Problem 24

We will start this problem by proving a useful lemma.

Lemma 1. *Let (X, d) be a metric space. Then, if every sequence in X has a Cauchy subsequence, X is totally bounded.*

Proof. We will prove the contrapositive. Suppose X is not totally bounded. Let $\epsilon > 0$, and let $x_0 \in X$. Since X is not totally bounded, $B(x_0; \epsilon)$ does not cover X. Thus there exists an $x_1 \in X$ such that $x_1 \notin B(x_0; \epsilon)$. Continuing in this manner, we can choose $x_n \in X$ by

$$x_n \notin \bigcup_{i=0}^{n-1} B(x_i; \epsilon)$$

for all $n \in \mathbb{N}$. If for any n we fail to find such an x_n, this means we have created a finite ($n-1$ element) ϵ-net, which contradicts our assumption that X is not totally bounded.

With this, we have created a sequence $\{x_n\}_{n=0}^{\infty}$ in A such that $d(x_i, x_j) \geq \epsilon$ for all $i, j \in \mathbb{N}$, from which it follows that $\{x_n\}_{n=0}^{\infty}$ has no Cauchy subsequence. We have now proven the contrapositive, and our proof is complete. □

Definition 1 (Closed Box)**.** A closed box $A \subseteq \mathbb{R}^n$ is a set that can be written as $\Pi_{i=1}^{n}[a_i, b_i]$ where for all $i \in \{1, ..., n\}$ and for some $l > 0$, we have $b_i - a_i = l$. We say this box has width l.

Lemma 2. *Any closed box A in $\mathbb{R}^n$ can be written as the union of 2^n closed boxes of width $l/2$.*

Proof. Consider first the one dimensional case, $n = 1$. Then $A = [a, b]$ for some $a, b \in \mathbb{R}$ such that $b - a = l$. Then, $A = [a, a + l/2] \cup [a + l/2, b]$, and we have written A as the union of two closed boxes of width $l/2$.

Now, utilizing what we just showed for the one dimensional case, we can see that for any $n \in \mathbb{N}$,

$$\begin{aligned} A &= \Pi_{i=1}^{n}([a_i, b_i]) \\ &= \Pi_{i=1}^{n}([a_i, a_i + l/2] \cup [a_i + l/2, b_i]). \end{aligned}$$

Let $X = \{\Pi_{i=1}^{n} x_i : x_i \in \{[a_i, a_i + l/2], [a_i + l/2, b_i]\}\}$. Clearly, every element of X is a closed box of width $l/2$. By elementary combinatorics, we can also see that X has 2^n elements. We propose that $A = \bigcup_{x \in X} x$. We have

$$\begin{aligned} a \in \Pi_{i=1}^{n}([a_i, a_i + l/2] \cup [a_i + l/2, b_i]) &\iff (\forall i \in \{1, ..., n\})(p_i(a) \in ([a_i, a_i + l/2] \cup [a_i + l/2, b_i])) \\ &\iff (\forall i \in \{1, ..., n\})(p_i(a) \in [a_i, a_i + l/2] \vee p_i(a) \in [a_i + l/2, b_i]) \\ &\iff (\forall i \in \{1, ..., n\})(\exists x_i \in \{[a_i, a_i + l/2], [a_i + l/2, b_i]\})(p_i(a) \in x_i) \\ &\iff a \in \bigcup_{x \in X}, \end{aligned}$$

and our proof is complete. □

Theorem 1. *Let $A \subset \mathbb{R}^n$ be bounded. Then A is totally bounded.*

Proof. Let $x \in A$, and let $\epsilon > 0$. For two points x, y in a box of width l, we have

$$\begin{aligned} d(x, y) &= \sqrt{\sum_{i=1}^{n}(p_i(x) - p_i(y))^2} \\ &\leq \sqrt{\sum_{i=1}^{n} \max\{(p_j(x) - p_j(y))^2 : j \in \{1, ..., n\}\}} \\ &= \sqrt{n} \max\{|(p_j(x) - p_j(y)| : j \in \{1, ..., n\}\}\} \\ &\leq l\sqrt{n}. \end{aligned}$$

Thus, for any points x, y in an n dimensional box of width $l = \frac{\epsilon}{\sqrt{n}}$, we have

$$d(x, y) \leq \epsilon. \tag{1}$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in A. By Lemma 1, it will suffice to show that $\{x_n\}_{n=1}^{\infty}$ has a Cauchy Subsequence. Since A is bounded, there exists an $r > 0$ such that $d(x, x_1) \leq r$ for all $x \in A$. We have, for all $x \in A$,

$$\begin{aligned} r &\geq d(x, x_1) \\ &= \sqrt{\sum_{i=1}^{n} (p_i(x) - p_i(x_1))^2} \\ &\geq |p_i(x) - p_i(x_1)| \text{ for all } i \in \{1, ..., n\}. \end{aligned}$$

Thus, we have $A \subseteq b_1$, where $b_1 \subseteq \mathbb{N}^n$ is a box centered at x_1 with width $l = r/2$. From Lemma 2, we can divide split b_1 into 2^n boxes of width $l/2 = r/4$. Since $\{x_n\}_{n=1}^{\infty}$ is infinite, one of these boxes must have an infinite subsequence of $\{x_n\}_{n=1}^{\infty}$ inside of it; denote this box b_2. Following this procedure, we can define a set of boxes $\{b_n : n \in \mathbb{N}\}$ where b_n is a closed box of width $l = \frac{r}{2^n}$ that results from splitting the previous box into 2^n boxes and choosing one that contains an infinite number of terms in the sequence $\{x_n\}_{n=1}^{\infty}$. By design, $b_{n+1} \subseteq b_n$ for all $n \in \mathbb{N}$.

