1 Topology

1.1 Compact Topological Spaces

1.1.1 Definition Basics

Definition 1.1. *) (Compactness)*

A Hausdorff topological space X is said to be **compact** if each open covering of X has a finite subcovering.

Definition 1.2. *(Finite Intersection Property)*

We say a topological space (X, τ) has the **finite intersection property** if for each collection $\{K_\alpha\}_{\alpha \in A}$ such that $\bigcap_{\alpha \in A} K_\alpha = \phi$, then there is a finite subcollection F_{α_i} such that $i \in N_n$ where $\bigcap_{i=1}^{n} F_{\alpha_i} = \phi$

Theorem 1. *X is compact iff the finite intersection property holds on X*

Proof. Apply De Morgan's law and recognize $\bigcap_{\alpha \in A} K_\alpha = \phi \iff \bigcup_{\alpha \in A} \complement K_\alpha = X$

□

Lemma 1.1. *(Compact Basis)*

Let X be a topological space. Then X is compact iff every open cover of X consisting only of elements of a basis $\mathcal{B}$ for τ_X has a finite subcover

Proof. Compactness implying covers of basis elements having finite subcovers is trivial since basis elements are open.

Now let each open cover of basis elements have a finite sub cover.

Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X. Then for each $\alpha \in A$, $U_\alpha = \bigcup_{\beta \in b} B_\beta^\alpha$ where $B_\beta \in \mathcal{B}$. Hence $\{B_\beta^\alpha\}_{\alpha \in A; \beta \in b}$ is a cover for X of basis elements. Hence it has a finite subcover.

$\{B_{\beta_i}^{\alpha_i}\}_{i=1}^{n}$. This implies $\{U_{\alpha_i}\}_{i=1}^{n}$ is a finite sub cover of $\{U_\alpha\}_{\alpha \in A}$, so X is compact.

□

Theorem 2. *(Compact facts)*

Let X and Y be topological spaces. Let $f : X \to Y$ be continuous.

1. *If X is compact then $f(X)$ is compact.*
2. *If X is Hausdorff and $A \subset X$ is compact, the A is closed.*
3. *If X is Compact and $A \subset X$, then, A is closed $\implies$ A is compact.*
4. *Let $\{X_k\}_{k=1}^{n}$ be a collection of topological spaces. Then $\prod_{i=1}^{n} X_k$ is a compact space iff each X_k is compact.*

Proof. (1)

Let $\{V_\alpha\}_{\alpha\in A}$ be an open cover for f(X). Then $\{f^{-1}(V_\alpha)\}_{\alpha\in A}$ openly covers X. Hence there is a finite subcovering $\{f^{-1}(V_{\alpha_i})\}_{i=1}^n$. Thus

$$f(X) = f(\bigcup_{i=1}^n f^{-1}(V_{\alpha_i})) = (\bigcup_{i=1}^n f(f^{-1}V_{\alpha_i})) \subset \bigcup_{i=1}^n V_{\alpha_i}$$

Hence $\{V_{\alpha_i}\}_{i=1}^n$ is a finite subcovering of f(X).

□

Proof. (2) How would you begin this proof?

□

Proof. (3) trivial. See if you can do it.

□

Proof. (4) ($\Longrightarrow$)

The projection mapping is continuous and surjective. Badda bing, Badda boom.

□

Proof. (4) ($\Longleftarrow$) Let X_i be compact for each $i \in N_n$

Define $M = \{n \in \mathbb{N} : \prod_{i=1}^n X_i \text{ is compact}\}$.

1 is the trivial case, so we should show $2 \in M$.

Let $\{B_\alpha\}_{\alpha\in A}$ be an open cover for $X_1 \times X_2$.

□

Theorem 3. *Let X be a compact topological space. Let Y be a Hausdorff space. If $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.*

Proof. Let $A \subset X$ be closed. Then A is compact $\implies$ $f(A)$ is compact. Hence since Y is Hausdorff, $f(A)$ is closed

□

Definition 1.3. *(Locally compact)*

Let X be a topological space. We say that X is a **Locally compact topological space** if

$$(\forall x \in X)(\exists U_x \in \tau_X) \pitchfork (\overline{U}_x \textit{ is compact})$$

1.1.2 Metric spaces convergence

Definition 1.4. *(Convergence in metric spaces)*

Let (X, d) be a metric space. Let $\{x_n\}_{n\in\mathbb{N}} \subset X$. We say that $\{x_n\}$ **converges** to a point $x \in X$, and we write $x_n \to x$, if

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)(d(x_n, x) < \epsilon)$$

Definition 1.5. *(Topological Convergence)*

Let (X, τ) be topological space. Let $\{x_n\}_{n\in\mathbb{N}} \subset X$. We say that x_n **τ-converges** to a point $x \in X$, and we write $x_n \to x$ if

$$(\forall U_x)K(\exists N \in \mathbb{N})(\forall n \geq N)(x_n \in U_x)$$

Theorem 4. *(Metric Topological Convergence)*

Let (X, d) be a metric space. Let $\{x_n\} \subset X$. $\{x_n\}$ τ-converges to a point $x \in X$ if and only if x_n converges to $x \in X$ wrt metric d.

