On rigid circuit graphs

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1. Introduction

The graphs considered in this paper may have loops (German: Schlinge) and multiple edges and they may be infinite, except where the contrary is stated. The axiom of choice is assumed throughout.

Definitions 1. If Ω is a circuit and x and y are two distinct vertices of Ω which are not joined by any edge belonging to Ω, then an edge which joins x and y is called a *chord* of Ω. A graph in which every circuit with more than three vertices has at least one chord, is called a *rigid circuit graph*, or ri. cir. graph for short.

All trees and all cliques (see below) are ri. cir. graphs, and so are the interval graphs investigated by G. HAJÓS [2]. Theorems on ri. cir. graphs have been given by A. HAJNAL and J. SURÁNYI [3], and by C. BERGE [1].

In the present paper a new characterisation of ri. cir. graphs will be established (Theorem 1) and the results of the above mentioned authors, and some new results, will be simply derived.

Definitions 2. If Γ is a connected graph and $\mathfrak{S}$ is a set of vertices contained in Γ, then $\mathfrak{S}$ is called a *cut-set* of Γ if $\Gamma - \mathfrak{S}$ is disconnected. A cut set $\mathfrak{S}$ is called a *minimal cut-set* of Γ if no proper subset of $\mathfrak{S}$ is a cut-set of Γ, and it is called a *relatively minimal cut-set* of Γ if Γ contains two vertices which are separated by $\mathfrak{S}$ but by no proper subset of $\mathfrak{S}$.

A graph in which every pair of distinct vertices are connected by at least one edge is called a *clique*. A graph having one vertex is counted as a clique, but the empty graph is not. A clique with μ vertices is called a *μ-clique*.

2. The properties of rigid circuit graphs

If Γ is a ri. cir. graph and $\mathfrak{S}$ is a proper subset of the set of vertices of Γ, then $\Gamma - \mathfrak{S}$ is a ri. cir. graph. This follows immediately from the definition.

Theorem 1. *A graph is a ri. cir. graph if and only if every pair of vertices which belong to the same relatively minimal cut-set are joined by at least one edge.*

(In other words, if $\mathfrak{S}$ is a relatively minimal cut-set, then the vertices of $\mathfrak{S}$ are the vertices of a clique.)

Proof. First suppose that Γ is a connected ri. cir. graph, that $\mathfrak{S}$ is a relatively minimal cut-set of Γ, and that i and i' are two vertices which belong to $\mathfrak{S}$ and are not joined by an edge. Let a_1 and a_2 be two vertices which are separated by $\mathfrak{S}$, but by no proper subset of $\mathfrak{S}$, and let Γ_1' and Γ_2' denote the connected components of $\Gamma - \mathfrak{S}$ to which a_1 and a_2 respectively belong. i is joined by an edge to at least one vertex of Γ_1' and to at least one vertex of Γ_2', and so is i', because no proper subset of $\mathfrak{S}$ separates a_1 and a_2. For $j = 1, 2$, let Y_j be a path with the least number of vertices among the paths which have i and i' as their end-vertices, and whose intermediate vertices all belong to Γ_j'. Y_1 and Y_2 exist because Γ_1' and Γ_2' are connected. $Y_1 \cup Y_2$ is then a circuit with at least four vertices, and it has no chord. This contradicts the assumption that Γ is a ri. cir. graph.

Secondly suppose that Γ is a connected graph in which every pair of vertices which belong to the same relatively minimal cut-set are joined by at least one edge, but Γ is not a ri. cir. graph. Let Ω denote a circuit contained in Γ which has more than three vertices, but no chord. Let w_1 and w_2 denote two vertices of Ω which are not joined by any edge belonging to Ω, and let the two paths connecting w_1 and w_2 which together make up Ω be denoted by Y and Y'. w_1 and w_2 are not joined to each other by any edge in Γ, because Ω has no chord. Consequently Γ contains at least one cut-set separating w_1 and w_2 (all vertices which are adjacent to w_1 form such a cut-set, for example). Therefore Γ contains a cut set $\mathfrak{S}$ such that $\mathfrak{S}$, but no proper subset of $\mathfrak{S}$, separates w_1 and w_2. $\mathfrak{S}$ is a relatively minimal cut-set, therefore, by hypothesis, every pair of vertices belonging to $\mathfrak{S}$ are joined by at least one edge. Y and Y' each contain at least one vertex belonging to $\mathfrak{S}$, because $\mathfrak{S}$ separates w_1 and w_2, and $(Y \cap \mathfrak{S}) \cap (Y' \cap \mathfrak{S}) = \emptyset$ because $Y \cup Y' = \Omega$. Every edge joining a vertex of $Y \cap \mathfrak{S}$ to a vertex of $Y' \cap \mathfrak{S}$ is however obviously a chord of Ω, and this contradicts the hypothesis that Ω has no chord. Theorem 1 is now proved.

