Problem 5

Theorem 1. *Let T be a linear operator on a finite dimensional vector space V, let $v \in V$ be nonzero, and let W be the T-cyclic subspace of V of dimension k generated by v*

$$W = Span\{v, T(v), \cdots, T^{k-1}(v)\}.$$

If S is a T-invariant subspace of V, then $W \subseteq S$.

Solution

Proof. Since $v \in S$, the definition of an invariant subspace tells us that $T(v) \in S$. Now suppose that for some natural $n \geq 0$ that $T^n(v) \in S$. Then, by invariance, we have $T(T^n(v)) = T^{n+1}(v) \in S$. Thus, for all nonnegative integers n, we have

$$T^n(v) \in S.$$

Since S is a subspace, it also contains the span of these vectors, and we can conclude that $W \subseteq S$. □