Problem 1

Theorem 1. *Let T be a normal operator on a finite-dimensional inner product space V, and let W be a subspace of V. Then W is T-invariant if and only if W is also T^*-invariant.*

Solution

Proof. Suppose W is T-invariant, and let $\{w_1, \cdots, w_k\}$ be an orthonormal basis for W. Extend this to an orthonormal basis of V:

$$\beta = \{w_1, \cdots, w_k, v_1, \cdots, v_n\}.$$

Then, since W is T-invariant, we know $[T]_\beta$ has the form

$$[T]_\beta = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A is an $k \times k$ matrix, and B is an $n \times k$ matrix, and C is an $n \times n$ matrix. With this notation, we have

$$[T^*]_\beta = \begin{bmatrix} A^* & 0 \\ B^* & C^* \end{bmatrix}.$$

Thus, to show that W is T^* invariant, we must show that $B^* = 0$. Now, since T is normal, we have

$$||Tw_i|| = ||T^*w_i|| \implies ||Tw_i||^2 = ||T^*w_i||^2$$

for each i. Writing these in terms of our matrices, we see

$$||Tw_i||^2 = \sum_{j=1}^{k} |A_{ji}|^2,$$

and

$$\begin{aligned} ||T^*w_i||^2 &= \sum_{j=1}^{k} |A^*_{ji}|^2 + \sum_{j=1}^{n} |B^*_{ji}|^2 \\ &= \sum_{j=1}^{k} |\overline{A_{ij}}|^2 + \sum_{j=1}^{n} |\overline{B_{ij}}|^2 \\ &= \sum_{j=1}^{k} |A_{ij}|^2 + \sum_{j=1}^{n} |B_{ij}|^2. \end{aligned}$$

As is, these do not allow for any immediately useful comparison, as the two sums are running over different elements of A. To ensure that we run over each element of A, we will sum over the square magnitude of the images of each w_i. Jumping right in, we see

$$\sum_{i=1}^{k} ||Tw_i||^2 = \sum_{i=1}^{k} \sum_{j=1}^{k} |A_{ji}|^2,$$

and

$$\begin{aligned} \sum_{i=1}^{k} ||T^*w_i||^2 &= \sum_{i=1}^{k} \sum_{j=1}^{k} |A_{ij}|^2 + \sum_{i=1}^{k} \sum_{j=1}^{n} |B_{ij}|^2 \\ &= \sum_{i=1}^{k} \sum_{j=1}^{k} |A_{ji}|^2 + \sum_{i=1}^{k} \sum_{j=1}^{n} |B_{ij}|^2 \\ &= \sum_{i=1}^{k} ||Tw_i||^2 + \sum_{i=1}^{k} \sum_{j=1}^{n} |B_{ij}|^2. \end{aligned}$$

Since the second summation in the equation above must be zero, and each element is nonnegative, it follows that each term in the sum is 0. However, this summation is adding together the complex square of each element of B^*, which means we can conclude that $B^* = 0$, as desired.

Now suppose that W is T^*-invariant. Since $T^{**} = T$, and T^* is normal, it follows immediately from the section above that W is T-invariant. □

Problem 2

A linear operator T on a finite-dimensional inner product space over $\mathbb{R}$ or $\mathbb{C}$ is called *positive definite* (resp. *positive semi- definite*) if T is self-adjoint and $\langle T(x), x\rangle > 0$ (resp. $\langle T(X), x\rangle \geq 0$) for all $x \neq 0$. A matrix A is called positive (semi)definite if L_A is positive (semi)definite.

Theorem 2. *Let T be hermitian. Then, the following are true*

Part (a):

T is positive (semi-)definite if and only if all of its eigenvalues are > 0 (≥ 0).

Part (b):

T is positive semi-definite if and only if there is an operator U such that $T = U^*U$.

