Problem 1

Theorem 1. *Let B be an $(m+1)\times(n+1)$ matrix, where $B_{11}=1$, and $B_{1j}=B_{i1}=0$ for $2\le j\le n+1$ and $2\le i\le m+1$. Define B' to be the $m\times n$ submatrix of B obtained by removing the first column and the first row of B. Then, if rank$(B)=r$, then rank$(B')=r-1$.*

Solution

Proof. For this proof, we will use the fact that rank of a matrix is equal to the dimension of its column space. Since the first element of $B_{\cdot 1}$ is non zero, and the first elements of each vector in the set of remaining column vectors $\{B_{\cdot 2},\dots,B_{\cdot(n+1)}\}$ are zero, it follows that

$$B_{\cdot 1}\notin \text{Span}\left(\{B_{\cdot 2},\dots,B_{\cdot(n+1)}\}\right).$$

Thus, we can conclude that

$$\begin{aligned}\dim\left(\text{Span}\left(\{B_{\cdot 2},\dots,B_{\cdot(n+1)}\}\right)\right)&=\dim\left(\text{Span}\left(\{B_{\cdot 1},\dots,B_{\cdot(n+1)}\}\right)\right)-1\\&=r-1.\end{aligned}$$

Now, define $\{v_1,...,v_{r-1}\}$ to be the resulting set of reducing $\{B_{\cdot 2},\dots,B_{\cdot(n+1)}\}$ to a basis of Span $\left(\{B_{\cdot 2},\dots,B_{\cdot(n+1)}\}\right)$. Now, define a new set of vectors $\{v_1',...,v_{r-1}'\}\subseteq\{B'_{\cdot 1},\dots,B'_{\cdot n}\}$ such that for all $1\le i\le r-1$, v_i' is the vector obtained by removing the first element of v_i. All that remains is to that $\{v_1',...,v_{r-1}'\}$ is a basis of Span $(B'_{\cdot 1},\dots,B'_{\cdot n}\})$.

To see that $\{v_1',...,v_{r-1}'\}$ is linearly independent, let $\{a_1,\dots,a_{r-1}\}\subseteq\mathbb{F}$ be such that

$$\sum_{i=1}^{r-1}a_iv_i'=0.$$

Then, since the first element of each vector in $\{v_1,...,v_{r-1}\}$ is zero, we have

$$\sum_{i=1}^{r-1}a_iv_i=0.$$

By linear independence of $\{v_1,...,v_{r-1}\}$, we can conclude that $a_i=0$ for each i, and we have shown that $\{v_1',...,v_{r-1}'\}$ is linearly independent.

To end the proof, we must now show that

$$\text{Span}\left(\{v_1,...,v_{r-1}\}\right)=\text{Span}\left(\{B'_{\cdot 1},\dots,B'_{\cdot n}\}\right).$$

One side of this equality follows directly from the fact that $\{v_1',...,v_{r-1}'\}\subseteq\{B'_{\cdot 1},\dots,B'_{\cdot n}\}$. For the other direction, let $v'\in$ Span $(\{B'_{\cdot 1},\dots,B'_{\cdot n}\})$. Define $v\in$ Span $\left(\{B_{\cdot 2},\dots,B_{\cdot(n+1)}\}\right)$ by adding a zero to the top of v'. Then, there exist scalars $\{a_1,\dots,a_{r-1}\}\subseteq\mathbb{F}$ such that

$$\sum_{i=1}^{r-1}a_iv_i=v.$$

Since vector addition is defined element wise, it follows immediately that

$$\sum_{i=1}^{r-1}a_iv_i'=v',$$

and our proof is complete. □