MA 585 (Probability): Take Home Final Exam

Name:____________________________________ Score: _________/100

Due by 2:00p.m. on Tuesday, April 26, 2022

This is an open book, open notes, no electronics exam. Please work out all seven problems by yourself for a maximum of 100 points. Please show your work in a well organized way. No work, no credit.

Honor Pledge: I pledge my honor that I have not violated the Honor Code during this examination. Signature: ______________________

1. (15 points, 5+10) Suppose that a random vector (X, Y) has joint density function

$$f_{X,Y}(x,y) = \begin{cases} c(1-y), & \text{if } 0 < y < x < 1 \\ 0, & \text{otherwise} . \end{cases}$$

(1). Determine the constant c.

(2). Let $U = XY$. Determine $f_{Y|U=u}(y)$, the conditional density function of Y, given $U = u$ for some $0 < u < 1$.

2. (20 points, 4 points for each part) Suppose that a random vector (X, Y) has joint density function

$$f_{X,Y}(x,y) = \begin{cases} cy^2, & \text{if } 0 < 2x < y < 2, \\ 0, & \text{otherwise }. \end{cases}$$

(1). Determine the constant c.

(2). Find $f_X(x)$, the marginal density function of X.

(3). Find $f_{Y|X=x}(y)$, the conditional density function of Y, given $X = x$.

(4). Determine $\mathbb{E}[Y|X]$.

(5). Compute $\mathbb{E}[Y]$.

3. (15 points, 10+5) Suppose we flip a coin independently and repeatedly for n many times. Let X_n be the number of heads we get. Consider the following model

$$X_n | P = p \sim \text{Bin}(n, p) \quad \text{with} \quad P \sim \text{Unif}\,[0.2,\, 0.5]\,.$$

(1). Find $\mathbb{P}(X_3 = 2)$.

(2). Find $f_{P|X_3=2}(p)$ and determine the maximum likelihood estimator for the probability of getting a head in one coin toss.

4. (15 points, 4+4+7) The life time T (hours) of the lightbulb in an overhead projector follows an $\mathrm{Exp}(1/u)$-distribution, given $U = u$, where U is a random variable uniformly distributed on $[0.4, 0.5]$.

(1). Find $\mathbb{E}[T]$ and $\mathrm{Var}(T)$.

(2). Find the density of T.

5. (10 points) Suppose that X is a random variable such that

$$\mathbb{E}[X^k] = \frac{1}{3} + 3^{k-2} + 5^{k-1}, \quad k = 1, 2, \ldots.$$

Determine the distribution of X.

6. (15 points, 5+5+5) Let $0 < q = 1 - p < 1$. Suppose that $X_1, X_2, \ldots$ are independent $\mathrm{Ge}(q)$-distributed random variables and that $N \sim \mathrm{Ge}(p)$ which is independent of X_k's. Define

$$Z = \sum_{k=1}^{N} X_k.$$

Determine the distribution of Z, $\mathbb{E}[Z]$ and $\mathrm{Var}(Z)$.

7. (10 points) Let $X_1, \ldots, X_n$ be n many i.i.d. standard normal random variables. Prove that

$$\sum_{j=1}^{n} X_j^2 \sim \Gamma\left(\frac{n}{2}, 2\right).$$

(Remark: A $\Gamma\left(\frac{n}{2}, 2\right)$ r.v. is also called a χ^2 r.v. with degrees of freedom n, denoted by $\chi^2(n)$, which is one of the most important distributions in statistics.)