MA 585 (Probability): Take Home Final Exam

Name: Jeremiah Givens Score: ______ /100

Due by 2:00p.m. on Tuesday, April 26, 2022

This is an open book, open notes, no electronics exam. Please work out all seven problems by yourself for a maximum of 100 points. Please show your work in a well organized way. No work, no credit.

Honor Pledge: I pledge my honor that I have not violated the Honor Code during this examination. Signature: Jeremiah Givens

1. (15 points, 5+10) Suppose that a random vector (X, Y) has joint density function

$$f_{X,Y}(x,y) = \begin{cases} c(1-y), & \text{if } 0 < y < x < 1 \\ 0, & \text{otherwise .} \end{cases}$$

(1). Determine the constant c.

(2). Let $U = XY$. Determine $f_{Y|U=u}(y)$, the conditional density function of Y, given $U = u$ for some $0 < u < 1$.

(1) Normalize: $1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$

$= \int_0^1 \int_0^x c(1-y)\,dy\,dx = c\int_0^1 (y - \frac{1}{2}y^2)\Big|_0^x\,dx$

$= c\int_0^1 (x - \frac{1}{2}x^2)\,dx = c\left(\frac{1}{2}x^2 - \frac{1}{6}x^3\right)\Big|_0^1$

$= c\left(\frac{1}{2} - \frac{1}{6}\right) = c\left(\frac{3}{6} - \frac{1}{6}\right) = \frac{c}{3} \Rightarrow \boxed{c = 3}$

(2) Let $U = XY$ and $V = Y$. Then, $X = \frac{U}{V}$, and $Y = V$.
Then, $|\det(J)| = \left|\left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{matrix}\right|\right| = \left|\left|\begin{matrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1\end{matrix}\right|\right| = \frac{1}{v}$.

$f_{U,V}(u,v) = f_{X,Y}(\frac{u}{v}, v)\cdot\frac{1}{v}$, for $0 \le v \le \frac{u}{v} < 1$.
Now we integrate out v to find $f_U(u)$:
$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv = 3\int_u^{\sqrt{u}} (1-v)\frac{1}{v}\,dv = 3\int_u^{\sqrt{u}} (\frac{1}{v} - 1)\,dv$

$= 3\ln\left(\frac{\sqrt{u}}{u}\right) - 3\sqrt{u} + 3u = 3u - 3\sqrt{u} - \frac{3}{2}\ln(u)$.

M1.2 Continued...

Then, we have

$$f_{Y|U=u}(y) = \frac{f_{U,Y}(u,y)}{f_U(u)} = \frac{3\left(\frac{1}{y}-1\right)}{3\left(u-\sqrt{u}-\frac{1}{2}\ln(u)\right)}$$ for $0<y\leq \frac{u}{y}<1$.

$$= \boxed{\frac{\frac{1}{y}-1}{u-\sqrt{u}-\frac{1}{2}\ln(u)}}$$

2. (20 points, 4 points for each part) Suppose that a random vector (X, Y) has joint density function

$$f_{X,Y}(x,y) = \begin{cases} cy^2, & \text{if } 0 < 2x < y < 2, \\ 0, & \text{otherwise} . \end{cases}$$

(1). Determine the constant c.

(2). Find $f_X(x)$, the marginal density function of X.

(3). Find $f_{Y|X=x}(y)$, the conditional density function of Y, given $X = x$.

(4). Determine $\mathbf{E}[Y|X]$.

(5). Compute $\mathbf{E}[Y]$.

(1) $1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = c\int_0^2\int_0^{\frac{y}{2}} y^2\,dx\,dy$

$= c\int_0^2 \frac{y^3}{2}\,dy = c\,\frac{1}{8}y^4\Big|_0^2 = 2c \Rightarrow \boxed{c = \frac{1}{2}}$

(2) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \frac{1}{2}\int_{2x}^{2} y^2\,dy = \frac{1}{6}y^3\Big|_{2x}^{2}$

$= \frac{8}{6} - \frac{8x^3}{6} = \boxed{\frac{4}{3}(1-x^3)}$ for $0 < x \le 1$.

(3) $f_{Y|X=x}(y) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{\frac{1}{2}y^2}{\frac{4}{3}(1-x^3)} = \boxed{\frac{3}{8}\cdot\frac{y^2}{(1-x^3)}}$, for $2x \le y < 2$

(4) $E(Y|X) = \int_{2x}^{2} f_{Y|X=x}(y)\,dy = \frac{3}{8}\int_{2x}^{2}\frac{y^2}{1-x^3}\,dy$

$= \frac{1}{8}\,\frac{y^3}{(1-x^3)}\Big|_{2x}^{2} = \frac{1}{8}\cdot\frac{8-8x^3}{1-x^3} = \boxed{1}$.

