CS 70 Spring 2017

Discrete Mathematics and Probability Theory

FINAL SOLUTIONS

INSTRUCTIONS

- You have 3 hours to complete the exam.
- Mark your answers **on the exam itself**. We will *not* grade answers written on scratch paper.

Last name	
First name	
Student ID number	
BearFacts email (`_@berkeley.edu`)	
TA	
Room	
Seat	
Name of the person to your left	
Name of the person to your right	
All the work on this exam is my own. **(please sign)**	

1. (5 points) High Expectations

Prove that $E[E[X|Y;Z]|Y] = E[X|Y]$.

$$\begin{aligned}
E[E[X|Y;Z]|Y=y] &= \sum_z E[X|Y=y,Z=z]Pr(Z=z|Y=y) \\
&= \sum_z \sum_x xPr(X=x|Y=y,Z=z)Pr(Z=z|Y=y) \\
&= \sum_{z,x} x\frac{Pr(X=x,Y=y,Z=z)}{Pr(Y=y,Z=z)}\frac{Pr(Z=z,Y=y)}{Pr(Y=y)} \\
&= \sum_{z,x} x\frac{Pr(X=x,Y=y,Z=z)}{Pr(Y=y)} \\
&= \sum_x x\frac{Pr(X=x,Y=y)}{P(Y=y)} \\
&= \sum_x xPr(X=x|Y=y) \\
&= E[X|Y=y]
\end{aligned}$$

Note the equivalence holds for all y, so $E[E[X|Y;Z]|Y] = E[X|Y]$.

2. (5 points) Parting Ways

Consider a finite undirected graph $G = (V, E)$, and a particle traversing this graph. At each time step, the particle on some node will transition to one of the node's neighbors with uniform probability. Notice that this is a Markov Chain. Consider a distribution π, with a probability for each node v, where $\pi[v] = \frac{d(v)}{2|E|}$ ($d(v)$ is the degree of v). Prove that π is a stationary distribution of this Markov Chain

Firstly, note that every $\pi[v] \geq 0$, and that $\sum\limits_{v \in V} \pi[v] = 1$, which follows from the Handshaking Lemma. Now we prove that it is stationary.

Consider the transition matrix P. If π is a stationary distribution, then $\pi P = \pi$. Without loss of generality, consider some entry $\pi[v] = \frac{d(v)}{2|E|}$. When we multiply π against P, the new entry we get is the dot product of π and the column of P corresponding to v. This is $\sum\limits_{u \in V} \frac{d(u)}{2|E|} P(u \to v)$, where $P(u \to v)$ is the probability of the particle transitioning from u to v. Note that this probability is equal to $\frac{1}{d(u)}$ if u is a neighbor of v, and 0 otherwise.

Then the dot product becomes $\sum\limits_{u \in V:(u,v) \in E} \frac{d(u)}{2|E|} \frac{1}{d(u)} = \sum\limits_{u \in V:(u,v) \in E} \frac{1}{2|E|} = \frac{d(v)}{2|E|}$.

This is the precisely $\pi[v]$, and this holds for all $v \in V$. *Quid est demonstratum.*

3. (5 points) Bomb Bounds

(a) Let X be the sum of 20 i.i.d. Poisson Random Variables $X_1 \ldots X_{20}$ with $\mathrm{E}[X_i] = 1$. Use Markov's inequality and Chebyshev's inequality to find two upper bounds on $Pr(X \geq 26)$

Using Markovs inequality, we have $20/26 = 10/13$. We note that since these are Poisson RVs, that the variance will also be 1. Using Chebyshevs inequality we are bounding the probability that $|x - 20| \geq 6$, thus have $\frac{20}{36} = \frac{5}{9}$

(b) Let Y be a Binomial Random Variable with n trials and an unknown probability of success p. Given that we dont know p, give the tightest bound possible using Chebyshev's on the probability that $Y \geq 5$

Since p is between 1 and 0, and since the variance of a Binomial RV is $np(1-p)$, the p that maximises the variance is $\frac{1}{2}$, so we know that $Var(Y) \leq \frac{n}{4}$, so we can use this in chebyshev's to get an upper bound even without knowing p. By using this value as variance, we get $\frac{n}{100}$ as our bound.

(c) Let Z be a normally distributed Random Variable (with mean μ and SD σ). Use Chebyshev's inequality to bound as tightly as possible the probability of falling more than $k \in R$ standard deviations above the mean.

By plugging in $k * \sigma$ as our α, and knowing that $sigma^2$ is our variance, we get $\frac{1}{k}$. We can tighten this bound by noting that a normal distribution is symmetric, so if we want the probability that we are specifically above a certain point (as we do in this case) we can divide the result of Chebyshev's in half, leaving us with $\frac{1}{2k}$ as the tightest bound that we can get using Chebyshev's.

4. (5 points) Continuously Raising Expectations

(a) A target is made of 3 concentric circles of radii $\frac{1}{\sqrt{3}}$, 1 and $\sqrt{3}$ feet. Shots within the inner circle are given 4 points, shots within the next ring are given 3 points, and shots within the third ring are given 2 points. Shots outside the target are given 0 points.Let X be the distance of the hit from the center (in feet), and let the probability density function of X be $f(x) = \frac{2}{\pi(1+x^2)}, x > 0, 0$ otherwise. What is the expected value of the score of a single shot? Dont solve; just set it up.

