Problem 4.

Theorem 1. *Let $(X, \mathcal{T})$ be a topological space and let $B \subseteq X$. Then*

$$\bar{B} = \mathbb{C}[\operatorname{int}(\mathbb{C}B)].$$

Proof. We have

$$\begin{aligned}
p \in \bar{B} &\iff (\forall U(p) \in \mathcal{T})(U(p) \cap B \neq \emptyset) && \text{We proved this in class.}\\
&\iff (\forall U \in \mathcal{T})(p \in U \implies U \cap B \neq \emptyset) && \text{Definition of neighborhood}\\
&\iff (\forall U \in \mathcal{T})(U \cap B = \emptyset \implies p \notin U) && \text{Contrapositive}\\
&\iff (\forall U \in \mathcal{T})(U \cap B = \emptyset \implies p \in \mathbb{C}U) && \\
&\iff (\forall U \in \mathcal{T})(U \subseteq \mathbb{C}B \implies p \in \mathbb{C}U) && U \cap B = \emptyset \iff U \subseteq \mathbb{C}B\\
&\iff (\forall U \in \mathcal{T})(p \in U \implies U \nsubseteq \mathbb{C}B) && \text{Contrapositive}\\
&\iff p \notin \operatorname{int}(\mathbb{C}B) && \text{By definition of interior}\\
&\iff p \in \mathbb{C}[\operatorname{int}(\mathbb{C}B)],
\end{aligned}$$

and we can conclude that $\bar{B} = \mathbb{C}[\operatorname{int}(\mathbb{C}B)]$. □

Problem 5.

Theorem 2. *Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Then*

$$\partial A = \bar{A} \setminus \operatorname{int}(A)$$

Proof. Proceeding as usual,

$$\begin{aligned}
p \in \partial A &\iff p \in \bar{A} \cap \overline{\mathbb{C}A} &&\\
&\iff p \in \bar{A} \cap \mathbb{C}[\operatorname{int}(\mathbb{C}(\mathbb{C}A))] && \text{Directly applying Theorem 1 from Problem 4}\\
&\iff p \in \bar{A} \cap \mathbb{C}[\operatorname{int}(A)] && \text{Basic property of complements}\\
&\iff p \in \bar{A} \setminus \operatorname{int}(A), && \text{Definition of set difference}
\end{aligned}$$

and we have shown $\partial A = \bar{A} \setminus \operatorname{int}(A)$. □

Problem 6.

Theorem 3. *Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Then* $\operatorname{int}(A)$ *is open.*

Proof. By definition of a set's interior, we have $(\forall p \in \operatorname{int}(A))(\exists U(p) \in \mathcal{T})(U(p) \subseteq A)$. From this we can define the set

$$\{U(p) \in \mathcal{T} | U(p) \subseteq A \text{ for some } p \in \operatorname{int}(A)\}.$$

We propose that

$$\operatorname{int}(A) = \bigcup \{U(p) \in \mathcal{T} | U(p) \subseteq A \text{ for some } p \in \operatorname{int}(A)\}.$$

We have

$$\begin{aligned}
p \in \operatorname{int}(A) &\iff (\exists U(p) \in \mathcal{T})(U(p) \subseteq A)\\
&\iff p \in \bigcup \{U(p) \in \mathcal{T} | U(p) \subseteq A \text{ for some } p \in \operatorname{int}(A)\}.
\end{aligned}$$

Since any union of sets in $\mathcal{T}$ is also in $\mathcal{T}$, it follows that $\operatorname{int}(A)$ is open. □