Problem 1

Find the singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Solution

We start by computing A^*A:

$$\begin{aligned} A^*A &= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}. \end{aligned}$$

Now, we find the eigenvalues of this matrix:

$$\begin{aligned} \begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} &= (3-\lambda)^2 - 4 \\ &= \lambda^2 - 6\lambda + 5 \\ &= (\lambda - 5)(\lambda - 1). \end{aligned}$$

Thus, we have $\lambda_1 = 5$ and $\lambda_2 = 1$. Now we find the corresponding eigenvectors to these eigenvalues. We have

$$\begin{aligned} \begin{bmatrix} 3-\lambda_1 & 2 \\ 2 & 3-\lambda_1 \end{bmatrix} &= \begin{bmatrix} 3-5 & 2 \\ 2 & 3-5 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \\ &\to \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} && -R_1 \\ &\to \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} && R_2 - R_1 \\ &\to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} && R_2 - R_1, \end{aligned}$$

and we can see that

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is a normalized eigenvector corresponding to eigenvalue λ_1. Doing the same for our remaining eigenvalue, we have

$$\begin{aligned} \begin{bmatrix} 3-\lambda_2 & 2 \\ 2 & 3-\lambda_2 \end{bmatrix} &= \begin{bmatrix} 3-1 & 2 \\ 2 & 3-1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \\ &\to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

and we can see that

$$v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

is a normalized eigenvector corresponding to eigenvalue λ_2.

Now, the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = \sqrt{5}$ and $\sigma_2 = \sqrt{\lambda_2} = 1$, and we can compute the column vectors of u:

$$\begin{aligned}
u_1 &= \frac{1}{\sigma_1} A v_1 \\
&= \frac{1}{5\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{5\sqrt{2}} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
u_2 &= \frac{1}{\sigma_2} A v_2 \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
A &= U \Sigma V^* \\
&= \begin{bmatrix} \frac{2}{5\sqrt{2}} & 0 \\ \frac{1}{5\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{5\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{5\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.
\end{aligned}$$

Problem 2

If $A = (a_i j)$ is an $m \times n$ real matrix, recall that the Frobenius norm of A is

$$||A||_F = \sqrt{\operatorname{Tr}(A^* A)}.$$

Show that $||A||_F = \sum \sigma_i^2$, the sum of the squares of the singular values of A.

Solution

Proof. Suppose A has k singular values. Define $\sigma_i = 0$ for $k \leq i \leq n$. As we saw in the proof of the singular value theorem, $A^* A$ is diagonlizable and has eigenvalues σ_i^2. Thus, there exists an invertible matrix Q such that

$$QA^*AQ^{-1} = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix}.$$

Finally, we have

$$\begin{aligned}
||A||_F^2 &= \mathrm{Tr}(A^*A) \\
&= \mathrm{Tr}(QA^*AQ^{-1}) && \text{Proved in part (c) of problem 3 of HW4} \\
&= \mathrm{Tr}\left(\begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix}\right) \\
&= \sum_{i=1}^{n} \sigma_i^2 \\
&= \sum_{i=1}^{k} \sigma_i^2,
\end{aligned}$$

as desired. □

Problem 3

Find the polar decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}.$$

Solution

Starting as usual,

$$\begin{aligned}
A^*A &= \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}.
\end{aligned}$$

Now we find the eigenvalues:

$$\begin{aligned}
\begin{vmatrix} 5-\lambda & -3 \\ -3 & 5-\lambda \end{vmatrix} &= (5-\lambda)^2 - 9 \\
&= \lambda^2 - 10\lambda + 16 \\
&= (\lambda - 2)(\lambda - 8).
\end{aligned}$$

Thus, $\sigma_1^2 = 8$ and $\sigma_2^2 = 2$. Finding the corresponding eigenvectors:

$$\begin{bmatrix} 5-8 & -3 \\ -3 & 5-8 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix},$$

and we can see we have a normalized eigenvector of

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}.$$

The other one we can easily get by looking:

$$v_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

We will now compute our u_i:

$$\begin{aligned}
u_1 &= \frac{1}{\sigma_1} A v_1 \\
&= \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{aligned}$$

We know u_2 is orthonormal to this, so we immediately have

$$u_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Following the construction used in the proof of the polar decomposition theorem, we have

$$\begin{aligned} W &= UV^* \\ &= V^* \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} P &= V\Sigma V^* \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}. \end{aligned}$$

Thus, we have found the unitary matrix W and positive semidefinite matrix P such that $A = WP$, which means we have found a polar decomposition of A.

