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Is Mean-Variance Analysis Vacuous: Or was Beta Still Born?

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Abstract. We show in any economy trading options, with investors having mean-variance preferences, that there are arbitrage opportunities resulting from negative prices for out of the money call options. The theoretical implication of this inconsistency is that mean-variance analysis is vacuous. The practical implications of this inconsistency are investigated by developing an option pricing model for a CAPM type economy. It is observed that negative call prices begin to appear at strikes that are two standard deviations out of the money. Such out-of-the money options often trade. For near money options, the CAPM option pricing model is shown to permit estimation of the mean return on the underlying asset, its volatility and the length of the planning horizon.

The model is estimated on S&P 500 futures options data covering the period January 1992–September 1994. It is found that the mean rate of return though positive, is poorly identified. The estimates for the volatility are stable and average 11%, while those for the planning horizon average 0.95. The hypothesis that the planning horizon is a year can not be rejected. The one parameter Black–Scholes model also marginally outperforms the three parameter CAPM model with average percentage errors being respectively, 3.74% and 4.5%. This out performance of the Black–Scholes model is taken as evidence consistent with the mean-variance analysis being vacuous in a practical sense as well.

The capital asset pricing model (CAPM) of Sharpe (1964), Lintner (1965) and Mossin (1966) was derived as a general equilibrium consequence of the portfolio decisions made by investors, with mean variance preferences, investing in a single risk free asset and a finite number of risky assets whose joint probability distribution is known to all investors. The CAPM model has had a long history of use and statistical evaluation in the finance literature, with the notable recent contributions of Fama and French (1992, 1995), Jaganathan and Wang (1995), Kothari, Shanken and Sloan (1995) and Roll and Ross (1994). In particular, the recent paper by Fama and French (1995) argues that ‘Beta is dead’ as it has no explanatory value in explaining stock price returns. This paper argues that the CAPM is vacuous as it implies the existence of arbitrage opportunities for deep out-of-the money options. Hence the title ‘was beta still born?’.

The existence of a CAPM equilibrium was first established in the finite asset context by Hart (1974), Nielsen (1990) and for an infinite asset economy by Dana

(1994). This paper uses a result of Dana (1994) to show that the CAPM implies the existence of arbitrage opportunities for economies trading options on the market portfolio with an unbounded sequence of strikes, when the distribution for the market portfolio's return is unbounded. The problem arises because mean variance preferences imply that the pricing kernel is linear in the return on the market portfolio, with a negative slope, thereby forcing call options with high strikes to have negative prices.

This inconsistency between the capital asset pricing model and option trading has been previously observed by Dybvig and Ingersoll (1982) (Theorem l) in the context of a complete market. The fact that their result requires market completeness, gives CAPM enthusiasts an 'out'. The 'out' is that with returns having a continuum support (the positive half line), markets are incomplete. The contribution of this paper is to show that market completeness is not required for this inconsistency. It is the mean variance preferences themselves that are inconsistent with the absence of arbitrage opportunities. The traditional sources for mean-variance preferences – either quadratic utility or return distributions that are multivariate normal or elliptical – do not concern us here. The consequence of this result, then, is that CAPM enthusiasts do not have an 'out'. From a theoretical perspective, with traded deep out of the money options, mean-variance analysis is vacuous.

The practical implication of this inconsistency critically depends on whether options on the market portfolio trade for the strikes admitting arbitrage opportunities. To investigate this possibility we develop an option pricing formula based on the CAPM for an arbitrary and discrete planning horizon $h > 0$. This is in contrast to the Black–Scholes equilibrium which is consistent with no arbitrage and a CAPM with an instantaneous planning horizon, Merton (1973).[1] This distinction between a finite and instantaneous planning horizon is crucial to the existence of the arbitrage opportunities proven below.

We find that the general shape of observed Black–Scholes implied volatilities, as for example in Derman and Kani (1994), are consistent with a CAPM model with call option strikes about two standard deviations out-of-the money generating arbitrage opportunities. Such options do trade, although most of the actively traded options are nearer to the money. We estimate our CAPM option pricing model on S&P 500 futures options data for the period January 1992–September 1994. We find the planning horizon h to be fairly consistently estimated at just under a year, with an average value of 0.95 years and a standard deviation of 0.04. The null hypothesis of the planning horizon being a year is not rejected. Mean rates of return, though positive are found to be poorly identified, while the implied CAPM volatility estimate is fairly stable. The average pricing error is 4.5% which is slightly inferior to the Black–Scholes model with an average percentage error of 3.74%. This outperformance of the Black–Scholes model is taken as evidence consistent with the mean-variance analysis being vacuous in a practical sense as well.

An outline for this paper is as follows. The economy is described in Section 1. Section 2 demonstrates the existence of the arbitrage opportunities. Section 3 derives an explicit option pricing formula consistent with a mean-variance preference CAPM. Section 4 studies the behavior of Black–Scholes implied volatilities for options trading in a CAPM type economy. Results of estimating the CAPM option pricing model on S&P 500 futures option data are presented in Section 5. Section 6 comments on the implications for other asset pricing theories invoking mean variance considerations. Finally, Section 7 concludes the paper.