Proof. ($\Longleftarrow$) Let $U_x \in \tau$. $\implies$ $(\exists \epsilon > 0) \pitchfork (B(x;\epsilon) \subset U_x)$
$\implies$ $(\exists N \in \mathbb{N})(\forall n \geq N)(x_n \in B(x;\epsilon) \subset U_x)$
$\implies$ $(\forall n \geq N)(x_n \in U_x)$
$\therefore x_n \to x$ in the topological sense.

□

Proof. ($\implies$)

□

Proposition 1.1. *(Metric Closed Characterization)*

Let (X,d) be a metric space and let $A \subset X$. Then A is closed iff each convergent sequence in A has a limit in A

Proof. Notice that $\overline{A}$ is the collection of points for which there exists a sequence in A that converges to it. (This isn't the proof but its the idea of it. I'm tired of typing rn and I know you won't get stuck on this theorem.) .

□

1.2 Continuity in Metric Spaces

1.2.1 Definition/Equivalence

Theorem 5. *(Metric Continuity Facts)*

Let (X, d) and (Y, p) be metric spaces. Let $f : X \to Y$. Pick $x_0 \in X$. Then the following are equivalent.

1.) $(\forall \epsilon > 0)(\exists \delta > 0) \pitchfork (d(x, x_0) < \delta \implies p(f(x), f(x_0)) < \epsilon)$.
2.) $(\forall \epsilon > 0)(\exists \delta > 0) \pitchfork (f(B(x_0;\delta) \subset B(f(x_0);\epsilon))$
3.) $(\forall \{x_n\}_{n\in\mathbb{N}})(x_n \to x_0 \implies f(x_n) \to f(x_0))$

4.) f is continuous at x_0

Proof. $(1 \implies 2)$

Pick $\epsilon > 0$. Then $\exists \delta > 0$ such that $d(x, x_0) < \delta \implies p(f(x), f(x_0)) < \epsilon$
$\iff x \in B(x_0, \delta) \implies f(x) \in B(f(x_0), \epsilon)$
$\iff f(B(x_0; \delta)) \subset B(f(x_0); \epsilon)$

□

Proof. $(2 \implies 3)$ Ask yourself, what do we need to show? and the next question would be, how do I proceed to prove this claim?

□

Proof. $(3 \implies 4)$ (contra-positive)

Suppose

$$(\exists V(f(x_0)) \pitchfork (\forall U(x_0))(f(U(x_0)) \not\subset V(f(x_0))).$$

$$\Leftrightarrow (\exists V(f(x_0)) \pitchfork (\forall U(x_0))(f(U(x_0)) \cap \complement V(f(x_0)) \neq \phi)$$
$$\Rightarrow (\forall n \in \mathbb{N})(\exists x_n \in B(x_0; \tfrac{1}{n})) \pitchfork (f(x_n) \in \complement V(f(x_0)))$$

Clearly $x_n \to x_0$ while the sequence $\{f(x_n)\}$ stays away from $V(f(x_0)))$.

□

Proof. $(4 \implies 1)$

Let $\epsilon > 0$. For each $B(f(x_0); \epsilon) \in \tau_Y$, $\exists U(x_0) \in \tau_X \pitchfork f(U) \subset B(f(x_0); \epsilon)$. Since $U(x_0)$ is open, $\exists \delta > 0 \pitchfork B(x_0; \delta) \subset U$. Thus $f(B(x_0; \delta)) \subset f(U) \subset B(f(x_0); \epsilon)$, and this is equivalent to

$$d(x, x_0) < \delta \implies p(f(x), f(x_0)) < \epsilon.$$

□

1.2.2 Uniform Continuity

Definition 1.6. *(Uniform Continuity)*

Let $f : (X, d) \to (X, \rho)$. We say f is uniformly continuous if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in X)(d(x, y) < \delta \implies \rho(f(x), f(y)) < \epsilon)$$

Examples 1. *Let $a < b \in \mathbb{R}$. Then $T : [a, b] \to \mathbb{R}$ defined by $T(x) = x + a$ is uniformly continuous*

Definition 1.7. *(Distance from a set)*

Let (X,d) be a metric space. Let $A, B \subset X$. Then

d(x,A)=inf$\{d(x, y) : y \in A\}$

Theorem 6. *Let (X, d) be a metric space. Let $\phi \neq A \subset X$. Define $f : X \to \mathbb{R}$ by $f(x) = d(x, A)$. Then f is uniformly continuous on X.*