Picard's Existence and Uniqueness Theorem

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1 Preliminaries

Lemma 1. *Let $x : [a,b] \to \mathbb{R}$ be continuous. Then, x is bounded, and x achieves it's maximum.*

Proof. We have

$$\begin{aligned}
[a,b] \text{ is compact} &\implies x([a,b]) \text{ is compact} && \text{continuous image of compact set} \\
&\implies x([a,b]) \text{ is closed and bounded} && \text{by the Heine-Borel Theorem, since } x([a,b]) \subseteq \mathbb{R} \\
&\implies x \text{ is bounded.}
\end{aligned}$$

Now we need to show that x achieves its maximum. We know from our argument above, that x has an upper bound. Thus, the least upper bound property of the real numbers tells us that the supremum $M = \sup\{x(t) : t \in [a,b]\}$ exists. By the definition of a supremum, we can choose a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $M - \frac{1}{n} \leq x(t_n) \leq M$. Clearly, the sequence $\{x(t_n)\}_{n\in\mathbb{N}}$ converges to M. Since $[a,b]$ is compact, and therefore sequentially compact, we have that $\{t_n\}_{n\in\mathbb{N}}$ has a subsequence $\{t_{n_i}\}_{i\in\mathbb{N}}$ that converges to some $t \in [a,b]$. Since x is continuous, $\{x(t_{n_i})\}_{i\in\mathbb{N}}$ converges to $x(t)$. Since a subsequence of any convergent sequence converges to the same limit, we have that $\{x(t_{n_i})\}_{i\in\mathbb{N}}$ converges to M. Therefore, x achieves its maximum at t. ☐

Lemma 2. *Let C equal the set of all continuous real-valued functions on the closed interval $[a,b]$. Then, C forms a metric space with the metric d defined by*

$$d(x,y) = \max_{t\in[a,b]} |x(t) - y(t)|.$$

Proof. Since we know every norm induces a metric, it will suffice to show that $||\cdot|| : C \to \mathbb{R}$ defined by

$$||x|| = \max_{t\in[a,b]} |x(t)|$$

forms a norm on C. Since the absolute value function is continuous, and the composition of two continuous functions is continuous, we can conclude from Lemma 1 that $||x||$ is well defined for all $x \in C$.

Let $s \in \mathbb{R}$, and $x \in C$. We have

$$\begin{aligned}
||sx|| &= \max_{t\in[a,b]} |sx(t)| \\
&= \max_{t\in[a,b]} |s||x(t)| \\
&= |s| \max_{t\in[a,b]} |x(t)| && \text{since } s \text{ has no dependence on } t \\
&= |s|||x||,
\end{aligned}$$

and we have shown that $||\cdot||$ obeys the absolute homogeneity property of a norm.

Next, we must show that for all $x \in C$, $||x|| = 0 \implies x = 0$. Let $x \in C$. Proving the contrapositive, we have

$$\begin{aligned}
x \neq 0 &\implies (\exists t_0 \in [a,b])(x(t_0) \neq 0) \\
&\implies \max_{t\in[a,b]} |x(t)| \geq |x(t_0)| && \text{by definition of a max} \\
&\implies \max_{t\in[a,b]} |x(t)| > 0 \\
&\implies ||x|| > 0.
\end{aligned}$$

Finally, we must show that the triangle inequality holds for $||\cdot||$. Let $x, y \in C$. By Lemma 1, we have that $|x|$, $|y|$, and $|x+y|$ achieve their maximums on $[a,b]$. Thus, we can define $t_1, t_2, t_3 \in [a,b]$ to be such that $|x(t_1)| = ||x||$, $|y(t_2)| = ||y||$, and $|x(t_3) + y(t_3)| = ||x+y||$. With this, we have

$$\begin{aligned}
||x+y|| &= |x(t_3) + y(t_3)| \\
&\leq |x(t_3)| + |y(t_3)| && \text{Since the absolute value function forms a norm on the real numbers} \\
&\leq |x(t_1)| + |y(t_3)| && \text{Since } |x(t_1)| \geq |x(t_3)| \text{ by definition of a max} \\
&\leq |x(t_1)| + |y(t_2)| && \text{Since } |y(t_2)| \geq |y(t_3)| \text{ by definition of a max} \\
&= ||x|| + ||y||,
\end{aligned}$$

and our proof is complete. ☐

Lemma 3. *The metric space (C,d) from the previous lemma is complete.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in C. Define $f:[a,b]\to\mathbb{R}$ by $f(t)=\lim_{n\to\infty}x_n(t)$. We need to show that f is well defined, that f is continuous, and that $f=\lim_{n\to\infty}x_n$.

