MA 653-01 Homework 3

1. We know that f is measurable, then for every $a \in \mathbb{R}^1$, the set $f^{-1}(\{a\})$ is measurable. Use the following function f to show that the converse of this result is not true, where $f : \mathbb{R}^1 \to \mathbb{R}^1$ defined by

$$f(x) = \begin{cases} e^x & \text{if } x \in E, \\ -e^x & \text{if } x \in E^c, \end{cases}$$

   where $E \subset \mathbb{R}^1$ is a nonmeasurable set.

2. Let $E \subset \mathbb{R}^1$ be measurable. Show that if $f : E \to [-\infty, \infty]$ is increasing, then f is measurable.

3. If f is differentiable on $[a, b]$, then f' is measurable. (Note: f' might not be continuous.)

4. Prove that the sum and product of two simple functions on E are still simple functions on E.

5. Assume that $|E| < \infty$ and f be a measurable function on E which is finite a.e. in E. Show that for every $\varepsilon > 0$, there exists a closed set $F \subset E$ such that $|E - F| < \varepsilon$ and that f is bounded on F.

6. Assume that $|E| < \infty$ and f is measurable on E. Show that there are at most countably many real number y such that $|f^{-1}(\{y\})| > 0$.

7. Problem 4 on Page 76 of the textbook.

8. Let $f(x, t)$ be a function on $[a, b] \times \mathbb{R}^1$. Assume that (i) for each fixed $x \in [a, b]$, $f(x, t)$ is continuous as a function of $t \in \mathbb{R}^1$; (ii) for each fixed $t \in \mathbb{R}^1$, $f(x, t)$ is measurable as a function of $x \in [a, b]$. Let $g(x)$ be a measurable function on $[a, b]$. Show that $F(x) = f(x, g(x))$ is measurable on $[a, b]$.

   Hint: Approximate f by a sequence of simple functions.