

## Problem 1.1

Show that if  $X \in C(0, 1)$ , then so is  $1/X$ .

### Solution

Since  $X \in C(0, 1)$ , we have

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \text{ for } -\infty < x < \infty.$$

Define  $Y = \frac{1}{X}$ . Then,  $X = \frac{1}{Y}$ , and we have

$$|\det(J)| = \left| \frac{dx}{dy} \right| = \frac{1}{y^2}.$$

Thus,

$$\begin{aligned} f_Y(y) &= |\det(J)| f_X(1/y) \\ &= \frac{1}{y^2} \cdot \frac{1}{\pi} \cdot \frac{1}{1+(1/y)^2} \\ &= \frac{1}{\pi} \cdot \frac{1}{1+y^2}, \end{aligned}$$

and we have shown that  $Y \in C(0, 1)$ .

## Problem 1.8

Show that if  $X, Y$  are independent  $N(0, 1)$ -distributed random variables, then  $X/Y \in C(0, 1)$ .

### Solution

We have

$$\begin{aligned} f_{X,Y}(x, y) &= \frac{1}{2\pi} e^{-x^2/2} e^{-y^2/2} \\ &= \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}. \end{aligned}$$

Define  $U = X/Y$ . We introduce the auxiliary variable  $V = X$ . Then we have the inverse relations

$$\begin{aligned} X &= V \\ Y &= V/U. \end{aligned}$$

With this, we can find our Jacobian determinant:

$$\begin{aligned} \det(J) &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \\ &= \begin{vmatrix} 0 & 1 \\ \frac{-v}{u^2} & \frac{1}{u} \end{vmatrix} \\ &= \frac{v}{u^2}. \end{aligned}$$

Thus,

$$\begin{aligned} f_{U,V}(u, v) &= |\det(J)| f_{X,Y}(v, v/u) \\ &= \frac{|v|}{u^2} \cdot \frac{1}{2\pi} e^{-\frac{v^2+(v/u)^2}{2}} \\ &= \frac{|v|}{u^2} \cdot \frac{1}{2\pi} e^{-v^2 \frac{1+(1/u)^2}{2}}. \end{aligned}$$

We want the distribution of  $U$ , so

$$\begin{aligned}
 f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u,v) dv \\
 &= \int_{-\infty}^{\infty} \frac{|v|}{u^2} \cdot \frac{1}{2\pi} e^{-v^2 \frac{1+(1/u)^2}{2}} dv \\
 &= \frac{1}{u^2 \pi} \int_0^{\infty} v \cdot e^{-v^2 \frac{1+(1/u)^2}{2}} dv \\
 &= \frac{1}{\pi} \cdot \frac{1}{1+u^2}.
 \end{aligned}$$

Thus, we have shown that  $U \in C(0,1)$ .

## Problem 1.11

Show that if  $X$  and  $Y$  are independent  $\text{Exp}(a)$ -distributed random variables, then  $X/Y \in F(2,2)$ .

### Solution

We have

$$\begin{aligned}
 f_{X,Y}(x,y) &= \frac{1}{a^2} e^{-x/a} e^{-y/a} \\
 &= \frac{1}{a^2} e^{-\frac{x+y}{a}},
 \end{aligned}$$

for  $0 \leq x, y < \infty$ . Define  $U$  and  $V$  as we did in the previous problem. Then,

$$\begin{aligned}
 f_{U,V}(u,v) &= |\det(J)| f_{X,Y}(v, v/u) \\
 &= \frac{v}{u^2} \cdot \frac{1}{a^2} e^{-\frac{v+v/u}{a}} \\
 &= \frac{v}{u^2} \cdot \frac{1}{a^2} e^{-v \frac{1+1/u}{a}}.
 \end{aligned}$$

Then, we can integrate to find the distribution of  $U$ :

$$\begin{aligned}
 f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u,v) dv \\
 &= \frac{1}{a^2 u^2} \int_0^{\infty} v e^{-v \frac{1+1/u}{a}} dv \\
 &= \frac{1}{(u+1)^2}.
 \end{aligned}$$

After looking up Fisher's distribution, we can conclude that  $X/Y \in F(2,2)$ .

## Problem 1.12

Let  $X, Y$  be independent random variables such that  $X \in U(0,1)$  and  $Y \in U(0,\alpha)$ . Find the density function of  $Z = X + Y$ .

### Solution

This is a test