Metric Spaces Fall 2020 Lecture Notes

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Contents

1 Foundations: Basic Ideas from Set Theory (Cantor 1800s)

Definition 1.1. ***(Set).*** *A **set** is a collection of objects. The objects contained in a set are called **elements**.*

Definition 1.2. ***(Element Inclusion).*** *An element a is said to be **included** in a set A if it is contained in A. Otherwise, a is said to be **not included** in A.*

$$a \in A \Leftrightarrow a \text{ is in } A$$

$$a \notin A \Leftrightarrow a \text{ is not in } A$$

Example 1.1. $A = \{0, -1, 1, -2, 2, \ldots\}$

$$2 \in A, \text{ but } \frac{1}{2} \notin A$$

Definition 1.3. ***(Subset).*** *A is a **subset** of B if every element in A is also an element in B.*

$$A \subseteq B \Leftrightarrow (\forall x)(x \in A \Rightarrow x \in B)$$

Definition 1.4. ***(Set Intersection).*** *The **intersection** between two sets A and B is the set of all elements contained in both A and B.*

$$x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$$

Definition 1.5. ***(Set Union).*** *The **union** between two sets A and B is the set of all elements contained either in A or B.*

$$x \in A \cup B \Leftrightarrow x \in A \vee x \in B$$

Definition 1.6. ***(Set Difference).*** *The **difference** between two sets A and B is the set of elements contained in A but not contained in B.*

$$x \in A - B \Leftrightarrow x \in A \wedge x \notin B$$

Definition 1.7. ***(Set Complement).*** *The* ***complement*** *of a set A is the set of elements in the universe that is not contained in A.*

$$x \in \complement A \Leftrightarrow x \notin A$$

Definition 1.8. ***(Collection).*** *A* ***collection*** *is a set whose elements are also sets. This is also sometimes referred to as a* ***family****.*

Definition 1.9. ***(Power Set).*** *The* ***power set*** *of a set A is defined to be the collection of all subsets of A.*

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

Definition 1.10. ***(Cardinality of a Set).*** *The* ***cardinality*** *of a set A is defined to be the number of elements contained in A. Denoted as $|A|$.*

Definition 1.11. ***(Multiple Unions and Intersections).***

$\bigcup_{\alpha \in \mathscr{A}} A_\alpha = \{x : x \in A_\alpha \text{ for some } \alpha\}$, *where α is a member of some well defined set of indices.*

$\bigcap_{\alpha \in \mathscr{A}} A_\alpha = \{x : x \in A_\alpha,\ \forall \alpha\}$, *where α is a member of some well defined set of indices.*

Definition 1.12. ***(Equality of Sets).*** *Two sets A and B are said to be* ***equal*** *if A is a subset of B and B is a subset of A.*

$$A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$$

Example 1.2. *Let $A_n = \left[0, \frac{1}{n}\right] = \{x \in \mathbb{R} : 0 \leq x \leq \frac{1}{n}\}$*

$$\bigcup_{n \in \mathbb{N}} A_n = [0, 1]$$

$$\bigcap_{n \in \mathbb{N}} A_n = \{0\}$$

2 Theory of Functions

Definition 2.1. ***(Image of a Set).*** *Let $A \subseteq X$. Then*

$$f(A) = \{y \in Y : y = f(x) \text{ for some } x \in A\} \text{ and } f^{-1}(G) = \{x \in X : f(x) \in G\}$$

Definition 2.2. ***(Point in the Inverse or Image).*** *Let $f : X \Rightarrow Y$ be a function between two topological spaces. Then for subsets $A \subseteq X$ and $G \subseteq Y$,*

$$y \in f(A) \Leftrightarrow y = f(x) \text{ for some } x \in A$$

$$x \in f^{-1}(G) \Leftrightarrow f(x) \in G$$

Theorem 1. *Let X, Y be nonempty sets and let $f : X \to Y$. Then,*

1. $f(A \cup B) = f(A) \cup f(B)$
2. $A \subseteq B \Rightarrow f(A) \subseteq f(B)$
3. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
4. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
5. $f(A \cap B) \subseteq f(A) \cap f(B)$
6. $f^{-1}(\mathbb{C}A) = \mathbb{C}(f^{-1}(A))$
7. $f(\mathbb{C}A)???\mathbb{C}(f(A))$ *(this is an open question)*

Proof. Proof here. □

Definition 2.3. ***(Injective).*** *$f : X \to Y$ is **injective** if $\forall x_1, x_2 \in X$, where $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. This is equivalent to say that for all $x_1, x_2 \in X$, such that*

$$f(x) = f(y) \Rightarrow x_1 = x_2.$$

Theorem 2. *f is injective if and only if $f(A \cap B) = f(A) \cap f(B)$, for all $A, B \subseteq X$*

Proof. Proof here. □

Definition 2.4. ***(Surjective).*** *A function $f : X \to Y$ is surjective if every $y \in Y$ has at least one preimage in X.*

Theorem 3. *Let $f : X \to Y$. Then,*

1. *$f(f^{-1}(B)) \subseteq B$, for every $B \subseteq Y$*
2. *$A \subseteq f^{-1}(f(A))$, for every $A \subseteq X$*
3. *f is injective $\Leftrightarrow A = f^{-1}(f(A))$, for every $A \subseteq X$*
4. *f is surjective $\Leftrightarrow f(f^{-1}(B)) = B$, for every $B \subseteq Y$*

Proof. Proof here. □