Supporting Lemmas

Lemma 1. *Let $x : [a, b] \to \mathbb{R}$ be continuous. Then, x is bounded, and x achieves it's maximum.*

Lemma 2. *Let C equal the set of all continuous real-valued functions on the closed interval $[a, b]$. Then, C forms a metric space with the metric d defined by*

$$d(x, y) = \max_{t \in [a,b]} |x(t) - y(t)|.$$

Lemma 3. *The metric space (C, d) from the previous lemma is complete.*

Lemma 4. *Let $x_0 \in \mathbb{R}$, and let $c \in (0, \infty)$. Let $\tilde{C}$ be a subspace of the metric space from Lemma 1, consisting of all functions $x \in C$ such that*

$$d(x, x_0) \leq c.$$

Then, $\tilde{C}$ is closed.

Lemma 5. *Let C be a complete metric space. Let $A \subseteq C$ be closed. Then, A is complete.*

Main Topic

The focus of this paper is on solutions to explicit ordinary differential equations of the form

$$x' = f(t, x) \tag{1}$$

where $x : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and the prime denotes differentiation of x with respect to t. Specifically, we are interested in initial value problems, where

$$x(t_0) = x_0.$$

Theorem 1. *Let f be continuous on a rectangle*

$$R = \{(t, x) \in R^2 : |t - t_0| \leq a \wedge |x - x_0| \leq b\},$$

and thus bounded on R, say

$$|f(t, x)| \leq c, \text{ for all } (t, x) \in R. \tag{2}$$

Suppose f satisfies a Lipschitz condition on R with respect to it's second argument, that is, there is a constant k (Lipschitz constant) such that for $(t, x), (t, v) \in R$

$$|f(t, x) - f(t, v)| \leq k|x - v|. \tag{3}$$

Then the initial value problem (1) has a unique solution. This solution exists on an interval $[t_0 - \beta, t_0 + \beta]$ where

$$\beta < \min\left\{a, \frac{b}{c}, \frac{1}{k}\right\} \tag{4}$$