RSA, Polynomials, Secret Sharing, Erasure Errors 4

Computer Science Mentors 70

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1 Polynomials

1.1 Introduction

1. There is a unique polynomial of degree $n-1$ such that $P(i) = m_i$ for each packet $m_1, \ldots, m_n$
2. To account for errors we send $c_1 = P(1), \ldots, c_{n+j} = P(n+j)$
3. If polynomial $P(x)$ has degree $n-1$ then we can uniquely reconstruct it from any n distinct points.
4. If a polynomial $P(x)$ has degree $n-1$ then it can be uniquely described by its n coefficients

1.2 Questions

1. Define the sequence of polynomials by $P_0(x) = x+12$, $P_1(x) = x^2-5x+5$ and $P_n(x) = xP_{n-2}(x) - P_{n-1}(x)$. (For instance, $P_2(x) = 17x-5$ and $P_3(x) = x^3-5x^2-12x+5$.)
   (a) Show that $P_n(7) \equiv 0 \pmod{19}$ for every $n \in N$.

Solution:

(a) Prove using strong induction.

Base Case There are two base cases because each polynomial is defined in terms of the two previous ones except for P_0 and P_1.

$$P_0(7) \equiv 7 + 12 \equiv 19 \equiv 0 \pmod{19}$$
$$P_1(7) \equiv 7^2 - 5 \cdot 7 + 5 \equiv 49 - 35 + 5 \equiv 19 \equiv 0 \pmod{19}$$

Inductive Hypothesis Assume $P_n(7) \equiv 0 \pmod{19}$ for every $n \leq k$.

Inductive Step Using the definition of P_{k+1}, we have that

$$\begin{aligned} P_{k+1}(7) &\equiv xP_{k-1}(7) - P_k(7) \pmod{19} \\ &\equiv x \cdot 0 - 0 \pmod{19} \\ &\equiv 0 \pmod{19} \end{aligned}$$

Therefore, $P_n(7) \equiv 0 \pmod{19}$ for all natural numbers n.

(b) Show that, for every prime q, if $P_{2013}(x) \not\equiv 0 \pmod{q}$, then $P_{2013}(x)$ has at most 2013 roots modulo q.

Solution: This question asks to prove that, for all prime numbers q, if $P_{2013}(x)$ is a non-zero polynomial $\pmod{q}$, then $P_{2013}(x)$ has at most 2013 roots $\pmod{q}$.

The proof of Property 1 of polynomials (a polynomial of degree d can have at most d roots) still works in the finite field $GF(q)$. Therefore we need only show that P_{2013} has degree at most 2013. We prove that $\deg(P_n) \leq n$ for $n > 1$ by strong induction.

Base cases There are 4:

$$\begin{aligned} \deg(P_0) &= \deg(x + 12) = 1 \\ \deg(P_1) &= \deg(x^2 - 5x + 5) = 2 \\ \deg(P_2) &= \deg(xP_0(x) - P_1(x)) \leq 2 \\ \deg(P_3) &= \deg(xP_1(x) - P_2(x)) \leq 3 \end{aligned}$$

Inductive Hypothesis Assume $\deg(P_n) \leq n$ for all $2 \leq n \leq k$.

Inductive Step Then

$$\begin{aligned} &\deg(P_{k+1}(x)) \\ &\quad \leq \max\{\deg(xP_{k-1}(x)), \deg(P_k(x))\} \\ &\quad = \max\{1 + \deg(P_{k-1}(x)), \deg(P_k(x))\} \\ &\quad \leq \max\{1 + k - 1, k\} \\ &\quad \leq k \\ &\quad \leq k + 1 \end{aligned}$$

2 Secret Sharing

2.1 Questions

1. Suppose the Oral Exam questions are created by 2 TAs and 3 Readers. The answers are all encrypted and we know that:
   (a) Both TAs should be able to access the answers
   (b) All 3 Readers can also access the answers
   (c) One TA and one Reader should also be able to do the same

   Design a secret sharing scheme to make this work.

   Solution: Use a 2 degree polynomial which requires at least 3 shares to recover the polynomial. Generate a total of 7 shares, give each Reader a share, and each TA 2 shares. Then, all possible combinations will have at least 3 shares to recover the answer key. Basically the point of this problem is to assign different weights to different classes of people. If we give one share to everyone, then 2 Readers can also recover the secret and the scheme is broken.

