Problem 10

Theorem 1. Let X be an arbitrary topological space. Suppose for each $x \in X$,

$$\bigcap \{ \overline{U_x} | U_x \text{ is a nbh of } x \} = \{x\}.$$

Then, $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$.

Proof. Let $x \in X$. Since the closure of any set is closed, Proposition 1.1 tells us that $\{x\}$ is closed in X, because $\{x\}$ is written as an intersection of closed sets. We have

$$\{x\} \text{ is closed} \iff \mathbb{C}\{x\} \text{ is open}$$

$$\implies \mathbb{C}\{x\} \times \mathbb{C}\{x\} \text{ is open in } X \times X$$

$$\iff \{(y,z): y,z \in \mathbb{C}\{x\}\} \text{ is open in } X \times X$$

$$\iff \{(y,z): y,z \in X \land y \neq x \land z \neq x\} \text{ is open in } X \times X$$

$$\iff \mathbb{C}\{(y,z): y,z \in X \land y \neq x \land z \neq x\} \text{ is closed in } X \times X$$

$$\iff \{(y,z): y,z \in X \land y = z = x\} \text{ is closed in } X \times X$$

$$\iff \{(x,x)\} \text{ is closed in } X \times X.$$

Thus, $\{(x,x)\}$ is closed in $X\times X$ for all $x\in X$. Then, again invoking Proposition 1.1, we have that

$$\Delta = \bigcap_{x \in X} \{(x, x)\} \text{ is closed in } X \times X,$$

and our proof is complete.

Problem 11

Theorem 2. Let X, Y be metric spaces with metrics d_X and d_Y respectively, and let $f, g: X \to Y$ be continuous functions. Let $h: X \to Y \times Y$ be a function defined by h(x) = (f(x), g(x)). Then, h is continuous.

Proof. We first remark that $Y \times Y$ forms a metric space with metric $d: Y \times Y \to \mathbb{R}$ defined by

$$d((x_1, y_1), (x_2, y_2)) = d_Y(x_1, x_2) + d_Y(y_1, y_2).$$

Let $x \in X$, let $\epsilon > 0$, and let $\{x_k\}_{k=1}^{\infty}$ be a sequence in X that converges to x. Since f is continuous, We have

$$(\exists K_f \in \mathbb{N})(k \geq K_f \implies d_Y(f(x_k), f(x)) < \epsilon/2),$$

by problem 8 of homework 3. Likewise, we have

$$(\exists K_a \in \mathbb{N})(k \ge K_a \implies d_Y(g(x_k), g(x)) < \epsilon/2).$$

Define $K := \max\{K_f, K_q\}$. Then,

$$k \ge K \implies [d_Y(f(x_k), f(x)) < \epsilon/2] \land [d_Y(g(x_k), g(x)) < \epsilon/2].$$

With this, we have

$$\begin{aligned} d(h(x_k), h(x)) &= d((f(x_k), g(x_k)), (f(x), g(x))) \\ &= d_Y(f(x_k), f(x)) + d_Y(g(x_k), g(x)) \\ &\leq \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

Since this is true for any sequence in X that converges to x, we can conclude that h is continuous at x. Since x was arbitrary, we have that h is continuous for all $x \in X$, as desired.

September 20, 2021 1

Problem 12

To start this problem, we will prove a useful Lemma:

Lemma 1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous. Then

- 1. af + bg is continuous for all $a, b \in \mathbb{R}$.
- 2. fg is continuous.
- 3. If $f(\mathbb{R}) \cap \{0\} = \emptyset$, then $\frac{1}{f}$ is continuous.

Proof. 1. Let $a, b, t_0 \in \mathbb{R}$. The case were either a or b is 0 is trivial, so we will assume $a \neq 0$, and $b \neq 0$. Let $\epsilon > 0$. Since f and g are continuous at t_0 , there exists a $\delta > 0$ such that

$$|t-t_0|<\delta \implies (|f(t)-f(t_0)|<\frac{\epsilon}{2|a|})\wedge (|g(t)-g(t_0)|<\frac{\epsilon}{2|b|}).$$

Then, for all $t \in \mathbb{R}$ with $|t - t_0| < \delta$, we have

$$\begin{aligned} |(af(t) + bg(t)) - (af(t_0) + bg(t_0))| &= |a(f(t) - f(t_0)) + b(g(t)) - g(t_0))| \\ &\leq |a(f(t) - f(t_0))| + |b(g(t)) - g(t_0))| \\ &= |a||f(t) - f(t_0)| + |b||g(t)) - g(t_0)| \\ &< |a| \frac{\epsilon}{2|a|} + |b| \frac{\epsilon}{2|b|} \\ &= \epsilon. \end{aligned}$$

Since ϵ and t_0 are arbitrary, we have that af + bg is continuous.

