Chapter Seventeen

The Sortino Ratio and Extreme Value Theory: An Application to Asset Allocation

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17.1 INTRODUCTION

Modern portfolio theory has its genesis in the seminal works of Markowitz (1952) and Roy (1952).[1] Before their works, the notion of considering investments in the portfolio context was known by academicians and practitioners but lacked a coherent theory.[2] Combining stocks and other assets to create diversified portfolios with desired risk and return characteristics quickly caught on and was extended to describe the behavior of the stock market using the capital asset pricing model

[1]In 1990, Markowitz won the Nobel Prize for his work and Roy did not. Markowitz (1999) attributes this outcome to the difference in the visibility of their total work in this area. He subsequently published many articles derived from his initial insights. See in particular Markowitz (1959, 1987). Roy, however, appears to have stopped contributing to this area after his initial article.

[2]See, for example, Hicks (1935), Markschak (1938), and Williams (1938). Rubinstein (2002) suggests that Fisher (1906) may have been the first to suggest that variance be used as a measure of economic risk. Much earlier, Shakespeare (1600) alludes to diversification in the *Merchant of Venice* by having Antonio, who is involved in maritime trade, comment that his business is spread across different ships, locations, and time.

Extreme Events in Finance: A Handbook of Extreme Value Theory and its Applications,
First Edition. Edited by François Longin.

(CAPM) (Sharpe, 1964; Lintner, 1965; Mossin, 1966). This model (often referred to simply as the CAPM), provides the theoretical basis for the Sharpe ratio (Sharpe, 1966, 1994), a performance measure that relates the return of a portfolio in excess of the risk-free rate to the risk of the portfolio as measured by its volatility, or more precisely, the standard deviation of its returns. This ratio has remained popular since its introduction and has proven to be robust in many environments.

Following the finance orthodoxy initiated by Bachelier (1900), the CAPM and the Sharpe ratio assume that the returns distribution is Gaussian, a distribution that is completely described by its mean and variance. This assumption is heroic, because the tails of the observed distributions are typically fatter (thicker) than those associated with a normal distribution (e.g., Mandelbrot, 1963; Fama, 1965a; Brock et al., 1991; Mandelbrot, 1997). Markowitz (1952) recognized that asset price distributions may not be normal and mentioned the possible need to consider the notion of left-tail skewness. Roy (1952) independently formalized this concept by promoting the safety-first criterion for portfolio construction. Following the influence of Mandelbrot (1963), Fama (1965b) considered the situation of portfolio selection when stock returns are described as a stable Paretian distribution. Arzac and Bawa (1977) and Harlow and Rao (1989) extended the safety-first notion of portfolio choice to asset pricing. Eventually, the label of "safety first" was replaced with the more general (and perhaps descriptive) term "downside risk."

Asset allocation is an aggregate version of portfolio construction. It involves the creation of an overall portfolio using instruments of different asset classes (e.g., bonds, stocks, real estate, commodities, money market instruments, and so forth) as inputs. Many researchers have explored the notion of downside risk in the asset allocation framework (e.g., Harlow, 1991; Lucas and Klaassen, 1998; Neftci, 2000; Booth and Broussard, 2002; and more recently Ornelas et al., 2012). Practitioners are also very concerned with the issue of downside risk, as can be witnessed by the recent development and usage of value-at-risk methods and stress testing (e.g., Longin, 2000; Berkowitz and O'Brien, 2002; Jorion, 2007). The concern of both groups is rooted in the notion that the typical remedies for diversification are inadequate in periods of stress and turmoil because large price swings appear to correspond to agents attempting to adjust their portfolios in the same manner.[3]

Brian Rom (Investment Technologies) created what is now known as the Sortino ratio in 1983.[4] He named it after Frank A. Sortino, who was an early promoter of using downside risk to measure performance. The Sortino ratio is a modification of the Sharpe ratio and requires as input minimum acceptable return (MAR) – often referred to as the target return (TR). Its numerator is the portfolio return in excess of the TR.[5] Its denominator is the square root of the

[3]This observation is supported by the empirical findings of Gabaix et al. (2003) and is consistent with the notion that agents learn from each other and behave accordingly. See, for example, Duffy and Feltovich (1999), Hong et al. (2005), and Booth et al. (in press). Using an agent-based model, Feng et al. (2012) show that this characteristic is consistent with the presence of fat-tailed return distributions.

[4]Examples of Sortino's work include Sortino and van der Meer (1991), Sortino and Price (1994), and Sortino (2001, 2010).

[5]In some early versions of the Sortino ratio, the target return is replaced by the risk-free rate.

lower second moment around the target rate. Like the Sharpe ratio, the Sortino ratio can be expressed in ex post and ex ante terms. The ex post version can be used as a performance measure, while the ex ante formulation can be employed to construct an expected optimal portfolio.

The purpose of this chapter is to apply extreme value theory (EVT) techniques when using the Sortino ratio for selecting optimal portfolio weights. We restrict our analysis to two assets for simplicity, with one of the assets being a portfolio of the U.S. real estate investment trusts (REIT) and the other a broad-based portfolio of U.S. equities represented by the S&P 500 stock index.[6] Both of these asset classes are well represented in the overall investment holdings of institutional investors such as foundations, pension funds, and insurance companies.

Our approach is threefold. First, we use an extensive daily dataset to calculate the Sortino ratio using various two-asset allocation weights. This requires calculating the second lower partial moment for selected MARs using discrete daily data. Second, we repeat the calculations after replacing the empirical data by a fitted generalized Pareto distribution (GPD). Third, we compare the Sortino ratio asset allocation results provided by the various scenarios to the portfolio suggested by the Sharpe ratio.

In our study period, we find that optimal asset allocation is strongly dependent on the risk tolerance of the investor. We also find that the choice of performance measure matters. Depending on the risk level and measure chosen, the appropriate portfolio allocation ranges from 100% S&P 500 to 81% REIT. Our allocation results differ markedly from those indicated by the Sharpe ratio, which does not explicitly consider downside risk. Our results also show, not surprisingly, that the recent financial crisis highlights the potential volatility embedded in real estate investments, and this volatility may have underpinned the long-held notion that real estate should only comprise a small part of an investor's investible assets.

The remainder of this chapter is divided into four sections. First, we describe and discuss the data, paying particular attention to its statistical properties. Second, we mathematically define the Sharpe ratio and Sortino ratio, and emphasize the specification of the second lower partial moment. In this section we also present our statistical methods, tools, and statistical considerations. Third, we present the results of our analyses, including a comparison of the allocation implications of the Sortino and Sharpe risk measures. Finally, we conclude by summarizing our findings, commenting on their relationships to previous work, and providing some suggestions for further research.

