CHAPTER 6

Sortino, Omega, Kappa: The Algebra of Financial Asymmetry

6.1 Extracting Downside Risk Measures from Lower Partial Moments

This chapter traces the development of entire families of downside risk measures from partial statistical moments. The Sortino, omega, and kappa ratios provide credible, workable single-factor measures of financial dispersion below mean return. At a minimum, specifying these ratios provides a useful contrast with conventional, two-tailed measures such as the Sharpe and Treynor ratios. Because it is based on downside semideviation, the square root of the lower partial second moment, the Sortino ratio is particularly easy to reconcile with the more traditional and more familiar tools of modern portfolio theory. Indeed, closer examination of the Sortino ratio reveals Pythagorean relationships between single-sided risk measures and their counterparts within the conventional capital asset pricing model (CAPM). These relationships allow single-sided measures of volatility to be evaluated with trigonometric tools.

At a higher level of generality, the progression from the Sortino ratio to its mathematical cousins, omega and kappa, represents a pivotal step in the postmodern revitalization of portfolio theory. Although I ultimately embrace a different approach leading to a four-moment CAPM, the

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Sortino, omega, and kappa ratios reveal the extent of information embedded in the statistical moments of the distribution of returns. Their value lies in demonstrating that single-sided risk measures are not merely feasible but very well suited to financial markets whose returns are abnormal and whose participants behave in less than fully rational, predictable fashion.

6.2 The Sortino Ratio

Among the many statistics generated by the complete specification of single-sided beta, upside and downside semideviation may be the simplest and easiest to deploy. In particular, downside semideviation, which "defines risks as volatility below [a market] benchmark," can be "articulated … into a very simple asset pricing model."[1] Substituting "standard semideviation … for standard deviation to measure portfolio risk" retains "the fundamental structure of the capital asset-pricing model."[2] In particular, replacing standard deviation with downside semideviation preserves the basic logic of the Sharpe ratio,[3] which transforms the statistical practice of standard scoring into a gauge of financial performance.

The pursuit of a single risk measure reflecting both volatility and skewness has a natural starting point: downside semideviation. In the early 1990s, Frank Sortino proposed a ratio measuring the ratio of reward (defined by returns on an asset exceeding some minimal acceptable benchmark) to downside risk (as expressed through target semideviation):[4]

$$\text{Sortino ratio} = \frac{R_a - R_b}{\sigma_-}$$

Or, in notation congruent with the specifications provided in Chap. 5: $\dfrac{x_a - \tau}{\sigma_{a,-}}$.

The development of the Sortino ratio may be regarded as the birth of postmodern portfolio theory, or at least a pivotal moment in the early history of the movement to isolate downside risk and to allocate assets in a more behaviorally sensitive manner. Following Sortino, other sources have also endorsed the use of downside semideviation as a risk measure.[5] At an absolute minimum, the Sortino ratio heralded the profusion of alternatives to the stalwart risk measures of modern portfolio theory, the Sharpe and Treynor ratios. An entire family of downside risk measures has arisen in the past quarter-century, all drawn from lower partial moments of the distribution of returns.

We should recall that downside semivariance is the measure that Harry Markowitz originally envisioned as modern portfolio theory's appropriate surrogate for risk, but ultimately forswore in favor of full standard deviation because of the formidable computational obstacles.[6] When it becomes feasible to gather and process large amounts of data, the ability to engage in raw computation of market outcomes deprecates simpler analytical shortcuts. Moore's Law all but dictates consideration of empirically derived alternatives to modern portfolio theory's computationally convenient but behaviorally deficient surrogates for risk.[7] Shortly after Sortino published his measure, Markowitz finally implemented his original idea of measuring risk through semideviation in 1993.[8] The nearly contemporaneous appearance of these ideas with the original formulation of Fama and French's three-factor model in the early 1990s[9] is no mere coincidence. Rather, the confluence of these ideas in finance arguably represents an iconic instance of multiple discovery,[10] which in turn may be characterized as the typical model by which science advances.[11]

The crucial innovation in the Sortino ratio is the substitution of downside volatility (σ_-) for standard deviation or beta as surrogates for risk in the Sharpe and Treynor ratios. Downside volatility, in turn, is defined as semideviation, the square root of the target semivariance:

$$\sigma_- = \sqrt{\int_{-\infty}^{\tau} (\tau - x)^2 \ f(x) \ dx}$$

where τ represents a targeted minimum acceptable rate of return (equivalent to d, Roy's minimally acceptable "disastrous" rate of return),[12] and $f(x)$ represents the probability density function of the returns. The integration function within the downside risk formula may be formally described as the second-order lower partial moment about τ.[13]

6.3 Comparing the Treynor, Sharpe, and Sortino Ratios

This formal definition of the Sortino ratio enables us to redefine the three leading risk-adjusted metrics of modern and postmodern portfolio theory according to a shared set of formal mathematical definitions. Table 6.1

restates the Treynor, Sharpe, and Sortino ratios in the simplest terms possible (Table 6.1):

The numerators of all three ratios are set forth in the second column as $R_a - R_b$, or the return on an asset minus the baseline return. The third column restates all three ratios in more formal terms as $\mu - \tau$, or expected return (μ) minus target return (τ). Whatever the notation, all three ratios have the same numerator. All three ratios begin with the difference between actual returns and some sort of target, whether it is the "disastrous" minimum acceptable return of Roy's safety-first criterion or the broader market-based benchmark by which an investor or professional manager measures her performance. Table 6.2 makes transparent the structural similarity between the Sharpe ratio, in particular, and the formula for a standard score.

We must overcome the superficial confusion arising from competing uses of the parameter μ. In the Sharpe ratio (and, for that matter, the Treynor and Sortino ratios), μ represents the expected return. It is the difference between that value and the baseline, τ, that counts. The definition of a standard score also uses the parameter μ, in its more traditional sense as the mean of a distribution. But the determination of a standard score z depends on a parallel process of seeking the difference between some independent variable and a fixed point of reference. In these risk-adjusted measures of performance, μ represents the relevant independent variable. By contrast, in standard scoring, μ represents the point of reference, a role assumed by τ in the Treynor, Sharpe, and Sortino ratios. Once we overcome this potential source of confusion, we can readily see that the Sharpe ratio is nothing more than a stylized application of standard

Table 6.1 The Treynor, Sharpe, and Sortino ratios

Treynor ratio	$\frac{R_a - R_b}{\beta}$	$\frac{\mu - \tau}{\beta}$
Sharpe ratio	$\frac{R_a - R_b}{\sigma}$	$\frac{\mu - \tau}{\sigma}$
Sortino ratio	$\frac{R_a - R_b}{\sigma_-}$	$\frac{\mu - \tau}{\sigma_-}$

Table 6.2 Comparing the Sharpe ratio with a standard score

Sharpe ratio	$\frac{\mu - \tau}{\sigma}$
Standard score	$z = \frac{x - \mu}{\sigma}$

scoring techniques to the problem of evaluating capital market returns. In mathematical jargon, the Sharpe ratio is a special case—a contextually applied version—of the more general formula for standard scoring.

