Problem 2.8

The density function of the two-dimensional random variable (X, Y) is

$$f_{X,Y}(x,y) = \begin{cases} \frac{x^2}{2y^3}e^{-\frac{x}{y}} & 0 \leq x \leq \infty, \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the distribution of Y.

(b) Find the conditional distribution of X given that $Y = y$.

(c) Use the results from (a) and (b) to compute EX and $\text{Var}X$

Solution

(a) For $0 \leq y \leq 1$, we have

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \\ &= \int_0^{\infty} \frac{x^2}{2y^3}e^{-\frac{x}{y}}\,dx \\ &= \frac{1}{2y^3}\left(\int_0^{\infty}\left(\frac{\partial}{\partial x}(-yx^2e^{-\frac{x}{y}}) + 2xye^{-\frac{x}{y}}\right)dx\right) \\ &= \frac{1}{y^2}\int_0^{\infty} xe^{-\frac{x}{y}}\,dx \\ &= \frac{1}{y}\int_0^{\infty} e^{-\frac{x}{y}}\,dx \\ &= -e^{-\frac{x}{y}}\Big|_0^{\infty} \\ &= 1. \end{aligned}$$

(b) From the definition of conditional distributions, we have

$$\begin{aligned} f_{X|Y=y}(x) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= f_{X,Y}(x,y) \\ &= \begin{cases} \frac{x^2}{2y^3}e^{-\frac{x}{y}} & 0 \leq x \leq \infty, \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(c)

$$\begin{aligned}
EX &= \int_0^1 \int_0^\infty x f_{X,Y}(x,y)dxdy \\
&= \int_0^1 \int_0^\infty \frac{x^3}{2y^3} e^{-\frac{x}{y}} dxdy \\
&= \frac{3}{2} && \text{after integration by parts 3 times} \\
EX^2 &= \int_0^1 \int_0^\infty x^2 f_{X,Y}(x,y)dxdy \\
&= \int_0^1 \int_0^\infty \frac{x^4}{2y^3} e^{-\frac{x}{y}} dxdy \\
&= 4 \\
\mathrm{Var}X &= EX^2 - (EX)^2 \\
&= 4 - (\frac{3}{2})^2 \\
&= \frac{16-9}{4} \\
&= \frac{7}{4}
\end{aligned}$$

Problem 2.9

The density of a random vector $(X, Y)'$ is

$$f_{X,Y}(x,y) = \begin{cases} cx & x \geq 0, \text{ and } y \geq 0, \text{ and } x + y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute

(a) c,

(b) the conditional expectations $E(Y|X = x)$ and $E(X|Y = y)$.

Solution

(a) Normalization yields

$$\begin{aligned}
1 &= \int_0^\infty \int_0^\infty f_{X,Y}(x,y)dxdy \\
&= \int_0^1 \int_0^{1-y} cxdxdy \\
&= \int_0^1 \frac{c}{2}(1-y)^2 dxdy
\end{aligned}$$