Fall 2016

MOCK Final

1.

POTPOURRI

Consider an infinite dimensional hypercube:

Can we define an algorithm to enumerate the vertices of the hypercube?

Can we define an algorithm to enumerate the vertices of the hypercube?

No: think about the bitstring representations of the vertices, they have one bit per dimension, which we can prove to be unenumerable by Cantor's diagonalization argument.

Find a bijection between two distinct sets of vertices of the hypercube.

Find a bijection between two distinct sets of vertices of the hypercube.

There are many solutions. A bijection uniquely pairs elements of the set. For example $f(0x) = 1x$, $f(1x) = 0x$ where $0x$ is the vertex with a bit string representation starting with 0 followed by a bitstring x.

Using the bitstring labels for vertices: how many of the vertices have labels whose digits sum to 5?

Using the bitstring labels for vertices: how many of the vertices have labels whose digits sum to 5?

If there were finite dimensions, this would be n CHOOSE 5 (since we must pick 5 digits to be 1, the rest are 0). With infinite dimensions, there are an infinite number of vertices. It is countable since we can represent each choice as a subset of N^5 (the indices of the 5 1s) (we exclude vectors with repeated elements).

Define a bijection between one of N or R to the *vertices* of the hypercube

Define a bijection between one of N or R to the *vertices* of the hypercube

The bit-string labels of the vertices are infinite length bit-strings, which we know to be bijective with R (established in the setup for the Cantor diagonalization argument in the notes).

Define a bijection between one of N or R to the *edges* of the hypercube.

Define a bijection between one of N or R to the *edges* of the hypercube.

We represent each edge as a pair (v, n) $\in V \times N$ (one of the vertices along with the bit that flips to get the adjoining vertex). We now take n to represent the whole number portion of a real, while v, the infinite length bitstring represents the binary decimal portion. The representation (v, n) is not unique, there are precisely two (v, n) for each $e = (v_1, v_2)$, however the two are bijective since the sets are infinite.

2.
High Expectations

Prove that $E[E[X|Y;Z] \mid Y] = E[X|Y]$.

Prove that E[E[X|Y;Z] | Y] = E[X|Y].

$$
\begin{aligned}
E[E[X|Y;Z]|Y=y] &= \sum_z E[X|Y=y, Z=z]Pr(Z=z|Y=y) \\
&= \sum_z \sum_x xPr(X=x|Y=y, Z=z)Pr(Z=z|Y=y) \\
&= \sum_{z,x} x \frac{Pr(X=x, Y=y, Z=z)}{Pr(Y=y, Z=z)} \frac{Pr(Z=z, Y=y)}{Pr(Y=y)} \\
&= \sum_{z,x} x \frac{Pr(X=x, Y=y, Z=z)}{Pr(Y=y)} \\
&= \sum_x x \frac{Pr(X=x, Y=y)}{P(Y=y)} \\
&= \sum_x xPr(X=x|Y=y) \\
&= E[X|Y=y]
\end{aligned}
$$

3.

Bomb Bounds

Let X be the sum of 20 i.i.d. Poisson Random Variables $X_1, \ldots, X_{20}$ with $E[X_i] = 1$. Use Markov's inequality and Chebyshev's inequality to find two upper bounds on $\Pr(X \geq 26)$

Let X be the sum of 20 i.i.d. Poisson Random Variables $X_1, \ldots, X_{20}$ with $E[X_i] = 1$. Use Markov's inequality and Chebyshev's inequality to find two upper bounds on $\Pr(X \geq 26)$

Using Markov's inequality, we have 20/26 = 10/13. We note that since these are Poisson RVs, that the variance will also be 1. Using Chebyshev's inequality we are bounding the probability that $|x-20| \geq 6$, thus have 20/36 = 5/9

Let Y be a Binomial Random Variable with n trials and an unknown probability of success p. Given that we don't know p, give the tightest bound possible using Chebyshev's on the probability that $Y \geq 5$

Let Y be a Binomial Random Variable with n trials and an unknown probability of success p. Given that we don't know p, give the tightest bound possible using Chebyshev's on the probability that $|Y - E[Y]| \geq 5$

Since p is between 1 and 0, and since the variance of a Binomial RV is np(1-p), the p that maximises the variance is ½, so we know that $Var(Y) \leq n/4$, so we can use this in Chebyshev's to get an upper bound even without knowing p. By using this value as variance, we get n/100 as our bound.

Let Z be a normally distributed Random Variable (with mean μ and SD σ). Use Chebyshev's inequality to bound as tightly as possible the probability of falling more than $k \in R$ standard deviations above the mean.

