Theorem 1. *Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous bijection. Then f is a homeomorphism iff either*

1. *f is open*
2. *f is closed.*

Proof. Suppose f is a homeomorphism. Let $G \in \mathcal{T}_X$. Then, the inverse of f^{-1} is continuous, and

$$f(G) = (f^{-1})^{-1}(G)$$

which is open, by definition. The argument works if you switch open for closed in the above.

Suppose now that f is open. Let $X \in \mathcal{T}_X$. Then $f(X) = (f^{-1})^{-1}(X)$ is open in Y. Therefore, f has a continuous inverse, and is therefore a homeomorphism. Swapping open for closed in the above argument will complete our proof. $\square$

Theorem 2. *Let $\{X_k\}_{k=1}^n$ be a finite collection of topological spaces, and let X be a topological space. Let $f : X \to \Pi_{k=1}^n X_k$. Then the following are equivalent:*

1. *f is continuous*
2. *$p_j \circ f$ is continuous for all $j \in \{1, .., n\}$.*

Proof. (1) $\implies$ (2). Suppose that f is continuous. We have by definition that p_j is continuous. Let $U_j \in \mathcal{T}_{X_j}$. Then, since p_j is continuous, $p_j^{-1}(U_j)$ is open in $\Pi_{k=1}^n X_k$. Then, since f is continuous, $f^{-1}(p_j^{-1}(U_j)) \in \mathcal{T}_X$. Since $f^{-1}(p_j^{-1}(U_j)) = (p_j \circ f)^{-1}(U_j)$, we can conclude that $p_j \circ f$ is continuous for all $j \in \{1, .., n\}$.

(2) $\implies$ (1). Now suppose $p_j \circ f$ is continuous for all $j \in \{1, .., n\}$. By one of our many continuity theorems, we have that f is continuous if and only if $f^{-1}(U)$ is open for all members U of a subbasis of $\Pi_{k=1}^n X_k$. Let $U_j \in \mathcal{T}_j$. Then, $p^{-1}(U_j)$ is, by definition of the product topology, a member of a subbasis of $\Pi_{k=1}^n X_k$. We have

$$f^{-1}(p^{-1}(U_j)) = (p_j \circ f)^{-1}(U_j),$$

which by initial assumption we know is continuous. $\square$