Theorem 1. *Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$. Then*

1. $A \subseteq B \implies \mathring{A} \subseteq \mathring{B}$
2. $\mathring{A} \cap \mathring{B} \subseteq \operatorname{int}(A \cup B)$
3. $\operatorname{int}(A \cap B) \subseteq \mathring{A} \cap \mathring{B}$
4. $\mathring{A} = \bigcup\{U : U \in \mathcal{T} \wedge U \subseteq A\}$.

Proof. 1. Suppose $A \subseteq B$. Then

$$\begin{aligned} p \in \mathring{A} &\iff (\exists U(p) \in \mathcal{T})(U(p) \subseteq A) \\ &\implies (\exists U(p) \in \mathcal{T})(U(p) \subseteq B) && \text{Since } A \subseteq B \\ &\iff p \in \mathring{B}, \end{aligned}$$

as desired.

2. We have

$$\begin{aligned} p \in \mathring{A} \cap \mathring{B} &\iff p \in \mathring{A} \wedge p \in \mathring{B} \\ &\iff (\exists U_1(p) \in \mathcal{T})(U_1(p) \subseteq A) \wedge (\exists U_2(p) \in \mathcal{T})(U_2(p) \subseteq B) \\ &\implies (\exists U(p) \in \mathcal{T})(U(p) \subseteq A \cup B), \text{ e.g., } U(p) = U_1(p) \\ &\iff p \in \operatorname{int}(A \cup B). \end{aligned}$$

3. In a similar manner

$$\begin{aligned} p \in \operatorname{int}(A \cap B) &\iff (\exists U(p) \in \mathcal{T})(U(p) \subseteq A \cap B) \\ &\iff (\exists U(p) \in \mathcal{T})(U(p) \subseteq A \wedge U(p) \subseteq B) \\ &\iff (\exists U(p) \in \mathcal{T})(U(p) \subseteq A) \wedge (\exists U(p) \in \mathcal{T})(U(p) \subseteq A) \\ &\iff p \in \mathring{A} \wedge p \in \mathring{B} \\ &\iff p \in \mathring{A} \cap \mathring{B}. \end{aligned}$$

4. This follows directly from the definition of a sets interior.

$\square$

Lemma 1. *Let $(X, \mathcal{T})$ be a topological space, let $A, Y \subseteq X$, and let $p \in X$. Then*

$$(\exists U(p) \in \mathcal{T}_Y) \implies (p \in Y). \tag{1}$$

Proof. We have, by definition of a relative topology, that $(\exists V \in \mathcal{T})(U(p) = V \cap Y)$. By definition of a sets intersection, we must have $p \in Y$. $\square$

Theorem 2. *Let $(X, \mathcal{T})$ be a topological space, and let $Y \subseteq X$. Then*

1. $A \subseteq X \implies (\bar{A_Y} = \bar{A} \cap Y) \wedge (A'_Y = A' \cap Y)$
2. $\operatorname{int}(A) \cap Y \subseteq \operatorname{int}_Y(A)$
3. $\partial_Y A \subseteq \partial A \cap Y$.

Proof. 1. Suppose $A \subseteq X$. We have

$p \in \bar{A_Y} \iff (\forall U(p) \in \mathcal{T}_Y)(U(p) \cap A \neq \emptyset)$
$\iff (\forall V(p) \in \mathcal{T})(V(p) \cap Y \cap A \neq \emptyset) \wedge p \in Y$, the first because $V(p) \cap Y \in \mathcal{T}_Y$ is a neighborhood of p, and $p \in Y$ by Lemma 1
$\iff (\forall V(p) \in \mathcal{T})(V(p) \cap A \neq \emptyset) \wedge p \in Y$
$\iff p \in \bar{A} \wedge p \in Y$
$\iff p \in \bar{A} \cap Y.$

Thus, $\bar{A_Y} = \bar{A} \cap Y$. We also have

$p \in A'_Y \iff (\forall U(p) \in \mathcal{T}_Y)(U(p) \cap (A\backslash\{p\}) \neq \emptyset)$
$\iff (\forall V(p) \in \mathcal{T})(V(p) \cap Y \cap (A\backslash\{p\}) \neq \emptyset) \wedge p \in Y$, the first because $V(p) \cap Y \in \mathcal{T}_Y$ is a neighborhood of p, and $p \in Y$ by Lemma 1
$\iff (\forall V(p) \in \mathcal{T})(V(p) \cap (A\backslash\{p\}) \neq \emptyset) \wedge p \in Y$
$\iff p \in A' \wedge p \in Y$
$\iff p \in A' \cap Y.$

Therefore, we have proven (1.).

2. We have

$$\begin{aligned} p \in \mathrm{int}(A) \cap Y &\iff (\exists U(p) \in \mathcal{T})(U(p) \subseteq A) \wedge p \in Y \\ &\implies (\exists V(p) \in \mathcal{T}_Y)(V(p) \subseteq A), \text{ e.g., } V(p) = U(p) \cap Y \\ &\iff p \in \mathrm{int}_Y(A), \end{aligned}$$

which allows us to conclude that $\mathrm{int}(A) \cap Y \subseteq \mathrm{int}_Y(A)$.

3. Following as before

$p \in \partial_Y A \iff p \in \bar{A}_Y \cap \overline{\mathbb{C}A}_Y$
$\iff (\forall U_1(p) \in \mathcal{T}_Y)(U_1(p) \cap A \neq \emptyset) \wedge (\forall U_2(p) \in \mathcal{T}_Y)(U_2(p) \cap \mathbb{C}A \neq \emptyset)$
$\iff (\forall V_1(p) \in \mathcal{T})(V_1(p) \cap A \neq \emptyset) \wedge (\forall V_2(p) \in \mathcal{T})(V_2(p) \cap \mathbb{C}A \neq \emptyset) \wedge p \in Y,$ By the same argument as in part (1.)
$\iff p \in \bar{A} \wedge p \in \bar{\mathbb{C}A} \wedge p \in Y$
$\iff p \in \partial A \cap Y$

and we have proven an even stronger condition of part (3.),

$$\partial_Y A = \partial A \cap Y.$$

Finally, we have completed the proof. $\square$

DO NOT FORGET TO BRING UP 1.3 AND 2.3 IN CLASS