Problem 19

We will start this problem by proving some useful lemmas:

Lemma 1. *Let X be a topological space, and let $A \subseteq X$ be connected. Then, for any $K \subset X$ with $A \subseteq K \subseteq \bar{A}$, K is connected.*

Proof. Let $K \subset X$ with $A \subseteq K \subseteq \bar{A}$. Suppose, for sake of contradiction, that K is not connected. Then, there exists a continuous function $f : K \to \bar{2}$ that is surjective. Suppose, without loss of generality, that $f(A) = \{0\}$. Then, by continuity and surjectivity of f, we have $f^{-1}(\{1\}) \subseteq K$ is open and $\exists x \in f^{-1}(\{1\})$. Since $x \in \bar{A}$, we have

$$f^{-1}(\{1\}) \cap A \neq \emptyset \implies (\exists a \in A)(f(a) = 1)$$

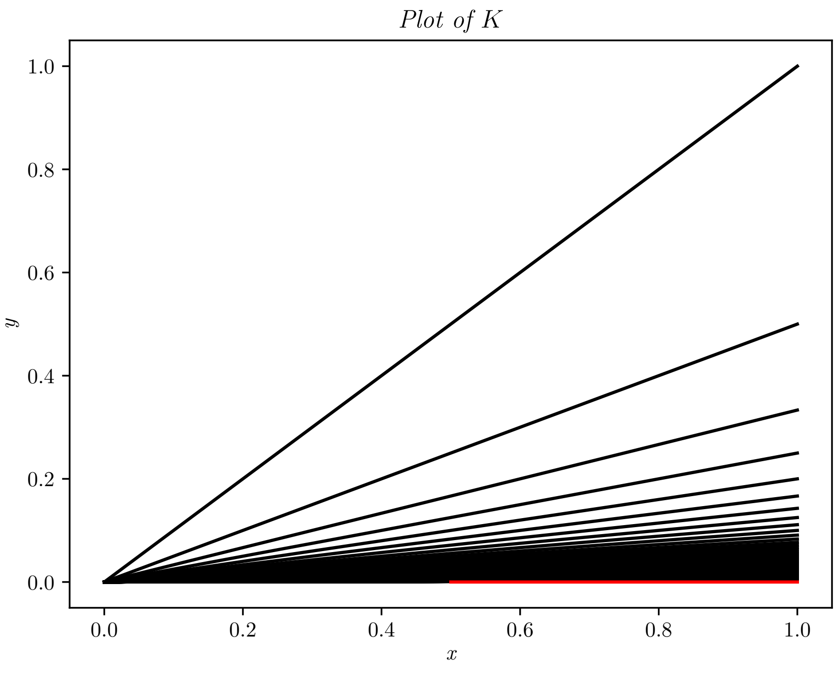
which contradicts our assumption that $f(A) = \{0\}$. Therefore, K is connected. □

Lemma 2. *If X is a path connected space, then X is connected.*

Proof. Suppose X is not connected. Then, there exists a continuous surjection $f : X \to \bar{2}$. Since f is surjective, there exist $a, b \in X$ such that $f(a) = 0$ and $f(b) = 1$. Let $\phi : [0, 1] \to X$ be such that $\phi(0) = a$ and $\phi(1) = b$. Then, $f \circ \phi : [0, 1] \to \bar{2}$ is a surjection. Since $[0, 1]$ is connected, $f \circ \phi$ is not continuous, which implies ϕ is not continuous. From this we can conclude that X is not path connected, and we have proven the contrapositive. □

Theorem 1. *Let $K = \{(t, t/n) : t \in [0, 1] \wedge n \in \mathbb{N}\} \cup \{(t, 0) : t \in [1/2, 1]\}$ in $\mathbb{R}^2$. Then*

1. *K is connected*
2. *K is not path connected*



Proof. 1. We will first show that $A = K \setminus \{(t, 0) : t \in [1/2, 1]\}$ is connected. As we proved in class, if there is a path from $(0, 0)$ to x for all $x \in A$, then A is path connected, and therefore connected (by Lemma 2). Let $(t, t/n) \in K$ for some $t \in [0, 1]$ and some $n \in \mathbb{N}$. Then, we have a continuous path $\phi : [0, 1] \to A$ between $(0, 0)$ and $(t, t/n)$ defined by

$$\phi(x) = (xt, xt/n).$$

Therefore, A is connected.

