1 Topology

2 Metric spaces convergence

Definition 2.1. *(Convergence in metric spaces)*

Let (X, d) be a metric space. Let $\{x_n\}_{n\in\mathbb{N}} \subset X$. We say that $\{x_n\}$ **converges** to a point $x \in X$, and we write $x_n \to x$, if

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)(d(x_n, x) < \epsilon)$$

Definition 2.2. *(Topological Convergence)*

Let (X, τ) be topological space. Let $\{x_n\}_{n\in\mathbb{N}} \subset X$. We say that x_n **τ-converges** to a point $x \in X$, and we write $x_n \to x$ if

$$(\forall U_x)(\exists N \in \mathbb{N})(\forall n \geq N)(x_n \in U_x)$$

Theorem 1. *(Metric Topological Convergence)*

Let (X, d) be a metric space. Let $\{x_n\} \subset X$. $\{x_n\}$ τ-converges to a point $x \in X$ if and only if x_n converges to $x \in X$ wrt metric d.

Proof. ($\Longleftarrow$) Let $U_x \in \tau$. $\implies (\exists \epsilon > 0) \pitchfork (B(x;\epsilon) \subset U_x)$

$\implies (\exists N \in \mathbb{N})(\forall n \geq N)(x_n \in B(x;\epsilon) \subset U_x)$

$\implies (\forall n \geq N)(x_n \in U_x)$

$\therefore x_n \to x$ in the topological sense.

$\square$

Proof. ($\implies$)

$\square$

Proposition 2.1. *(Metric Closed Characterization)*

Let (X,d) be a metric space and let $A \subset X$. Then A is closed iff each convergent sequence in A has a limit in A

Proof. Notice that $\overline{A}$ is the collection of points for which there exists a sequence in A that converges to it.

$\square$

2.1 Continuity in Metric Spaces

2.1.1 Definition/Equivalence

Theorem 2. *(Metric Continuity Facts)*

Let (X, d) and (Y, ρ) be metric spaces. Let $f : X \to Y$. Pick $x_0 \in X$. Then the following are equivalent.

4.) f is continuous at x_0

2.) $(\forall \epsilon > 0)(\exists \delta > 0) \pitchfork (f(B(x_0; \delta) \subset B(f(x_0); \epsilon))$

3.) $(\forall \{x_n\}_{n \in \mathbb{N}})(x_n \to x_0 \implies f(x_n) \to f(x_0))$

Proof. $(1 \implies 2)$

Pick $\epsilon > 0$. Then $\exists \delta > 0$ such that $d(x, x_0) < \delta \implies p(f(x), f(x_0)) < \epsilon$

$\iff x \in B(x_0, \delta) \implies f(x) \in B(f(x_0), \epsilon)$

$\iff f(B(x_0; \delta)) \subset B(f(x_0); \epsilon)$

□

Proof. $(2 \implies 3)$ Ask yourself, what do we need to show? and the next question would be, how do I proceed to prove this claim?

□

Proof. $(3 \implies 1)$ exercise

□

Proof. $(2 \implies 1)$

Let $\epsilon > 0$. For each $B(f(x_0); \epsilon) \in \tau_Y$, $\exists U(x_0) \in \tau_X \pitchfork f(U) \subset B(f(x_0); \epsilon)$. Since $U(x_0)$ is open, $\exists \delta > 0 \pitchfork B(x_0; \delta) \subset U$. Thus $f(B(x_0; \delta)) \subset f(U) \subset B(f(x_0); \epsilon)$, and this is equivalent to

$$d(x, x_0) < \delta \implies p(f(x), f(x_0)) < \epsilon.$$

□

2.2 Uniform Continuity

Definition 2.3. *(Uniform Continuity)*

Let $f : (X, d) \to (X, \rho)$. We say f is uniformly continuous if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in X)(d(x, y) < \delta \implies \rho(f(x), f(y)) < \epsilon)$$

Examples 1. *Let $a < b \in \mathbb{R}$. Then $T : [a, b] \to \mathbb{R}$ defined by $T(x) = x + a$ is uniformly continuous*

Definition 2.4. *(Distance from a set)*

Let (X,d) be a metric space. Let $A, B \subset X$. Then

d(x,A)=inf$\{d(x, y) : y \in A\}$

Theorem 3. *Let (X, d) be a metric space. Let $\phi \neq A \subset X$. Define $f : X \to \mathbb{R}$ by $f(x) = d(x, A)$. Then f is uniformly continuous on X.*