Thursday, March 23, 2023 11:33 AM

Further Discussion of Absolute Continuity.

Problem 4: Let $f : [a, b] \longrightarrow \mathbb{R}$ be increasing and $f'(x)$ exists with finite value for $\forall x \in [a, b]$. Then f is a.c. on $[a, b]$.

Pf: We know that

$$\int_a^b f'(t)dt \leq f(b) - f(a).$$

and also if $\int_a^b f'(t)dt = f(b) - f(a)$

then $\int_a^x f'(t)dt = f(x) - f(a)$, $\forall x \in [a, b]$.

and hence f is a.c. on $[a, b]$.

So it suffices to show that

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Given $\varepsilon > 0$, we prove that

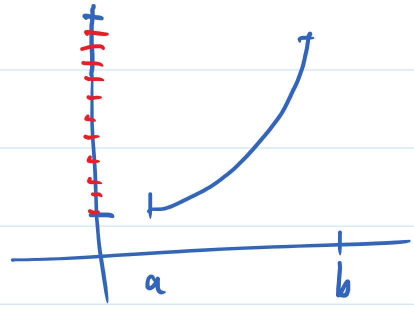
$$f(b) - f(a) \leq \int_a^b f'(x)dx + \varepsilon(b-a).$$

- Since f' exists on $[a, b]$, $\Rightarrow$ f is cont. on $[a, b]$.

and hence

$$f(a) = \min_{[a,b]} f$$

$$f(b) = \max_{[a,b]} f$$



$$\Rightarrow \quad f([a, b]) = [f(a), f(b)]$$

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$$\Rightarrow \quad |f([a,b])| = |[f(a), f(b)]| = f(b) - f(a)$$

Note: $[a,b] = \{x : f'(x) \text{ exists and is finite}\}$

$$= \bigcup_{k=1}^{\infty} \{x : (k-1)\varepsilon \le f'(x) < k\varepsilon\}$$

$$= \bigcup_{k=1}^{\infty} E_k .$$

- E_k is m'ble
- E_k are disjoint
- on E_k:

$(k-1)\varepsilon \le f'(x) < k\varepsilon .$

Recall we proved for increasing function:

if $f' \le p$ $(p > 0)$ on $E \subset [a,b]$

then $|f(E)|_e \le p\,|E|_e .$

$$\Rightarrow \quad |f(E_k)|_e \le k\varepsilon\,|E_k|$$

$\Rightarrow$ Note $[a,b] = \cup E_k$

$$\Rightarrow \quad f([a,b]) = f\Big(\bigcup_{k=1}^{\infty} E_k\Big)$$

$$= \bigcup_{k=1}^{\infty} f(E_k)$$

$\Rightarrow$

$$f(b) - f(a) = |f([a,b])| = \Big|\bigcup_{k=1}^{\infty} f(E_k)\Big|$$

$$\le \sum_{k=1}^{\infty} |f(E_k)|_e \le \sum_{k=1}^{\infty} k\varepsilon\,|E_k|$$

$$= \sum \left[(k-1)\,\varepsilon\,|E_k| + \varepsilon |E_k| \right]$$

$$= \sum_{k=1}^{\infty} (k-1)\,\varepsilon\,|E_k| + \varepsilon \sum_{k=1}^{\infty} |E_k|$$

$$\le \sum_{k=1}^{\infty} \int_{E_k} f'(x)\,dx + \varepsilon \left| \bigcup_{k=1}^{\infty} E_k \right|$$

additive property

$$= \int_{\cup E_k} f'(x)\,dx + \varepsilon\,|[a,b]|$$

$$= \int_a^b f'(x)\,dx + \varepsilon(b-a).$$

This shows (*). Let $\varepsilon \to 0$ in (*):

$$f(b) - f(a) \le \int_a^b f'(x)\,dx.$$

<u>Hence</u>

$$f(b) - f(a) = \int_a^b f'(x)\,dx. \qquad \square$$

Lemma: Let $f : [a,b] \to \mathbb{R}$ and let $E \subseteq [a,b]$.

If $f'(x)$ exists $\forall x \in E$ and $|f'(x)| \le p$ $\forall x \in E$ for some $p > 0$,

then

$$|f(E)|_e \le p\,|E|_e.$$

(ii) Null-function

(i) Null-function

Def: (N-function) Let $f:[a,b]\to\mathbb{R}$. Then f is said to be a N-function if

$\forall\ E\subset[a,b]$ with $|E|=0$

then $|f(E)|=0$.

