Definition 1. Let $(X, \mathcal{T})$, $(Y, \mathcal{S})$ be topological spaces, and let $f : X \to Y$. We say that f is continuous if and only if

$$(\forall G \in \mathcal{S})(f^{-1}(G) \in \mathcal{T}).$$

Definition 2 (Local Continuity)**.** f is said to be continuous at $x_0 \in X$ if and only if

$$(\forall W(f(x_0))(\exists U(x_0))(f(U(x_0)) \subseteq W(f(x_0)).$$

Lemma 1. *Let X and Y be topological spaces. Let $f : X \to Y$, and let $B \subseteq Y$. Then*

$$f(f^{-1}(B)) \subseteq B.$$

Proof. We have

$$\begin{aligned} p \in f(f^{-1}(B)) &\iff (\exists x \in f^{-1}(B))(f(x) = p) && \text{Definition of a set's image} \\ &\iff (\exists x \in X)(f(x) = p \wedge f(x) \in B) && \text{Definition of a set's preimage} \\ &\implies p \in B, \end{aligned}$$

which proves that $f(f^{-1}(B)) \subseteq B$. □

Lemma 2. *Let X and Y be topological spaces. Let $f : X \to Y$, and let $A \subseteq X$. Then*

$$A \subseteq f^{-1}(f(A))$$

Proof. We have

$$\begin{aligned} p \in A &\implies (\exists x \in A)(f(p) = f(x)) && \text{Just let } x = p \\ &\iff f(p) \in f(A) && \text{Definition of a set's image} \\ &\iff p \in f^{-1}(f(A)), && \text{Definition of a set's preimage} \end{aligned}$$

which proves that $A \subseteq f^{-1}(f(A))$. □

Lemma 3. *Let $f : X \to Y$, and let $A \in Y$. Then*

$$\complement f^{-1}(A) = f^{-1}(\complement A).$$

Proof. We have

$$\begin{aligned} p \in \complement f^{-1}(A) &\iff p \notin f^{-1}(A) \\ &\iff f(p) \notin A \\ &\iff f(p) \in \complement A \\ &\iff p \in f^{-1}(\complement A), \end{aligned}$$

and we have proven $\complement f^{-1}(A) = f^{-1}(\complement A)$. □

Lemma 4. *Let $f : X \to Y$, and let $A, B \subseteq Y$. Then,*

$$A \cap B \neq \emptyset \implies f^{-1}(A) \cap f^{-1}(B) \neq \emptyset.$$

Proof. We have

$$\begin{aligned} f^{-1}(A) \cap f^{-1}(B) \neq \emptyset &\iff (\exists p \in X)(p \in f^{-1}(A) \wedge p \in f^{-1}(B)) \\ &\iff (\exists p \in X)(f(p) \in A \cap f(p) \in B) \\ &\iff A \cap B \neq \emptyset. \end{aligned}$$

□

Lemma 5. *Let $f : X \to Y$, and let $A, B \subseteq X$. Then,*

$$A \cap B \neq \emptyset \implies f(A) \cap f(B) \neq \emptyset.$$

Proof. We have

$$
\begin{aligned}
f(A) \cap (B) \neq \emptyset &\iff (\exists p \in Y)(p \in f(A) \wedge p \in f(B)) \\
&\implies (\exists p \in Y)(p \in f(A) \cap f(B)) \\
&\implies (\exists x \in X)(f(x) \in A \cap B) \\
&\implies A \cap B \neq \emptyset.
\end{aligned}
$$

□

Lemma 6. *Let $(X, \mathcal{T})$ be a metric space, and let $A \subseteq X$. Then*

$$A \in \mathcal{T} \iff A \cap \overline{\mathbb{C}A} = \emptyset.$$

Proof. We have

$$
\begin{aligned}
A \notin \mathcal{T} &\iff (\exists x \in A)(\forall U(x) \in \mathcal{T})(U(x) \not\subseteq A) \\
&\iff (\exists x \in A)(\forall U(x) \in \mathcal{T})(\exists q \in U(x))(q \notin A) \\
&\iff (\exists x \in A)(\forall U(x) \in \mathcal{T})(\exists q \in U(x))(q \in \mathbb{C}A) \\
&\iff (\exists x \in A)(\forall U(x) \in \mathcal{T})(U(x) \cap \mathbb{C}A \neq \emptyset) \\
&\iff (\exists x \in A)(x \in \overline{\mathbb{C}A}) \\
&\iff A \cap \overline{\mathbb{C}A} \neq \emptyset.
\end{aligned}
$$

□

Theorem 1 (Global Continuity Facts)**.** *Let X and Y be topological spaces. Let $f : X \to Y$. Then the following are equivalent:*

1. *f is continuous.*
2. *$f^{-1}(F)$ is closed in X for all F closed in Y.*
3. *$f^{-1}(U)$ is open in X for all U members of a subbasis of $\mathcal{T}_Y$.*
4. *$f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.*
5. *$\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$ for all $B \subseteq Y$.*

Proof. ((1) $\implies$ (2)). Suppose f is continuous. Let F be closed in Y. We have, by definition, that $\mathbb{C}F$ is open in Y. By continuity of f we have $f^{-1}(\mathbb{C}F) \in \mathcal{T}_X$. By

$$
\begin{aligned}
F \text{ is closed in } Y &\iff \mathbb{C}F \text{ is open } Y && \text{By definition of a closed set} \\
&\implies f^{-1}(\mathbb{C}F) \in \mathcal{T}_X && \text{Since } f \text{ is continuous} \\
&\iff \mathbb{C}f^{-1}(F) \in \mathcal{T}_X && \text{By Lemma 3} \\
&\iff f^{-1}(F) \text{ is closed in } X && \text{By definition of a closed set,}
\end{aligned}
$$

and we have proven ((1) $\implies$ (2)).

