

**Theorem 1.** Let  $A_n = [0, \frac{1}{n}] = \{x \in \mathbb{R} : 0 \leq x \leq \frac{1}{n}\}$ . Then,

$$\bigcap_{n \in \mathbb{N}} A_n = \{0\}.$$

*Proof.* We have

$$\begin{aligned} x \in \{0\} &\implies x = 0 \\ &\implies (\forall n \in \mathbb{N})(0 \leq x \leq \frac{1}{n}) \\ &\implies (\forall n \in \mathbb{N})(x \in A_n) \\ &\implies x \in \bigcap_{n \in \mathbb{N}} A_n. \end{aligned}$$

Therefore,

$$\{0\} \subseteq \bigcap_{n \in \mathbb{N}} A_n.$$

Suppose

$$x \in \bigcap_{n \in \mathbb{N}} A_n,$$

and for the sake of contradiction that  $x \notin \{0\}$ . We can see

$$\begin{aligned} x \in \bigcap_{n \in \mathbb{N}} A_n \wedge x \notin \{0\} &\implies (\exists n \in \mathbb{N})(x \in A_n \wedge x \notin \{0\}) \\ &\implies (\exists n \in \mathbb{N})(x \in A_n \wedge x \neq 0) \\ &\implies (\exists n \in \mathbb{N})(0 < x \leq \frac{1}{n}) \\ &\implies (\exists m \in \mathbb{N})(\frac{1}{m} < x) \\ &\implies (\exists m \in \mathbb{N})(x \notin A_m) \\ &\implies x \notin \bigcap_{n \in \mathbb{N}} A_n, \end{aligned}$$

which is a contradiction. Therefore,  $x \in \{0\}$ , and we can conclude that

$$\bigcap_{n \in \mathbb{N}} A_n = \{0\}.$$

□

**Theorem 2.** Let  $f : X \rightarrow Y$ . Then

1.  $f(A \cup B) = f(A) \cup f(B)$
2.  $A \subseteq B \implies f(A) \subseteq f(B)$
3.  $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
4.  $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

5.  $f(A \cap B) \subseteq f(A) \cap f(B)$
6.  $f^{-1}(\mathbb{C}A) = \mathbb{C}(f^{-1}(A))$
7.  $f(\mathbb{C}A) \subseteq \mathbb{C}(f(A))$  (*this is an open question*)

*Proof.* 1. Applying the definition of a set's image and the definition of unions of sets, we see

$$\begin{aligned}
 y \in f(A \cup B) &\iff (\exists x \in A \cup B)(f(x) = y) \\
 &\iff (\exists x \in A)(f(x) = y) \vee (\exists x \in B)(f(x) = y) \\
 &\iff y \in f(A) \vee y \in f(B) \\
 &\iff y \in f(A) \cup f(B).
 \end{aligned}$$

From this, we can conclude that  $f(A \cup B) = f(A) \cup f(B)$ .

2. Suppose  $A \subseteq B$ . We have

$$\begin{aligned}
 y \in f(A) &\iff (\exists x \in A)(f(x) = y) \\
 &\implies (\exists x \in B)(f(x) = y) && \text{By initial assumption} \\
 &\iff y \in f(B),
 \end{aligned}$$

which means  $f(A) \subseteq f(B)$ .

3. By definition of a set's preimage and the definition of a union of sets, we have

$$\begin{aligned}
 x \in f^{-1}(C \cup D) &\iff f(x) \in C \cup D \\
 &\iff f(x) \in C \vee f(x) \in D \\
 &\iff x \in f^{-1}(C) \vee x \in f^{-1}(D) \\
 &\iff x \in f^{-1}(C) \cup f^{-1}(D).
 \end{aligned}$$

Therefore,  $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$ .

4. From the definition of a set's preimage and the definition of an intersection of two sets,

$$\begin{aligned}
 x \in f^{-1}(C \cap D) &\iff f(x) \in C \cap D \\
 &\iff f(x) \in C \wedge f(x) \in D \\
 &\iff x \in f^{-1}(C) \wedge x \in f^{-1}(D) \\
 &\iff x \in f^{-1}(C) \cap f^{-1}(D).
 \end{aligned}$$

Thus,  $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$ .

5. From the definition of a set's image and the definition of an intersection of sets,

$$\begin{aligned}
 y \in f(A \cap B) &\iff (\exists x \in A \cap B)(f(x) = y) \\
 &\iff (\exists x)(x \in A \wedge x \in B \wedge f(x) = y) \\
 &\implies (\exists x \in A)(f(x) = y) \wedge (\exists x \in B)(f(x) = y) \\
 &\iff y \in f(A) \wedge y \in f(B) \\
 &\iff y \in f(A) \cap f(B).
 \end{aligned}$$

Thus,  $f(A \cap B) \subseteq f(A) \cap f(B)$ .

6. From the definitions of a set's preimage and the complement of a set, we have

$$\begin{aligned}
 x \in f^{-1}(\mathbb{C}A) &\iff f(x) \in \mathbb{C}A \\
 &\iff f(x) \in Y \wedge f(x) \notin A \\
 &\iff x \in f^{-1}(Y) \wedge x \notin f^{-1}(A) \\
 &\iff x \in \mathbb{C}f^{-1}(A).
 \end{aligned}$$

Therefore,  $f^{-1}(\mathbb{C}A) = \mathbb{C}(f^{-1}(A))$ .

7. Expanding the sets by definition, can see

$$\begin{aligned}
 y \in \mathbb{C}f(A) &\iff y \in f(X) \wedge y \notin f(A) \\
 &\iff (\exists x \in X)(y = f(x) \wedge y \notin f(A)) \\
 &\iff (\exists x \in X)(f(x) \notin f(A) \wedge f(x) = y) \\
 &\implies (\exists x \in X)(x \notin A \wedge f(x) = y) \\
 &\iff (\exists x \in \mathbb{C}A)(f(x) = y) \\
 &\iff y \in f(\mathbb{C}A),
 \end{aligned}$$

which leads us to conclude that  $\mathbb{C}f(A) \subseteq f(\mathbb{C}A)$ .  
 Finally, our proof is complete. □