1 Topology

1.1 Continuity

1.1.1 Definitions/Equivalence

Definition 1.1. *(Global Continuity)*

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f : X \to Y$. We say that f is **continuous** if

$$(\forall V \in \mathcal{T}_Y)(f^{-1}(V) \in \mathcal{T}_X)$$

Definition 1.2. *(Local Continuity)*

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f : X \to Y$. Let $x_0 \in X$. We say f is **continuous at** x_0 if

$$(\forall V_{f(x_0)} \in \mathcal{T}_Y)(\exists U_{x_0} \in \mathcal{T}_X) \pitchfork (f(U_{x_0}) \subset V_{f(x_0)})$$

Theorem 1. *(Global Local Continuity)*

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f : X \to Y$. Then the following are equivalent.

1.) f is globally continuous.

2.) $(\forall x_0 \in X)(f$ is continuous at $x_0)$.

Proof. $(1 \implies 2)$

Pick $x_0 \in X$. Let $f(x_0) \in V \in \mathcal{T}_Y$. Since f is continuous, $f^{-1}(V) \in \mathcal{T}_X$. Thus $f^{-1}(V)$ is a neighborhood of x_0 such that $f(f^{-1}(V)) \subset V$, so f is continuous at x_0.

□

Proof. $(2 \implies 1)$

Let $V \in \mathcal{T}_Y$. I must show $f^{-1}(V) \in \S_X$. Let $x \in f^{-1}(V)$. Since f is continuous at x, $(\exists U_x \in \mathcal{T}_X) \pitchfork f(U_x) \subset V$

$\iff U_x \subset f^{-1}(V)$. Thus an arbitrary element of $f^{-1}(V)$ has a neighborhood contained within $f^{-1}(V)$. Hence $f^{-1}(V) \in \mathcal{T}_X$. Since V was an arbitrary element of $\mathcal{T}_Y$, we conclude that this is true for all open sets in Y, and thus f is globally continuous.

□

1.1.2 Basic Facts

Theorem 2. *(Composition Continuity)*

Let X, Y, and Z be topological spaces. Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then $g \circ f : X \to Z$ is continuous.

Proof. let $V \in \mathcal{T}_Z$.

$\implies g^{-1}(V) \in \mathcal{T}_Y$.

$\implies f^{-1}(g^{-1}(V)) \in \mathcal{T}_X$

$\iff (g \circ f)^{-1}(V) \in \mathcal{T}_X$.

□

Theorem 3. *(Global Continuity Facts)*

Let X and Y be topological spaces. Let $f : X \to Y$. Then the following are equivalent.

1.) f is continuous.

2.) $f^{-1}(F)$ is closed in X for all F closed in Y.

3.) $f^{-1}(U)$ is open in X for all U members of a subbasis of $\mathcal{T}_Y$.

4.) $f(\overline{A}) \subset \overline{f(A)} (\forall A \subset X)$

5.) $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$

Proof. $(1 \implies 2)$

Let F be closed in Y. Then $Y \setminus \complement F \in \mathcal{T}_Y$.

$\implies \complement f^{-1}(F) = f^{-1}(\complement F) \in \mathcal{T}_X$

$\implies f^{-1}(F)$ is closed in X.

$\square$

Proof. $2 \implies 3$.

$\square$

Proof. $(3 \implies 4)$

$\square$

Proof. $(4 \implies 1)$ A good challenging question for you to work on.

$\square$

1.2 Continuity in Metric Spaces

Definition 1.3. *Let (X, d) and (Y, ρ) be metric spaces, and let $f : X \to Y$. We say that f is continuous at $x_0 \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$f(B(x_0; \delta)) \subset B(f(x_0; \epsilon)).$$

Theorem 4. *Let (X, d) and (Y, ρ) be metric spaces, and let $f : X \to Y$. Then the following are equivalent :*

1. *f is continuous at $x_0 \in X$*
2. *$(\forall \epsilon > 0)(\exists \delta > 0)(f(B(x_0; \delta)) \subset B(f(x_0); \epsilon))$.*

Proposition 1. *Let (X, d) and (Y, ρ) be metric spaces, and let $f : X \to Y$. Then the following are equivalent :*

