Problem 1

Suppose the number of customers who enter a post office over $[0, 1]$ has a Poisson(λ) distribution. Let X be the time that the first customer enters the post office. Find $\mathbb{P}\{X \leq 1\}$.

Solution

Let $N \sim$ Poisson(λ) be the number of people that have entered the post office over $[0, 1]$. Then, we have the probability mass function of N is

$$p_N(k) = e^{-\lambda}\frac{\lambda^k}{k!}.$$

Now, another way of phrasing $\mathbb{P}\{X \leq 1\}$ is: what is the probability that at least one person enters the post office? Thus,

$$\begin{aligned}
\mathbb{P}\{X \leq 1\} &= \mathbb{P}\{N > 0\} \\
&= 1 - \mathbb{P}\{N = 0\} \\
&= 1 - p_N(0) \\
&= 1 - e^{-\lambda}.
\end{aligned}$$

Problem 2

Let $f(x)$ be the density function of a continuous r.v. X defined by $f(x) = c|x|$ for $-2 \leq x \leq 4$, and $f(x) = 0$ otherwise. (1) Determine the constant c. (2) Find the mean and median(s) of X.

Solution

(1) We need to normalize the density function:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} f(x)dx \\
&= \int_{-2}^{4} c|x|dx \\
&= -c\int_{-2}^{0} xdx + c\int_{0}^{4} xdx \\
&= -c\frac{1}{2}x^2\Big|_{-2}^{0} + c\frac{1}{2}x^2\Big|_{0}^{4} \\
&= c(2 + 8) \\
&= 10c \\
c &= \frac{1}{10}.
\end{aligned}$$

(2) For the mean, we have

$$\begin{aligned}
\mathrm{E}X &= \int_{-\infty}^{\infty} xf(x)dx \\
&= \int_{-2}^{4} cx|x|dx \\
&= -\frac{1}{10}\int_{-2}^{0} x^2dx + c\int_{0}^{4} x^2dx \\
&= \frac{1}{10}\left(-\frac{1}{3}x^3\bigg|_{-2}^{0} + \frac{1}{3}x^3\bigg|_{0}^{4}\right) \\
&= \frac{1}{10}\left(-\frac{8}{3} + \frac{64}{3}\right) \\
&= \frac{56}{60} \\
&= \frac{14}{15}.
\end{aligned}$$

Now, let M denote the median of X. Then, we have by definition

$$\begin{aligned}
\frac{1}{2} &= \int_{-\infty}^{M} f(x)dx \\
&= \frac{1}{10}\left(-\int_{-2}^{0} xdx + \int_{0}^{M} xdx\right) \\
&= \frac{1}{10}\left(2 + \frac{M^2}{2}\right) \\
5 &= 2 + \frac{M^2}{2} \\
6 &= M^2 \\
M &= \sqrt(6).
\end{aligned}$$

Problem 3

A fair coin is tossed twice. Let X be the number of heads, and Y the indicator function of the event $\{X = 2\}$. Find the joint probability mass function of (X, Y).

Solution

Due to the small number of possible outcomes, we can easily list them out:

$$\begin{aligned}
\{\mathrm{HH}\} &= (2,1) \\
\{\mathrm{TH}\} &= (1,0) \\
\{\mathrm{HT}\} &= (1,0) \\
\{\mathrm{TT}\} &= (0,0).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
p_{X,Y}(0,0) &= \frac{1}{4} \\
p_{X,Y}(1,0) &= \frac{2}{4} = \frac{1}{2} \\
p_{X,Y}(2,1) &= \frac{1}{4},
\end{aligned}$$

and $p_{X,Y}(x, y) = 0$ otherwise.

Problem 4 (Uncorrelated random variables need not be independent)

Let $X \sim N(0,1)$. Let Y be a discrete random variable independent of X with $\mathbb{P}(Y=1) = \mathbb{P}(Y=-1) = \frac{1}{2}$, and define $Z = XY$. Show that X and Z are uncorrelated but not independent.

Solution

To show that X and Z are dependent, we will show that the probability density of Z is different than the probability of Z given X. So, we start with finding the probability distribution of Z:

$$\begin{aligned}
F_Z(z) &= \mathbb{P}(Z \leq z) \\
&= \mathbb{P}(XY \leq z) \\
&= \mathbb{P}(XY \leq z | Y = 1)\mathbb{P}(Y=1) + \mathbb{P}(XY \leq z | Y=-1)\mathbb{P}(Y=-1) \\
&= \mathbb{P}(X \leq z)\mathbb{P}(Y=1) + \mathbb{P}(-X \leq z)\mathbb{P}(Y=-1) \\
&= \mathbb{P}(X \leq z)\mathbb{P}(Y=1) + \mathbb{P}(X > -z)\mathbb{P}(Y=-1) \\
&= \frac{1}{2}\left(\int_{-\infty}^{z} f_X(x)dx + \int_{-z}^{\infty} f_X(x)dx\right) \\
&= \frac{1}{2\sqrt{2\pi}}\left(\int_{-\infty}^{z} e^{-x^2}dx + \int_{-z}^{\infty} e^{-x^2}dx\right) \\
&= \frac{1}{2\sqrt{2\pi}}\left(\int_{-\infty}^{z} e^{-x^2}dx + \int_{-\infty}^{z} e^{-x^2}dx\right) && \text{By symmetry of gaussian function} \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-x^2}dx \\
&= F_X(z).
\end{aligned}$$

Thus, $Z \sim N(0,1)$. However, we can see

$$\begin{aligned}
f_{Z|X=x}(z) &= \mathbb{P}(Y=1)\delta(z-x) + \mathbb{P}(Y=-1)\delta(z+x) \\
&= \frac{1}{2}\delta(z-x) + \frac{1}{2}\delta(z+x),
\end{aligned}$$

where δ is the Dirac-Delta function. Therefore,z has a different distribution when the value of X is known, and we can conclude that Z and X are not independent.

To show that X and Z are not correlated, we first note that because $X, Z \sim N(0,1)$, we have $\mathrm{E}X = \mathrm{E}Z = 0$. Then,

$$\begin{aligned}
\mathrm{Cov}(X,Z) &= \mathrm{E}(X - \mathrm{E}X)(Z - \mathrm{E}Z) \\
&= \mathrm{E}(XZ) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xz f_{X,Y}(x,y)dydx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 y f_X(x) f_Y(y)dydx && \text{By independence} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 y f_X(x)(\frac{1}{2}\delta(y-1) + \frac{1}{2}\delta(y+1))dydx \\
&= \int_{-\infty}^{\infty}\left(\frac{1}{2}x^2 f_X(x) - \frac{1}{2}x^2 f_X(x)\right)dx && \text{By the sifting property of the Dirac-Delta} \\
&= \int_{-\infty}^{\infty}(0)dx \\
&= 0.
\end{aligned}$$

Thus, we have shown that X and Y are not correlated.