Problem 2.8

The density function of the two-dimensional random variable (X, Y) is

$$f_{X,Y}(x, y) = \begin{cases} \frac{x^2}{2y^3} e^{-\frac{x}{y}} & 0 \le x \le \infty, \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the distribution of Y.

(b) Find the conditional distribution of X given that $Y = y$.

(c) Use the results from (a) and (b) to compute EX and $\text{Var}X$

Solution

(a) For $0 \le y \le 1$, we have

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx \\ &= \int_0^{\infty} \frac{x^2}{2y^3} e^{-\frac{x}{y}} dx \\ &= \frac{1}{2y^3} \left(\int_0^{\infty} \left(\frac{\partial}{\partial x} (-yx^2 e^{-\frac{x}{y}}) + 2xye^{-\frac{x}{y}} \right) dx \right) \\ &= \frac{1}{y^2} \int_0^{\infty} xe^{-\frac{x}{y}} dx \\ &= \frac{1}{y} \int_0^{\infty} e^{-\frac{x}{y}} dx \\ &= -e^{-\frac{x}{y}} \Big|_0^{\infty} \\ &= \boxed{1}. \end{aligned}$$

(b) From the definition of conditional distributions, we have

$$\begin{aligned} f_{X|Y=y}(x) &= \frac{f_{X,Y}(x, y)}{f_Y(y)} \\ &= \frac{f_{X,Y}(x, y)}{f_Y(y)} \\ &= f_{X,Y}(x, y) \\ &= \boxed{\begin{cases} \frac{x^2}{2y^3} e^{-\frac{x}{y}} & 0 \le x \le \infty, \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}} \end{aligned}$$

(c)

$$
\begin{aligned}
EX &= \int_0^1 \int_0^\infty x f_{X,Y}(x,y)dxdy \\
&= \int_0^1 \int_0^\infty \frac{x^3}{2y^3} e^{-\frac{x}{y}} dxdy \\
&= \boxed{\frac{3}{2}} && \text{after integration by parts 3 times} \\
EX^2 &= \int_0^1 \int_0^\infty x^2 f_{X,Y}(x,y)dxdy \\
&= \int_0^1 \int_0^\infty \frac{x^4}{2y^3} e^{-\frac{x}{y}} dxdy \\
&= 4 \\
\text{Var}X &= EX^2 - (EX)^2 \\
&= 4 - (\frac{3}{2})^2 \\
&= \frac{16-9}{4} \\
&= \boxed{\frac{7}{4}}
\end{aligned}
$$

Problem 2.9

The density of a random vector $(X,Y)'$ is

$$
f_{X,Y}(x,y) = \begin{cases} cx & x \geq 0, \text{ and } y \geq 0, \text{ and } x+y \leq 1 \\ 0 & \text{otherwise.} \end{cases}
$$

Compute

(a) c,

(b) the conditional expectations $E(Y|X=x)$ and $E(X|Y=y)$.

Solution

(a) Normalization yields

$$
\begin{aligned}
1 &= \int_0^\infty \int_0^\infty f_{X,Y}(x,y)dxdy \\
&= \int_0^1 \int_0^{1-y} cxdxdy \\
&= \int_0^1 \frac{c}{2}(1-y)^2 dy \\
&= \int_0^1 \frac{c}{2}(y^2-2y+1)dy \\
&= \frac{c}{2}(\frac{1}{3}y^3 - y^2 + y)\Big|_0^1 \\
&= \frac{c}{6} \\
c &= \boxed{6}.
\end{aligned}
$$

(b) In order to calculate $E(Y|X=x)$ and $E(X|Y=y)$, we will need to first find $f_{Y|X=x}(y)$ and $f_{X|Y=y}(x)$. To find these, we will first need to compute our marginal probability density functions. For $x \in [0,1]$, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \\ &= \int_0^{1-x} 6xdy \\ &= 6x(1-x) \end{aligned}$$

Likewise, for $y \in [0,1]$, we have

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \\ &= \int_0^{1-y} 6xdx \\ &= 3(1-y)^2. \end{aligned}$$

With this, we have

$$\begin{aligned} f_{X|Y=y}(x) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{6x}{3(1-y)^2} \\ &= \frac{2x}{(1-y)^2} \end{aligned}$$

and

$$\begin{aligned} f_{Y|X=x}(y) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{6x}{6x(1-x)} \\ &= \frac{1}{1-x}, \end{aligned}$$

on the support of $f_{X,Y}(x,y)$. Finally, we have

$$\begin{aligned} E(X|Y=y) &= \int_{-\infty}^{\infty} x f_{X|Y=y}(x)dx \\ &= \int_0^{1-y} \frac{2x^2}{(1-y)^2}dx \\ &= \frac{2(1-y)^3}{3(1-y)^2} \\ &= \boxed{\frac{2}{3}(1-y)}, \end{aligned}$$

and

$$\begin{aligned} E(Y|X=x) &= \int_{-\infty}^{\infty} y f_{Y|X=x}(y)dy \\ &= \int_0^{1-x} \frac{y}{1-x}dy \\ &= \frac{(1-x)^2}{2(1-x)} \\ &= \boxed{\frac{1}{2}(1-x)}. \end{aligned}$$

Problem 2.33

Suppose that the random variable X is uniformly distributed symmetrically around zero, but in such a way that the parameter is uniform on $(0, 1)$; that is suppose that

$$X|A = a \in U(-a, a) \text{ with } A \in U(0, 1).$$

Find the distribution of X, EX, and $\text{Var}X$.

