Problem 1

Before we jump into this problem, we will first prove an easy lemma.

Lemma 1. *Let F, B and E be sets with $F \subseteq B \subseteq E$. Then,*

$$(E - F) \cap B = B - F.$$

Proof. We have

$$\begin{aligned} x \in (E-F) \cap B &\iff (x \in E - F) \wedge x \in B \\ &\iff (x \in E) \wedge (x \notin F) \wedge (x \in B) \\ &\iff (x \in E \cap B) \wedge (x \notin F) \\ &\iff (x \in B) \wedge (x \notin F) && \text{Since } B \subseteq E \\ &\iff x \in B - F, \end{aligned}$$

and our proof is complete. □

Theorem 1. *$E \subseteq \mathbb{R}^n$ is measurable if and only if for every $\epsilon > 0$ there exists a measurable set $B \subseteq E$ such that $|E - B|_e < \epsilon$.*

Solution

Proof. Suppose E is measurable, and let $\epsilon > 0$. By Lemma 3.22 in our textbook, there exists a closed set $B \subseteq E$ such that $|E - B|_e < \epsilon$. Since all closed sets are measurable, B is measurable, and we have proven one direction of the theorem.

Now suppose that for each $\epsilon > 0$, there exists a measurable set $B \subseteq E$ such that $|E - B|_e < \epsilon$. Thus, let $\epsilon > 0$, and fix a measurable $B \subseteq E$ such that $|E - B|_e < \frac{\epsilon}{2}$. Since B is measurable, Lemma 3.22 tells us that there exists a closed set $F \subseteq B$ such that $|B - F|_e < \frac{\epsilon}{2}$. Now, by Caratheory's theorem, we have

$$\begin{aligned} |E - F|_e &= |(E-F) \cap B|_e + |(E-F) - B|_e \\ &= |(E-F) \cap B|_e + |E - B|_e && \text{Since } F \subseteq B \\ &= |B - F|_e + |E - B|_e && \text{By Lemma 1} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus, utilizing Lemma 3.22 one last time, it follows that E is measurable, and our proof is complete. □

Problem 2

Theorem 2. *Let $\{I_1, I_2, ..., I_N\}$ be a finite family of closed intervals in $\mathbb{R}^1$ such that $\mathbb{Q} \cap [0,1] \subseteq \bigcup_{j=1}^N I_j$. Then $\sum_{j=1}^N |I_j| \geq 1$. Furthermore, if this family of intervals is infinite, this may not be true.*

Solution

Proof. Suppose there exists an $x \in [0,1] \backslash \mathbb{Q}$ such that $x \notin \bigcup_{j=1}^N I_j$. Then, since each I_j is closed, there exist $\delta_j > 0$ such that

$$(\forall y \in I_j)(|x - y| \geq \delta_j).$$

Define $\delta = \min\{\delta_j | j = 1, 2, ..., N\}$, which exists because there are only finitely many I_j. By construction, we have that $\delta > 0$. By density of the rational numbers, there exists a rational number $q \in [0,1]$ such that $|x - q| < \delta$. Thus, $q \notin \bigcup_{j=1}^N I_j$, and we have a contradiction.

Therefore, we have that $[0,1] \subseteq \bigcup_{j=1}^N I_j$. Now, by definition of Lebesgue measure, we have

$$\begin{aligned} 1 &= |[0,1]| \\ &\leq \sum_{j=1}^N |I_j|, \end{aligned}$$

and our proof is complete.

For the last remark, let $\{q_1, q_2, ...\}$ be an enumeration of $\mathbb{Q} \cap [0,1]$. Define an infinite family of closed intervals $\{I_1, I_2, ...\}$ such that

$$I_k = [q_k, q_k + \frac{\epsilon}{2}]$$

for some $0 < \epsilon < 1$. Clearly, we have $\mathbb{Q} \cap [0,1] \subseteq \bigcup_{j=1}^{\infty} I_j$, but we also have

$$\begin{aligned}\sum_{j=1}^{\infty} |I_j| &= \sum_{j=1}^{\infty} |[q_k, q_k + \frac{\epsilon}{2^k}]| \\ &= \sum_{j=1}^{\infty} \frac{\epsilon}{2^k} \\ &= \epsilon \\ &< 1,\end{aligned}$$

and we have shown that the statement does not hold for an infinite family of closed intervals. □

Problem 3

Theorem 3. Part (i):
Let $A, B \subseteq \mathbb{R}^n$, with A measurable. If $A \cap B = \emptyset$, then $|A \cup B|_e = |A| + |B|_e$

Part (ii):
If $F \subseteq E$ is closed such that $|E|_e - |F| < \epsilon$ and $|F| < \infty$, then $|E - F|_e < \epsilon$.