Let Z be a normally distributed Random Variable (with mean μ and SD σ). Use Chebyshev's inequality to bound as tightly as possible the probability of falling more than $k \in R$ standard deviations above the mean.

By plugging in $k*\sigma$ as our α, and knowing that σ^2 is our variance, we get $1/k^2$. We can tighten this bound by noting that a normal distribution is symmetric, so if we want the probability that we are specifically above a certain point (as we do in this case) we can divide the result of Chebyshev's in half, leaving us with $1/2k^2$ as the tightest bound that we can get using Chebyshev's.

4.
Parting Ways

Consider a finite undirected graph $G = (V, E)$, and a particle traversing this graph. At each time step, the particle on some node will transition to one of the node's neighbors with uniform probability. Notice that this is a Markov Chain. Consider a distribution π, with a probability for each node v, where

$$\pi[v] = d(v)/(2|E|) \quad (d(v) \text{ is the degree of } v).$$

Prove that π is a stationary distribution of this Markov Chain

π[*v*] =d(*v*)/2|E| (d(*v*) is the degree of *v*).

Prove that π is a stationary distribution of this Markov Chain

Firstly, note that every π[*v*] ≥ 0, and that $\sum_{v \in V} \pi[v] = 1$, which follows from the Handshaking Lemma. Now we prove that it is stationary.

Consider the transition matrix P. If π is a stationary distribution, then π P = π. Without loss of generality, consider some entry π[*v*] = d(*v*)/(2|E|). When we multiply π against P, the new entry we get is the dot product of π and the column of P corresponding to *v*.

This is $\sum_{u \in V} \frac{d(u)}{2|E|} P(u \to v)$, where $P(u \to v)$ is the probability of the particle transitioning from *u* to *v*. Note that this probability is equal to 1/d(*u*) if *u* is a neighbor of *v*, and 0 otherwise.

Then the dot product becomes $\sum_{u \in V:(u,v) \in E} \frac{d(u)}{2|E|} \frac{1}{d(u)} = \sum_{u \in V:(u,v) \in E} \frac{1}{2|E|} = \frac{d(v)}{2|E|}$

This is the precisely π[*v*], and this holds for all *v* $\in$ V. QED

5.

Continuously Raising Expectations

A target is made of 3 concentric circles of radii $1/\sqrt{3}$, 1 and $\sqrt{3}$ feet. Shots within the inner circle are given 4 points, shots within the next ring are given 3 points, and shots within the third ring are given 2 points. Shots outside the target are given 0 points.Let X be the distance of the hit from the center (in feet), and let the probability density function of X be $f(x) = 2/(\pi(1+x^2))$, $x > 0$, 0 otherwise.

What is the expected value of the score of a single shot?

A target is made of 3 concentric circles of radii 1/√3, 1 and √3 feet. Shots within the inner circle are given 4 points, shots within the next ring are given 3 points, and shots within the third ring are given 2 points. Shots outside the target are given 0 points.Let X be the distance of the hit from the center (in feet), and let the probability density function of X be f(x) = $2/(\pi(1+x^2))$, x > 0, 0 otherwise.

What is the expected value of the score of a single shot?

The expected value is

$$\int_0^{\frac{1}{\sqrt{3}}} 4\frac{2}{\pi(1+x^2)} + \int_{\frac{1}{\sqrt{3}}}^{1} 3\frac{2}{\pi(1+x^2)} + \int_1^{\sqrt{3}} 2\frac{2}{\pi(1+x^2)}$$

Consider the following game. Spin a wheel and wait until it comes to rest at some x between 0 and 359. The amount of money won is x/36 - 6 dollars. Let Y be a random variable for your winnings. First, define a probability density function. From there, calculate the expectation and variance.

For the probability density function, we know that the area under the curve must be 1. We know that we can either lose a maximum of -6 dollars or win a maximum of ~4 dollars, so our random variable Y can only range in between -6 and 4. What will be the value at those points? We have 10 values and we need the area of this rectangle to be 1, so f(y) is 1/10 in this area between -6 ≤ y ≤ 4, and 0 elsewhere.

Expectation: $\int_{-\infty}^{\infty} y f(y) dy$ which is actually $\int_{-6}^{4} \frac{y}{10} dy$ which evaluates to1/20 * (16-36) = -1.

Variance: $\int_{-\infty}^{\infty} (y - E[Y])^2 f(y) dy$ which is actually $\int_{-6}^{4} \frac{(y+1)^2}{10} dy$ which evaluates to 1/30 * (125+125) = 25/3.

6.
Linearly Estimate Me

The random variables X, Y, Z are i.i.d. N (0, 1)

(Recall the normal distribution has mean 0 and variance 1).

