Problem 1

Show that every positive definite matrix must be invertible. Also show that if a matrix is positive definite, then it is unitary if and only if all of it's eigenvalues are 1.

Solution

Proof. Let A be a positive definite $n \times n$. By definition, this matrix is hermitian, and therefore there exists an orthonormal basis $\beta = \{v_1, \cdots, v_n\}$ of our vector space consisting of eigenvectors of A, with corresponding eigenvalues λ_i. By problem 2.a of HW2, we have that $\lambda_i > 0$ for all i. Let $x \in \mathbb{F}^n$ be nonzero. Then, there exist $a_i \in \mathbb{F}$, at least one of which is nonzero, such that

$$x = \sum_{i=1}^{n} a_i v_i.$$

Then

$$Ax = \sum_{i=1}^{n} a_i A v_i$$
$$= \sum_{i=1}^{n} a_i \lambda_i v_i.$$

Since each λ_i and at least one a_i is nonzero, the linear independence of β tells us that Ax is nonzero. Thus, the dimension of the nullspace of A is zero, and we can conclude that A is invertible.

Now, suppose A is also unitary. Then, we have

$$\langle Av_i, Av_i \rangle = \lambda_i^2$$
$$= \langle x, x \rangle \quad \text{Property of unitary operators} \quad = 1 \quad \text{Since our basis is orthonormal.}$$

Thus, since λ_i is real and nonnegative, we can conclude $\lambda_i = 1$ for each i.

For the other direction, suppose As eigenvalues are all 1. Then, defining x as we did above, we have

$$\langle Ax, Ax \rangle = \sum_{i=1}^{n} |a_i|^2 \lambda_i^2$$
$$= \sum_{i=1}^{n} |a_i|$$
$$= \langle x, x \rangle,$$

and we have shown that A is unitary. $\qquad \square$

Problem 2

Use question 1 and the previous paragraph to solve the following generalized eigenvalue problem $Ax = \lambda Bx$.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x = \lambda \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} x$$

Solution

Since B has a nonzero determinant, we can see it is invertible. Thus, we will find it's inverse:

$$\begin{bmatrix} B & I \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 1 & 0 & 1 \end{bmatrix} \qquad\qquad R_1 + 2R_2$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 1 & 3 \end{bmatrix} \qquad\qquad R_2 + R_1$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \end{bmatrix} \qquad\qquad \frac{1}{2}R_2$$

$$\rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \end{bmatrix} \qquad\qquad R_1 - R_2$$

Thus, we have

$$B^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}.$$

Now we compute $B^{-1}A$:

$$B^{-1}A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 2 \end{bmatrix}.$$

Our generalized eigenvalue problem reduces to finding the eigenvectors of this matrix:

$$|B^{-1}A - \lambda I| = \begin{vmatrix} \frac{1}{2} - \lambda & 1 \\ \frac{1}{2} & 2 - \lambda \end{vmatrix}$$

$$= (\frac{1}{2} - \lambda)(2 - \lambda) - \frac{1}{2}$$

$$= \lambda^2 - \frac{5}{2}\lambda + \frac{1}{2}.$$

From the quadratic formula, we have eigenvalues

$$\lambda_1 = \frac{1}{4}(5 - \sqrt{17}), \text{ and } \lambda_2 = \frac{1}{4}(5 + \sqrt{17}).$$

Now we find our eigenvectors:

$$\begin{bmatrix} \frac{1}{2} - \lambda_1 & 1 \\ \frac{1}{2} & 2 - \lambda_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{4}(5 - \sqrt{17}) & 1 \\ \frac{1}{2} & 2 - \frac{1}{4}(5 - \sqrt{17}) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-3}{4} + \frac{1}{4}\sqrt{17} & 1 \\ \frac{1}{2} & 2 - \frac{1}{4}(5 - \sqrt{17}) \end{bmatrix}.$$

Thus, looking at the first row, we have an eigenvector

$$v_1 = \begin{bmatrix} 4 \\ 3 - \sqrt{17} \end{bmatrix}.$$

For the next one, we follow the same procedure:

$$\begin{bmatrix} \frac{1}{2} - \lambda_2 & 1 \\ \frac{1}{2} & 2 - \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{4}(5 + \sqrt{17}) & 1 \\ \frac{1}{2} & 2 - \frac{1}{4}(5 + \sqrt{17}) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-3}{4} - \frac{1}{4}\sqrt{17} & 1 \\ \frac{1}{2} & 2 - \frac{1}{4}(5 - \sqrt{17}) \end{bmatrix}.$$

Again, the first row gives us the eigenvector

$$v_2 = \begin{bmatrix} 4 \\ 3 + \sqrt{17} \end{bmatrix},$$

and we have solved the generalized eigenvalue problem.