(5) $E(Y) = E(E(Y|X)) = \int_0^1 dx = \boxed{1}$

3. (15 points, 10+5) Suppose we flip a coin independently and repeatedly for n many times. Let X_n be the number of heads we get. Consider the following model

$$X_n | P = p \sim \text{Bin}(n, p) \quad \text{with} \quad P \sim \text{Unif}[0.2, 0.5].$$

(1). Find $\mathbb{P}(X_3 = 2)$.

(2). Find $f_{P|X_3=2}(p)$ and determine the maximum likelihood estimator for the probability of getting a head in one coin toss.

(1) We have $p_{X_n|P=p}(k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0, \ldots, n$.

Now, $p_{X_3|P=p}(k) = \binom{3}{k} p^k (1-p)^{3-k} \Rightarrow p_{X_3|P=p}(2) = \binom{3}{2} p^2 (1-p)^1$

$= 3(p^2 - p^3)$. Integrating against $f_P(p)$, we have

$$\mathbb{P}(X_3 = 2) = \frac{3}{0.3} \int_{0.2}^{0.5} (p^2 - p^3)\, dp = 10\left(\tfrac{1}{3}p^3 - \tfrac{1}{4}p^4\right)\Big|_{0.2}^{0.5}$$

$$= 10\left(\tfrac{1}{3}\left(\left(\tfrac{1}{2}\right)^3 - \left(\tfrac{1}{5}\right)^3\right) - \tfrac{1}{4}\left(\left(\tfrac{1}{2}\right)^4 - \left(\tfrac{1}{5}\right)^4\right)\right)$$

$$= \boxed{\frac{951}{4000} \approx 0.23775}$$

(2) We have

$$p_{X_3}(k) = \int_{-\infty}^{\infty} p_{X_3|P=p}(k) \cdot f_P(p)\, dp.$$

We already found this for $k = 2$, so we just need to find it for $k = 0, 1, 3$:

$$p_{X_3}(0) = \frac{10}{3} \int_{0.2}^{0.5} (1-p)^3\, dp = \boxed{0.28925}$$

$$p_{X_3}(1) = \frac{10}{3}(3) \int_{0.2}^{0.5} p(1-p)^2\, dp = \boxed{0.42225}$$

$$p_{X_3}(3) = 1 - p_{X_3}(0) - p_{X_3}(1) - p_{X_3}(2) = \boxed{0.05075}$$

3.2 Continued...

Finally,

$$f_{P|X_3=2}(p) = \frac{f_P(p)\cdot p_{X_3|P=p}(2)}{p_{X_3}(2)} = \frac{\frac{10}{3}\left(3(p^2-p^3)\right)}{\frac{951}{4000}}$$

$$= \boxed{\frac{4\times10^4}{951}(p^2-p^3)} \quad \text{for} \quad 0.2 \le p \le 0.5$$

We will have the maximum likelyhood of flipping heads in one flip when $p=0.5$.

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4. (15 points, 4+4+7) The life time T (hours) of the lightbulb in an overhead projector follows an Exp$(1/u)$-distribution, given $U = u$, where U is a random variable uniformly distributed on $[0.4, 0.5]$.

(1). Find $\mathbb{E}[T]$ and $\text{Var}(T)$.

(2). Find the density of T.

$$f_{T|U=u}(t) = u e^{-ux}, \text{ for } x > 0.$$

I'm going to start with (2):

(2) $$f_{T,U}(t,u) = f_U(u)\, f_{T|U=u}(t)$$

$$f_T(t) = \frac{1}{0.1}\int_{0.4}^{0.5} u e^{-ut}\, du = -10\, \frac{e^{-ut}(ut+1)}{t^2}\Big|_{0.4}^{0.5}$$

$$= -10\left(\frac{e^{-0.5t}(0.5t+1)}{t^2} - \frac{e^{-0.4t}(0.4t+1)}{t^2}\right)$$

$$= \boxed{\frac{10}{t^2}\left(e^{-0.4t}(0.4t+1) - e^{-0.5t}(0.5t+1)\right)}$$

Now I see why we were supposed to find that after... I do not want to integrate that. So:

(1) $$ET = E\,E(T|U) = \int_{0.4}^{0.5} 10\left(\int_0^\infty t u e^{-ut}\, dt\right) du$$

$$= \int_{0.4}^{0.5} 10\left(\frac{1}{u}\right) du = 10 \ln\left(\frac{0.5}{0.4}\right) = \boxed{10 \ln\left(\frac{5}{4}\right)}$$

$$ET^2 = E(T^2|U) = \int_{0.4}^{0.5} 10\left(\int_0^\infty t^2 u e^{-ut}\, dt\right) du$$

$$= \int_{0.4}^{0.5} \frac{20}{u^2}\, du = -\frac{20}{u}\Big|_{0.4}^{0.5} = -40 + \frac{200}{4} = 10$$

$$\text{Var}\, T = ET^2 - (ET)^2 = \boxed{10 - 100\left(\ln\left(\frac{5}{4}\right)\right)^2}.$$

5. (10 points) Suppose that X is a random variable such that

$$\mathbf{E}[X^k] = \frac{1}{3} + 3^{k-2} + 5^{k-1}, \quad k = 1, 2, \ldots.$$

Determine the distribution of X.

We have, $\psi_X(t) = Ee^{tX}$. Thus,

$$\psi_X(t) = E\sum_{n=0}^{\infty} \frac{X^n}{n!} t^n = 1 + \sum_{k=1}^{\infty} \frac{EX^k}{k!} t^n$$

$$= 1 + \sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{1}{3} + 3^{k-2} + 5^{k-1}\right) t^k$$

$$= 1 + \frac{1}{45}\left(-29 + 15e^t + 5e^{3t} + 9e^{5t}\right)$$

$$= 1 + \left(\frac{-29}{45}\right) + \frac{1}{3}e^t + \frac{1}{9}e^{3t} + \frac{1}{5}e^{5t}$$

$$= \frac{16}{45}e^{(0)t} + \frac{1}{3}e^{(1)t} + 0\cdot e^{2t} + \frac{1}{9}e^{3t} + 0\cdot e^{4t} + \frac{1}{5}e^{5t}.$$

Thus, We have $p_X(0) = \frac{16}{45}$, $p_X(1) = \frac{1}{3}$, $p_X(3) = \frac{1}{9}$, $p_X(5) = \frac{1}{5}$, and $p_X(k) = 0$ for all other k.

6. (15 points, 5+5+5) Let $0 < q = 1 - p < 1$. Suppose that $X_1, X_2, \dots$ are independent Ge(q)-distributed random variables and that $N \sim$ Ge(p) which is independent of X_k's. Define

$$Z = \sum_{k=1}^{N} X_k.$$

Determine the distribution of Z, $\mathrm{E}[Z]$ and $\mathrm{Var}(Z)$.

We have $g_X(t) = \sum_{k=0}^{\infty} t^k q(1-q)^k = \frac{q}{1-pt}$,

and $g_N(t) = \sum_{k=0}^{\infty} t^k p(1-p)^k = \frac{p}{1-qt}$.

From theorem 3.6.1, we have

$$g_Z(t) = g_N(g_X(t)) = \frac{p}{1-q\left(\frac{q}{1-pt}\right)} = \frac{p}{\frac{1-pt-q^2}{1-pt}}$$

$$= \frac{p(1-pt)}{1-pt-q^2} = \frac{p(1-pt)}{1-pt-(1-p)^2} = \frac{p(1-pt)}{1-1+2p-pt+p^2}$$

$$= \frac{p(1-pt)}{p(2-t+p)} = \frac{1-pt}{2-t+p} = (1-pt)(2-t+p)^{-1}$$

Now we find our derivatives:

$$g_Z^{(1)}(t) = -p(2-t+p)^{-1} + \frac{(2+p)(1-pt)}{(2-t+p)^2}$$

$$= \frac{-2p+tp-p^2+2+p-2pt-p^2t}{(2-t+p)^2}$$

$$= \frac{-p-pt-p^2(1-t)+2}{(2-t+p)^2} = \frac{(t-1)(p^2-p)+2}{(2-t+p)^2}$$

$$g_Z^{(2)}(t) = \frac{p^2-p}{(2-t+p)^2} + \frac{2(t-1)(p^2-p)+4}{(2-t+p)^3}$$

Plugging in $t=0$, we get:

$$g_Z^{1}(0) = \frac{p-p^2+2}{(2+p)^2}, \quad \text{and} \quad g_Z^{2}(0) = \frac{p^2-p}{(2+p)^2} + \frac{2p-4p^2+4}{(2+p)^3}$$