1. The Economic Model

This section presents the economic model. Consider a one period economy with dates $t = 0, 1$. Let $(\Omega, \mathcal{F}, P)$ be a probability space with the set of events Ω. All uncertainty is resolved at time 1. The economy has n investors, indexed by $i = 1, \ldots, n$, who know the probability measure P on $\mathcal{F}$.

Traded at time 0 in the economy is a closed subspace, Z, of P-square integrable, $\mathcal{F}$-measurable real-valued functions on Ω. The investors' endowments are given by non-negative elements $\varepsilon_i \in Z$, for $i = 1, \ldots, n$. The aggregate endowment is $\varepsilon = \sum_{i=1}^{n} \varepsilon_i$. We let ε have a distribution under P with unbounded support on the positive half line. It follows under the above structure that all endowments have finite mean and variance under P.

We let Z include the constant payoff $\mathbf{1}_\Omega$ (a riskless asset), and a sequence of call options on the aggregate endowment ε with time one payoffs $(\varepsilon - k_j)^+$ for an unbounded increasing sequence of strikes k_j. The closed subspace Z therefore has infinite dimension.

The investors have mean-variance preferences given by a utility function of the form

$$u_i(z) = U_i(E[z], \text{var}(z)), \tag{1}$$

for $z \in Z$ and $i = 1, \ldots, n$, where E is the expectation operator under the measure P and var(z) is the variance of z under P which equals $E[z^2] - (E[z])^2$. We suppose, as in Dana (1994), that the partials U_{i1} and U_{i2} are respectively positive and negative, reflecting a preference for mean and an aversion to variance.[2]

A competitive equilibrium for this economy is given by a continuous linear pricing functional ϕ on the space of traded claims Z and a vector $\bar{c} = (\bar{c}_i, \ldots, \bar{c}_n) \in Z^n$ denoting the consumption of the n investors, such that (a) each investor's utility is maximized subject to their budget constraint, i.e. $\phi[\bar{c}_i] \leq \phi[\varepsilon_i]$ and if $\phi[c_i] \leq \phi[\varepsilon_i]$ for $c_i \in Z$ then $u_i(c_i) \leq u_i(\bar{c}_i)$, and (b) markets are cleared, i.e. $\sum_{i=1}^{n} \bar{c}_i = \varepsilon$. The existence of such an equilibrium for this economy is proven in Dana (1994).

2. The Arbitrage Opportunity

This section shows that the above equilibrium contains an arbitrage opportunity. As a result, the mean-variance analysis is vacuous.

First note that the space Z is a closed subspace of the space of square integrable random variables. It follows from the Riesz representation theorem, that the continuous linear functional ϕ has a representation in terms of product moments, specifically, for some $\pi \in Z$,

$$\phi[z] = E[\pi z], \quad \text{for all } z \in Z. \tag{2}$$

Dana (1994) characterizes π as linear in the aggregate endowment ε. We reproduce this proof here in somewhat greater detail.

Theorem 1. The pricing functional π has the form

$$\pi = a - b\varepsilon \tag{3}$$

for scalars a and b, where b is positive.

Proof. Let u be an arbitrary element of Z and define for each $i = 1, \ldots, n$, the real valued function of the scalar argument t, $f_i(t)$ by

$$f_i(t) = u_i(\bar{c}_i + tu - t(\phi[u]/\phi[\mathbf{1}_\Omega])\mathbf{1}_\Omega) - u_i(\bar{c}_i). \tag{4}$$

Observe by construction that $\phi[\bar{c}_i + tu - t(\phi[u]/\phi[\mathbf{1}_\Omega])\mathbf{1}_\Omega] = \phi[\bar{c}_i] \leq \phi[\varepsilon_i]$ and hence it follows that $\bar{c}_i + tu - t(\phi[u]/\phi[\mathbf{1}_\Omega])\mathbf{1}_\Omega$ is budget feasible for investor i. Therefore by the defining property of $\bar{c}_i$ we have that $u_i(\bar{c}_i + tu - t(\phi[u]/\phi[\mathbf{1}_\Omega])\mathbf{1}_\Omega) \leq u_i(\bar{c}_i)$ or that $f_i(t) \leq 0$ for all t. We therefore have that $f_i'(0) = 0$.