Problem 3

Solve the generalized eigenvalue problem from Question 2 using this second method.

Solution

We have

$$
\begin{aligned}
|A - \lambda B| &= \left| \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \right| \\
&= \left| \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3\lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix} \right| \\
&= \begin{vmatrix} 1 - 3\lambda & 1 + \lambda \\ \lambda & 1 - \lambda \end{vmatrix} \\
&= (1 - 3\lambda)(1 - \lambda) - \lambda(1 + \lambda) \\
&= 3\lambda^2 - 4\lambda + 1 - \lambda - \lambda^2 \\
&= 2\lambda^2 - 5\lambda + 1 \\
&= 0 \\
0 &= \lambda^2 - \frac{5}{2}\lambda + \frac{1}{2}.
\end{aligned}
$$

As before, this is the same equation we had before (as it clearly should be), so we have eigenvalues

$$
\lambda_1 = \frac{1}{4}(5 - \sqrt{17}), \text{ and } \lambda_2 = \frac{1}{4}(5 + \sqrt{17}).
$$

Now we find the eigenvectors. First, we do a little reduction

$$
\begin{aligned}
\begin{bmatrix} 1 - 3\lambda & 1 + \lambda \\ \lambda & 1 - \lambda \end{bmatrix} &\rightarrow \begin{bmatrix} 1 - 2\lambda & 2 \\ \lambda & 1 - \lambda \end{bmatrix} && R_1 + R_2 \\
&\rightarrow \begin{bmatrix} \frac{1}{2} - \lambda & 1 \\ \lambda & 1 - \lambda \end{bmatrix} && \frac{1}{2}R_1 \\
&\rightarrow \begin{bmatrix} \frac{1}{2} - \lambda & 1 \\ \frac{1}{2} & 2 - \lambda \end{bmatrix} && R_2 + R_1.
\end{aligned}
$$

We reduced this to the same matrix we had set equal to $B^{-1}A - \lambda I$ in problem 2. Since we have already shown the eigenvalues to be the same, we can conclude that the eigenvectors are the same (as they should be).

Problem 4

For two $n \times n$ complex matrices A and B, what is the maximum number of distinct generalized eigenvalues we would expect? Justify your answer. Then solve the generalized eigenvalue problem corresponding with the matrix pencil

$$
\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right).
$$

Solution

Theorem 1. *For two $n \times n$ complex matrices A and B, the maximum number of distinct generalized eigenvalues is n.*

Proof. We claim that the determinant of $A - \lambda B$ is at most an nth degree polynomial of λ, from which the conclusion follows. We will proceed by induction on n. For $n = 2$, this is immediately clear.

Now, suppose for some $n \geq 2$ that for any two $n \times n$ complex matrices, the maximum number of distinct generalized eigenvalues is n. Let A and B be $n + 1 \times n + 1$ complex matrices. For an $m \times m$ matrix C, let $\overline{C}_{ij}$ denote the $m - 1 \times m - 1$ submatrix of C obtained by deleting the ith row and j^{th} column of C. Then, we have

$$
|A - \lambda B| = \sum_{j=1}^{n+1} (A_{1j} - \lambda B_{1j}) |\overline{A - \lambda B}_{1j}|.
$$

By the inductive hypothesis, $|\overline{A - \lambda B}_{1j}|$ is an at most $n \times n$ polynomial of λ. Since $(A_{1j} - \lambda B_{1j})$ is an at most first degree polynomial, we have that each term of the above sum is an at most $n + 1^{\text{th}}$ degree polynomial, and we have completed our proof. $\qquad\square$

Now, we solve the generalized eigenvalue problem:

$$|A - \lambda B| = \begin{vmatrix} -\lambda & 1 \\ 0 & 0 \end{vmatrix}$$
$$= 0$$

Thus, every nonzero vector is a generalized eigenvector with corresponding eigenvalue 0.

Problem 5

Theorem 2. *Let $\hat{E}_\lambda$ be the collection of all generalized eigenvectors associated with the eigenvalue λ. Then $\hat{E}_\lambda$ is a subspace of V.*

Proof. We have

$$\hat{E}_\lambda = \{x \in V | (T - \lambda U)x = 0\}.$$

That is, $\hat{E}_\lambda$ is the nullspace of the linear operator $(T - \lambda U)$, and is therefore a subspace of V. $\qquad\square$

Problem 6

For distinct generalized eigenvalues λ and ν it is not necessarily the case that $\hat{E}_\lambda \cap \hat{E}_\nu = \{0\}$. Given a regular matrix pencil (A, B) and distinct nonzero eigenvalues $\lambda \neq \nu$, what is $\hat{E}_\lambda \cap \hat{E}_\nu$?