The equivalence of the numerators in these three ratios invites comparative analysis of the only meaningful difference among them, their denominators. We know that both the Treynor and Sharpe ratios rely on variance. β is the more complicated measure, being the ratio of (1) the covariance of an asset with the overall portfolio to (2) the variance of the portfolio as a whole—or, alternatively, correlated relative volatility, in the sense of the ratio volatility of an asset to the volatility of the whole portfolio, times the correlation between asset-specific and portfolio-wide returns. σ is straightforwardly the standard deviation, which is in turn the positive square root of the variance: $\sigma = |\sqrt{\sigma^2}|$.

The formal definitions of mean, variance, and standard deviation as mathematical moments help us express the downside risk concept in the denominator of the Sortino ratio in terms that connect it to the Sharpe ratio. The *n*th moment of a distribution $f(x)$ about the value c is defined as:[14]

$$\mu_n' = \int_{-\infty}^{\infty} (x-c)^n \ f(x) \ dx$$

The mean of a distribution is the first raw moment (the first moment about 0). The variance is the second moment about the mean; the standard deviation is the square root of that value. Inasmuch as all risk-adjusted measures of performance turn on the expected value of a portfolio relative to a baseline, the numerator is uniformly $\mu - \tau$, or expected return (μ) minus target return (τ). $\mu - \tau$ can be expressed as the first moment about the target return:

$$\mu - \tau = \mu_1'(\tau) = \int_{-\infty}^{\infty} (\mu - \tau)^1 \ f(x) \ dx$$

In a 1994 reconsideration of his own ratio after nearly three decades of widespread use and commentary, William Sharpe acknowledged that the risk-free return—the target return baseline of his index—could vary over time.[15] The effect of this revision was that the denominator, σ, would formally represent the positive square root of the variance about τ as a variable rather than a presumed constant of 0.[16] This revision effectively enabled both the numerator and the denominator of the Sharpe ratio to be expressed as transformations of the second moment about τ, the target return. In the most formal terms possible, the denominator of the Sharpe ratio is the positive square root of the second moment about τ:

$$\sigma = \sqrt{\mu_2'(\tau)} = \sqrt{\int_{-\infty}^{\infty} (\mu - \tau)^2 \;\; f(x) \; dx}$$

The Sortino ratio makes use of partial moments, particularly the lower partial moment about the target return about τ. The nth-order lower partial moment with respect to reference point r is defined as[17]:

$$\mu_n^-(r) = \int_{-\infty}^{r} (r - x)^n \;\; f(x) \;\; dx$$

In the interest of completeness, we should note that the nth *upper* partial moment about r is:

$$\mu_n^+(r) = \int_{-\infty}^{r} (x - r)^n \;\; f(x) \;\; dx$$

Substituting μ for r and τ for x and then taking the square root of the second moment about τ yields the definition of downside volatility, the denominator of the Sortino ratio:

$$\sigma_- = \sqrt{\mu_2^-(\tau)} = \sqrt{\int_{-\infty}^{\tau} (\tau - \mu)^2 \;\; f(x) \;\; dx}$$

The Sortino ratio measures risk through strictly downside volatility. Whereas the Sharpe ratio divides the distribution of returns about a target τ by σ, the Sortino ratio divides that same numerator by the semideviation in the same distribution below target τ.

Table 6.3's comparison of the definitions of the Sharpe and Sortino ratio at multiple levels of mathematical formality reveals the closeness—and the distinctiveness—of these celebrated measures of risk-adjusted performance in modern and postmodern portfolio theory.

This is one sense, though by no means the only one, in which all measurements of risk-adjusted performance converge into a single class of distributions, represented by the original and still iconic Sharpe ratio.[18]

6.4 Pythagorean Extensions of Second-Moment Measures: Triangulating Deviation About a Target Not Equal to the Mean

In formulating the semivariation-based ratio that now bears his name, Frank Sortino consistently took pains to emphasize that the target return could and should vary by investor and by tailored, individual circumstance. He consistently spoke not of the risk-free rate or of mean returns, but rather of "minimal acceptable return."[19] By contrast, I have spoken thus far of mean and variance exclusively as the first and second *central* moments of the distribution of returns. But those values are merely special cases of moments about a value τ. We can extend the quantitative apparatus of Chap. 5 by relaxing the assumption that $\tau = \mu$. The generalization of this model is surprisingly easy and elegant. It also connects this elaboration of mean-variance analysis to a different set of mathematical tools: the Pythagorean theorem and trigonometry.

Table 6.3 A closer comparison of the Sharpe and Sortino ratios

Sharpe	$\frac{\mu - \tau}{\sigma}$	$\frac{\mu_1'(\tau)}{\sqrt{\mu_2'(\tau)}}$	$\frac{\int_{-\infty}^{\infty} (\mu - \tau)^1 \; f(x) \; dx}{\sqrt{\int_{-\infty}^{\infty} (\mu - \tau)^2 \; f(x) \; dx}}$
Sortino	$\frac{\mu - \tau}{\sigma_-}$	$\frac{\mu_1'(\tau)}{\sqrt{\mu_2^-(\tau)}}$	$\frac{\int_{-\infty}^{\infty} (\mu - \tau)^1 \; f(x) \; dx}{\sqrt{\int_{-\infty}^{\tau} (\tau - \mu)^2 \; f(x) \; dx}}$

In certain circumstances, we may wish to set a benchmark return, or "target semivariance,"[20] at some quantity other than mean return.[21] Such a benchmark, also known as target return or minimum acceptable return, is "customized to the investor's tolerance for periodic losses" and "can be different than the mean return."[22] We can designate the difference between τ and μ as κ. To wit: $\kappa = \mu - \tau$ and $\tau = \mu - \kappa$. The general definition of moments of a statistical distribution about τ yields formulas for the first and second moments about $\tau = \mu - \kappa$:

$$\begin{aligned}
&\tau = \mu - \kappa;\ \kappa = \mu - \tau \\
&\sigma_\tau^2 = \left\langle (x-\tau)^2 \right\rangle = \left\langle (x-\mu-\kappa)^2 \right\rangle \\
&\sigma_\tau^2 = \left\langle x^2 - 2\mu x + \mu^2 + \kappa^2 - 2\kappa x + 2\kappa\mu \right\rangle
\end{aligned}$$

This seemingly unruly series is easily reevaluated:

$$\sigma_\tau^2 = \left\langle \left(x^2 - 2\mu x + \mu^2\right) - 2\kappa(x-\mu) + \kappa^2 \right\rangle$$