Corollary to Theorem 1. *In a ri. cir. graph every pair of vertices belonging to the same minimal cut-set are joined by at least one edge.*

For a minimal cut-set is also relatively minimal.

But the converse is not true. A simple counterexample is the graph with the eight vertices $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$, and the eight edges (x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_1), (x_i, y_i), $i = 1, 2, 3, 4$. In this graph every minimal cut-set contains just one vertex, but the graph is not a ri. cir. graph. A generalisation of this counterexample is the graph which contains the $\alpha\beta$ vertices x_{pq} ($p = 1, \ldots, \alpha$; $q = 1, \ldots, \beta$), $\alpha \geq 4$, $\beta \geq 2$, and the edges (x_{pq}, x_{pr}) ($p = 1, \ldots, \alpha; q = 1, \ldots, \beta; r = 1, \ldots, \beta$; $q \neq r$) and (x_{st}, x_{s+1t}) ($s = 1, \ldots, \alpha - 1$; $t = 1, \ldots, \beta - 1$), and $(x_{\alpha t}, x_{1t})$ ($t = 1, \ldots, \beta - 1$).

Theorem 2. *If Γ_1 and Γ_2 are ri. cir. graphs and $\Gamma_1 \cap \Gamma_2$ is a clique or empty, then $\Gamma_1 \cup \Gamma_2$ is a ri. cir. graph.*

Proof. If $\Gamma_1 \cup \Gamma_2$ contains no circuit with more than three vertices then it is a ri. cir. graph. If $\Gamma_1 \cup \Gamma_2$ contains such circuits, then let one of them be denoted by Ω. If $\Omega \subseteq \Gamma_1$ or $\Omega \subseteq \Gamma_2$, then Ω has a chord. If $\Omega \nsubseteq \Gamma_1$ and $\Omega \nsubseteq \Gamma_2$, then Ω contains at least one vertex from each of $\Gamma_1 - (\Gamma_1 \cap \Gamma_2)$ and $\Gamma_2 - (\Gamma_1 \cap \Gamma_2)$, w_1 and w_2 respectively say. w_1 and w_2 are separated by the vertices of $\Gamma_1 \cap \Gamma_2$. It now follows, exactly as in the second part of the proof of Theorem 1, that Ω has a chord. Thus every circuit of $\Gamma_1 \cup \Gamma_2$ with more than three vertices has a chord.

Theorem 1 shows that any ri. cir. graph which is not a clique can be constructed out of two smaller mutually disjoint ri. cir. graphs by identifying a clique in one with a similar clique in the other. It follows that any ri. cir. graph which is not a clique, can be obtained by applications of this process starting from a set of cliques. Theorem 2 shows that conversely, whenever the process is applied to two mutually disjoint ri. cir. graphs, the result is a ri. cir. graph. But the union of two ri. cir. graphs whose intersection is neither empty nor a clique may of course be a ri. cir. graph.

The theorems of Berge [1] and of Hajnal and Surányi [3] will now be deduced from Theorem 1.

Definitions 3. A graph is called *$\varkappa$-colourable*, $\varkappa$ a positive integer, if the vertices of the graph can be devided into $\varkappa$ mutually disjoint (colour) classes in such a way that no two vertices belonging to the same class are joined by an edge. It is called $\varkappa$-chromatic if it is $\varkappa$-colourable and not $(\varkappa - 1)$-colourable.

Theorem 3. (Berge [1]). *If a ri. cir. graph is not $\varkappa$-colourable, then it contains a $(\varkappa + 1)$-clique.*

Proof. It is sufficient to prove the theorem for finite ri. cir. graphs, because every infinite graph which is not $\varkappa$-colourable contains a finite subgraph which is not $\varkappa$-colourable [4]. Suppose that the theorem does not hold for all finite ri. cir. graphs. Then for some value of $\varkappa \geq 2$ there exists a finite ri. cir. graph which is not $\varkappa$-colourable and does not contain a $(\varkappa + 1)$-clique, but every ri. cir. graph with fewer vertices which is not $\varkappa$-colourable, contains a $(\varkappa + 1)$-clique.

Γ is not a clique, hence it contains a minimal cut-set, $\mathfrak{S}$ say. Let $\Gamma - \mathfrak{S} = \Gamma_1' \cup \Gamma_2'$, where $\Gamma_1' \neq \emptyset$, $\Gamma_2' \neq \emptyset$ and $\Gamma_1' \cap \Gamma_2' = \emptyset$. Let $\Gamma - \Gamma_2' = \Gamma_1$ and $\Gamma - \Gamma_1' = \Gamma_2$. Then $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \Gamma - (\Gamma - \mathfrak{S})$. By Theorem 1 $\Gamma_1 \cap \Gamma_2$ is a clique. Γ_1 and Γ_2 are ri. cir. graphs and contain fewer vertices than Γ, therefore they are $\varkappa$-colourable. Because $\Gamma_1 \cap \Gamma_2$ is a clique, it follows at once that Γ is $\varkappa$-colourable. This contradiction proves the theorem.

