Problem 1

Theorem 1. *Suppose that f is a continuous function a.e. in a measurable set $E \in \mathbb{R}^n$. Show that f is measurable on E.*

Solution

Proof. By Theorem 4.5 in our textbook, we can assume, without loss of generality, that f is continuous on all of E. Let $a \in \mathbb{R}$. We have

$$\begin{aligned} x \in \{f > a\} &\iff f(x) > a \\ &\iff (f(x) - a > 0). \end{aligned}$$

Define $\epsilon_x = f(x) - a$. Since$f$ is continuous, there exists some δ_x such that $f(B(x;\delta_x)) \subseteq B(f(x);\epsilon_x)$. Let $y \in B(x;\delta_x)$. Then

$$\begin{aligned} f(y) \in B(f(x);\epsilon_x) &\implies |f(x) - f(y)| < \epsilon_x \\ &\implies f(y) > f(x) - \epsilon_x \\ &\implies f(y) > f(x) - (f(x) - a) \\ &\implies f(y) > a \\ &\implies y \in \{f > a\}. \end{aligned}$$

Thus, $B(x;\delta_x) \subseteq \{f > a\}$, and we have shown that $\{f > a\}$ is an open set. Since every open set is measurable, we have shown that $\{f > a\}$ is measurable. Since a was arbitrary, we have shown that f is measurable, and our proof is complete. □

Problem 2

Theorem 2. *Let f_k $(k = 1, 2, ...)$ be measurable functions and finite a.e. on E, with $|E| < \infty$. Let f be a function on E. Assume that for any $\delta > 0$, there is a measurable set $F \subseteq E$ such that $|E - F| < \delta$ and $f_k \to f$. Then $f_k \to f$ a.e. on E and f is measurable on E.*

Solution

Proof. Construct a family of measurable sets $\{F_j\}$ such that for each $j \in \mathbb{N}$, $F_j \subseteq E$ with $|E - F_j| < \frac{1}{j}$, and $f_k \to f$ on F_j. Define a new family of measurable sets $\{E_j\}$ by $E_j = \bigcup_{i=1}^{j} F_j$. Then we have that $E_j \nearrow E$ and $E_j \nearrow \bigcup_{j=1}^{\infty} F_j$. By theorem 3.26 in our textbook

$$\left|\bigcup_{j=1}^{\infty} F_j\right| = \lim_{k\to\infty} |E_k| = |E|.$$

Thus, since $|E| < \infty$, we have that

$$\left|E - \bigcup_{j=1}^{\infty} F_j\right| = 0.$$

Since $f_k \to f$ on F_j for all j, we have that $f_k \to f$ on $\bigcup_{j=1}^{\infty} F_j$, and our proof is complete. □

Problem 3

Theorem 3. *Let $f_k = \chi_{|x|<k}$ for all $k \in \mathbb{N}$. Then,*

Part (i):

$f_k \to 1$ for all $x \in \mathbb{R}^n$.

Part (ii):

f_k does not converge uniformly outside any ball $|x| > m$ for $m > 0$.

Part (iii):

The conclusion in Egorov's theorem does not hold for $\{f_k\}$.

Solution

Proof. Part (i):

Let $x \in \mathbb{R}^n$. Define $K \in \mathbb{N}$ to be the smallest natural number such that $K \geq |x|$. Then, we have that $f_k(x) = 1$ for all $k \geq K$. Since we can do this for any x, we have that $f_k \to 1$.

Part (ii):

Suppose, for sake of contradiction, that f_k does converge uniformly to f on some unbounded subset of $\mathbb{R}^n$. Then, there exists some natural K such that $|f_k(x) - 1| < 1$ for all $k \geq K$. Now let $x \in \mathbb{R}^n$ be such that $|x| > K$. Then, $f_K(x) = 0$, and we have a contradiction. Thus, f_k does not converge uniformly on any unbounded set.

