

Problem 2.8

The density function of the two-dimensional random variable (X, Y) is

$$f_{X,Y}(x, y) = \begin{cases} \frac{x^2}{2y^3} e^{-\frac{x}{y}} & 0 \leq x \leq \infty, \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Determine the distribution of Y .
- (b) Find the conditional distribution of X given that $Y = y$.
- (c) Use the results from (a) and (b) to compute EX and $\text{Var}X$

Solution

- (a) For $0 \leq y \leq 1$, we have

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx \\ &= \int_0^{\infty} \frac{x^2}{2y^3} e^{-\frac{x}{y}} dx \\ &= \frac{1}{2y^3} \left(\int_0^{\infty} \left(\frac{\partial}{\partial x} (-yx^2 e^{-\frac{x}{y}}) + 2xy e^{-\frac{x}{y}} \right) dx \right) \\ &= \frac{1}{y^2} \int_0^{\infty} x e^{-\frac{x}{y}} dx \\ &= \frac{1}{y} \int_0^{\infty} e^{-\frac{x}{y}} dx \\ &= -e^{-\frac{x}{y}} \Big|_0^{\infty} \\ &= \boxed{1}. \end{aligned}$$

- (b) From the definition of conditional distributions, we have

$$\begin{aligned} f_{X|Y=y}(x) &= \frac{f_{X,Y}(x, y)}{f_Y(y)} \\ &= \frac{f_{X,Y}(x, y)}{f_Y(y)} \\ &= f_{X,Y}(x, y) \\ &= \boxed{\begin{cases} \frac{x^2}{2y^3} e^{-\frac{x}{y}} & 0 \leq x \leq \infty, \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}} \end{aligned}$$

(c)

$$\begin{aligned}
EX &= \int_0^1 \int_0^\infty x f_{X,Y}(x,y) dx dy \\
&= \int_0^1 \int_0^\infty \frac{x^3}{2y^3} e^{-\frac{x}{y}} dx dy \\
&= \boxed{\frac{3}{2}} && \text{after integration by parts 3 times} \\
EX^2 &= \int_0^1 \int_0^\infty x^2 f_{X,Y}(x,y) dx dy \\
&= \int_0^1 \int_0^\infty \frac{x^4}{2y^3} e^{-\frac{x}{y}} dx dy \\
&= 4 \\
\text{Var}X &= EX^2 - (EX)^2 \\
&= 4 - \left(\frac{3}{2}\right)^2 \\
&= \frac{16-9}{4} \\
&= \boxed{\frac{7}{4}}
\end{aligned}$$

Problem 2.9

The density of a random vector $(X, Y)'$ is

$$f_{X,Y}(x,y) = \begin{cases} cx & x \geq 0, \text{ and } y \geq 0, \text{ and } x+y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute

- (a) c ,
 (b) the conditional expectations $E(Y|X=x)$ and $E(X|Y=y)$.

Solution

- (a) Normalization yields

$$\begin{aligned}
1 &= \int_0^\infty \int_0^\infty f_{X,Y}(x,y) dx dy \\
&= \int_0^1 \int_0^{1-y} cx dx dy \\
&= \int_0^1 \frac{c}{2} (1-y)^2 dy \\
&= \int_0^1 \frac{c}{2} (y^2 - 2y + 1) dy \\
&= \frac{c}{2} \left(\frac{1}{3} y^3 - y^2 + y \right) \Big|_0^1 \\
&= \frac{c}{6} \\
c &= \boxed{6}.
\end{aligned}$$

- (b) In order to calculate $E(Y|X=x)$ and $E(X|Y=y)$, we will need to first find $f_{Y|X=x}(y)$ and $f_{X|Y=y}(x)$. To find these, we will first need to compute our marginal probability density functions. For $x \in [0, 1]$, we

have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \\ &= \int_0^{1-x} 6xdy \\ &= 6x(1-x) \end{aligned}$$

Likewise, for $y \in [0, 1]$, we have

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \\ &= \int_0^{1-y} 6xdx \\ &= 3(1-y)^2. \end{aligned}$$

With this, we have

$$\begin{aligned} f_{X|Y=y}(x) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{6x}{3(1-y)^2} \\ &= \frac{2x}{(1-y)^2} \end{aligned}$$

and

$$\begin{aligned} f_{Y|X=x}(y) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{6x}{6x(1-x)} \\ &= \frac{1}{1-x}, \end{aligned}$$

on the support of $f_{X,Y}(x,y)$. Finally, we have

$$\begin{aligned} E(X|Y=y) &= \int_{-\infty}^{\infty} xf_{X|Y=y}(x)dx \\ &= \int_0^{1-y} \frac{2x^2}{(1-y)^2}dx \\ &= \frac{2(1-y)^3}{3(1-y)^2} \\ &= \boxed{\frac{2}{3}(1-y)}, \end{aligned}$$

and

$$\begin{aligned} E(Y|X=x) &= \int_{-\infty}^{\infty} yf_{Y|X=x}(y)dy \\ &= \int_0^{1-x} \frac{y}{1-x}dy \\ &= \frac{(1-x)^2}{2(1-x)} \\ &= \boxed{\frac{1}{2}(1-x)}. \end{aligned}$$

