Problem 1

Use Holder's inequality to show

$$\int_0^1 \sqrt{x}(1-x)^{-1/3}dx \le \frac{2}{5^{1/3}}.$$

Solution

Using Holder's inequality, let $p = 3$, and let $p' = 3/2$. Then, we have

$$\begin{aligned}
\int_0^1 \sqrt{x}(1-x)^{-1/3}dx &\le \left(\int_0^1 \sqrt{x}^3 dx\right)^{1/3} \left(\int_0^1 ((1-x)^{-1/3})^{3/2}dx\right)^{2/3} \\
&= \left(\int_0^1 x^{3/2}dx\right)^{1/3} \left(\int_0^1 u^{-1/2}du\right)^{2/3} \\
&= \left(\frac{2}{5}\right)^{1/3} 2^{2/3} \\
&= \frac{2}{5^{1/3}},
\end{aligned}$$

as desired.

Problem 2

Let $E \subseteq \mathbb{R}^n$ be measurable with $|E| = 1$. Let $h \ge 0$ be measurable on E. Let $A = \int_E h dx$. Show that

$$\sqrt{1+A^2} \le \int_E \sqrt{1+h^2}dx \le 1 + A.$$

Solution

(I've not yet figured out the proof of the first inequality, which would go here.)

For the second inequality, we have

$$\begin{aligned}
\int_E \sqrt{1+h^2}dx &\le \int_E \sqrt{1+h^2+2h}dx \\
&= \int_E \sqrt{(1+h)^2}dx \\
&= \int_E 1 + h dx \\
&= 1 + A.
\end{aligned}$$

Problem 3

Find all nonnegative functions $g \in L^3(0,1)$ such that

$$\left(\int_0^1 xg(x)dx\right)^3 = \frac{4}{25}\int_0^1 g^3(x)dx$$

Solution

Using Holders inequality, we can see that if $p = \frac{3}{2}$, and $p' = 3$, then

$$\begin{aligned}
\int_0^1 xg(x)dx &\leq \left(\int_0^1 x^{3/2}dx\right)^{2/3} \left(\int_0^1 g^3(x)dx\right)^{1/3} \\
&= \left(\frac{2}{5}\right)^{2/3} \left(\int_0^1 g^3(x)dx\right)^{1/3} \\
&= \left(\frac{4}{25}\int_0^1 g^3(x)dx\right)^{1/3} \\
\left(\int_0^1 xg(x)dx\right)^3 &= \frac{4}{25}\int_0^1 g^3(x)dx.
\end{aligned}$$

Now, as we proved in class, the equality holds if and only if $\alpha x^{3/2} = g^3(x)$ almost everywhere for some real α. Thus, we have

$$g(x) = \alpha x^{\frac{1}{2}}$$

for some nonnegative real number α and for almost every x.

Problem 4

Let $f \in L^\infty(0,1)$ and $||f||_\infty \leq 1$. Show that

$$\int_0^1 \sqrt{1-f^2(x)}dx \leq \sqrt{1-\left(\int_0^1 f(x)dx\right)^2},$$

and describe the class of functions f for which equality takes place.

Solution

We have

$$\begin{aligned}
\int_0^1 \sqrt{1-f^2(x)}dx &= \int_0^1 \sqrt{(1-f)(1+f)}dx \\
&= \int_0^1 \sqrt{1-f}\sqrt{1+f}dx \\
&\leq \sqrt{\int_0^1 (1-f)dx}\sqrt{\int_0^1 (1+f)dx} && \text{Cauchy-Schwarz's inequality} \\
&= \sqrt{1-\int_0^1 fdx}\sqrt{1+\int_0^1 fdx} \\
&= \sqrt{1-\left(\int_0^1 f(x)dx\right)^2}.
\end{aligned}$$

From the proof of Holder's Inequality, we know that the equality holds iff there exists some $\alpha \in \mathbb{R}$ such that $1 - f = \alpha(1 + f)$. Thus,

$$\begin{aligned}
1 - f = \alpha(1+f) &\implies 1 - f = \alpha + \alpha f \\
&\implies 1 - \alpha = (1+\alpha)f \\
&\implies f = \frac{1-\alpha}{1+\alpha}.
\end{aligned}$$

Problem 5

Theorem 1. *Prove that*

$$\int_0^\infty e^{-x}\sqrt{x^4+3x^2+2}dx \leq \sqrt{12},$$

and that the equality does not hold.