The foregoing rearrangement may be more intuitively understood if we see this relationship:

$$\begin{aligned}
&\sigma_\tau^2 = \left\langle (x-\tau)^2 \right\rangle = \left\langle (x-\mu-\kappa)^2 \right\rangle = \left\langle \left[(x-\mu)-\kappa\right]^2 \right\rangle \\
&\left\langle \left[(x-\mu)-\kappa\right]^2 \right\rangle = \left\langle (x-\mu)^2 - 2\kappa(x-\mu) + \kappa^2 \right\rangle
\end{aligned}$$

The expectation operator, $\langle\ \rangle$, allows us to compress even further:

$$\sigma_\tau^2 = \sigma_{\mu-\kappa}^2 = \left\langle (x-\mu)^2 - 2\kappa(x-\mu) + \kappa^2 \right\rangle = \left\langle (x-\mu)^2 \right\rangle - \left\langle 2\kappa(x-\mu) \right\rangle + \left\langle \kappa^2 \right\rangle$$

Since the expectation of the middle term, $2\kappa(x-\mu)$, equals zero, we can simplify the entire expression:

$$\sigma_{\mu-\kappa}^2 = \left\langle (x-\mu)^2 \right\rangle - \left\langle 2\kappa(x-\mu) \right\rangle + \left\langle \kappa^2 \right\rangle = \left\langle (x-\mu)^2 \right\rangle + \left\langle (\mu-\tau)^2 \right\rangle$$

And ultimately:

$$\sigma_{\mu-\kappa}^2 = \left\langle (x-\mu)^2 \right\rangle + \left\langle (\mu-\tau)^2 \right\rangle = \sigma^2 + \kappa^2$$

A final exercise in rearrangement reveals the formula for the deviation of the distribution about $\tau = \mu - \kappa$, which is merely the square root of variance about that value about $\tau = \mu - \kappa$:

$$\sigma_{\mu-\kappa} = \sqrt{\sigma^2 + \kappa^2}$$

The relationship between standard variance and variance about a value $p = \mu - \kappa$, which in turn has been defined by reference to μ as the first central moment, is governed by the Pythagorean theorem. Having expressed mean, variance, and deviation relative to $\tau = \mu - \kappa$, it becomes a simple (if somewhat tedious) exercise to define covariance, correlation, the coefficient of determination, and beta for that benchmark.

The separation of each of these measures into upside and downside components is comparably straightforward. That exercise does more than extend the two-moment model for assessing portfolio risk to any benchmark besides mean return. The relationship between standard variance or deviation, on the one hand, and these second-moment measures' upside and downside components, on the other, lends itself to evaluation in Pythagorean and trigonometric terms. I now turn to that subject.

6.5 Further Pythagorean extensions: Triangulating Semivariance and Semideviation

To facilitate the proper use of semideviation as a risk measure, I will now elaborate the mathematical relationship between upside semideviation, downside semideviation, and standard deviation. Sources within the literature on single-sided measures of risk sometimes err in describing this relationship as one of simple arithmetic, as though "the lower semideviation" were equal to "half the standard deviation" in a purely symmetrical distribution of returns.[23] Proper mathematical evaluation of semivariance and semideviation contradicts this assertion. Rather, upside and downside semideviation are related to standard deviation according to the Pythagorean theorem.

Recall the general definitions of upside and downside semicovariance between two portfolios, *p* and *q*:

$$\sigma_{+}(p,\ q) = \operatorname{cov}(p,\ q \mid x_p > \mu_p,\ x_q > \mu_q)$$
$$\sigma_{-}(p,\ q) = \operatorname{cov}(p,\ q \mid x_p < \mu_p,\ x_q < \mu_q)$$

It should be evident from this definition that upside and downside covariance are straightforwardly *additive*. In other words, overall covariance is the sum of upside and downside covariance:

$$\operatorname{cov}(p,\ q) = \sigma\ (p,\ q) = \sigma_{+}\ (p,\ q) + \sigma_{-}\ (p,\ q)$$

Since the variance of a single distribution is merely a special case of covariance, where both variables are the same, the same additive relationship holds for upside and downside semivariance:

$$\operatorname{cov}(p,\ p) = \sigma_p^2 = \sigma_{p,+}^2 + \sigma_{p,-}^2$$

Volatility in any of its guises is the positive square root of the corresponding form of variance. This insight confirms what should be evident from the foregoing equation: the relationship between upside and downside *semideviation* is exactly that of the legs of a right triangle to the hypotenuse under the Pythagorean theorem. The sum of the squares of the upside and downside semideviation is equal to the square of standard deviation, or overall variance.

The same relationship can also be revealed through decomposition of variance as the second central moment of the distribution of returns:

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\ dx$$

$$\sigma^2 = \int_{-\infty}^{\mu} (\mu-x)^2 f(x)\ dx + \int_{\mu}^{\infty} (x-\mu)^2 f(x)\ dx$$

$$\sigma = \sqrt{\int_{-\infty}^{\mu} (\mu-x)^2 f(x)\ dx + \int_{\mu}^{\infty} (x-\mu)^2 f(x)\ dx}$$

Or more simply:

$$\sigma = \sqrt{\sigma_{-}^{2} + \sigma_{+}^{2}}$$

Taking the square of both sides reveals the Pythagorean relationship between variance and both halves of semivariance as the upper and lower partial second moments of the distribution of returns:

$$\sigma^{2} = \sigma_{-}^{2} + \sigma_{+}^{2}$$

The applicability of the Pythagorean theorem to semideviation subjects single-sided measures of volatility to the entire apparatus of trigonometry. This property proves extremely useful for evaluating asymmetrical financial returns.

6.6 Single-Sided Risk Measures in Popular Financial Reporting

Many popular financial news sources report the Sortino ratio alongside traditional modern portfolio theory measures. Whereas the Sharpe ratio is that of excess over mean return to standard deviation, the Sortino ratio is that of excess over mean return to downside semideviation. That much is apparent from this chapter's specification of the Sortino ratio. What is not as readily apparent from popular reporting of risk and return on either side of a target is the full extent of the mathematical relationship between the Sharpe and Sortino ratios.

Morningstar provides a case in point. That company has recently added two related measures of upside and downside potential to its proprietary Investor Return data series:

> Upside capture ratios for funds are calculated by taking the fund's monthly return during months when the benchmark had a positive return and dividing it by the benchmark return during that same month. Downside capture ratios are calculated by taking the fund's monthly return during the periods of negative benchmark performance and dividing it by the benchmark return.[24]

Morningstar's upside and downside capture ratios seek to communicate the movement of an investment alongside its benchmark during times of gain and times of loss. Dividing the upside capture ratio by the downside capture ratio rapidly, if crudely, expresses the extent to which these related measures express the combined effect of a security's correlation with upward and with downward movements in a broader market benchmark. Intuitively, a "best-of-both-worlds" investment is one that rises more aggressively than its benchmark during good times, but mutes losses relative to that benchmark during bad times. Such an investment would offer greater benefit to the investor than either an investment that matched only the upside potential or only the downside limitation on loss.

