Problem 1

Theorem 1. Part (a):

The skew symmetric matrices form a subspace of $M_{n\times n}(\mathbb{R})$.

Part (b):

If W denotes the subspace of symmetric matrices, and W_- denotes the subspace of antisymmetric matrices, then $M_{n\times n}(\mathbb{R}) = W + W_-$ and $W \cap W_- = \{0\}$.

Solution

Proof. Part (a):

Since each entry of the zero matrix is zero, we have $0_{ij} = -0_{ji}$, and can conclude that the zero matrix is skew symmetric. Now, let $w, x \in M_{n\times n}(\mathbb{R})$ be skew symmetric. Then, for $1 \leq i, j \leq n$,

$$\begin{aligned}
(w+x)_{ji} &= w_{ji} + x_{ji} && \text{Basic property of matrix addition} \\
&= -w_{ij} - x_{ij} && \text{Both vectors are skew symmetric} \\
&= -(w+x)_{ij},
\end{aligned}$$

and we have shown that $w + x$ is skew symmetric.

Finally, let $a \in \mathbb{R}$ and $v \in M_{n\times n}(\mathbb{R})$ be skew symmetric. For $1 \leq i, j \leq n$, we have

$$\begin{aligned}
(ax)_{ji} &= a(x_{ji}) \\
&= a(-x_{ij}) && x \text{ is skew symmetric} \\
&= -(ax)_{ij},
\end{aligned}$$

and we have proven that ax is skew symmetric. With this, we have proven that the subset of skew symmetric matrices forms a subspace.

Part (b):

Let $v \in M_{n\times n}(\mathbb{R})$. Define $w, x \in M_{n\times n}(\mathbb{R})$ by

$$w_{ij} = \frac{1}{2}(v_{ij} - v_{ji}), \text{ and } x_{ij} = \frac{1}{2}(v_{ij} + v_{ji})$$

for $1 \leq i, j \leq n$. Then, we have

$$\begin{aligned}
w_{ji} &= \frac{1}{2}(v_{ji} - v_{ij}) \\
&= -\frac{1}{2}(v_{ij} - v_{ji}) \\
&= -w_{ij},
\end{aligned}$$

and we have that $w \in W_-$. Similarly,

$$\begin{aligned}
x_{ji} &= \frac{1}{2}(v_{ji} + v_{ij}) \\
&= \frac{1}{2}(v_{ij} + v_{ji}) \\
&= x_{ij},
\end{aligned}$$

and we have shown that $x \in W$.

From the definition of x and w, we have

$$\begin{aligned}
w_{ij} + x_{ij} &= \frac{1}{2}(v_{ij} - v_{ji}) + \frac{1}{2}(v_{ij} + v_{ji}) \\
&= v_{ij},
\end{aligned}$$

and we can conclude that $v = w + x$. Since v was arbitrary, we have shown any $W + W_- = V$.

Finally, we have

$$\begin{aligned}
v \in W \cap W_- &\implies (\forall 1 \leq i, j \leq n)(v_{ij} = v_{ji} \wedge v_{ij} = -v_{ji}) \\
&\implies (\forall 1 \leq i, j \leq n)(v_{ij} = 0) \\
&\implies v = 0.
\end{aligned}$$

Thus, we have shown $W \cap W_- = \{0\}$, and our proof is complete. $\square$

Problem 2

Theorem 2. *Let T be an invertible linear transformation $T : V \to W$, and let $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ be a basis for V. Then $\{T(v_1), T(v_2), ..., T(v_n)\}$ is a basis for W.*

Solution

Proof. For this proof, we will show that $\{T(v_1), T(v_2), ..., T(v_n)\}$ spans V and is linearly independent. Let $w \in W$. Then, since T is invertible, it is surjective, which implies there exists a $v \in V$ such that $T(v) = w$. Since $\mathcal{B}$ is a basis of V, there exist $a_i \in \mathbb{F}$ for $1 \leq i \leq n$ such that

$$v = \sum_{i=1}^{n} a_i v_i.$$

Then, using the fact that T is linear, we have

$$\begin{aligned} w &= T(v) \\ &= T\left(\sum_{i=1}^{n} a_i v_i\right) \\ &= \sum_{i=1}^{n} T(a_i v_i) \\ &= \sum_{i=1}^{n} a_i T(v_i), \end{aligned}$$

and we have shown that $\{T(v_1), T(v_2), ..., T(v_n)\}$ spans V.

Finally, let $a_i \in \mathbb{F}$ for $1 \leq i \leq n$, be such that

$$\sum_{i=1}^{n} a_i T(v_i) = 0. \tag{1}$$

Then, we have

$$\begin{aligned} 0 &= \sum_{i=1}^{n} a_i T(v_i) \\ &= \sum_{i=1}^{n} T(a_i v_i) \\ &= T\left(\sum_{i=1}^{n} a_i v_i\right). \end{aligned}$$

By linearity of T, $T(0) = 0$. Furthermore, since T is injective (as a result of invertibility), we can conclude

$$\sum_{i=1}^{n} a_i v_i = 0.$$

By linear independence of $\mathcal{B}$, we have $a_i = 0$ for all i. Thus, the only solution to (1) is the trivial one, implying that $\{T(v_1), T(v_2), ..., T(v_n)\}$ is linearly independent. With this, our proof is complete. □