Solution

Proof. Part (i):
Since A is measurable, we have

$$\begin{aligned}|A \cup B|_e &= |(A \cup B) \cap A|_e + |(A \cup B) - A|_e && \text{By Caratheory's theorem} \\ &= |A| + |(A \cup B) - A|_e && \text{Since } A \cap B = \emptyset \\ &= |A| + |B|_e && \text{Since } (A \cup B) - A = B\end{aligned}$$

and we have completed the proof of this part.

Part (ii):
Since F is closed (and therefore measurable), we can use Caratheodory's theorem once more, to see

$$\begin{aligned}|E|_e &= |E \cap F|_e + |E - F|_e \\ &= |F| + |E - F|_e. && \text{Since } F \subseteq E\end{aligned}$$

Since $|F|$ is finite, we can subtract it from both sides to yield

$$\begin{aligned}|E - F|_e &= |E|_e - |F| \\ &< \epsilon,\end{aligned}$$

as desired. □

Problem 4

Theorem 4. *Let A and B be subsets of $\mathbb{R}^n$. Then*

$$|A \cup B|_e + |A \cap B|_e \leq |A|_e + |B|_e.$$

Solution

Proof. Let $\epsilon > 0$. By theorem 3.6 in our textbooks, there exist open sets G_A, G_B such that $A \subseteq G_A$, $B \subseteq G_B$, $|G_A| \leq |A|_e + \frac{\epsilon}{2}$, and $|G_B| \leq |B|_e + \frac{\epsilon}{2}$. With this, we have

$$\begin{aligned} |A \cup B|_e + |A \cap B|_e &\leq |G_A \cup G_B| + |G_A \cap G_B| && \text{Since } A \cup B \subseteq G_A \cup G_B \text{ and } A \cap B \subseteq G_A \cap G_B \\ &= |G_A| + |G_B| && \text{From HW2 Problem 4} \\ &\leq |A|_e + \frac{\epsilon}{2} + |B|_e + \frac{\epsilon}{2} \\ &= |A|_e + |B|_e + \epsilon. \end{aligned}$$

Since this is true for any $\epsilon > 0$, we have

$$|A \cup B|_e + |A \cap B|_e \leq |A|_e + |B|_e,$$

and our proof is complete. ☐

Problem 5

Theorem 5. *If $E \subseteq \mathbb{R}$ is measurable and $|E| > 0$, then there are $x, y \in E$, with $x \neq y$, such that $x - y$ is a rational number.*

Solution

Proof. Since E has nonzero measure, there exists a bounded subset $F \subseteq E$ such that $|F| > 0$. Since F is bounded, there exists integers $a < b$ such that $F \subseteq [a, b]$. Thus, it will suffice to show that there are $x, y \in F$ such that $x - y$ is a rational number.

Let $\{q_1, q_2, ...\}$ be an enumeration of the rational numbers contained in $[0, 1]$. Define a family of sets $\{F_k\}$ by

$$F_k = F + q_1,$$

where we have just translated F some rational number in $[0, 1]$. Now, suppose that $F_j \cap F_k = \emptyset$ if $j \neq k$. Using the fact that $\bigcup_{k=1}^{\infty} F_k \subseteq [a, b + 1]$, we have

$$\begin{aligned} |[a, b + 1]| &\geq |\bigcup_{k=1}^{\infty} F_k| \\ &= \sum_{k=1}^{\infty} |F_k| && \text{Since the sets are disjoint, by assumption} \\ &= \sum_{k=1}^{\infty} |F| && \text{By translation invariance of Lebesgue measure} \\ &= \infty, \end{aligned}$$

which is a contradiction, since $|[a, b + 1]|$ is finite. Thus, there exist numbers $j, k \in \mathbb{N}$, with $j \neq k$, such that $F_j \cap F_k \neq \emptyset$. Thus, there exist $f_j, f_k \in F$ such that

$$f_j + q_j = f_k + q_k.$$

Then, we have

$$f_j - f_k = q_k - q_j.$$

Since the rational numbers are closed under subtraction, we have that $f_j - f_k$ is rational, and our proof is complete. ☐

