Problem 1

Theorem 1. *Let $f : X \to Y$ be a mapping. Then*

$$f(f^{-1}(B)) \subseteq B, \text{ for every } B \subseteq Y.$$

Proof. Let $B \subseteq X$. By definition of a set's image and a set's preimage, we have

$$\begin{aligned} y \in f(f^{-1}(B)) &\iff (\exists x \in f^{-1}(B))(y = f(x)) \\ &\iff (\exists x \in X)(y = f(x) \wedge f(x) \in B) \\ &\implies y \in B. \end{aligned}$$

Therefore, $f(f^{-1}(B)) \subseteq B$, for every $B \subseteq Y$. □

Problem 2

We will start this problem by proving a lemma that we will need for the main theorem.

Lemma 1. *Let $f : X \to Y$ be a mapping. Then*

$$f(A \cap B) \subseteq f(A) \cap f(B), \text{ for all } A, B \subseteq X.$$

Proof. Let $A, B \subseteq X$. By definition of a set's image and the intersection of two sets, we have

$$\begin{aligned} y \in f(A \cap B) &\iff (\exists x \in A \cap B)(y = f(x)) \\ &\iff (\exists x \in A)(x \in B \wedge f(x) = y) \\ &\implies (\exists x \in A)(f(x) = y) \wedge (\exists x \in B)(f(x) = y) \\ &\iff y \in f(A) \wedge y \in f(B) \\ &\iff y \in f(A) \cap f(B), \end{aligned}$$

and our proof is complete. □

Theorem 2. *Let $f : X \to Y$ be a mapping. Then*

$$f \text{ is injective if and only if } f(A \cap B) = f(A) \cap f(B), \text{ for all } A, B \subseteq X. \tag{1}$$

Proof. Let $A, B \subseteq X$. We have already proven that for any function $f : X \to Y$, $f(A \cap B) \subseteq f(A) \cap f(B)$. Therefore, we need to show f is injective if and only if $f(A) \cap f(B) \subseteq f(A \cap B)$.

Suppose f is injective. Then

$$\begin{aligned} y \in f(A) \cap f(B) &\iff y \in f(A) \wedge y \in f(B) \\ &\iff (\exists x \in A)(f(x) = y) \wedge (\exists x \in B)(f(x) = y) \\ &\iff (\exists x \in A \cap B)(f(x) = y) && \text{Injectivity of } f \\ &\iff y \in f(A \cap B). \end{aligned}$$

Thus, $f(A) \cap f(B) \subseteq f(A \cap B)$.

We will now prove the contrapositive of the reverse direction. That is, we will prove that

$$\text{f is not injective } \implies f(A) \cap f(B) \not\subseteq f(A \cap B). \tag{2}$$

Suppose f is not injective. Then, $(\exists x_1, x_2 \in X)(f(x_1) = f(x_2) \wedge x_1 \neq x_2)$. Define $A = \{x_1\}$ and $B = \{x_2\}$. We have $A, B \subseteq X$, and $A \cap B = \emptyset$. By definition of a sets image, $f(A \cap B) = f(\emptyset) = \emptyset$. However, $f(A) \cap f(B) = \{f(x_1)\} \cap \{f(x_2)\} = \{f(x_1)\}$, since $f(x_1) = f(x_2)$. Therefore, $f(A) \cap f(B) \not\subseteq f(A \cap B)$, and our proof is complete. □

Problem 3

We will start this problem with some lemmas that we will use for the proof of the main theorem.

Lemma 2. *Define* $d : \mathbb{R} \to \mathbb{R}$ *by* $d(x, y) = |x - y|$. *Then* d *is a metric for* $\mathbb{R}$.

Proof. Let $x, y \subseteq \mathbb{R}$. By definition of absolute value, we have $d(x, y) = x - y \iff x \geq y$, and $d(x, y) = y - x \iff y > x$. By ordering of the reals, we have $d(x, y) \geq 0$. Suppose $x = y$. Then, $d(x, y) = x - y = x - x = 0$. Now suppose $d(x, y) = 0$. We can assume, without loss of generality, that $x \leq y$. Then, from the algebraic properties of $\mathbb{R}$,

$$\begin{aligned} d(x, y) &= 0 \\ |x - y| &= 0 \\ y - x &= 0 \\ y &= x. \end{aligned}$$

Therefore, d satisfies the first requirement of a metric.

Assume, without loss of generality, that $x \leq y$. Then $d(x, y) = y - x$. Now, swap x and y in the above definition of absolute value to see (trivially) that $d(y, x) = y - x$. Therefore, $d(x, y) = d(y, x)$, and d fits the second criteria of a metric.

