Problem 16

We will start this problem with an easy corollary to the theorem we have about all intervals in $\mathbb{R}$ being connected:

Corollary 1. *Let $I \subseteq \mathbb{R}$, and let $a, b \in \mathbb{R}$. Then, the spaces $\{(a, y) : y \in I\}$ and $\{(a, y) : y \in I\}$ are connected.*

Proof. Define $f : I \to \{(a, y) : y \in I\}$ by $f(y) = (a, y)$. Clearly f is continuous. Since I is connected, the topological invariance of connectedness tells us that $f(I)$ is connected. Since $f(I) = \{(a, y) : y \in I\}$, we have that $\{(a, y) : y \in I\}$ is connected. The argument for $\{(a, y) : y \in I\}$ is nearly identical. □

Theorem 1. *For some $z \in \mathbb{R}^2$, the space $\mathbb{R}^2 \backslash seg[0, z]$ is connected.*

Proof. Let $z \in \mathbb{R}^2$, and let $|z| = d(0, z)$ denote the distance between the origin and z. Then, via a rotation about the origin, $\mathbb{R}^2 \backslash \text{seg}[0, z]$ is homeomorphic to $\mathbb{R}^2 \backslash \text{seg}[(0, 0), (|z|, 0)]$. Thus, by the topological invariance of connectedness, it will suffice to show that $\mathbb{R}^2 \backslash \text{seg}[(0, 0), (|z|, 0)]$ is connected.

Let $A_r = \{(x, r) : x \in \mathbb{R}\}$ be a horizontal line for $r \in \mathbb{R} \backslash \{0\}$. Let $L = \{(x, 0) \in \mathbb{R}^2 : x < 0\}$, and let $R = \{(x, 0) \in \mathbb{R}^2 : x > |z|\}$. Define $H = \{A : A = A_r \text{ for some } r \in \mathbb{R} \backslash \{0\} \vee A = L \vee A = R\}$. We have constructed H in such a way that

$$\bigcup_{A \in H} A = \mathbb{R}^2 \backslash \text{seg}[(0, 0), (|z|, 0)].$$

By Corollary 1, we have that A is connected for all $A \in H$. Taking advantage of Corollary 1 once more, we define two connected sets:

$$v_1 = \{(-1, y) : y \in \mathbb{R}\}$$
$$v_2 = \{(|z| + 1, y) : y \in \mathbb{R}\}.$$

We have $v_1 \cap A \neq \emptyset$ for all $A \in H \backslash \{R\}$, and we have $v_2 \cap A \neq \emptyset$ for all $A \in H \backslash \{L\}$. Thus, as we proved in class, $v_1 \cup A$ is connected for all $A \in H \backslash \{R\}$, and $v_2 \cup A$ is connected for all $A \in H \backslash \{L\}$. Define

$$F_1 = \{v_1 \cup A : A \in H \backslash \{R\}$$
$$F_2 = \{v_2 \cup A : A \in H \backslash \{L\}.$$

By construction, $A \cap B \neq \emptyset$ for all $A, B \in F_1$, and $A \cap B \neq \emptyset$ for all $A, B \in F_2$. Thus, for

$$F_3 = \bigcup_{A \in F_1} A, \text{ and}$$
$$F_4 = \bigcup_{A \in F_2} A,$$

we have F_3, F_4 are connected. In a similar manner, we have that $F_3 \cap F_4 \neq \emptyset$, since, for example, $A_1 \subseteq F_3$ and $A_1 \subseteq F_4$. Therefore, $F_3 \cup F_4$ is connected. By design, we have

$$F_3 \cup F_4 = \mathbb{R}^2 \backslash \text{seg}[(0, 0), (|z|, 0)],$$

which leads us to conclude that $\mathbb{R}^2 \backslash \text{seg}[(0, 0), (|z|, 0)]$ is connected. □

Remark 1. The intuition behind this proof is breaking $\mathbb{R}^2 \backslash \text{seg}[(0, 0), (|z|, 0)]$ into connected horizontal lines, and then joining them together in such a way to show $\mathbb{R}^2 \backslash \text{seg}[(0, 0), (|z|, 0)]$ is connected. This method is laborious, though fairly straightforward.

Problem 17

Theorem 2. *Let X, Y be topological spaces and let $f : X \to Y$ be a continuous mapping. Then $f(C(x))$ is not, in general, a component of $f(x)$ if $C(x)$ is a component of x.*

Proof. Let $(a, b) \subseteq \mathbb{R}$ be an interval, and consider the function $f : (a, b) \to \mathbb{R}$, defined by $f(x) = x$. We have that f is continuous (let $\delta = \epsilon$, and continuity follows). Also, (a, b) is a maximal connected subset of (a, b). Let $x \in (a, b)$. Then, $(a, b) = C(x)$. By design, $f((a, b)) = (a, b)$ is a connected set containing $f(x)$. However, $\mathbb{R}$ is a connected set containing $f(x)$, $f((a, b)) \subseteq \mathbb{R}$, and $f((a, b)) \neq \mathbb{R}$. Therefore, by our theorem on component facts, $f(C(x))$ is not a component of $f(x)$. □

Problem 18

Theorem 3. *$\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.*

Proof. Suppose, for the sake of contradiction, that there exists a homeomorphism $f : \mathbb{R} \to \mathbb{R}^2$. Let $x \in \mathbb{R}$. Since f^{-1} is continuous, $f^{-1}\big|_{\mathbb{R}^2 \setminus \{f(x)\}}$ is also continuous. We have $f^{-1}(\mathbb{R}^2 \setminus \{f(x)\}) = \mathbb{R} \setminus \{x\}$. By Theorem 1, $\mathbb{R}^2 \setminus \{f(x)\}$ is connected. Since $\mathbb{R} \setminus \{x\}$ is not an interval, it is not connected. Therefore, we have a continuous function mapping a connected space to a non-connected space, which is a contradiction. From this, we can conclude that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. $\square$