Problem 2.8

The density function of the two-dimensional random variable (X, Y) is

$$f_{X,Y}(x,y) = \begin{cases} \frac{x^2}{2y^3}e^{-\frac{x}{y}} & 0 \leq x \leq \infty, \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the distribution of Y.

(b) Find the conditional distribution of X given that $Y = y$.

(c) Use the results from (a) and (b) to compute EX and $\text{Var}X$

Solution

(a) For $0 \leq y \leq 1$, we have

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \\ &= \int_0^{\infty} \frac{x^2}{2y^3}e^{-\frac{x}{y}}\,dx \\ &= \frac{1}{2y^3}\left(\int_0^{\infty}\left(\frac{\partial}{\partial x}(-yx^2e^{-\frac{x}{y}}) + 2xye^{-\frac{x}{y}}\right)dx\right) \\ &= \frac{1}{y^2}\int_0^{\infty} xe^{-\frac{x}{y}}\,dx \\ &= \frac{1}{y}\int_0^{\infty} e^{-\frac{x}{y}}\,dx \\ &= -e^{-\frac{x}{y}}\Big|_0^{\infty} \\ &= \boxed{1}. \end{aligned}$$

(b) From the definition of conditional distributions, we have

$$\begin{aligned} f_{X|Y=y}(x) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= f_{X,Y}(x,y) \\ &= \boxed{\begin{cases} \frac{x^2}{2y^3}e^{-\frac{x}{y}} & 0 \leq x \leq \infty, \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}} \end{aligned}$$

(c)

$$\begin{aligned}
EX &= \int_0^1 \int_0^\infty x f_{X,Y}(x,y)dxdy \\
&= \int_0^1 \int_0^\infty \frac{x^3}{2y^3} e^{-\frac{x}{y}} dxdy \\
&= \boxed{\frac{3}{2}} && \text{after integration by parts 3 times} \\
EX^2 &= \int_0^1 \int_0^\infty x^2 f_{X,Y}(x,y)dxdy \\
&= \int_0^1 \int_0^\infty \frac{x^4}{2y^3} e^{-\frac{x}{y}} dxdy \\
&= 4 \\
\text{Var}X &= EX^2 - (EX)^2 \\
&= 4 - (\frac{3}{2})^2 \\
&= \frac{16-9}{4} \\
&= \boxed{\frac{7}{4}}
\end{aligned}$$

Problem 2.9

The density of a random vector $(X, Y)'$ is

$$f_{X,Y}(x,y) = \begin{cases} cx & x \geq 0, \text{ and } y \geq 0, \text{ and } x + y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute

(a) c,

(b) the conditional expectations $E(Y|X = x)$ and $E(X|Y = y)$.

Solution

(a) Normalization yields

$$\begin{aligned}
1 &= \int_0^\infty \int_0^\infty f_{X,Y}(x,y)dxdy \\
&= \int_0^1 \int_0^{1-y} cxdxdy \\
&= \int_0^1 \frac{c}{2}(1-y)^2 dy \\
&= \int_0^1 \frac{c}{2}(y^2 - 2y + 1)dy \\
&= \frac{c}{2}(\frac{1}{3}y^3 - y^2 + y)\Big|_0^1 \\
&= \frac{c}{6} \\
c &= \boxed{6}.
\end{aligned}$$

(b) In order to calculate $E(Y|X = x)$ and $E(X|Y = y)$, we will need to first find $f_{Y|X=x}(y)$ and $f_{X|Y=y}(x)$. To find these, we will first need to compute our marginal probability density functions. For $x \in [0, 1]$, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \\ &= \int_0^{1-x} 6xdy \\ &= 6x(1-x) \end{aligned}$$

Likewise, for $y \in [0,1]$, we have

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \\ &= \int_0^{1-y} 6xdx \\ &= 3(1-y)^2. \end{aligned}$$

With this, we have

$$\begin{aligned} f_{X|Y=y}(x) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{6x}{3(1-y)^2} \\ &= \frac{2x}{(1-y)^2} \end{aligned}$$

and

$$\begin{aligned} f_{Y|X=x}(y) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{6x}{6x(1-x)} \\ &= \frac{1}{1-x}, \end{aligned}$$

on the support of $f_{X,Y}(x,y)$. Finally, we have

$$\begin{aligned} E(X|Y=y) &= \int_{-\infty}^{\infty} x f_{X|Y=y}(x)dx \\ &= \int_0^{1-y} \frac{2x^2}{(1-y)^2}dx \\ &= \frac{2(1-y)^3}{3(1-y)^2} \\ &= \boxed{\frac{2}{3}(1-y)}, \end{aligned}$$

and

$$\begin{aligned} E(Y|X=x) &= \int_{-\infty}^{\infty} y f_{Y|X=x}(y)dy \\ &= \int_0^{1-x} \frac{y}{1-x}dy \\ &= \frac{(1-x)^2}{2(1-x)} \\ &= \boxed{\frac{1}{2}(1-x)}. \end{aligned}$$

Problem 2.33

Suppose that the random variable X is uniformly distributed symmetrically around zero, but in such a way that the parameter is uniform on $(0,1)$; that is suppose that

$$X|A = a \in U(-a,a) \text{ with } A \in U(0,1).$$

Find the distribution of X, EX, and $\text{Var}X$.