Part (iii):

Let $f \subseteq \mathbb{R}^n$ be closed, with $|E - F| < \epsilon$ for some finite $\epsilon > 0$. Now, F must be unbounded, for any bounded subset of $\mathbb{R}^n$ would have finite measure, and thus we would have $|E - F| = \infty$. Therefore, by part (ii), f_k cannot converge uniformly to 1 on F, and the conclusion of Ergorov's theorem does not hold. □

Problem 4

Theorem 4. Part (i):

Assume that $f_k \overset{m}{\to} f$ on E and $f_k \leq f_{k+1}$ for natural k. Then, $f_k \to f$ a.e. on E.

Part (ii):

Assume that $f_k \overset{m}{\to} f$ on E and $f_k \leq M$ a.e. on E for each natural k. Then $f \leq M$ a.e. on E.

Solution

Proof. Part (i):

By theorem 4.22 in our textbook, there exists a subsequence f_{k_j} such that $f_{k_j} \to f$ a.e. on E. Let $\epsilon > 0$, and let $x \in \mathbb{R}^n$ be such that $f_{k_j}(x) \to f(x)$. Then, there exists some $J \in \mathbb{N}$ such that for all $j \geq J$

$$|f_{k_j}(x) - f(x)| < \frac{\epsilon}{3}.$$

Let $k \geq k_J$. Then, there exist $j_1, j_2 \geq J$ such that $k_{j_1} \leq k \leq k_{j_2}$. Then, we have

$$\begin{aligned}
|f_k(x) - f(x)| &\leq |f_k(x) - f_{k_{j_1}}(x)| + |f_{k_{j_1}}(x) - f(x)| && \text{By the triangle inequality} \\
&= f_k(x) - f_{k_{j_1}}(x) + |f_{k_{j_1}}(x) - f(x)| && \text{Since } k \geq k_1 \\
&\leq f_{k_{j_2}}(x) - f_{k_{j_1}}(x) + |f_{k_{j_1}}(x) - f(x)| && \text{Since } k \leq k_2 \\
&= |f_{k_{j_2}}(x) - f_{k_{j_1}}(x)| + |f_{k_{j_1}}(x) - f(x)| \\
&= 2|f_{k_{j_1}}(x) - f(x)| + |f_{k_{j_2}}(x) - f(x)| && \text{By the triangle inequality} \\
&< 2\frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

Part (ii):

Suppose that $f > M$ on a set $F \subseteq E$ with nonzero measure. We can write F as a countable union of disjoint measurable sets

$$F = \bigcup_{j=1}^{\infty} \{f > M + \frac{1}{j}\}.$$

One of these sets must have nonzero measure. Thus, there exists some $\epsilon > 0$ such that $\{f > M + \epsilon\}$ has non zero measure. Since $f_k \leq M$ a.e. on E, we have that for all k

$$\{f > M + \epsilon\} - Z \subseteq \{x \in E : |f(x) - f_k(x)| > \epsilon\},$$

where Z is a set of measure zero, where each of the $f_k > M$. With this, we have that

$$\lim_{k \to \infty} |\{x \in E : |f(x) - f_k(x)| > \epsilon\}| \geq |\{f > M + \epsilon\}| > 0,$$

showing that $f_k \overset{m}{\not\to} f$, and proving the contrapositive. □

Problem 5

Theorem 5. *Let $f_k(x) = \frac{x}{k}$ for $x \in \mathbb{R}$ and $k \in \mathbb{N}$. Then f_x does not converge in measure on $\mathbb{R}$.*

