Problem 1

Suppose one is rolling a six-sided die. Let $\mathcal{F}$ the smallest σ-field containing $A = \{5, 6\}$ and $B = \{2, 4, 6\}$. Prove that $\{5\}$, $\{1, 3\}$ and $\{1, 2, 3, 4\}$ are events for the measurable space $(\Omega, \mathcal{F})$.

Solution

Proof. Since $\mathcal{F}$ is a σ-algebra, we have that $\mathcal{F}$ is closed under complements and countable unions (and countable intersections, by D'Morgan's Law). In addition, $\mathcal{F}$ contains $\Omega = \{1, ..., 6\}$, and the empty set $\emptyset$. Thus, if we can show that each of the three sets in the problem can be written as a finite union and/or intersection of A, B and their complements, then our proof will be complete.

We have

$$\begin{aligned} B^c \cap (A \cup B) &= \{1, 3, 5\} \cap \{2, 4, 5, 6\} \\ &= \{5\}, \end{aligned}$$

and

$$\begin{aligned} B^c \cap A^c &= \{1, 3, 5\} \cap \{1, ..., 4\} \\ &\{1, 3\}, \end{aligned}$$

and

$$A^c = \{1, 2, 3, 4\}.$$

With this, our proof is complete. □

Problem 2

Let A and B be a pair of independent events. Prove that A^c and B^c are independent.

Solution

Proof. We can see

$$\begin{aligned} P(A^c \cap B^c) &= P((A \cup B)^c) && \text{D'Morgan's Law} \\ &= 1 - P(A \cup B) \\ &= 1 - (P(A) + P(B) - P(A \cap B)) && \text{We derived this in class} \\ &= 1 - (P(A) + P(B) - P(A)P(B)) && \text{By independence of } A, B \\ &= 1 - (P(A) + P(B)(1 - P(A)) \\ &= 1 - (P(A) + P(B)P(A^c)) \\ &= 1 - P(A) - P(B)P(A^c) \\ &= P(A^c) - P(B)P(A^c) \\ &= P(A^c)(1 - P(B)) \\ &= P(A^c)P(B^c), \end{aligned}$$

and we can conclude that A^c and B^c are independent. □

Problem 3

Consider a coin-die experiment: One flips a fair coin at first. If he gets a head, then he will roll a 6-sided fair die; otherwise, he will roll a 4-sided unfair die, which has probability $\frac{5-i}{10}$ to get the i^{th} face up, where $i \in \{1, ..., 4\}$. If one gets a 2 face up, what is the probability that they got a tail when they flipped the coin?

Solution

Let us denote the result of the coin flip as X. Then, either $X = H$ or $X = T$ for a given experiment. Let us denote the result of the die roll by the random variable Y, defined in the obvious way. Let $A = \{X = T\}$ and $B = \{Y = 2\}$ be two events. By definition of conditional probability, we have

$$P(A|B) = \frac{P(A \cup B)}{P(B)}, \text{ and } P(B|A) = \frac{P(A \cup B)}{P(A)} \implies P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

Now, since $\{X = H\}$ and $\{X = T\}$ partitions our sample space, we have that

$$P(B) = P(B \cap \{X = H\}) + P(B \cap \{X = T\}) = P(B|X = H)P(X = H) + P(B|X = T)P(X = T).$$

Thus, the equation of interest becomes

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|X = H)P(X = H) + P(B|X = T)P(X = T)}. \tag{1}$$

The probabilities involved are all essentially given in the problem statement:

$$\begin{aligned}
&P(A) = \frac{1}{2} = P(X = T) = P(X = H) && \text{because we flip a fair coin} \\
&P(B|X = H) = P(Y = 2|X = H) = \frac{1}{6} && \text{because we roll a fair die} \\
&P(B|A) = P(B|X = T) = P(Y = 2|X = T) = \frac{5-2}{10} = \frac{3}{10} && \text{because flip our weighted die}
\end{aligned}$$

Plugging these into (1), we have

$$\begin{aligned}
P(A|B) &= \frac{\frac{3}{10}\frac{1}{2}}{\frac{1}{6}\frac{1}{2} + \frac{3}{10}\frac{1}{2}} \\
&= \frac{\frac{3}{20}}{\frac{1}{12} + \frac{3}{20}} \\
&= \frac{3}{\frac{5+9}{3}} \\
&= \frac{9}{14}.
\end{aligned}$$

For a sanity check, the code for our numerical simulation can be found HERE, which supports our conclusion.