From Equation (1) we may write

$$f_i(t) = U_i(E[\bar{c}_i + tu - t(\phi[u]/\phi[\mathbf{1}_\Omega])\mathbf{1}_\Omega], \text{var}[\bar{c}_i + tu]) \tag{5}$$

Let $a_i = U_{i1}(E[\bar{c}_i], \text{var}[\bar{c}_i]) > 0$ and $b_i = -U_{i2}(E[\bar{c}_i], \text{var}[\bar{c}_i]) > 0$ then we may write

$$0 = f_i'(0) = a_i E[u - (\phi[u]/\phi[\mathbf{1}_\Omega])\mathbf{1}_\Omega] - 2b_i E[(\bar{c}_i - E[\bar{c}_i]\mathbf{1}_\Omega)u]. \tag{6}$$

Using the characterization of ϕ given by Equation (2), whereby $\phi(u) = E[\pi u]$, Equation (6) may be rewritten in the form

$$E[(a_i(\mathbf{1}_\Omega - 1/\phi[\mathbf{1}_\Omega]\pi) - 2b_i(\bar{c}_i - E[\bar{c}_i]\mathbf{1}_\Omega))u] = 0. \tag{7}$$

Since Equation (7) holds for all arbitrary $u \in Z$ we must have that

$$a_i(\mathbf{1}_\Omega - 1/\phi[\mathbf{1}_\Omega]\pi) - 2b_i(\bar{c}_i - E[\bar{c}_i]\mathbf{1}_\Omega) = 0, \tag{8}$$

and hence that

$$\bar{c}_i = (a_i/2b_i + E[\bar{c}_i])\mathbf{1}_\Omega - (a_i/(2b_i\phi[\mathbf{1}_\Omega]))\pi. \tag{9}$$

Summing (9) over all i and employing the market clearing condition yields

$$\varepsilon = a'\mathbf{1}_\Omega - b'\pi \tag{10}$$

where $a' = \sum_{i=1}^{n}(a_i/2b_i + E[\bar{c}_i])$ and $b' = \sum_{i=1}^{n}(a_i/(2b_i\phi[\mathbf{1}_\Omega]))$. Note that b' is positive. Defining $a = a'/b'$ and $b = 1/b' > 0$ we can rewrite (10) as (3). □

It follows from Theorem 1 that π is negative for ε sufficiently large. Hence the market price of a call option on ε with a strike k_j, for sufficiently large k_j is

$$c = E[(a - b\varepsilon)(\varepsilon - k_j)^+] < 0, \tag{11}$$

when the distribution of ε has unbounded support. But as a call option is a limited liability asset with a non-negative cash flow, a negative price for such an asset constitutes an arbitrage opportunity.

It follows then that mean-variance preferences are inconsistent with the absence of arbitrage opportunities, for any equilibrium trading options on the endowment with an unbounded sequence of strikes, when the endowment also has a distribution with unbounded support. As adding this arbitrage opportunity to the investor's consumption is feasible, it must not increase his utility, or the equilibrium could not exist. In fact, this problem with mean variance preferences is quite dramatic as the preferences are typically not monotone. That is given a random variable x with utility

$$u_i(x) = U_i(E[x], \text{var}[x])$$

one can add a nonnegative variable y of high volatility to construct $z = x + y$ so that $U_i(E[z], \text{var}(z)) < U_i(E[x], \text{var}[x])$. Hence z dominates x uniformly over the whole space (y is an arbitrage opportunity) but x is preferred to z by the mean-variance preference ordering.

This inconsistency is for a finite horizon $h > 0$. For a continuous trading economy, the inconsistency does not appear, as demonstrated by Merton (1973) and Black and Scholes (1973). From the work of Merton (1973) on the theory of asset pricing in continuous time with diffusion uncertainties, we know that expected utility maximizers with concave utility functions behave as if their instantaneous preferences are mean variance. Furthermore, when risky asset prices have a constant investment opportunity set and a riskless asset trades, then all investors hold a linear combination of the risk free asset and a unique mean-variance efficient risky asset portfolio. The asset holdings in this portfolio match those from static mean-variance theory, and are constant through time. The market portfolio must also be efficient and if the riskless asset is in zero net supply then the market portfolio must be the mean-variance efficient portfolio. In this setting, the number of shares outstanding fluctuates to keep the price processes log normal as noted by Merton (1975). The state price density induced by the Black–Scholes economy, and shared with the Merton continuous time equilibrium, is a nonnegative process given by the stochastic exponential of a locally linear position in the market portfolio.

Consequently, it is the non-monotone, mean variance preferences over a finite planning horizon $h > 0$ that is inconsistent with no arbitrage in the presence of option trading. For mean-variance preferences over a finite planning horizon, the state price density is linear in the market portfolio itself.

Although the finite planning horizon mean variance theory may have been reasonable prior to 1973 when option trading was essentially nonexistent, its continued relevance today is questionable given the active trading in derivatives. However, we recognize that theoretical inconsistencies may not be relevant in practice. The next section studies the practical implications of this inconsistency.

