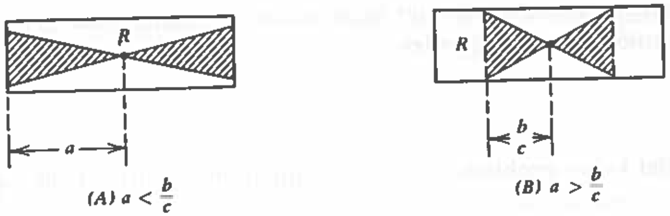

Fig. 54. Geometric illustration of inequality (2) for (A) relatively small c, (B) relatively large c. The solution curve must remain in the shaded region bounded by straight lines with slopes $\pm c$.

that for $(t, x), (t, v) \in R$

$$|f(t, x) - f(t, v)| \leqq k\, |x - v|. \tag{3}$$

Then the initial value problem (1) *has a unique solution. This solution exists on an interval* $[t_0 - \beta, t_0 + \beta]$, *where*[1]

$$\beta < \min\left\{a, \frac{b}{c}, \frac{1}{k}\right\}. \tag{4}$$

Proof. Let $C(J)$ be the metric space of all real-valued continuous functions on the interval $J = [t_0 - \beta, t_0 + \beta]$ with metric d defined by

$$d(x, y) = \max_{t \in J} |x(t) - y(t)|.$$

$C(J)$ is complete, as we know from 1.5-5. Let $\tilde{C}$ be the subspace of $C(J)$ consisting of all those functions $x \in C(J)$ that satisfy

$$|x(t) - x_0| \leqq c\beta. \tag{5}$$

It is not difficult to see that $\tilde{C}$ is closed in $C(J)$ (cf. Prob. 6), so that $\tilde{C}$ is complete by 1.4-7.

By integration we see that (1) can be written $x = Tx$, where $T\colon \tilde{C} \longrightarrow \tilde{C}$ is defined by

$$Tx(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\, d\tau. \tag{6}$$

[1] In the classical proof, $\beta < \min\{a, b/c\}$, which is better. This could also be obtained by a modification of the present proof (by the use of a more complicated metric); cf. A.

Indeed, T is defined for all $x \in \tilde{C}$, because $c\beta < b$ by (4), so that if $x \in \tilde{C}$, then $\tau \in J$ and $(\tau, x(\tau)) \in R$, and the integral in (6) exists since f is continuous on R. To see that T maps $\tilde{C}$ into itself, we can use (6) and (2), obtaining

$$|Tx(t) - x_0| = \left|\int_{t_0}^{t} f(\tau, x(\tau))\, d\tau\right| \leqq c\, |t - t_0| \leqq c\beta.$$

We show that T is a contraction on $\tilde{C}$. By the Lipschitz condition (3),

$$\begin{aligned} |Tx(t) - Tv(t)| &= \left|\int_{t_0}^{t} [f(\tau, x(\tau)) - f(\tau, v(\tau))]\, d\tau\right| \\ &\leqq |t - t_0| \max_{\tau \in J} k\, |x(\tau) - v(\tau)| \\ &\leqq k\beta\, d(x, v). \end{aligned}$$

Since the last expression does not depend on t, we can take the maximum on the left and have

$$d(Tx, Tv) \leqq \alpha d(x, v) \qquad \text{where} \qquad \alpha = k\beta.$$

From (4) we see that $\alpha = k\beta < 1$, so that T is indeed a contraction on $\tilde{C}$. Theorem 5.1-2 thus implies that T has a unique fixed point $x \in \tilde{C}$, that is, a continuous function x on J satisfying $x = Tx$. Writing $x = Tx$ out, we have by (6)

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\, d\tau. \tag{7}$$

Since $(\tau, x(\tau)) \in R$ where f is continuous, (7) may be differentiated. Hence x is even differentiable and satisfies (1). Conversely, every solution of (1) must satisfy (7). This completes the proof. ∎

Banach's theorem also implies that the solution x of (1) is the limit of the sequence $(x_0, x_1, \cdots)$ obtained by the *Picard iteration*

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(\tau, x_n(\tau))\, d\tau \tag{8}$$