Problem 1

Show that every positive definite matrix must be invertible. Also show that if a matrix is positive definite, then it is unitary if and only if all of it's eigenvalues are 1.

Solution

Proof. Let A be a positive definite $n \times n$. By definition, this matrix is hermitian, and therefore there exists an orthonormal basis $\beta = \{v_1, \cdots, v_n\}$ of our vector space consisting of eigenvectors of A, with corresponding eigenvalues λ_i. By problem 2.a of HW2, we have that $\lambda_i > 0$ for all i. Let $x \in \mathbb{F}^n$ be nonzero. Then, there exist $a_i \in \mathbb{F}$, at least one of which is nonzero, such that

$$x = \sum_{i=1}^{n} a_i v_i.$$

Then

$$\begin{aligned} Ax &= \sum_{i=1}^{n} a_i A v_i \\ &= \sum_{i=1}^{n} a_i \lambda_i v_i. \end{aligned}$$

Since each λ_i and at least one a_i is nonzero, the linear independence of β tells us that Ax is nonzero. Thus, the dimension of the nullspace of A is zero, and we can conclude that A is invertible.

Now, suppose A is also unitary. Then, we have

$$\begin{aligned} \langle Av_i, Av_i \rangle &= \lambda_i^2 & \\ &= \langle x, x \rangle & \text{Property of unitary operators} \\ &= 1 & \text{Since our basis is orthonormal.} \end{aligned}$$

Thus, since λ_i is real and nonnegative, we can conclude $\lambda_i = 1$ for each i.

For the other direction, suppose As eigenvalues are all 1. Then, defining x as we did above, we have

$$\begin{aligned} \langle Ax, Ax \rangle &= \sum_{i=1}^{n} |a_i|^2 \lambda_i^2 \\ &= \sum_{i=1}^{n} |a_i| \\ &= \langle x, x \rangle, \end{aligned}$$

and we have shown that A is unitary. □

Problem 2

Use question 1 and the previous paragraph to solve the following generalized eigenvalue problem $Ax = \lambda Bx$.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x = \lambda \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} x$$

Solution

Since B has a nonzero determinant, we can see it is invertible. Thus, we will find it's inverse:

$$
\begin{aligned}
\begin{bmatrix} B & I \end{bmatrix} &= \begin{bmatrix} 3 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 1 & 0 & 1 \end{bmatrix} && R_1 + 2R_2 \\
&\to \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 1 & 3 \end{bmatrix} && R_2 + R_1 \\
&\to \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \end{bmatrix} && \frac{1}{2}R_2 \\
&\to \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \end{bmatrix} && R_1 - R_2
\end{aligned}
$$

Thus, we have

$$
B^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}.
$$

Now we compute $B^{-1}A$:

$$
\begin{aligned}
B^{-1}A &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 2 \end{bmatrix}.
\end{aligned}
$$

Our generalized eigenvalue problem reduces to finding the eigenvectors of this matrix:

$$
\begin{aligned}
|B^{-1}A - \lambda I| &= \begin{vmatrix} \frac{1}{2} - \lambda & 1 \\ \frac{1}{2} & 2 - \lambda \end{vmatrix} \\
&= (\frac{1}{2} - \lambda)(2 - \lambda) - \frac{1}{2} \\
&= \lambda^2 - \frac{5}{2}\lambda + \frac{1}{2}.
\end{aligned}
$$

From the quadratic formula, we have eigenvalues

$$
\lambda_1 = \frac{1}{4}(5 - \sqrt{17}), \text{ and } \lambda_2 = \frac{1}{4}(5 + \sqrt{17}).
$$

Now we find our eigenvectors:

$$
\begin{aligned}
\begin{bmatrix} \frac{1}{2} - \lambda_1 & 1 \\ \frac{1}{2} & 2 - \lambda_1 \end{bmatrix} &= \begin{bmatrix} \frac{1}{2} - \frac{1}{4}(5 - \sqrt{17}) & 1 \\ \frac{1}{2} & 2 - \frac{1}{4}(5 - \sqrt{17}) \end{bmatrix} \\
&= \begin{bmatrix} \frac{-3}{4} + \frac{1}{4}\sqrt{17} & 1 \\ \frac{1}{2} & 2 - \frac{1}{4}(5 - \sqrt{17}) \end{bmatrix}.
\end{aligned}
$$

Thus, looking at the first row, we have an eigenvector

$$
v_1 = \begin{bmatrix} 4 \\ 3 - \sqrt{17} \end{bmatrix}.
$$

For the next one, we follow the same procedure:

$$
\begin{aligned}
\begin{bmatrix} \frac{1}{2} - \lambda_2 & 1 \\ \frac{1}{2} & 2 - \lambda_2 \end{bmatrix} &= \begin{bmatrix} \frac{1}{2} - \frac{1}{4}(5 + \sqrt{17}) & 1 \\ \frac{1}{2} & 2 - \frac{1}{4}(5 + \sqrt{17}) \end{bmatrix} \\
&= \begin{bmatrix} \frac{-3}{4} - \frac{1}{4}\sqrt{17} & 1 \\ \frac{1}{2} & 2 - \frac{1}{4}(5 - \sqrt{17}) \end{bmatrix}.
\end{aligned}
$$

Again, the first row gives us the eigenvector

$$
v_2 = \begin{bmatrix} 4 \\ 3 + \sqrt{17} \end{bmatrix},
$$

and we have solved the generalized eigenvalue problem.

