Linear Algebra, Fall 2023

Written Homework 5

Due: 20 Sep 2023

Please answer the following questions on a separate sheet of paper and turn it in.

1. Let A denote the $k \times k$ matrix

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{bmatrix},$$

   where the a_i's are arbitrary scalars. Prove that the characteristic polynomial $f(t)$ of A is $f(t) = (-1)^k(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k)$. (hint: Use induction and cofactor expansion along the first row)

2. Let T be a linear operator on a finite-dimensinoal vector space V. Prove that T is diagonalizable if and only if V is the direct sum of one-dimensional T-invariant subspaces.

3. Let A be a $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

   (a) Prove that if A is invertible, then

$$A^{-1} = (-1/a_0)[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n]$$

   (b) Use (a) to compute A^{-1} for

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$