Solution

Proof. Clearly, $f_k \to 0$. Let $\epsilon > 0$, and let $k \in \mathbb{N}$. Then, there exists and $X \in \mathbb{R}$ such that for all $x \geq X$, we have $\frac{x}{k} > \epsilon$. Therefore,

$$\{x \in E : |f_k(x) - 0| > \epsilon\} \subseteq \{x \geq X\}$$

has infinite measure. Since this is true for all k, we can conclude that

$$\lim_{k\to\infty} |\{x \in E : |f_k(x) - 0| > \epsilon\}| = \infty,$$

and we have shown that f_k does not converge in measure. □

Problem 6

Theorem 6. *$f_k \xrightarrow{m} f$ on E if and only if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $k \geq N$,*

$$|\{x \in E : |f_k(x) - 0| > \epsilon\}| < \epsilon.$$

Solution

Proof. Suppose first that $f_k \xrightarrow{\mathrm{m}} f$. Then, by definition, we have that for every $\epsilon > 0$,

$$\lim_{k\to\infty} |\{x \in E : |f_k(x) - 0| > \epsilon\}| = 0.$$

By directly applying the definition of a limit, there exists some $N \in \mathbb{N}$ such that for all $k \geq N$,

$$|\{x \in E : |f_k(x) - 0| > \epsilon\}| < \epsilon.$$

With this, we have proven the forward direction.

Now suppose that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $k \geq N$,

$$|\{x \in E : |f_k(x) - 0| > \epsilon\}| < \epsilon.$$

Fix some $\epsilon_0 > 0$, and let $\epsilon > 0$ be such that $\epsilon < \epsilon_0$. Then,

$$\{x \in E : |f_k(x) - 0| > \epsilon_0\} \subseteq \{x \in E : |f_k(x) - 0| > \epsilon\} \implies |\{x \in E : |f_k(x) - 0| > \epsilon_0\}| \leq |\{x \in E : |f_k(x) - 0| > \epsilon\}|.$$

By initial assumption, there exists $N \in \mathbb{N}$ such that for any $k \geq N$,

$$|\{x \in E : |f_k(x) - 0| > \epsilon\}| < \epsilon.$$

Thus, for any $k \geq N$,

$$|\{x \in E : |f_k(x) - 0| > \epsilon_0\}| < \epsilon,$$

and we have shown that

$$\lim_{k\to\infty} |\{x \in E : |f_k(x) - 0| > \epsilon_0\}| = 0.$$

Thus, we have shown that $f_k \xrightarrow{\mathrm{m}} f$, and our proof is complete. □

Problem 7

Theorem 7. *Let E be measurable and $|E| < \infty$. Then $f_k \xrightarrow{m} f$ if and only if any subsequence $\{f_{k_n}\}$ has a subsequence $\{f_{k_{n_j}}\}$ that is convergent to f a.e. on E.*

Solution

Proof. Suppose first that $f_k \xrightarrow{\text{m}} f$, and let $\{f_{k_n}\}$ be a subsequence of f_k. Let $\epsilon > 0$. Since $f_k \xrightarrow{\text{m}} f$, there exists a $K \in \mathbb{N}$ such that for all $k \geq K$,

$$|\{x \in E : |f_k(x) - f(x)| > \epsilon\}| < \epsilon.$$

Thus, since $k_n \geq K$ for all $n \geq K$, we have that

$$|\{x \in E : |f_{n_k}(x) - f(x)| > \epsilon\}| < \epsilon$$

for all $n \geq K$. Thus, $\{f_{k_n}\}$ converges in measure to f. By theorem 4.22 in our textbook, we can conclude that $\{f_{k_n}\}$ has a subsequence $\{f_{k_{n_j}}\}$ that is convergent to f a.e. on E.