Problem 2.33

Suppose that the random variable X is uniformly distributed symmetrically around zero, but in such a way that the parameter is uniform on $(0, 1)$; that is suppose that

$$X|A = a \in U(-a, a) \text{ with } A \in U(0, 1).$$

Find the distribution of X , EX , and $\text{Var}X$.

Solution

We have

$$f_{X|A=a}(x) = \frac{f_{X,A}(x, a)}{f_A(a)} \implies f_{X,A}(x, a) = f_{X|A=a}(x) \cdot f_A(a),$$

thus

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,A}(x, a) da \\ &= \int_{-\infty}^{\infty} f_{X|A=a}(x) \cdot f_A(a) da. \end{aligned}$$

At this point, in order to correctly integrate over the support of $f_{X,A}(x, a)$, we will break this into case 1 where $-1 \leq x \leq 0$, and case 2 where $0 \leq x \leq 1$.

(1) Satisfying the conditions of our support, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,A}(x, a) da \\ &= \int_{-x}^1 \frac{1}{2a} da \\ &= -\frac{1}{2} \ln(-x). \end{aligned}$$

(2) Satisfying the conditions of our support once more, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,A}(x, a) da \\ &= \int_x^1 \frac{1}{2a} da \\ &= -\frac{1}{2} \ln(x). \end{aligned}$$

We can succinctly write this as

$$\boxed{f_X(x) = \frac{1}{2} \ln |x|},$$

for $-1 \leq x \leq 1$.

Now, since we can see that $f_X(x)$ is even, we have that $xf_X(x)$ is odd, and we can automatically conclude that $\boxed{EX = 0}$. Thus, we have

$$\begin{aligned} \text{Var}X &= EX^2 - (EX)^2 \\ &= EX^2. \end{aligned}$$

Now computing EX^2 directly with $f_X(x)$ requires some moderately difficult integration, so we will instead use our helper variable:

$$\begin{aligned}
 EX^2 &= E_A(E_{X|A=a}(X)) \\
 &= \int_0^1 f_A(a) \cdot (E_{X|A=a}(X)) da \\
 &= \int_0^1 f_A(a) \cdot \left(\int_{-a}^a x^2 \cdot f_{X|A=a}(x) dx \right) da \\
 &= \int_0^1 \int_{-a}^a \frac{x^2}{2a} dx da \\
 &= \int_0^1 \int_0^a \frac{x^2}{a} dx da \\
 &= \int_0^1 \frac{a^2}{3} da \\
 &= \boxed{\frac{1}{9}}.
 \end{aligned}$$

Problem 2.34

Consider the following model:

$$X_n|Y = y \in \text{Bin}(n, y) \text{ with } f_Y(y) = 6y(1-y), \quad 0 < y < 1.$$

- (a) Compute EX_n and $\text{Var}X_n$.
- (b) Determine the distribution of X_n .

Solution

- (a) Since $X_n|Y = y \in \text{Bin}(n, y)$, we have

$$p_{X_n|Y=y}(k) = \binom{n}{k} y^k (1-y)^{n-k} \text{ for } k = 0, 1, \dots, n.$$

Thus,

$$\begin{aligned}
 EX_n &= E_Y(E_{X_n|Y=y}(X_n)) \\
 &= \int_0^1 \left(\sum_{k=0}^n k \cdot p_{X_n|Y=y}(k) \right) \cdot f_Y(y) dy \\
 &= \int_0^1 \left(\sum_{k=0}^n k \binom{n}{k} y^k (1-y)^{n-k} \right) 6y(1-y) dy \\
 &= \int_0^1 \left(\sum_{k=0}^n 6k \binom{n}{k} y^{k+1} (1-y)^{n-k+1} \right) dy \\
 &= - \int_0^1 6n(y-1)y^2 dy, \text{ found by entering into wolfram alpha} \\
 &= -6n \int_0^1 (y^3 - y^2) dy \\
 &= -6n \left(\frac{1}{4} - \frac{1}{3} \right) \\
 &= 6n \left(\frac{4}{12} - \frac{3}{12} \right) \\
 &= \boxed{\frac{n}{2}}.
 \end{aligned}$$

