Problem 1

Theorem 1. Let $f:(a,b)\to\mathbb{R}$ be such that for all $x_0\in(a,b)$, there is a support line

$$l_{x_0}(x) = f(x_0) + m(x - x_0)$$

for some $m \in \mathbb{R}$ such that

$$f(x) \ge l_{x_0}(x)$$

for all $x \in (a, b)$. Then f is convex on (a, b).

Solution

Proof. Suppose that f is not convex on (a,b). Then, for some $[x_1,x_2] \subset (a,b)$, there exists an $x_0 \in [x_1,x_2]$ such that $f(x_0) > L(x_0)$, where L is the straight line such that $L(x_1) = f(x_1)$ and $L(x_2) = f(x_2)$. We have $l_{x_0}(x_1) \leq f(x_1) = L(x_1)$ and $l_{x_0}(x_2) \leq f(x_2) = L(x_2)$. Since L and l_{x_0} are both straight lines, we have that $l_{x_0}(x) \leq L(x)$ for all $x \in [x_1, x_2]$. However, we have

$$l_{x_0}(x_0) = f(x_0) + m(x_0 - x_0) = f(x_0) > L(x_0),$$

and we have reached a contradiction. Thus, we have proven that f is convex.

Problem 2

Theorem 2. Let $f:[a,b] \to \mathbb{R}$ be absolutely continuous and f'(x) be increasing except on a zero measure subset of [a,b]. Then f is convex on [a,b].

Solution

We will start by proving a lemma which will aid in the proof of this theorem:

Lemma 1. Let $f:[a,b] \to \mathbb{R}$ be continuous and satisfy the midpoint convexity

$$f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}.$$

for any $x_1, x_2 \in [a, b]$. Then f is convex on [a, b].

Proof. We will first prove, that for any $n \in \mathbb{N}$, if $x_1, ..., x_{2^n} \in [a, b]$, then

$$f\left(\frac{x_1 + \dots + x_{2^n}}{2^n}\right) \le \frac{f(x_1) + \dots + f(x_{2^n})}{2^n}.$$

To prove this, we note that the base case (n = 1) is true by midpoint convexity of f. Now, suppose for some $n \in \mathbb{N}$ that if $x_1, ..., x_{2^n} \in [a, b]$, then

$$f\left(\frac{x_1 + \dots + x_{2^n}}{2^n}\right) \le \frac{f(x_1) + \dots + f(x_{2^n})}{2^n}.$$

Let $x_1, ..., x_{2^{n+1}} \in [a, b]$. Then, we have

$$\begin{split} f\left(\frac{x_1+\ldots+x_{2^{n+1}}}{2^{n+1}}\right) &= f\left(\frac{\frac{x_1+\ldots+x_{2^n}}{2^n}+\frac{x_{2^n+1}+\ldots+x_{2^{n+1}}}{2^n}}{2}\right) \\ &\leq \frac{f\left(\frac{x_1+\ldots+x_{2^n}}{2^n}\right)+f\left(\frac{x_{2^n+1}+\ldots+x_{2^{n+1}}}{2^n}\right)}{2} \\ &= \frac{f(x_1)+\ldots+f(x_{2^{n+1}})}{2^{n+1}}, \end{split}$$

and our induction is complete.

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From elementary analysis, we have that rational numbers of the form $\frac{m}{2^n}$ with $1 \le m \le 2^n$ are dense in [0,1]. Let $[a_1,b_2] \subseteq [a,b]$. Fix some $n \in \mathbb{N}$ and some $1 \le m \le 2^n$. Setting $x_i = a_i$ for $1 \le i \le m$, and $x_i = b_i$ for $m+1 \le i \le 2^n$, we see from the above argument that

$$f\left(\frac{x_1 + \dots + x_{2^n}}{2^n}\right) \le \frac{f(x_1) + \dots + f(x_{2^n})}{2^n}$$

$$= \frac{mf(a_i) + (2^n - m)f(b_i)}{2^n}$$

$$= \frac{m}{2^n}f(a_i) + (1 - \frac{m}{2^n})f(b_i).$$

Finally, let $\theta \in [0,1]$ be a real number. For each $n \in \mathbb{N}$, define $1 \le m_n \le 2^n$ to be the largest number such that $\frac{m_n}{2^n} \ge \theta$. Then, $\frac{m_n}{2^n} \to \theta$, and it follows from the continuity of f that

$$f(\theta a_i + (1 - \theta)b_i) \le \theta f(a_i) + (1 - \theta)f(b_i),$$

and we have shown that f is convex.