Solution

Proof. Part (a):

Suppose T is positive (semi-)definite, and let v be an eigenvector of T with corresponding eigenvalue λ. Then, we have

$$\begin{aligned} 0 > (\geq)&\langle Tv, v\rangle \\ &= \langle \lambda v, v\rangle \\ &= \lambda\langle v, v\rangle. \end{aligned}$$

Since $\langle v, v\rangle > 0$, we can conclude that $\lambda > (\geq)0$.

Now suppose all of Ts eigenvalues are > 0 (≥ 0). Since T is self-adjoint, there exists an orthonormal basis of V consisting of the eigenvectors of T,

$$\beta = \{v_1, ..., v_n\}$$

with corresponding eigenvalues λ_i. Let $x \in V$ be nonzero. Then, there exist $a_i \in \mathbb{F}$ (atleast one non-zero) such that

$$x = \sum a_i v_i.$$

Then, we have

$$\begin{aligned} \langle Tx, x\rangle &= \langle T\left(\sum a_i v_i\right), \sum a_i v_i\rangle \\ &= \langle \sum \lambda_i a_i v_i, \sum a_i v_i\rangle \\ &= \sum \lambda_i a_i \langle v_i, \sum a_i v_i\rangle \\ &= \sum \lambda_i a_i \overline{a_i} \\ &= \sum \lambda_i |a_i|^2. \end{aligned}$$

Since each of these terms is nonnegative, and at least one a_i is nonzero, we have $\langle T(x), x\rangle > 0$ (resp. $\langle T(X), x\rangle \geq 0$), and our proof of (a) is complete.

Part (b):

Suppose T is positive semi-definite. Let T_i and λ_i and k be defined as in the spectral theorem. Define an operator U on V by

$$U = \sum_{i=1}^{k} \sqrt{\lambda_i} T_i.$$

From part(a) of this theorem, we have that U is well defined, as each of the eigenvalues is a non-negative real number. Then, we have

$$\begin{aligned} U^* &= \left(\sum_{i=1}^{k} \sqrt{\lambda_i} T_i\right)^* \\ &= \sum_{i=1}^{k} \overline{\sqrt{\lambda_i}} T_i^* \\ &= \sum_{i=1}^{k} \sqrt{\lambda_i} T_i && \text{Orthogonal projections are hermitian.} \end{aligned}$$

Thus,

$$\begin{aligned} U^*U &= \sum_{i=1}^{k} \lambda_i T_i && \text{By part (c) of the spectral theorem} \\ &= T && \text{By part (e) of the spectral theorem.} \end{aligned}$$

Now suppose that $T = U^*U$ for some operator U on V. Clearly, T is self-adjoint, since

$$\begin{aligned} T^* &= (U^*U)^* \\ &= U^*U^{**} \\ &= U^*U \\ &= T. \end{aligned}$$

Then, for any $x \in V$, we have

$$\begin{aligned} \langle Tx, x\rangle &= \langle U^*Ux, x\rangle \\ &= \langle Ux, Ux\rangle \\ &\geq 0, \end{aligned}$$

and we can conclude that T is positive semi-definite. □

Problem 3

Theorem 3. *Let T be a normal operator on a complex finite dimensional inner product space V, and let U be any operator on V. If the eigenspaces of T are U-invariant, $TU = UT$.*

Solution

Proof. Let W_i, λ_i, and k be defined as in the spectral theorem, and let $x \in V$. By the spectral theorem, there exist $x_i \in W_i$ such that

$$x = \sum_{i=1}^{k} x_i.$$

Then, we have

$$\begin{aligned}
UTx &= UT\sum_{i=1}^{k} x_i \\
&= \sum_{i=1}^{k} UTx_i \\
&= \sum_{i=1}^{k} U\lambda_i x_i && x_i \in W_i \\
&= \sum_{i=1}^{k} \lambda_i U x_i \\
&= \sum_{i=1}^{k} TUx_i && Ux_i \in W_i \\
&= TU\sum_{i=1}^{k} x_i \\
&= TUx
\end{aligned}$$

Since this is true for all $x \in V$, we have shown that $TU = UT$. □

Problem 4

Suppose that T is normal on the finite dimensional complex inner product space. Use the spectral decomposition of T,

$$T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k,$$

to prove the following results.

