Problem 1.6

Theorem 1. *Let $\alpha > 0$. Then $f(x) = x^\alpha$ is absolutely continuous on every subinterval $[a,b] \subseteq [0,\infty)$.*

Solution

Proof. We have that f is differentiable on $(0,\infty)$ with derivative

$$f'(x) = \alpha x^{\alpha-1}.$$

Thus, f is differentiable almost everywhere on $[0,\infty)$. Now, let $[a,b] \subseteq [0,\infty)$. Since f' is continuous a.e. on $[a,b]$, f' is integrable on $[a,b]$. Furthermore, we have for any $x \in [a,b]$,

$$\begin{aligned}\int_a^x f'(x)dx &= \int_a^x \alpha x^{\alpha-1} dx \\ &= \alpha x^\alpha|_a^x \\ &= b^\alpha - x^\alpha \\ &= f(x) - f(a).\end{aligned}$$

Thus, by theorem 7.29 in our textbook, we have that f is absolutely continuous on any subinterval of $[0,\infty)$. □

Problem 1.7

Theorem 2. *A function f is absolutely continuous on $[a,b]$ if and only if given $\epsilon > 0$, there exists $\delta > 0$ such that $|\sum[f(b_i) - f(a_i)]| < \epsilon$ for any finite collection $\{[a_i,b_i]\}$ of nonoverlapping subintervals of $[a,b]$ with $\sum(b_i - a_i) < \delta$.*

Solution

Proof. Suppose f is absolutely continuous on $[a,b]$, and let $\epsilon > 0$. Since f is absolutely continuous, there exists a $\delta > 0$ such that $\sum |[f(b_i) - f(a_i)]| < \epsilon$ for any finite collection $\{[a_i,b_i]\}$ of nonoverlapping subintervals of $[a,b]$ with $\sum(b_i - a_i) < \delta$. Thus, if we let $\{[a_i,b_i]\}$ be a set of nonoverlapping subintervals of $[a,b]$ with $\sum(b_i - a_i) < \delta$, we have

$$\begin{aligned}\epsilon &> \sum |[f(b_i) - f(a_i)]| \\ &\geq |\sum[f(b_i) - f(a_i)]|, && \text{Basic property of absolute value}\end{aligned}$$

and we have proven the forward direction.

Now, suppose that if we are given $\epsilon > 0$, there exists $\delta > 0$ such that $|\sum[f(b_i) - f(a_i)]| < \epsilon$ for any finite collection $\{[a_i,b_i]\}$ of nonoverlapping subintervals of $[a,b]$ with $\sum(b_i - a_i) < \delta$. Let $\epsilon > 0$, and choose $\delta > 0$ such that $|\sum[f(b_i) - f(a_i)]| < \frac{\epsilon}{2}$ for any finite collection $\{[a_i,b_i]\}$ of nonoverlapping subintervals of $[a,b]$ with $\sum(b_i - a_i) < \delta$. Let $\{[a_i,b_i]\}$ be a set of nonoverlapping subintervals of $[a,b]$ with $\sum(b_i - a_i) < \delta$. We have

$$\sum_{i\in\{i:f(b_i)\geq f(a_i)\}} (b_i - a_i) < \delta,$$

which means that

$$\begin{aligned}\frac{\epsilon}{2} &> \left| \sum_{i\in\{i:f(b_i)\geq f(a_i)\}} [f(b_i) - f(a_i)] \right| \\ &= \sum_{i\in\{i:f(b_i)\geq f(a_i)\}} |f(b_i) - f(a_i)|.\end{aligned}$$

Similarly, we have

$$\sum_{i\in\{i:f(b_i)< f(a_i)\}} (b_i - a_i) < \delta,$$

which implies

$$\begin{aligned}\frac{\epsilon}{2} &> \left| \sum_{i\in\{i: f(b_i)<f(a_i)\}} [f(b_i)-f(a_i)] \right| \\ &= \sum_{i\in\{i: f(b_i)<f(a_i)\}} |f(b_i)-f(a_i)|.\end{aligned}$$

Finally, we have

$$\begin{aligned}\sum_i |f(b_i)-f(a_i)| &= \sum_{i\in\{i: f(b_i)<f(a_i)\}} |f(b_i)-f(a_i)| + \sum_{i\in\{i: f(b_i)\geq f(a_i)\}} |f(b_i)-f(a_i)| \\ &\leq \frac{\epsilon}{2}+\frac{\epsilon}{2} \\ &= \epsilon,\end{aligned}$$

and we have shown that f is absolutely continuous. With this, our proof is complete. □

Problem 1.8

Theorem 3. *If f is of bounded variation on $[a,b]$, and if the function $V(x) = V[a,x]$ is absolutely continuous on $[a,b]$, then f is absolutely continuous on $[a,b]$.*

Solution

Proof. Let $\epsilon > 0$. Since $V(x)$ is absolutely continuous, we have that there exists $\delta > 0$ such that $\sum |V(b_i) - V(a_i)| < \epsilon$ for any finite collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a,b]$ with $\sum(b_i - a_i) < \delta$. Let $\{[a_i, b_i]\}$ be a collection of nonoverlapping subintervals of $[a,b]$ with $\sum(b_i - a_i) < \delta$. From Theorem 2.2 (part i) in our textbook, since f is of bounded variation, and $V(x)$ is finite for all $x \in [a,b]$. Furthermore, from theorem 2.2 (part ii), we have

$$V[a,b] = V[a,a_i] + V[a_i,b] \qquad\qquad = V[a,b_i] + V[b_i,b].$$

Then,

$$\begin{aligned}V(b_i) - V(a_i) &= V[a,b_i] - V[a,a_i] \\ &= V[a_i,b] - V[b_i,b] \\ &\geq V[a_i,b] \\ &\geq V[a_i,b_i] && \text{Theorem 2.2 part i} \\ &\geq |f(b_i) - f(a_i)|.\end{aligned}$$

Finally, we have

$$\begin{aligned}\epsilon &> \sum |V(b_i) - V(a_i)| \\ &\geq \sum |f(b_i) - f(a_i)|,\end{aligned}$$

and we have proven that f is absolutely continuous. □