Theorem 4. *If Γ is a finite connected ri. cir. graph which is not a clique and $\mathfrak{S}$ is a cut-set of Γ with the property that $\Gamma - (\Gamma - \mathfrak{S})$ is a clique, and if $\Gamma - \mathfrak{S}$ consists of ν connected components, then Γ contains a set of ν vertices $v_1, \ldots, v_\nu$, such that no two of them are joined by an edge, and if $v_i(\Gamma)$ denotes the set of vertices of Γ which are adjacent to v_i, then $\Gamma - (\Gamma - v_i(\Gamma))$ is a clique, for $i = 1, \ldots, \nu$.*

Proof. By induction over the number of vertices μ. The theorem is clearly true for $\mu = 3$. (A connected ri. cir. graph with three vertices which is not a clique is a path with three vertices, and the two end-vertices of this path have the required property.) Let Γ be any finite connected ri. cir. graph with $\mu \geq 4$ vertices which is not a clique, and assume the theorem true when the number of vertices is less than μ.

Let $\mathfrak{S}$ be any cut-set of Γ such that $\Gamma - (\Gamma - \mathfrak{S})$ is a clique, and let $\Gamma'_1, \ldots, \Gamma'_\nu$ denote the connected components of $\Gamma - \mathfrak{S}$. Let $\Gamma_i = \Gamma - \bigcup_{j=1,\ldots,\nu,\, j \neq i} \Gamma'_j$. Then $\Gamma = \bigcup_{i=1,\ldots,\nu} \Gamma_i$, and $\Gamma_i \cap \Gamma_j = \Gamma - (\Gamma - \mathfrak{S})$ if $1 \leq i < j \leq \nu$. $\Gamma_1, \ldots, \Gamma_\nu$ are all ri. cir. graphs, because each of them is obtained from Γ by deleting vertices.

If Γ_i $(1 \leq i \leq \nu)$ is a clique, then let v_i denote some vertex of Γ'_i. If Γ_i is not a clique, then it contains a minimal cut-set $\mathfrak{S}_i$, by Theorem 1. $\Gamma_i - (\Gamma_i - \mathfrak{S}_i)$ is a clique, therefore by our induction hypothesis Γ_i contains two vertices v_i and v'_i not joined by an edge, and such that $\Gamma_i - (\Gamma_i - v_i(\Gamma_i))$ and $\Gamma_i - (\Gamma_i - v'_i(\Gamma_i))$ are cliques. At most one of v_i and v'_i belongs to $\mathfrak{S}$ since $\Gamma - (\Gamma - \mathfrak{S})$ is a clique, suppose that $v_i \notin \mathfrak{S}$. Then $v_i \in \Gamma'_i$, so that $v_i(\Gamma_i) = v_i(\Gamma)$ and $\Gamma_i - (\Gamma_i - v_i(\Gamma_i)) = \Gamma - (\Gamma - v_i(\Gamma))$, consequently $\Gamma - (\Gamma - v_i(\Gamma))$ is a clique. No two of the vertices v_i, $i = 1, \ldots, \nu$, are joined by an edge, since $v_i \in \Gamma'_i$. Theorem 4 is therefore true for Γ, and consequently it is true generally.

Theorem 4 is not necessarily true if Γ is an infinite ri. cir. graph. It is not true for an infinite path. More generally, if Γ is any graph and Γ^* is constructed from Γ by attaching an infinite path $Y(a)$ to each vertex a of Γ so that $Y(a) \cap Y(a') = \emptyset$ if $a \neq a'$, then Γ^* contains no vertex v such that $\Gamma^* - (\Gamma^* - v(\Gamma^*))$ is a clique.

Definitions 4. A set of vertices in a graph are called *mutually independent* if no two of them are joined by an edge. Two graphs are called *complementary* if they have the same vertices and their union is a clique.

Theorem 5 (HAJNAL and SURÁNYI [3]). *If a ri. cir. graph does not contain $\alpha + 1$ mutually independent vertices, where α is a positive integer, then it contains a set of $\leq \alpha$ mutually disjoint cliques which together include all its vertices.*

Alternative forms of Theorem 5:

1. *If a ri. cir. graph does not contain $\alpha + 1$ mutually independent vertices, then its complementary graph is α-colourable.*

2. *If the complementary graph of a ri. cir. graph is not α-colourable, then it contains an $(\alpha + 1)$-clique.*

Theorem 5 will first be proved for all ri. cir. graphs with a finite number of vertices by induction over the number of vertices μ, and then its truth for all ri. cir. graphs will be deduced.