3. Option Pricing in Finite Horizon Mean-Variance Equilibria

As a first step in assessing the practical importance of this inconsistency, we derive an option pricing model for the traded calls in this economy. We term this new option pricing model the CAPM option pricing model as it is the pricing model that would prevail in an economy with mean-variance preferences. Consider an economy over the interval $[0, T]$, with investors preferences expressed as functions of the mean and variance of the wealth accumulation at time T. For this economy, the pricing functional was shown in theorem 1 to be linear in the time T endowment of the economy. If at each time t, $0 \leq t < T$, the economy equilibrates with mean variance preferences for the time T position, then the time t pricing functional is by Theorem 1 again of the form

$$\pi_{t,T} = a_t - b_t \varepsilon_T \tag{12}$$

where a_t, b_t are scalars and ε_T is the time T endowment, a random variable in the set of traded assets Z. The measure change provided by employing Equation (12) as the time t pricing functional is that of a signed measure.

Suppose that a money market account paying a continuous interest rate of r is available for riskless investment in this economy. Any contingent claim paying $c(s)$ time s for $t < s \leq T$ is equivalent to one paying $c(s)e^{r(T-s)}$ at time T. The time t price of such a claim is then given by applying the price functional to its equivalent time T payoff. This is because preferences focus on time T. The time t price of such a claim is therefore

$$\Phi_t[c(\cdot)] = E[(a_t - b_t \varepsilon_T)c(s)e^{r(T-s)} | \mathcal{F}_t] \tag{13}$$

Let the endowment (under P) be a geometric Brownian motion with mean μ and variance σ^2, then for all t, $0 \leq u \ \leq T$

$$\varepsilon_u = \varepsilon_0 e^{\mu u + \sigma b(u) - \sigma^2 u/2}, \tag{14}$$

where $b(u)$ is a standard Brownian motion.

Substituting from (14) into (13) and evaluating expectations conditional on $\mathcal{F}_s$, we obtain

$$\Phi_t[c(\cdot)] = E[(a_t - b_t\varepsilon_s e^{\mu(T-s)})c(s)e^{r(T-s)}|\mathcal{F}_t]. \tag{15}$$

Consider now the case of a call option trading at t, maturing at s, $t < s \leq T$, and written on the endowment value at time s, ε_s with a strike of K. In this case

$$c(s) = (\varepsilon_s - K)^+. \tag{16}$$

The time t price of the call option denoted $w_t(\varepsilon_t, K, \mu, \sigma, t, s, T, r)$ is:

$$w_t(\varepsilon_t, K, \mu, \sigma, t, s, T, r) = E[(a_t - b_t\varepsilon_s e^{\mu(T-s)})(\varepsilon_s - K)^+ e^{r(T-s)}|\mathcal{F}_t]. \tag{17}$$

Before evaluating expression (17), note that the coefficients a_t and b_t can be determined from the conditions analogous to (17) for both the time t price of a bond paying a unit face value at T, and a security paying the endowment at T, with time t value denoted V_t. Defining $h = (T - t)$ as the time horizon these conditions give:

$$e^{-rh} = a_t b_t V_t e^{\mu h} b_t \tag{18a}$$

and

$$V_t = V_t e^{\mu h} a_t - V_t^2 e^{2\mu h} + \sigma^2 h b_t \tag{18b}$$

The solution for the coefficients a_t and b_t is therefore

$$a_t = \frac{e^{(\sigma^2 - r)h} - e^{-\mu h}}{e^{\sigma^2 h} - 1}, \tag{19a}$$

and

$$b_t = \frac{e^{(\mu - r)h} - 1}{V_t e^{2\mu h}(e^{\sigma^2 h} - 1)}. \tag{19b}$$

Standard integration techniques yield the option price of Equation (17) in terms of $\tau = (s - t)$, the time to maturity, and $h = (T - t)$ the time horizon, as:

$$\begin{aligned} w_t(V_t, K, \mu, \sigma, \tau, h, r) \; = \; & (a_t e^{r(h-\tau)} + b_t K e^{(\mu+r)(h-\tau)}) V_t e^{\mu\tau} N(d_1) \\ & - a_t e^{r(h-\tau)} K N(d_2) \\ & - b_t e^{(\mu+r)(h-\tau)} V_t^2 e^{2\mu\tau + \sigma^2\tau} N(d_3), \end{aligned} \tag{20}$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln(V_t/K)}{\sigma\sqrt{\tau}} + \left(\frac{\mu}{\sigma} + \frac{\sigma}{2}\right)\sqrt{\tau} \\
d_2 &= d_1 - \sigma\sqrt{\tau} \\
d_3 &= d_1 + \sigma\sqrt{\tau}.
\end{aligned} \tag{21}
$$

Similar to the derivation of Equation (20), one can derive the value of a put option on the endowment at time s as:

$$
\begin{aligned}
w_t'(V_t, K, \mu, \sigma, \tau, h, r) \;=\; & -(a_t e^{r(h-\tau)} + b_t K e^{(\mu+r)(h-\tau)}) V_t e^{\mu\tau} N(-d_1) \\
& + a_t e^{r(h-\tau)} K N(-d_2) \\
& + b_t e^{(\mu+r)(h-\tau)} V_t^2 e^{2\mu\tau+\sigma^2\tau} N(-d_3),
\end{aligned} \tag{22}
$$

