Definition 1. Let $(X, \mathcal{T})$, $(Y, \mathcal{S})$ be topological spaces, and let $f : X \to Y$. We say that f is continuous if and only if

$$(\forall G \in \mathcal{S})(f^{-1}(G) \in \mathcal{T}).$$

Definition 2 (Local Continuity)**.** f is said to be continuous at $x_0 \in X$ if and only if

$$(\forall W(f(x_0))(\exists U(x_0))(f(U(x_0)) \subseteq W(f(x_0)).$$

Problem 7

Lemma 1. *Let X and Y be topological spaces. Let $f : X \to Y$, and let $B \subseteq Y$. Then*

$$f(f^{-1}(B)) \subseteq B.$$

Proof. We have

$$\begin{aligned} p \in f(f^{-1}(B)) &\iff (\exists x \in f^{-1}(B))(f(x) = p) && \text{Definition of a set's image} \\ &\iff (\exists x \in X)(f(x) = p \wedge f(x) \in B) && \text{Definition of a set's preimage} \\ &\implies p \in B, \end{aligned}$$

which proves that $f(f^{-1}(B)) \subseteq B$. □

Lemma 2. *Let X and Y be topological spaces. Let $f : X \to Y$, and let $A \subseteq X$. Then*

$$A \subseteq f^{-1}(f(A))$$

Proof. We have

$$\begin{aligned} p \in A &\implies (\exists x \in A)(f(p) = f(x)) && \text{Just let } x = p \\ &\iff f(p) \in f(A) && \text{Definition of a set's image} \\ &\iff p \in f^{-1}(f(A)), && \text{Definition of a set's preimage} \end{aligned}$$

which proves that $A \subseteq f^{-1}(f(A))$. □

Theorem 1 (Global Continuity Facts)**.** *Let X and Y be topological spaces. Let $f : X \to Y$. Then the following are equivalent:*

1. *f is continuous.*
2. *$f^{-1}(F)$ is closed in X for all F closed in Y.*
3. *$f^{-1}(U)$ is open in X for all U members of a subbasis of $\mathcal{T}_Y$.*
4. *$f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.*
5. *$\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$ for all $B \subseteq Y$.*

Proof. ((4) $\implies$ (5)). Suppose $f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \in X$, and let $B \subseteq Y$. We have

$$\begin{aligned} p \in \overline{f^{-1}(B)} &\iff p \in f^{-1}(f(\overline{f^{-1}(B)})) && \text{By Lemma 2} \\ &\iff p \in f^{-1}(\overline{f(f^{-1}(B))}) && \text{By initial assumption} \\ &\iff p \in f^{-1}(\bar{B}), && \text{By Lemma 1} \end{aligned}$$

and we have proven $\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$. Since B was arbitrary, we have proven (4) $\implies$ (5). □

Problem 8

Theorem 2. *Let (X, d) and (Y, ρ) be metric spaces, and let $f : X \to Y$. Then the following are equivalent:*

1. *f is continuous at $x_0 \in X$.*
2. *$(\forall\{x_k\})(x_k \to x_0 \implies f(x_k) \to f(x_0))$.*

Proof. ((2) $\implies$ (1)). We will prove the contrapositive. We have

$$\begin{aligned} f \text{ is not continuous at } x_0 \in X &\iff \neg(\forall\epsilon > 0)(\exists\delta > 0)(f(B(x_0; \delta)) \subseteq B(f(x_0); \epsilon)) \\ &\iff (\exists\epsilon > 0)(\forall\delta > 0)(f(B(x_0; \delta)) \not\subseteq B(f(x_0); \epsilon)) \\ &\iff (\exists\epsilon > 0)(\forall\delta > 0)(\exists x \in B(x_0; \delta))(f(x) \notin B(f(x_0); \epsilon)) \\ &\iff (\exists\epsilon > 0)(\forall\delta > 0)(\exists x \in B(x_0; \delta))(\rho(f(x), f(x_0)) \geq \epsilon). \end{aligned}$$

Fix this ϵ, and define $\delta_n = 1/n$ for $n \in \mathbb{N}$. Using the Axiom of Choice, we can define a sequence $\{x_k\}_{k=1}^{\infty}$ where $x_k \in B(x_0; \delta_k)$, and $\rho(f(x_k), f(x_0)) \geq \epsilon$. Thus, we have created a sequence $\{x_k\}_{k=1}^{\infty}$ that converges to x_0, and we have a sequence $\{f(x_k)\}_{k=1}^{\infty}$ which does not converge to x_0. Therefore, we have proven the contrapositive. $\square$

Problem 9

Theorem 3. *Let $(\mathbb{R}, d)$ be a metric space, and let $A \subseteq \mathbb{R}$ be a bounded. Then* $\sup A \in \bar{A}$.

Proof. Since A is bounded, the least upper bound property of the real numbers leads us to conclude that $\sup A$ exists. Let $s = \sup A$. Suppose, for sake of contradiction, that $\sup A \notin \bar{A}$. Then, $(\exists U(s) \in \mathcal{T})(U(s) \cap A = \emptyset)$. Since this $U(s)$ is open, we can find a $\delta > 0$ such that $B(s; \delta) \subseteq U(s)$. This implies that there is no element in A between s and $s - \delta$, which means that $s - \delta$ is an upper bound of A. Since $s - \delta$ is an upper bound of A, and $s - \delta < s$, s cannot be the supremum of A, and we have reached a contradiction. Thus, $\sup A \in \bar{A}$ $\square$