Problem 1

Theorem 1. Let x, y be elements of the inner product space V. Then

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2.$$

Solution

Proof. Using the axioms of an inner product space and theorem 6.1, we have

$$\begin{aligned} ||x+y||^2 &= \langle x+y, x+y \rangle \\ &= \langle x, x+y \rangle + \langle y, x+y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, y \rangle + \langle y, x \rangle \\ &= ||x||^2 + ||y||^2 + \langle x, y \rangle + \langle y, x \rangle. \end{aligned}$$

Similarly,

$$\begin{aligned} ||x - y||^2 &= \langle x - y, x - y \rangle \\ &= \langle x, x - y \rangle - \langle y, x - y \rangle \\ &= \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle \\ &= ||x||^2 + ||y||^2 - \langle x, y \rangle - \langle y, x \rangle. \end{aligned}$$

Thus, we have

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2,$$

as desired.

Problem 2

Let V be a real or complex finite dimensional vector space, and let $\mathcal{B} = \{v_1, v_2, \dots, v_k\}$ be a basis for V. Then for any $x, y \in V$, we may write

$$x = \sum_{i} a_i v_i$$
 and $y = \sum_{j} b_j v_j$.

Define

$$\langle x, y \rangle = \sum_{i=1}^{k} a_i \overline{b_i}.$$

Prove that $\langle \cdot, \cdot \rangle$ is an inner product on V and that \mathcal{B} is an orthonormal basis for V. Conclude that every real or complex vector space may be regarded as an inner product space.

Solution

Proof. Two show properties (a) and (b), let $z = \sum_i c_i v_i$ and let $c \in \mathbb{F}$. Then, we have

$$\langle cx + z, y \rangle = \sum_{i=1}^{k} (ca_i + c_i) \overline{b_i}$$

$$= c \sum_{i=1}^{k} a_i \overline{b_i} + \sum_{i=1}^{k} c_i \overline{b_i}$$

$$= c \langle x, y \rangle + \langle z, y \rangle.$$

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For property c, we have

$$\overline{\langle x, y \rangle} = \overline{\sum_{i=1}^{k} a_i \overline{b_i}}$$

$$= \sum_{i=1}^{k} \overline{a_i \overline{b_i}}$$

$$= \sum_{i=1}^{k} \overline{a_i} b_i$$

$$= \sum_{i=1}^{k} b_i \overline{a_i}$$

$$= \langle y, x \rangle.$$

Finally, to verify condition (d), let $x \neq 0$. We have

$$\langle x, x \rangle = \sum_{i=1}^{k} a_i \overline{a_i}$$

= $\sum_{i=1}^{k} |a_i|^2$.

Now, since each term is a complex square, this is a sum of nonnegative real numbers, and is therefore real and nonnegative. Furthermore, since $x \neq 0$, this sum has at least one term that is nonzero. Thus, this sum is a positive real number, and we have shown this function is an inner product on V.

Problem 3

Theorem 2. Let $V \in \mathbb{F}^n$ with that standard inner product and $A \in M_{n \times n}(\mathbb{F})$. Then the following statements are true:

Part (a):

$$\langle x, Ay \rangle = \langle A^*x, y \rangle$$

Part (b):

If for some $B \in M_{n \times n}(\mathbb{F})$ we have $\langle x, Ay \rangle = \langle Bx, y \rangle$ for all $x, y \in V$, then $B = A^*$.

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Solution

Proof. Part (a):

Using the definition of our standard inner product, we have

$$\langle x, Ay \rangle = \sum_{i=1}^{n} x_{i} \overline{(Ay)_{i}}$$

$$= \sum_{i=1}^{n} x_{i} \overline{\sum_{j=1}^{n} A_{i,j} y_{j}}$$

$$= \sum_{i=1}^{n} x_{i} \sum_{j=1}^{n} \overline{A_{i,j} y_{j}}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} x_{i} \overline{A_{i,j}} \overline{y_{j}}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} A_{j,i}^{*} x_{i} \overline{y_{j}}$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} A_{j,i}^{*} x_{i} \right) \overline{y_{j}}$$

$$= \sum_{j=1}^{n} (A^{*}x)_{j} \overline{y_{j}}$$

$$= \langle A^{*}x, y \rangle,$$

as desired.

Part (b):

For any two vectors $x, y \in V$, we have

$$\langle Bx, y \rangle = \langle x, Ay \rangle$$

= $\langle A^*x, y \rangle$

by part (a). Thus, for any $x \in V$, we have

$$0 = \langle Bx, Bx - A^*x \rangle - \langle A^*x, Bx - A^*x \rangle$$
$$= \langle Bx - A^*x, Bx - A^*x \rangle.$$

By taking the contrapositive of property (d) of inner product, we have that

$$Bx - A^*x = 0 \implies Bx = A^*x.$$

Since this is true for any $x \in V$, we can conclude that $B = A^*$.

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