Find $L[X^2 + Y^2 \mid X + Y]$

The random variables X, Y, Z are i.i.d. N (0, 1)

(Recall the normal distribution has mean 0 and variance 1).

Find $L[X^2 + Y^2 \mid X + Y]$

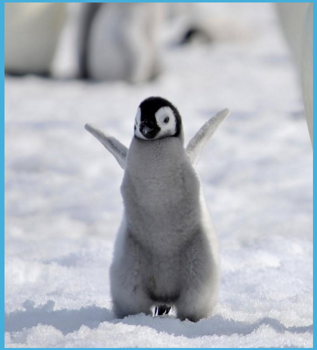
$$
\begin{aligned}
E((X^2+Y^2)(X+Y)) &= E(X^3 + X^2Y + XY^2 + Y^3) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
Cov(X^2+Y^2, X+Y) &= 0 \\
L[X^2+Y^2|X+Y] &= E(X^2+Y^2) \\
&= 2
\end{aligned}
$$

Find $L[X + 2Y \mid X + 3Y + 4Z]$

Find L[X + 2Y |X + 3Y + 4Z]

$$\begin{aligned} cov(X+2Y, X+3Y+4Z) &= E((X+2Y)(X+3Y+4Z)) \\ &= 1+6 \\ &= 7 \\ var(X+3Y+4Z) &= 1+9+16 \\ &= 26 \\ L[X+2Y|X+3Y+4Z] &= 1+9+16 \\ &= 26 \\ L[X+2Y|X+3Y+4Z] &= \frac{7}{26}(X+3Y+4Z) \end{aligned}$$



7.

Diving Into Distributions

Find the distribution of:

$\mathrm{Min}(U_1, U_2)$ where $U_1, U_2 \sim \mathrm{Uniform}[0, 1]$

Find the distribution of:

Min(U_1, U_2) where U_1, U_2 ~ Uniform[0, 1]

$$P(min(U_1, U_2) > x) = P(U_1 > x)P(U_2 > x)$$
$$= (1 - x)^2$$

Find the distribution of:

The sum of N i.i.d Geometric random variables with parameter p

Find the distribution of:

The sum of N i.i.d Geometric random variables with parameter p

$$P(X_N = x) = P(\text{success on the } x^{th} \text{ trial})P(N-1 \text{ success in } x-1 \text{ trials})$$
$$= p\binom{N-1}{x-1}p^{N-1}(1-p)^{x-N}$$
$$= p^N q^{x-N}\binom{N-1}{x-1}$$

$E[U_1 \mid U_1 < U_2]$ where $U_1, U_2 \sim \text{Uniform}[0,1]$

$E[U_1 \mid U_1 < U_2]$ where $U_1, U_2 \sim \text{Uniform}[0,1]$

Notice that this is the same as $\text{Min}(U_1, U_2)$, and we calculated the PDF already, so we can just find the expected value, which is ⅓.

One other way is to use symmetry, which we will use in the next part.

$E[U_1 \mid U_1 > U_2]$ where $U_1, U_2 \sim \text{Uniform}[0, 1]$

$E[U_1 \mid U_1 > U_2]$ where $U_1, U_2 \sim \text{Uniform}[0, 1]$

This is $\max(U_1, U_2)$. Note that

$$E[\max(U_1, U_2) + \min(U_1, U_2)] = E[U_1 + U_2] = 1.$$

Therefore $E[\max(U_1, U_2)] = 1 - \frac{1}{3} = \frac{2}{3}$

$$\mathrm{E}[(X-t)^2] = \mathrm{E}[(X-\mu)^2] + (t-\mu)^2 = Var(X) + (t-\mu)^2$$

$$\mathrm{E}[(X-t)^2] = \mathrm{E}[(X-\mu)^2] + (t-\mu)^2 = Var(X) + (t-\mu)^2$$

$$\begin{aligned} \mathrm{E}[(X-t)^2] &= \mathrm{E}[((X-\mu)-(t-\mu))^2] \\ &= \mathrm{E}[(X-\mu)^2] + (t-\mu)^2 \end{aligned}$$

by multiplying out the terms and using linearity

Find t such that the quantity $g(t) = E[(X - t)^2]$ is minimized.

Find t such that the quantity $g(t) = E[(X - t)^2]$ is minimized.

$g(t) = Var(X) + c$, where $c = (t-\mu)^2$.

Evaluating at $t = \mu$, we find that $c = 0$, which minimizes the above expression.

THANKS!

Good luck!

Solutions will be posted on the CSM 70 Piazza

Apply to be a Junior Mentor for CSM 70 next semester: http://bit.do/sp17_jm_70