Problem 4

Use the pseudoinverse to find the least-square solution to the following system of equations:

$$x + y + z = 5, \text{ and } 2x - y + z = 2.$$

Solution

The matrix for this system of equations is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}.$$

To find the psuedoinverse, we will first find the matrices used in the singular value decomposition of A. We have

$$\begin{aligned} A^*A &= \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -1 & 3 \\ -1 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}. \end{aligned}$$

Now we find the eigenvalues:

$$\begin{aligned} \begin{vmatrix} 5-\lambda & -1 & 3 \\ -1 & 2-\lambda & 0 \\ 3 & 0 & 2-\lambda \end{vmatrix} &= 3\begin{vmatrix} -1 & 2-\lambda \\ 3 & 0 \end{vmatrix} + (2-\lambda)\begin{vmatrix} 5-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)[-9 + (5-\lambda)(2-\lambda) - 1] \\ &= (2-\lambda)[-10 + 10 - 7\lambda + \lambda^2] \\ &= \lambda(2-\lambda)(\lambda - 7). \end{aligned}$$

Thus, $\sigma_1^2 = 7$, and $\sigma_2^2 = 2$. Now we find the orthonormal basis (by finding our eigenvectors):

$$\begin{aligned}
\begin{bmatrix} 5-7 & -1 & 3 \\ -1 & 2-7 & 0 \\ 3 & 0 & 2-7 \end{bmatrix} &= \begin{bmatrix} 5-7 & -1 & 3 \\ -1 & 2-7 & 0 \\ 3 & 0 & 2-7 \end{bmatrix} \\
&= \begin{bmatrix} -2 & -1 & 3 \\ -1 & -5 & 0 \\ 3 & 0 & -5 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 14 & 3 \\ -1 & -5 & 0 \\ 3 & 0 & -5 \end{bmatrix} && R_1 - 3R_2 \\
&\to \begin{bmatrix} 1 & 14 & 3 \\ 0 & 9 & 3 \\ 0 & -42 & -14 \end{bmatrix} && R_3 - 3R_1 \\
&\to \begin{bmatrix} 1 & 14 & 3 \\ 0 & 1 & \frac{1}{3} \\ 0 & -42 & -14 \end{bmatrix} && \frac{1}{9}R_2 \\
&\to \begin{bmatrix} 1 & 0 & -\frac{5}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} && R_3 + 42R_2.
\end{aligned}$$

Thus, the vector

$$x = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$$

is an eigenvector with eigenvalue 7. Normalizing, we have

$$v_1 = \begin{bmatrix} \frac{5}{\sqrt{35}} \\ \frac{-1}{\sqrt{35}} \\ \frac{3}{\sqrt{35}} \end{bmatrix}.$$

Doing the same for our next eigenvalue:

$$\begin{bmatrix} 5-2 & -1 & 3 \\ -1 & 2-2 & 0 \\ 3 & 0 & 2-2 \end{bmatrix} = \begin{bmatrix} -3 & -1 & 3 \\ -1 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}.$$

Thus, the vector

$$y = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

is an eigenvector with corresponding eigenvalue 2. Normalizing, we have

$$v_2 = \begin{bmatrix} 0 \\ \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}.$$

To find our last eigenvector, we will use the cross product, as we have done enough row reduction for a lifetime:

$$\begin{aligned}
z &= \begin{vmatrix} i & j & k \\ 5 & -1 & 3 \\ 0 & 3 & 1 \end{vmatrix} \\
&= i(-1-9) - 5(j - 3k) \\
&= -10i - 5j + 15k.
\end{aligned}$$

Normalizing, we have

$$v_3 = \begin{bmatrix} \frac{-2}{\sqrt{14}} \\ \frac{-1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix}.$$

Now we find our u_i:

$$\begin{aligned} u_1 &= \frac{1}{\sigma_1} A v_1 \\ &= \frac{1}{\sqrt{7}} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{\sqrt{35}} \\ \frac{-1}{\sqrt{35}} \\ \frac{3}{\sqrt{35}} \end{bmatrix} \\ &= \frac{1}{7\sqrt{5}} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} \\ &= \frac{1}{7\sqrt{5}} \begin{bmatrix} 7 \\ 14 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} u_2 &= \frac{1}{2\sqrt{5}} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{bmatrix}. \end{aligned}$$

Finally, we have

$$\begin{aligned} A^\dagger &= V\Sigma^\dagger U^* \\ &= \begin{bmatrix} \frac{5}{\sqrt{35}} & 0 & \frac{-2}{\sqrt{14}} \\ \frac{-1}{\sqrt{35}} & \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{14}} \\ \frac{3}{\sqrt{35}} & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{14}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{7}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & 0. \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix}. \end{aligned}$$

Thus, (assuming we somehow did not make a book keeping mistake) our least squares solution is given by

$$\begin{bmatrix} \frac{5}{\sqrt{35}} & 0 & \frac{-2}{\sqrt{14}} \\ \frac{-1}{\sqrt{35}} & \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{14}} \\ \frac{3}{\sqrt{35}} & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{14}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{7}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & 0. \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 9 \\ 15 \\ 11 \end{bmatrix}.$$

As a test, we can plug it in, and see how the result looks:

$$\frac{1}{7} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 15 \\ 11 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

We have a solution, which is a very good sign we have done this correctly. *Breathes sigh of relief that this homework is finally over.*