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[6]REITs are functionally similar to closed-end mutual funds and are authorized and governed by the Real Estate investment Act of 1960. The purpose of this legislation is to increase real estate investment by institutions and wealthy individuals by creating an investment vehicle that is tax free at the corporate level but taxable to the investor when the generated income is distributed. REITs became a major investment category only after the enactment of the Tax Reform Act of 1986, which significantly reduced the financial advantages of real estate partnerships. A brief history of REITs is provided by Woolley et al. (1997). This chapter also summarizes some of the industry's major business strategies. See also Brock (1998).

17.2 DATA DEFINITIONS AND DESCRIPTION

We use daily observations of the S&P 500 index and a U.S. REIT index. Daily data are used because studies have shown that persistence is not present in empirical trading data (Menkhoff, 2010) and that fund managers put very little emphasis on intraday trading (Eisler and Kertèsz, 2007). Daily returns are expressed as the first difference of the natural logarithm of the ratio of successive index values. Natural logarithms convert the price indexes to a series that theoretically contains infinitely small and infinitely large values, and first differencing produces returns that are continuous and stationary in the mean.

The S&P 500 index is a capitalization-weighted index that consists of 500 large company stocks, which represent the leading stocks in major U.S. industries. Over $5 trillion in financial paper is benchmarked to this index, and this index itself accounts for approximately 80% of the capitalization of all U.S. stocks. Thus, academics and practitioners often consider this index to be a proxy for the U.S. market in general.

U.S. REITs account for roughly half of the global supply of these instruments.[7] We construct our REIT index from REIT stocks extracted from the daily CRSP files using "Share Code = 18" or "SICCD = 6798." These REITs, although real estate oriented, often differ in capital structure and investment strategy. For example, some REITs are funded by a mixture of debt and equity, while others are fully funded by equity. Additionally, some provide mortgage financing, other REITs concentrate on construction projects, and still others focus on real estate ownership. We calculate daily market values for all REITs by multiplying share price by shares outstanding. These market values are used to generate daily weightings for each REIT's returns. Summing the daily products of all REIT's weights and returns generates a daily weighted market value REIT return index.

Our data begin on January 2, 1987, which is the first full year following the 1986 Tax Reform Act (see footnote 6), and end on December 31, 2012. The sample contains 6555 daily observations. This 26-year span includes the October 1987 crash as well as the recent financial crisis that began in 2007. Figure 17.1 displays the time-series plots of the daily observations for both the return series. Figure 17.2 condenses these data and illustrates the nature of the left-hand tails of the two distributions. Table 17.1 presents some sample descriptive statistics for five portfolios: mean, variance, skewness, kurtosis, minimum value, maximum value, a measure of linear dependence (Ljung–Box statistic), and a measure of nonlinear dependence (Lagrange multiplier statistic). The five S&P/REIT portfolios are labeled 100%/0%, 75%/25%, 50%/50%, 25%/75%, and 0%/100%, with the first number being the S&P 500 weight and the second the REIT weight. The five portfolios are used to illustrate numerically our results throughout the remainder of this chapter. Figure 17.3 provides a scatter plot of the returns with the S&P 500 return on the x-axis and the REIT return on the y-axis.

[7] Over 40 countries have some type of tradable real estate security that is similar to the U.S. model. Major non-US players include the UK, France, Australia, and Japan. Smaller nations such as Finland, Singapore, and Turkey also have viable REIT markets.

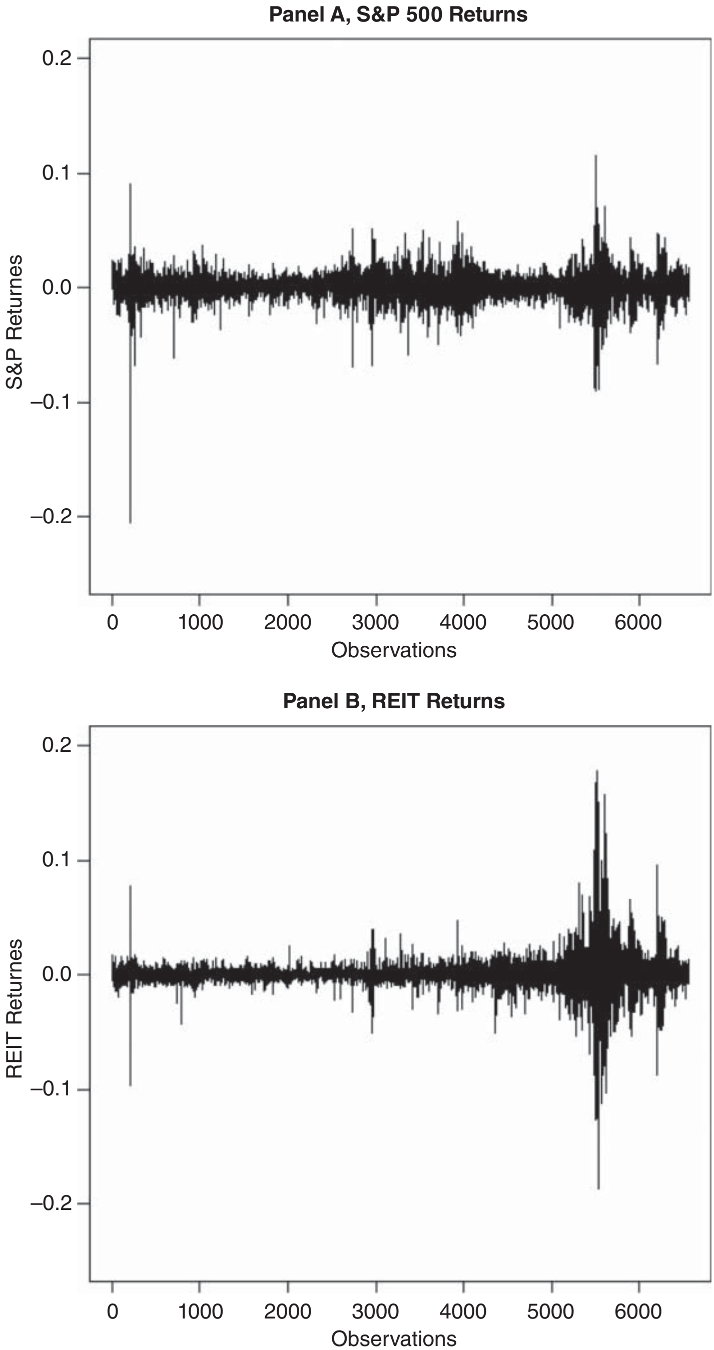


FIGURE 17.1 REIT and S&P 500 daily returns. 6555 observations from January 1987 to December 2012.