To show that f is well defined, it will suffice to show that for all $t\in[a,b]$, the sequence $\{x_n(t)\}_{n=1}^{\infty}$ is Cauchy, since $\mathbb{R}$ is complete. Let $\epsilon>0$. Since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy, there exists an $N\in\mathbb{N}$ such that for all $n,m\geq N$,

$$\begin{aligned}||x_n-x_m||<\epsilon &\iff \max_{t\in[a,b]}|x_n(t)-x_m(t)|<\epsilon\\ &\implies (\forall t\in[a,b])(|x_n(t)-x_m(t)|<\epsilon).\end{aligned}$$

From this, we can conclude that f is well defined.

We will now show that f is continuous. Let $\epsilon>0$, and let $p_0\in[a,b]$. Since $\{x_n(p)\}_{n\in\mathbb{N}}$ converges to $f(p)$ for all $p\in[a,b]$, there exists an $n\in\mathbb{N}$ such that $|f(p)-x_n(p)|<\frac{\epsilon}{3}$. Since x_n is continuous at p_0, there exists a $\delta>0$ such that for all $p\in[a,b]$

$$|p-p_0|<\delta \implies |x_n(p)-x_n(p_0)|<\frac{\epsilon}{3}.$$

With this, we have that for all $p\in[a,b]$ with $|p-p_0|<\delta$

$$\begin{aligned}|f(p)-f(p_0)| &\leq |f(p)-x_n(p)|+|f(p_0)-x_n(p)| && \text{triangle inequality}\\ &\leq |f(p)-x_n(p)|+|f(p_0)-x_n(p_0)|+|x_n(p_0)-x_n(p)| && \text{triangle inequality}\\ &< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}\\ &=\epsilon.\end{aligned}$$

Thus, we have that f is continuous, and therefore $f\in C$.

Finally, all that remains is to show that $f=\lim_{n\to\infty}x_n$. Let $\epsilon>0$. Since $\{x_n(p)\}_{n\in\mathbb{N}}$ is Cauchy, there exists an $N\in\mathbb{N}$ such that for all $m,n\geq N$, we have $||x_n-x_m||<\epsilon$. Then, for all $t\in[a,b]$ and $m\geq N$, we have

$$\begin{aligned}|f(t)-x_m(t)| &= |\lim_{n\to\infty}x_n(t)-x_m(t)| && \text{by definition of } f\\ &=\lim_{n\to\infty}|x_n(t)-x_m(t)| && \text{by elemetary limit laws}\\ &\leq \lim_{n\to\infty}||x_n-x_m|| && \text{by definition of our norm}\\ &\leq \epsilon. && \text{by definition of } N \text{ above}\end{aligned}$$

Thus, since ϵ was arbitrary, we have shown that $f=\lim_{n\to\infty}x_n$, and that C is complete. □

Lemma 4. *Let $x_0\in\mathbb{R}$, and let $c\in(0,\infty)$. Let $\tilde{C}$ be a subspace of the metric space from Lemma 1, consisting of all functions $x\in C$ such that*

$$d(x,x_0)\leq c.$$

Then, $\tilde{C}$ is closed.

Proof. Let $f\in\bar{\tilde{C}}$. We have

$$\begin{aligned}f\in\bar{\tilde{C}} &\iff (\forall\epsilon>0)(B(f;\epsilon)\cap\tilde{C}\neq\emptyset)\\ &\iff (\forall\epsilon>0)(\exists x\in\tilde{C})(d(f,x)<\epsilon)\\ &\iff (\forall n\in\mathbb{N})(\exists x_n\in\tilde{C})(d(f,x_n)<\frac{1}{n}).\end{aligned}$$

With this, we have

$$\begin{aligned}d(f,x_0) &\leq d(f,x_n)+d(x_n,x_0) && \text{triangle inequality}\\ &<\frac{1}{n}+d(x_n,x_0)\\ &\leq\frac{1}{n}+c && \text{since } x_n\in\tilde{C}\end{aligned}$$

Since this is true for all $n\in\mathbb{N}$, we have $d(f,x_0)\leq c$ which implies $f\in\tilde{C}$. □

Lemma 5. *Let C be a complete metric space. Let $A \subseteq C$ be closed. Then, A is complete.*