Morningstar also supplies traditional modern portfolio statistics such as mean return (μ), standard deviation (σ), beta (β), the Sharpe ratio, and r^2 (the coefficient of determination derived by squaring the correlation between the returns of an individual asset or an asset class and the returns on a benchmark portfolio). Intriguingly, Morningstar's tables on individual securities also supply the Sortino ratio, in seeming tribute to the original postmodern alternative to the Sharpe ratio. Morningstar's inclusion of the Sortino ratio in its reports effectively reports negative semideviation as the lower partial second moment of each security's distribution of returns around its target benchmark.

Although Morningstar and other popular sources typically do not report downside semideviation separately from the Sortino ratio, σ_- is easily recovered from the interaction of the Sortino ratio with other traditional measures of portfolio risk and performance:

$$\text{Sharpe} = \frac{x-\mu}{\sigma}; \quad \text{Sortino} = \frac{x-\mu}{\sigma_-}$$

$$\sigma_- = \sigma \cdot \frac{\text{Sharpe}}{\text{Sortino}}$$

Slightly more effort yields information on *upside* semideviation. Even where financial news sources neglect to report upside volatility, that value awaits extraction by analytical means from any report that includes downside semideviation. Modest rearrangement, combined with the Pythagorean relationship of upside to downside semideviation, reveals a method for recovering upside semideviation from standard deviation and the Sharpe and Sortino ratios:

$$\sigma_+ = \sigma \cdot \sqrt{1 - \left(\frac{\text{Sharpe}}{\text{Sortino}}\right)^2}$$

The trigonometric manifestation of the Pythagorean theorem makes it possible, and arguably more desirable on strictly aesthetic grounds, to recover upside semideviation by treating either the arcsine or the arccosine of the ratio of the Sharpe to the Sortino ratio as an angle in radians:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sigma_+ = \sigma \cdot \sin\left[\cos^{-1}\left(\frac{\text{Sharpe}}{\text{Sortino}}\right)\right]$$

6.7 The Trigonometry of Semideviation

To repeat: the applicability of the Pythagorean theorem to semideviation renders single-sided measures of volatility susceptible to analysis through trigonometry. Trigonometric tools enable us to evaluate the relationship between upside (σ_+) and downside (σ_-) semideviation in any number of interesting and useful ways. To the extent that financial returns are negatively skewed,[25] we may expect downside semideviation to exceed its upside counterpart.

A simple stylized example illustrates the point in easily grasped graphic terms. Imagine a negatively skewed distribution in which downside semideviation exceeds upside semideviation by a ratio of 4 to 3:

$$\sigma_- = \frac{4}{3}\sigma_+$$

Under the Pythagorean theorem, standard deviation would exceed downside semideviation by a ratio of 5 to 4 and upside semideviation by a ratio of 5 to 3 (Fig. 6.1):

$$\sigma = \sqrt{\sigma_-^2 + \sigma_+^2}$$

$$\sigma = \frac{5}{4}\sigma_-$$

$$\sigma = \frac{5}{3}\sigma_+$$

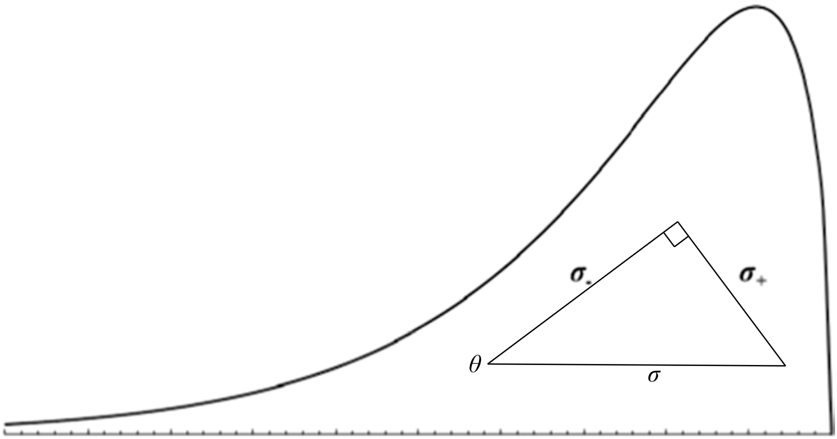

Fig. 6.1 The Pythagorean relationship between upside and downside semideviation

As the foregoing graphic shows, we may arrange upside and downside semideviation as the legs of a right triangle whose hypotenuse represents overall volatility.

On the foregoing assumptions, the ratio of upside to downside semideviation provides a crude gauge of asymmetry in volatility: σ_+ / σ_-. To like effect, we could divide the ratio of the standard deviation to downside semideviation by $\sqrt{2}$, which is the expected ratio of standard deviation to either downside or upside semideviation where volatility on either side of expected return is perfectly symmetrical: $\frac{\sigma}{\sigma_- \sqrt{2}}$. This value reflects the assumption that perfectly symmetrical volatility would generate a standard deviation (hypotenuse) that is $\sqrt{2}$ times as large as either the upside or the downside semideviation. This follows from the trigonometric properties of an isosceles right triangle:

$$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}; \quad \sec\left(\frac{\pi}{4}\right) = \sqrt{2}$$

Finally, the Pythagorean relationship between standard deviation and upside and downside semideviation enables us to express asymmetry in volatility according to angular terms. The angle that is formed by the downside semideviation and the standard deviation, θ, can be derived from the ratio between semideviation and standard deviation:

$$\cos\theta = \frac{\sigma_-}{\sigma}$$

$$\theta = \cos^{-1}\left(\frac{\sigma_-}{\sigma}\right)$$

Equivalently, in terms making use of upside semideviation:

$$\theta = \sin^{-1}\left(\frac{\sigma_+}{\sigma}\right) = \tan^{-1}\left(\frac{\sigma_+}{\sigma_-}\right)$$

Comparing θ to the hypothetical angle formed by perfectly symmetrical measures of semideviation on either side of expected return gives us a final way to express asymmetrical volatility:

$$\frac{\theta}{\pi/4} = \frac{4}{\pi}\cdot\cos^{-1}\left(\frac{\sigma_-}{\sigma}\right)$$

Table 6.4 summarizes the prospective measures of asymmetry in volatility in this stylized example.

