1 Topology

1.0.1 Countability in Metric Spaces

Proposition 1. *Let X be a metric Space. Then X is first countable*

Proof. Pick $x \in X$ then the set $B_x = \{B(x; \frac{1}{n}) : n \in \mathbb{N}\}$ is a nbh base for τ in X

□

Theorem 1. *Let X be a metric space. Then the following are equivalent.*

1.) X is separable.

2.) There exists a countable collection $\mathscr{K}_o$ of open subsets of X such that each open set

$$G = \bigcup_i \{A_{n_i} : A_{n_i} \in \mathscr{K}_0\}.$$

3.) Every open covering of X contains a countable subcovering of X.

Proof. $(1 \implies 2)$ Let $D = \{x_i : i \in \mathbb{N}\}$ be a countable dense set. Then

$$\mathscr{K} = \{B(x_i; r) : r \in \mathbb{Q} \wedge i \in \mathbb{N}\} \text{ is countable.}$$

Let G be open and let $x \in G$. Then

$$(\exists r \in \mathbb{Q}, r > 0) \pitchfork \; B(x; r) \subset G.$$

Since D is dense, there exists $x_j \in D \cap B(x; \frac{r}{3})$ such that

$$x \in B(x_j; \frac{2r}{3}) \subset B(x, r) \subset G.$$

Consequently $G = \cup_j \{B(x_j; r) : r \in \mathbb{Q}\}$. Since G was arbitrary, $\mathscr{K}$ is a countable basis.

□

Proof. $(2 \implies 3)$ Let $\mathscr{K} = \{U_\alpha : U_\alpha \text{ is open}, \alpha \in \mathscr{A}\}$, but each U_α is a union of elements of $\mathscr{K}_o$. Then $\{A_{n_i} : j \in \mathbb{N}\}$ is an open covering. By choosing $U_{\alpha_j} \supset A_{n_j}$ for each $j \in \mathbb{N}$

$$\mathcal{S} = \{U_{\alpha_j} : j \in \mathbb{N}\} \text{ is a subcovering of } \mathscr{K} \text{ for } X.$$

□

Proof. $(3 \implies 1)$. We say that a set A is ϵ-dense if for each $x \in X$, $d(x, A) < \epsilon$. For each $\epsilon > 0$ there exists a countable ϵ-dense set denoted by $A(\epsilon)$. For the open covering $\{B(x; \epsilon) : x \in X\}$, we may extract a countable subcovering $\{B(x_i; \epsilon) : i \in \mathbb{N}\}$ and let $A(\epsilon) = \{x_i : i \in \mathbb{N}\}$. Define

$$D = \bigcup_{n=1}^{\infty} A(\frac{1}{n}).$$

□

1.1 Compactness in Metric Spaces

Let us begin with some facts on finite dimensional spaces.

Theorem 2. *(Heine-Berel Theorem) A closed and bounded subset of $\mathbb{R}^n$ is compact.*

Corollary 1. *(Bolzano-Weierstrass) Theorem) Let $A \subset \mathbb{R}^n$ be bounded and infinite. Then $A' \neq \phi$.*

Proof. Suppose $A' = \phi$. Then A is closed and bounded. Hence, by the Heine-Borel Theorem, A is compact. For every $x \in A$, there exists a nbh $U(x)$ such that

$$U(x) \cap (A \setminus \{x\}) = \phi.$$

Then $\mathscr{K} = \{U(x) : x \in A\}$ is an open covering of A, we may extract a finite subcovering, say, $\mathscr{K}_0 = \{U(x_i) : i = 1, ..., n\}$. Since each $U(x_i)$ contains only the point x_i of A, it follows that A is finite, which is a contradiction! □

1.1.1 Total Boundedness

Definition 1.1. *(Totally Bounded Metric Space)*

A metric space (X, d) is said to be **totally bounded** if $(\forall \epsilon > 0)$, the open covering

$$\{B(x; \epsilon) : x \in X\}$$

has a finite subcovering

$$X = \bigcup_{i=1}^{n} B(x_i; \epsilon)$$

We call the set $\{x_i\}_{i=1}^{n}$ an $\epsilon - net$.

Theorem 3. *If (X, d) is totally bounded, then X is separable.*

Proof. For each $n \in \mathbb{N}$, let D_n be a $\frac{1}{n}$ net. Then $J = \bigcup_{n \in \mathbb{N}} D_n$ is countable.
Complete the proof...

□

Theorem 4. *Let (X,d) be a metric space. Let $A \subset X$. If A is totally bounded then $\overline{A}$ is totally bounded.*

Proof. Let $\epsilon > 0$. Then A has an $\frac{\epsilon}{2}$-net E.

Let $x \in \overline{A}$.
$\implies (\exists y \in A) \pitchfork (d(x, y) < \frac{\epsilon}{4})$
$\implies (\exists x_i \in E) \pitchfork (d(x, x_i) \leq d(x, y) + d(y, x_i) \leq \frac{\epsilon}{4} + \frac{\epsilon}{2} < \epsilon)$
Hence E is an ϵ-net for $\overline{A}$.

□

Definition 1.2. *Diameter of a set.*

Let (X, d) be a metric space. We define the following

1. *diam(ϕ) $= 0$.*
2. *diam(A) $=$ sup$\{d(x,y) : x, y \in A\}$ if A is bounded.*
3. *diam(A) $= \infty$ if A is unbounded*

Theorem 5. *Let (X, d) be a metric space and let $A \subset X$. Then the following hold*

1. *If A is Totally bounded, then A is bounded*
2. *If $X = \mathbb{R}^m$, then A bounded implies A totally bounded*

Proof. (1)

Let $D_1 = \{x_i\}_{i=1}^n$ be a 1-net of A. Define

$$\eta = \sup_{(i,j) \in N_n \times N_n} \{d(x_i, x_j)\}.$$

Let $x, y \in A$. Then $(\exists i, j \in N_n)$ such that $d(x, x_i) < 1$ and $d(y, x_j) < 1$. Hence

$$d(x, y) \leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) \leq 2 + \eta$$

$\square$

Proof. (2) exercise

$\square$