Theorem 1. *f is injective if and only if $f(A \cap B) = f(A) \cap f(B)$, for all $A, B \subseteq X$.*

Proof. Let $A, B \subseteq X$. We have already proven that for any function $f : X \to Y$, $f(A \cap B) \subseteq f(A) \cap f(B)$. Therefore, we need to show f is injective if and only if $f(A) \cap f(B) \subseteq f(A \cap B)$.

Suppose f is injective. Then

$$\begin{aligned} y \in f(A) \cap f(B) &\iff y \in f(A) \wedge y \in f(B) \\ &\iff (\exists x \in A)(f(x) = y) \wedge (\exists x \in B)(f(x) = y) \\ &\iff (\exists x \in A \cap B)(f(x) = y) && \text{Injectivity of } f \\ &\iff y \in f(A \cap B). \end{aligned}$$

Thus, $f(A) \cap f(B) \subseteq f(A \cap B)$.

We will now prove the contrapositive of the reverse direction. Suppose f is not injective. Then, $(\exists x_1, x_2 \in X)(f(x_1) = f(x_2) \wedge x_1 \neq x_2)$. Define $A = \{x_1\}$ and $B = \{x_2\}$. Clearly $A, B \subseteq X$, and $A \cap B = \emptyset$. By definition of a sets image, $f(A \cap B) = \emptyset$ However, $f(A) \cap f(B) = \{f(x_1)\} \cap \{f(x_2)\} = \{f(x_1)\}$. Therefore, $f(A) \cap f(B) \not\subseteq f(A \cap B)$, we have proven the contrapositive, and our proof is complete. $\square$