2. An officer stored an important letter in her safe. In case she is killed in battle, she decides to share the password with her troops. Everyone knows there are 3 spies among the troops, but no one knows who they are except for the three spies themselves.The 3 spies can coordinate with each other and they will either lie and make people not able to open the safe, or will open the safe themselves if they can. Therefore, the officer would like a scheme to share the password that satisfies the following conditions:

   1. When M of them get together, they are guaranteed to be able to open the safe even if they have spies among them.
   2. The 3 spies must not be able to open the safe all by themselves.

   Please help the officer to design a scheme to share her password. What is the scheme? What is the smallest M? Show your work and argue why your scheme works and any smaller M couldnt work.

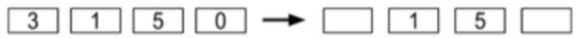
Solution: The key insight is to realize that both polynomial-based secret-sharing and polynomial-based error correction work on the basis of evaluating an underlying polynomial at many points and then trying to recover that polynomial. Hence they can be easily combined. Suppose the password is s. The officer can construct a polynomial $P(x)$ such that $s = P(0)$ and share $(i, P(i))$ to the i-th person in her troops. Then the problem is: what should the degree of $P(x)$ be and what is the smallest M? First, the degree of polynomial d should not be less than 3. It is because when $d < 3$, the 3 spies can decide the polynomial $P(x)$ uniquely. Thus, n will be at least 4 symbols. Lets choose a polynomial $P(x)$ of degree 3 such that $s = P(0)$. We now view the 3 spies as 3 general errors. Then the smallest $M = 10$ since n is at least 4 symbols and we have $k = 3$ general errors, leading us to a codeword of $4 + 2 \cdot 3 = 10$ symbols (or people in our case). Even though the 3 spies are among the 10 people and try to lie on their numbers, the 10 people can still be able to correct the $k = 3$ general errors by the Berlekamp-Welch algorithm and find the correct $P(x)$.

Alternative solution: Another valid approach is making $P(x)$ of degree $M1$ and adding 6 public points to deal with 3 general errors from the spies. In other words, in addition to their own point $(i, P(i))$, everyone also knows the values of 6 more points, $(t+1, P(t+1)), (t+2, P(t+2)), \ldots, (t+6, P(t+6))$, where t is the number of the troops. The spies have access to total of $3 + 6 = 9$ points so the degree $M - 1$ must be at least 9 to prevent the spies from opening the safe by themselves. Therefore, the minimum M is 10.

3 Erasure Errors

3.1 Introduction

We want to send n packets and we know that k packets could get lost.



How many more points does Alice need to send to account for k possible errors? ___

Solution: k

What degree will the resulting polynomial be? ___

Solution: $n - 1$

How large should q be if Alice is sending n packets with k erasure errors, where each packet has b bits?

Solution: Modulus should be larger than $n + k$ and larger than 2^b and be prime

What would happen if Alice instead send $n+k-1$? Why will Bob be unable to recover the message?

Solution: Bob will receive $n - 1$ distinct points and needs to reconstruct a polynomial of degree $n - 1$. By Fact #3 this is impossible. There are q polynomials of at most degree $n - 1$ in $GF(q)$ that go through the $n - 1$ points that Alice sent.

3.2 Questions

1. Suppose $A = 1$, $B = 2$, $C = 3$, $D = 4$, and $E = 5$. Assume we want to send a message of length 3. Recover the lost part of the message, or explain why it can not be done.

1. C_AA

> **Solution:** $P(0) = 3, P(2) = 1, P(3) = 1$. Once we interpolate the polynomial over $\mod 7$, as E is 5, we get $3x^2 + 3x + 3$. Now, once we evaluate this at 1, we get 2. So, in the end, its CBAA.

2. CE_ _

> **Solution:** Impossible. In order to get the original degree 2 polynomial, we need at least $3 > 2$ points.

2. Suppose we want to send n packets, and we know $p = 20\%$ of the packets will be erased. How many extra packets should we send? What happens if p increases (say to 90%)?

> **Solution:** We want to have $(1-p)*(n+k) = n$, where k is the number of additional packets we send. Solving for k, we get $\frac{n}{1-p} - n$. When p is large, we have to send many times the number of original packets. (fraction packets not erased)*(how many packets are sent) = (number packets in original message)