Corollary 1. *Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$. Then*

$$A \textit{ is closed} \iff A = \bar{A}$$

Proof. By definition, we have $\bar{A} = A \cup A'$. Also, if A is closed, then $A' \subseteq A$. Thus, we have

$$\begin{aligned} A \text{ is closed} &\iff A' \subseteq A \\ &\iff A \cup A' = A \\ &\iff \bar{A} = A, \end{aligned}$$

and our proof is complete. □

Theorem 1 ((Closure))**.** *Let $(X, \mathcal{T})$ be a topological space, and let $A, B \subseteq X$. Then*

1. $A \subseteq B \implies \bar{A} \subseteq \bar{B}$
2. $\overline{A \cup B} = \bar{A} \cup \bar{B}$
3. $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$
4. $\bar{A} = \bigcap\{F : A \subseteq F \wedge F \textit{ is closed}\}$

Proof. 1. Suppose $A \subseteq B$. We have

$$\begin{aligned} p \in \bar{A} &\iff p \in A \cup A' \\ &\iff p \in B \cup A' && \text{Since } A \subseteq B \\ &\iff p \in B \cup B' && \text{Since } (\forall U)(U \cap (A \backslash \{p\}) \subseteq U \cap (B \backslash \{p\})) \\ &\iff p \in \bar{B}, \end{aligned}$$

which proves that $\bar{A} \subseteq \bar{B}$.

2. We have

$$\begin{aligned} \overline{A \cup B} &= (A \cup B) \cup (A \cup B)' \\ &= (A \cup B) \cup \{p \in X | (\forall U(p))(U(p) \cap [(A \cup B) \backslash \{p\}] \neq \emptyset\}. \end{aligned}$$

Now suppose for some $p \in X$, there exists $U_1(p), U_2(p) \in \mathcal{T}$ such that $U_1(p) \cap [(A \cup B) \backslash \{p\}] \neq \emptyset$, $U_2(p) \cap [(A \cup B) \backslash \{p\}] \neq \emptyset$, $U_1(p) \cap (A \backslash \{p\}) = \emptyset$, and $U_2(p) \cap (B \backslash \{p\}) = \emptyset$. Then, $p \in U_1(p) \cap U_2(p) = \{p\} \in \mathcal{T}$, but $\{p\} \cap [(A \cup B) \backslash \{p\}] = \emptyset$, which is a contradiction. Therefore, with all that we can conclude

$$\begin{aligned} \overline{A \cup B} &= (A \cup B) \cup \{p \in X | (\forall U(p))(U(p) \cap (A \backslash \{p\} \neq \emptyset \vee U(p) \cap (B \backslash \{p\}) \neq \emptyset\} \\ &= A \cup B) \cup \{p \in X | p \in A' \vee p \in B'\} \\ &= A \cup B \cup A' \cup B' \\ &= A \cup A' \cup B \cup B' \\ &= \bar{A} \cup \bar{B}, \end{aligned}$$

as desired.

3. In a similar manner

$$\begin{aligned}
p \in \overline{A \cap B} &\iff p \in (A \cap B) \wedge (\forall U(p))(U(p) \cap [(A \cap B)\backslash\{p\}] \neq \emptyset\} \\
&\implies p \in (A \cap B) \wedge (\forall U(p))(U(p) \cap (A\backslash\{p\}) \neq \emptyset \wedge U(p) \cap (B\backslash\{p\}) \neq \emptyset\} \\
&\iff p \in (A \cap B) \cup (A' \cap B') \\
&\iff p \in (A \cup A') \cap (B \cup B') \\
&\iff p \in \bar{A} \cap \bar{B},
\end{aligned}$$

which means that $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$.

$\square$