Linear Algebra, Fall 2023

Written Homework 6

Due: 20 Sep 2023

Please answer the following questions on a separate sheet of paper and turn it in.

1. Prove the *Parallelogram Law* for an inner product space V.

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

2. Let V be a real or complex finite dimensional vector space, and let $\mathcal{B} = \{v_1, v_2, \ldots, v_k\}$ be a basis for V. Then for any $x, y \in V$, we may write

$$x = \sum_i a_i v_i \quad \text{and} \quad y = \sum_j b_j v_j.$$

   Define

$$\langle x, y \rangle = \sum_{i=1}^{k} a_i \overline{b_i}.$$

   Prove that $\langle \cdot, \cdot \rangle$ is an inner product on V and that $\mathcal{B}$ is an orthonormal basis for V. Conclude that every real or complex vector space may be regarded as an inner product space.

3. One way to think about the standard inner product $\langle x, y \rangle = \sum_i x_i \overline{y_i}$ is to visualize x and y as $n \times 1$ column vectors, then $\langle x, y \rangle = y^* x$ (note the similarity to the Frobenius norm).

   Let $V = \mathbb{F}^n$ with the standard inner product and $A \in M_{n\times n}(\mathbb{F})$.

   (a) Prove that $\langle x, Ay \rangle = \langle A^* x, y \rangle$

   (b) Suppose for some $B \in M_{n\times n}(\mathbb{F})$ we have that $\langle x, Ay \rangle = \langle Bx, y \rangle$ for all $x, y \in V$. Prove that $B = A^*$.