Theorem 1. *Let* $A_n = \left[0, \frac{1}{n}\right] = \{x \in \mathbb{R} : 0 \leq x \leq \frac{1}{n}\}$. *Then,*

$$\bigcap_{n\in\mathbb{N}} A_n = \{0\}.$$

Proof. We have

$$\begin{aligned} x \in \{0\} &\implies x = 0 \\ &\implies (\forall n \in \mathbb{N})(0 \leq x \leq \frac{1}{n}) \\ &\implies (\forall n \in \mathbb{N})(x \in A_n) \\ &\implies x \in \bigcap_{n\in\mathbb{N}} A_n. \end{aligned}$$

Therefore,

$$\{0\} \subseteq \bigcap_{n\in\mathbb{N}} A_n.$$

Suppose

$$x \in \bigcap_{n\in\mathbb{N}} A_n,$$

and for the sake of contradiction that $x \notin \{0\}$. We can see

$$\begin{aligned} x \in \bigcap_{n\in\mathbb{N}} A_n \wedge x \notin \{0\} &\implies (\exists n \in \mathbb{N})(x \in A_n \wedge x \notin \{0\}) \\ &\implies (\exists n \in \mathbb{N})(x \in A_n \wedge x \neq 0) \\ &\implies (\exists n \in \mathbb{N})(0 < x \leq \frac{1}{n}) \\ &\implies (\exists m \in \mathbb{N})(\frac{1}{m} < x) \\ &\implies (\exists m \in \mathbb{N})(x \notin A_m) \\ &\implies x \notin \bigcap_{n\in\mathbb{N}} A_n, \end{aligned}$$

which is a contradiction. Therefore, $x \in \{0\}$, and we can conclude that

$$\bigcap_{n\in\mathbb{N}} A_n = \{0\}.$$

□

Theorem 2. *Let* $f : X \to Y$. *Then*

1. $f(A \cup B) = f(A) \cup f(B)$
2. $A \subseteq B \implies f(A) \subseteq f(B)$
3. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
4. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

5. $f(A \cap B) \subseteq f(A) \cap f(B)$
6. $f^{-1}(\mathbb{C}A) = \mathbb{C}(f^{-1}(A))$
7. $f(\mathbb{C}A)???\mathbb{C}(f(A))$ *(this is an open question)*

Proof. 1. Applying the definition of a set's image and the definition of unions of sets, we see

$$\begin{aligned} y \in f(A \cup B) &\iff (\exists x \in A \cup B)(f(x) = y) \\ &\iff (\exists x \in A)(f(x) = y) \vee (\exists x \in B)(f(x) = y) \\ &\iff y \in f(A) \vee y \in f(B) \\ &\iff y \in f(A) \cup f(B). \end{aligned}$$

From this, we can conclude that $f(A \cup B) = f(A) \cup f(B)$.

2. Suppose $A \subseteq B$. We have

$$\begin{aligned} y \in f(A) &\iff (\exists x \in A)(f(x) = y) \\ &\implies (\exists x \in B)(f(x) = y) && \text{By initial assumption} \\ &\iff y \in f(B), \end{aligned}$$

which means $f(A) \subseteq f(B)$.

3. By definition of a set's preimage and the definition of a union of sets, we have

$$\begin{aligned} x \in f^{-1}(C \cup D) &\iff f(x) \in C \cup D \\ &\iff f(x) \in C \vee f(x) \in D \\ &\iff x \in f^{-1}(C) \vee x \in f^{-1}(D) \\ &\iff x \in f^{-1}(C) \cup f^{-1}(D). \end{aligned}$$

Therefore, $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

4. From the definition of a set's preimage and the definition of an intersection of two sets,

$$\begin{aligned} x \in f^{-1}(C \cap D) &\iff f(x) \in C \cap D \\ &\iff f(x) \in C \wedge f(x) \in D \\ &\iff x \in f^{-1}(C) \wedge x \in f^{-1}(D) \\ &\iff x \in f^{-1}(C) \cap f^{-1}(D). \end{aligned}$$

Thus, $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

5. From the definition of a set's image and the definition of an intersection of sets,

$$\begin{aligned} y \in f(A \cap B) &\iff (\exists x \in A \cap B)(f(x) = y) \\ &\iff (\exists x)(x \in A \wedge x \in B \wedge f(x) = y) \\ &\implies (\exists x \in A)(f(x) = y) \wedge (\exists x \in B)(f(x) = y) \\ &\iff y \in f(A) \wedge y \in f(B) \\ &\iff y \in f(A) \cap f(B). \end{aligned}$$

Thus, $f(A \cap B) \subseteq f(A) \cap f(B)$.

6. From the definitions of a set's preimage and the complement of a set, we have

$$\begin{aligned} x \in f^{-1}(\complement A) &\iff f(x) \in \complement A \\ &\iff f(x) \in Y \wedge f(x) \notin A \\ &\iff x \in f^{-1}(Y) \wedge x \notin f^{-1}(A) \\ &\iff x \in \complement f^{-1}(A). \end{aligned}$$

   Therefore, $f^{-1}(\complement A) = \complement(f^{-1}(A))$.

7. Expanding the sets by definition, can see

$$\begin{aligned} y \in \complement f(A) &\iff y \in f(X) \wedge y \notin f(A) \\ &\iff (\exists x \in X)(y = f(x) \wedge y \notin f(A)) \\ &\iff (\exists x \in X)(f(x) \notin f(A) \wedge f(x) = y) \\ &\implies (\exists x \in X)(x \notin A \wedge f(x) = y) \\ &\iff (\exists x \in \complement A)(f(x) = y) \\ &\iff y \in f(\complement A), \end{aligned}$$

   which leads us to conclude that $\complement f(A) \subseteq f(\complement A)$.

Finally, our proof is complete. $\square$