Linear Algebra, Fall 2023

Exam 2

Due: 15 November 2023

Please complete the following questions and return the exam to me by the beginning of class Wednesday, 15 November 2023. You may only use your textbook, notes, and the resources available on Canvas to complete the following questions. Please do not collaborate with other students; email me if you have any questions. As a forewarning, I will not help you as much on these questions as I would on a homework assignment.

Finding the characteristic polynomial and eigenvalues of matrices and operators is of central importance to linear algebra in all of its forms. In this project we will look at generalizations of linear operators on a finite dimensional complex inner product space.

Suppose that T and U are operators on a finite dimensional complex inner product space. A "generalized eigenvalue problem" asks us to find a "generalized eigenvector" $x \in \mathbb{C}^n$ and a "generalized eigenvalue" λ such that $T(x) = \lambda U(x)$. Note that this problem is not all that interesting if U is unitary (or even invertible), whence we may transform the generalized eigenvalue problem into a more familiar format:

$$U^*T(x) = \lambda x \text{ or } U^{-1}T(x) = \lambda x, \text{ respectively.}$$

(Question 1): Show that every positive definite matrix must be invertible. Under which conditions is a positive definite matrix unitary?

(Question 2): Use Question 1 and the previous paragraph to solve the following generalized eigenvalue problem $Ax = \lambda Bx$.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x = \lambda \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} x$$

We can, however, still work to solve this problem in the traditional way, i.e. looking for the values of λ such that $N(T - \lambda U) \supsetneq \{0\}$, i.e. by taking the determinant and looking for the values of λ which give a zero determinant.

(Question 3): Solve the generalized eigenvalue problem from Question 2 using this second method. The advantage of this second method is that we are not limited to focusing only on invertible and/or unitary operators. We can also reformulate a generalized eigenvalue problem into an "operator form,"

$$(T - \lambda U)(x),$$

and the solutions of the generalized eigenvalue problem correspond with elements of $N(T - \lambda U)$. When T and U are matrices, we call $T - \lambda U$ a "matrix pencil," which is sometimes denoted (T, U)

(Question 4): Consult Question 3; for two $n \times n$ complex matrices A and B, what is the maximum number of distinct generalized eigenvalues we would expect, justify your guess. Then solve the generalized eigenvalue problem corresponding with the matrix pencil

$$\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right).$$

We will limit ourselves to matrix pencils for which the determinant of the pencil is not identically equal to 0, which we call "regular" matrix pencils. For regular matrix pencils, we can define "generalized eigenspaces" in the same way that we defined eigenspaces for linear operators.

(Question 5): Let $\hat{E}_\lambda$ be the collection of all generalized eigenvectors associated with the eigenvalue λ. Show that $\hat{E}_\lambda$ is a subspace of V.

(Question 6): For distinct generalized eigenvalues λ and ν it is not necessarily the case that $\hat{E}_\lambda \cap \hat{E}_\nu = \{0\}$. Given a regular matrix pencil (A, B) and distinct nonzero eigenvalues $\lambda \neq \nu$, what is $\hat{E}_\lambda \cap \hat{E}_\nu$.

In class, we showed that if T is a projection on a finite dimensional vector space, then we my write $V = \{x : T(x) = x\} \oplus N(T)$ and $\{x : T(x) = x\} = R(T)$

(Question 7): Let V be an n-dimensional vector space over C. Suppose that P and Q are two orthogonal projections. Suppose that $L = R(P)$, $N = R(Q)$; show that we may write L as follows

$$L = (L \cap N) \oplus (L \cap N^\perp).$$

(There is a similar decomposition for $L^\perp$.) Use these two decompositions to write

$$V = (L \cap N) \oplus (L \cap N^\perp) \oplus (L^\perp \cap N) \oplus (L^\perp \cap N^\perp).$$

Using this decomposition of V, we may examine the action of the operator $(P - \lambda Q)$ on $x \in V$, namely, if $x = x_1 + x_2 + x_3 + x_4$, then $(P - \lambda Q)(x) = (1 - \lambda)x_1 + x_2 - \lambda x_3$. From here we can study the subspaces of $R(P - \lambda Q)$. We need to be careful though,

(Question 8): What happens if to the decomposition you gave in the first part of Question 7 if you only assume that P is a projection, but not an orthogonal projection? Give a concrete example of such a projection onto a subspace of $\mathbb{C}^2$