Table 6.4 Trigonometric measurements of asymmetry in single-sided volatility

Formula	*Description*	*Value in this example*
σ_+ / σ_-	Ratio of upside to downside semideviation	$3/4 = 0.75$
$\dfrac{\sigma}{\sigma_-\sqrt{2}}$	Ratio of standard deviation to downside semideviation, relative to $\sec(\pi/4)$	$\dfrac{5}{4\sqrt{2}} \oplus 0.8839$
$\dfrac{\theta}{\pi/4} = \dfrac{4}{\pi}\cdot\cos^{-1}\left(\dfrac{\sigma_-}{\sigma}\right)$	Ratio of the semideviation angle, relative to $\cos^{-1}\left(1/\sqrt{2}\right) = \pi/4$	$\dfrac{4}{\pi}\cos^{-1}\left(\dfrac{4}{5}\right) \approx 0.8193$

6.8 Omega

The Sortino ratio has spawned a host of comparable risk measures. Indeed, all of these measures are mathematically related. Unlike the Sharpe ratio, which "measures only the sign and magnitude of the average risk premium relative to the risk incurred in achieving it," the downside risk measures heralded by the Sortino ratio disavow the assumption of normally distributed returns and "focus[] [instead] on the *likelihood* of not meeting some target return."[26] These measures are all designed to measure the ratio between upside potential and downside risk. Their common objective is to interpret, if not necessarily to report, skewness in the distribution of returns. These measures share a primary goal of anticipating the reaction of investors to losses and a secondary goal of retaining sensitivity to prospective gains. In other words, the postmodern quest in portfolio theory is heightened vigilance against downside risk, without complete sacrifice of upside potential.

One possibility is a direct comparison of the upper and lower partial moments of a single probability distribution, each defined according to a shared point of reference. William Shadwick and Con Keating have developed what they call the omega measure, consisting of the ratio of the first-order upper and lower partial moments of the distribution of returns, each about the target return, τ:[27]

$$\Omega(\tau) = \frac{\int_{\tau}^{\infty} \left[1 - F(x)\right] \, dx}{\int_{-\infty}^{\tau} F(x) \, dx}$$

where τ is the target return and $F(x)$ is the cumulative distribution function of returns about that target. Rendering this equation in the terms used to define other measures of risk-adjusted performance requires the expression of the denominator and the numerator into moments (partial and complete) of the probability density function:[28]

$$\Omega(\tau) = \frac{\int_{\tau}^{\infty} (\mu - \tau) f(x) \, dx}{\int_{-\infty}^{\tau} (\tau - \mu) f(x) \, dx}$$

The omega ratio is a straightforward comparison of the first-order upper and lower partial moments for returns. $\Omega(\tau) > 1$ indicates a bias in favor of upside gains, while $\Omega(\tau) < 1$ indicates a preponderance of losses. A symmetrical distribution, with τ set at μ itself, would return an omega ratio of $\Omega(\tau) = 1$.

The numerator of the omega ratio introduces the upper partial moment, the mirror image of the lower partial moment that has figured so prominently in the specification of these postmodern alternatives to conventional measures of risk:[29]

$$\mu_n^+(r) = \int_{\tau}^{\infty} (x - r)^n f(x)\ dx$$

It should be transparent that the omega ratio, for any targeted reference point, τ, is simply the ratio of the upper partial first moment about τ to the lower partial first moment about τ:

$$\Omega(\tau) = \frac{\int_{-\tau}^{\infty} (\mu - \tau) f(x)\ dx}{\int_{-\infty}^{\tau} (\tau - \mu) f(x)\ dx} = \frac{\mu_1^+(\tau)}{\mu_1^-(\tau)}$$

6.9 Kappa

In mathematical terms, the omega ratio and the Sortino ratio are very closely related. Paul Kaplan and James Knowles have restated the omega ratio in terms of the lower partial moments used to define the Sortino ratio:[30]

$$\Omega(\tau) = \frac{\mu - \tau + \int_{-\infty}^{\tau} (\tau - \mu) f(x)\ dx}{\int_{-\infty}^{\tau} (\tau - \mu) f(x)\ dx} = \frac{\mu - \tau}{\int_{-\infty}^{\tau} (\tau - \mu) f(x)\ dx} + 1$$

Kaplan and Knowles treat the Sortino ratio as a specialized, second-order case of a more general family of downside risk-adjusted performance measures called kappa:

$$\kappa_n(\tau) = \frac{\mu - \tau}{\sqrt[n]{\int_{-\infty}^{\tau} (\tau - \mu) f(x)\ dx}}$$

By this definition, the Sortino ratio is simply κ_2. The omega ratio is $\kappa_1 + 1$.[31] A third measure, designated by its creators as the "Sharpe-omega ratio,"[32] turns out to be, straightforwardly, κ_1.[33]

Like omega, kappa can be stated in terms of complete and partial moments about τ. Kappa to any degree uses the same numerator as the Sharpe, Treynor, and Sortino ratios: $\mu - \tau$. As I have already demonstrated, that expression may be rendered as the first complete moment about τ, or $\mu_1{'}(\tau)$. Kappa's denominator is a generalized version of the Sortino ratio's denominator:

$$\kappa_n(\tau) = \frac{\int_{-\infty}^{\infty} (\tau - \mu)^1 f(x) dx}{\sqrt[n]{\int_{-\infty}^{\tau} (\tau - \mu)^n f(x) dx}} = \frac{\mu_1^{'}(\tau)}{\sqrt[n]{\mu_n^{-}(\tau)}}$$

The relationship between the omega ratio and first-order kappa arises from the fact that the complete first moment about τ is the upper partial first moment *minus* the lower partial first moment:

$$\mu - \tau = \mu_1^{'}(\tau) = \int_{-\infty}^{\infty} (\mu - \tau)^1 f(x)\ dx$$

$$\int_{-\infty}^{\infty} (\mu - \tau)^1 f(x) dx = \int_{\tau}^{\infty} (\mu - \tau)^1 f(x) dx + \int_{-\infty}^{\tau} (\mu - \tau)^1 f(x) dx$$

$$\int_{-\infty}^{\infty} (\mu - \tau)^1 f(x) dx = \int_{\tau}^{\infty} (\mu - \tau)^1 f(x) dx - \int_{-\infty}^{\tau} (\tau - \mu)^1 f(x) dx$$

$$\therefore \quad \mu_1^{'}(\tau) = \mu_1^{+}(\tau) - \mu_1^{-}(\tau)$$

$$\kappa_1(\tau) = \frac{\mu_1^{'}(\tau)}{\mu_1^{-}(\tau)}$$

$$\kappa_1(\tau) = \frac{\mu_1^+(\tau) - \mu_1^-(\tau)}{\mu_1^-(\tau)}$$

$$\kappa_1(\tau) = \frac{\mu_1'(\tau)}{\mu_1^-(\tau)} - 1$$

$$\kappa_1(\tau) = \Omega(\tau) - 1$$

$$\Omega(\tau) = \kappa_1(\tau) + 1$$

Kappa is readily scaled. Kappa enables a portfolio manager to calibrate the degree to which downside risk can, will, or should affect decision-making. κ_n need not be adjusted by integers. A value of $n = 1.5$, for instance, achieves some but not all of the magnification of downside risk that takes place under the Sortino ratio, where $n = 2$.

