MA 653-01 Midterm Test (10/14/22)

1. Prove that $E \subset \mathbb{R}^n$ is measurable if and only if for every $\epsilon > 0$ there exists a measurable set $B \subset E$ such that $|E - B|_e < \epsilon$.

2. Let $\{I_1, I_2, \cdots, I_N\}$ be a finite family of closed intervals in $\mathbb{R}^1$ such that $Q \cap [0,1] \subset \cup_{j=1}^N I_j$. Prove that $\sum_{j=1}^N |I_j| \geq 1$. Is this true if the family of intervals is infinite? Justify your answer.

3. (i) Let A, $B \in \mathbb{R}^n$ such that A is measurable. Show that if $A \cap B = \emptyset$, then $|A \cup B|_e = |A| + |B|_e$.

   (ii) Show that if $F \subset E$ is closed such that $|E|_e - |F| < \varepsilon$ and $|F| < \infty$, then $|E - F|_e < \varepsilon$.

4. Let A and B be subsets of $\mathbb{R}^n$. Show that $|A \cup B|_e + |A \cap B|_e \leq |A|_e + |B|_e$.

5. Show that if $E \subset \mathbb{R}$ is measurable and $|E| > 0$, then there are $x, y \in E$ such that $x - y$ is a rational number. (Hint: You may use a similar argument used in the proof of the nonmeasurable set given in the lecture.)

6. **(Bonus)** Let E be the nonmeasurable set in $I := [0,1]$ constructed in the lecture. Show the following:

   (i) $|E|_i = 0$.

   (ii) $|I| < |E|_e + |I - E|_e$.

   (iii) $|I| > |E|_i + |I - E|_i$.

   (Hint: For (i), show by a similar argument used in the proof of the nonmeasurable set given in the lecture that any closed set in E has measure zero. For (ii) and (iii), use the result: If A is measurable and B is a subset of A, then $|A| = |B|_i + |A - B|_e$.)

7. Let $f(x) = x^3$. Show that if E is a zero measure set of $\mathbb{R}$, then $|f(E)| = 0$.

8. Show that if f and g are continuous functions on $\mathbb{R}^n$ and are equal a.e. in $\mathbb{R}^n$, then $f \equiv g$ on $\mathbb{R}^n$.

9. Let (f_k) be a sequence of measurable functions defined on a measurable set E with $|E| < \infty$. If $|f_k(x)| \leq M_x < \infty$ for all k for each $x \in E$, show that given $\varepsilon > 0$, there is a closed $F \subset E$ and a finite $M > 0$ such that $|E - F| < \varepsilon$ and $|f_k(x)| \leq M$ for all k and all $x \in F$.

10. Let f be a measurable function and finite a.e on $E \subset \mathbb{R}^n$. Show that there is a sequence of continuous functions g_k on $\mathbb{R}^n$ such that $g_k \to f$ a.e. on E.

    Hint: Use the alternative version of the Lusin's theorem.

11. Let f be a measurable function and finite a.e on $[a,b]$. Show that there is a sequence of polynomials p_k such that $p_k \to f$ a.e. on $[a,b]$.