The theorem is obviously true for $\mu = 1$ and for all cliques. Assume it to be true for ri. cir. graphs with fewer than μ vertices, and suppose that Γ is a ri. cir. graph with μ vertices which is not a clique. If Γ is disconnected, then it follows at once from the induction hypothesis that Theorem 5 holds for Γ. If Γ is connected, then it contains a minimal cut-set, $\mathfrak{S}$ say. By Theorem 1 $\Gamma - (\Gamma - \mathfrak{S})$ is a clique, therefore by Theorem 4 there are two vertices v_1 and v_2 in Γ, which are not joined by an edge and such that $\Gamma - (\Gamma - v_1(\Gamma))$ and $\Gamma - (\Gamma - v_2(\Gamma))$ are cliques. Let the clique contained in Γ whose set of vertices is $\{v_1\} \cup v_1(\Gamma)$ be denoted by $\mathfrak{V}_1$.

$\Gamma - \mathfrak{V}_1$ does not contain α mutually independent vertices, because any such set and v_1 would together constitute a set of $\alpha + 1$ mutually independent vertices in Γ. Therefore, by the induction hypothesis, $\Gamma - \mathfrak{V}_1$ contains a set of $\leq \alpha - 1$ mutually disjoint cliques which together include all its vertices. Such a set of cliques and $\mathfrak{V}_1$ together satisfy the requirements of the theorem, which is therefore true for all finite ri. cir. graphs.

Let Δ be an infinite ri. cir. graph which does not contain $\alpha + 1$ mutually independent vertices. If $\mathfrak{S}$ is a subset of the vertices of Δ such that $\Delta - \mathfrak{S}$ is finite, then it follows from Theorem 5 that $\Delta - \mathfrak{S}$ contains a set of $\leq \alpha$ mutually disjoint cliques which together include all the vertices of $\Delta - \mathfrak{S}$. Consequently the graph complementary to $\Delta - \mathfrak{S}$ is α-colourable. It follows that every finite subgraph of the graph complementary to Δ is α-colourable. Consequently the graph complementary to Δ is α-colourable, i. e. Theorem 5 holds for Δ [4].

The converse of Theorem 5 is not true, a graph which does not contain $\alpha + 1$ mutually independent vertices, and which contains α cliques which together include all its vertices, is not necessarily a ri. cir. graph. A circuit with an even number of vertices is a simple counterexample.

3. The construction of rigid circuit graphs by the addition of edges

Theorem 6. *From any graph with μ vertices which contains α mutually independent vertices, it is always possible to obtain a $(\mu - \alpha + 1)$-colourable ri. cir. graph by adding edges.*

Proof. Let $\mathfrak{A}$ denote a set of α mutually independent vertices of Γ. The graph obtained from Γ by making $\Gamma - \mathfrak{A}$ into a clique is a ri. cir. graph by Theorem 2, and is $(\mu - \alpha + 1)$-colourable.

For all $\varkappa \geq 1$ and all $\mu \geq \varkappa$ there exist $\varkappa$-chromatic graphs having μ vertices, for which Theorem 6 is best possible.

Proof. A graph with μ isolated vertices is 1-chromatic and a ri. cir. graph. For $\varkappa \geq 2$ and $\mu \geq \varkappa$ let $\Gamma(\mu, \varkappa)$ be a graph having μ vertices and such that its vertices fall into $\varkappa$ non empty mutually disjoint classes $\mathfrak{C}_1, \ldots, \mathfrak{C}_\varkappa$, and two vertices are joined by an edge if and only if they do not belong to the same class. In order to obtain a ri. cir. graph from $\Gamma(\mu, \varkappa)$ by adding edges, it is necessary to join every pair of distinct vertices by an edge in $\varkappa - 1$ of the classes $\mathfrak{C}_1, \ldots, \mathfrak{C}_\varkappa$.

Theorem 7. *From any $\varkappa$-chromatic graph it is always possible to obtain a $\varkappa$-chromatic graph in which every circuit with at least five vertices has chords, by adding edges.*

Proof. Colour the graph with $\varkappa$ colours, and join every pair of vertices which do not have the same colour, by an edge.

In Theorem 6 and 7 some or all of $\mu, \alpha, \varkappa$ may be infinite cardinals.

The graphs $\Gamma(\mu, \varkappa)$ show that the least number of new edges required to make a $\varkappa$-chromatic graph ,$\varkappa \geq 2$, with μ vertices into a ri. cir. graph may be as high as $\mu(\mu - \varkappa)(\varkappa - 1)\varkappa^{-2}$.

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Eingegangen am 30. 6. 1960