Now we are ready to prove the main theorem:

Proof. Let $[x_1, x_2] \subseteq [a, b]$. By the above lemma, it will suffice to show that

$$f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}.$$

Since f is absolutely continuous, we have

$$f\left(\frac{x_1+x_2}{2}\right) - f(x_1) = \int_{x_1}^{\frac{x_1+x_2}{2}} f'(x)dx,$$

and

$$f(x_2) - f\left(\frac{x_1 + x_2}{2}\right) = \int_{\frac{x_1 + x_2}{2}}^{x_2} f'(x)dx.$$

Now, since f' is increasing almost everywhere, we have

$$\int_{x_1}^{\frac{x_1+x_2}{2}} f'(x)dx \le \int_{\frac{x_1+x_2}{2}}^{x_2} f'(x)dx.$$

With this, we have

$$f\left(\frac{x_1+x_2}{2}\right) - f(x_1) \le f(x_2) - f\left(\frac{x_1+x_2}{2}\right)$$
$$f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1) + f(x_2)}{2}$$

and we have proven that f is midpoint-convex, and therefore convex.

Problem 3

Theorem 3. Let $f:[a,b] \to \mathbb{R}$ and let $E \subseteq [a,b]$. Assume that f'(x) exists with a finite value for any $x \in E$. Then,

$$|f(E)|_e \le \int_E |f'(x)| dx.$$

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Solution

Proof. We will use the fact that if f'(x) exists for all $x \in A \subseteq [a,b]$, and $|f(x)| \leq p$ for all $x \in A$, then

$$|f(A)|_e \leq p|A|_e$$
.

Let $\epsilon > 0$, and define $E_k = \{x \in E : (k-1)\epsilon \le |f'(x)| < k\epsilon\}$ for each natural k. Then, we have

$$|f(E)|_{e} = \left| f(\bigcup_{k \in \mathbb{N}} E_{k}) \right|_{e}$$

$$= \left| \bigcup_{k \in \mathbb{N}} f(E_{k}) \right|_{e}$$

$$\leq \sum_{k \in \mathbb{N}} |f(E_{k})|_{e}$$

$$\leq \sum_{k \in \mathbb{N}} k\epsilon |E_{k}|$$

$$= \sum_{k \in \mathbb{N}} (k-1)\epsilon |E_{k}| + \epsilon |E_{k}|$$

$$\leq \sum_{k \in \mathbb{N}} \int_{E_{k}} |f'(x)| dx + \epsilon \sum_{k \in \mathbb{N}} |E_{k}|$$

$$= \int_{E} |f'(x)| dx + \epsilon \left| \bigcup_{k \in \mathbb{N}} E_{k} \right|$$

$$= \int_{E} |f'(x)| dx + \epsilon |E|$$

$$\leq \int_{E} |f'(x)| dx + \epsilon |[a, b]|.$$

Thus, taking the limit as $\epsilon \to 0$, we have $|f(E)|_e \le \int_E |f'(x)| dx$, and our proof is complete.

Problem 4

Theorem 4. Let $f:[a,b] \to \mathbb{R}$ be continuous. If f'(x) exists as a finite number almost on all but a countable subset of [a,b] and $f' \in L([a,b])$, then f is absolutely continuous on [a,b].

Solution

Proof. By a theorem we recently proved in class, it will suffice to show that f is a null function. Let $N \subseteq [a,b]$ have measure zero. Define $Z \subseteq N$ to be the at most countable set where f' does not have a finite value. Then, we have

$$\begin{split} |f(N)|_e &= |f(N\backslash Z) \cup f(Z)|_e \\ &\leq |f(N\backslash Z)|_e + |f(Z)|_e \\ &\leq \int_{N\backslash Z} |f'(x)| dx + 0 \qquad \text{By Theorem 3 and the fact that } f(Z) \text{ is at most countable} \\ &= 0. \qquad \qquad \text{Since } N\backslash Z \text{ has measure zero, and } f' \text{ is finite on } N\backslash Z \end{split}$$

Thus, we have shown that f is absolutely continuous.

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