1 Topology

1.1 Separation/ Countability

1.1.1 Axioms of Separation

Definition 1.1. *(Separation axioms)*

Let X be a topological space. We give the following definitions and names

1.) We call X a T_1 space if $(\forall x \neq y)(\exists U_x, U_y) \pitchfork (U_x \cap \{y\} = \phi \wedge U_y \cap \{x\} = \phi)$

2.) We call X a T_2 , or **Hausdorff** space if $(\forall x \neq y)(\exists U_x, U_y) \pitchfork (U_x \cap U_y = \phi)$

3.) We call X a T_3 , or **regular** space, if X is T_1, and $(\forall$ closed $C \subset x)(\forall x \in \complement C)(\exists U_x, V \in \tau_X) \pitchfork (C \subset V \wedge U_x \cap V = \phi)$.

4.) We call X a T_4 , or **Normal** space if X is a T_1 space and $(\forall$ closed $C, K \subset X) \pitchfork (C \cap K = \phi)(\exists U, V \in \tau_X) \pitchfork (C \subset U \wedge K \subset V \wedge U \cap V = \phi)$

Proposition 1.1. *(Closed singletons)*

Let X be a T_1 space. Let $x \in X$. Then $\{x\}$ is closed.

Proof. (Exercise)

□

Theorem 1. *(Hausdorff Facts)*

Let $(X, \mathcal{T})$ be a topological space. The following are equivalent.

1.) X is $Hausdorff$.

2.) Let $x \in X$. Then for each $y \neq x$. $(\exists U_x) \pitchfork (y \notin \overline{U_x})$

3.) For each $x \in X$, $\bigcap\{\overline{U_x} : U_x$ is a nbh of $x\} = \{x\}$

4.) $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$.

Proposition 1.2. *(Dense set)*

Let X be a topological space. Then the following are equivalent.

1.) $D \subset X$ is dense

2.) $(\forall x \in X)(\forall U_x)(U_x \cap D \neq \phi)$

Theorem 2. *Let X, Y be topological spaces. Let Y be Hausdorff. Suppose $f, g : X \to Y$ are continuous. Let D be dense in X. Then the following hold.*

1.) $K = \{x \in X : f(x) = g(x)\}$ is closed.

2.) If $D \subset X$ and $f|_D = g|_D \implies f = g$.

3.) $\mathcal{G}(f) = \{(x, f(x)) : x \in X\}$ is closed in $X \times Y$.

4.) If f is injective, then X is Hausdorff

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Definition 1.2. *(Separable Space)*

We say a topological space X is **separable** if X contains a countable dense subset

1.1.2 Axioms of Countability

Definition 1.3. *(Axioms of Countability)*

Let X be a topological space. We give the following labels to X

1. *We say X is* 1^{st} **countable** if each element of X has a countable neighborhood base

2. We say X is 2^{nd} **countable** if X has a countable basis.

Note that second countability implies first countability, but the converse is not generally true.

Theorem 3. *(separable)*

Let X and Y be topological spaces. Let $T : X \to Y$ be continuous.

1.) If X is separable then T(X) is separable.

2.) X and Y are separable iff $X \times Y$ is separable.

3.) There exists a separable space which has a non separable subspace.

1.2 Connected Topological Spaces

1.2.1 Basics

Definition 1.4. *(Connectedness)*

Let $(X,\mathcal{T})$ be a topological space. We say that X is **Connected** if X is not the union of two nonempty disjoint sets open in X.

Theorem 4. *Let X be a topological space. Then the following are equivalent*

1.) X is connected.

2.) X and ϕ are the only closed and open sets in X.

3.) There is no continuous surjection $f : X \to \overline{2}$ where $\overline{2} = \{0, 1\}$ and $\tau_{\overline{2}} = \{\{0, 1\}, \{1\}, \{0\}, \phi\}$

Proof. $(1 \implies 2)$

Suppose $\phi \neq U \subset X$ is closed and open. Then $\complement U$ is clopen, and $X = U \cup \complement U$.