The expected value is

$$\int_0^{\frac{1}{\sqrt{3}}} 4\frac{2}{\pi(1+x^2)} + \int_{\frac{1}{\sqrt{3}}}^1 3\frac{2}{\pi(1+x^2)} + \int_1^{\sqrt{3}} 2\frac{2}{\pi(1+x^2)}$$

(b) Consider the following game. Spin a wheel and wait until it comes to rest at some x between 0 and 359. The amount of money won is $\frac{x}{36-6}$ dollars. Let Y be a random variable for your winnings. First, define a probability density function. From there, calculate the expectation and variance.

For the probability density function, we know that the area under the curve must be 1. We know that we can either lose a maximum of -6 dollars or win a maximum of 4 dollars, so our random variable Y can only range in between there. What will be the value at those points? We have 10 values and we need the area of this rectangle to be 1, so $f(y)$ is $\frac{1}{10}$ in this area between $-6 \leq y \leq 4$, and 0 elsewhere.

Expectation: $\int_{-\infty}^{\infty} yf(y)dy$ which is actually $\int_{-6}^{4} \frac{y}{10}dy$ which evaluates to $\frac{1}{20} * (16 - 36) = -1$.

Variance: $\int_{-\infty}^{\infty}(y - E[Y])^2 f(y)dy$ which is actually $\int_{-6}^{4} \frac{(y+1)^2}{10}dy$ which evaluates to $\frac{1}{30} * (125 + 125) = \frac{25}{3}$.

5. (5 points) Linearly Estimate Me

The random variables X, Y, Z are i.i.d. $N(0,1)$ (Recall the normal distribution has mean 0 and variance 1).

(a) Find $L[X^2 + Y^2 | X + Y]$

$$\begin{aligned} E((X^2+Y^2)(X+Y)) &= E(X^3 + X^2Y + XY^2 + Y^3) \\ &= 0 \\ Cov(X^2+Y^2, X+Y) &= 0 \\ L[X^2+Y^2|X+Y] &= E(X^2+Y^2) \\ &= 2 \end{aligned}$$

(b) Find $L[X + 2Y | X + 3Y + 4Z]$

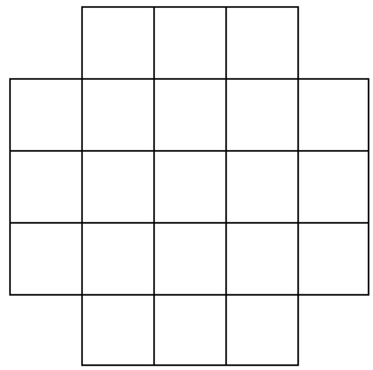
$$\begin{aligned} cov(X+2Y, X+3Y+4Z) &= E((X+2Y)(X+3Y+4Z)) \\ &= 1+6 \\ &= 7 \\ var(X+3Y+4Z) &= 1+9+16 \\ &= 26 \\ L[X+2Y|X+3Y+4Z] &= 1+9+16 \\ &= 26 \\ L[X+2Y|X+3Y+4Z] &= \frac{7}{26}(X+3Y+4Z) \end{aligned}$$

6. (5 points) One, two, three

(a) How many ways can we roll 3 dice, such that they are in a strictly decreasing sequence?

This would simply be $\binom{6}{3}$, because given any combination there is exactly 1 strictly decreasing sequence (and we can get all the strictly decreasing sequences).

(b) How many rectangles are in the following grid?



We count the number of rectangles in the 5 x 5 grid, then subtract each of the cases. We have $\binom{6}{2}^2 - 4^3 - 4^2 - 0 - 1 = \square 144$.

7. (5 points) Diving Into Distributions

(a) Find the distribution of:

i. $Min(U_1, U_2)$ where $U_1, U_2 \sim Uniform[0, 1]$

$$\begin{aligned} P(min(U_1, U_2) > x) &= P(U_1 > x)P(U_2 > x) \\ &= (1 - x)^2 \end{aligned}$$

Therefore PDF $= -2(1 - x)$

ii. The sum of N i.i.d Geometric random variables with parameter p

$$\begin{aligned} P(X_N = x) &= P(\text{success on the } x^{th} \text{ trial})P(N - 1 \text{ success in } x - 1 \text{ trials}) \\ &= p\binom{N-1}{x-1}p^{N-1}(1-p)^{x-N} \\ &= p^N q^{x-N}\binom{N-1}{x-1} \end{aligned}$$

(b) Expectations of distributions

i. $E[U_1|U_1 < U_2]$ where $U_1, U_2 \sim Uniform[0,1]$
Notice that this is the same as $Min(U_1, U_2)$, and we calculated the PDF already, so we can just find the expected value, which is $\frac{1}{3}$. One other way is to use symmetry, which we will use in the next part.

ii. $E[U_1|U_1 > U_2]$ where $U_1, U_2 \sim Uniform[0,1]$
This is $max(U_1, U_2)$. Note that $E[max(U_1, U_2) + min(U_1, U_2)] = E[U_1 + U_2] = 1$. Therefore $E[max(U_1, U_2)] = 1 - \frac{1}{3} = \frac{2}{3}$

iii. Show that $\mathrm{E}[(X-t)^2] = \mathrm{E}[(X-\mu)^2] + (t-\mu)^2 = Var(X) + (t-\mu)^2$
$\mathrm{E}[(X-t)^2] = \mathrm{E}[((X-\mu)-(t-\mu))^2] = \mathrm{E}[(X-\mu)^2] + (t-\mu)^2$ by multiplying out the terms and using linearity.

iv. Find t such that the quantity $g(t) = \mathrm{E}[(X-t)^2]$ is minimized.
$g(t) = Var(X) + c$, where $c = (t-\mu)^2$. Evaluating at $t = \mu$, we find that $c = 0$, which minimizes the above expression.