Subtracting (22) from (21) we observe that

$$
\begin{aligned}
w_t - w_t' \;=\; & (a_t e^{r(h-\tau)} + b_t K e^{(\mu+r)(h-\tau)}) V_t e^{\mu\tau} \\
& -(a_t e^{r(h-\tau)} K + b_t V_t^2 e^{2\mu\tau+\sigma^2\tau} e^{(\mu+r)(h-\tau)}),
\end{aligned} \tag{23}
$$

and it follows from Equations (18a) and (18b) that when $\tau = h$, put-call parity holds as

$$
w_t - w_t' = V_t - K e^{-r\tau}. \tag{24}
$$

In general, however, put-call parity fails to hold for h exceeding τ. To see this, using Equations (18) we obtain that

$$
\begin{aligned}
w_t - w_t' \;=\; & V_t - K e^{-r\tau} + V_t(e^{(r-\mu)(h-\tau)} - 1) \\
& + b_t V_t^2 e^{2\mu h+\sigma^2 h} e^{(r-\mu)(h-\tau)} (1 - e^{-\sigma^2(h-\tau)}).
\end{aligned} \tag{25}
$$

The difference between the call and put option prices is not equal to the endowment price less the price of a bond with a face value equal to the option strike prices and maturing along with the options. The discrepancy, however, is small for h close to τ.

This violation is understood by noting that we do not have a martingale type condition holding for the direct sale of the endowment at time $s = (t + \tau)$. If trading in the endowment at time s was allowed, and the endowment sold for V_s as given by our lognormal process, then we must have the pricing condition

$$
V_t = E[(a_t - b_t V_T) V_s e^{r(T-s)} | \mathcal{F}_t]. \tag{26}
$$

This yields on evaluation that

$$V_t = a_t V_t e^{\mu\tau + r(h-\tau)} - b_t V_t^2 e^{2\mu\tau + \sigma^2\tau + (\mu+r)(h-\tau)}, \tag{27}$$

which if substituted into (23) yields (24). But it easy to see that (27) holding for all τ is inconsistent with (18b). Hence in our economy, the endowment is analogous to a closed end fund's net asset value process following a geometric Brownian motion on which options are written. For a closed end fund, at each t there is a market only for the net asset value at T, and no direct market for the net asset values at time t itself.[3]

4. Black–Scholes Implied Volatilities in the Finite Horizon CAPM Equilibria

The CAPM option pricing model of Section 3 is inconsistent with the absence of arbitrage opportunities. We wish to assess here whether this inconsistency is merely theoretical or whether it also has relevance in practice. In practice, a pricing kernel that goes negative may price all of a finite set of limited liability traded assets positively, and given the linearity of the pricing rule, there will be no arbitrage opportunities. The arbitrage opportunity detected in Section 2 presumed that calls of sufficiently high strikes trade. It may be that these strikes are outside the range of strikes actually traded in practice, and an arbitrage free CAPM equilibrium holds in practice.

To study this issue, we compute the implied volatilities using a Black–Scholes formula for option prices given by the CAPM option pricing model of Section 3. For this purpose we calibrate CAPM option prices using a spot price of 100, a mean rate of return on the risky asset of 15%, a volatility of 20%, and an interest rate of 5%. The implied volatilities are graphed for comparison with reported implied volatility graphs, as for example in Derman and Kani (1994). We graph implied volatilities for a range of strike prices, and a maturity of 0.5 or half a year. We also present a graph of implied volatilities against maturity for options struck at the money. The time horizon of the CAPM economy is set at one year.

Figure 1 graphs implied volatilities against the option strike, for strikes ranging from 80 to 120, or one standard deviation in each direction. The general shape of the implied volatility function, in Figure 1 is similar to that reported in Derman and Kani (1994), though the absolute magnitude of the fall in implied volatilities appears somewhat less.

Figure 2 graphs implied volatilities against the option maturity. Here we observe a slight reduction, as opposed to the considerable increase reported in Derman and Kani (1994). The decrease is to be expected, however, as an increase in maturity raises the effective volatility, and gives a relatively greater weighting to regions where the pricing kernel goes negative, thereby lowering the CAPM price and hence the implied Black Scholes volatility.

The CAPM call option price does not go negative until we reach a strike of 140, or an option that is out of the money by two standard deviations. The

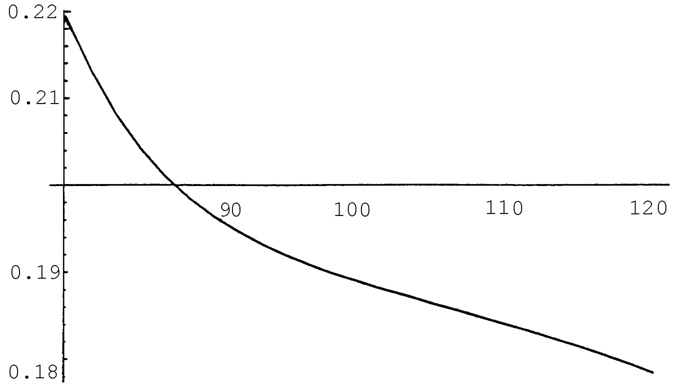

Figure 1. Black–Scholes implied volatilities for CAPM equilibrium option prices, implied volatilities as a function of the option strike.