□

Proof. $(2 \implies 3)$

Suppose there is a continuous surjection $f : X \to \overline{2}$. Then $f^{-1}(\{0\}) \in \tau_X$. Also $\complement f^{-1}(\{0\}) = f^{-1}(\complement\{0\}) = f^{-1}(\{1\}) \in \tau_X$. Thus $\complement\complement f^{-1}(\{0\}) = f^{-1}(\{0\})$ is closed, a contradiction because $f^{-1}(\{0\}) \neq X$.

□

Proof. $(3 \implies 1)$

Suppose X is not connected. Then $X = A \cup B$ which are disjoint and open. Then χ_A is continuous and surjective., which is a contradiction!

□

Theorem 5. *(Shared Point Connectedness)*

Let X be a topological space. Let $\{C_\alpha\}_{\alpha \in \mathscr{A}}$ be a family of connected sets such that there exists $x_0 \in \bigcap_{\alpha \in \mathscr{A}} C_\alpha$ Then $C = \bigcup_{\alpha \in \mathscr{A}} C_\alpha$ is connected.

Proof. Let $f : C \to \overline{2}$ be continuous. Then $(\forall \alpha \in \mathscr{A})(f|_{C_\alpha})$ is non surjective. Let $\beta \in \mathscr{A}$. Suppose $f_\beta(x_0) = 0$. Then $f_\alpha(x_0) = 0$ for all $\alpha \in \mathscr{A}$. Therefore f is non surjective.

□

Theorem 6. *(Connectedness Topological invariance)*

Let X, Y be topological spaces. Let $f : X \to Y$ be continuous. Let X be connected. Then f(X) is connected.

Proof. Suppose there exists a continuous surjective mapping $g : f(X) \to \overline{2}$. The composition of the two mappings reaches a contradiction.

□

Theorem 7. *Any interval Y in $\mathbb{R}$ is connected*

Proof. Suppose Y is not connected. Then $Y = A \cup B$ where $A \cap B = \phi$ and $\phi \neq A, B \in \tau_Y$.

WLOG, suppose $a < b$ with $a \in A$, $b \in B$.

Define $\alpha = sup\{x : [a, x) \subset A\}$. Then $\alpha \leq b$ $(\forall b \in B)$ because if $\alpha > b$ for some b, then $b \in A$.

Since Y is an interval, $\alpha \in Y$, and $\alpha \in \overline{A}_Y$

Since $A = \complement_Y B$ is closed in Y, $\alpha \in A$.

However, $A \in \tau_Y$, so $(\exists \epsilon > 0) \pitchfork (\alpha - \epsilon, \alpha + \epsilon) \subset A$, a contradiction.

□

Corollary 1.1. *(Intermediate value theorem)*

Let X be a topological space. Let $f : X \to \mathbb{R}$ be continuous. Suppose A is a connected subset of X. Let $a, b \in A$. Let $f(a) < c < f(b)$.

Then $(\exists x_0 \in A)(f(x_0) = c)$.

Proof. A is connected so $f(A)$ is an interval. Thus $c \in f(A)$

□

Lemma 1.1. *(Closure connected)*

Let X be a topological space and let $A \subset X$ be connected. Then for each K such that $A \subset K \subset \overline{A}$, K is connected.

Proof. Suppose K is not connected.

□

1.2.2 Components

Definition 1.5. *(Connected Components)*

Let X be a topological space. Let $x \in X$. We define C(x) to be the union of all connected subsets containing x. We call C(x) the **Component** of x.

Theorem 8. *(Component properties)*

Let X be a topological space. Let $x \in X$ then

1. *C(x) is a maximal connected subset of X*
2. *The family of all distinct components in X partitions X*
3. *C(x) is closed in X*

Proof. (1)

This follows directly from the definition. □

Proof. (2) □

Proof. (3) □