Solution

Proof. We have

$$\begin{aligned}
\int_0^\infty e^{-x}\sqrt{x^4+3x^2+2}dx &= \int_0^\infty \sqrt{e^{-2x}(x^4+3x^2+2)}dx \\
&= \int_0^\infty \sqrt{e^{-x}(x^2+2)e^{-x}(x^2+1)}dx \\
&= \int_0^\infty \sqrt{e^{-x}(x^2+2)}\sqrt{e^{-x}(x^2+1)}dx \\
&\leq \sqrt{\int_0^\infty e^{-x}(x^2+2)dx}\sqrt{\int_0^\infty e^{-x}(x^2+1)dxdx} && \text{Cauchy-Schwarz's inequality} \\
&= \sqrt{2+2}\sqrt{2+1} && \text{Apply integration by parts twice} \\
&= \sqrt{12}.
\end{aligned}$$

Assume that the equality holds. Then, there exists some $\alpha \in \mathbb{R}$ such that for almost every $x \in [0,\infty)$, we have

$$e^{-x}(x^2+2) = \alpha e^{-x}(x^2+1).$$

With this, we see

$$\begin{aligned}
x^2+2 = \alpha x^2 + \alpha &\implies x^2(1-\alpha) = 2+\alpha \\
&\implies x^2 = \frac{1-\alpha}{2+\alpha}.
\end{aligned}$$

Thus, for any given α, this can only be true for at most two points in $[0,\infty)$, which contradicts our assumption that this is true almost everywhere. Therefore, we can conclude that the equality does not hold, as desired. □

Problem 6

Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and bounded. Show that

$$||f||_\infty = \sup\{|f(x)| : x \in \mathbb{R}^n\}.$$

Solution

Let $M = \sup\{|f(x)| : x \in \mathbb{R}^n\}$, and let $\alpha < M$. Then, by definition of supremum, we have

$$\begin{aligned}
\{x \in \mathbb{R}^n : |f(x)| > M\} = \emptyset &\implies |\{x \in \mathbb{R}^n : |f(x)| > M\}| = 0 \\
&\implies M \geq ||f||_\infty.
\end{aligned}$$

Thus, we just need to show that $M \leq ||f||_\infty$, and our proof will be complete. To do this, let's suppose for sake of contradiction that $M > ||f||_\infty$. Then, there exists some nonnegative $\alpha < M$ such that

$$|\{x \in \mathbb{R}^n||f(x)| > \alpha\}| = 0.$$

We have

$$\{x \in \mathbb{R}^n||f(x)| > \alpha\} = |f|^{-1}((\alpha,\infty)).$$

Since f is continuous, $|f|$ is continuous. Thus, since (α,∞) is open, we can conclude that $|f|^{-1}((\alpha,\infty))$ is open. Now, by definition of supremum, we have that there exists some $x \in \mathbb{R}^n$ such that $\alpha < |f(x)| \leq M$. Thus, $|f|^{-1}((\alpha,\infty))$ is nonempty. Since nonempty open subsets of $\mathbb{R}^n$ have positive measure, we can conclude that

$$|\{x \in \mathbb{R}^n||f(x)| > \alpha\}| > 0$$

and we have reached a contradiction. Thus, $M \leq ||f||_\infty$, and our proof is complete.

Problem 7

Theorem 2. *Let $\{f_k\} \subset L^p(E)$ for some $1 \leq p < \infty$. Assume that $|E| < \infty$, $||f_k||_p \leq A$ for each natural k, and that*

$$\lim_{k\to\infty} f_k(x) = f(x)$$

for almost every $x \in E$. Then $f_k \to f$ in L^1.

Solution

To prove this, we will start by proving an intuitive lemma.

Lemma 1. *Let $E \subseteq \mathbb{R}^n$, and let $f \in L(E)$. Then for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $A \subseteq E$ with $|A| < \delta$, then*

$$\int_A |f| < \epsilon.$$

Informally, this is known as the fact that integrals over small sets are small.

Proof. Let $\epsilon > 0$, and let $A \subseteq E$. Let g be a simple function with $0 \leq g \leq |f|$. Then

$$\int_A (|f| - g) \leq \int_E (|f| - g) \implies \int_A |f| \leq \int_A g + \int_E |f| - \int_E g.$$

By theorem 4.13 in our textbook, and the monotone convergence theorem, we can choose g such that

$$\int_E |f| - \int_E g < \frac{\epsilon}{2}.$$

If we define $M = \max g$, then

$$\int_A g \leq |A|M.$$

Thus, if we let $\delta = \frac{\epsilon}{2M}$, and A is such that $|A| < \delta$, then we have

$$\begin{aligned}\int_A |f| &\leq \int_A g + \int_E |f| - \int_E g \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon,\end{aligned}$$

and our proof is complete. □

Now we are ready for the proof of the main theorem.

