Problem 1

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous. Then $|f|$ is absolutely continuous on $[a, b]$, and*

$$\left|\frac{d}{dx}|f(x)|\right| \leq |f'(x)|$$

for almost every $x \in [a, b]$.

Solution

Proof. Let $\epsilon > 0$. Since f is absolutely continuous, there exists a $\delta > 0$ such for any collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$, we have

$$\sum |f(b_i) - f(a_i)| < \epsilon \text{ if } \sum (b_i - a_i) < \delta.$$

Thus, let $\{[a_i, b_i]\}$ be a collection of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$. Then, we have

$$\begin{aligned} \sum ||f(b_i)| - |f(a_i)|| &\leq \sum |f(b_i) - f(a_i)| && \text{The reverse triangle inequality} \\ &< \epsilon. \end{aligned}$$

Thus, $|f|$ is absolutely continuous.

By theorem 7.29 in our textbook, the derivatives of f and $|f|$ exists almost everywhere in $[a, b]$. Let $Z, Z' \subseteq [a, b]$ be the sets of measure zero where the derivatives of f and $|f|$ respectively do not exist. Then, $Z \cup Z'$ has measure zero, and the derivatives of $|f|$ and f exists on $[a, b] \backslash Z \cup Z'$. If we let $x \in [a, b] \backslash Z \cup Z'$, then we have

$$\begin{aligned} \left|\frac{d}{dx}|f(x)|\right| &= \left|\lim_{n \to 0} \frac{|f(x + n)| - |f(x)|}{n}\right| \\ &= \lim_{n \to 0} \frac{||f(x + n)| - |f(x)||}{|n|} \\ &\leq \lim_{n \to 0} \frac{|f(x + n)| - |f(x)|}{|n|} && \text{Reverse triangle inequality} \\ &= \left|\lim_{n \to 0} \frac{f(x + n) - f(x)}{n}\right| \\ &= |f'(x)|, \end{aligned}$$

as desired. □

Problem 2

Use Fubini's theorem and the relation

$$\frac{1}{x} = \int_0^\infty e^{-xy} dy$$

to prove that

$$\lim_{A \to \infty} \int_0^A \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Solution

Proof. Consider the function

$$f(x, y) = \sin(x) e^{-xy}.$$

We have that f is continuous, which implies that f is initegrable on $(0, A) \times (0, \infty)$. Thus, by Fubini's theorem, we have

$$\int_0^A \frac{\sin x}{x} dx = \int_0^\infty \left(\int_0^A \sin(x) e^{-xy} dx\right) dy.$$

By performing integration by parts twice, or by recognizing that

$$\sin(x)e^{-xy} = -\frac{1}{1+y^2}\frac{d}{dx}(y\sin(x)e^{-xy} + \cos(x)e^{-xy}),$$

we have

$$\begin{aligned}\int_0^A \sin(x)e^{-xy}dx &= \int_0^A \left(-\frac{1}{1+y^2}\frac{d}{dx}(y\sin(x)e^{-xy} + \cos(x)e^{-xy})\right)dx \\ &= \frac{1}{1+y^2}[1 - e^{-Ay}(y\sin A + \cos A)].\end{aligned}$$

Plugging this into our original integral, we have

$$\begin{aligned}\int_0^\infty \left(\int_0^A \sin(x)e^{-xy}dx\right)dy &= \int_0^\infty \frac{1}{1+y^2}[1 - e^{-Ay}(y\sin A + \cos A)]dy \\ &= \int_0^\infty \frac{1}{1+y^2}dy - \int_0^\infty \frac{e^{-Ay}(y\sin A + \cos A)}{1+y^2}dy \\ &= \tan^{-1}(y)\big|_0^\infty - \int_0^\infty \frac{e^{-Ay}(y\sin A + \cos A)}{1+y^2}dy \\ &= \frac{\pi}{2} - \int_0^\infty \frac{e^{-Ay}(y\sin A + \cos A)}{1+y^2}dy.\end{aligned}$$

Thus, we have

$$\lim_{A\to\infty}\int_0^A \frac{\sin x}{x}dx = \frac{\pi}{2} - \lim_{A\to\infty}\int_0^\infty \frac{e^{-Ay}(y\sin A + \cos A)}{1+y^2}dy.$$

We can see that

$$\left|\frac{e^{-Ay}(y\sin A + \cos A)}{1+y^2}\right| \leq e^{-Ay}(y+1).$$

Thus, since

$$\int_0^\infty e^{-Ay}(y+1)dy = \frac{1}{A} + \frac{1}{A^2},$$

we have that $e^{-Ay}(y+1)$ is integrable. By the Lebesgue dominated convergence theorem,

$$\begin{aligned}\lim_{A\to\infty}\int_0^\infty \frac{e^{-Ay}(y\sin A + \cos A)}{1+y^2}dy &= \int_0^\infty \lim_{A\to\infty}\frac{e^{-Ay}(y\sin A + \cos A)}{1+y^2}dy \\ &= \int_0^\infty (0)dy \\ &= 0.\end{aligned}$$

With this, our proof is complete. □

Problem 3

Theorem 2. *Let $E = [1,\infty)$ and $f \in L^2(E)$. Assume that $f \geq 0$ almost every where on E. Let*

$$g(x) = \int_E f(y)e^{-xy}dy$$

for all $x \in E$. Then $g \in L^1(E)$ and

$$||g||_1 \leq c||f||_2$$

for some $c < 1$. (Also Estimate c).

Solution

Proof. □