MA 653-01 Homework 3

1. We know that f is measurable, then for every $a \in \mathbb{R}^1$, the set $f^{-1}(\{a\})$ is measurable. Use the following function f to show that the converse of this result is not true, where $f : \mathbb{R}^1 \to \mathbb{R}^1$ defined by

$$f(x) = \begin{cases} e^x & \text{if } x \in E, \\ -e^x & \text{if } x \in E^c, \end{cases}$$

where $E \subset \mathbb{R}^1$ is a nonmeasurable set.

2. Let $E \subset \mathbb{R}^1$ be measurable. Show that if $f : E \to [-\infty, \infty]$ is increasing, then f is measurable.

3. If f is differentiable on $[a, b]$, then f' is measurable. (Note: f' might not be continuous.)

   Hint: $f'(t) = \lim_{k\to\infty} k(f(t + \frac{1}{k}) - f(t))$ for $t \in [a, b)$.

4. Prove that the sum and product of two simple functions on E are still simple functions on E.

5. Assume that $|E| < \infty$ and f be a measurable function on E which is finite a.e. in E. Show that for every $\varepsilon > 0$, there exists a closed set $F \subset E$ such that $|E - F| < \varepsilon$ and that f is bounded on F.

6. Assume that $|E| < \infty$ and f is measurable on E. Show that there are at most countably many real number y such that $|f^{-1}(\{y\})| > 0$.

7. Problem 4 on Page 76 of the textbook.

8. Let $f(x, t)$ be a function on $E \times \mathbb{R}$ where E is a measurable set in $\mathbb{R}^n$. Assume that (i) for almost every $x \in E$, $f(x, t)$ is continuous as a function of $t \in \mathbb{R}$; (ii) for every fixed $t \in \mathbb{R}^1$, $f(x, t)$ is measurable as a function of $x \in E$. Such conditions (i) and (ii) are called carathéodory's conditions.

   Let $g : E \to \mathbb{R}$ be a measurable function. Show that $F(x) = f(x, g(x))$ is measurable on E.

   Hint: First show the case that g is a simple function and then approximate g by a sequence of simple functions.

9. Let $f : [0, \infty) \to \mathbb{R}$ be continuous. For each $k \in \mathbb{N}$, divide the interval $[0, k)$ into k^2 disjoint subintervals

$$[\frac{j-1}{k}, \frac{j}{k}), \qquad j = 1, 2, \cdots, k^2,$$

and define the step function

$$f_k(x) = \begin{cases} f(\frac{j-1}{k}), & x \in [\frac{j-1}{k}, \frac{j}{k}), \quad j = 1, 2, \cdots, k^2, \\ f(k), & x \geq k. \end{cases}$$

Show that $f_k(x) \to f(x)$ for every $x \in [0, \infty)$