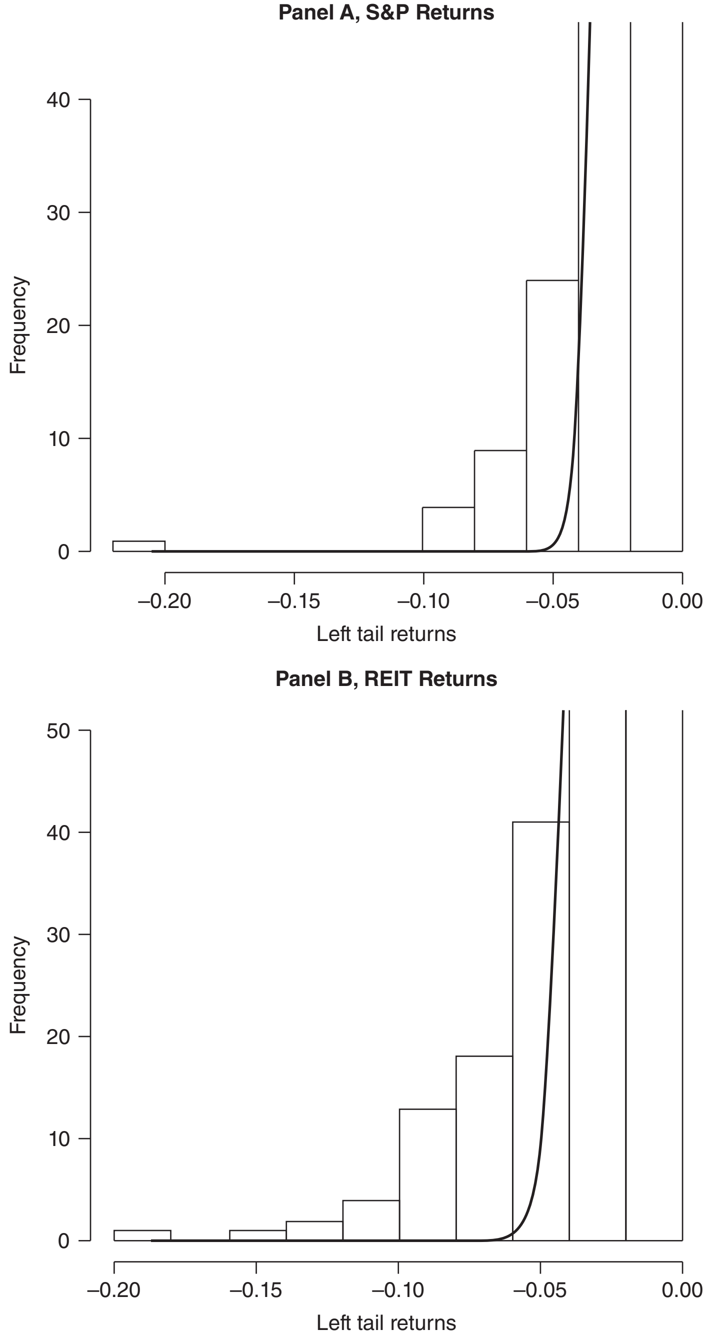


FIGURE 17.2 REIT and S&P 500 return left-tail histograms, as represented by bars, with normal distribution overlay, as represented by line.

TABLE 17.1 Descriptive statistics for portfolios with various REIT weights

	100/0 SP/REIT	75/25 SP/REIT	50/50 SP/REIT	25/75 SP/REIT	0/100 SP/REIT
Mean $\times 10^3$	0.343	0.374	0.405	0.435	0.466
Variance $\times 10^3$	0.144	0.136	0.146	0.173	0.217
Skewness	−0.829	−0.617	−0.243	0.137	0.413
Skewness *t*-stat.	−27.4	−20.4	−8.04	4.53	13.6
Kurtosis	17.8	17.4	19.7	23.7	26.8
Kurtosis *t*-stat.	294.	287.	325.	392.	442.
Min.	−0.205	−0.178	−0.151	−0.163	−0.189
Max.	0.116	0.123	0.138	0.153	0.179
Ljung–Box	14.7	36.4	78.1	128.	168.
Lagrange multiplier	169.	262.	520.	742.	835.

Headings correspond to portfolio weightings. The first number represents the weight of the S&P 500, while the second number represents the weight REITs. The individual null hypotheses that there is no skewness, no kurtosis, no linear dependence (Ljung–Box), and no GARCH effects (Lagrange multiplier) are rejected at least at the $p = 0.00015$ level. The correlation coefficient between the REIT and S&P 500 portfolios is 0.63.

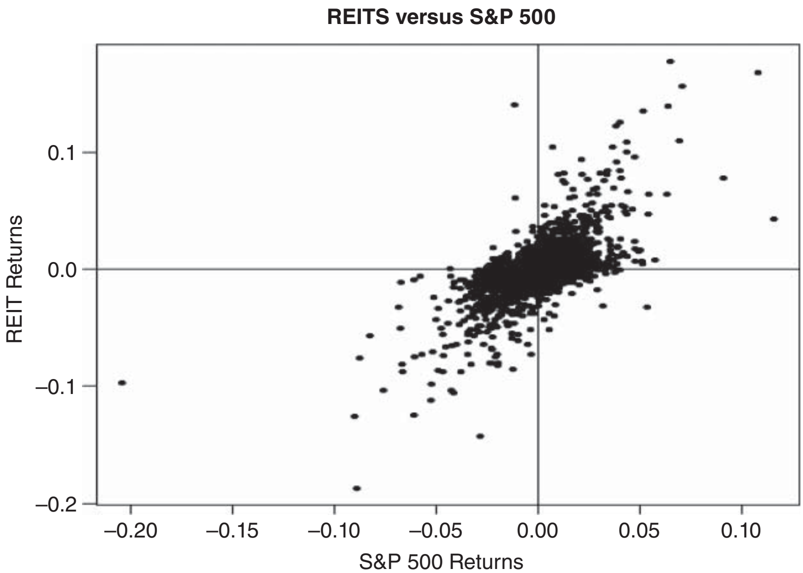


FIGURE 17.3 REIT returns plotted against S&P 500 returns. Correlation coefficient between the REIT and S&P 500 is 0.63.