Problem 3

Solve the generalized eigenvalue problem from Question 2 using this second method.

Solution

We have

$$\begin{aligned}
|A-\lambda B| &= \left|\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} - \lambda\begin{bmatrix}3 & -1\\ -1 & 1\end{bmatrix}\right| \\
&= \left|\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} - \begin{bmatrix}3\lambda & -\lambda\\ -\lambda & \lambda\end{bmatrix}\right| \\
&= \begin{vmatrix}1-3\lambda & 1+\lambda\\ \lambda & 1-\lambda\end{vmatrix} \\
&= (1-3\lambda)(1-\lambda) - \lambda(1+\lambda) \\
&= 3\lambda^2 - 4\lambda + 1 - \lambda - \lambda^2 \\
&= 2\lambda^2 - 5\lambda + 1 \\
&= 0 \\
0 &= \lambda^2 - \frac{5}{2}\lambda + \frac{1}{2}.
\end{aligned}$$

As before, this is the same equation we had before (as it clearly should be), so we have eigenvalues

$$\lambda_1 = \frac{1}{4}(5-\sqrt{17}), \text{ and } \lambda_2 = \frac{1}{4}(5+\sqrt{17}).$$

Now we find the eigenvectors. First, we do a little reduction

$$\begin{aligned}
\begin{bmatrix}1-3\lambda & 1+\lambda\\ \lambda & 1-\lambda\end{bmatrix} &\to \begin{bmatrix}1-2\lambda & 2\\ \lambda & 1-\lambda\end{bmatrix} && R_1 + R_2 \\
&\to \begin{bmatrix}\frac{1}{2}-\lambda & 1\\ \lambda & 1-\lambda\end{bmatrix} && \frac{1}{2}R_1 \\
&\to \begin{bmatrix}\frac{1}{2}-\lambda & 1\\ \frac{1}{2} & 2-\lambda\end{bmatrix} && R_2 + R_1.
\end{aligned}$$

We reduced this to the same matrix we had set equal to $B^{-1}A - \lambda I$ in problem 2. Since we have already shown the eigenvalues to be the same, we can conclude that the eigenvectors are the same (as they should be).

Problem 4

For two $n \times n$ complex matrices A and B, what is the maximum number of distinct generalized eigenvalues we would expect? Justify your answer. Then solve the generalized eigenvalue problem corresponding with the matrix pencil

$$\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right).$$

Solution

Theorem 1. *For two $n \times n$ complex matrices A and B, the maximum number of distinct generalized eigenvalues is n.*

Proof. We claim that the determinant of $A - \lambda B$ is at most an n^{th} degree polynomial of λ, from which the conclusion follows. We will proceed by induction on n. For $n = 2$, this is immediately clear.

Now, suppose for some $n \geq 2$ that for any two $n \times n$ complex matrices, the maximum number of distinct generalized eigenvalues is n. Let A and B be $n+1 \times n+1$ complex matrices. For an $m \times m$ matrix C, let $\overline{C}_{ij}$ denote the $m-1 \times m-1$ submatrix of C obtained by deleting the i^{th} row and j^{th} column of C. Then, we have

$$|A - \lambda B| = \sum_{j=1}^{n+1} (A_{1j} - \lambda B_{1j})|\overline{A - \lambda B}_{1j}|.$$

By the inductive hypothesis, $|\overline{A - \lambda B}_{1j}|$ is an at most $n \times n$ polynomial of λ. Since $(A_{1j} - \lambda B_{1j})$ is an at most first degree polynomial, we have that each term of the above sum is an at most $n + 1^{\text{th}}$ degree polynomial, and we have completed our proof. $\square$

Now, we solve the generalized eigenvalue problem:

$$\begin{aligned}|A - \lambda B| &= \begin{vmatrix} -\lambda & 1 \\ 0 & 0 \end{vmatrix} \\ &= 0\end{aligned}$$

Thus, every nonzero vector is a generalized eigenvector with corresponding eigenvalue 0.

Problem 5

Theorem 2. *Let $\hat{E}_\lambda$ be the collection of all generalized eigenvectors associated with the eigenvalue λ. Then $\hat{E}_\lambda$ is a subspace of V.*

Proof. We have

$$\hat{E}_\lambda = \{x \in V | (T - \lambda U)x = 0\}.$$

That is, $\hat{E}_\lambda$ is the nullspace of the linear operator $(T - \lambda U)$, and is therefore a subspace of V. $\square$

Problem 6

For distinct generalized eigenvalues λ and ν it is not necessarily the case that $\hat{E}_\lambda \cap \hat{E}_\nu = \{0\}$. Given a regular matrix pencil (A, B) and distinct nonzero eigenvalues $\lambda \neq \nu$, what is $\hat{E}_\lambda \cap \hat{E}_\nu$?

