Problem 13

Theorem 1. *The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x+2}$ is continuous at $x = 1$.*

Proof. Let $\epsilon > 0$. Let $\delta_0 > 0$. Then, for all $x \in \mathbb{R}$ with $|x - 1| < \delta_0$, we have

$$\begin{aligned}
|f(x) - f(1)| &= |\frac{1}{x+2} - \frac{1}{1+2}| && \text{This is well defined if we impose } \delta_0 \leq 3 \text{ because } \delta_0 \leq 3 \implies x \neq -2 \\
&= |\frac{1}{x+2} - \frac{1}{3}| \\
&= |\frac{3 - x - 2}{3(x+2)}| \\
&= |\frac{1 - x}{3(x+2)}| \\
&= \frac{|1 - x|}{|3(x+2)|} \\
&< \frac{\delta_0}{|3(x+2)|} \\
&\leq \frac{\delta_0}{3((1-\delta)+2)} && \text{Now imposing further that } \delta_0 \leq 1 \\
&= \frac{\delta_0}{9 - 3\delta_0}.
\end{aligned}$$

Now what we want to do is choose a δ_0 such that this expression is less than or equal to ϵ:

$$\begin{aligned}
\frac{\delta_0}{9 - 3\delta_0} &\leq \epsilon \\
\delta_0 &\leq 9\epsilon - 3\epsilon\delta_0 \\
\delta_0(1 + 3\epsilon) &\leq 9\epsilon \\
\delta_0 &\leq \frac{9\epsilon}{1 + 3\epsilon}.
\end{aligned}$$

Assigning $\delta = \min\{1, \frac{9\epsilon}{1+3\epsilon}\}$, we have shown that

$$(\forall x \in \mathbb{R})(|x - 1| < \delta \implies |f(x) - f(1)| < \epsilon).$$

Therefore, since ϵ was arbitrary, we have shown that f is continuous at $x = 1$. □

Problem 14.

Before we jump into this problem, we are going to prove a useful lemma that allows us to describe the product topology in terms of a simple basis (as apposed to the standard subbasis definition).

Lemma 1. *Let $\Pi_{k=1}^n X_k$ be a product space. Then, the set $\mathcal{B} = \{\Pi_{k=1}^n U_k : U_k \in \mathcal{T}_k\}$ forms a basis for the product topology.*

Proof. By the usual definition we have

$$\Sigma = \{p_k^{-1}(U_k) : U_k \in \mathcal{T}_k\}$$

generates the product topology. Since any basis is automatically a subbasis, it will suffice to show that every element of $\mathcal{B}$ can be written as a union of finite intersections of elements in Σ(to show that $\mathcal{B}$ generates a coarser topology than Σ does), and that every element of Σ is an element of $\mathcal{B}$(to show that $\mathcal{B}$ generates a finer topology than Σ does).

Let $p_k^{-1}(U_k)$ be an element of Σ. Written out, this element is

$$p_k^{-1}(U_k) = X_1 \times ...X_{k-1} \times U_k \times X_{k+1} \times ... \times X_n.$$

Thus, since $X_j \in \mathcal{T}_j$ for $j \in \{1, .., n\}$, and $U_k \in \mathcal{T}_k$, we have that $p_k^{-1}(U_k) \in \mathcal{B}$.

Now let $\Pi_{k=1}^n U_k \in \mathcal{B}$. Then, we have easily that $\Pi_{k=1}^n U_k = \bigcap_{k=1}^n p_k^{-1}(U_k)$. We have now shown that $\mathcal{B} = \{\Pi_{k=1}^n U_k : U_k \in \mathcal{T}_k\}$ is a basis for the product topology generated by Σ. □

Theorem 2. *Let X, Y be topological spaces and let $f : X \to Y$ be continuous. Show that the mapping $h : X \times Y \to Y \times Y$ defined by $h(x, y) = (f(x), y)$ is continuous.*

Proof. Let $\mathcal{B} = \{U \times V : U, V \in T_Y\}$ be a basis for the product topology on $Y \times Y$. By our continuity facts theorem, it will suffice to show that $h^{-1}(A)$ is open in $X \times X$ for each $A \in B$. Let $U \times V \in \mathcal{B}$. Then

$$\begin{aligned} h^{-1}(U \times V) &= \{(a, b) : a \in X \wedge b \in V \wedge f(a) \in U\} \\ &= f^{-1}(U) \times V. \end{aligned}$$

By continuity of f, $f^{-1}(U) \in \mathcal{T}_X$. By initial assumption, $V \in \mathcal{T}_Y$. Thus, $h^{-1}(U \times V) \in \mathcal{T}_X \times \mathcal{T}_Y$, and is therefore open. From this, we can conclude that h is continuous. □

Problem 15

Theorem 3. *Let $\{X_k\}_{k=1}^n$ be a family of topological spaces. Suppose $\Pi_{k=1}^n X_k$ is second countable. Then, X_k is second countable for some X_k for some $k \in \{1, ..., n\}$.*

Proof. Suppose $\Pi_{k=1}^n X_k$ is second countable. Then, there exists a countable basis $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ of $\Pi_{k=1}^n X_k$. We propose that $\{p_j(B_n) : n \in \mathbb{N}\}$ forms a countable basis of X_j for each $j \in \{1, ..., n\}$.

Let $U_j \in \mathcal{T}_j$. Then, there exists an indexing set $I \subseteq \mathbb{N}$ such that

$$p_j^{-1}(U_j) = \bigcup_{i \in I} B_i.$$

Then, we have

$$\begin{aligned} U_j &= p_j(\bigcup_{i \in I} B_i) \\ &= \bigcup_{i \in I} p_j(B_i), \end{aligned}$$

and our proof is complete. □