To find our variance, we compute EX^2 in a very similar manner

$$\begin{aligned} EX^2 &= \int_0^1 \left(\sum_{k=0}^n 6k^2 \binom{n}{k} y^{k+1} (1-y)^{n-k+1} \right) dy \\ &= -6 \int_0^1 (y-1)y^2((n-1)ny+n)dy \\ &= \frac{1}{10}n(3n+2). \end{aligned}$$

Thus,

$$\begin{aligned} \text{Var}X &= EX^2 - (EX)^2 \\ &= \frac{1}{10}n(3n+2) - \left(\frac{n}{2}\right)^2 \\ &= \boxed{\frac{n^2}{20} + \frac{n}{5}} \end{aligned}$$

Problem 2.35

Let X and Y be jointly distributed random variables such that $Y|X=x \in \text{Bin}(n, x)$ with $X \in U(0, 1)$. Compute EY , $\text{Var}Y$, and $\text{Cov}(X, Y)$.

Solution

Similar to the previous problem, we know we have

$$p_{Y|X=x}(k) = \binom{n}{k} x^k (1-x)^{n-k} \text{ for } k = 0, 1, \dots, n.$$

We also have

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1, \text{ and } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Similar to our previous problem, we have

$$\begin{aligned} EY &= E_X(E_{Y|X=x}(Y)) \\ &= \int_0^1 \left(\sum_{k=0}^n \binom{n}{k} k \cdot x^k (1-x)^{n-k} \right) dx \\ &= \int_0^1 nx dx \\ &= \boxed{\frac{n}{2}}, \end{aligned}$$

and

$$\begin{aligned} EY^2 &= E_X(E_{Y|X=x}(Y^2)) \\ &= \int_0^1 \left(\sum_{k=0}^n \binom{n}{k} k^2 \cdot x^k (1-x)^{n-k} \right) dx \\ &= \int_0^1 x(n + (-1+n)nx) dx \\ &= \frac{1}{6}n(2n+1) \\ &= \frac{n^2}{3} + \frac{n}{6}. \end{aligned}$$

Thus, we have

$$\begin{aligned}\text{Var}Y &= EY^2 - (EY)^2 \\ &= \frac{n^2}{3} + \frac{n}{6} - \frac{n^2}{4} \\ &= \boxed{\frac{n^2}{12} + \frac{n}{6}}.\end{aligned}$$

Finally, we have

$$\begin{aligned}\text{Cov}(X, Y) &= E((X - E(X))(Y - E(Y))) \\ &= \int_0^1 \sum_{k=0}^n (x - E(X))(k - E(Y)) \cdot p_{Y|X=x}(k) \cdot f_X(x) dx \\ &= \int_0^1 \sum_{k=0}^n \left(x - \frac{1}{2}\right) \left(k - \frac{n}{2}\right) \binom{n}{k} x^k (1-x)^{n-k} dx \\ &= \int_0^1 \frac{1}{4} n (1-2x)^2 dx && \text{Let } u = 1 - 2x \\ &= -\frac{1}{8} n \int_1^{-1} u^2 du \\ &= \boxed{\frac{n}{12}}.\end{aligned}$$

Problem 2.37

Let X be the number of coin tosses until heads is obtained. Suppose that the probability of heads is unknown in the sense that we consider it to be a random variable $Y \in U(0, 1)$.

- Find the distribution of X .
- The expected value of an Fs-distributed random variable exists and is well known. What about EX ?
- Suppose that the value $X = n$ has been observed. Find the posterior distribution of Y , that is, the distribution of $Y|X = n$.

Solution

- For $x = 1, 2, \dots$, we have

$$\begin{aligned}p_X(x) &= \int_0^1 f_Y(y) \cdot p_{X|Y=y} dy \\ &= \int_0^1 (1-y)^{x-1} y dy \\ &= \int_0^1 \frac{1}{x} (1-y)^x dy && \text{Integration by parts} \\ &= -\frac{1}{x(x+1)} (1-y)^{x+1} \Big|_0^1 \\ &= \boxed{\frac{1}{x(x+1)}}.\end{aligned}$$

- We have

$$\begin{aligned}EX &= \sum_{x=1}^{\infty} x p_X(x) \\ &= \sum_{x=1}^{\infty} \frac{1}{x+1}.\end{aligned}$$

Employing the integral test, we see

$$\int_1^{\infty} \frac{1}{x+1} dx = \ln(x+1)|_1^{\infty} \rightarrow \infty,$$

thus EX does not exist.

(c) Finally, we have

$$\begin{aligned} f_{Y|X=n} &= \frac{f_{X,Y}(n,y)}{f_X(n)} \\ &= \frac{f_Y(y) \cdot f_{X|Y=y}(n)}{f_X(n)} \\ &= \boxed{(1-y)^{n-1} y n(n+1), \text{ for } y = 1, 2, \dots}. \end{aligned}$$