Problem 1

Theorem 1. *Let x, y be elements of the inner product space V. Then*

$$||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2.$$

Solution

Proof. Using the axioms of an inner product space and theorem 6.1, we have

$$\begin{aligned} ||x+y||^2 &= \langle x+y, x+y\rangle \\ &= \langle x, x+y\rangle + \langle y, x+y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, y\rangle + \langle y, x\rangle \\ &= ||x||^2 + ||y||^2 + \langle x, y\rangle + \langle y, x\rangle. \end{aligned}$$

Similarly,

$$\begin{aligned} ||x-y||^2 &= \langle x-y, x-y\rangle \\ &= \langle x, x-y\rangle - \langle y, x-y\rangle \\ &= \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\ &= ||x||^2 + ||y||^2 - \langle x, y\rangle - \langle y, x\rangle. \end{aligned}$$

Thus, we have

$$||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2,$$

as desired. □

Problem 2

Let V be a real or complex finite dimensional vector space, and let $\mathcal{B} = \{v_1, v_2, \ldots, v_k\}$ be a basis for V. Then for any $x, y \in V$, we may write

$$x = \sum_i a_i v_i \quad \text{and} \quad y = \sum_j b_j v_j.$$

Define

$$\langle x, y\rangle = \sum_{i=1}^{k} a_i \overline{b_i}.$$

Prove that $\langle \cdot, \cdot \rangle$ is an inner product on V and that $\mathcal{B}$ is an orthonormal basis for V. Conclude that every real or complex vector space may be regarded as an inner product space.

Solution

Proof. Two show properties (a) and (b), let $z = \sum_i c_i v_i$ and let $c \in \mathbb{F}$. Then, we have

$$\begin{aligned} \langle cx+z, y\rangle &= \sum_{i=1}^{k} (ca_i + c_i)\overline{b_i} \\ &= c\sum_{i=1}^{k} a_i \overline{b_i} + \sum_{i=1}^{k} c_i \overline{b_i} \\ &= c\langle x, y\rangle + \langle z, y\rangle. \end{aligned}$$

For property c, we have

$$\begin{aligned}
\overline{\langle x, y\rangle} &= \overline{\sum_{i=1}^{k} a_i \overline{b_i}} \\
&= \sum_{i=1}^{k} \overline{a_i \overline{b_i}} \\
&= \sum_{i=1}^{k} \overline{a_i} b_i \\
&= \sum_{i=1}^{k} b_i \overline{a_i} \\
&= \langle y, x\rangle.
\end{aligned}$$

Finally, to verify condition (d), let $x \neq 0$. We have

$$\begin{aligned}
\langle x, x\rangle &= \sum_{i=1}^{k} a_i \overline{a_i} \\
&= \sum_{i=1}^{k} |a_i|^2.
\end{aligned}$$

Now, since each term is a complex square, this is a sum of nonnegative real numbers, and is therefore real and nonnegative. Furthermore, since $x \neq 0$, this sum has at least one term that is nonzero. Thus, this sum is a positive real number, and we have shown this function is an inner product on V. □

Problem 3

Theorem 2. *Let $V \in \mathbb{F}^n$ with that standard inner product and $A \in M_{n\times n}(\mathbb{F})$. Then the following statements are true:*

Part (a):

$\langle x, Ay\rangle = \langle A^* x, y\rangle$

Part (b):

If for some $B \in M_{n\times n}(\mathbb{F})$ we have $\langle x, Ay\rangle = \langle Bx, y\rangle$ for all $x, y \in V$, then $B = A^*$.

Solution

Proof. Part (a):

Using the definition of our standard inner product, we have

$$\begin{aligned}
\langle x, Ay\rangle &= \sum_{i=1}^{n} x_i \overline{(Ay)_i} \\
&= \sum_{i=1}^{n} x_i \overline{\sum_{j=1}^{n} A_{i,j} y_j} \\
&= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} \overline{A_{i,j} y_j} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \overline{A_{i,j}}\, \overline{y_j} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} A^*_{j,i} x_i \overline{y_j} \\
&= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} A^*_{j,i} x_i \right) \overline{y_j} \\
&= \sum_{j=1}^{n} (A^* x)_j \overline{y_j} \\
&= \langle A^* x, y\rangle,
\end{aligned}$$

as desired.

Part (b):

For any two vectors $x, y \in V$, we have

$$\begin{aligned}
\langle Bx, y\rangle &= \langle x, Ay\rangle \\
&= \langle A^* x, y\rangle
\end{aligned}$$

by part (a). Thus, for any $x \in V$, we have

$$\begin{aligned}
0 &= \langle Bx, Bx - A^* x\rangle - \langle A^* x, Bx - A^* x\rangle \\
&= \langle Bx - A^* x, Bx - A^* x\rangle.
\end{aligned}$$

By taking the contrapositive of property (d) of inner product, we have that

$$Bx - A^* x = 0 \implies Bx = A^* x.$$

Since this is true for any $x \in V$, we can conclude that $B = A^*$. □