Metric Spaces Fall 2020 Lecture Notes

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1 Theory of Functions

Definition 1.1. ***(Image of a Set).*** *Let $A \subseteq X$. Then*

$$f(A) = \{y \in Y : y = f(x) \text{ for some } x \in A\} \text{ and } f^{-1}(G) = \{x \in X : f(x) \in G\}$$

Definition 1.2. ***(Point in the Inverse or Image).*** *Let $f : X \Rightarrow Y$ be a function between two topological spaces. Then for subsets $A \subseteq X$ and $G \subseteq Y$,*

$$y \in f(A) \Leftrightarrow y = f(x) \text{ for some } x \in A$$

$$x \in f^{-1}(G) \Leftrightarrow f(x) \in G$$

Theorem 1. *Let X, Y be nonempty sets and let $f : X \to Y$. Then,*

1. $f(A \cup B) = f(A) \cup f(B)$
2. $A \subseteq B \Rightarrow f(A) \subseteq f(B)$
3. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
4. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
5. $f(A \cap B) \subseteq f(A) \cap f(B)$
6. $f^{-1}(\mathbb{C}A) = \mathbb{C}(f^{-1}(A))$
7. $f(\mathbb{C}A)???\mathbb{C}(f(A))$ *(this is an open question)*

Proof. Proof here. ☐

Definition 1.3. ***(Injective).*** *$f : X \to Y$ is* ***injective*** *if $\forall x_1, x_2 \in X$, where $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. This is equivalent to say that for all $x_1, x_2 \in X$, such that*

$$f(x) = f(y) \Rightarrow x_1 = x_2.$$

Theorem 2. *f is injective if and only if $f(A \cap B) = f(A) \cap f(B)$, for all $A, B \subseteq X$*

Proof. Proof here. ☐

Definition 1.4. ***(Surjective).*** *A function $f : X \to Y$ is surjective if every $y \in Y$ has at least one preimage in X.*

Theorem 3. *Let $f : X \to Y$. Then,*

1. $f(f^{-1}(B)) \subseteq B$, *for every $B \subseteq Y$*

2. $A \subseteq f^{-1}(f(A))$, *for every* $A \subseteq X$

3. f *is injective* $\Leftrightarrow A = f^{-1}(f(A))$, *for every* $A \subseteq X$

4. f *is surjective* $\Leftrightarrow f(f^{-1}(B)) = B$, *for every* $B \subseteq Y$

Proof. Proof here. □

2 Distance and Metric Spaces

Definition 2.1. *Let X be a nonempty set, and let $d : X \times X \to \mathbb{R}$. We say that d is a metric for X if the following holds :*

1. $d(x, y) \geq 0$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if* $x = y$

2. $d(x, y) = d(y, x)$ *for all* $x, y \in X$

3. $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

The pair (X, d) is called a metric space.

Definition 2.2. *Let (X, d) be a metric space and let A be a subset of X. A is said to be bounded if there exists $r > 0$ such that*

$$d(x, x_0) \leq r \text{ for all} x \in A \text{ and some } \; x_0 \in X.$$

Examples :

1. Let X be a nonempty set. Define

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

2. $d(x, y) = |x - y|$ is a metric for $\mathbb{R}$

3. Let $X = \mathbb{R}^n$. Define

$$d(x, y) = \big[\sum_{k=1}^{n} (x_k - y_k)^2\big]^{1/2}.$$

2.1 Space of Sequences

4. Let X be the space of bounded sequences

   1. $l^\infty = \{\{x_k\} : \{x_k\} \text{ is bounded}\}$ with the metric of supremum, defined as follows :

$$d(s, t) = \sup_k \{|s_k - t_k|\}.$$

2. $l^2 = \left\{ \{x_k\} : \sum_{k=1}^{\infty} |x_k|^2 < \infty \right\}$ with the metric defined as:

$$d(s,t) = \sum_{k=1}^{\infty} |s_k - t_k|^2.$$

3. $l^1 = \left\{ \{x_k\} : \sum_{k=1}^{\infty} |x_k| < \infty \right\}$ with the metric defined as:

$$d(s,t) = \sum_{k=1}^{\infty} |s_k - t_k|.$$

5. $C[a,b] = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$ is a metric space under the following metric:

$$d(f,g) = \sup_{t \in [a,b]} \{|f(t) - g(t)|\}.$$

Learning Activity : *Show that the addition and scalar multiplication for each one these spaces is well-defined.*

3 Topology

3.1 Topological Basics

3.1.1 Definition/Motivation

Definition 3.1. *(Topology)*

Let $X \neq \phi$. Let $\mathcal{T} \subset 2^X$. Further, Suppose the following.