6.10 An Overview of Single-Sided Measures of Risk Based on Lower Partial Moments

Table 6.5 provides a complete overview of risk-adjusted measures of financial performance drawn from lower partial moments of the distribution of returns.

Among these measures, the unfiltered omega ratio provides the most information about skewness within a distribution of returns. $\Omega(\tau)$ is a direct measure of the ratio of the skewness of a distribution about a chosen target, τ. As a ratio, $\Omega(\tau)$ communicates by reference to the intuitive baseline of 1 whether a distribution is skewed toward gains or toward losses. As a guide to downside risk-adjusted performance, omega has proved quite productive in the literature on quantitative finance.[34]

To be sure, postmodern portfolio theory strives to account for the impact of behavioral psychology on the perception of risk and on the formulation of appropriate responses to risk. To the extent that we wish to apply some sort of magnifying glass to account for behavioral reactions to risk or, by contrast, to achieve the opposite in order to neutralize the impact of behavior psychology, we can borrow Kaplan and Knowles's twist of applying exponents to convert a special-case ratio into a generalized family of risk-adjusted measurements of performance capable of being calibrated separately for risk-seeking and risk-averse behavior. Their kappa

Table 6.5 Risk-adjusted measures of financial performance drawn from lower partial moments of the distribution of returns

Measure	*Description*	*Partial moment formula*	*Full formula*
Sharpe	$\frac{\mu-\tau}{\sigma}$	$\frac{\mu_1'(\tau)}{\sqrt{\mu_2'(\tau)}}$	$\frac{\int_{-\infty}^{\infty}(\mu-\tau)^1 \, f(x)\,dx}{\sqrt{\int_{-\infty}^{\infty}(\mu-\tau)^2 \, f(x)\,dx}}$
Sortino	$\frac{\mu-\tau}{\sigma_-}$	$\frac{\mu_1'(\tau)}{\sqrt{\mu_2^-(\tau)}}$	$\frac{\int_{-\infty}^{\infty}(\mu-\tau)^1 \, f(x)\,dx}{\sqrt{\int_{-\infty}^{\tau}(\tau-\mu)^2 \, f(x)\,dx}}$
Kappa$_n$	$\frac{\mu-\tau}{\sqrt[n]{\mu_n^-}}$	$\frac{\mu_1'(\tau)}{\sqrt[n]{\mu_n^-(\tau)}}$	$\frac{\int_{-\infty}^{\infty}(\mu-\tau)^1 \, f(x)\,dx}{\sqrt[n]{\int_{-\infty}^{\tau}(\tau-\mu)^n \, f(x)\,dx}}$
Omega	$\frac{UPM_1(\tau)}{LPM_1(\tau)}$	$\frac{\mu_1^+(\tau)}{\mu_1^-(\tau)}$	$\frac{\int_{\tau}^{\infty}(\mu-\tau)\, f(x)\,dx}{\int_{-\infty}^{\tau}(\tau-\mu)\, f(x)\,dx}$

measure, it bears remembering, "is defined for any [order] exceeding zero."[35] These adjustments may provide us a more finely grained account of the full range of conduct predicted by prospect theory:

$$\Omega_{\varsigma,\alpha}(\tau) = \frac{\left[\int_{\tau}^{\infty}(\mu-\tau)f(x)\,dx\right]^{\varsigma}}{\left[\int_{-\infty}^{\tau}(\tau-\mu)f(x)\,dx\right]^{\alpha}}$$

where the exponent $\varsigma \geq 1$ in the numerator indicates risk-seeking behavior motivated by a desire for gains over a target benchmark, and the exponent

$\alpha \geq 1$ in the denominator indicates risk-averse behavior motivated by fear of downside risk of failure to attain gains matching or exceeding the same benchmark.[36]

6.11 Noninteger Exponents Versus Ordinary Polynomial Representations

David Nawrocki has praised the use of partial moments raised to noninteger exponents, as typified by the work of W.V. Harlow and Ramesh Rao,[37] as a great advance in the flexibility of single-sided risk measures. The ability to derive lower partial moments of noninteger orders, Nawrocki said at the turn of the century, catapulted the use of semivariance from "silent black and white film" to "wide screen" Technicolor film with digital surround sound."[38]

Single-factor risk measures drawn from lower partial moments, such as omega and kappa, continue to hold their allure. In a pivotal 2010 article identifying the alternative approach of using higher moments to specify a CAPM of an order greater than 2—an approach that Chap. 10 of this book describes and ultimately embraces—Campbell Harvey, John Liechty, Merrill Liechty, and Peter Müller acknowledged their debt to William Shadwick, Con Keating, and Ana Cascon, the original proponents of the omega measure, for "advocat[ing] the use of a summary function … that represents all of the relevant information contained within the observed data" through "a complicated, nonlinear utility function which can accommodate higher order moments."[39]

As appealing as it sounds to be able to calibrate partial moments for noninteger orders, very similar results can be obtained through ordinary polynomial equations with appropriate coefficients. Consider the problem of devising a polynomial equation that best fits the equation $y = x^{2.6}$ over the range $x = 0$ to 3. With strikingly good results, we can render a very close substitute for this equation as a polynomial taking the form, $y = kx^3 + (1 - k)x^2$, where the sum of coefficients k and $1 - k$ naturally is equal to 1. The general solution to this problem where the noninteger exponent is expressed as the sum of integer $a > 0$ and noninteger $0 < b < 1$ follows:

$$x^{a+b} \approx kx^{a+1} + (1-k)x^{a}$$

Let $z = x_{\max}$. Solving $z^{a+b} = kz^{a+1} + (1-k)z^{a}$ provides perfect matches for the noninteger value of the exponent, $a + b$, and the polynomial approximation of z^{a+b} at the opposite ends of the relevant range, $x = 0$ and $x = z$. The equation $x^{a+b} = kx^{a+1} + (1-k)x^a$ necessarily intersects at $x = 1$, since 1 raised to any power is 1 and the sum of the coefficients is also 1. Dividing both sides of the equation, $z^{a+b} = kz^{a+1} + (1-k)z^a$, by z^a quickly yields the two coefficients, k and $1 - k$:

$$z^b = kz + (1-k)$$
$$z^b - 1 = k(z-1)$$
$$k = \frac{z^b - 1}{z - 1}$$
$$1 - k = \frac{z - z^b}{z - 1}$$

For $z = 1$, or the value of these equations over the range $x = 0$ to 1, avoidance of division by zero provides an even simpler solution to the coefficients: $k = b$; $1 - k = 1 - b$. equation such as $y = x^{2.6}$ over a defined range can be easily emulated by a third-order polynomial with the right coefficients (Fig. 6.2).[40] For $0 \leq \times \leq 3$, $x^{2.6} \approx 0.46651\, x^3 + 0.533409\, x^2$.