Solution

Let $x \in \hat{E}_\lambda \cap \hat{E}_\nu$. Then, we have

$$
\begin{aligned}
(A - \lambda B)x &= 0 \\
&= (A - \nu B)x \\
0 &= (A - \lambda B)x - (A - \nu B)x \\
&= Ax - \lambda Bx - Ax + \nu Bx \\
&= (\nu - \lambda)Bx,
\end{aligned}
$$

and we can conclude that $x \in N(B)$. With this, we have

$$
\begin{aligned}
0 &= (A - \lambda B)x \\
&= Ax - \lambda Bx \\
&= Ax.
\end{aligned}
$$

Thus, we also have that $x \in N(A)$. Therefore, we have $x \in N(A) \cap N(B)$ which implies $\hat{E}_\lambda \cap \hat{E}_\nu \subseteq N(A) \cap N(B)$. Supposing that $x \in N(A) \cap N(B)$, it follows immediately that $x \in \hat{E}_\lambda \cap \hat{E}_\nu$. With this, we can conclude that $\hat{E}_\lambda \cap \hat{E}_\nu = N(A) \cap N(B)$.

Problem 7

Let V be an n-dimensional vector space over $\mathbb{C}$. Suppose that P and Q are two orthogonal projections. Suppose that $L = R(P)$, and $N = R(Q)$. Then we may write L as

$$L = (L \cap N) \oplus (L \cap N^\perp).$$

There is an equivalent decomposition for $L^\perp$. With these two decompositions we can write

$$V = (L \cap N) \oplus (L \cap N^\perp) \oplus (L^\perp \cap N) \oplus (L^\perp \cap N^\perp)$$

Solution

Proof. Since Q is an orthogonal projection, we have

$$V = N \oplus N^\perp.$$

Thus, we have

$$
\begin{aligned}
x \in L \cap V &\iff x \in L \cap (N + N^\perp) \\
&\iff x \in L \wedge x \in \{x_1 + x_2 | x_1 \in N \wedge x_2 \in N^\perp\} \\
&\iff x \in \{x_1 + x_2 | x_1 \in L \cap N \wedge x_2 \in L \cap N^\perp\} \\
&\iff x \in (L \cap N) + L \cap N^\perp.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
x \in (L \cap N) \cap (L \cap N^\perp) &\iff x \in L \cap (N \cap N^\perp) \\
&\iff x \in L \cap \{0\} \qquad \text{Since } Q \text{ is an orthogonal projection} \\
&\iff x = 0.
\end{aligned}
$$

Thus, we have shown that

$$L = (L \cap N) \oplus (L \cap N^\perp).$$

In a nearly identical argument, we can derive the equality

$$L^\perp = (L^\perp \cap N) \oplus (L^\perp \cap N^\perp).$$

Since P is an orthogonal projection, we have

$$
\begin{aligned}
V &= L \oplus L^\perp \\
&= (L \cap N) \oplus (L \cap N^\perp) \oplus (L^\perp \cap N) \oplus (L^\perp \cap N^\perp),
\end{aligned}
$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Problem 8

What happens to the decomposition in the first part of question 7 if we only assume that Q is a projection, but not an orthogonal projection? Give a concrete example of such a projection onto a subspace of $\mathbb{C}^2$.

Note for Dr. Bossaller: you wrote P for this problem instead of Q, but I am pretty certain you meant Q.

Solution

If Q was not an orthogonal projection, then we would have $N^\perp \neq N(Q)$. Thus, we would not be able to say that $V = N \oplus N^\perp$, which means we could not say, in general, that $L = (L \cap N) \oplus (L \cap N^\perp)$.

To see this, let $V = \mathbb{C}^2$, and consider the projection Q onto $\mathrm{Span}(1, 1)$ given by

$$Q(x_1, x_2) = (x_1, x_1).$$

Then, we have

$$
\begin{aligned}
\langle Qx, y \rangle &= \langle (x_1, x_1), (y_1, y_2) \rangle \\
&= x_1 \overline{y_1} + x_1 \overline{y_2} \\
&= x_1 (\overline{y_1} + \overline{y_2}) \\
&= x_1 \overline{(y_1 + y_2)}.
\end{aligned}
$$

From this, we can see that $N^\perp = \{(y, -y) | y \in \mathbb{C}\}$. Furthermore, we have

$$
\begin{aligned}
N(Q) &= \{(0, x) | x \in \mathbb{C}\} \\
&\neq N^\perp.
\end{aligned}
$$