Proof. Let q be the conjugate exponent of p, and let $F \subseteq E$ be measurable. If $p > 1$, we have

$$\begin{aligned}\int_F |f_k| &\leq \left(\int_F 1\right)^{1/q} \left(\int_F |f|^p\right)^{1/p} && \text{By Holder's inequality} \\ &\leq \left(\int_F 1\right)^{1/q} \left(\int_E |f|^p\right)^{1/p} \\ &= |F|^{1/q} ||f_k||_p \\ &\leq |F|^{1/q} A.\end{aligned}$$

Let $\epsilon > 0$. By Ergorov's theorem, there exists a closed set $F \subseteq E$ such that f_k converges uniformly to f on F, and $|E \backslash F| < \epsilon$. We have

$$\begin{aligned}
\lim_{k\to\infty} \int_E |f_k - f| &= \lim_{k\to\infty} \int_F |f_k - f| + \int_{E\backslash F} |f_k - f| \\
&= 0 + \lim_{k\to\infty} \int_{E\backslash F} |f_k - f| && \text{By the uniform convergence theorem} \\
&\leq \int_{E\backslash F} |f_k| + \int_{E\backslash F} |f| \\
&\leq |E\backslash F|^{1/q} A + \int_{E\backslash F} \liminf |f_k| \\
&\leq |E\backslash F|^{1/q} A + \liminf \int_{E\backslash F} |f_k| && \text{By Fatou's Lemma} \\
&\leq 2|E\backslash F|^{1/q} A \\
&\leq 2|\epsilon|^{1/q} A.
\end{aligned}$$

Thus, taking the limit as $\epsilon \to 0$, we have proven that f_k converges to f in L^1 if $p > 1$.

Now consider the case of $p = 1$. Then, we have

$$\begin{aligned}
\lim_{k\to\infty} \int_E |f_k - f| &= \lim_{k\to\infty} \int_F |f_k - f| + \int_{E\backslash F} |f_k - f| \\
&= 0 + \lim_{k\to\infty} \int_{E\backslash F} |f_k - f| && \text{By the uniform convergence theorem} \\
&\leq \int_{E\backslash F} |f_k| + \int_{E\backslash F} |f| \\
&\leq \int_{E\backslash F} |f_k| + \int_{E\backslash F} \liminf |f_k| \\
&\leq \int_{E\backslash F} |f_k| + \liminf \int_{E\backslash F} |f_k| && \text{By Fatou's Lemma.}
\end{aligned}$$

Thus, taking the limit as $\epsilon \to 0$, the above lemma allows us to conclude that f_k converges to f in L^1, and our proof is complete. $\square$

Problem 8

Theorem 3. *Let $1 \leq p < \infty$ and assume that $f \in L^p(\mathbb{R})$. Then*

$$\lim_{t\to\infty} \int_t^{t+1} f(x)dx = 0.$$

Solution

Proof. We have

$$\begin{aligned}
\lim_{t\to\infty} \int_t^{t+1} |f(x)|dx = 0 &\implies \lim_{t\to\infty} \left| \int_t^{t+1} f(x)dx \right| = 0 \\
&\implies \lim_{t\to\infty} \int_t^{t+1} f(x)dx = 0,
\end{aligned}$$

so it will suffice to show that $\lim_{t\to\infty} \int_t^{t+1} f(x)dx = 0$.

Suppose, for sake of contradiction, that $\lim_{t\to\infty} \int_t^{t+1} f(x)dx \neq 0$. Then, there exists some $\epsilon > 0$ such that for any natural N, there exists an $t \geq N$ such that

$$\int_t^{t+1} f(x)dx > \epsilon.$$

Let t_0 be any natural number such that

$$\int_{t_0}^{t_0+1} f(x)dx > \epsilon,$$

and define a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \geq t_{n-1} + 1$, and

$$\int_{t_n}^{t_n+1} f(x)dx > \epsilon.$$

Then, the intervals $[t_n, t_n + 1]$ are non overlapping, and we have

$$\begin{aligned}\int_{\mathbb{R}} |f(x)|dx &\geq \sum_{n=0}^{\infty} \int_{t_n}^{t_n+1} f(x)dx \\ &> \sum_{n=0}^{\infty} \epsilon \\ &= \infty.\end{aligned}$$

Thus, $f \notin L^1$ and by theorem 8.2 $f \notin L^p$, which is a contradiction. Thus, we have proven the theorem. □

Problem 9

Theorem 4. *Let $p, q, r \in [1, \infty)$ such that*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

