Problem 1

Theorem 1. *Let f be measurable on a measurable set E in $\mathbb{R}^n$. Define an extension of f on $\mathbb{R}$ by*

$$\bar{f}(x) = \begin{cases} f(x) & x \in E \\ 0 & \text{if } x \in \mathbb{R}^n - E \end{cases}$$

Then

Part (i):
$\bar{f}$ is measurable on $\mathbb{R}^n$, and

Part (i):
If $f \in L(E)$, then $\bar{f} \in L(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \bar{f} = \int_E f.$$

Solution

Proof. Part (i):
Let $a \in \mathbb{R}$. We have

$$\begin{aligned} \{\bar{f} > a\} &= \{x \in \mathbb{R}^n | \bar{f}(x) > a\} \\ &= \{x \in E | f(x) > a\} \cup \{x \in E^c | 0 > a\}. \end{aligned}$$

The fist set in the union above is measurable, since f is measurable. The second set is either the empty set (if $a \geq 0$), or is E^c if $a < 0$. Since the empty set and the complement of a measurable set are both measurable, and the union of two measurable sets is measurable, we have shown that $\{\bar{f} > a\}$ is measurable.

Part (ii):
Suppose $f \in L(E)$. By theorem 5.24, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \bar{f} &= \int_{\mathbb{R}^n - E} \bar{f} + \int_E \bar{f} \\ &= \int_{\mathbb{R}^n - E} (0) + \int_E f \\ &= \int_E f. \end{aligned}$$

Thus, since $f \in L(E)$, $\int_E f$ is finite, and we can conclude that $\int_{\mathbb{R}^n} \bar{f}$ is finite. Thus, we have shown that $f \in L(\mathbb{R}^n)$. □

Problem 2

Theorem 2. *Let f_k $(k = 1, 2, ...)$ and f be measurable and finite a.e. on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} |f_k - f| dx \to 0$ as $k \to \infty$. Then f_k converges to f in measure.*

Solution

Proof. Suppose that f_k does not converge to f in measure. Then, there exists some $\epsilon > 0$ such that

$$\lim_{k \to \infty} |\{x \in \mathbb{R}^n : |f(x) - f_k(x)| > \epsilon\}| \neq 0.$$

Thus, there exists some $\delta > 0$, such that for all $K \in \mathbb{N}$, there exists a $k \geq K$ such that

$$\lim_{k \to \infty} |\{x \in \mathbb{R}^n : |f(x) - f_k(x)| > \epsilon\}| > \delta.$$

Define $E = \{x \in \mathbb{R}^n : |f(x) - f_k(x)| > \epsilon\}$. Then, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |f_k - f|dx &= \int_{\mathbb{R}^n - E} |f_k - f|dx + \int_E |f_k - f|dx && \text{By theorem 5.7} \\
&\geq \int_E |f_k - f|dx && \text{Since } |f_k - f| \geq 0 \\
&\geq \int_E \epsilon dx && \text{By theorem 5.5} \\
&\geq |E|\epsilon && \text{By corollary 5.5} \\
&\geq \delta\epsilon
\end{aligned}$$

Thus, for every $K \in \mathbb{N}$, there exists a $k \geq K$ such that $\int_{\mathbb{R}^n} |f_k - f|dx \geq \delta\epsilon$, and we have shown that $\int_{\mathbb{R}^n} |f_k - f|dx$ does not go to 0. With this, we have proven the contrapositive, and our proof is complete. □