A visual inspection of Figure 17.1 reveals that both the S&P 500 and REIT return series show volatility that changes over time and this phenomenon is aperiodically clustered. Both series also contain positive and negative spikes, that is, large abrupt changes. Particularly noticeable in both series is the October 1987 crash and its speedy recovery as well as the prolonged market turmoil associated with the 3-year financial crisis that began around 2007. This return behavior suggests that the unconditional distribution may have tails that are thicker than those associated with a Gaussian distribution. Figure 17.2 supports this observation and suggests that the left-tail probability mass of the REIT distribution is greater than that of the S&P 500 distribution. There also appears to be some asymmetric behavior leaning toward the presence of negative returns for the S&P 500 returns and of positive returns for the REIT returns.

These observations are supported by the statistics contained in Table 17.1. As indicated by the kurtosis statistics, both return series have fat tails, that is, they are thicker than those that characterize a Gaussian distribution, and they are skewed. The S&P 500 returns are negatively skewed, while the REIT returns are positively skewed. Both series have large negative spikes. Particularly noticeable is the 20% daily loss in 1987 for the S&P 500 returns and the 19% loss for the REIT series in the midst of the 2007–2009 financial crisis. The Ljung–Box and Lagrange multiplier tests indicate the strong presence of linear and nonlinear dependence, respectively, although both types of dependency appear stronger in the REIT series. Finally, not only is the REIT mean return higher than its S&P 500 counterpart, but also its standard deviation is higher. Nevertheless, the latter series yields more return per unit of volatility as measured by the standard deviation.

Asset allocation depends not only on the return and return volatility of the assets under consideration but also on the way in which the returns of these assets are related to each other. Figure 17.3, which plots the daily REIT returns against the corresponding S&P 500 returns, provides some insights into the co-movement of the returns of these two assets. The denser concentration of returns in the lower left and upper right quadrants as compared to the upper left and lower right suggests some sort of positive relationship between the two return series. Indeed, the linear correlation between the two return series is 0.63. This measure, however, is dominated by the majority of observations falling approximately between −0.03 and 0.03. For values that lie outside this fuzzy range, there are numerous instances of plots that fall far from this linear relationship. Again, the crash of October 1987 serves as an example. At this time, the REIT index experienced a 10% loss, or about one-half of that associated with the S&P 500.

The above statistics and visual patterns suggest that some performance enhancement may be possible through diversification. Comparing the descriptive statistics displayed in Table 17.1 for the five sample portfolios that contain different mixtures of the S&P 500 and REIT indexes as well as the indexes themselves supports this contention. For example, although the 100% REIT portfolio exhibits the most positive skew, the 75%/25% S&P/REIT portfolio has the thinnest tails. Further, The 100% REIT portfolio has the largest maximum return, but the smallest minimum return is associated with the 50%/50% S&P/REIT portfolio. Finally, the 50%/50% portfolio has the largest return per unit of volatility.

17.3 PERFORMANCE RATIOS AND THEIR ESTIMATIONS

We begin our formal analysis by specifying our performance ratios, which express the performance per unit of risk. As mentioned above, our benchmark is the Sharpe ratio, where the numerator is the mean portfolio return ($\overline{R}$) in excess of the risk-free rate (R_F), and the denominator is the square root of the variance (standard deviation) of the portfolio returns (Var). In other words

$$\text{Sharpe} = \frac{\left(\overline{R} - R_F\right)}{(\text{Var})^{1/2}}. \tag{17.1}$$

According to the CAPM, the goal of the investor is to select the portfolio that generates the largest feasible Sharpe ratio value, which is equivalent to maximizing performance. Thus, portfolios can be ranked on the basis of the Sharpe ratio and their relative performances judged accordingly.

The Sortino ratio also measures portfolio performance, but it is structured to focus on downside risk, that is, the left tail of the distribution of returns. The numerator is the portfolio return that is in excess of a MAR (R_{MAR}), while the denominator is a measure of downside risk. The downside risk is defined as the square root of the second lower partial moment (LPM_2), which is the average sum of squares of the returns that are less than the MAR. In mathematical terms

$$\text{Sortino} = \frac{\left(\overline{R} - R_{\text{MAR}}\right)}{\left(\text{LPM}_2\right)^{1/2}} \tag{17.2}$$

and

$$\text{LPM}_2 = \frac{1}{n}\sum_{i=1}^{n}\left(R_{\text{MAR}} - R_i\right)^2, \quad \forall R_i \leq R_{\text{MAR}}, \tag{17.3}$$

where n is the number of observations in the left tail as defined by R_{MAR}.

One drawback of the definition depicted in Eq. (17.3) is that it depends heavily on the returns observed during a specific performance period. In other words, this discrete version LPM_2 measures only what is observed and not what might be observed. For ex post performance assessment, this is acceptable because it measures what actually happened. For deciding ex ante portfolio allocations, this formulation ignores the notion of what might happen. Following Harlow (1991), Booth and Broussard (2002), and Choffray and Mortanges (2017), among others, we remedy this drawback by recasting Eq. (17.3) in probabilistic terms.

Two possibilities are usually considered: the generalized extreme value distribution (GEV) and the GPD. The GEV is estimated using the block maxima method. This approach splits the time series in question to sequential segments. In each segment, the smallest value is selected to be included in the estimation process. The GEV is then fitted to the sample of sequential minimum observations.

The GPD is estimated using the threshold exceedance approach. The threshold defines the tail of the distribution to contain (i) all of the sample observations that fall at or below a predetermined value, or (ii) a specified number of observations beginning with the smallest. We use the threshold exceedance approach for two reasons. First, it intuitively mirrors the notion of a MAR. Second, it makes sure that all of the large negative return spikes are included in the analysis.

In particular, we assume that the left tail of the return distribution can be modeled as a GPD. The cumulative density function (F) of this distribution is given by

$$F(R) = 1 - \left[1 + \tau\left(\frac{R-\lambda}{\sigma}\right)\right]^{-1/\tau}, \tag{17.4}$$

where $\lambda =$ location and $\in (-\infty, \infty)$, $\sigma =$ scale and $\in (0, \infty)$, and $\tau =$ tail shape and $\in (-\infty, \infty)$.[8] Its accompanying probability distribution function is

$$f(R) = \frac{1}{\sigma}\left[1 + \tau\left(\frac{R-\lambda}{\sigma}\right)\right]^{-((1/\tau)+1)}. \tag{17.5}$$

Location represents the MAR and is determined exogenously by the investor. The other two parameters, τ and σ, are endogenously estimated. Increasing τ, given a value of σ, steepens the slope in the central part of the density function and increases the tail. Increasing σ, given a value of τ, flattens the slope in the central portion of the density function but also increases the tail. Various tail index estimation techniques are discussed in Beirlant et al. (2017).