1. T is invertible if and only if $\lambda_i \neq 0$ for $1 \leq i \leq k$.
2. T is a projection if and only if every eigenvalue of T is 1 or 0.
3. $T = -T^*$ if and only if every eigenvalue λ_i is imaginary.

Solution

Proof. 1. For the forward direction, we will prove the contrapositive. Suppose that there exists an i such that $\lambda_i = 0$. Let W_j be the eigenspace of T corresponding to λ_i. Then, if $x_i \in W_i$, we have

$$\begin{aligned}
Tx_i &= \lambda_1 T_1 x_i + \cdots + \lambda_i T_i x_i + \cdots + \lambda_k T_k x_i \\
&= \lambda_i x_i && T_j \text{ are orthogonal projections} \\
&= 0.
\end{aligned}$$

Thus, x_i is a non-zero vector in the nullspace of T, which means T is non-invertible, proving the contrapositive.

For the other direction, suppose every eigenvalue of T is nonzero, and let $x \in V$ be nonzero. By the spectral theorem, there exist $x_i \in W_i$, not all zero, such that

$$x = x_1 + \cdots + x_k.$$

Then,

$$\begin{aligned}
Tx &= \lambda_1 T_1 x + \cdots + \lambda_k T_k x \\
&= \lambda_1 x_1 + \cdots + \lambda_k x_k.
\end{aligned}$$

Thus, since any of these vectors which are nonzero are linearly independent (and we know at least one is nonzero), and the eigenvalues are nonzero, we can conclude that $Tx \neq 0$. Since this is true for any nonzero x, we can conclude that the only vector in Ts nullspace is the zero vector, implying that T is invertible.

2. Using part (c) of the spectral theorem, we have

$$\begin{aligned}
T \text{ is a projection} &\iff T^2 = T \\
&\iff (\lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k)(\lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k) = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k \\
&\iff \lambda_1^2 T_1 + \lambda_2^2 T_2 + \cdots + \lambda_k^2 T_k = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k \\
&\iff (\forall 1 \le i \le k)(\lambda_i = 1) \vee (\forall 1 \le i \le k)(\lambda_i = 0),
\end{aligned}$$

where the last line follows from the fact that these orthogonal operators are linearly independent in the vector space of linear operators on V. To see that these are in fact linearly independent, let a_i be such that

$$\sum_i^k a_i T_i = 0.$$

Letting $x_i \in W_i$ be nonzero, we see

$$\begin{aligned}
0 &= 0x_i \\
&= \sum_i^k a_i T_i x_i \\
&= a_i x_i,
\end{aligned}$$

and we can conclude that $a_i = 0$ for all i, as desired.

3. Let $\lambda_j = a_j + ib_j$. Then,

$$\begin{aligned}
T &= \lambda_1 T_1 + \cdots + \lambda_k T_k \\
&= (a_1 + ib_1)T_1 + \cdots + (a_k + ib_k)T_k
\end{aligned}$$

and

$$\begin{aligned}
-T^* &= -(\lambda_1 T_1 + \cdots + \lambda_k T_k)^* \\
&= -(\overline{\lambda_1} T_1 + \cdots + \overline{\lambda_k} T_k) && \text{Orthogonal projections are hermitian} \\
&= -(\overline{(a_1 + ib_1)} T_1 + \cdots + \overline{(a_k + ib_k)} T_k) \\
&= -((a_1 - ib_1)T_1 + \cdots + (a_k - ib_k)T_k) \\
&= (-a_1 + ib_1)T_1 + \cdots + (-a_k + ib_k)T_k.
\end{aligned}$$

Thus, by linear independence of each T_k, we have

$$\begin{aligned}
T = -T^* &\iff (\forall 1 \le i \le k)(a_i = 0) \\
&\iff (\forall 1 \le i \le k)(\lambda_i \text{ is imaginary}),
\end{aligned}$$

and our proof is complete. □