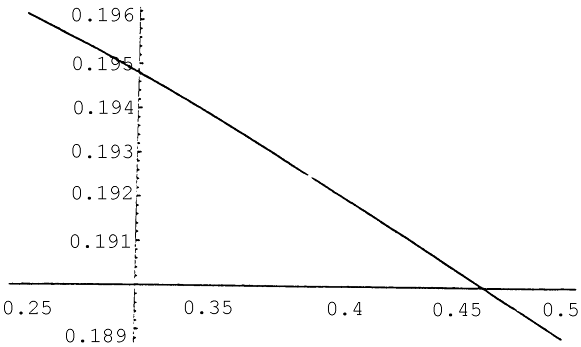

Figure 2. Black–Scholes implied volatilities for CAPM equilibrium option prices, implied volatilities as a function of the option maturity.

Black–Scholes price of the 140 strike, half year maturity option is 8 cents and as prices are often quoted as low as as 1/16 or 1/32, options of a 140 strike on a 20% volatility are not out side the range of derivatives actually traded. Hence, we conclude that mean variance preferences do pose an empirically serious problem for economies trading derivatives. To further evaluate this inconsistency, the next section empirically compares the CAPM option pricing model to Black–Scholes.

5. Estimation Results

This section compares the CAPM and Black–Scholes option pricing models. One may employ the CAPM option pricing model to infer estimates of its three parameters from current option prices. These parameters are the drift on the underlying asset price, its volatility, and the length of the planning horizon $h = T - t$.[4] In this section, we report the results of estimating both the CAPM and Black–Scholes option pricing models on S&P 500 futures option price data.

The data for the estimation was obtained from the Financial Futures Institute in Washington D.C. The data consists of time stamped price observations on S&P 500 futures options from January 1992 to September 1994. Closing prices on index futures were also available as was the level of the spot index. Daily data on the three month Treasury Bill rate was obtained from the Federal Reserve in Washington D.C. and a series of dividend yields was constructed using spot forward arbitrage. The options were viewed as written on the underlying spot index.

Up to 16 options were selected for each day ensuring that prices were time stamped at near the daily close. Up to four strikes for each of four maturities were selected. The strikes chosen were those that had the highest trading volume. There were 2824 option prices selected from the 1992 data file, 3010 from the 1993 file, and 2411 from 1994. This gave us a set of 8245 option prices. The estimation of the finite horizon CAPM option pricing model was done weekly for 140 weeks. The number of option price observations for each week varied from 40 to 80.

The parameters of both the CAPM and Black Scholes option pricing models were estimated using nonlinear least squares on the percentage error of the model price relative to the observed market price. The results are reported in Table I. For the CAPM model the average estimated value of μ, the mean return on the index, is 8.92% across the 140 weekly estimates, with an average t statistic of 0.25%. There is also a considerable instability in the estimation of this parameter and this is reflected in Figure 3 that graphs the weekly mean return and volatility estimates. On the other hand the volatility and horizon estimates are fairly stable and significant with average values of 11% and 0.9533 years, respectively. Graphs of the weekly horizon estimates are presented in Figure 4. The Black–Scholes model volatility estimates are also stable, comparable to those of the CAPM model, with an average value of 11.2%. The Black–Scholes model outperforms the CAPM model, even though it has two fewer parameters. The Black–Scholes average percentage error is 3.74%, compared to the 4.5% error of the CAPM model. Figure 5 plots the weekly mean percentage pricing errors from the Black–Scholes model and the CAPM model. We observe that the performance of the CAPM option pricing model is quite consistently inferior to that of the Black–Scholes model even though it has two more parameters. This evidence is consistent with the empirical inconsistency of the mean-variance analysis.

6. Implications for Other Asset Pricing Theories

The CAPM is not the only asset pricing theory to embody the implications of mean variance preferences. The Ross (1976) Arbitrage Pricing Theory, as demonstrated by Chamberlain and Rothschild (1983) and Jarrow (1988), is essentially equivalent to assuming the continuity of the market pricing functional in the L^2 norm on the space of cash flows viewed as finite second moment random variables. This requires that if means and variances of a sequence of cash flows converge to zero, then so do their market prices. It follows, as shown for example in Madan and Milne