Solution

Let $x \in \hat{E}_\lambda \cap \hat{E}_\nu$. Then, we have

$$\begin{aligned}(A - \lambda B)x &= 0 \\ &= (A - \nu B)x \\ 0 &= (A - \lambda B)x - (A - \nu B)x \\ &= Ax - \lambda Bx - Ax + \nu Bx \\ &= (\nu - \lambda)Bx,\end{aligned}$$

and we can conclude that $x \in N(B)$. With this, we have

$$\begin{aligned}0 &= (A - \lambda B)x \\ &= Ax - \lambda Bx \\ &= Ax.\end{aligned}$$

Thus, we also have that $x \in N(A)$. Therefore, we have $x \in N(A) \cap N(B)$ which implies $\hat{E}_\lambda \cap \hat{E}_\nu \subseteq N(A) \cap N(B)$. Supposing that $x \in N(A) \cap N(B)$, it follows immediately that $x \in \hat{E}_\lambda \cap \hat{E}_\nu$. With this, we can conclude that $\hat{E}_\lambda \cap \hat{E}_\nu = N(A) \cap N(B)$.

Problem 7

Let V be an n-dimensional vector space over $\mathbb{C}$. Suppose that P and Q are two orthogonal projections. Suppose that $L = R(P)$, and $N = R(Q)$. Then we may write L as

$$L = (L \cap N) \oplus (L \cap N^\perp).$$

There is an equivalent decomposition for $L^\perp$. With these two decompositions we can write

$$V = (L \cap N) \oplus (L \cap N^\perp) \oplus (L^\perp \cap N) \oplus (L^\perp \cap N^\perp)$$

Solution

Proof. By theorem 6.6 in our textbook, we have

$$V = N \oplus N^{\perp},$$

since the range of Q is a subspace of V. Thus, we have

$$\begin{aligned} x \in L \cap V &\iff x \in L \cap (N + N^{\perp}) \\ &\iff x \in L \wedge x \in \{x_1 + x_2 | x_1 \in N \wedge x_2 \in N^{\perp}\} \\ &\iff x \in \{x_1 + x_2 | x_1 \in L \cap N \wedge x_2 \in L \cap N^{\perp}\} \\ &\iff x \in (L \cap N) + L \cap N^{\perp}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} x \in (L \cap N) \cap (L \cap N^{\perp}) &\iff x \in L \cap (N \cap N^{\perp}) \\ &\iff x \in L \cap \{0\} && \text{Since } Q \text{ is an orthogonal projection} \\ &\iff x = 0. \end{aligned}$$

Thus, we have shown that

$$L = (L \cap N) \oplus (L \cap N^{\perp}).$$

In a nearly identical argument, we can derive the equality

$$L^{\perp} = (L^{\perp} \cap N) \oplus (L^{\perp} \cap N^{\perp}).$$

Since P is an orthogonal projection, we have

$$\begin{aligned} V &= L \oplus L^{\perp} \\ &= (L \cap N) \oplus (L \cap N^{\perp}) \oplus (L^{\perp} \cap N) \oplus (L^{\perp} \cap N^{\perp}), \end{aligned}$$

as desired. □

Problem 8

What happens to the decomposition in the first part of question 7 if we only assume that Q is a projection, but not an orthogonal projection? Give a concrete example of such a projection onto a subspace of $\mathbb{C}^2$.

Note for Dr. Bossaller: you wrote P for this problem instead of Q, but I am pretty certain you meant Q.

Solution

By theorem 6.6 in our textbook, we have $V = W \oplus W^{\perp}$ for any subspace of W of V. If Q was not an orthogonal projection, then we would still be able to write

$$V = N \oplus N^{\perp},$$

as N would still be a subspace of V. In fact, this would be true if Q were any operator on V, since the range of any operator is a subspace of V. The notable unique quality of orthogonal projections is that $R(Q)^{\perp} = N(Q)$, but the decomposition above does not explicitly use the null space of Q, thus the decomposition holds.

To see a concrete example, let $V = \mathbb{C}^2$, and consider the projection Q onto $\operatorname{Span}(1, 1)$ given by

$$Q(x_1, x_2) = (x_1, x_1).$$

Then, we have

$$\begin{aligned} \langle Qx, y \rangle &= \langle (x_1, x_1), (y_1, y_2) \rangle \\ &= x_1 \overline{y_1} + x_1 \overline{y_2} \\ &= x_1 (\overline{y_1} + \overline{y_2}) \\ &= x_1 \overline{(y_1 + y_2)}. \end{aligned}$$

From this, we can see that $N^\perp = \{(y, -y) | y \in \mathbb{C}\} = \text{Span}(1, -1)$. Furthermore, we have

$$\begin{aligned} N(Q) &= \text{Span}(0, 1) \\ &\neq N^\perp, \end{aligned}$$

which means the projection we have chosen is not orthogonal. Finally, we can see that $(1, -1)$ and $(1, 1)$ are linearly independent, and if follows that

$$V = N \oplus N^\perp,$$

as we expect.