1.) $\{X, \phi\} \subset \mathcal{T}$.

2.) $\{G_\alpha\}_{\alpha \in \mathscr{A}} \subset \mathcal{T} \implies \bigcup_{\alpha \in \mathscr{A}} G_\alpha \in \mathcal{T}$

3.) $\{G_i\}_{i=1}^n \subset \mathcal{T} \implies \bigcap_{i=1}^n G_i \in \mathcal{T}$

Then we call $\mathcal{T}$ a **Topology** on X and we call the pair $(X, \mathcal{T})$ a **topological space**

If $U \in \mathcal{T}$, we call U an **Open Set.**

Examples 1. $X = \{0, 1\}$

$\mathcal{T}_1 = \{\emptyset, X\}$ *is a topology for X, the* ***indiscrete topology***.

$\mathcal{T}_2 = \{\emptyset, X, \{0\}, \{1\}\}$ *is a topology for X, the* ***discrete topology***.

$\mathcal{T}_3 = \{\emptyset, X, \{0\}\}$ *is a topology for X, the* ***Sierpinski topology***.

Theorem 4. *Let (X, d) be a metric space. Then the collection*

$$\mathcal{T} = \{G : G \text{ is open}\} \text{ is a topology for } X$$

known as the topology induced by the metric, and denoted by $\mathcal{T}(d)$.

Proof. Done in class. $\square$

Definition 3.2. *(Neighborhood)*

Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$. A **neighborhood** of X is a set U_x such that $x \in U_x \in \mathcal{T}$.

Notation: A set U_x or V_y means $x \in U_x \in \mathcal{T}$ or $y \in V_y \in \mathcal{T}$ respectively.

The Purpose of Topology as a study is to determine what properties of spaces are preserved under continuous mappings. This motivates the following definition.

3.1.2 Accumulation point

Definition 3.3. *(Accumulation Point)*

Let $(X, \mathcal{T})$ be a topological space. Let $A \subset X$. A point $x \in X$ is said to be an accumulation point of A if

$$(\forall U_x)(U_x \cap A \setminus \{x\} \neq \phi)$$

We denote the set of accumulation points of A with A'.

Learning Activities : *1. Negate definition of accumulation point.*
2. Create examples with finite, as well as infinite, number of accumulation points.
3. Build an example of an infinite bounded set of $\mathbb{R}^2$ with no accumulation points.

Definition 3.4. *(Closed set)*

Let $(X, \mathcal{T})$ be a topological space. A set F is said to be a **Closed set** if $F' \subset F$.

Learning Activity : *Make up nontrivial examples of closed sets in $\mathbb{R}$, as well as, $\mathbb{R}^2$.*

Theorem 5. *(Closed characterization)*

Let $(X, \mathcal{T})$ be a topological space. Let $A \subset X$. Then

$$A \text{ is closed} \iff \mathbb{C}A \in \mathcal{T}.$$

Proof. $(\Rightarrow)$. Let $p \in \mathbb{C}A$

$\square$

Proof. $(\Leftarrow)$ Let $p \in A'$

$\square$

Proposition 3.1. *(Closed Union/ Closed intersection)*

Let (X, τ) $\{S_\alpha\}_{\alpha \in \mathscr{A}}$ be a collection of closed sets in X. Then the following hold

1.) $\bigcap_{\alpha \in \mathscr{A}} S_\alpha$ is closed

2.) Finite unions of elements of $\{S_\alpha\}_{\alpha \in \mathscr{A}}$ are closed.

Proof. De Morgan's Law

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Proposition 3.2. *Let $(X, \mathcal{T})$ be a topological space. Let $G \subset X$. Then*

$$G \in \mathcal{T} \iff (\forall x \in G)(\exists U_x \subset G)$$

Proof. ($\Rightarrow$) Obvious

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Proof. ($\Leftarrow$) Write G

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