Homework 2 (due 9/18)

1. If $E \subseteq [0,1]$ has measure 1, then E is dense in $[0,1]$.
2. Let F be a proper closed subset in $[0,1]$. Show that $|F| < 1$.
3. Let $E_k \subseteq [0,1]$ be measurable with $\sum_{k=1}^N |E_k| > N-1$. Prove that $|\cap_{k=1}^N E_k| > 0$.

   (HINT: show that $|[0,1] \setminus \cap_{k=1}^N E_k| < 1$.)
4. Problem 10 on P. 59 in the textbook.
5. Suppose that the measurable sets $A_1, A_2, \cdots$ are "almost disjoint" in the sense that $|A_i \cap A_j| = 0$ if $i \neq j$.

   (i) Use the mathematical induction to show that for any $N > 1$,

   $$|\cup_{k=1}^N A_k| = \sum_{k=1}^N |A_k|.$$

   (ii) Show that

   $$|\cup_{k=1}^\infty A_k| = \sum_{k=1}^\infty |A_k|.$$
6. **(Bonus)** Problem 13 on P. 59 in the textbook.
7. **(Bonus)** Problem 15 on P. 59 in the textbook.
8. Problem 30 on P. 60 in the textbook.
9. Problem 32 on P. 60 in the textbook.
10. Problem 34 on P. 60 in the textbook.

1. Let I_1 and I_2 be two intervals in $\mathbb{R}^2$. Show the following elementary fact that $I_1 \cup I_2$ is a union of finitely many non-overlapping intervals J_k for $k = 1, \cdots, N$, with
$$\sum_{k=1}^{N} v(J_k) \leq v(I_1) + v(I_2).$$
Hint. All intervals mean closed intervals. You may prove by cases with the help of sketching figures. Note that this fact is used in the proof of Theorem 3.2.

2. (i) Show that if $f : [a, b] \to \mathbb{R}$ is continuous where $-\infty < a < b < \infty$, then its graph $E = \{(x, f(x)) : x \in [a, b]\}$ as a subset of $\mathbb{R}^2$ has the Lebesgue measure zero.

   (ii) Show that that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then its graph $E = \{(x, f(x)) : x \in \mathbb{R}\}$ as a subset of $\mathbb{R}^2$ has the Lebesgue measure zero.

3. Let $A = \{r_1, r_2, \cdots\}$ be the set of rational numbers in $(0, 1)$. Given $\varepsilon \in (0, 1/2)$, let
$$A_k = (r_k - \frac{\varepsilon}{2^k}, r_k + \frac{\varepsilon}{2^k}) \cap (0, 1), \qquad k = 1, 2, \cdots,$$
and
$$E = \cup_{k=1}^{\infty} A_k.$$
Show that $0 < |E| \leq 2\varepsilon$ and $|\partial E| = 1$.

   (Note: This gives an example that the Lebesgue measure of the boundary set of an open set may not be zero.)

4. (i) Show that if $E \subset \mathbb{R}$ is such that $|E| = 0$, then $\mathring{E} = \emptyset$.

   (ii) Show that if $E \in [0, 1]$ is Lebesgue measurable with $|E| = 1$, then E is dense in $[0, 1]$.