1. *f is continuous at $x_0 \in X$*
2. *$(\forall \epsilon > 0)(\exists \delta > 0)(x \in B(x_0; \delta) \Rightarrow f(x) \in B(f(x_0); \epsilon))$.*

Definition 1.4. *Let (X, d) be a metric space and let $\{x_k\}$ be a sequence in X. We say that $\{x_k\}$ is a convergent sequence to a point x_0 if for every $\epsilon > 0$ there exists $K \in \mathbb{N}$ such that $d(x_k, x_0) < \epsilon$ for all $k \geq K$.*

Theorem 5. *Let (X,d) and (Y,ρ) be metric spaces, and let $f : X \to Y$. Then the following are equivalent :*

1. *f is continuous at $x_0 \in X$*

2. *$(\forall \{x_k\})(x_k \to x_0 \Rightarrow f(x_k) \to f(x_0))$.*

Proof. (1) $\Rightarrow$ (2) Let $\epsilon > 0$. Since f is continuous at x_0, There exists $\delta > 0$ so that

$$f(B(x_0;\delta)) \subset B(f(x_0;\epsilon)).$$

Since $x_k \to x_0$, there exists $K \in \mathbb{N}$ such that $d(x_k, x_0) < \delta$ for all $k \geq K$. This means,

$$x_k \in B(x_0;\delta) \text{ for all } k \geq K.$$

Consequently,

$$f(x_k) \in B(f(x_0);\epsilon) \text{ for all } \; k \geq K.$$

(2) $\Rightarrow$ (1) exercise

$\square$

1.3 Semicontinuity

Definition 1.5. *(Semicontinuity)*

Let X be a topological space. Let $f : X \to \mathbb{R}$.

We say f is upper semicontinuous if $(\forall \alpha \in \mathbb{R})(f^{-1}(-\infty,\alpha)) \in \tau_X)$

We say that f is lower semicontinuous if $(\forall \alpha \in \mathbb{R})(f^{-1}(\alpha,+\infty) \in \tau_X)$

Theorem 6. *Let X be a topological space. Let $f, g : X \to \mathbb{R}$. be continuous. Then*

1.) $|f|$ is continuous.

2.) $af + bg$ is continuous $(\forall a, b \in \mathbb{R})$

3.) fg is continuous.

4.) If $f(X) \cap \{0\} = \phi$, then $\frac{1}{f}$ is continuous.

You should prove it using lower and upper semicontinuity.

Proof. (3) Use the fact $4ab = (a+b)^2 - (a-b)^2$

$\square$

Proof. (4) Formal assessment : Combine results on continuity.

$\square$

1.3.1 Pasting Lemma

Theorem 7. *(The Pasting lemma)*

Let X and Y be topological spaces such that X=$A \cup B$ where A and B are closed. Let $f : A \to Y$ and let $g : B \to Y$ be continuous such that $f|_{A \cap B} = g|_{A \cap B}$. Define $h : X \to Y$ by $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ for $x \in B$. (That is, $h = f \cup g$) as sets.

Then h is continuous.

Proof. Let K be closed in Y. Then $h^{-1}(K) = f^{-1}(K) \cup g^{-1}(K)$ which is closed in X.

□

1.3.2 Homeomorphisms

Definition 1.6. *(Homeomorphism)*

Let X and Y be topological spaces. Let $h : X \to Y$. We say h is a homeomorphism if h is continuous, bijective, and h^{-1} is continuous.

Definition 1.7. *Let X and Y be topological spaces. and let $f : X \to Y$.*

1.) f is said to be open if $f(G)$ is open in Y for every G open in X.

2.) f is said to be closed if $f(C)$ is closed in Y for every C closed in X..

Theorem 8. *Let X and Y be topological spaces. Let $f : X \to Y$ be a continuous bijection. Then f is a homeomorphism if either of the following hold.*

1.) f is an open mapping

2.) f is a closed mapping.

Proof. Define $g : Y \to X$ by $g = f^{-1}$. Then for all $B \subset X$, we have $g^{-1}(B) = f(B)$. Our conclusion follows directly.

□