1 Topology

Definition 1.1. *(Topology)*

Let $X \neq \phi$. Let $\mathcal{T} \subset 2^X$. Further, Suppose the following.

1.) $\{X, \phi\} \subset \mathcal{T}$.

2.) $\{G_\alpha\}_{\alpha \in \mathscr{A}} \subset \mathcal{T} \implies \bigcup_{\alpha \in \mathscr{A}} G_\alpha \in \mathcal{T}$

3.) $\{G_i\}_{i=1}^n \subset \mathcal{T} \implies \bigcap_{i=1}^n G_i \in \mathcal{T}$

Then we call $\mathcal{T}$ a **Topology** on X and we call the pair $(X, \mathcal{T})$ a **topological space**

If $U \in \mathcal{T}$, we call U an **Open Set.**

Examples 1. $X = \{0, 1\}$

$\mathcal{T}_1 = \{\emptyset, X\}$ *is a topology for X, the* ***indiscrete topology****.*

$\mathcal{T}_2 = \{\emptyset, X, \{0\}, \{1\}\}$ *is a topology for X, the* ***discrete topology****.*

$\mathcal{T}_3 = \{\emptyset, X, \{0\}\}$ *is a topology for X, the* ***Sierpinski topology****.*

Definition 1.2. *(Neighborhood)*

Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$. A **neighborhood** of X is a set U_x such that $x \in U_x \in \mathcal{T}$.

Notation: A set U_x or V_y means $x \in U_x \in \mathcal{T}$ or $y \in V_y \in \mathcal{T}$ respectively.

The Purpose of Topology as a study is to determine what properties of spaces are preserved under continuous mappings. This motivates the following definition.

1.0.1 Open Set; Closed Set

Definition 1.3. *(Closed set)*

Let $(X, \mathcal{T})$ be a topological space. A set C is said to be a **Closed set** if $\complement C \in \tau$.

Proposition 1.1. *(Closed Union/ Closed intersection)*

Let (X, τ) $\{S_\alpha\}_{\alpha \in \mathscr{A}}$ be a collection of closed sets in X. Then the following hold

1.) $\bigcap_{\alpha \in \mathscr{A}} S_\alpha$ *is closed* *2.) Finite unions of elements of* $\{S_\alpha\}_{\alpha \in \mathscr{A}}$ *are closed.*

Proof. De Morgan's Law

□

Proposition 1.2. *Let $(X, \mathcal{T})$ be a topological space. Let $G \subset X$. Then*

$$G \in \mathcal{T} \iff (\forall x \in G)(\exists U_x \subset G)$$

Proof. $(\rightarrow)$ Obvious

□

Proof. $\leftarrow$. Write G as the union $\bigcup_{x \in G} U_x$. Easy peasy.

□

1.0.2 Interior; Closure

Definition 1.4. *(Interior of a set)*
Let $(X, \mathcal{T})$ be a topological space. Let $A \subset X$. We say that p is an **interior point** of A if

$$(\exists U_p \subset A)$$

We refer to the set of interior points of A as the interior of A, and denote it by int(A).

Definition 1.5. *(Accumulation Point)*
Let $(X, \mathcal{T})$ be a topological space. Let $A \subset X$. A point $x \in X$ is said to be an accumulation point of A if

$$(\forall U_x)(U_x \cap A \setminus \{x\} \neq \phi)$$

We denote the set of accumulation points of A with A'.

Theorem 1. *(Closed characterization)*
Let $(X, \mathcal{T})$ be a topological space. Let $A \subset X$. Then A is closed $\iff$ $A' \subset A$.

Proof. $(\rightarrow)$. Let A be closed

$\square$

Proof. $(\leftarrow)$ Let $A' \subset A$

$\square$

Definition 1.6. *(Closure of a set)*
Let $(X, \mathcal{T})$ be a topological space. Let $A \subset X$. We define the **closure** of A, denoted by $\overline{A}$, to be $\overline{A} = A \cup A'$.

Corollary 1.1. *(Closed Characterization)*
Let (X,τ) be a T.S., and let $A \subset X$. Then A is closed if and only if $A = \overline{A}$.

Proof. $(\implies)$ Suppose A is closed...

$\square$

Proof. $(\leftarrow)$ Suppose $A = \overline{A}$...

$\square$

Proposition 1.3. *Let X be a topological space and let $A \subset X$. Then $\overline{A}$ is closed.*

Theorem 2. *(Closure Properties)*
Let $(X, \mathcal{T})$ be a topological space. Let $A, B \subset X$. Then the following hold.