The value of this simple algebraic exercise will become apparent once this book develops a four-moment CAPM approximated by a fourth-order Taylor series. By definition, a Taylor series expansion takes the form of a polynomial. In due course, we shall see the wisdom that Harvey, Liechty, Liechty, and Müller exhibited in distinguishing their approach from the single-exponent methodology underlying the omega and kappa ratios: even as we embrace "the premise that investors have utilities which accommodate higher order moments of the predictive distribution," and "strongly" at that, it will make analytical sense to "restrict[] our focus to utility functions which can be approximated by a third- [or higher] order Taylor series."[41]

In the meanwhile, it suffices to make the philosophical or even the aesthetic case for analytical simplicity. Although quantitative finance may not quite achieve the "great mathematical beauty" often ascribed to Albert Einstein's theory of general relativity ($E = mc^2$)[42] or Euler's identity ($e^i\pi + 1 = 0$),[43] there is more than enough elegance in portfolio theory to motivate us to perform financial "mathematics for the beauty of it."[44] As appealing as it seems to be able to adjust a risk measure "rheostatically," as

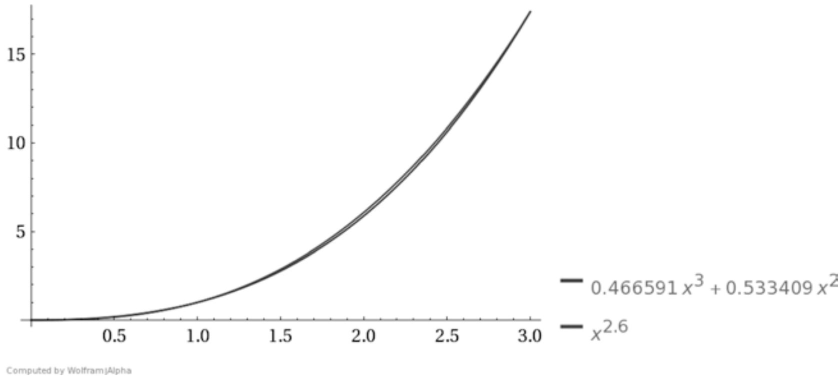

Fig. 6.2 Fitting an ordinary polynomial function to the equation $y = x^{a+b}$, where the exponent $a+b$ is a noninteger. The coefficients for each term in the polynomial equation are analytically rather than computationally determined

it were, with a single noninteger exponent reflecting investor preference for upside potential as well as investor aversion to downside risk, retaining the proper polynomial form, with exponents restricted to integers, preserves the analytical structure that has defined portfolio theory since its "modern" origins. The coefficients corresponding to each order of the polynomial function define distinct factor loadings. They accommodate all of the empirical or computational nuance promised by measures such as noninteger orders of kappa, while retaining the analytical power and interpretive ease of ordinary polynomial equations.

Notes

1. Javier Estrada, *The Cost of Equity of Internet Stocks: A Downside Risk Approach*, 10 Eur. J. Fin. 239–254, 241 (2004).
2. Hogan & Warren, Chap. 5, *supra* note 33, at 10; *accord* James Chong & G. Michael Phillips, *Measuring Risk for Cost of Capital: The Downside Beta Approach*, 4 J. Corp. Treas. Mgmt. 344–352, 347 (2012); *see also* Bawa & Lindenberg, Chap. 5, *supra* note 35, at 197 (noting that a "mean-lower partial moment framework … is identical in form to the traditional Capital Asset Pricing Model obtained in the

mean-variance (MV) framework," with the substitution of conditional beta for beta across the full spectrum of returns).

3. *See* William F. Sharpe, *Mutual Fund Performance*, 39 J. Bus. 119–138, 123 (1966); William F. Sharpe, *Adjusting for Risk in Portfolio Performance Measurement*. 1:2 J. Portfolio Mgmt. 29–34 (Winter 1975).
4. *See* Frank A. Sortino & Robert van der Meer, *Downside Risk*, 17:4 J. Portfolio Mgmt. 27–31 (Summer 1991). *See generally* Frank A. Sortino & Stephen Satchell, Measuring Downside Risk in Financial Markets: Theory, Practice and Implementation (2001).
5. *See*, e.g., James Clash, *Focus on the Downside*, Forbes, Feb. 12, 1999, at 162–163; Estrada, *Cost of Equity in Internet Stocks*, *supra* note 1, at 241.
6. *See supra* § 5.1, at 59–60.
7. "Moore's law is the observation that over the history of computing hardware, the number of transistors on integrated circuits doubles approximately every two years." http://en.wikipedia.org/wiki/Moore's_law. Computing power is often assumed to double every 18 months, based on "a doubling in chip performance," which in turn combines the effects of more transistors and greater processing speed. *Id*. *See generally* Gordon E. Moore, *Cramming More Components onto Integrated Circuits*, Electronics Mag., Aug. 1965, at 4–7 (available at http://www.monolithic3d.com/uploads/6/0/5/5/6055488/gordon_moore_1965_article.pdf). For a journalistic assessment of the economic and sociological impact of Moore's law, see Jonathan Rauch, *The New Old Economy: Oil, Computers, and the Reinvention of the Earth*, Atlantic Monthly, January 2001, at 35–49 (available at http://www.theatlantic.com/past/docs/issues/2001/01/rauch.htm).
8. *See* Markowitz, Todd, Xu & Yamane, Chap. 5, *supra* note 9.
9. *See* Fama & French, *The Cross-Section of Stock Returns*, Chap. 4, *supra* note 4; Fama & French, *Size and Book-to-Market Factors*, Chap. 4, *supra* note 49.
10. *See generally*, e.g., David Lamb & Susan M. Easton, Multiple Discovery: The Pattern of Scientific Progress (1984); Robert K. Merton, *Resistance to the Systematic Study of Multiple Discoveries in Science*, 4 Eur. J. Sociol. 237–282 (1963), *reprinted in* Robert K. Merton, The Sociology of Science: Theoretical and Empirical Investigations 371–382 (1973).