For the reverse direction, we will prove the contrapositive. Suppose then, that $f_k \stackrel{\text{m}}{\not\to} f$. Then, there exists some $\epsilon > 0$ such that

$$\lim_{k\to\infty} |\{x \in E : |f_k(x) - f(x)| > \epsilon\}| \neq 0.$$

Then, for some $\delta > 0$, we have that for all $K \in \mathbb{N}$, there exists a $k \geq K$ such that

$$|\{x \in E : |f_k(x) - f(x)| > \epsilon\}| > \delta.$$

Thus, we can define a subsequence $\{f_{k_n}\}$ of $\{f_k\}$ such that

$$|\{x \in E : |f_{k_n}(x) - f(x)| > \epsilon\}| > \delta$$

for all $n \in \mathbb{N}$. Define $E_n \subseteq E$ by

$$E_n = \{x \in E : |f_{k_n}(x) - f(x)| > \epsilon\}.$$

Let $\{f_{k_{n_j}}\}$ be a subsequence of $\{f_{k_n}\}$, and suppose for sake of contradiction that $f_{k_{n_j}} \to f$ a.e.. Then, there exists some set $Z \subseteq E$ such that $|Z| = 0$ and $f_{k_{n_j}} \to f$ on $E - Z$. Thus, there exists some $J \in \mathbb{N}$ such that for all $j \geq J$, and all $x \in E - Z$

$$|f_{k_j}(x) - f(x)| < \epsilon.$$

However, since $|E_{n_j}| > 0$, we must have that $E_{n_j} \cap (E - Z) \neq \emptyset$. Thus, there exists an $x \in E - Z$ such that

$$|f_{k_j}(x) - f(x)| > \epsilon,$$

and we have reached a contradiction. Thus, $f_{k_{n_j}} \not\to f$ a.e.. Since this is true for any subsequence of $\{f_{k_n}\}$, our proof is complete. □

Problem 8

Theorem 8. *Assume that $f_k \xrightarrow{m} f$ on E with $|E| < \infty$. Then, $f_k^2 \xrightarrow{m} f^2$ on E.*

Solution

Before we jump into this proof, we will need to prove a useful lemma.

Lemma 1. *Let $\{a_k\}$ and $\{b_k\}$ be two sequences in R with limits A and B respectively. Then,*

$$a_k b_k \to AB.$$

Proof. Let $\epsilon > 0$. Assume first that A and B are nonzero, as otherwise the proof would be trivial. Using the handy equality

$$a_k b_k - AB = (a_k - A)B + (b_k - B)A + (a_k - A)(b_k - B),$$

we have that for all k,

$$|a_k b_k - AB| \leq |a_k - A||B| + |b_k - B||A| + |a_k - A||b_k - B|.$$

Define $N_1 \in \mathbb{N}$ to be such that for all $n \geq N_1$,

$$|a_k - A| < \min\left\{\frac{\epsilon}{3|B|}, \sqrt{\frac{\epsilon}{3}}\right\}.$$

Furthermore, define $N_2 \in \mathbb{N}$ to be such that for all $n \geq N_2$,

$$|b_k - B| < \min\left\{\frac{\epsilon}{3|A|}, \sqrt{\frac{\epsilon}{3}}\right\}.$$

Finally, define $N \in \mathbb{N}$ to be the larger of N_1 and N_2, and we can see that for all $k \geq N$, we have

$$\begin{aligned} |a_k b_k - AB| &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

With this, we have completed the proof. $\square$

Now we can jump into the main proof.

Proof. Let $\{f_{k_n}^2\}$ be a subsequence of $\{f_k^2\}$. By the result of problem 7, and the fact that $f_k \overset{\text{m}}{\to} f$, we have that there exists a subsequence $\{f_{k_{n_j}}\}$ of $\{f_{k_n}\}$ that converges almost everywhere to f. Thus, for almost every $x \in E$, we have that

$$f_{k_{n_j}}(x) \to f(x).$$

Thus, by Lemma 1, we can see that for almost every $x \in E$,

$$f_{k_{n_j}}(x)^2 \to f(x)^2.$$

Thus, $\{f_{k_{n_j}}^2\}$ converges almost every where to f^2. Utilizing the other direction of what we proved in problem 7, we have shown that $f_k^2 \overset{\text{m}}{\to} f^2$ on E, and our proof is complete. $\square$