Let $\{x_{n_j}\}_{j=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_j} \in b_j$ for all $j \in \mathbb{N}$. Picking an $N \in \mathbb{N}$ large enough that $\frac{r}{2^n} \leq \frac{\epsilon}{\sqrt{n}}$, we have from (1) that $d(x_{n_j}, x_{n_k}) \leq \epsilon$ for all $j, k \geq N$. Thus, since ϵ was arbitrary, $\{x_{n_j}\}_{j=1}^{\infty}$ is a Cauchy subsequence of $\{x_n\}_{n=1}^{\infty}$, and we can conclude that A is totally bounded. □

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Theorem 2. *Every sequentially compact metric space is separable.*

Proof. Let A be a sequentially compact metric space. Define $B_x^n = B(x; \frac{1}{n})$ for $x \in A$ and $n \in \mathbb{N}$. Let $x_0^n \in A$ be arbitrary for all $n \in \mathbb{N}$. Choose $x_i^n \in A$ such that $x_i^n \in A \backslash (\bigcup_{j=0}^{i-1} B_{x_j^n}^n)$, for $i \in \mathbb{N}$. Suppose we can choose these for all $i \in \mathbb{N}$. Then, we'd have a sequence $\{x_i^n\}_{i=0}^{\infty}$. By design, we have $d(x_i^n, x_j^n) \geq \frac{1}{n}$ for all $i, j \in \mathbb{N}$. However, since A is sequentially compact, this must have a Cauchy subsequence, which is a contradiction. Therefore, there exists some largest $m_n \in \mathbb{N}$ such that $x_{m_n}^n$ exists.

Define $D \subseteq A$ by

$$D = \{x_i^n : n \in \mathbb{N} \wedge i \in \{1, ..., m_n\}\}.$$

Clearly, since D is a countable union of finite sets, D is countable. We propose that $\bar{D} = A$. Since $\bar{D} \subseteq A$, it will suffice to show that $A \subseteq \bar{D}$.

Let $x \in A$, and let $n \in \mathbb{N}$. Suppose, for the sake of contradiction, that for all $i \in \{1, ..., m_n\}$ that $x_i^n \notin B_x^n$. Then,

$$\begin{aligned} d(x_i^n, x) \geq \frac{1}{n} &\implies x \notin B_{x_i^n}^n \\ &\implies x \notin \bigcup_{j=0}^{m_n} B_{x_j^n}^n \\ &\implies x \in A \backslash \left(\bigcup_{j=0}^{m_n} B_{x_j^n}^n \right) \\ &\implies x_{m_n+1}^n \text{ is well defined,} \end{aligned}$$

which is a contradiction. Therefore, there exists some $x_i^n \in B_x^n$. Since this is true for all $n \in \mathbb{N}$, we have that $x \in \bar{D}$, and we have proven that A is separable. □

Problem 24

Theorem 3. *Let X be compact and let $f : X \to \mathbb{R}$ be upper semi-continuous. Then f attains its supremum.*

Proof. Since f is upper semi-continuous, $f^{-1}((-\infty, \alpha)) \in \mathcal{T}$, for all $\alpha \in \mathbb{R}$. Let $x \in X$. Then, it follows that

$$\begin{aligned} f(x) \in (-\infty, f(x) + 1) &\implies x \in f^{-1}((-\infty, f(x) + 1)) \\ &\implies (\forall x \in X)(\exists \alpha \in \mathbb{R})(x \in f^{-1}((\infty, \alpha))) \\ &\implies \{f^{-1}((\infty, \alpha)) : \alpha \in \mathbb{R}\} \text{ forms an open cover of } X. \end{aligned}$$

Since X is compact, there exists a finite subcover $\{f^{-1}((-\infty, \alpha_i)) : i \in \{1, ..., n\}\}$ of X. Thus, we have that $\max\{\alpha_i : i \in \{1, ..., n\}\}$ is an upper bound of $f(X)$. By the least upper bound property of the real numbers, the supremum

$$M = \sup\{f(x) : x \in X\}$$

exists.

Now lets suppose, for sake of contradiction, that f never attains its supremum. Then, we have that

$$\{f^{-1}((-\infty, M - \frac{1}{n})) : n \in \mathbb{N}\}$$

forms an open covering of X. By compactness of X, there exists a finite open subcovering

$$\{f^{-1}((-\infty, M - \frac{1}{n_i})) : i \in \{1, ..., n\}\}$$

of X. Define $m = \max\{n_i : i \in \{1, ..., n\}\}$. Then, we have that for all $x \in X$, $f(x) < M - \frac{1}{m}$. However, $M - \frac{1}{m} < M$, so we have just contradicted the fact that M is a supremum. Therefore, f must attain it's supremum. $\square$