Now let $x, y, z \in \mathbb{R}$. Once more, we can assume that $x \leq z$. Then, $d(x, z) = z - x$. We have three cases to consider:

Case 1. $y < x \leq z$**.** Then,

$$\begin{aligned} d(x, y) + d(y, z) &= (x - y) + (z - y) \\ &> (x - y) + (z - y) + 2(y - x) && \text{because } (y - x) < 0 \\ &= z - x \\ &= d(x, z). \end{aligned}$$

Case 2. $x \leq y \leq z$**.** In a similar fashion,

$$\begin{aligned} d(x, y) + d(y, z) &= (y - x) + (z - y) \\ &= z - x \\ &= d(x, z). \end{aligned}$$

Case 3. $x \leq z < y$**.** Once more,

$$\begin{aligned} d(x, y) + d(y, z) &= (y - x) + (y - z) \\ &> (y - x) + (y - z) + 2(z - y) && \text{because } (z - y) < 0 \\ &= z - x \\ &= d(x, z). \end{aligned}$$

In all of these cases, $d(x, z) \leq d(x, y) + d(y, z)$, which means that d fits the final criteria of a metric. □

Theorem 3. *Let* $X = \{f : [a, b] \to \mathbb{R} | f \text{ is continuous}\}$. *Define*

$$d(f, g) = \sup\{|f(t) - g(t)| : t \in [a, b]\}.$$

Then d *is a metric for* X.

Proof. To show that d is a metric, we will show that it fits the three requirements of a metric. Let $f, g \in X$. By Lemma 2, we have that $|f(t) - g(t)| \geq 0$ for all $t \in [a, b]$. Thus, $d(f, g) \geq 0$. Now suppose that $f = g$. By definition of equality of functions, we have

$$(\forall t \in [a, b])(f(t) = g(t)).$$

Thus, by Lemma 2, we have

$$\begin{aligned} \{|f(t) - g(t)| : t \in [a, b]\} &= \{0\} \\ d(f, g) &= \sup\{|f(t) - g(t)| : t \in [a, b]\} \\ &= \sup\{0\} \\ &= 0. \end{aligned}$$

Now suppose that $\sup\{|f(t) - g(t)| : t \in [a, b]\} = 0$. Then, $(\forall t \in [a, b])(|f(t) - g(t)| \leq 0)$. By Lemma 2, we have $(\forall t \in [a, b])(|f(t) - g(t)| \geq 0)$. Thus, $f = g$, and we have shown that d fits the first criteria of a metric.

By Lemma 2, we have

$$\begin{aligned} d(f, g) &= \sup\{|f(t) - g(t)| : t \in [a, b]\} \\ &= \sup\{|g(t) - f(t)| : t \in [a, b]\} \\ &= d(g, f), \end{aligned}$$

and d fits the second criteria of a metric. $\square$

Let $f, g, h \in X$. By Lemma 2, we have

$$(\forall t \in [a, b])(|f(t) - g(t)| \leq |f(t) - h(t)| + |g(t) - h(t)|),$$

from which it follows that

$$\begin{aligned} d(f, g) &= \sup\{|f(t) - g(t)| : t \in [a, b]\} \\ &\leq \sup\{|f(t) - h(t)| + |g(t) - h(t)| : t \in [a, b]\}. \end{aligned}$$

Define $s_1 := \sup\{|f(t)-h(t)|+|g(t)-h(t)| : t \in [a, b]\}$, $s_2 := \sup\{|f(t)-h(t)| : t \in [a, b]\}$, and $s_3 := \sup\{|g(t) - h(t)| : t \in [a, b]\}$. By the Extreme Value Theorem, there exist $t_1, t_2, t_3 \in [a, b]$ such that $|f(t_1) - h(t_1)| + |g(t_1) - h(t_1)| = s_1$, $|f(t_2) - h(t_2)| = s_2$, and $|g(t_3) - h(t_3)| = s_3$. Now suppose, for sake of contradiction, that $s_1 > s_2 + s_3$. Then,

$$\begin{gathered} s_1 > s_2 + s_3 \\ |f(t_1) - h(t_1)| + |g(t_1) - h(t_1)| > |f(t_2) - h(t_2)| + |g(t_3) - h(t_3)| \\ |f(t_1) - h(t_1)| + |g(t_3) - h(t_3)| > |f(t_2) - h(t_2)| + |g(t_3) - h(t_3)| \\ |f(t_1) - h(t_1)| > |f(t_2) - h(t_2)|, \end{gathered}$$

which contradicts our assumption that $|f(t) - h(t)|$ attains a maximum at t_2. Therefore $s_1 \leq s_2 + s_3$, and d fits the triangle equality

$$d(f, g) \leq d(h, g) + d(f, h).$$

Finally, we can conclude that d is a metric for X.