Homework 4

Thursday, March 23, 2023 10:33 PM

1. Let $f : (a, b) \to \mathbb{R}$ such that $\forall\, x_0 \in (a, b)$ there is a support line

$$l_{x_0}(x) = f(x_0) + m(x - x_0)$$

for some $m \in \mathbb{R}$ such that

$$f(x) \geq l_{x_0}(x) = f(x_0) + m(x - x_0), \quad \forall\, x \in (a, b).$$

Show that f is convex on (a, b).

2. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous and $f'(x)$ is increasing exept on a zero measure set of $[a, b]$. Show that f is convex on $[a, b]$.

(Hint: use, $\forall\, x, y \in [a, b]$,
$$f(y) = f(x) + \int_0^1 f'((1-\theta)x + \theta y)\, d\theta\, (y - x)$$
)

3. Let $f : [a, b] \to \mathbb{R}$ and let $E \subseteq [a, b]$. Assume that $f'(x)$ exists with a finite value for any $x \in E$. Show that

$$|f(E)| \leq \int |f'(x)|\, dx.$$

$$|f(E)|_e \leq \int_E |f'(x)|\,dx .$$

(Hint: Use the following result:

If $f'(x)$ exists for all $x \in A \subseteq [a,b]$ and $|f'(x)| \leq p$, $\forall x \in A$, then

$$|f(A)|_e \leq p|A|_e$$)

4. Let $f : [a,b] \to \mathbb{R}$ be continuous. Assume that $f'(x)$ exists (as a finite number) except on a countable subset of $[a,b]$ and assume $f' \in L([a,b])$. Show that f is absolutely continuous on $[a,b]$.

(Hint: Use the result of problem 3 to show that f is a N-function.)