Thm. Let $f:[a,b]\to\mathbb{R}$ be continuous. Then f is a N-function $\iff$ for $\forall\ E\subset[a,b]$ m'ble, $f(E)$ is m'ble.

Proof ($\Rightarrow$) Let $E\subset[a,b]$ be m'ble. Then

We can write $E=H\cup Z$,

where H is a F_σ set and $|Z|=0$.

Write $H=\bigcup_{k=1}^{\infty}F_k$, where F_k are closed.

We may also assume that F_k are compact.

(First we assume that $F_k\nearrow$. then
$\tilde{F}_k=F_k\cap\{|x|\le k\}$. $\Rightarrow H=\bigcup\tilde{F}_k$.)

Hence $f(E)=f(H\cup Z)=f(H)\cup f(Z)$

$$=f\left(\bigcup_{k=1}^{\infty}F_k\right)\cup f(Z)$$

$$=\left(\bigcup_{k=1}^{\infty}f(F_k)\right)\cup f(Z).$$

Note: $\cdot$ $f(F_k)$ is compact. $\Rightarrow f(E_k)$ is m'ble.

$\cdot$ $|Z|=0 \Rightarrow |f(Z)|=0 \Rightarrow f(Z)$ is m'ble

$\Rightarrow$ $f(E)$ is m'ble. $\boxed{\cdot}$

Thm. Let $f:[a,b]\to\mathbb{R}$. Then f is a.c. on $[a,b]$ if and only if the following 4 conditions hold:

(1) f is continuous;

(2) f' exists a.e.;

(3) $f'\in L[a,b]$;

(4) f is a N-function.

((4) can be replaced by:

(4'): $\quad f(x)=f(a)+\int_a^x f'(t)\,dt \quad \forall x\in[a,b]$.)

Proof: $(\Rightarrow)$. (1)-(3) have been proved before.

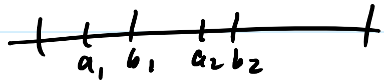
(4) is proved from a HW problem.

$(\Leftarrow)$ Assume (1)-(4).

Let $A=\{x\in[a,b]:\ f'(x)$ does not exist$\}$.

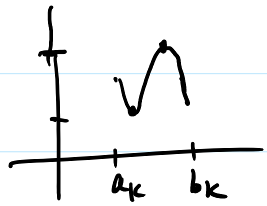
Then $|A|=0$ from (2) $\Rightarrow |f(A)|=0$

Let $\{[a_k, b_k]\}_{k=1}^{N}$ be a finite collection of nonoverlapping intervals in $[a, b]$.



Since f is cont on $[a,b]$ and hence on $[a_k, b_k]$,

$$\Rightarrow \quad f([a_k, b_k]) = [m_k, M_k]$$



$$\Rightarrow \quad \underline{|f(b_k) - f(a_k)|} \le M_k - m_k$$

$$= |f([a_k, b_k])|$$

$$= |f([a_k, b_k] - A) \cup f(A \cap [a_k, b_k])|$$

$$\le |f([a_k, b_k] - A)|_e + |f(A)|_e$$

$$= |f([a_k, b_k] - A)|_e$$

$|f(A)| = 0$ since $|A| = 0$

$$\le \int_{[a_k, b_k] - A} |f'(x)|\, dx \le \int_{[a_k, b_k]} |f'(x)|$$

$$\Rightarrow \quad |f(b_k) - f(a_k)| \le \int_{[a_k, b_k]} |f'(x)|\, dx$$

Hence $\displaystyle\sum^{N} |f(b_k) - f(a_k)| \le \sum_{k=1}^{N} \int_{[a_k, b_k]} |f'(x)|\, dx$

$$= \int |f'(x)|\, dx$$

$$= \int_{\bigcup [a_k, b_k]} |f'(x)|\, dx$$

Then from the absolute continuity of the integral

$$\int_{[a,b]} |f'(x)|\, dx$$

we conclude that f is a.c. on $[a,b]$.

Corollary: Let $f: [a,b] \to \mathbb{R}$ be continuous. Assume that f' exists except on a countable set of $[a,b]$, and $f' \in L[a,b]$. Then f is a.c. on $[a,b]$ and hence

$$f(x) = f(a) + \int_a^x f'(t)\,dt \quad \forall x \in [a,b].$$