Problem 1

Let T be a linear operator on a finite dimensional vector space V with Jordan Canonical Form

$$\left[\begin{array}{ccc|cc|cc} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right]$$

Answer the following questions

1. Find the characteristic polynomial of T
2. Find the dot diagram corresponding to each eigenvalue of T
3. For which eigenvalues λ_i, if any, does $E_{\lambda_i} = K_{\lambda_i}$
4. For each eigenvalue λ_i find the smallest postive integer p_i for which $K_{\lambda_i} = N((T_{\lambda_i} I)^{p_i})$
5. Compute the following numbers for each i where U_i denotes the restriction of $T - \lambda_i$ to K_{λ_i}
   (a) $\text{rank}(U_i)$
   (b) $\text{rank}(U_i^2)$
   (c) $\dim(N(U_i))$
   (d) $\dim(N(U_i^2))$

Solution

1. The characteristic polynomial for T is immediately obtainable from its Jordan Canonical Form:
$$f(t) = (2-t)^5(3-\lambda)^2.$$
2. For the eigenvalue $\lambda = 2$, we can see from our JCF of T that we have one cycle of length 3, and another cycle of length 2. Thus, the dot diagram is
$$\begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \end{bmatrix}.$$
   For the eigenvalue $\lambda = 3$, we have two cycles of length 1, so we have dot diagram
$$\begin{bmatrix} \bullet & \bullet \end{bmatrix}.$$
3. For $\lambda_i = 3$ we have that $K_{\lambda_i} = E_{\lambda_i}$, as all of the vectors in K_{λ_i} are eigenvalues of T.
4. These can immediately be obtained from the dot diagrams. If we let $\lambda_1 = 2$, and $\lambda_2 = 3$, we have
$$p_1 = 3, \text{ and } p_2 = 1.$$
5. (a) We have that $\text{rank}(U_i)$ is equal to the number of dots in the dot diagram that are not the bottom dot for their column. Thus, we have
$$\text{rank}(U_1) = 3, \text{ and } \text{rank}(U_2) = 0.$$
   (b) We have that $\text{rank}(U_i^2)$ is equal to the number of dots in the dot diagram that are not in the bottom two dots for their column. Thus,
$$\text{rank}(U_1^2) = 1, \text{ and } \text{rank}(U_2^2) = 0.$$
   (c) The number of dots in the dot diagram is equal to the dimension of K_{λ_i}. Thus, using the rank nullity theorem, and our results from (a), we have
$$\dim N(U_1) = 5 - 3 = 2, \text{ and } \dim N(U_2) = 2 - 0 = 2.$$
   (d) Using the results from part (b), we have
$$\dim N(U_1^2) = 5 - 1 = 4, \text{ and } \dim N(U_2^2) = 2 - 0 = 2.$$

Problem 2

Find the Jordan Canonical Form for the following matrix.

$$A = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 3 & -2 \\ 0 & 2 & 2 & -1 \end{bmatrix}$$

Solution

We start by finding the finding the characteristic polynomial for A:

$$\begin{aligned} |A - tI| &= \begin{vmatrix} 1-t & 0 & 2 & 2 \\ 0 & 1-t & 0 & 0 \\ 0 & 2 & 3-t & -2 \\ 0 & 2 & 2 & -1-t \end{vmatrix} \\ &= (1-t)^2(-(3-t)(1+t)+4) \\ &= (1-t)^2(t^2-2t+1) \\ &= (1-t)^4. \end{aligned}$$

Thus, we have one unique eigenvalue, $\lambda = 1$. Since this eigenvalue has multiplicity 4, we have $\dim(K_1) = 4$. We will now find the dot diagram, but finding how many dots go in each row:

$$\begin{aligned} r_1 &= \dim(V) - \operatorname{rank}(A - \lambda I) \\ &= 4 - \operatorname{rank}\left(\begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & -2 \\ 0 & 2 & 2 & -2 \end{bmatrix} \right) \\ &= 4 - \operatorname{rank}\left(\begin{bmatrix} 0 & 2 & 2 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