Problem 6

Theorem 6. *Let E be the nonmeasurable set in $I = [0, 1]$ constructed in the lecture. Then*

Part (i):
$|E|_i = 0$.

Part (ii):
$|I| < |E|_e + |I - E|_e$.

Part (iii):
$|I| > |E|_i + |I - E|_i$.

Solution

Proof. Part (i):
Let $F \subseteq E$ be closed. Then, we have that F is measurable, and that $|F| \leq 1$. Let $\{1_1, q_2, ...\}$ be an enumeration of the rational numbers in $[0, 1]$. Define the translated sets $\{F_k\}$ by

$$F_k = F + q_k.$$

Suppose that there exist $j, k \in \mathbb{N}$ such that $F_j \cap F_k \neq \emptyset$. Then, there exists some $x, y \in F$ such that

$$x + q_j = y + q_k.$$

This implies that $x - y = q_k - q_j$, which is a rational number. From this, we have that y is in the equivalence class of x. However, since each element of E (and therefore F) belong to distinct equivalent classes, we have that $x = y$, which implies that $j = k$. Thus, we can conclude that the sets in $\{F_k\}$ are pairwise disjoint. Furthermore, by translation invariance of the Lebesgue measure, each of these sets is measurable, with measure $|F|$. Since the union of these sets is contained in $[0, 2]$, we have

$$\begin{aligned}
2 &= |[0,2]| \\
&\geq \left| \bigcup_{k=1}^{\infty} F_k \right| \\
&= \sum_{k=1}^{\infty} |F_k| && \text{Since these sets are disjoint} \\
&= \sum_{k=1}^{\infty} |F| \\
&= \infty \cdot |F|.
\end{aligned}$$

Thus, we can conclude that $|F|$ has measure zero. Since this is true for any closed subset of E, we have shown that

$$|E|_i = 0.$$

Part (ii):
As we proved on Problem 7 of Homework 2, we have that

$$|I| = |E|_i + |I - E|_e.$$

By part (i), we have that $|I| = |I - E|_e$. Furthermore, we have that $|E|_e > 0$, since sets of zero outer measure are measurable, and we have already proven that E is not measurable. Thus, since $|I|$ is finite, it follows that

$$|I| < |E|_e + |I - E|_e.$$

Part (iii):
As we showed above, $|I| = |I - E|_e$. As we also showed on Homework 2, $|I - E|_e \geq |I - E|_i$. Suppose, for sake of contradiction, that $|I - E|_e = |I - E|_i$. Then, since $|I - E|_e$ is finite, this would imply that $I - E$ is measurable (which we also proved on Homework 2). However, we have

$$I - (I - E) = E,$$

and since the set difference of two measurable sets is measurable, we have reached a contradiction. Thus, $|I - E|_e > |I - E|_i$, and we have reached the desired conclusion. □

Problem 7

Theorem 7. *Let $f(x) = x^3$. If $E \subseteq \mathbb{R}$ is a zero measure set, then $|f(E)| = 0$.*

Solution

Proof. Suppose first that E is bounded. Then, as I proved on homework 1, f is uniformly continuous. Thus, there exists some $M > 0$ such that for all $x, y \in E$, we have

$$|f(x) - f(y)| \leq M|x - y|.$$

Furthermore, it follows that for any interval I, we have

$$|f(I)| \leq M|I|.$$

Now let $\epsilon > 0$. Since E is of measure zero, there exists a cover of $\{I_1, I_2, ...\}$ of E consisting of closed intervals such that

$$\sum_{j=1}^{\infty} |I_j| < \frac{\epsilon}{M}.$$

We have

$$f(E) = \bigcup_{j=1}^{\infty} f(I_j),$$

therefore

$$\begin{aligned}
|f(E)| &\leq \sum_{j=1}^{\infty} |f(I_j)| \\
&\leq \sum_{j=1}^{\infty} M|I_j| \\
&< M\frac{\epsilon}{M} \\
&= \epsilon.
\end{aligned}$$

Since this is true for any ϵ, we can conclude that $|f(E)| = 0$.