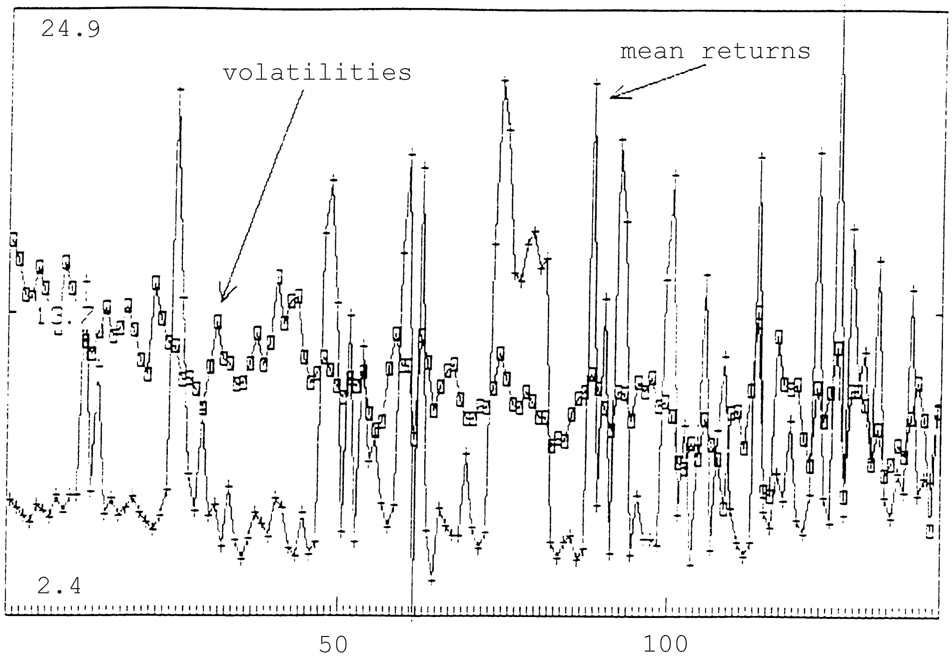

Figure 3. Weekly estimates of mean returns and volatilities, January 1992–September 1994.

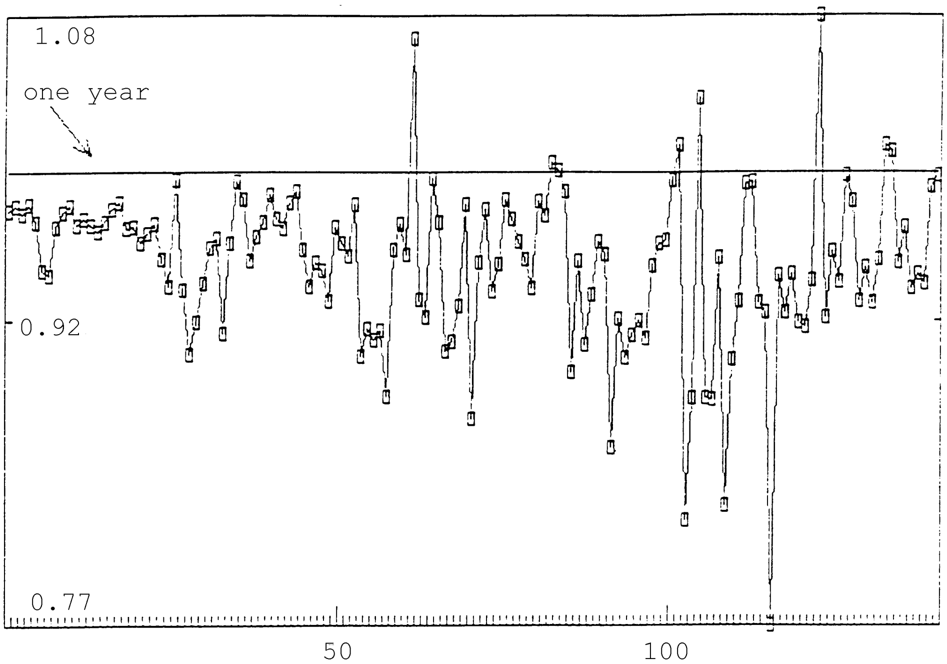

Figure 4. Weekly time horizon estimates in years, January 1992–September 1994.

Table I. Results on estimating the CAPM and Black–Scholes option pricing models

	Parameter values and t-statistics	
	CAPM model	Black–Scholes model
μ	8.92%	–
	(0.25)	
σ	11%	11.2%
	(6.98)	(5.52)
h	0.9533	–
	(2.01)	
Mean percentage error	4.5%	3.74%

