Problem 1.1

Show that if $X \in C(0,1)$, then so is $1/X$.

Solution

Since $X \in C(0,1)$, we have

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \text{ for } -\infty < x < \infty.$$

Define $Y = \frac{1}{X}$. Then, $X = \frac{1}{Y}$, and we have

$$|\det(J)| = \left|\frac{dx}{dy}\right| = \frac{1}{y^2}.$$

Thus,

$$\begin{aligned} f_Y(y) &= |\det(J)| f_X(1/y) \\ &= \frac{1}{y^2} \cdot \frac{1}{\pi} \cdot \frac{1}{1+(1/y)^2} \\ &= \frac{1}{\pi} \cdot \frac{1}{1+y^2}, \end{aligned}$$

and we have shown that $Y \in C(0,1)$.

Problem 1.8

Show that if X, Y are independent $N(0,1)$-distributed random variables, then $X/Y \in C(0,1)$.

Solution

We have

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{1}{2\pi} e^{-x^2/2} e^{-y^2/2} \\ &= \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}. \end{aligned}$$

Define $U = X/Y$. We introduce the auxiliary variable $V = X$. Then we have the inverse relations

$$\begin{aligned} X &= V \\ Y &= V/U. \end{aligned}$$

With this, we can find our Jacobian determinant:

$$\begin{aligned} \det(J) &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \\ &= \begin{vmatrix} 0 & 1 \\ \frac{-v}{u^2} & \frac{1}{u} \end{vmatrix} \\ &= \frac{v}{u^2}. \end{aligned}$$

Thus,

$$\begin{aligned} f_{U,V}(u,v) &= |\det(J)| f_{X,Y}(v, v/u) \\ &= \frac{|v|}{u^2} \cdot \frac{1}{2\pi} e^{-\frac{v^2+(v/u)^2}{2}} \\ &= \frac{|v|}{u^2} \cdot \frac{1}{2\pi} e^{-v^2 \frac{1+(1/u)^2}{2}}. \end{aligned}$$

We want the distribution of U, so

$$\begin{aligned}
f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u,v)dv \\
&= \int_{-\infty}^{\infty} \frac{|v|}{u^2} \cdot \frac{1}{2\pi} e^{-v^2 \frac{1+(1/u)^2}{2}} dv \\
&= \frac{1}{u^2\pi} \int_0^{\infty} v \cdot e^{-v^2 \frac{1+(1/u)^2}{2}} dv \\
&= \frac{1}{\pi} \cdot \frac{1}{1+u^2}.
\end{aligned}$$

Thus, we have shown that $U \in C(0,1)$.

Problem 1.11

Show that if X and Y are independent Exp(a)-distributed random variables, then $X/Y \in F(2,2)$.

Solution

We have

$$\begin{aligned}
f_{X,Y}(x,y) &= \frac{1}{a^2} e^{-x/a} e^{-y/a} \\
&= \frac{1}{a^2} e^{-\frac{x+y}{a}},
\end{aligned}$$

for $0 \le x, y < \infty$. Define U and V as we did in the previous problem. Then,

$$\begin{aligned}
f_{U,V}(u,v) &= |\det(J)| f_{X,Y}(v, v/u) \\
&= \frac{v}{u^2} \cdot \frac{1}{a^2} e^{-\frac{v+v/u}{a}} \\
&= \frac{v}{u^2} \cdot \frac{1}{a^2} e^{-v\frac{1+1/u}{a}}.
\end{aligned}$$

Then, we can integrate to find the distribution of U:

$$\begin{aligned}
f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u,v)dv \\
&= \frac{1}{a^2u^2} \int_0^{\infty} v e^{-v\frac{1+1/u}{a}} dv \\
&= \frac{1}{(u+1)^2}.
\end{aligned}$$

After looking up Fisher's distribution, we can conclude that $X/Y \in F(2,2)$.

Problem 1.12

Let X, Y be independent random variables such that $X \in U(0,1)$ and $Y \in U(0,\alpha)$. Find the density function of $Z = X + Y$.

Solution

Since X and Y are independent, we have

$$\begin{aligned}
f_{X,Y}(x,y) &= f_X(x) f_Y(y) \\
&= \begin{cases} \frac{1}{\alpha} & 0 \le x \le 1 \text{ and } 0 \le y \le \alpha \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Define $U = X + Y$, and introduce the auxiliary variable $V = X$. Then, we have the inverse relations $X = V$ and $Y = U - V$. With this, our Jacobian determinant becomes

$$\begin{aligned}\det(J) &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \\ &= \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \\ &= -1.\end{aligned}$$

Thus,

$$\begin{aligned}f_{U,V}(u,v) &= f_{X,Y}(v, u-v) \\ &= \begin{cases} \frac{1}{\alpha} & 0 \le v \le 1 \text{ and } 0 \le u - v \le \alpha \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} \frac{1}{\alpha} & 0 \le v \le 1 \text{ and } v \le u \le \alpha + v \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Thus, to find the distribution of U, we integrate over all v

$$\begin{aligned}f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u,v)dv \\ &= \begin{cases} \int_0^1 \frac{1}{\alpha}dv = \frac{1}{\alpha} & 1 \le u \le \alpha + 1 \\ \int_0^u \frac{1}{\alpha}dv = \frac{u}{\alpha} & 0 \le u \le 1 \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$