Solution

We have

$$f_{X|A=a}(x) = \frac{f_{X,A}(x,a)}{f_A(a)} \implies f_{X,A}(x,a) = f_{X|A=a}(x) \cdot f_A(a),$$

thus

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,A}(x,a)da \\ &= \int_{-\infty}^{\infty} f_{X|A=a}(x) \cdot f_A(a)da. \end{aligned}$$

At this point, in order to correctly integrate over the support of $f_{X,A}(x,a)$, we will break this into case 1 where $-1 \le x \le 0$, and case 2 where $0 \le x \le 1$.

(1) Satisfying the conditions of our support, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,A}(x,a)da \\ &= \int_{-x}^{1} \frac{1}{2a}da \\ &= -\frac{1}{2}\ln(-x). \end{aligned}$$

(2) Satisfying the conditions of our support once more, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,A}(x,a)da \\ &= \int_{x}^{1} \frac{1}{2a}da \\ &= -\frac{1}{2}\ln(x). \end{aligned}$$

We can succinctly write this as

$$\boxed{f_X(x) = \frac{1}{2}\ln|x|},$$

for $-1 \le x \le 1$.

Now, since we can see that $f_X(x)$ is even, we have that $xf_X(x)$ is odd, and we can automatically conclude that $\boxed{EX = 0}$. Thus, we have

$$\begin{aligned} \text{Var}X &= EX^2 - (EX)^2 \\ &= EX^2. \end{aligned}$$

Now computing EX^2 directly with $f_X(x)$ requires some moderately difficult integration, so we will instead use our helper variable:

$$\begin{aligned}
EX^2 &= E_A(E_{X|A=a}(X)) \\
&= \int_0^1 f_A(a) \cdot (E_{X|A=a}(X)) da \\
&= \int_0^1 f_A(a) \cdot \left(\int_{-a}^{a} x^2 \cdot f_{X|A=a}(x) dx \right) da \\
&= \int_0^1 \int_{-a}^{a} \frac{x^2}{2a} dx da \\
&= \int_0^1 \int_0^a \frac{x^2}{a} dx da \\
&= \int_0^1 \frac{a^2}{3} da \\
&= \boxed{\frac{1}{9}}.
\end{aligned}$$

Problem 2.34

Consider the following model:

$$X_n | Y = y \in \text{Bin}(n, y) \text{ with } f_Y(y) = 6y(1-y),\ 0 < y < 1.$$

(a) Compute EX_n and $\text{Var}X_n$.

(b) Determine the distribution of X_n.

Solution

(a) Since $X_n|Y = y \in \text{Bin}(n, y)$, we have

$$p_{X_n|Y=y}(k) = \binom{n}{k} y^k (1-y)^{n-k} \text{ for } k = 0, 1, ..., n.$$

Thus,

$$\begin{aligned}
EX_n &= E_Y(E_{X_n|Y=y}(X_n)) \\
&= \int_0^1 \left(\sum_{k=0}^{n} k \cdot p_{X_n|Y=y}(k) \right) \cdot f_Y(y) dy \\
&= \int_0^1 \left(\sum_{k=0}^{n} k \binom{n}{k} y^k (1-y)^{n-k} \right) 6y(1-y) dy \\
&= \int_0^1 \left(\sum_{k=0}^{n} 6k \binom{n}{k} y^{k+1} (1-y)^{n-k+1} \right) dy \\
&= -\int_0^1 6n(y-1)y^2 dy, \text{ found by entering into wolfram alpha} \\
&= -6n \int_0^1 (y^3 - y^2) dy \\
&= -6n(\frac{1}{4} - \frac{1}{3}) \\
&= 6n(\frac{4}{12} - \frac{3}{12}) \\
&= \boxed{\frac{n}{2}}.
\end{aligned}$$

To find our variance, we compute EX^2 in a very similar manner

$$\begin{aligned}
EX^2 &= \int_0^1 \left(\sum_{k=0}^{n} 6k^2 \binom{n}{k} y^{k+1}(1-y)^{n-k+1} \right) dy \\
&= -6 \int_0^1 (y-1)y^2((n-1)ny+n)dy \\
&= \frac{1}{10} n(3n+2).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{Var}X &= EX^2 - (EX)^2 \\
&= \frac{1}{10} n(3n+2) - \left(\frac{n}{2}\right)^2 \\
&= \boxed{\frac{n^2}{20} + \frac{n}{5}}
\end{aligned}$$

Problem 2.35

Let X and Y be jointly distributed random variables such that $Y|X = x \in \mathrm{Bin}(n, x)$ with $X \in U(0, 1)$. Compute EY, $\mathrm{Var}Y$, and $\mathrm{Cov}(X, Y)$.