Now we would like to show that $\{(t,0) : t \in [1/2,1]\} \in \bar{A}$. Let $(t_1, 0) \in \{(t,0) : t \in [1/2,1]\}$, and let $\epsilon > 0$. We have, for any $n \in \mathbb{N}$,

$$\begin{aligned} d((t_1,0),(t_1,t_1/n)) &= \sqrt{(t_1-t_1)^2+(0-t_1/n)^2} \\ &= t_1/n. \end{aligned}$$

Thus, letting $n > 1/\epsilon$, we have

$$\begin{aligned} d((t_1,0),(t_1,t_1/n)) &< t_1/(1/\epsilon) \\ &= \epsilon. \end{aligned}$$

Therefore,

$$\begin{aligned} (t_1,t_1/n) \in B((t_1,0);\epsilon) &\implies B((t_1,0);\epsilon) \cap A \neq \emptyset \\ &\implies (t_1,0) \in \bar{A} && \text{since } \epsilon \text{ is arbitrary} \\ &\implies \{(t,0) : t \in [1/2,1]\} \subseteq \bar{A} && \text{since } (t_1,0) \text{ is arbitrary} \\ &\implies K \subseteq \bar{A}. \end{aligned}$$

Since we have, by design, that $A \subseteq K$, we can conclude from Lemma 1 that K is connected.

2. Suppose, for the sake of contradiction, that there exists a continuous $f : [0,1] \to K$ with $f(0) = (0,0)$ and $f(1) = (1,0)$. Define $f_x : [0,1] \to p_x(K)$ and $f_y : [0,1] \to p_y(K)$ by

$$f_x = p_x \circ f \text{ and } f_y = p_y \circ f.$$

Since f is continuous, we have that f_x, and f_y are continuous.

Define $t_0 = \sup\{t \in [0,1] : f_x(t) \leq \frac{1}{3}\}$. Since $\{t \in [0,1] : f_x(t) \leq \frac{1}{3}\}$ has an upper bound, the least upper bound property of the reals tells us that t_0 exists. Furthermore, since $f_x(0) = 0$, and $f_x(1) = 1$, the intermediate value theorem tells us that $f(t_0) = \frac{1}{3}$.

Define $R : [t_0,1] \to \mathbb{R}$ by $R(t) = \frac{f_y(t)}{f_x(t)}$. We have that $f_x(t) \neq 0$ on the interval $[t_0,1]$. This, combined with the fact that f_x and f_y are continuous, means R is continuous. We have that $f_y(t_0) = \frac{f_x(t_0)}{n} = \frac{1}{3n}$ for some $n \in \mathbb{N}$, because $f(t_0) \notin \{(t,0) : t \in [1/2,1]\}$. Thus, $R(t_0) = \frac{1}{n}$. Also, by initial assumption of f, we have $R(1) = 0$. By the Intermediate Value Theorem, there exists some $x_0 \in [t_0,1]$ such that $R(x_0) = \frac{\sqrt{2}}{2n}$. However, for each $x \in (0,1]$, there exists an $m \in \mathbb{N}$ such that $R(x) = \frac{1}{m}$ whenever $f_y(x) > 0$. Thus, $R(x_0) = \frac{1}{m}$ for some $m \in \mathbb{N}$. However, this yields

$$\frac{\sqrt{2}}{2n} = \frac{1}{m},$$

and we have equated an irrational number and a rational number, which is a contradiction. With this, we have shown that K is not path connected.

□

Problem 20

Theorem 2. *The function $f : [0,\infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is uniformly continuous.*

Proof. Let $\epsilon > 0$, and let $x_1, x_2 \in \mathbb{R}$, with $x_1 \leq x_2$. Then

$$\begin{aligned} d(f(x_1),f(x_2)) &= |\sqrt{x_1} - \sqrt{x_2}| \\ d(f(x_1),f(x_2))^2 &= |\sqrt{x_1} - \sqrt{x_2}|^2 \\ &= |x_1 + x_2 - 2\sqrt{x_1}\sqrt{x_2}| \\ &= |x_1 + x_2 - 2\sqrt{x_1 x_2}| \\ &\leq |x_1 + x_2 - 2\sqrt{x_1^2}| && \text{Since } x_1 \leq x_2 \\ &= |x_1 + x_2 - 2x_1| \\ &= x_2 - x_1 \\ &= d(x_1,x_2). \end{aligned}$$

Taking the square root of both sides, we have

$$d(f(x_1), f(x_2)) \leq \sqrt{d(x_1, x_2)}.$$

Therefore, letting $\delta = \epsilon^2$, we have

$$|x_1 - x_2| < \delta \implies d(f(x_1), f(x_2)) < \sqrt{\epsilon^2} \implies d(f(x_1), f(x_2)) < \epsilon.$$

Since our x_1 and x_2 are arbitrary, we have shown that f is uniformly continuous. □

Problem 21

Theorem 3. *Let $f : (X, d) \to (X, \rho)$ be a homeomorphism. Define*

$$d^* : X \times X \to \mathbb{R} \text{ by } d^*(x, y) = \rho(f(x), f(y)).$$

Then d^* is a metric on X which is equivalent to d.

