Problem 1.1

Show that if $X \in C(0,1)$, then so is $1/X$.

Solution

Since $X \in C(0,1)$, we have

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \text{ for } -\infty < x < \infty.$$

Define $Y = \frac{1}{X}$. Then, $X = \frac{1}{Y}$, and we have

$$|\det(J)| = \left|\frac{dx}{dy}\right| = \frac{1}{y^2}.$$

Thus,

$$\begin{aligned} f_Y(y) &= |\det(J)| f_X(1/y) \\ &= \frac{1}{y^2} \cdot \frac{1}{\pi} \cdot \frac{1}{1+(1/y)^2} \\ &= \frac{1}{\pi} \cdot \frac{1}{1+y^2}, \end{aligned}$$

and we have shown that $Y \in C(0,1)$.

Problem 1.8

Show that if X, Y are independent $N(0,1)$-distributed random variables, then $X/Y \in C(0,1)$.

Solution

We have

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{1}{2\pi} e^{-x^2/2} e^{-y^2/2} \\ &= \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}. \end{aligned}$$

Define $U = X/Y$. We introduce the auxiliary variable $V = X$. Then we have the inverse relations

$$\begin{aligned} X &= V \\ Y &= V/U. \end{aligned}$$

With this, we can find our Jacobian determinant:

$$\begin{aligned} \det(J) &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \\ &= \begin{vmatrix} 0 & 1 \\ \frac{-v}{u^2} & \frac{1}{u} \end{vmatrix} \\ &= \frac{v}{u^2}. \end{aligned}$$

Thus,

$$\begin{aligned} f_{U,V}(u,v) &= |\det(J)| f_{X,Y}(v, v/u) \\ &= \frac{|v|}{u^2} \cdot \frac{1}{2\pi} e^{-\frac{v^2+(v/u)^2}{2}} \\ &= \frac{|v|}{u^2} \cdot \frac{1}{2\pi} e^{-v^2 \frac{1+(1/u)^2}{2}}. \end{aligned}$$

We want the distribution of U, so

$$\begin{aligned}
f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u,v)dv \\
&= \int_{-\infty}^{\infty} \frac{|v|}{u^2} \cdot \frac{1}{2\pi} e^{-v^2 \frac{1+(1/u)^2}{2}} dv \\
&= \frac{1}{u^2\pi} \int_0^{\infty} v \cdot e^{-v^2 \frac{1+(1/u)^2}{2}} dv \\
&= \frac{1}{\pi} \cdot \frac{1}{1+u^2}.
\end{aligned}$$

Thus, we have shown that $U \in C(0,1)$.

Problem 1.11

Show that if X and Y are independent Exp(a)-distributed random variables, then $X/Y \in F(2,2)$.

Solution

We have

$$\begin{aligned}
f_{X,Y}(x,y) &= \frac{1}{a^2} e^{-x/a} e^{-y/a} \\
&= \frac{1}{a^2} e^{-\frac{x+y}{a}},
\end{aligned}$$

for $0 \le x, y < \infty$. Define U and V as we did in the previous problem. Then,

$$\begin{aligned}
f_{U,V}(u,v) &= |\det(J)| f_{X,Y}(v, v/u) \\
&= \frac{v}{u^2} \cdot \frac{1}{a^2} e^{-\frac{v+v/u}{a}} \\
&= \frac{v}{u^2} \cdot \frac{1}{a^2} e^{-v \frac{1+1/u}{a}}.
\end{aligned}$$

Then, we can integrate to find the distribution of U:

$$\begin{aligned}
f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u,v)dv \\
&= \frac{1}{a^2u^2} \int_0^{\infty} v e^{-v\frac{1+1/u}{a}} dv \\
&= \frac{1}{(u+1)^2}.
\end{aligned}$$

After looking up Fisher's distribution, we can conclude that $X/Y \in F(2,2)$.

Problem 1.12

Let X, Y be independent random variables such that $X \in U(0,1)$ and $Y \in U(0,\alpha)$. Find the density function of $Z = X + Y$.

Solution

Since X and Y are independent, we have

$$\begin{aligned}
f_{X,Y}(x,y) &= f_X(x) f_Y(y) \\
&= \begin{cases} \frac{1}{\alpha} & 0 \le x \le 1 \text{ and } 0 \le y \le \alpha \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Define $U = X + Y$, and introduce the auxiliary variable $V = X$. Then, we have the inverse relations $X = V$ and $Y = U - V$. With this, our Jacobian determinant becomes

$$\begin{aligned}\det(J) &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \\ &= \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \\ &= -1.\end{aligned}$$

Thus,

$$\begin{aligned}f_{U,V}(u,v) &= f_{X,Y}(v, u-v) \\ &= \begin{cases} \frac{1}{\alpha} & 0 \le v \le 1 \text{ and } 0 \le u - v \le \alpha \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} \frac{1}{\alpha} & 0 \le v \le 1 \text{ and } -u \le -v \le \alpha - u \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} \frac{1}{\alpha} & 0 \le v \le 1 \text{ and } u - \alpha \le v \le u \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Thus, to find the distribution of U, we integrate over all v. However to do this, we need to take care to only integrate over the support of our distribution. To make this easier, we will consider the cases where $\alpha \le 1$, and the case where $\alpha > 1$. For the case where $\alpha \le 1$, we have

$$\begin{aligned}f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u,v)dv \\ &= \begin{cases} \int_0^u \frac{1}{\alpha} dv = \frac{u}{\alpha} & 0 \le u \le \alpha \\ \int_{u-\alpha}^u \frac{1}{\alpha} dv = 1 & \alpha \le u \le 1 \\ \int_{u-\alpha}^1 \frac{1}{\alpha} dv = \frac{1}{\alpha}(1 + \alpha - u) & 1 \le u \le 1 + \alpha \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Integrating this f_U over all u, we get unity, which is a great sanity check.

Now, we consider the case where $\alpha > 1$. This changes the bounds that satisfy our inequality in a fairly straightforward way:

$$\begin{aligned}f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u,v)dv \\ &= \begin{cases} \int_0^u \frac{1}{\alpha} dv = \frac{u}{\alpha} & 0 \le u \le 1 \\ \int_0^1 \frac{1}{\alpha} dv = \frac{1}{\alpha} & 1 \le u \le \alpha \\ \int_{u-\alpha}^1 \frac{1}{\alpha} dv = \frac{1}{\alpha}(1 + \alpha - u) & \alpha \le u \le 1 + \alpha \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Once again, a quick integration verifies that this distribution is normalized, which excludes any obvious errors.