((2) $\implies$ (3)). Suppose that $f^{-1}(F)$ is closed in X for all F closed in Y. Let $\{U_\alpha \in Y\}$ be a subbasis of $\mathcal{T}_Y$. We have that the union of any finite intersections of this set are open, we can conclude that any member U of this subbasis is open in Y. With this, we have

$$
\begin{aligned}
U \in \mathcal{T}_Y &\iff \mathbb{C}U \text{ is closed in } Y \\
&\implies f^{-1}(\mathbb{C}U) \text{ is closed in } X \\
&\iff \mathbb{C}f^{-1}(U) \text{ is closed in } X \\
&\iff f^{-1}(U) \in \mathcal{T}_X,
\end{aligned}
$$

and we have proven ((2) $\implies$ (3)).

((3) $\implies$ (4)). Suppose $f^{-1}(U)$ is open in X for all U members of a subbasis of $\mathcal{T}_Y$. Let $A \subseteq X$. We want to show that $f(\bar{A}) \subseteq \overline{f(A)}$. Let $p \in \bar{A} \iff (\exists x \in \bar{A})(f(x) = p)$. We have

$$x \in \bar{A} \implies f(x) \in f(\bar{A}).$$

If we let $V(f(x)) \in \mathcal{T}_Y$ be a neighborhood of $f(x)$, then we can easily define a subbasis of $\mathcal{T}_Y$ of which V is a member. By initial assumption, $f^{-1}(V) \in \mathcal{T}_X$, and we have by definition that $f^{-1}(V)$ is a neighborhood of $f(x)$. Then,

$$\begin{aligned}
x \in \bar{A} &\implies f^{-1}(V) \cap A \neq \emptyset \\
&\iff f(f^{-1}(V)) \cap f(A) \neq \emptyset && \text{By Lemma 5} \\
&\implies V \cap f(A) \neq \emptyset && \text{By Lemma 1} \\
&\implies f(x) \in \overline{f(A)} \\
&\implies p \in \overline{f(A)},
\end{aligned}$$

and we have shown that $f(\bar{A}) \subseteq \overline{f(A)}$. Thus, ((3) $\implies$ (4)).

((4) $\implies$ (5)). Suppose $f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \in X$, and let $B \subseteq Y$. We have

$$\begin{aligned}
p \in \overline{f^{-1}(B)} &\implies p \in f^{-1}(f(\overline{f^{-1}(B)})) && \text{By Lemma 2} \\
&\implies p \in f^{-1}(\overline{f(f^{-1}(B))}) && \text{By initial assumption} \\
&\implies p \in f^{-1}(\bar{B}), && \text{By Lemma 1}
\end{aligned}$$

and we have proven $\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$. Since B was arbitrary, we have proven (4) $\implies$ (5).

((5) $\implies$ (4)). Suppose $\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$ for all $B \subseteq Y$. We want to show that $f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \in X$. Let $A \subseteq X$. We have

$$\begin{aligned}
p \in f(\bar{A}) &\implies p \in f(\overline{f^{-1}(f(A))}) && \text{By Lemma 2} \\
&\implies p \in f(f^{-1}(\overline{f(A)})) && \text{By initial assumption} \\
&\implies p \in \overline{f(A)}, && \text{By Lemma 1}
\end{aligned}$$

and we have shown that $f(\bar{A}) \subseteq \overline{f(A)}$.

((4) $\implies$ (1)). Suppose $f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \in X$. We want to show that $(\forall G \in \mathcal{T}_Y)(f^{-1}(G) \in \mathcal{T}_X)$. Let $G \in \mathcal{T}_Y$. We have

$$\begin{aligned}
p \in \overline{\mathbb{C}f^{-1}(G)} &\implies f(p) \in f(\overline{\mathbb{C}f^{-1}(G)}) \\
&\implies f(p) \in \overline{f(\mathbb{C}f^{-1}(G))} && \text{By initial assumption} \\
&\implies f(p) \in \overline{f(f^{-1}(\mathbb{C}G))} && \text{By Lemma 3} \\
&\implies f(p) \in \overline{\mathbb{C}G} && \text{By Lemma 1} \\
&\implies f(p) \in \mathbb{C}G && \text{Since } \mathbb{C}G \text{ is closed} \\
&\implies f(p) \notin G \\
&\iff p \notin f^{-1}(G),
\end{aligned}$$

which leads us to conclude that $f^{-1}(G) \cap \overline{\mathbb{C}f^{-1}(G)} = \emptyset$. By Lemma 6, $f^{-1}(G) \in \mathcal{T}_X$, and we have that f is continuous.

Finally, the proof is complete. $\square$