Proof. We will start by showing that $d*$ is a metric. Let $x, y \in X$. Since ρ is a metric, it follows immediately that $d^*(x, y) \geq 0$. Since f is a homeomorphism, f is injective. By this, we have that $x = y \iff f(x) = f(y)$. Since ρ is a metric, we have that $\rho(f(x), f(y)) = 0 \iff f(x) = f(y)$. From this, we have $d^*(x, y) = 0 \iff x = y$, and d^* fits the first criteria of a metric.

We have

$$\begin{aligned} d^*(x, y) &= \rho(f(x), f(y)) \\ &= \rho(f(y), f(x)) && \text{Since } \rho \text{ is a metric} \\ &= d^*(y, x), \end{aligned}$$

and d^* fits the second criteria of a metric.

Now let $x, y, z \in X$. Then

$$\begin{aligned} d^*(x, z) &= \rho(f(x), f(z)) \\ &\leq \rho(f(x), f(y)) + \rho(f(y), f(z)) && \text{Since } \rho \text{ is a metric} \\ &= d^*(x, y) + d^*(y, z), \end{aligned}$$

And we can finally conclude that d^* is a metric on X.

We will now show that d and d^* are equivalent. To do this, we will first show that $\mathcal{T}_d \subseteq \mathcal{T}_{d^*}$. Let $U \in \mathcal{T}_d$, and let $x_0 \in U$. By definition of openness in a metric space, there exists a radius $r_1 \in \mathbb{R}$ such that $B_d(x_0; r_1) \subseteq U$. Since f is a homeomorphism, and all homemorphisms are open, $f(B_d(x_0; r_1)) \in \mathcal{T}_\rho$. Again, using the definition of openness, there exists a radius $r_2 \in \mathbb{R}$ such that $B_\rho(f(x_0); r_2) \subseteq f(B_d(x_0; r_1))$. By surjectivity of f, for all $y \in B_\rho(f(x_0); r_2))$, there exists $x \in X$ such that $f(x) = y$. Thus,

$$\begin{aligned} B_\rho(f(x_0); r_2) &= \{f(x) \in X : \rho(f(x_0), f(x)) \leq r_2\} \\ &= f(\{x \in X : d^*(x_0, x) \leq r_2\}) \\ &= f(B_{d^*}(x_0; r_2)). \end{aligned}$$

With this, we have

$$\begin{aligned} f(B_{d^*}(x_0; r_2)) \subseteq f(B_d(x_0; r_1)) &\implies f^{-1}(f(B_{d^*}(x_0; r_2))) \subseteq f^{-1}(f(B_d(x_0; r_1))) \\ &\implies B_{d^*}(x_0; r_2) \subseteq B_d(x_0; r_1) \\ &\implies B_{d^*}(x_0; r_2) \subseteq U, && \text{Since } B_d(x_0; r_1) \subseteq U \end{aligned}$$

and we can conclude that $U \in \mathcal{T}_{d^*}$, which means $\mathcal{T}_d \subseteq \mathcal{T}_{d^*}$.

Now let $U \in \mathcal{T}_{d^*}$, and let $x_0 \in \mathcal{T}_{d^*}$. Then, there exists a radius $r \in \mathbb{R}$ such that $B_{d^*}(x_0; r) \subseteq U$. Now we have

$$\begin{aligned} B_{d^*}(x_0; r) &= \{x \in X : d^*(x_0, x) < r\} \\ &= \{x \in X : \rho(f(x_0), f(x)) < r\} \\ &= f^{-1}(\{f(x) \in X : \rho(f(x_0), f(x)) < r\}) \\ &= f^{-1}(B_\rho(f(x_0); r)), \end{aligned}$$

thus, $f^{-1}(B_\rho(f(x_0); r)) \subseteq U$. By continuity of f, we have $f^{-1}(B_\rho(f(x_0); r)) \in \mathcal{T}_d$. Thus, we have have found an open neighborhood in $\mathcal{T}_d$ containing x_0 that is a subset of U, and we've shown that $\mathcal{T}_{d^*} \subseteq \mathcal{T}_d$. From this, we have that $\mathcal{T}_{d^*} = \mathcal{T}_d$, which implies $d^* \sim d$. □