Linear Algebra, Fall 2023

Written Homework 4

Due: 13 Sep 2023

Please answer the following questions on a separate sheet of paper and turn it in.

1. Let A be a diagonalizable matrix. Prove that for any positive integer m, A and A^m, if the invertible matrix Q diagonalizes A, then Q diagonalizes A^m. Hence, A and A^m have the same basis of eigenvectors.

2. The trace of a matrix is defined to be the sum of the elements along the main diagonal,

$$\mathrm{Tr}(A) = \sum_i a_{ii}.$$

   (a) Given a complex monic polynomial,

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

   show that $-a_{n-1}$ is the sum of the complex roots of $p(z)$.

   (b) Given a complex matrix $A \in M_{n\times n}(\mathbb{C})$, with characteristic polynomial

$$g(z) = (-1)^n(z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0),$$

   Show that $-a_{n-1}$ is a sum of the eigenvalues (with multiplicity) of A.

   (c) Use the fact that similar matrices have the same characteristic polynomial (you proved this in your last homework) to show that for a diagonalizable complex matrix A,

$$\mathrm{Tr}(A) = -a_{n-1}.$$

3. Hertz Rent-a-Car has a fleet of 2000 cars distributed across three locations in the Huntsville area: 1) Airport, 2) Downtown Huntsville, and 3) Madison Blvd. The movement of cars from one location to another over the course of a week can be described as a Markov process with transition matrix

$$\begin{array}{cccc|l} & \multicolumn{3}{c|}{\text{Rented from:}} & \\ & \text{Air} & \text{Dtwn} & \text{Mad} & \text{Returned to:} \\ \hline A = & .90 & .01 & .09 & \text{Air} \\ & .01 & .90 & .01 & \text{Dtwn} \\ & .09 & .09 & .90 & \text{Mad} \end{array}$$

   You may use a computer to help with completing these questions.

(a) Find a diagonalizing matrix Q and diagonal matrix D so that $Q^{-1}AQ = D$

(b) What are the eigenvalues of the transition matrix?

(c) Find $\lim_{n\to\infty} A^n$.

(d) How many cars should be allocated at each location to minimize the numbers of cars which need to be moved from one lot to another at week's end?