Solution

Similar to the previous problem, we know we have

$$p_{Y|X=x}(k) = \binom{n}{k} x^k (1-x)^{n-k} \text{ for } k = 0, 1, ..., n.$$

We also have

$$f_X(x) = \begin{cases} 1 & 0 \le x \le \infty, \text{ and } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Similar to our previous problem, we have

$$\begin{aligned}
E_Y &= E_X(E_{Y|X=x}(Y)) \\
&= \int_0^1 \left(\sum_{k=0}^{n} \binom{n}{k} k \cdot x^k (1-x)^{n-k} \right) dx \\
&= \int_0^1 nx dx \\
&= \boxed{\frac{n}{2}},
\end{aligned}$$

and

$$\begin{aligned}
EY^2 &= E_X(E_{Y|X=x}(Y^2)) \\
&= \int_0^1 \left(\sum_{k=0}^{n} \binom{n}{k} k^2 \cdot x^k (1-x)^{n-k} \right) dx \\
&= \int_0^1 x(n + (-1+n)nx) dx \\
&= \frac{1}{6} n(2n+1) \\
&= \frac{n^2}{3} + \frac{n}{6}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\text{Var}Y &= EY^2 - (EY)^2 \\
&= \frac{n^2}{3} + \frac{n}{6} - \frac{n^2}{4} \\
&= \boxed{\frac{n^2}{12} + \frac{n}{6}}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\text{Cov}(X,Y) &= E((X - E(X))(Y - E(Y))) \\
&= \int_0^1 \sum_{k=0}^{n} (x - E(X))(k - E(Y)) \cdot p_{Y|X=x}(k) \cdot f_X(x) dx \\
&= \int_0^1 \sum_{k=0}^{n} \left(x - \frac{1}{2}\right)\left(k - \frac{n}{2}\right)\binom{n}{k} x^k (1-x)^{n-k} dx \\
&= \int_0^1 \frac{1}{4} n (1-2x)^2 dx && \text{Let u = 1 - 2x} \\
&= -\frac{1}{8} n \int_1^{-1} u^2 du \\
&= \boxed{\frac{n}{12}}.
\end{aligned}$$

Problem 2.37

Let X be the number of coin tosses until heads is obtained. Suppose that the probability of heads is unknown in the sense that we consider it to be a random variable $Y \in U(0,1)$.

(a) Find the distribution of X.

(b) The expected value of an Fs-distributed random variable exists and is well known. What about EX?

(c) Suppose that the value $X = n$ has been observed. Find the posterior distribution of Y, that is, the distribution of $Y|X = n$.

Solution

(a) For $x = 1, 2, \ldots$, we have

$$\begin{aligned}
p_X(x) &= \int_0^1 f_Y(y) \cdot p_{X|Y=y} dy \\
&= \int_0^1 (1-y)^{x-1} y dy \\
&= \int_0^1 \frac{1}{x}(1-y)^x dy && \text{Integration by parts} \\
&= -\frac{1}{x(x+1)}(1-y)^{x+1}\Big|_0^1 \\
&= \boxed{\frac{1}{x(x+1)}}.
\end{aligned}$$

(b) We have

$$\begin{aligned}
EX &= \sum_{x=1}^{\infty} x p_X(x) \\
&= \sum_{x=1}^{\infty} \frac{1}{x+1}.
\end{aligned}$$

Employing the integral test, we see

$$\int_1^\infty \frac{1}{x+1}dx = ln(x+1)|_1^\infty \to \infty,$$

thus EX does not exist.

(c) Finally, we have

$$\begin{aligned} f_{Y|X=n} &= \frac{f_{X,Y}(n,y)}{f_X(n)} \\ &= \frac{f_Y(y) \cdot f_{X|Y=y}(n)}{f_X(n)} \\ &= \boxed{(1-y)^{n-1}yn(n+1), \text{ for y = 1, 2, ...}}. \end{aligned}$$

Problem 2.40

Let the joint distribution of X and Y be given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{, for } |x|+|y| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that Y has constant regression with respect to X, i.e., that $E(Y|X) = EY$, but that X and Y are not independent.

Solution

We will begin, by finding $f_X(x)$. We have for $-1 \leq x \leq 1$,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^\infty f_{X,Y}(x,y)dy \\ &= \int_{|x|-1}^{1-|x|} \frac{1}{2}dy \\ &= \frac{1}{2}(1-|x|-|x|+1) \\ &= 1-|x|. \end{aligned}$$

Thus, by symmetry, we have

$$f_Y(y) = 1-|y|,$$

for $-1 \leq y \leq 1$. With this, we can see that $f_{X,Y}(x,y) \neq f_X(x) \cdot f_Y(y)$, and we can conclude that X and Y are not independent.

Since $f_Y(y)$ is even, we can conclude that $EY = 0$. We must now find $f_{Y|X=x}$ and use it to show that $E(Y|X) = EY$. Thus,

$$\begin{aligned} f_{Y|X=x}(y) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{\frac{1}{2}}{1-|x|}, \end{aligned}$$

for all $-(1-|x|) \leq y \leq (1-|X|)$. Thus, we have that $f_{Y|X=x}(y)$ is even, which implies that $y \cdot f_{Y|X=x}(y)$ is odd, and integration over symmetric bounds leads us to conclude that

$$\begin{aligned} E(Y|X) &= 0 \\ &= EY. \end{aligned}$$