Proof. Let A be closed, and let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in A. Since X is complete, we know that the limit converges to some point x in X. We have

$$\begin{aligned}
\lim_{n\to\infty} x_n = x &\implies (\forall\epsilon > 0)(\exists N \in \mathbb{N})(n \geq N \implies d(x_n, x) \leq \epsilon) \\
&\implies (\forall\epsilon > 0)(\exists x_n \in A)(x_n \in B(x;\epsilon)) \\
&\implies (\forall\epsilon > 0)(B(x;\epsilon) \cap A \neq \emptyset) \\
&\implies x \in \bar{A} \\
&\implies x \in A. && \text{since } A \text{ is closed}
\end{aligned}$$

Since this is true for any Cauchy sequence in A, A is complete. □

2 Main Topic

The focus of this paper is on solutions to explicit ordinary differential equations of the form

$$x' = f(t, x) \tag{1}$$

where $x : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and the prime denotes differentiation of x with respect to t. Specifically, we are interested in initial value problems, where

$$x(t_0) = x_0.$$

Theorem 1. *Let f be continuous on a rectangle*

$$R = \{(t, x) \in R^2 : |t - t_0| \leq a \wedge |x - x_0| \leq b\},$$

and thus bounded on R, say

$$|f(t, x)| \leq c, \text{ for all } (t, x) \in R. \tag{2}$$

Suppose f satisfies a Lipschitz condition on R with respect to it's second argument, that is, there is a constant k (Lipschitz constant) such that for $(t, x), (t, v) \in R$

$$|f(t, x) - f(t, v)| \leq k|x - v|. \tag{3}$$

Then the initial value problem (1) has a unique solution. This solution exists on an interval $[t_0 - \beta, t_0 + \beta]$ where

$$\beta < \min\left\{a, \frac{b}{c}, \frac{1}{k}\right\} \tag{4}$$

Proof. Let $C(J)$ be the metric space of all real-valued continuous functions on the interval $J = [t_0 - \beta, t_0 + \beta]$ with metric d defined by

$$d(x, y) = \max_{t\in J} |x(t) - y(t)|.$$

By Lemma 2 we know that $C(J)$ is a metric space, and by Lemma 3 we know that $C(J)$ is complete. Let $\tilde{C}$ be a subspace of $C(J)$ consisting of all functions $x \in C(J)$ such that

$$|x(t) - x_0| \leq c\beta. \tag{5}$$

By Lemma 4, we have that $\tilde{C}$ is closed, so by Lemma 5 $\tilde{C}$ is complete.

From the fundamental theorem of calculus, (1) can be written in the form $x = Tx$, where $T : \tilde{C} \to \tilde{C}$ is defined by

$$Tx(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau \tag{6}$$

We have $\tau \in J \implies |\tau - t_0| \le a$ (by 4), and

$$\begin{aligned} x \in \tilde{C} &\implies |x(\tau) - x_0| \le c\beta && \text{by (5)} \\ &\implies |x(\tau) - x_0| \le b && \text{by (4).} \end{aligned}$$

Thus, $(\tau, x(\tau)) \in R$ and the integral in (6) exists since f is continuous on R. To see that T maps $\tilde{C}$ into itself,

$$\begin{aligned} |Tx(t) - x_0| &= \left| \int_{t_0}^{t} f(\tau, x(\tau)) d\tau \right| \\ &\le c|t - t_0| && \text{by (2)} \\ &\le c\beta && \text{since } t \in [t_0 - \beta, t_0 + \beta]. \end{aligned}$$

Thus, by (5) we have that $Tx \in \tilde{C}$.

We will now show that T is a contraction on $\tilde{C}$. We have

$$\begin{aligned} |Tx(t) - Tv(t)| &= \left| \int_{t_0}^{t} f(\tau, x(\tau)) - f(\tau, v(\tau)) d\tau \right| \\ &\le |t - t_0| \max_{\tau \in J} k|x(\tau) - v(\tau)| && \text{by the Lipschitz condition (3)} \\ &\le k\beta d(x, v) && \text{by (4) and by definition of a the metric } d. \end{aligned}$$

Since the right hand side does not depend on t, we can take the max on both sides to yield

$$d(Tx, Tv) \le k\beta d(x, v).$$

By (4), we have that $k\beta < 1$. Thus, we have shown that T is a contraction mapping on $\tilde{C}$.

By the Banach Fixed Point Theorem, T has a unique fixed point $x \in \tilde{C}$, such that $Tx = x$. That is, there exists a unique $x \in \tilde{C}$ such that

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau)) d\tau. \tag{7}$$

Since $(\tau, x(\tau)) \in R$ where f is continuous, we have that both sides of (7) are differentiable, and x satisfies (1). Since every solution x to (1) must satisfy (7), we have that x is unique, and our proof is complete. $\square$