Now, to find the number of dots in the second row, we will need to square $A - \lambda I$:

$$\begin{aligned} (A - \lambda I)^2 &= \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & -2 \\ 0 & 2 & 2 & -2 \end{bmatrix}^2 \\ &= \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & -2 \\ 0 & 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & -2 \\ 0 & 2 & 2 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 8 & 8 & -8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Thus, we have

$$\begin{aligned} r_2 &= r_1 - \operatorname{rank}(A - \lambda I)^2 \\ &= 2 - 1 \\ &= 1. \end{aligned}$$

Since the number of dots must equal the multiplicity of λ, we have $r_3 = 1$. Thus, the dot diagram is

$$\begin{bmatrix} \bullet & \bullet \\ \bullet & \\ \bullet & \end{bmatrix}.$$

With this, we can see that the JCF of A is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 3

Suppose that V is the real vector space of functions spanned by the set of real-valued functions $\beta_0 = \{e^t, te^t, t^2e^t, e^{2t}\}$. Suppose that T is the linear operator on V defined by $T(f) = f'$. Find a Jordan Canonical Form J of T and a Jordan basis β of generalized eigenvectors for V.

Solution

We will start by expressing T as a matrix in this given basis. We have

$$\begin{aligned} Te^t &= e^t \\ Tte^t &= e^t + te^t \\ Tt^2e^t &= 2te^t + t^2e^t \\ Te^{2t} &= 2e^{2t}. \end{aligned}$$

Thus, we have

$$[T]_{\beta_0} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Now, we will find the characteristic polynomial:

$$\begin{aligned} |T - tI| &= \begin{vmatrix} 1-t & 1 & 0 & 0 \\ 0 & 1-t & 2 & 0 \\ 0 & 0 & 1-t & 0 \\ 0 & 0 & 0 & 2-t \end{vmatrix} \\ &= (1-t)^3(2-t). \end{aligned}$$

Thus, we have two unique eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ with respective multiplicities 3 and 1. Now we will find matrix form of $T - \lambda_1 I$:

$$\begin{aligned} T - \lambda_1 I &= \begin{bmatrix} 1-\lambda_1 & 1 & 0 & 0 \\ 0 & 1-\lambda_1 & 2 & 0 \\ 0 & 0 & 1-\lambda_1 & 0 \\ 0 & 0 & 0 & 2-\lambda_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

This has rank 3, which means that there is only 1 dot in the first row of our dot diagram, which means we can conclude that the dot diagram corresponding to λ_1 is

$$\begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix}.$$

The dot diagram for λ_2 is trivial, thus we can conclude that a JCF of T is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Now, we will find the nullspace of $(T - \lambda_1 I)^2$. We have

$$\begin{aligned} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^2 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Thus, the a basis for this null space is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Now we find the null space for $(T - \lambda_1 I)^3$:

$$\begin{aligned} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^3 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Thus, we see that a basis for this null space is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Now, we want the one vector that is not in the nullspace of $(T - \lambda I)^2$, so we know we want the third one. Call this vector v_3. We can use this to find the next vector in our basis that will put T in JCF:

$$\begin{aligned} v_2 &= (T - \lambda I)v_3 \\ &= Tt^2e^t - t^2e^t \\ &= 2te^t + t^2e^t - t^2e^t \\ &= 2te^t. \end{aligned}$$

Finally, we use this to get the eigenvector:

$$\begin{aligned} v_1 &= (T - \lambda I)v_2 \\ &= 2Tte^t - 2te^T \\ &= 2e^t + 2te^t - 2te^t \\ &= 2e^t. \end{aligned}$$

Putting it all together, our basis for putting T into JCF is

$$\beta = \{t^2e^t, 2te^t, 2e^t, e^{2t}\}.$$

As a sanity check, we will express T in this basis. First, we see

$$\begin{aligned}
Tt^2e^t &= t^2e^t + 2te^t \\
T2te^t &= 2te^t + 2e^t \\
T2e^t &= 2e^t \\
T2e^{2t} &= 2e^{2t}.
\end{aligned}$$

With this, we can see

$$[T]_\beta = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

as desired.