Problem 10

Theorem 1. *Let X be an arbitrary topological space. Suppose for each $x \in X$,*

$$\bigcap \{\overline{U_x} | U_x \textit{ is a nbh of } x\} = \{x\}.$$

Then, $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$.

Proof. Let $x \in X$. Since the closure of any set is closed, Proposition 1.1 tells us that $\{x\}$ is closed in X, because $\{x\}$ is written as an intersection of closed sets. We have

$$\begin{aligned} \{x\} \text{ is closed} &\iff \mathbb{C}\{x\} \text{ is open} \\ &\implies \mathbb{C}\{x\} \times \mathbb{C}\{x\} \text{ is open in } X \times X \\ &\iff \{(y, z) : y, z \in \mathbb{C}\{x\}\} \text{ is open in } X \times X \\ &\iff \{(y, z) : y, z \in X \wedge y \neq x \wedge z \neq x\} \text{ is open in } X \times X \\ &\iff \mathbb{C}\{(y, z) : y, z \in X \wedge y \neq x \wedge z \neq x\} \text{ is closed in } X \times X \\ &\iff \{(y, z) : y, z \in X \wedge y = z = x\} \text{ is closed in } X \times X \\ &\iff \{(x, x)\} \text{ is closed in } X \times X. \end{aligned}$$

Thus, $\{(x, x)\}$ is closed in $X \times X$ for all $x \in X$. Then, again invoking Proposition 1.1, we have that

$$\Delta = \bigcap_{x \in X} \{(x, x)\} \text{ is closed in } X \times X,$$

and our proof is complete. □

Problem 11

Theorem 2. *Let X, Y be metric spaces with metrics d_X and d_Y respectively, and let $f, g : X \to Y$ be continuous functions. Let $h : X \to Y \times Y$ be a function defined by $h(x) = (f(x), g(x))$. Then, h is continuous.*

Proof. We first remark that $Y \times Y$ forms a metric space with metric $d : Y \times Y \to \mathbb{R}$ defined by

$$d((x_1, y_1), (x_2, y_2)) = d_Y(x_1, x_2) + d_Y(y_1, y_2).$$

Let $x \in X$, let $\epsilon > 0$, and let $\{x_k\}_{k=1}^{\infty}$ be a sequence in X that converges to x. Since f is continuous, We have

$$(\exists K_f \in \mathbb{N})(k \geq K_f \implies d_Y(f(x_k), f(x)) < \epsilon/2),$$

by problem 8 of homework 3. Likewise, we have

$$(\exists K_g \in \mathbb{N})(k \geq K_g \implies d_Y(g(x_k), g(x)) < \epsilon/2).$$

Define $K := \max\{K_f, K_g\}$. Then,

$$k \geq K \implies [d_Y(f(x_k), f(x)) < \epsilon/2] \wedge [d_Y(g(x_k), g(x)) < \epsilon/2].$$

With this, we have

$$\begin{aligned} d(h(x_k), h(x)) &= d((f(x_k), g(x_k)), (f(x), g(x))) \\ &= d_Y(f(x_k), f(x)) + d_Y(g(x_k), g(x)) \\ &\leq \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

Since this is true for any sequence in X that converges to x, we can conclude that h is continuous at x. Since x was arbitrary, we have that h is continuous for all $x \in X$, as desired. □

Problem 12

To start this problem, we will prove a useful Lemma:

Lemma 1. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous. Then*

1. *$af + bg$ is continuous for all $a, b \in \mathbb{R}$.*
2. *fg is continuous.*
3. *If $f(\mathbb{R}) \cap \{0\} = \emptyset$, then $\frac{1}{f}$ is continuous.*

Proof. 1. Let $a, b, t_0 \in \mathbb{R}$. The case were either a or b is 0 is trivial, so we will assume $a \neq 0$, and $b \neq 0$. Let $\epsilon > 0$. Since f and g are continuous at t_0, there exists a $\delta > 0$ such that

$$|t - t_0| < \delta \implies (|f(t) - f(t_0)| < \frac{\epsilon}{2|a|}) \wedge (|g(t) - g(t_0)| < \frac{\epsilon}{2|b|}).$$

Then, for all $t \in \mathbb{R}$ with $|t - t_0| < \delta$, we have

$$\begin{aligned}
|(af(t) + bg(t)) - (af(t_0) + bg(t_0))| &= |a(f(t) - f(t_0)) + b(g(t)) - g(t_0))| \\
&\leq |a(f(t) - f(t_0))| + |b(g(t)) - g(t_0))| \\
&= |a||f(t) - f(t_0)| + |b||g(t)) - g(t_0)| \\
&< |a|\frac{\epsilon}{2|a|} + |b|\frac{\epsilon}{2|b|} \\
&= \epsilon.
\end{aligned}$$

Since ϵ and t_0 are arbitrary, we have that $af + bg$ is continuous.