11. *See* Robert K. Merton, *Singletons and Multiples in Scientific Discovery: a Chapter in the Sociology of Science*, 105 PROC. AM. PHIL. SOC'Y 470–486 (1961), *reprinted in* MERTON, THE SOCIOLOGY OF SCIENCE, *supra* note 10, at 343–370.
12. *Compare* Roy, *Safety First*, Chap. 5, *supra* note 10, *with* Fishburn, Chap. 5, *supra* note 39.
13. *See* http://en.wikipedia.org/wiki/Moment_(mathematics)#Partial_moments.
14. *See* http://en.wikipedia.org/wiki/Moment_(mathematics).
15. *See* William F. Sharpe, *The Sharpe Ratio*, 21:1 J. PORTFOLIO MGMT. 49–58 (Fall 1994).
16. *See id.*
17. *See* Bawa, *Optimal Rules*, Chap. 5, *supra* note 39; Fishburn, Chap. 5, *supra* note 39; W.V. Harlow, *Asset allocation in a Downside Risk Framework*, 47:5 FIN. ANALYSTS J. 28–40, 30 (Sept./Oct. 1991); Harlow & Rao, Chap. 5, *supra* note 37; http://en.wikipedia.org/wiki/Moment_(mathematics).
18. *See* Li Chen, Simai He & Shuzhong Zhang, *When All Risk-Adjusted Performance Measures Are the Same: In Praise of the Sharpe Ratio*, 11 QUANT. FIN. 1439–1447 (2011).
19. *See*, e.g., Frank A. Sortino, *From Alpha to Omega*, MANAGING DOWNSIDE RISK IN FINANCIAL MARKETS, Chap. 5, *supra* note 19, at 3–25, 10; Sortino, van der Meer & Plantinga, Chap. 4, *supra* note 36. Others have also endorsed the use of downside semideviation as a risk measure. *See*, e.g., Clash, *supra* note 5; Estrada, *Cost of Equity in Internet Stocks*, *supra* note 1, at 241.
20. Robert Libby & Peter C. Fishburn, *Behavioral Models of Risk Taking in Business Decisions*, 15 J. ACCOUNTING RESEARCH 272–292, 277 (1977); *accord* Harlow & Rao, Chap. 5, *supra* note 37, at 292.
21. *See*, e.g., Bawa & Lindenberg, Chap. 5, *supra* note 35, at 192 n.3 (acknowledging that portfolio optimization according to semivariance "can be solved for any fixed point"); Harlow & Rao, Chap. 5, *supra* note 37, at 286 (devising a "generalized Mean-Lower Partial Moment" model "consistent with *any* prespecified target rate of return" [emphasis in original]); *id.* at 287 (obtaining portfolio equilibrium "for arbitrary τ" as part of "a generalized…asset pricing framework" making use of mean lower partial lower moments to any order n).
22. FEIBEL, Chap. 5, *supra* note 39, at 160.

23. E.g., Chong & Phillips, *supra* note 2, at 347. Other sources take pains to specify that it is semivariance rather than semideviation that is straightforwardly additive. *See*, e.g., Estrada, *An Alternative Behavioural Model*, Chap. 5, *supra* note 51, at 231, 237; Estrada, *Downside Risk and Capital Asset Pricing*, Chap. 5, *supra* note 42, at 177 n.4.
24. http://www.morningstar.com/InvGlossary/upside-downside-capture-ratio.aspx.
25. *See generally* sources cited, Chap. 4, *supra* note 14.
26. Paul D. Kaplan & James A. Knowles, *Kappa: A Generalized Downside Risk-Adjusted Performance Measure* 2 (2004) (available at http://corporate.morningstar.com/NO/documents/MethodologyDocuments/ResearchPapers/KappaADownsideRisk_AdjustedPerformanceMeasure_PK.pdf).
27. *See* William F. Shadwick & Con Keating, *A Universal Performance Measure*, 6:3 J. Performance Measurement 59–84 (Spring 2002).
28. *See* Kaplan & Knowles, *supra* note 26, at 15.
29. *See* http://en.wikipedia.org/wiki/Moment_(mathematics).
30. *See* Kaplan & Knowles, *supra* note 26, at 15.
31. *See id*. at 3.
32. *See* Hossein Kazemi, Thomas Schneeweis & Raj Gupta, *Omega as a Performance Measure* (June 15, 2003) (available at http://faculty.fuqua.duke.edu/~charvey/Teaching/BA453_2006/Schneeweis_Omega_as_a.pdf) (developing a closely related measure called Sharpe-Omega).
33. *See* Kaplan & Knowles, *supra* note 26, at 3 n.1.
34. *See*, e.g., Philippe Bertrand & Jean-Luc Prigent, *Omega Performance Measures and Portfolio Insurance*, 35 J. Banking & Fin. 1811–1823 (2011); Theofanis Darsinos & Stephen Satchell, *Generalising Universal Performance Measures*, Risk, June 2004, at 80–84 (available at http://www.risk.net/data/Pay_per_view/risk/technical/2004/0604_tech_investment.pdf) (proposing an entire family of measures, like the omega ratio, that directly compare upper and lower partial moments of the same degree); S.J. Kane, M.C. Bartholomew-Biggs, M. Cross & M. Dewar, *Optimizing Omega*, 5 J. Global Optimization 153–167 (2009); Helmut Mausser, David Saunders & Luis Seco, *Optimising Omega*, Risk, Nov. 2006, at 88–92 (available at http://www.risk.net/data/risk/pdf/technical/risk_1106_Mausser.pdf).

35. Kaplan & Knowles, *supra* note 26, at 3.
36. Cf. Denisa Cumova & David Nawrocki, *Portfolio Optimization in an Upside Potential and Downside Risk Framework* 9 (Oct. 2003) (available at http://www90.homepage.villanova.edu/michael.pagano/DN%20upm%20lpm%20measures.pdf) (proposing the possibility of applying separate exponents to upper and lower partial moments to reflect different levels of risk-seeking or risk-averse behavior in individual investors).
37. *See* Harlow & Rao, Chap. 5, *supra* note 37.
38. David Nawrocki, *A Brief History of Downside Risk Measures*, 8:3 J. INVESTING 9–25 (Fall 1999).
39. Campbell R. Harvey, John C. Liechty, Merrill W. Liechty & Peter Müller, *Portfolio Selection with Higher Moments*, 10 QUANT. FIN. 469–485, 471 (2010).
40. *See* http://www.wolframalpha.com/input/?i=plot+%283%5E.6-1%29%2F%283-1%29*x%5E3+%2B+%283-3%5E.6%29%2F%283-1%29*x%5E2+and+x%5E2.6+for+x%3D0+to+3.
41. Harvey, Liechty, Liechty & Müller, *supra* note 39, at 471.
42. SUBRAHMANYAN CHANDRASEKHAR, TRUTH AND BEAUTY: AESTHETICS AND MOTIVATIONS IN SCIENCE 148 (1987).
43. *See* RICHARD P. FEYNMAN, THE FEYNMAN LECTURES ON PHYSICS 22 (1977) (describing Euler's identity as "our jewel"—indeed, as "the most remarkable formula in mathematics").
44. SERGE LANGE, THE BEAUTY OF MATHEMATICS: THREE PUBLIC DIALOGUES 3 (1985).