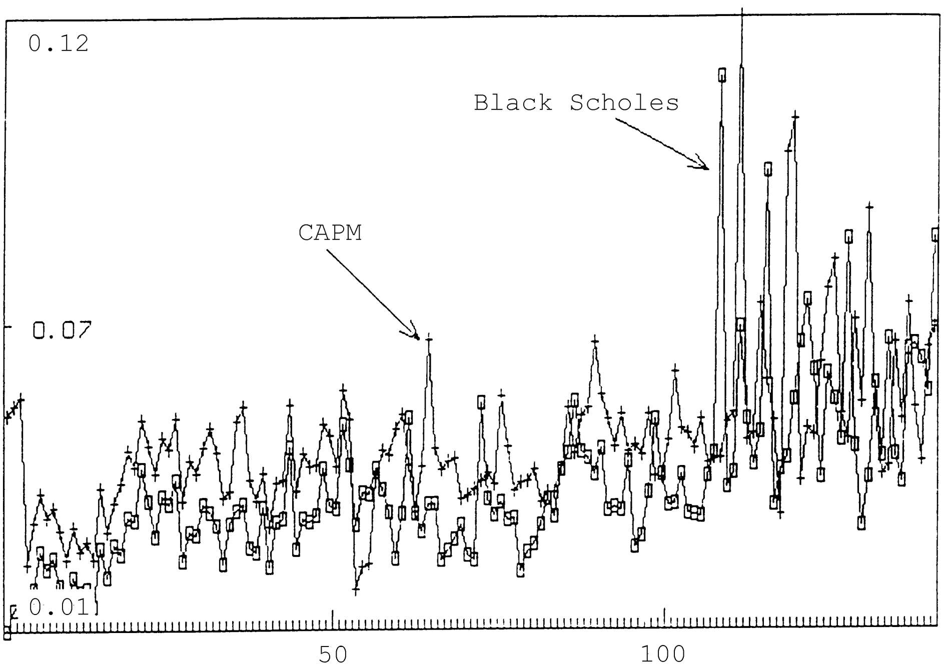

Figure 5. CAPM and Black–Scholes percentage pricing errors, January 1992–September 1994.

(1994), that under assumptions of separability, the pricing kernel may be written as a linear combination of an infinite sequence of orthonormal basis elements with basis element prices as coefficients. Typically, the entire sequence of basis elements would be involved in attaining a positive pricing kernel, and this gives us an asset pricing model with a countable infinity of factors. There is no inconsistency here.

In contrast, equilibrium arbitrage pricing theories that deliver an exact finite factor asset pricing model, using for the factors a finite set of portfolios, once again yield pricing kernels that are linear in a set portfolio returns. Such finite factor pricing kernels are problematic for economies trading derivatives, on the so-called factor mimicking portfolios. A problem of consistency with no arbitrage arises if the support of the return distribution extends in to the region where the pricing kernel is negative and options with strikes in this range trade. Essentially, if the pricing kernel is linear in a portfolio return then non-negativity of the kernel will require boundedness of the portfolio return.

Mean variance preferences are also employed to formulate hedging strategies in incomplete markets contexts, and lead to the minimal martingale law as the candidate for the pricing kernel (see Follmer and Schweizer (1990), Duffie and Richardson (1991)). As shown in Colwell and Elliott (1993) and Runggaldier and Schweizer (1996) for continuous time models with jump discontinuities, and Elliott and Madan (1996) for discrete time models, the resulting minimal martingale law pricing kernel is no longer non-negative. This again yields the inconsistency for various traded derivatives, and casts doubt on the usefulness of the minimal martingale measure as a pricing kernel.

7. Conclusion

This paper shows that in any equilibrium economy in which investors have mean-variance preferences and a wide set of derivatives trade, arbitrage opportunities will exist. Such equilibria are therefore inconsistent with the absence of arbitrage. The inconsistency is due to the linearity and negativity of the pricing kernel for large realizations of the endowment. This implies that mean-variance analysis is vacuous. The practical implication of this violation is investigated by developing an option pricing model for a CAPM type economy that trades options on the market portfolio. It is observed that negative call prices begin to appear at strikes that are two standard deviations out of the money. As such options typically trade, this is a serious concern.

To further illustrate this concern, the CAPM option model is estimated on S&P 500 futures options data covering the period January 1992–September 1994. It is found that the mean rate of return estimates, though variable and poorly identified, are consistently positive. The estimates for the volatility are stable and average 11%, while those for the planning horizon average 0.95. The hypothesis that the planning horizon is a year is not rejected. The CAPM option model gives a mean pricing error of 4.5%, which is inferior to the performance of the Black Scholes model, with a comparable error of 3.74%.

We conclude that the use of mean variance preferences is vacuous. Although analytically attractive in simplifying the issues of risk measurement through an asset's beta, the CAPM has serious theoretical drawbacks. Unfortunately, this inconsistency also extends to finite factor Ross style arbitrage pricing theories, when the factor realizations are unbounded.

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Notes

[1] The Black–Scholes formula may also be derived in discrete time models using particular utility functions, wealth and return distributions as in Brennan (1979). Here the subspace of assets with normally distributed returns are priced as if by mean-variance, but preferences in general are not mean variance.

[2] We are not concerned here with the foundations for mean variance preferences from either quadratic utility or normally distributed returns. It is their universal relevance that is the source of inconsistency. As noted earlier, their partial relevance for normally distributed instantaneous returns can be consistent with a Black–Scholes equilibrium.

[3] The market value of a closed end fund may be viewed as the price today of the net asset value at the future liquidation date, say time T. The net asset value today is not a traded asset.

[4] The ability to observe h has been noted earlier in another context by Merton (1975).

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