1.) $A \subset B \implies \overline{A} \subset \overline{B}$
2.) $\overline{A \cup B} = \overline{A} \cup \overline{B}$
3.) $\overline{A} = \bigcap\{f : f \text{ is closed and } A \subset f\}$

Proof. (1) Do it yourself

□

Proof. (2) Test your progress

□

Proof. (3) It is about proving the equality of two sets.

□

Proposition 1.4. *(Interior Closure complement)*

Let X be a topological space. Let $F \subset X$. Then $\complement\overline{F} = int(\complement F)$.

Proof. □

Definition 1.7. *(Boundary of a set)*

Let (X, τ) be a topological space. Let $A \subset X$. We define the **Boundary** of A, denoted by ∂A, to be the set

$$\partial A = \overline{A} \cap \overline{\complement A}$$

An Intuitive Approach to Topology

2 Topology and Metric Spaces

Definition 2.1. ***(Metric Space).*** *Let $X \neq \emptyset$ and let $d : X \times X \to \mathbb{R}$. We say that d is a **metric** for X and the pair (X, d) is a **metric space** if*

1. *$d(x, y) \geq 0$, and that $d(x, y) = 0 \Leftrightarrow x = y$*
2. *$d(x, y) = d(y, x)$*
3. *$d(x, y) \leq d(x, z) + d(z, y)2$*

Theorem 3. *Every metric space induces a topology.*

Proof. Proof here. □

Example 2.1. ***(Euclidean Topology).***

*In $\mathbb{R}^n$, $\mathcal{E} = \{G : G \text{ is open}\} \cup \{\emptyset\}$, where G is open is equivalent to say that $(\forall x \in \mathbb{R}^n)(\exists B(x; \epsilon))(B(x; \epsilon) \subseteq G)$. $(\mathbb{R}^n, \mathcal{E})$ is called the **Euclidean topology**. In this topology, we have defined a metric, $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}$. This leads to the definition of an open ball using the metric: $B(x, \epsilon) = \{y \in \mathbb{R}^n : d(x, y) < \epsilon\}$*

3 Closure and Interior Points

Theorem 4. *Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$, then*

$$A \textit{ is closed} \Leftrightarrow A' \subseteq A$$

Proof. Proof here. □

Definition 3.1. ***(Closure of a Set).*** *The **closure** of a set A is defined as the union between the accumulation points of A and A.*

$$cl(A) = \overline{A} = A \cup A'$$

Theorem 5. *$\overline{A}$ is a closed set.*

Proof. Proof here. □

Theorem 6. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. Proof here. □

Theorem 7. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

Proof. Proof here. □

Definition 3.2. ***(Boundary Points).*** *Let $(X, \mathcal{T})$ be a topological space. Let $A \subseteq X$. Then the **boundary** of the set A is the intersection between the closure of A and the closure of the complement of A.*

$$\partial(A) = \overline{A} \cap \overline{\mathbb{C}A}$$

Definition 3.3. ***(Interior Points).*** *Let $(X, \mathcal{T})$ be a topological space. Let $A \subseteq X$. Then the **interior** of the set A is the set of all points x such that there exists a neighborhood of x such that $U(x) \subseteq A$.*

$$int(A) = \dot{A} = \{x \in A : (\exists U(x))(U(x) \subseteq G)\}$$

Theorem 8. $int(A) \subseteq A \subseteq \overline{A}$

Proof. Proof here. □

Theorem 9. $\overline{A} = int(A) \cup \partial A$

Proof. Proof here. $\square$

Theorem 10. *G is open* $\Leftrightarrow int(G) = G$.

Proof. Proof here. $\square$

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Theorem 11. $\mathbb{C}\overline{A} = int(\mathbb{C}A) \Leftrightarrow \overline{A} = \mathbb{C}(int(\mathbb{C}A))$

Proof. Proof here. $\square$

Theorem 12. $x \in \overline{A} \Leftrightarrow (\forall U(x))(U(x) \cap A \neq \emptyset)$

Proof. Proof here. $\square$

Theorem 13. $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$

Proof. Proof here. $\square$

Theorem 14. $A \subseteq B \Rightarrow int(A) \subseteq int(B)$

Proof. Proof here. $\square$

Theorem 15. $\overline{A} = \bigcap\{F : F \text{ is closed and } A \subseteq F\}$

Proof. Proof here. $\square$

Theorem 16. $int(A) = \bigcup\{G : G \text{ is open and } G \subseteq A\}$

Proof. Proof here. $\square$

Theorem 17. $\mathbb{C}(int(A)) = \overline{\mathbb{C}A}$

Proof. Proof here. $\square$

Theorem 18. $\partial A = \partial(\mathbb{C}A)$

Proof. Proof here. $\square$