

Problem 1

Theorem 1. Let B be an $(m+1) \times (n+1)$ matrix, where $B_{11} = 1$, and $B_{1j} = B_{i1} = 0$ for $2 \leq j \leq n+1$ and $2 \leq i \leq m+1$. Define B' to be the $m \times n$ submatrix of B obtained by removing the first column and the first row of B . Then, if $\text{rank}(B) = r$, then $\text{rank}(B') = r - 1$.

Solution

Proof. For this proof, we will use the fact that rank of a matrix is equal to the dimension of its column space. Since the first element of $B_{\cdot 1}$ is non zero, and the first elements of each vector in the set of remaining column vectors $\{B_{\cdot 2}, \dots, B_{\cdot (n+1)}\}$ are zero, it follows that

$$B_{\cdot 1} \notin \text{Span}(\{B_{\cdot 2}, \dots, B_{\cdot (n+1)}\}).$$

Thus, we can conclude that

$$\begin{aligned} \dim(\text{Span}(\{B_{\cdot 2}, \dots, B_{\cdot (n+1)}\})) &= \dim(\text{Span}(\{B_{\cdot 1}, \dots, B_{\cdot (n+1)}\})) - 1 \\ &= r - 1. \end{aligned}$$

Now, define $\{v_1, \dots, v_{r-1}\}$ to be the resulting set of reducing $\{B_{\cdot 2}, \dots, B_{\cdot (n+1)}\}$ to a basis of $\text{Span}(\{B_{\cdot 2}, \dots, B_{\cdot (n+1)}\})$. Now, define a new set of vectors $\{v'_1, \dots, v'_{r-1}\} \subseteq \{B'_{\cdot 1}, \dots, B'_{\cdot n}\}$ such that for all $1 \leq i \leq r-1$, v'_i is the vector obtained by removing the first element of v_i . All that remains is to show that $\{v'_1, \dots, v'_{r-1}\}$ is a basis of $\text{Span}(B'_{\cdot 1}, \dots, B'_{\cdot n})$.

To see that $\{v'_1, \dots, v'_{r-1}\}$ is linearly independent, let $\{a_1, \dots, a_{r-1}\} \subseteq \mathbb{F}$ be such that

$$\sum_{i=1}^{r-1} a_i v'_i = 0.$$

Then, since the first element of each vector in $\{v_1, \dots, v_{r-1}\}$ is zero, we have

$$\sum_{i=1}^{r-1} a_i v_i = 0.$$

By linear independence of $\{v_1, \dots, v_{r-1}\}$, we can conclude that $a_i = 0$ for each i , and we have shown that $\{v'_1, \dots, v'_{r-1}\}$ is linearly independent.

To end the proof, we must now show that

$$\text{Span}(\{v_1, \dots, v_{r-1}\}) = \text{Span}(\{B'_{\cdot 1}, \dots, B'_{\cdot n}\}).$$

One side of this equality follows directly from the fact that $\{v'_1, \dots, v'_{r-1}\} \subseteq \{B'_{\cdot 1}, \dots, B'_{\cdot n}\}$. For the other direction, let $v' \in \text{Span}(\{B'_{\cdot 1}, \dots, B'_{\cdot n}\})$. Define $v \in \text{Span}(\{B_{\cdot 2}, \dots, B_{\cdot (n+1)}\})$ by adding a zero to the top of v' . Then, there exist scalars $\{a_1, \dots, a_{r-1}\} \subseteq \mathbb{F}$ such that

$$\sum_{i=1}^{r-1} a_i v_i = v.$$

Since vector addition is defined element wise, it follows immediately that

$$\sum_{i=1}^{r-1} a_i v'_i = v',$$

and our proof is complete. □

Problem 2

Calculate the determinant of the following matrix whose entries are in \mathbb{C} :

$$\begin{bmatrix} i & 2+i & 0 \\ -1 & 3 & i \\ 0 & -1 & 1-i \end{bmatrix}$$

Solution

Proceeding with cofactor expansion on the first row, we have

$$\begin{aligned}
 \begin{vmatrix} i & 2+i & 0 \\ -1 & 3 & i \\ 0 & -1 & 1-i \end{vmatrix} &= i \begin{vmatrix} 3 & i \\ -1 & 1-i \end{vmatrix} - (2+i) \begin{vmatrix} -1 & i \\ 0 & 1-i \end{vmatrix} \\
 &= i(3 - 3i + i) - (2+i)(i-1) \\
 &= 3i + 2 + (2+i)(1-i) \\
 &= 3i + 2 + 2 + i - 2i + 1 \\
 &= 5 + 2i
 \end{aligned}$$

Problem 3

Theorem 2. Let T be a linear operator on a vector space V over the field \mathbb{F} , and let $g(t)$ be a polynomial with coefficients from \mathbb{F} . Define $g(T)$ to be the operator obtained by plugging T in for the variable t , and letting the constant term a_0 be replaced with a_0I , where I is the identity operator on V . Then,

Part (a):

If $v \in V$ is an eigenvector of T with eigenvalue λ , then $g(T)(v) = g(\lambda)v$.

Part (b):

If $g(t)$ is the characteristic polynomial of T , then $g(T)(v) = 0$.

Solution

Part (a):

We have

$$\begin{aligned}
 g(T)(v) &= \left(\sum_{i=0} a_i IT^i \right) (v) \\
 &= \sum_{i=0} a_i IT^i(v) \\
 &= \sum_{i=0} a_i \lambda^i v \\
 &= \left(\sum_{i=0} a_i \lambda^i \right) (v) \\
 &= g(\lambda)v,
 \end{aligned}$$

as desired.

Part (b):

Now suppose $g(t)$ is the characteristic polynomial of T . Then, if v is an eigen vector of T with eigenvalue λ , we have

$$\begin{aligned}
 g(T)(v) &= g(\lambda)v && \text{by Part (a)} \\
 &= (0)v,
 \end{aligned}$$

with the last equality being true because the eigenvalues of T are the roots to the characteristic polynomial of T .

Problem 4

Theorem 3. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(\lambda) = (-1)^n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0.$$

Then, $f(0) = a_0 = \det(A)$, and we can deduce that $\det(A)$ is invertible if and only if $a_0 \neq 0$.

Solution

Proof. We have

$$f(\lambda) = \det(A - \lambda I).$$

Therefore

$$\begin{aligned} f(0) &= \det(A - 0I) \\ &= \det(A) \\ &= (-1)^n(0)^n + a_{n-1}(0)^{n-1} + \cdots + a_1(0) + a_0 \\ &= a_0, \end{aligned}$$

as desired. Since A is invertible if and only if it has determinant zero, and we have $a_0 = \det(A)$, it follows that A is invertible if and only if $a_0 \neq 0$. \square