7. **(Column sum criterion)** To the metric in (1) there corresponds the condition (5). If we use on X the metric d_1 defined by

$$d_1(x, z) = \sum_{j=1}^{n} |\xi_j - \zeta_j|,$$

show that instead of (5) we obtain the condition

$$\sum_{j=1}^{n} |c_{jk}| < 1 \qquad (k = 1, \cdots, n). \tag{15}$$

8. **(Square sum criterion)** To the metric in (1) there corresponds the condition (5). If we use on X the Euclidean metric d_2 defined by

$$d_2(x, z) = \left[\sum_{j=1}^{n} (\xi_j - \zeta_j)^2\right]^{1/2},$$

show that instead of (5) we obtain the condition

$$\sum_{j=1}^{n} \sum_{k=1}^{n} c_{jk}^2 < 1. \tag{16}$$

9. **(Jacobi iteration)** Show that for the Jacobi iteration the sufficient convergence conditions (5), (15) and (16) take the form

$$\sum_{\substack{k=1\\k\neq j}}^{n} \frac{|a_{jk}|}{|a_{jj}|} < 1, \qquad \sum_{\substack{j=1\\j\neq k}}^{n} \frac{|a_{jk}|}{|a_{jj}|} < 1, \qquad \sum_{\substack{j=1\\j\neq k}}^{n} \sum_{k=1}^{n} \frac{a_{jk}^2}{a_{jj}^2} < 1.$$

10. Find a matrix C which satisfies (5) but neither (15) nor (16).

5.3 Application of Banach's Theorem to Differential Equations

The most interesting applications of Banach's fixed point theorem arise in connection with function spaces. The theorem then yields existence and uniqueness theorems for differential and integral equations, as we shall see.

In fact, in this section let us consider an *explicit ordinary differential equation of the first order*

(1a) $$x' = f(t, x) \qquad (' = d/dt).$$

An **initial value problem** for such an equation consists of the equation and an *initial condition*

(1b) $$x(t_0) = x_0$$

where t_0 and x_0 are given real numbers.

We shall use Banach's theorem to prove the famous Picard's theorem which, while not the strongest of its type that is known, plays a vital role in the theory of ordinary differential equations. The idea of approach is quite simple: (1) will be converted to an integral equation, which defines a mapping T, and the conditions of the theorem will imply that T is a contraction such that its fixed point becomes the solution of our problem.

5.3-1 Picard's Existence and Uniqueness Theorem (Ordinary differential equations). *Let f be continuous on a rectangle* (Fig. 53)

$$R = \{(t, x) \mid |t - t_0| \leq a, |x - x_0| \leq b\}$$

and thus bounded on R, say (see Fig. 54)

$$|f(t, x)| \leq c \qquad \text{for all } (t, x) \in R \tag{2}$$

Suppose that f satisfies a **Lipschitz condition** *on R with respect to its second argument, that is, there is a constant k* (Lipschitz constant) *such*

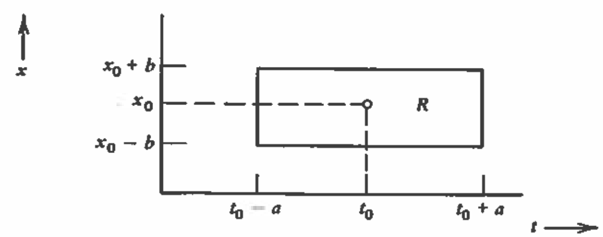

Fig. 53. The rectangle R