Problem 1

We know that if f is measurable, then for every $a \in \mathbb{R}^1$, the set $f^{-1}(\{a\})$ is measurable. Use the following function f to show that the converse of this result is not true, where $f : \mathbb{R}^1 \to \mathbb{R}^1$ is defined by

$$f = \begin{cases} e^x & x \in E \\ -e^x & x \in E^c \end{cases}$$

where $E \in \mathbb{R}^1$ is a nonmeasurable set.

Solution

Proof. We have that e^x is a positive function. Thus, $-e^x$ is a negative function. Thus, it follows that

$$\{f > 0\} = E.$$

Since E is nonmeasurable, this tells us that $\{f > 0\}$ is nonmeasurable. With this, we have shown that f is not measurable. □

Problem 2

Theorem 1. *Let $E \in \mathbb{R}^1$ be measurable. Show that if $f : E \to [-\infty, \infty]$ is increasing, then f is measurable.*

Proof. Let $a \in \mathbb{R}$. We have

$$\{f > a\} = \{x : E | f(x) > a\}.$$

Now, suppose there is no lower bound to $\{f > a\}$. Then,

$$\begin{aligned} (\forall x \in E)(\exists x_a \in \{f > a\})(x_a < x) &\implies (\forall x \in E)(\exists x_a \in \{f > a\})(f(x_a) \leq f(x)) && \text{Since } f \text{ is increasing} \\ &\implies (\forall x \in E)(f(x) > a) \\ &\implies \{f > a\} = E \\ &\implies \{f > a\} \text{ is measurable.} \end{aligned}$$

Suppose then, that $\{f > a\}$ has a lower bound. Then, by a fundamental property of the real numbers, $\{f > a\}$ has a greatest lower bound $l \in \mathbb{R}$. Suppose first that $l \in \{f > a\}$. Then, since f is increasing, we have

$$\begin{aligned} (\forall x \geq l)(f(x) > a) &\implies [l, \infty) \cap E = \{f > a\} \\ &\implies \{f > a\} \text{ Is measurable} && \text{Intersection of two measurable sets.} \end{aligned}$$

Similarly, if $l \notin \{f > a\}$, then

$$\begin{aligned} (\forall x > l)(f(x) > a) &\implies (l, \infty) \cap E = \{f > a\} \\ &\implies \{f > a\} \text{ Is measurable} && \text{Intersection of two measurable sets.} \end{aligned}$$

Therefore, we have shown that f is measurable. □

Problem 3

Theorem 2. *If f is differentiable on $[a, b]$, the f' is measurable.*

Proof. □