Problem 1

Use Holder's inequality to show

$$\int_0^1 \sqrt{x}(1-x)^{-1/3}dx \leq \frac{2}{5^{1/3}}.$$

Solution

Using Holder's inequality, let $p = 3$, and let $p' = 3/2$. Then, we have

$$\begin{aligned}
\int_0^1 \sqrt{x}(1-x)^{-1/3}dx &\leq \left(\int_0^1 \sqrt{x}^3 dx\right)^{1/3} \left(\int_0^1 ((1-x)^{-1/3})^{3/2}dx\right)^{2/3} \\
&= \left(\int_0^1 x^{3/2}dx\right)^{1/3} \left(\int_0^1 u^{-1/2}du\right)^{2/3} \\
&= \left(\frac{2}{5}\right)^{1/3} 2^{2/3} \\
&= \frac{2}{5^{1/3}},
\end{aligned}$$

as desired.

Problem 2

Let $E \subseteq \mathbb{R}^n$ be measurable with $|E| = 1$. Let $h \geq 0$ be measurable on E. Let $A = \int_E hdx$. Show that

$$\sqrt{1+A^2} \leq \int_E \sqrt{1+h^2}dx \leq 1 + A.$$

Solution

Show the first inequality here!!!!

Now, we have

$$\begin{aligned}
\int_E \sqrt{1+h^2}dx &\leq \int_E \sqrt{1+h^2+2h}dx \\
&= \int_E \sqrt{(1+h)^2}dx \\
&= \int_E 1 + hdx \\
&= 1 + A.
\end{aligned}$$

Problem 3

Find all nonnegative functions $g \in L^3(0,1)$ such that

$$\left(\int_0^1 xg(x)dx\right)^3 = \frac{4}{25}\int_0^1 g^3(x)dx$$

Solution

Using Holders inequality, we can see that if $p = \frac{3}{2}$, and $p' = 3$, then

$$\begin{aligned}
\int_0^1 xg(x)dx &\leq \left(\int_0^1 x^{3/2}dx\right)^{2/3} \left(\int_0^1 g^3(x)dx\right)^{1/3} \\
&= \left(\frac{2}{5}\right)^{2/3} \left(\int_0^1 g^3(x)dx\right)^{1/3} \\
&= \left(\frac{4}{25}\int_0^1 g^3(x)dx\right)^{1/3} \\
\left(\int_0^1 xg(x)dx\right)^3 &= \frac{4}{25}\int_0^1 g^3(x)dx.
\end{aligned}$$

Now, as we proved in class, the equality holds if and only if $\alpha x^{3/2} = g^3(x)$ almost everywhere for some real α. Thus, we have

$$g(x) = \alpha x^{\frac{1}{2}}$$

for some nonnegative real number α and for almost every x.