 9, 10, 11, 12$). If we observe that the average of the n samples (n is large enough) is 15.5, develop a 99% confidence interval for the value of k.

3 Continuous Probability

3.1 Questions

1. Given the following density functions, identify if they are valid random variables. If yes, find the expectation and variance. If not, what rules does the variable violate?

   (a) $f(x) = \begin{cases} \frac{1}{4} & \text{if } x \in \{\frac{1}{2}, \frac{9}{2}\} \\ 0 & \text{otherwise} \end{cases}$

   (b) $f(x) = \begin{cases} x - \frac{1}{2} & x \in \{0, \infty\} \end{cases}$

2. For a discrete random variable X we have $\Pr[X \in [a, b]]$ that we can calculate directly by finding how many points in the probability space fall in the interval and how many total points are in the probability space. How do we find $\Pr[X \in [a, b]]$ for a continuous random variable?

3. Are there any values of a, b for the following functions which gives a valid pdf? If not, why? If yes, what values?

(a) $f(x) = -1,\ a < x < b$

(b) $f(x) = 0,\ a < x < b$

(c) $f(x) = 10000,\ a < x < b$

4. For what values of the parameters are the following functions probability density functions? What is the expectation and variance of the random variable that the function represents?

(a) $f(x) = \begin{cases} ax & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(b) $f(x) = \begin{cases} -2x & \text{if } a < x < b \ (a = 0 \vee b = 0) \\ 0 & \text{otherwise} \end{cases}$

5. Define a continuous random variable R as follows: we pick a random point on a disk of radius 1; the value of R is distance of this point from the center of the disk. We will find the probability density function of this random variable.

(a) What is (should be) the probability that R is between 0 and $\frac{1}{2}$? Why?

(b) What is (should be) the probability that R is between a and b, for any $0 \leq a \leq b \leq 1$?

(c) What is a function $f(x)$, for which $\int_a^b f(x)dx$ satisfies these same probabilities?

4 Continuous Distributions

4.1 Introduction

Uniform Distribution: $U(a, b)$ This is the distribution that represents an event that randomly happens at any time during an interval of time.

- $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$
- F(x) = 0 for $x < a$, $\frac{x-a}{b-a}$ for $a < x < b$, 1 for $x > b$
- E$(x) = \frac{a+b}{2}$
- Var$(x) = \frac{1}{12}(b-a)^2$

Exponential Distribution: $Expo(\lambda)$ This is the continuous analogue of the geometric distribution, meaning that this is the distribution of how long it takes for something to happen if it has a rate of occurrence of λ.

- memoryless
- $f(x) = \lambda * e^{-\lambda * x}$
- $F(x) = 1 - e^{\lambda x}$
- E$(x) = \frac{1}{\lambda}$

Gaussian (Normal) Distribution: $N(\mu, \sigma^2)$

- The CLT states that any unspecified distribution of events will converge to the Gaussian as n increases
- Mean: μ
- Variance: σ^2
- $f(x|\mu, \sigma^2) = \frac{1}{\sigma\sqrt{(2\pi)}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

4.2 Questions

1. There are on average 8 office hours in a day. The scores of an exam followed a normal distribution with an average of 50 and standard deviation of 6. If a student waits until an office hour starts, what is the expected value of the sum of the time they wait in hours and their score on the exam?

2. Every day, 100,000,000,000 cars cross the Bay Bridge, following an exponential distribution.

   (a.) What is the expected amount of time between any two cars crossing the bridge?

   (b.) Given that you havent seen a car cross the bridge for 5 minutes, how long should you expect to wait before the next car crosses?

3. There are certain jellyfish that dont age called hydra. The chances of them dying is purely due to environmental factors, which well call λ. On average, 2 hydras die within 1 day.

   (a) What is the probability you have to wait at least 5 days for a hydra dies?

   (b) Let X and Y be two independent discrete random variables. Derive a formula for expressing the distribution of the sum S = X + Y in terms of the distributions of X and of Y.

   (c) Use your formula in part (a) to compute the distribution of S = X +Y if X and Y are both discrete and uniformly distributed on 1,...,K.

   (d) Suppose now X and Y are continuous random variables with densities f and g respectively (X,Y still independent). Based on part (a) and your understanding of continuous random variables, give an educated guess for the formula of the density of S = X +Y in terms of f and g.

   (e) Use your formula in part (c) to compute the density of S if X and Y have both uniform densities on [0, a].

5 Conditional Expectation

5.1 Introduction

The **conditional expectation** of Y given X is defined by

$$\mathrm{E}[Y|X=x] = \sum_y y \cdot \mathrm{P}[Y=y|X=x] = \sum_y y \cdot \frac{\mathrm{P}[X=x, Y=y]}{\mathrm{P}[X=x]}$$

Properties of Conditional Expectation

$$\begin{aligned}
\mathrm{E}(a|Y)) &= a \\
\mathrm{E}(aX+bZ|Y) &= a \cdot \mathrm{E}(X|Y) + b \cdot \mathrm{E}(Z|Y) \\
\mathrm{E}(X|Y) &\geq 0 \text{ if } X \geq 0 \\
\mathrm{E}(X|Y) &= \mathrm{E}(X) \text{ if } X, Y \text{ independent} \\
\mathrm{E}(\mathrm{E}(X|Y)) &= \mathrm{E}(X)
\end{aligned}$$

5.2 Questions

1. Prove $\mathrm{E}(\mathrm{E}(Y|X)) = \mathrm{E}(Y)$

2. Prove $\mathrm{E}(h(X) \cdot Y|X) = h(X) \cdot \mathrm{E}(Y|X)$