To calculate the second lower partial moment, we numerically integrate the squared difference between MAR over a return multiplied by the GPD probability that the return occurs. The result is valued from R equals minus infinity to R_{MAR}, the minimum accepted return. More formally

$$\mathrm{LPM}_2 = \int_{-\infty}^{R_{\mathrm{MAR}}} \left(R_{\mathrm{MAR}} - R\right)^2 f(R)\, dR. \tag{17.6}$$

This version of LPM_2 replaces its counterpart in (17.3).

Before proceeding with the analysis, it is useful to point out the difference between Eqs (17.3) and (17.6). In Eq. (17.3), $(R_{\mathrm{MAR}} - R)^2$ is weighted by $1/n$. In other words, each exceedance in the calculation of LPM_2 is equally weighted. In contrast, if Eq. (17.6) is used, each exceedance is weighted by different probabilities. For large negative values, these probabilities are very small even if modeled by a GPD. Thus, the contribution to LPM_2 for a specific exceedance is typically less if Eq. (17.6) is used than if Eq. (17.3) is employed, which results in a larger Sortino ratio, *ceteris paribus*. This impact is mitigated by recalling that Eq. (17.6) uses an infinite instead of finite number of exceedances. Nevertheless, it is doubtful whether the calculated values of the Sortino ratio under both alternatives will

[8]The GPD and the GEV are closely related. Not only is the tail shape parameter the same for both distributions, but also the limiting distribution of the exceedances is GPD if and only if the corresponding distribution of the block maxima is GEV.

be of the same magnitude. Whether they provide similar optimal asset allocation information is an empirical question.

To estimate Eq. (17.5), we use **R** to estimate the GPD's probability density function associated with the left tail of the 101 return distributions.[9,10] **R** is a readily available statistical package that is well suited to our task. There is no precise way to determine where the tail of a distribution ends and its body begins. We investigate this issue by examining the sensitivity of the GPD to different left tail demarcations by first following the empirical rule derived by Loretan and Philips (1994), that is, $k = n^{2/3}/\ln[\ln(n)]$, where k is the number of observations in the left tail and n is the sample size. This calculation suggests a tail size of the 161 most negative observations. We then halve, double, and quadruple this number and pick the n that provides the best coherence between the estimated GPD and its associated observed value as measured by the midpoint of the histogram generated by the return data.[11]

We report the parameter estimation results for the 100% S&P 500 and 100% REIT portfolios in Table 17.2 for the four different tail sizes. The threshold (λ) is the return associated with the accompanying exceedance number. The estimated probability density function for the two portfolios for each of the four exceedance sizes is plotted in Figure 17.4. Superimposed on these plots are the respective empirical distributions that are constructed by connecting the midpoints of the empirical histograms.

A review of the two figures indicates that the 644 observation sample, or approximately 10% of the total number of observations, appear to provide the best overall fit. This is equivalent to defining the distributions left tail to start at approximately −0.01. The tail shape and scale parameters are highly significant, and the fitted distributions closely follow their associated empirical ones. The main difference between the four distributions for each portfolio is that the samples with the larger number observations appear to better depict the sharp drop off in the number of observations around −0.025. The better fit is seen even for much larger losses, but the discrepancy is much smaller. This is the case even if this loss amount is contained within the sample observations used to estimate the model's parameters. These relationships hold for the other 99 return distributions as well,

[9] Sweeting and Fotiou (2011) and Scarrott and MacDonald (2012) provide excellent reviews of the use and estimation of the GPD. Nadarajah and Chan (2017) provide a listing of other software platforms useful when estimating extreme value distribution parameters.

[10] An alternate approach is to link the two GPDs using a copula. There are two ways of creating this link. One is to consider only those cases where both series exhibit exceedances, and the other is where at least one of the series exhibits an exceedance. Our analysis is in the spirit of the latter. See Rootsén and Tajvidi (2006) for more details concerning the second approach, which we believe is more appropriate for portfolio allocation applications. For finance applications using copulas to address portfolio risk management issues, see, for example, Hu (2006), Palaro and Hotta (2006), and Bhatti and Nguyen (2012).

[11] Other tail size selection rubrics include the 10% and $n^{1/2}$ rules (e.g., DuMouchel, 1983; Ferreira et al., 2003, respectively). For our data, the second rule suggests a tail size of 81, while the first rule indicates a tail containing the 656 smallest observations. These sizes compare favorably to our smallest and largest tail sizes.

TABLE 17.2 GPD estimates for S&P 500 and REIT portfolios. Relationship estimated is from Eq. (17.4) $F(R) = 1 - \left[1 + \tau\left(\frac{R-\lambda}{\sigma}\right)\right]^{-(1/\tau)}$

Number of exceedances	Threshold (λ)	Tail shape (τ)	Scale (σ)
Panel A: S&P 500			
81	−0.03	0.3107	0.0112
		(1.97)	(5.38)
161	−0.0238	0.3542	0.0083
		(3.21)	(7.79)
322	−0.0179	0.2851	0.0075
		(4.09)	(11.89)
644	−0.0122	0.1887	0.0076
		(4.48)	(18.29)
Panel B: REIT			
81	−0.0397	-0.0584	0.029
		(0.60)	(6.79)
161	−0.0275	0.1507	0.0192
		(1.41)	(7.61)
322	−0.0169	0.2865	0.0135
		(3.52)	(10.4)
644	−0.0096	0.4022	0.009
		(6.68)	(14.64)

Note: *t*-values in parentheses.

