Covariance, LLSE, Conditional Expectation, Markov Chains 10

Computer Science Mentors 70

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1 Covariance

1.1 Introduction

The **covariance** of two random variables X and Y is defined as:

$$\text{Cov}(X, Y) := \text{E}((X - \text{E}(X)) \cdot (Y - \text{E}(Y)))$$

1.2 Warm Up

1. Prove that $\text{Cov}(X, X) = \text{Var}(X)$:

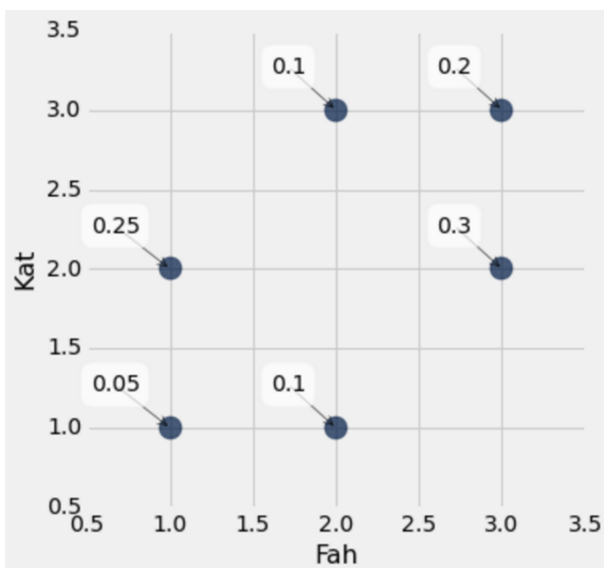
2. Prove that if X and Y are independent, then $\text{Cov}(X, Y) = 0$:

3. Prove that $\text{Cov}(X + Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$:

1.3 Questions

1. Roll 2 dice. Let A be the number of 6's you get, and B be the number of 5's, find $\text{Cov}(A, B)$

2. Consider the following distribution with random variables Fah and Kat:



Find the covariance of Fah and Kat.

2 LLSE

2.1 Introduction

Theorem: Consider two random variables, X, Y with a given distribution $\mathrm{P}[X = x, Y = y]$. Then

$$\mathrm{L}[Y|X] = \mathrm{E}(Y) + \frac{\mathrm{Cov}(X, Y)}{\mathrm{Var}(X)}(X - \mathrm{E}(X))$$

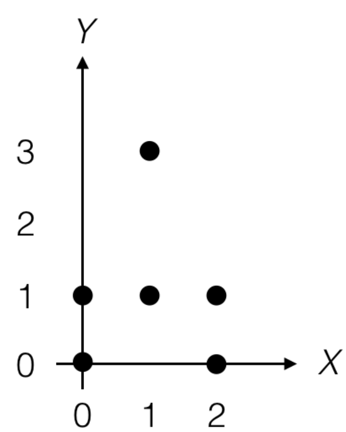
2.2 Questions

1. Assume that

$$Y = \alpha X + Z$$

   where X and Z are independent and $\mathrm{E}(X) = \mathrm{E}(Z) = 0$. Find $\mathrm{L}[X|Y]$.

2. The figure below shows the six equally likely values of the random pair (X, Y). Specify the functions of:
   - $L[Y \mid X]$
   - $E(X \mid Y)$
   - $L[X \mid Y]$
   - $E(Y \mid X)$



3 Conditional Expectation

3.1 Introduction

The **conditional expectation** of Y given X is defined by

$$\mathrm{E}[Y|X=x] = \sum_y y \cdot \mathrm{P}[Y=y|X=x] = \sum_y y \cdot \frac{\mathrm{P}[X=x, Y=y]}{\mathrm{P}[X=x]}$$

Properties of Conditional Expectation

$$\begin{aligned}
\mathrm{E}(a|Y)) &= a \\
\mathrm{E}(aX+bZ|Y) &= a\cdot\mathrm{E}(X|Y) + b\cdot\mathrm{E}(Z|Y) \\
\mathrm{E}(X|Y) &\geq 0 \text{ if } X \geq 0 \\
\mathrm{E}(X|Y) &= \mathrm{E}(X) \text{ if } X, Y \text{ independent} \\
\mathrm{E}(\mathrm{E}(X|Y)) &= \mathrm{E}(X)
\end{aligned}$$

3.2 Questions

1. Prove $\mathrm{E}(\mathrm{E}(Y|X)) = \mathrm{E}(Y)$

2. Prove $\mathrm{E}(h(X)\cdot Y|X) = h(X)\cdot\mathrm{E}(Y|X)$