Problem 3.1

The nonnegative, integer-valued, random variable X has generating function $g_X(t) = \ln(\frac{1}{1-qt})$. Determine $P(X = k)$ for $k = 0, 1, 2, ...$, and determine EX and $\text{Var}X$.

Solution

By definition of a generating function, we have

$$\begin{aligned} g_X(t) &= Et^X \\ &= \sum_{k=0}^{\infty} P(X = k)t^k. \end{aligned}$$

As we derived in class, we have

$$P(X = k) = \frac{g_X^{(k)}(0)}{k!},$$

and

$$EX(X-1)...(X-k+1) = g_X^{(k)}(1). \tag{1}$$

Thus, we should find the derivatives of g:

$$\begin{aligned} g_X^{(1)}(t) &= \frac{q}{1-qt} \\ g_X^{(2)}(t) &= \frac{q^2}{(1-qt)^2} \\ g_X^{(3)}(t) &= \frac{2q^3}{(1-qt)^3} \\ g_X^{(4)}(t) &= \frac{(3)(2)q^4}{(1-qt)^4} \\ &. \\ &. \\ &. \\ g_X^{(k)}(t) &= \frac{(k-1)! \cdot q^k}{(1-qt)^k}. \end{aligned}$$

Thus, we have

$$g_X^{(k)}(0) = (k-1)! \cdot q^k$$

for $k \in \mathbb{Z}^+$, and $g_X(0) = 0$. We also have

$$g_X^{(k)}(1) = \frac{(k-1)! \cdot q^k}{(1-q)^k}$$

for $k \in \mathbb{Z}^+$. With this, we have

$$\begin{aligned} P(X = k) &= \frac{g_X^{(k)}(0)}{k!} \\ &= \frac{(k-1)! \cdot q^k}{k!} \\ &= \frac{q^k}{k}. \end{aligned}$$

Now, normalization allows us to solve for q:

$$\begin{aligned}
1 &= \sum_{k=1}^{\infty} P(X = k) \\
&= \sum_{k=1}^{\infty} \frac{q^k}{k} \\
&= -\ln(1 - q) \text{ (Pulled from my big book of series)} \\
-1 &= \ln(1 - q) \\
e^{-1} &= 1 - q \\
q &= 1 - e^{-1}.
\end{aligned}$$

Thus, we have that

$$P(X = k) = \boxed{\frac{(1 - e^{-1})^k}{k}}.$$

Now, utilizing equation (1),

$$\begin{aligned}
EX &= g_X^{(1)}(1) \\
&= \frac{q}{1 - q} \\
&= \boxed{\frac{1 - e^{-1}}{e^{-1}}} \\
EX(X - 1) &= EX^2 - EX \\
&= g_X^{(2)} \\
&= \frac{(1 - e^{-1})^2}{e^{-2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{Var} X &= EX^2 - (EX)^2 \\
&= g_X^{(2)} + EX - (EX)^2 \\
&= \frac{(1 - e^{-1})^2}{e^{-2}} + \frac{1 - e^{-1}}{e^{-1}} - \left(\frac{1 - e^{-1}}{e^{-1}}\right)^2 \\
&= \boxed{\frac{1 - e^{-1}}{e^{-1}}}.
\end{aligned}$$

Problem 3.4

Suppose that Y is a random variable such that

$$EY^k = \frac{1}{4} + 2^{k-1}, \text{ for } k = 1, 2, ...$$

Determine the distribution of Y.

Solution