If $f \in L^p(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, then

$$||fg||_r \leq ||f||_p \cdot ||g||_q.$$

Solution

Proof. Define $p' = \frac{p}{r}$ and $q' = \frac{q}{r}$. Then p' and q' are conjugate exponents. We have

$$\begin{aligned}||fg||_r &= \left(\int_{\mathbb{R}^n} |fg|^r\right)^{1/r} \\ &= \left(\int_{\mathbb{R}^n} |f^r g^r|\right)^{1/r} \\ &\leq \left(\left(\int_{\mathbb{R}^n} |f^r|^{p'}\right)^{1/p'} \left(\int_{\mathbb{R}^n} |g^r|^{q'}\right)^{1/q'}\right)^{1/r} && \text{By Holder's inequality} \\ &= \left(\int_{\mathbb{R}^n} |f|^{rp'}\right)^{1/rp'} \left(\int_{\mathbb{R}^n} |g|^{rq'}\right)^{1/rq'} \\ &= \left(\int_{\mathbb{R}^n} |f|^p\right)^{1/p} \left(\int_{\mathbb{R}^n} |g|^q\right)^{1/q} \\ &= ||f||_p \cdot ||g||_q,\end{aligned}$$

and our proof is complete. □

Problem 10

Theorem 5. *Let $1 \leq p < r < q < +\infty$ and define $\theta \in (0, 1)$ by*

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}.$$

Let $f \in L^p \cap L^q$. Prove that

$$||f||_r \leq ||f||_p^{\theta} \cdot ||f||_q^{1-\theta}.$$

Solution

Proof. Define $p' = \frac{p}{\theta}$ and $q' = \frac{q}{1-\theta}$. Then, we have

$$\frac{1}{r} = \frac{1}{p'} + \frac{1}{q'}.$$

Utilizing the results of the previous problem, we have

$$\begin{aligned}
||f||_r &= ||f^\theta f^{1-\theta}||_r \\
&\leq ||f^\theta||_{p'} \cdot ||f^{1-\theta}||_{q'} \\
&= \left(\int_{\mathbb{R}^n} |f^\theta|^{p'}\right)^{1/p'} \left(\int_{\mathbb{R}^n} |f^{1-\theta}|^{q'}\right)^{1/q'} \\
&= \left(\int_{\mathbb{R}^n} |f|^{\theta p'}\right)^{1/p'} \left(\int_{\mathbb{R}^n} |f|^{(1-\theta)q'}\right)^{1/q'} \\
&= \left(\int_{\mathbb{R}^n} |f|^{p}\right)^{\theta/p} \left(\int_{\mathbb{R}^n} |f|^{q}\right)^{(1-\theta)/q} \\
&= ||f||_p^\theta \cdot ||f||_q^{1-\theta},
\end{aligned}$$

and our proof is complete. □

Problem 11

Theorem 6. *Let $0 < r < \infty$ and $f \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then the following statements are true:*

Part (i):

For all $p \in (r, \infty)$,

$$||f||_p \leq ||f||_r^{r/p} \cdot ||f||_\infty^{1-r/p}.$$

Part (ii):

$$\lim_{p\to\infty} ||f||_p = ||f||_\infty.$$

Solution

Proof. Part (i):

We have

$$\begin{aligned}
||f||_p &= ||f^{r/p} f^{1-r/p}||_p \\
&= \left(\int_{\mathbb{R}^n} |f^{r/p} f^{1-r/p}|^p dx\right)^{1/p} \\
&= \left(\int_{\mathbb{R}^n} |f^r f^{p-r}| dx\right)^{1/p} \\
&\leq \left(||f^{p-r}||_\infty \cdot \int_{\mathbb{R}^n} |f^r| dx\right)^{1/p} && \text{Holder's Inequality} \\
&= \left(||f||_\infty^{p-r} \cdot \int_{\mathbb{R}^n} |f|^r dx\right)^{1/p} \\
&= \left(||f||_\infty^{p-r} \cdot ||f||_r^r\right)^{1/p} \\
&= ||f||_r^{r/p} \cdot ||f||_\infty^{1-r/p},
\end{aligned}$$

as desired.

Part (ii):

Using the results from the previous part, we have

$$\begin{aligned}\lim_{p\to\infty} ||f||_p &\leq \lim_{p\to\infty} ||f||_r^{r/p} \cdot ||f||_\infty^{1-r/p} \\ &= ||f||_r^0 \cdot ||f||_\infty^{1-0} \\ &= ||f||_\infty.\end{aligned}$$

Thus, all we must do to complete the proof is show that $\lim_{p\to\infty} ||f||_p \geq ||f||_\infty$ (but I've not figured this part out yet). □