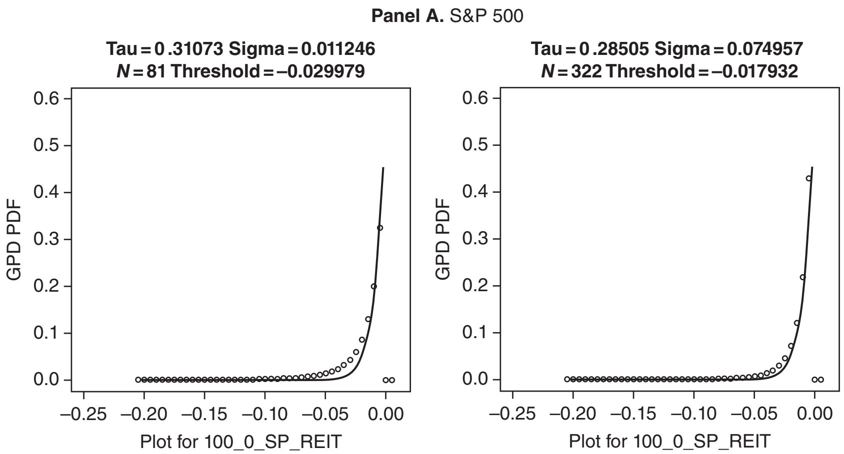
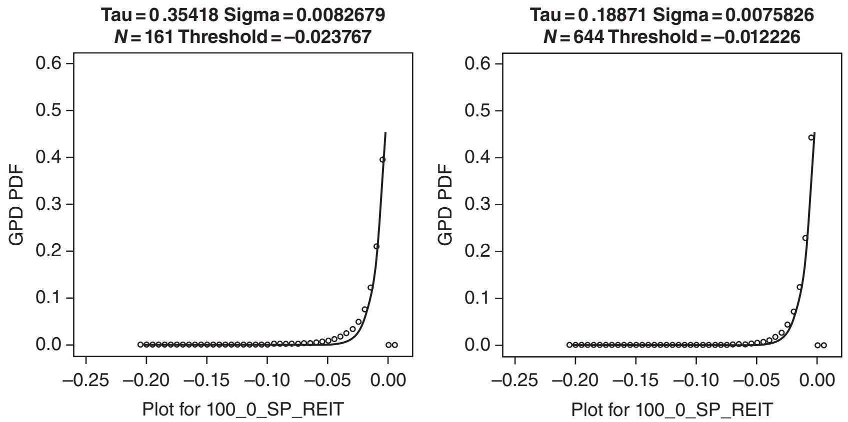


FIGURE 17.4 Plots of S&P 500 and REIT portfolios with various exceedances used for GPD estimation. GPD PDF generated from (17.5) $f(R) = \frac{1}{\sigma}\left[1 + \tau\left(\frac{R-\lambda}{\sigma}\right)\right]^{-((1/\tau)+1)}$. Panel A: S&P 500 and Panel B: REIT. Note: Circles indicate GPD fit. Line shows empirical fit using mid-point histogram plotting. *N* equals various exceedances used in the estimation. Threshold corresponds to the return associated with that exceedance observation.



Panel B. REIT

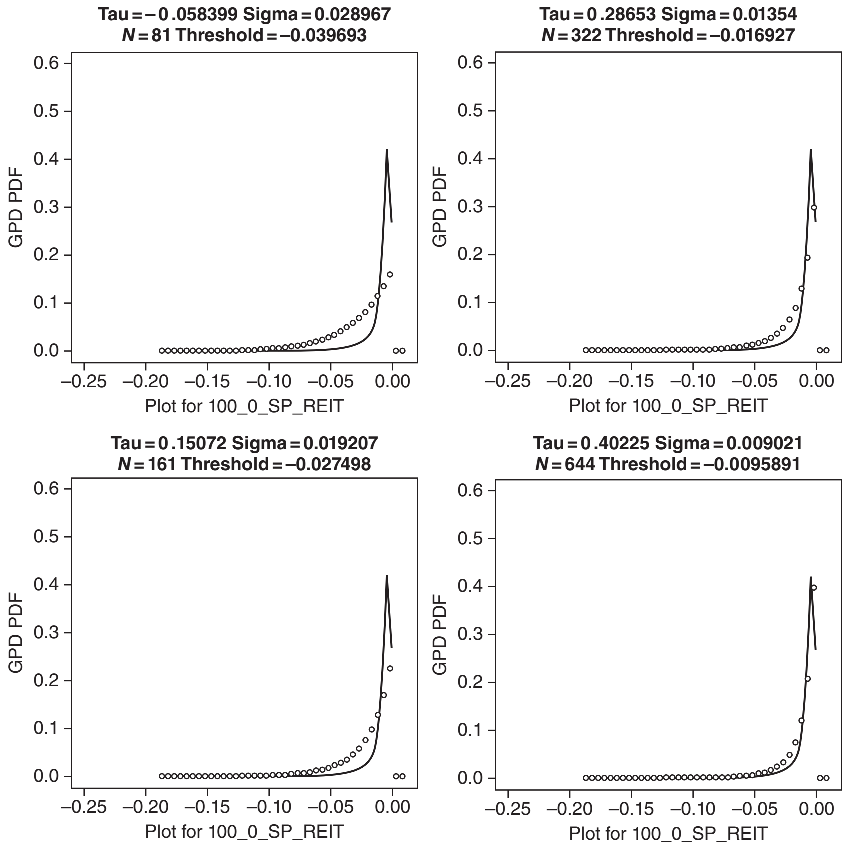


FIGURE 17.4 (*Continued*)

although we do not report their detail here. Because of the better fit, we proceed with our analysis using the GPD parameters provided by the 644 observation sample.

Before turning to the performance results, three points are worthy of note. First, although the 644 sample provides the best fit the S&P 500 and REIT returns, the physical nature of their left tails are not the same. The main difference is in the tail shape parameter (τ) of the REIT index, which is more than twice the size of the S&P 500's parameter while the scales (σ) of both extreme value distributions are similar, although the size of the REIT parameter is slightly larger. This indicates that the estimated return distribution of the former has longer tails but also a sharper bend in the middle of the distribution.