Solution

We have

$$f_{X|A=a}(x) = \frac{f_{X,A}(x, a)}{f_A(a)} \implies f_{X,A}(x, a) = f_{X|A=a}(x) \cdot f_A(a),$$

thus

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,A}(x, a)da \\ &= \int_{-\infty}^{\infty} f_{X|A=a}(x) \cdot f_A(a)da. \end{aligned}$$

At this point, in order to correctly integrate over the support of $f_{X,A}(x, a)$, we will break this into case 1 where $-1 \leq x \leq 0$, and case 2 where $0 \leq x \leq 1$.

(1) Satisfying the conditions of our support, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,A}(x, a)da \\ &= \int_{-x}^{1} \frac{1}{2a}da \\ &= -\frac{1}{2}\ln(-x). \end{aligned}$$

(2) Satisfying the conditions of our support once more, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,A}(x, a)da \\ &= \int_{x}^{1} \frac{1}{2a}da \\ &= -\frac{1}{2}\ln(x). \end{aligned}$$

We can succinctly write this as

$$\boxed{f_X(x) = \frac{1}{2}\ln|x|},$$

for $-1 \leq x \leq 1$.

Now, since we can see that $f_X(x)$ is even, we have that $xf_X(x)$ is odd, and we can automatically conclude that $\boxed{EX = 0}$. Thus, we have

$$\begin{aligned} \text{Var}X &= EX^2 - (EX)^2 \\ &= EX^2. \end{aligned}$$

Now computing EX^2 directly with $f_X(x)$ requires some moderately difficult integration, so we will instead use our helper variable:

$$\begin{aligned}
EX^2 &= E_A(E_{X|A=a}(X)) \\
&= \int_0^1 f_A(a) \cdot (E_{X|A=a}(X))da \\
&= \int_0^1 f_A(a) \cdot \left(\int_{-a}^a x^2 \cdot f_{X|A=a}(x)dx \right) da \\
&= \int_0^1 \int_{-a}^a \frac{x^2}{2a} dxda \\
&= \int_0^1 \int_0^a \frac{x^2}{a} dxda \\
&= \int_0^1 \frac{a^2}{3} da \\
&= \boxed{\frac{1}{9}}.
\end{aligned}$$

Problem 2.34

Consider the following model:

$$X_n|Y = y \in \text{Bin}(n, y) \text{ with } f_Y(y) = 6y(1-y),\ 0 < y < 1.$$

(a) Compute EX_n and $\text{Var}X_n$.

(b) Determine the distribution of X_n.

Solution

(a) Since $X_n|Y = y \in \text{Bin}(n, y)$, we have

$$p_{X_n|Y=y}(k) = \binom{n}{k} y^k (1-y)^{n-k} \text{ for } k = 0, 1, ..., n.$$

Thus,

$$\begin{aligned}
EX_n &= E_Y(E_{X_n|Y=y}(X_n)) \\
&= \int_0^1 \left(\sum_{k=0}^n k \cdot p_{X_n|Y=y}(k) \right) \cdot f_Y(y)dy \\
&= \int_0^1 \left(\sum_{k=0}^n k \binom{n}{k} y^k (1-y)^{n-k} \right) 6y(1-y)dy \\
&= \int_0^1 \left(\sum_{k=0}^n 6k \binom{n}{k} y^{k+1} (1-y)^{n-k+1} \right) dy \\
&= -\int_0^1 6n(y-1)y^2 dy, \text{ found by entering into wolfram alpha} \\
&= -6n \int_0^1 (y^3 - y^2)dy \\
&= -6n(\frac{1}{4} - \frac{1}{3}) \\
&= 6n(\frac{4}{12} - \frac{3}{12}) \\
&= \boxed{\frac{n}{2}}.
\end{aligned}$$

To find our variance, we compute EX^2 in a very similar manner

$$\begin{aligned}
EX^2 &= \int_0^1 \left(\sum_{k=0}^{n} 6k^2 \binom{n}{k} y^{k+1}(1-y)^{n-k+1} \right) dy \\
&= -6 \int_0^1 (y-1)y^2((n-1)ny+n)dy \\
&= \frac{1}{10} n(3n+2).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{Var} X &= EX^2 - (EX)^2 \\
&= \frac{1}{10} n(3n+2) - \left(\frac{n}{2}\right)^2 \\
&= \boxed{\frac{n^2}{20} + \frac{n}{5}}
\end{aligned}$$