# Problem 1

**Theorem 1.** *Let $B$ be an $(m+1)\times(n+1)$ matrix, where $B_{11}=1$, and $B_{1j}=B_{i1}=0$ for $2\le j\le n+1$ and $2\le i\le m+1$. Define $B'$ to be the $m\times n$ submatrix of $B$ obtained by removing the first column and the first row of $B$. Then, if rank$(B)=r$, then rank$(B')=r-1$.*

## Solution

*Proof.* For this proof, we will use the fact that rank of a matrix is equal to the dimension of its column space. Since the first element of $B_{\cdot 1}$ is non zero, and the first elements of each vector in the set of remaining column vectors $\{B_{\cdot 2},\dots,B_{\cdot(n+1)}\}$ are zero, it follows that

$$B_{\cdot 1}\notin \operatorname{Span}\left(\{B_{\cdot 2},\dots,B_{\cdot(n+1)}\}\right).$$

Thus, we can conclude that

$$\begin{aligned}\dim\left(\operatorname{Span}\left(\{B_{\cdot 2},\dots,B_{\cdot(n+1)}\}\right)\right)&=\dim\left(\operatorname{Span}\left(\{B_{\cdot 1},\dots,B_{\cdot(n+1)}\}\right)\right)-1\\&=r-1.\end{aligned}$$

Now, define $\{v_1,...,v_{r-1}\}$ to be the resulting set of reducing $\{B_{\cdot 2},\dots,B_{\cdot(n+1)}\}$ to a basis of $\operatorname{Span}\left(\{B_{\cdot 2},\dots,B_{\cdot(n+1)}\}\right)$. Now, define a new set of vectors $\{v_1',...,v_{r-1}'\}\subseteq\{B_{\cdot 1}',\dots,B_{\cdot n}'\}$ such that for all $1\le i\le r-1$, $v_i'$ is the vector obtained by removing the first element of $v_i$. All that remains is to that $\{v_1',...,v_{r-1}'\}$ is a basis of $\operatorname{Span}\left(B_{\cdot 1}',\dots,B_{\cdot n}'\}\right)$.

To see that $\{v_1',...,v_{r-1}'\}$ is linearly independent, let $\{a_1,\dots,a_{r-1}\}\subseteq\mathbb{F}$ be such that

$$\sum_{i=1}^{r-1}a_iv_i'=0.$$

Then, since the first element of each vector in $\{v_1,...,v_{r-1}\}$ is zero, we have

$$\sum_{i=1}^{r-1}a_iv_i=0.$$

By linear independence of $\{v_1,...,v_{r-1}\}$, we can conclude that $a_i=0$ for each $i$, and we have shown that $\{v_1',...,v_{r-1}'\}$ is linearly independent.

To end the proof, we must now show that

$$\operatorname{Span}\left(\{v_1,...,v_{r-1}\}\right)=\operatorname{Span}\left(\{B_{\cdot 1}',\dots,B_{\cdot n}'\}\right).$$

One side of this equality follows directly from the fact that $\{v_1',...,v_{r-1}'\}\subseteq\{B_{\cdot 1}',\dots,B_{\cdot n}'\}$. For the other direction, let $v'\in\operatorname{Span}\left(\{B_{\cdot 1}',\dots,B_{\cdot n}'\}\right)$. Define $v\in\operatorname{Span}\left(\{B_{\cdot 2},\dots,B_{\cdot(n+1)}\}\right)$ by adding a zero to the top of $v'$. Then, there exist scalars $\{a_1,\dots,a_{r-1}\}\subseteq\mathbb{F}$ such that

$$\sum_{i=1}^{r-1}a_iv_i=v.$$

Since vector addition is defined element wise, it follows immediately that

$$\sum_{i=1}^{r-1}a_iv_i'=v',$$

and our proof is complete. ∎

# Problem 2

Calculate the determinant of the following matrix whose entries are in $\mathbb{C}$:

$$\begin{bmatrix} i & 2+i & 0\\ -1 & 3 & i\\ 0 & -1 & 1-i\end{bmatrix}$$

### Solution

Proceeding with cofactor expansion on the first row, we have

$$\begin{aligned}\begin{vmatrix} i & 2+i & 0 \\ -1 & 3 & i \\ 0 & -1 & 1-i \end{vmatrix} &= i\begin{vmatrix} 3 & i \\ -1 & 1-i \end{vmatrix} - (2+i)\begin{vmatrix} -1 & i \\ 0 & 1-i \end{vmatrix} \\ &= i(3-3i+i)-(2+i)(i-1) \\ &= 3i+2+(2+i)(1-i) \\ &= 3i+2+2+i-2i+1 \\ &= 5+2i\end{aligned}$$

## Problem 3

**Theorem 2.** *Let $T$ be a linear operator on a vector space $V$ over the field $\mathbb{F}$, and let $g(t)$ be a polynomial with coefficients from $\mathbb{F}$. Define $g(T)$ to be the operator obtained by plugging $T$ in for the variable $t$, and letting the constant term $a_0$ be replaced with $a_0 I$, where $I$ is the identity operator on $V$. Then,*

Part (a):

*If $v \in V$ is an eigenvector of $T$ with eigenvalue $\lambda$, then $g(T)(v) = g(\lambda)v$.*

Part (b):

*If $g(t)$ is the characteristic polynomial of $T$, then $g(T)(v) = 0$.*

### Solution

*Part (a):*

We have

$$\begin{aligned} g(T)(v) &= \left(\sum_{i=0} a_0 I T^i\right)(v) \\ &= \sum_{i=0} a_0 I T^i(v) \\ &= \sum_{i=0} a_0 \lambda^i v \\ &= \left(\sum_{i=0} a_0 \lambda^i\right)(v) \\ &= g(\lambda)v, \end{aligned}$$

as desired.

*Part (b):*

Now suppose $g(t)$ is the characteristic polynomial of $T$. Then, if $v$ is an eigen vector of $T$ with eigenvalue $\lambda$, we have

$$\begin{aligned} g(T)(v) &= g(\lambda)v && \text{by Part (a)} \\ &= (0)v, \end{aligned}$$

with the last equality being true because the eigenvalues of $T$ are the roots to the characteristic polynomial of $T$.

## Problem 4

**Theorem 3.** *Let $A$ be an $n \times n$ matrix with characteristic polynomial*

$$f(\lambda) = (-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0.$$

*Then, $f(0) = a_0 = det(A)$, and we can deduce that $det(A)$ is invertible if and only if $a_0 \neq 0$.*

## Solution

*Proof.* We have

$$f(\lambda) = \det(A - \lambda I).$$

Therefore

$$\begin{aligned}
f(0) &= \det(A - 0I) \\
&= \det(A) \\
&= (-1)^n(0)^n + a_{n-1}(0)^{n-1} + \cdots + a_1(0) + a_0 \\
&= a_0,
\end{aligned}$$

as desired. Since $T$ is invertible if and only if it has determinant zero, and we have $a_0 = T$, it follows that $T$ is invertible if and only if $a_0 = 0$. $\square$