6.) Continued :

$$g_z^{2}(0) = \frac{p^2 - 2p + p^3 + 2p - 4p^2 + 4}{(2+p)^3}$$

$$= \frac{p^3 - 3p^2 + 4}{(2+p)^3}$$

$$g_z^{(0)}(0) = \frac{1}{2+p}$$

I'm not seeing a general pattern here, but the probability mass func can be found by continuing in this manner and plugging into

$$p_Z(k) = \frac{g_Z^{(k)}(0)}{k!}$$

We now also have the info to find EZ and $\text{Var } Z$:

$$EZ = g_Z^{(0)}(1) = \frac{1-p}{1+p} = \boxed{\frac{q}{1+p}}$$

$$EZ(Z+1) = EZ^2 + EZ = g_Z^{(1)}(1) \Rightarrow EZ^2 = g_Z^{(1)}(1) - EZ$$

$$\text{Var}(Z) = EZ^2 - (EZ)^2 = g_Z^{(1)}(1) - (EZ)^2 - EZ$$

$$= \frac{2}{(1+p)^2} - \frac{q^2}{(1+p)^2} - \frac{q}{(1+p)} = \frac{2 - q^2 - q - qp}{(1+p)^2}$$

$$= \frac{2 - (1-p)^2 - (1-p) - (1-p)(1+p)}{(1+p)^2}$$

$$= \frac{2 - 1 - p^2 + 2p - 1 + p - 1 + p^2}{(1+p)^2}$$

$$= \boxed{\frac{3p-1}{(1+p)^2}}$$

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7. (10 points) Let $X_1, \ldots, X_n$ be n many i.i.d. standard normal random variables. Prove that

$$\sum_{j=1}^{n} X_j^2 \sim \Gamma\left(\frac{n}{2}, 2\right).$$

(Remark: A $\Gamma\left(\frac{n}{2}, 2\right)$ r.v. is also called a χ^2 r.v. with degrees of freedom n, denoted by $\chi^2(n)$, which is one of the most important distributions in statistics.)

Let $S_n = \sum_{j=1}^{n} X_j^2$, and let $L \sim \Gamma\left(\frac{n}{2}, 2\right)$.

By theorem 3.1, it will suffice to show $M_{S_n}(t) = M_L(t)$, for $|t| < h$ for some $h > 0$.

By example in ~~out~~ our text book, we already have

$$M_L(t) = \frac{1}{(1-2t)^{n/2}}, \quad \text{for } t < \frac{1}{2}.$$

We also have

$$M_X(t) = e^{t^2/2}, \quad \text{for } -\infty < t < \infty.$$

However, we need $M_{X^2}(t)$. First, we find the distribution of X^2:

Set $U = X^2$. Then, ~~since~~ for $X > 0$, $X = \sqrt{U}$.

$$\frac{\partial X}{\partial U} = \frac{1}{2\sqrt{U}}, \quad f_{U^+}(u) = \frac{1}{2\sqrt{u}} f_X(\sqrt{u})$$

$$= \frac{1}{2\sqrt{2\pi u}} e^{-u/2} \quad \text{for } u > 0.$$

For $x < 0$, we have $x = -\sqrt{u}$. Then, $|\det(J)| = \frac{1}{2\sqrt{u}}$, and $f_{U^-}(u) = \frac{1}{2\sqrt{u}} f_X(-\sqrt{u}) = \frac{1}{2\sqrt{2\pi u}} e^{-u/2}$.

Then,

$$f_U(u) = f_{U^-}(u) + f_{U^+}(u) = \frac{1}{\sqrt{2\pi u}} e^{-u/2}$$

Now we find $M_U(t)$:

$$M_U(t) = E e^{Ut} = \int_0^\infty \frac{1}{\sqrt{2\pi u}} e^{ut} e^{-u/2} \, du$$

7.) Continued . . .

$$\Psi_u(t) = \int_0^\infty \frac{1}{\sqrt{2\pi u}} e^{u(t-\frac{1}{2})} du = \frac{1}{\sqrt{1-2t}}, \text{ for } t < \frac{1}{2}.$$

Then, using theorem 3.3.2, we have

$$\Psi_{S_n}(t) = \prod_{j=1}^{n} \Psi_{x_j^2}(t) = \prod_{j=1}^{n} \frac{1}{\sqrt{1-2t}}$$

$$= \left(\frac{1}{\sqrt{1-2t}}\right)^n = \boxed{\frac{1}{(1-2t)^{n/2}}}$$

Thus, our job is done.