2. Let $\epsilon, \epsilon_0 > 0$, and let $t_0 \in \mathbb{R}$. Since f is continuous, there exists a $\delta_f > 0$ such that

$$|t-t_0|<\delta_f \implies |f(t)-f(t_0)|<\epsilon_0.$$

Likewise, since g is continuous, there exists a $\delta_g > 0$ such that

$$|t - t_0| < \delta_f \implies |g(t) - g(t_0)| < \epsilon_0.$$

Define $\delta = \min\{\delta_f, \delta_g\}$. Then, we have that for all $t \in \mathbb{R}$ with $|t - t_0| < \delta_g$

$$|f(t)g(t) - f(t_0)g(t_0)| = |f(t)g(t) - f(t_0)g(t_0) + f(t)g(t_0) - f(t)g(t_0)|$$

$$= |f(t)(g(t) - g(t_0)) + g(t_0)(f(t) - f(t_0))|$$

$$\leq |f(t)||g(t) - g(t_0)| + |g(t_0)||f(t) - f(t_0)|$$

$$< (|f(t_0)| + \epsilon_0)|g(t) - g(t_0)| + |g(t_0)||f(t) - f(t_0)|$$

$$< (|f(t_0)| + \epsilon_0)\epsilon_0 + |g(t_0)|\epsilon_0$$

$$= \epsilon_0(|f(t_0)| + \epsilon_0 + |g(t_0)|).$$

This expression gets arbitrarily small as we make ϵ_0 arbitrarily small (and therefore we can make it smaller than ϵ). With this, we can can conclude that there exists a $\delta > 0$ such that

$$|t-t_0| < \delta_f \implies |f(t)g(t) - f(t_0)g(t_0)| < \epsilon.$$

Since ϵ and t_0 are arbitrary, we have shown that fg is continuous.

3. Let $\epsilon, \epsilon_0 > 0$, and let $t_0 \in \mathbb{R}$. Since f is continuous, there exists a $\delta > 0$ such that

$$|t - t_0| < \delta \implies |f(t) - f(t_0)| < \epsilon_0.$$

$$\begin{split} |\frac{1}{f(t)} - \frac{1}{f(t_0)}| &= |\frac{f(t_0) - f(t)}{f(t)f(t_0)}| \\ &= \frac{|f(t_0) - f(t)|}{|f(t)||f(t_0)|} \\ &< \frac{|f(t_0) - f(t)|}{(|f(t_0)| - \epsilon_0)|f(t_0)|} \\ &< \frac{\epsilon_0}{(|f(t_0)| - \epsilon_0)|f(t_0)|}. \end{split}$$

September 20, 2021 2

Once more, we can make ϵ_0 arbitrarily small, and in a similar fashion to part (2), we can conclude that $\frac{1}{f}$ is continuous.

Theorem 3. Let $K = \{(x,y) : x^2 + y^2 = y\}$. Let $h : \mathbb{R} \to K$ be defined such that h(t) is where the line segment from (t,0) to (0,1) meets K. Then, h(t) is continuous.

Proof. For $t \neq 0$, we have that the equation for the line segment from (t,0) to (0,1) is given by $y = -\frac{1}{t}x + 1$. Plugging this in for the expression $x^2 + y^2 = y$ yields

$$x^{2} + y^{2} = y$$

$$x^{2} + (-\frac{1}{t}x + 1)^{2} = -\frac{1}{t}x + 1$$

$$x^{2} + \frac{x^{2}}{t^{2}} - \frac{2}{t}x + 1 = -\frac{1}{t}x + 1$$

$$x^{2} + \frac{x^{2}}{t^{2}} - \frac{1}{t}x = 0$$

$$x(x + x\frac{1}{t^{2}} - \frac{1}{t}) = 0$$

$$x(1 + \frac{1}{t^{2}}) - \frac{1}{t} = 0$$

$$x = \frac{1}{t(1 + \frac{1}{t^{2}})}$$

$$x = \frac{1}{t + \frac{1}{t}}$$

$$x = \frac{t}{t^{2} + 1}.$$

Plugging this back into our expression for y, we have

$$y = -\frac{1}{t}x + 1$$

$$= -\left(\frac{1}{t}\right)\frac{t}{t^2 + 1} + 1$$

$$= 1 - \frac{1}{t^2 + 1}.$$

Therefore,

$$h(t) = (\frac{t}{t^2 + 1}, 1 - \frac{1}{t^2 + 1}).$$

Although we derived this expression under the assumption that $t \neq 0$, it is easy to see that this expression works for t = 0 as well.

Trivially, we have that the function f(t)=t is continuous. Using part 2 of Lemma 3, we have that $f(t)=t^2$ is continuous. From part 1 of Lemma 3, we have that $f(t)=t^2+1$ is continuous. From part 3 of Lemma 3, we have that $f(t)=\frac{1}{t^2+1}$ is continuous. From part 2 of Lemma 3, we have $f(t)=\frac{t}{t^2+1}$ is continuous. Finally, from part 1 of Lemma 3, we have that $f(t)=1-\frac{1}{t^2+1}$ is continuous.

We have just shown that the x and y components of h are continuous. From Theorem 2 in Problem 11, we can conclude that h is continuous, as desired.

September 20, 2021 3