Now, suppose that E is not bounded. We have that

$$E = \bigcup_{k=1} E \cap B(0; k),$$

where $B(0; k)$ is the open ball centered at the origin with radius k. Furthermore, we have

$$f(E) = \bigcup_{k=1} f(E \cap B(0; k)).$$

Now, since $E \cap B(0; k)$ is bounded for each k and has measure zero, the previous result allows us to conclude that $|f(E \cap B(0; k))| = 0$, and therefore that $|f(E)| = 0$, as desired. □

Problem 8

Theorem 8. *If f and g are continuous functions on $\mathbb{R}^n$ and are equal a.e. in $\mathbb{R}^n$, then $f \equiv g$ on $\mathbb{R}^n$.*

Solution

Proof. Suppose that g is continuous, and that $f \not\equiv g$. Then, there exists some point x_0 such that $f(x_0) \neq g(x_0)$. Since f and g are equal almost everywhere, we can construct a sequence $\{x_k\}$ such that $f(x_k) = g(x_k)$ and $|x_0 - x_k| < \frac{1}{k}$ for all $k \in \mathbb{N}$. By construction, we have that $x_k \to x_0$. By continuity of g, we have that $g(x_k) \to g(x_0)$. Furthermore, since $f(x_k) = g(x_k)$ for all k, we have that $f(x_k) \to g(x_0)$, implying that $f(x_k) \not\to f(x_0)$. Thus, we have shown that f is not continuous, and we have proven the contrapositive. □

Problem 9

Theorem 9. *Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set E with $|E| < \infty$. If $|f_k(x)| \leq M_x < \infty$ for all k for each $x \in E$, then given $\epsilon > 0$, there is a closed $f \subseteq E$ and a finite $M > 0$ such that $|E - F| < \epsilon$ and $|f_k(x)| \leq M$ for all k and all $x \in F$.*

Solution

Proof. By the remark after theorem 4.6 in our textbook, we have that $|f_k|$ is measurable for each k. By theorem 4.11 in our text book, $\sup_k |f_k(x)|$ is measurable. Furthermore, we have by assumption that $\sup_k |f_k(x)|$ is finite everywhere. With this, we can construct a sequence of measurable sets $\{E_k\}$ such that $E_k \nearrow E$ defined by

$$E_k = \{x \in E | \sup_k |f_k(x)| \leq k\}.$$

By theorem 3.26 in the book, we have

$$\lim_{k\to\infty} |E_k| = |E|.$$

Thus, there exists an $M > 0$ such that $|E| - |E_M| < \frac{\epsilon}{2}$. Using Lemma 3.22, we can find a closed $F \subseteq E_M$ such that $|E_M - F| < \frac{\epsilon}{2}$. Using Caratheory's theorem, we have

$$\begin{aligned}
|E_n| &= |E_n \cap F| + |E_n - F| && \\
&= |F| + |E_n - F| && \text{Since } F \subseteq E_n \\
|F| &= |E_n| - |E_n - F| && \text{Since } |E| < \infty \implies |E_n| < \infty \implies |F| < \infty,
\end{aligned}$$

and, using the exact same approach,

$$\begin{aligned}
|E - F| &= |E| - |E \cap F| \\
&= |E| - |F| \\
&= |E| - (|E_n| - |E_n - F|) \\
&= (|E| - |E_n|) + |E_n - F| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Finally, since $F \subseteq E_M$, we have that

$$(\forall x \in F)(\sup_k |f_k(x)| \leq M) \implies (\forall x \in F)(\forall k)(|f_k(x)| \leq M),$$

and our proof is complete. ☐

Problem 10

Theorem 10. *Let f be a measurable function and finite a.e. on $E \subseteq \mathbb{R}^n$. Show that there is a sequence of continuous functions g_k on $\mathbb{R}^n$ such that $g_k \to f$ a.e. on E.*

Solution

Proof. By the alternative version of Lusin's theorem, we can construct a sequence of closed sets $\{F_k^*\}$ such that $F_k^* \subseteq E$ and $|E - F_k^*| < \frac{1}{k}$, and a sequence of continuous functions $\{g_k^*\}$ such that $g_k^* : \mathbb{R}^n \to \mathbb{R}$ and $g_k^*(x) = f(x)$ for all $x \in F_k$.

