Problem 3.1

The nonnegative, integer-valued, random variable X has generating function $g_X(t) = \ln(\frac{1}{1-qt})$. Determine $P(X = k)$ for $k = 0, 1, 2, ...$, and determine EX and $\text{Var}X$.

Solution

By definition of a generating function, we have

$$\begin{aligned} g_X(t) &= Et^X \\ &= \sum_{k=0}^{\infty} P(X = k)t^k. \end{aligned}$$

As we derived in class, we have

$$P(X = k) = \frac{g_X^{(k)}(0)}{k!},$$

and

$$EX(X-1)...(X-k+1) = g_X^{(k)}(1). \tag{1}$$

Thus, we should find the derivatives of g:

$$\begin{aligned} g_X^{(1)}(t) &= \frac{q}{1-qt} \\ g_X^{(2)}(t) &= \frac{q^2}{(1-qt)^2} \\ g_X^{(3)}(t) &= \frac{2q^3}{(1-qt)^3} \\ g_X^{(4)}(t) &= \frac{(3)(2)q^4}{(1-qt)^4} \\ &. \\ &. \\ &. \\ g_X^{(k)}(t) &= \frac{(k-1)! \cdot q^k}{(1-qt)^k}. \end{aligned}$$

Thus, we have

$$g_X^{(k)}(0) = (k-1)! \cdot q^k$$

for $k \in \mathbb{Z}^+$, and $g_X(0) = 0$. We also have

$$g_X^{(k)}(1) = \frac{(k-1)! \cdot q^k}{(1-q)^k}$$

for $k \in \mathbb{Z}^+$. With this, we have

$$\begin{aligned} P(X = k) &= \frac{g_X^{(k)}(0)}{k!} \\ &= \frac{(k-1)! \cdot q^k}{k!} \\ &= \frac{q^k}{k}. \end{aligned}$$

Now, normalization allows us to solve for q:

$$\begin{aligned}
1 &= \sum_{k=1}^{\infty} P(X=k) \\
&= \sum_{k=1}^{\infty} \frac{q^k}{k} \\
&= -\ln(1-q) \text{ (Pulled from my big book of series)} \\
-1 &= \ln(1-q) \\
e^{-1} &= 1-q \\
q &= 1-e^{-1}.
\end{aligned}$$

Thus, we have that

$$P(X=k) = \boxed{\frac{(1-e^{-1})^k}{k}}.$$

Now, utilizing equation (1),

$$\begin{aligned}
EX &= g_X^{(1)}(1) \\
&= \frac{q}{1-q} \\
&= \boxed{\frac{1-e^{-1}}{e^{-1}}} \\
EX(X-1) &= EX^2 - EX \\
&= g_X^{(2)} \\
&= \frac{(1-e^{-1})^2}{e^{-2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{Var}X &= EX^2 - (EX)^2 \\
&= g_X^{(2)} + EX - (EX)^2 \\
&= \frac{(1-e^{-1})^2}{e^{-2}} + \frac{1-e^{-1}}{e^{-1}} - \left(\frac{1-e^{-1}}{e^{-1}}\right)^2 \\
&= \boxed{\frac{1-e^{-1}}{e^{-1}}}.
\end{aligned}$$

Problem 3.4

Suppose that Y is a random variable such that

$$EY^k = \frac{1}{4} + 2^{k-1}, \text{ for } k = 1, 2, ...$$

Determine the distribution of Y.

Solution

By definition of the moment generating function, we have

$$\psi_Y(t) = Ee^{tY} \tag{2}$$

$$= \sum_{k=0}^{\infty} \mathbb{P}(Y=k) \cdot e^{tk} \qquad \text{for discrete } Y \in \mathbb{N} \tag{3}$$

