Linear Algebra, Fall 2023

Written Homework 7

Due: 25 Oct 2023

Please answer the following questions on a separate sheet of paper and turn it in.

1. Let T be a normal operator on a complex finite-dimensional inner product space V, and let W be a subspace of V. Prove that W is T-invariant if and only if W is also T^*-invariant.

2. A linear operator T on a finite-dimensional inner product space over $\mathbb{R}$ or $\mathbb{C}$ is called *positive definite* (resp. *positive semi-definite*) if T is self-adjoint and $\langle T(x), x\rangle > 0$ (resp. $\langle T(X), x\rangle \geq 0$) for all $x \neq 0$. A matrix A is called positive (semi)definite if L_A is positive (semi)definite.

   Let T be a self-ajoint operator, prove the following.

   (a) Show that T is positive (semi-)definite if and only if all of its eigenvalues are > 0 (≥ 0).

   (b) Show that T is positive semi-definite if and only if there is an operator U such that $T = U^*U$. (Hint: use the spectral resolution).

3. Let T be a normal operatoor and let U be any operator on V. If the eigenspaces of T are U-invariant, show that T and U commute.

4. Suppose that T is normal on the finite dimensional complex inner product space. Use the spectral decomposition of T,

   $$T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k,$$

   to prove the following results.

   (a) T is invertible if and only if $\lambda_i \neq 0$ for $1 \leq i \leq k$.

   (b) T is a projection if and only if every eigenvalue of T is 1 or 0.

   (c) $T = -T^*$ if and only if every eigenvalue λ_i is imaginary.