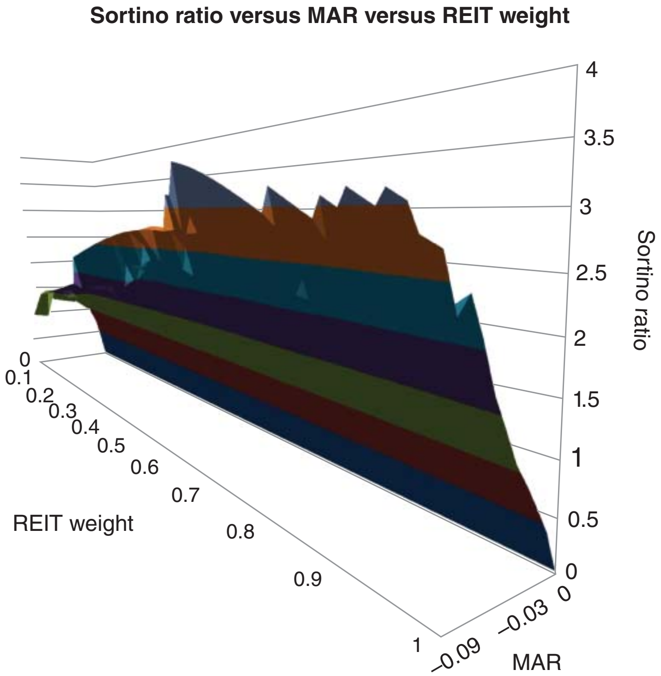
Second, the minimum acceptable rate of return is determined by the investor and reflects his/her risk–reward tradeoff. The rate, then, could be any real number that the investor deems to be minimally acceptable. However, we are interested in tail risk. Thus the maximum allowable target rate should be less than or approximately equal to the rate that separates the left tail from the rest of the distribution. If it is not, its use must involve the issues associated with out-of-sample prediction.

Third, for benchmark purposes we consider the entire 26-year period as a single unit. Over this span of time, the returns are stationary in the mean but not in the variance. This is typical of financial assets. Some have suggested that, before the stochastic process that determines the tail behavior of the series is estimated, the impact of the time-varying variance should be removed. Although it is possible to remove the conditional variance from the return series using generalized autoregressive conditional heteroskedastic (GARCH) methods, we do not do so for two related reasons. First, tail behavior and conditional variances are not independent. For example, in the GARCH setup, a large negative return will cause the next day's conditional variance to increase. Thus, removing the GARCH effects may inadvertently impact the magnitude of a future spike.[12] Second, we currently do not know when crashes or extended periods of turbulence caused by financial crises will occur. Along these lines, Booth and Broussard (2002) document that spikes are often statistically independent of each other. Nevertheless, recently some preliminary research has addressed the issue of whether financial stability can be predicted through the development of early warning systems. Some interesting examples of this ongoing research include Bussière and Fratzscher (2006), Alessi and Detken (2011), and Lo Duca and Peltonen (2013), to name but a few.

17.4 PERFORMANCE MEASUREMENT RESULTS AND IMPLICATIONS

Figure 17.5 depicts the Sortino ratio values, (17.2), which uses (17.3) to calculate its denominator, while Figure 17.6 also reports these performance values but

[12]Kozhan et al. (2013) suggest that the skew and variance premium are manifestations of the same underlying risk factor.



Panel B–Selected Sortino Ratios from Panel A					
	REIT Weights (boldface signifies largest row value)				
Minimum Acceptable Return	0	25	50	75	100
0.00	0.0271	0.0305	0.0321	**0.0322**	0.0315
–0.01	**0.6766**	0.6377	0.5526	0.4798	0.4149
–0.02	**1.0010**	0.9158	0.8129	0.7134	0.6535
–0.03	1.0747	**1.1021**	1.0631	0.9661	0.8777
–0.04	1.1788	1.2342	**1.2940**	1.2246	1.0778
–0.05	1.2824	1.4801	**1.5596**	1.4899	1.3797
–0.06	1.4406	1.5826	1.8772	**1.8787**	1.6668
–0.07	1.2357	1.7318	**2.2083**	2.0484	2.0823
–0.08	1.4290	1.7066	2.3336	2.3455	**2.4183**
–0.09	1.1142	1.7139	2.2687	**2.6422**	2.2647
–0.1	0.9587	1.7991	2.7168	**3.1551**	2.7191

FIGURE 17.5 Sortino ratio values using (17.3) versus various MAR and REIT weights. Panel A: Plots of 1111 Sortino ratio values. *X*-axis represents weight in REIT portfolio, *Z*-axis represents Minimal Acceptable Return (MAR), and *Y*-axis represents calculated Sortino ratio provided REIT weight and MAR. Panel B: Selected Sortino ratios from Panel A.

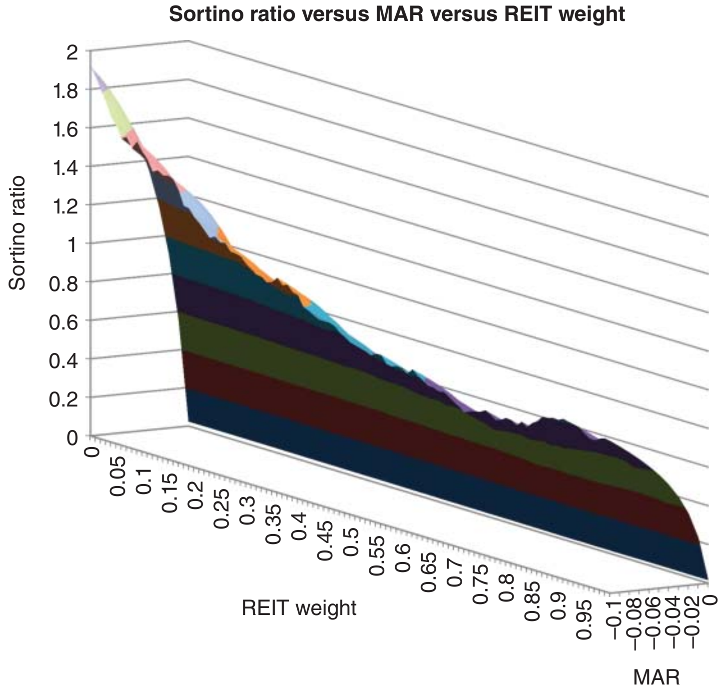
calculates the denominator of this ratio using (17.6). In both figures, Panel A visually illustrates the calculated performance results for 101 portfolios and numerous target rates. The portfolio weights run from 0.0 (100% S&P 500) to 1.0 (100% REIT) by 0.01, and the MARs run from −0.01 to −0.10 by −0.01, thereby creating 1111 data points. Panel B provides the numerical estimates for the aforementioned five S&P–REIT portfolios, that is, 100%/0%, 75%/25%, 50%/50%, 25%/75%, and 0%/100%.