Now, taking advantage of the Taylor series of the exponential function (also sometimes called the definition of the exponential function), and the linearity of expectation, we have

$$\begin{aligned}
\psi_Y(t) &= Ee^{tY} \\
&= E\sum_{k=0}^{\infty}\frac{Y^k}{k!}t^k \\
&= 1+\sum_{k=1}^{\infty}\frac{EY^k}{k!}t^k \\
&= 1+\sum_{k=1}^{\infty}\frac{\frac{1}{4}+2^{k-1}}{k!}t^k \\
&= 1+\frac{-3}{4}+\frac{1}{4}e^t+\frac{1}{2}e^{2t} && \text{Result from wolfram alpha} \\
&= \frac{1}{4}+\frac{1}{4}e^t+\frac{1}{2}e^{2t}.
\end{aligned}$$

Combining this with the result from equation (3), we have

$$\begin{aligned}
\mathbb{P}(Y=0) &= \boxed{\frac{1}{4}} \\
\mathbb{P}(Y=1) &= \boxed{\frac{1}{4}} \\
\mathbb{P}(Y=2) &= \boxed{\frac{1}{2}}
\end{aligned}$$

Problem 3.6

Show, by using the moment generating functions, that if $X \in L(1)$, then $X \stackrel{\mathrm{d}}{=} Y_1 - Y_2$, where Y_1 and Y_2 are independent, exponentially distributed random variables.

Solution

First, we compute the moment generating function of X:

$$\begin{aligned}
\psi_X(t) &= \int_{-\infty}^{\infty} f_X(x) \cdot e^{tx} dx \\
&= \int_{-\infty}^{\infty} f_X(x) \cdot e^{tx} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{2} e^{-|x|} \cdot e^{tx} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{2} \cdot e^{tx-|x|} dx \\
&= \int_{-\infty}^{0} \frac{1}{2} \cdot e^{tx+x} dx + \int_{0}^{\infty} \frac{1}{2} \cdot e^{tx-x} dx \\
&= \frac{1}{2} \int_{-\infty}^{0} e^{x(t+1)} dx + \frac{1}{2} \int_{0}^{\infty} e^{x(t-1)} dx \\
&= \frac{1}{2} \left(\frac{1}{t+1} e^{x(t+1)} \Big|_{-\infty}^{0} + \frac{1}{t-1} e^{x(t-1)} \Big|_{0}^{\infty} \right) \\
&= \frac{1}{2} \left(\frac{1}{t+1} - \frac{1}{t-1} \right) \text{ for } |t| < 1 \\
&= \frac{1}{2} \left(\frac{t-1-t-1}{(t-1)(t+1)} \right) \text{ for } |t| < 1 \\
&= \frac{1}{(1-t)(t+1)} \text{ for } |t| < 1 \\
&= \frac{1}{1-t^2} \text{ for } |t| < 1.
\end{aligned}$$

Now, we must compute the moment generating function of Y_1 and $-Y_2$:

$$\begin{aligned}
\psi_{Y_1}(t) &= \int_{-\infty}^{\infty} f_{Y_1}(y) \cdot e^{ty} dy \\
&= \int_{0}^{\infty} e^{-y} \cdot e^{ty} dy \\
&= \int_{0}^{\infty} e^{y(t-1)} dy \\
&= \frac{1}{t-1} e^{y(t-1)} \Big|_{0}^{\infty} \\
&= \frac{1}{t-1} \text{ for } |t| < 1,
\end{aligned}$$

and

$$\begin{aligned}
\psi_{-Y_2}(t) &= \int_{-\infty}^{\infty} f_{-Y_2}(y) \cdot e^{ty} dy \\
&= \int_{-\infty}^{0} e^{y} \cdot e^{ty} dy \\
&= \int_{-\infty}^{0} e^{y(t+1)} dy \\
&= \frac{1}{t+1} \text{ for } |t| < 1.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\psi_{Y_1-Y_2}(t) &= \psi_{Y_1}(t)\psi_{-Y_2}(t) \\
&= \frac{1}{t-1} \cdot \frac{1}{t+1} \\
&= \frac{1}{1-t^2} \\
&= \psi_X(t).
\end{aligned}$$

Thus, by Theorem 3.1, we have shown that $X \overset{\mathrm{d}}{=} Y_1 - Y_2$.