2. Let $\epsilon, \epsilon_0 > 0$, and let $t_0 \in \mathbb{R}$. Since f is continuous, there exists a $\delta_f > 0$ such that

$$|t - t_0| < \delta_f \implies |f(t) - f(t_0)| < \epsilon_0.$$

Likewise, since g is continuous, there exists a $\delta_g > 0$ such that

$$|t - t_0| < \delta_f \implies |g(t) - g(t_0)| < \epsilon_0.$$

Define $\delta = \min\{\delta_f, \delta_g\}$. Then, we have that for all $t \in \mathbb{R}$ with $|t - t_0| < \delta$,

$$\begin{aligned}
|f(t)g(t) - f(t_0)g(t_0)| &= |f(t)g(t) - f(t_0)g(t_0) + f(t)g(t_0) - f(t)g(t_0)| \\
&= |f(t)(g(t) - g(t_0)) + g(t_0)(f(t) - f(t_0))| \\
&\leq |f(t)||g(t) - g(t_0)| + |g(t_0)||f(t) - f(t_0)| \\
&< (|f(t_0)| + \epsilon_0)|g(t) - g(t_0)| + |g(t_0)||f(t) - f(t_0)| \\
&< (|f(t_0)| + \epsilon_0)\epsilon_0 + |g(t_0)|\epsilon_0 \\
&= \epsilon_0(|f(t_0)| + \epsilon_0 + |g(t_0)|).
\end{aligned}$$

This expression gets arbitrarily small as we make ϵ_0 arbitrarily small (and therefore we can make it smaller than ϵ). With this, we can can conclude that there exists a $\delta > 0$ such that

$$|t - t_0| < \delta_f \implies |f(t)g(t) - f(t_0)g(t_0)| < \epsilon.$$

Since ϵ and t_0 are arbitrary, we have shown that fg is continuous.

3. Let $\epsilon, \epsilon_0 > 0$, and let $t_0 \in \mathbb{R}$. Since f is continuous, there exists a $\delta > 0$ such that

$$|t - t_0| < \delta \implies |f(t) - f(t_0)| < \epsilon_0.$$

$$\begin{aligned}
|\frac{1}{f(t)} - \frac{1}{f(t_0)}| &= |\frac{f(t_0) - f(t)}{f(t)f(t_0)}| \\
&= \frac{|f(t_0) - f(t)|}{|f(t)||f(t_0)|} \\
&< \frac{|f(t_0) - f(t)|}{(|f(t_0)| - \epsilon_0)|f(t_0)|} \\
&< \frac{\epsilon_0}{(|f(t_0)| - \epsilon_0)|f(t_0)|}.
\end{aligned}$$

Once more, we can make ϵ_0 arbitrarily small, and in a similar fashion to part (2), we can conclude that $\frac{1}{f}$ is continuous.

$\square$

Theorem 3. *Let $K = \{(x, y) : x^2 + y^2 = y\}$. Let $h : \mathbb{R} \to K$ be defined such that $h(t)$ is where the line segment from $(t, 0)$ to $(0, 1)$ meets K. Then, $h(t)$ is continuous.*

Proof. For $t \neq 0$, we have that the equation for the line segment from $(t, 0)$ to $(0, 1)$ is given by $y = -\frac{1}{t}x + 1$. Plugging this in for the expression $x^2 + y^2 = y$ yields

$$\begin{aligned}
x^2 + y^2 &= y \\
x^2 + (-\frac{1}{t}x + 1)^2 &= -\frac{1}{t}x + 1 \\
x^2 + \frac{x^2}{t^2} - \frac{2}{t}x + 1 &= -\frac{1}{t}x + 1 \\
x^2 + \frac{x^2}{t^2} - \frac{1}{t}x &= 0 \\
x(x + x\frac{1}{t^2} - \frac{1}{t}) &= 0 \\
x(1 + \frac{1}{t^2}) - \frac{1}{t} &= 0 \\
x &= \frac{1}{t(1 + \frac{1}{t^2})} \\
x &= \frac{1}{t + \frac{1}{t}} \\
x &= \frac{t}{t^2 + 1}.
\end{aligned}$$

Plugging this back into our expression for y, we have

$$\begin{aligned}
y &= -\frac{1}{t}x + 1 \\
&= -\left(\frac{1}{t}\right)\frac{t}{t^2 + 1} + 1 \\
&= 1 - \frac{1}{t^2 + 1}.
\end{aligned}$$

Therefore,

$$h(t) = (\frac{t}{t^2 + 1}, 1 - \frac{1}{t^2 + 1}).$$

Although we derived this expression under the assumption that $t \neq 0$, it is easy to see that this expression works for $t = 0$ as well.

Trivially, we have that the function $f(t) = t$ is continuous. Using part 2 of Lemma 3, we have that $f(t) = t^2$ is continuous. From part 1 of Lemma 3, we have that $f(t) = t^2 + 1$ is continuous. From part 3 of Lemma 3, we have that $f(t) = \frac{1}{t^2+1}$ is continuous. From part 2 of Lemma 3, we have $f(t) = \frac{t}{t^2+1}$ is continuous. Finally, from part 1 of Lemma 3, we have that $f(t) = 1 - \frac{1}{t^2+1}$ is continuous.

We have just shown that the x and y components of h are continuous. From Theorem 2 in Problem 11, we can conclude that h is continuous, as desired. $\square$