Turning first to Figure 17.5, Panels A and B, we find that the Sortino ratio's value depends heavily on the choice of the MAR. Its value tends to increase as the investor increases his/her ability to tolerate risk but then decrease slightly if the target rate becomes too onerous. The same hoop-shaped pattern occurs as the portfolio changes from one that is 100% S&P 500 to one that is 100% REIT. The joint movement can be seen in Panel B, where the Sortino ratios for the listed minimal accepted returns and REIT weights are in boldface type. This pattern suggests that for the period being examined investors with a low tolerance for medium-large losses would have been better off on a risk–return basis if they had relatively more heavily allocated their investable resources to the S&P 500. In contrast, those individuals with a low tolerance for only very large losses would have been better off by investing more in the REIT portfolio.

As shown in Figure 17.6, Panels A and B, the results are markedly different if the GPD approach to calculate the Sortino ratio is used. In this case, the optimal portfolio appears to be 100% S&P 500 no matter what the choice of the minimum accepted return. We suspect that this difference in allocation involves the way in which the two versions of the Sortino ratio are calculated with the values in Figure 17.5 associated with the equal weighting of exceedances and those in Figure 17.6 with probability weighting. Our intuition is that the results associated with the GPD approach are a result of the REIT index having a longer left tail than the S&P 500 index as evinced by the tail shape and scale parameters (see Table 17.2) and the heavier probability mass in the left tail as documented in Figure 17.2.

Table 17.3 displays the specific optimal allocation weights for both versions of the Sortino ratio and the Sharpe ratio. The optimum REIT weight according to the Sortino ratio, that is, (17.3), ranges from 0.00 to 0.81, depending on the MAR chosen. The corresponding weights for the Sortino ratio, that is, (17.6), are always 0.00 at least to the precision of two decimal places. By way of comparison, the Sharpe ratio calculated the optimal REIT weight to be 0.56.[13] These results indicate that that performance measurement and asset allocation decisions to obtain future performance are extremely sensitive to the way in which risk is measured.

[13]The risk-free rate is assumed to be zero. Relaxing this assumption does not affect the performance ranking.



Panel B–Selected Sortino Ratios from Panel A					
	REIT Weights (boldface signifies largest row value)				
Minimum Acceptable Return	0	25	50	75	100
0.00	**0.0227**	0.0211	0.0177	0.0149	0.0147
–0.01	**0.5505**	0.4204	0.3089	0.2382	0.2388
–0.02	**0.9025**	0.6430	0.4620	0.3566	0.3695
–0.03	**1.1543**	0.7872	0.5614	0.4363	0.4596
–0.04	**1.3435**	0.8899	0.6342	0.4970	0.5284
–0.05	**1.4911**	0.9683	0.6921	0.5472	0.5847
–0.06	**1.6096**	1.0313	0.7407	0.5909	0.6331
–0.07	**1.7072**	1.0840	0.7834	0.6303	0.6761
–0.08	**1.7891**	1.1296	0.8219	0.6668	0.7153
–0.09	**1.8591**	1.1701	0.8576	0.7012	0.7520
–0.1	**1.9198**	1.2069	0.8913	0.7341	0.7867

FIGURE 17.6 Sortino ratio values using (17.6) versus various MAR and REIT weights. Panel A: Plots of 1111 Sortino ratio values. *X*-axis represents weight in REIT portfolio, *Z*-axis represents Minimal Acceptable Return (MAR), and *Y*-axis represents calculated Sortino ratio provided REIT weight and MAR. Panel B: Selected Sortino ratios from Panel A.

TABLE 17.3 Optimal REIT portfolio weights

Minimum acceptable return	Sortino ratio		Sharpe ratio
	Equation (17.3)	Equation (17.6)	
0.00	0.65	0.00	0.56
−0.01	0.01	0.00	0.56
−0.02	0.00	0.00	0.56
−0.03	0.18	0.00	0.56
−0.04	0.39	0.00	0.56
−0.05	0.63	0.00	0.56
−0.06	0.67	0.00	0.56
−0.07	0.46	0.00	0.56
−0.08	0.63	0.00	0.56
−0.09	0.81	0.00	0.56
−0.10	0.63	0.00	0.56

Notes: The Sharpe ratio is constant by construction, while the Sortino ratio calculated using (17.6) is constant, that is, 0.00 by calculation.

17.5 CONCLUDING REMARKS

Our study contributes to two strands of the performance and asset allocation literature. First, Ornelas et al. (2012) suggest that there may not be any material differences in the performance rankings between the Sharpe ratio and the Sortino ratio. In a study involving mutual funds and asset allocation, using monthly data, they find that the rankings of the Sharpe ratio and the Sortino ratio are highly correlated. The correlation is 0.998 if Spearman's rank correlation is used and 0.971 if Kendall's rank correlation is used. The correlation between the Sharpe ratio and conditional Sharpe ratio is 0.989 and 0.920, respectively. In contrast, our empirical results support the notion that the choice of the performance measure does matter.

Second, with respect to the performance of REITs, earlier studies such as Firstenberg and Ross (1988), Ennis and Burik (1991), and Kallberg et al. (1996) find that real estate is typically unrepresented in investor portfolios when compared to that suggested by modern portfolio theory. These and other studies find that investors should hold somewhere between 10% and 20% of their investible assets in real estate instead of the typical 3%. Wilshire (2012) using mean-variance optimization suggests no exposure to REITs for those investors who are retired or about to retire and around 15% for those just beginning their work-life. Booth and Broussard (2002) using the standard approach also find that the appropriate allocation is around 10%, but when they employ a downside risk framework using GEV probabilities they conclude that, depending on the target rate of return, investors should hold at least 40% of their investments in REITs and perhaps as much as 100% depending on their need to avoid substantial exposure to unwanted losses. These studies, however, with the exception of Wilshire (2012) and ours, do not consider the impact of the 2007–2009 financial crisis. It appears then that the old industry prescription of having no more than 3% of investible assets allocated to real estate may not be as out of line as has been recently thought.

We believe that future research should be directed in two directions. First, the implications of downside risk should be explored for asset allocation decisions involving more than two assets. This will better fit the needs of institutional investors that routinely deal with risk management decisions dealing with several or even more asset classes. It will also provide additional insights into the role of extant (especially REITs) and new asset classes that have been or yet to be proposed. Second, the notion of a conditional Sortino ratio should be fully explored. This suggests that whatever extreme value measure is used, it must not only be time dependent but also must discriminate between ordinary volatility caused by noise and the normal ebb and flow of business activity and extreme volatility caused by unpredictable financial crises. Such research can help the investment practitioner make better asset allocation decisions.

Acknowledgments

We thank Ishaq Bhatti for providing the initial version of the **R** code used to estimate the various GPDs. We also acknowledge the research help of Allison Braetzel and Ryan Timmer.

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