Definition 1. Let $(X, \mathcal{T})$, $(Y, \mathcal{S})$ be topological spaces, and let $f : X \to Y$. We say that f is continuous if and only if

$$(\forall G \in \mathcal{S})(f^{-1}(G) \in \mathcal{T}).$$

Definition 2 (Local Continuity)**.** f is said to be continuous at $x_0 \in X$ if and only if

$$(\forall W(f(x_0))(\exists U(x_0))(f(U(x_0)) \subseteq W(f(x_0)).$$

Problem 1

Theorem 1 (Global Continuity Facts)**.** *Let X and Y be topological spaces. Let $f : X \to Y$. Then the following are equivalent:*

1. *f is continuous.*
2. *$f^{-1}(F)$ is closed in X for all F closed in Y.*
3. *$f^{-1}(U)$ is open in X for all U members of a subbasis of $\mathcal{T}_Y$.*
4. *$f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \in X$.*
5. *$\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$ for all $B \subseteq Y$.*

Proof. ((4) $\implies$ (5)). Suppose $f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \in X$, and let $B \subseteq Y$. We have

$$p \in \overline{f^{-1}(B)} \iff (\forall U(p) \in \mathcal{T}_X)(U(p) \cap f^{-1}(B) \neq \emptyset)$$

□