Linear Algebra, Fall 2023

Written Homework 2

Due: 30 Aug 2023

Please answer the following questions on a separate sheet of paper and turn it in.

1. Let

$$B = \left[\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & B' \\ 0 & \end{array}\right]$$

where B' is a $m \times n$ sub-matrix of B. Prove that if $\mathrm{rank}(B) = r$, then $\mathrm{rank}(B') = r - 1$.

2. Calculate the determinant of the following matrix whose entries are from $\mathbb{C}$.

$$\begin{bmatrix} i & 2+i & 0 \\ -1 & 3 & 2i \\ 0 & -1 & 1-i \end{bmatrix}$$

3. Given a linear operator T on the vector space V over the field $\mathbb{F}$, and a polynomial $g(t)$ with coefficients from $\mathbb{F}$, we may define an "operator polynomial" $g(T)$ as the linear operator you get by "plugging in" the operator T in for the variable t, and considering the constant term a_0 as $a_0 I_V$ where I_V is the identity map on the vector space. For example, if $g(t) = t^2 - 2$, then $g(T) = T^2 - 2I_V$. This polynomial can then be applied to $v \in V$ via

$$g(T)(v) = T^2(v) - 2I_V(v)$$

   (a) Prove that if v is an eigenvector of T with corresponding eigenvalue λ, then $g(T)(v) = g(\lambda)v$, that is, v is an eigenvector of $g(T)$ with corresponding eigenvalue $g(\lambda)$.

   (b) Use the result of part (a) to show that if $g(t)$ is the characteristic polynomial of T, then $g(T)(v) = 0$, i.e. that the eigenvector v is contained in the nullity of $g(T)$.

4. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(\lambda) = (-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0.$$

Prove that $f(0) = a_0 = \det(A)$. Deduce that $\det(A)$ is invertible if and only if $a_0 \neq 0$.