Problem 2.8

The density function of the two-dimensional random variable (X, Y) is

$$f_{X,Y}(x,y) = \begin{cases} \frac{x^2}{2y^3}e^{-\frac{x}{y}} & 0 \le x \le \infty, \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the distribution of Y.

(b) Find the conditional distribution of X given that $Y = y$.

(c) Use the results from (a) and (b) to compute EX and $\mathrm{Var}X$

Solution

(a) For $0 \le y \le 1$, we have

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \\ &= \int_0^{\infty} \frac{x^2}{2y^3}e^{-\frac{x}{y}}\,dx \\ &= \frac{1}{2y^3}\left(\int_0^{\infty}\left(\frac{\partial}{\partial x}(-yx^2e^{-\frac{x}{y}}) + 2xye^{-\frac{x}{y}}\right)dx\right) \\ &= \frac{1}{y^2}\int_0^{\infty} xe^{-\frac{x}{y}}\,dx \\ &= \frac{1}{y}\int_0^{\infty} e^{-\frac{x}{y}}\,dx \\ &= -e^{-\frac{x}{y}}\Big|_0^{\infty} \\ &= 1. \end{aligned}$$

(b) From the definition of conditional distributions, we have

$$\begin{aligned} f_{X|Y=y}(x) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= f_{X,Y}(x,y) \\ &= \begin{cases} \frac{x^2}{2y^3}e^{-\frac{x}{y}} & 0 \le x \le \infty, \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(c)

$$\begin{aligned}
EX &= \int_0^1 \int_0^\infty x f_{X,Y}(x,y)dxdy \\
&= \int_0^1 \int_0^\infty \frac{x^3}{2y^3} e^{-\frac{x}{y}} dxdy \\
&= \frac{3}{2} && \text{after integration by parts 3 times} \\
EX^2 &= \int_0^1 \int_0^\infty x^2 f_{X,Y}(x,y)dxdy \\
&= \int_0^1 \int_0^\infty \frac{x^4}{2y^3} e^{-\frac{x}{y}} dxdy \\
&= 4 \\
\text{Var}X &= EX^2 - (EX)^2 \\
&= 4 - (\frac{3}{2})^2 \\
&= \frac{16-9}{4} \\
&= \frac{7}{4}
\end{aligned}$$

Problem 2.9

The density of a random vector $(X, Y)'$ is

$$f_{X,Y}(x,y) = \begin{cases} cx & x \geq 0, \text{ and } y \geq 0, \text{ and } x + y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute

(a) c,

(b) the conditional expectations $E(Y|X = x)$ and $E(X|Y = y)$.

Solution

(a) Normalization yields

$$\begin{aligned}
1 &= \int_0^\infty \int_0^\infty f_{X,Y}(x,y)dxdy \\
&= \int_0^1 \int_0^{1-y} cxdxdy \\
&= \int_0^1 \frac{c}{2}(1-y)^2 dy \\
&= \int_0^1 \frac{c}{2}(y^2 - 2y + 1)dy \\
&= \frac{c}{2}(\frac{1}{3}y^3 - y^2 + y)\Big|_0^1 \\
&= \frac{c}{6} \\
c &= 6.
\end{aligned}$$

(b) In order to calculate $E(Y|X = x)$ and $E(X|Y = y)$, we will need to first find $f_{Y|X=x}(y)$ and $f_{X|Y=y}(x)$. To find these, we will first need to compute our marginal probability density functions. For $x \in [0, 1]$, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \\ &= \int_0^{1-x} 6xdy \\ &= 6x(1-x) \end{aligned}$$

Likewise, for $y \in [0,1]$, we have

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \\ &= \int_0^{1-y} 6xdx \\ &= 3(1-y)^2. \end{aligned}$$

With this, we have

$$\begin{aligned} f_{X|Y=y}(x) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{6x}{3(1-y)^2} \\ &= \frac{2x}{(1-y)^2} \end{aligned}$$

and

$$\begin{aligned} f_{Y|X=x}(y) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{6x}{6x(1-x)} \\ &= \frac{1}{1-x}, \end{aligned}$$

on the support of $f_{X,Y}(x,y)$. Finally, we have

$$\begin{aligned} E(X|Y=y) &= \int_{-\infty}^{\infty} x f_{X|Y=y}(x)dx \\ &= \int_0^{1-y} \frac{2x^2}{(1-y)^2}dx \\ &= \frac{2(1-y)^3}{3(1-y)^2} \\ &= \frac{2}{3}(1-y), \end{aligned}$$

and

$$\begin{aligned} E(Y|X=x) &= \int_{-\infty}^{\infty} y f_{Y|X=x}(y)dy \\ &= \int_0^{1-x} \frac{y}{1-x}dy \\ &= \frac{(1-x)^2}{2(1-x)} \\ &= \frac{1}{2}(1-x). \end{aligned}$$