Now, this sequence of continuous functions does not necessarily converge, as $\{F_k^*\}$ is not necessarily an increasing sequence. One way to remedy this, is to define a new sequence, $\{F_k\}$ such that $F_1 = F_1^*$, and $F_k = F_k^* \cup F_{k-1}$ for $k > 1$. By design, we have that $F_k \nearrow E$, and each F_k is closed. Using this, we will define a new sequence of functions $\{g_k\}$, where $g_1 = g_1^*$, and $g_{k+1}(x) = g_{k+1}^*(x)$ when $x \in F_{k+1}^*$, and $g_{k+1}(x) = g_k(x)$ when $x \notin F_{k+1}^*$. All that remains is to show that each g_k is continuous, and our proof will be complete.

Since g_1^* is continuous, it follows that g_1 is continuous. Now suppose that for some $k \geq 1$ that g_k is continuous. Let $\{x_j\}$ be any convergent sequence in $\mathbb{R}^n$, and call it's limit x. Now, either $x \in F_{k+1}^*$, or $x \notin F_{k+1}^*$. Suppose first that $x \in F^*_{k+1}$. Since F^*_{k+1} is closed, there exists some $J \in \mathbb{N}$ such that $x_j \in F^*_{k+1}$ for all $j \geq J$. Thus, $g_{k+1}(x_j) = g^*_{k+1}(x_j)$ for all $j \geq J$. Then, by continuity of g^*_{k+1}, $g_{k+1}(x_j) \to g_{k+1}(x)$, and we have that g_{k+1} is continuous in F^*_{k+1}.

Now, suppose that $x \notin F^*_{k+1}$. Then, since F^*_{k+1} is closed, there exists some $J \in \mathbb{N}$ such that $x_j \notin F^*_{k+1}$ for all $j \geq J$. Thus, $g_{k+1}(x_j) = g_k(x_j)$ for all $j \geq J$. By continuity of g_k, we have that $g_{k+1}(x_j) \to g_{k+1}(x)$, and we have shown that g_{k+1} is continuous everywhere. Since we have already shown the base case of $k = 1$, the principle of mathematical induction allows us to conclude that g_k is continuous for all k, and our proof is complete. □

Problem 11

Theorem 11. *Let f be a measurable function and finite a.e. on $[a, b]$. Then, there is a sequence of polynomials p_k such that $p_k \to f$ a.e. on $[a, b]$.*

Solution

Proof. As we proved in the previous problem, there exists a sequence of continuous functions g_k on $\mathbb{R}^n$ such that $g_k \to f$ a.e. on $[a, b]$. By the Stone-Weierstrass theorem, we can construct a sequence of polynomials p_k such that for all $x \in [a, b]$, $|p_k(x) - g_k(x)| < \frac{1}{k}$. Now, let $x \in [a, b]$ be such that $g_k(x) \to f(x)$, and let $\epsilon > 0$. Choose $K_1 \in \mathbb{N}$ to be such that $\frac{1}{K_1} < \frac{\epsilon}{2}$. Additionally, choose $K_2 \in \mathbb{N}$ large enough that for all $k \geq K_2$, we have $|g_k(x) - f(x)| < \frac{\epsilon}{2}$. Then, if we define K as the maximum of K_1 and K_2, we have that for all $k \geq K$,

$$\begin{aligned}
|p_k(x) - f(x)| &\leq |p_k(x) + g_k(x)| + |g_k(x) - f(x)| && \text{By the triangle inequality} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Thus, since ϵ was arbitrary, we have shown that $p_k \to f$ a.e. on $[a, b]$, and our proof is complete. □