

## Problem 1

**Theorem 1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be absolutely continuous. Then  $|f|$  is absolutely continuous on  $[a, b]$ , and

$$\left| \frac{d}{dx} |f(x)| \right| \leq |f'(x)|$$

for almost every  $x \in [a, b]$ .

### Solution

*Proof.* Let  $\epsilon > 0$ . Since  $f$  is absolutely continuous, there exists a  $\delta > 0$  such for any collection  $\{[a_i, b_i]\}$  of nonoverlapping subintervals of  $[a, b]$ , we have

$$\sum |f(b_i) - f(a_i)| < \epsilon \text{ if } \sum (b_i - a_i) < \delta.$$

Thus, let  $\{[a_i, b_i]\}$  be a collection of nonoverlapping subintervals of  $[a, b]$  with  $\sum (b_i - a_i) < \delta$ . Then, we have

$$\begin{aligned} \sum ||f(b_i)| - |f(a_i)|| &\leq \sum |f(b_i) - f(a_i)| && \text{The reverse triangle inequality} \\ &< \epsilon. \end{aligned}$$

Thus,  $|f|$  is absolutely continuous.

By theorem 7.29 in our textbook, the derivatives of  $f$  and  $|f|$  exists almost everywhere in  $[a, b]$ . Let  $Z, Z' \subseteq [a, b]$  be the sets of measure zero where the derivatives of  $f$  and  $|f|$  respectively do not exist. Then,  $Z \cup Z'$  has measure zero, and the derivatives of  $|f|$  and  $f$  exists on  $[a, b] \setminus Z \cup Z'$ . If we let  $x \in [a, b] \setminus Z \cup Z'$ , then we have

$$\begin{aligned} \left| \frac{d}{dx} |f(x)| \right| &= \left| \lim_{n \rightarrow 0} \frac{|f(x+n)| - |f(x)|}{n} \right| \\ &= \lim_{n \rightarrow 0} \frac{||f(x+n)| - |f(x)||}{|n|} \\ &\leq \lim_{n \rightarrow 0} \frac{|f(x+n) - f(x)|}{|n|} && \text{Reverse triangle inequality} \\ &= \left| \lim_{n \rightarrow 0} \frac{f(x+n) - f(x)}{n} \right| \\ &= |f'(x)|, \end{aligned}$$

as desired. □

## Problem 2

Use Fubini's theorem and the relation

$$\frac{1}{x} = \int_0^\infty e^{-xy} dy$$

to prove that

$$\lim_{A \rightarrow \infty} \int_0^A \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

### Solution

*Proof.* Consider the function

$$f(x, y) = \sin(x)e^{-xy}.$$

We have that  $f$  is continuous, which implies that  $f$  is integrable on  $(0, A) \times (0, \infty)$ . Thus, by Fubini's theorem, we have

$$\int_0^A \frac{\sin x}{x} dx = \int_0^\infty \left( \int_0^A \sin(x)e^{-xy} dx \right) dy.$$

By performing integration by parts twice, or by recognizing that

$$\sin(x)e^{-xy} = -\frac{1}{1+y^2} \frac{d}{dx}(y \sin(x)e^{-xy} + \cos(x)e^{-xy}),$$

we have

$$\begin{aligned} \int_0^A \sin(x)e^{-xy} dx &= \int_0^A \left( -\frac{1}{1+y^2} \frac{d}{dx}(y \sin(x)e^{-xy} + \cos(x)e^{-xy}) \right) dx \\ &= \frac{1}{1+y^2} [1 - e^{-Ay}(y \sin A + \cos A)]. \end{aligned}$$

Plugging this into our original integral, we have

$$\begin{aligned} \int_0^\infty \left( \int_0^A \sin(x)e^{-xy} dx \right) dy &= \int_0^\infty \frac{1}{1+y^2} [1 - e^{-Ay}(y \sin A + \cos A)] dy \\ &= \int_0^\infty \frac{1}{1+y^2} dy - \int_0^\infty \frac{e^{-Ay}(y \sin A + \cos A)}{1+y^2} dy \\ &= \tan^{-1}(y) \Big|_0^\infty - \int_0^\infty \frac{e^{-Ay}(y \sin A + \cos A)}{1+y^2} dy \\ &= \frac{\pi}{2} - \int_0^\infty \frac{e^{-Ay}(y \sin A + \cos A)}{1+y^2} dy. \end{aligned}$$

Thus, we have

$$\lim_{A \rightarrow \infty} \int_0^A \frac{\sin x}{x} dx = \frac{\pi}{2} - \lim_{A \rightarrow \infty} \int_0^\infty \frac{e^{-Ay}(y \sin A + \cos A)}{1+y^2} dy.$$

We can see that

$$\left| \frac{e^{-Ay}(y \sin A + \cos A)}{1+y^2} \right| \leq e^{-Ay}(y+1).$$

Thus, since

$$\int_0^\infty e^{-Ay}(y+1) dy = \frac{1}{A} + \frac{1}{A^2},$$

we have that  $e^{-Ay}(y+1)$  is integrable. By the Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_{A \rightarrow \infty} \int_0^\infty \frac{e^{-Ay}(y \sin A + \cos A)}{1+y^2} dy &= \int_0^\infty \lim_{A \rightarrow \infty} \frac{e^{-Ay}(y \sin A + \cos A)}{1+y^2} dy \\ &= \int_0^\infty (0) dy \\ &= 0. \end{aligned}$$

With this, our proof is complete. □

### Problem 3

**Theorem 2.** Let  $E = [1, \infty)$  and  $f \in L^2(E)$ . Assume that  $f \geq 0$  almost every where on  $E$ . Let

$$g(x) = \int_E f(y)e^{-xy} dy$$

for all  $x \in E$ . Then  $g \in L^1(E)$  and

$$\|g\|_1 \leq c\|f\|_2$$

for some  $c < 1$ . (Also Estimate  $c$ ).

**Solution**

*Proof.* We have

$$\begin{aligned}
\|g\|_1 &= \int_E |g| dx \\
&= \int_1^\infty \left| \int_E f(y) e^{-xy} dy \right| dx \\
&= \int_1^\infty \int_1^\infty f(y) e^{-xy} dy dx \\
&= \int_1^\infty f(y) \left( \int_0^\infty e^{-xy} dx - \int_0^1 e^{-xy} dx \right) dy \\
&= \int_1^\infty f(y) \left( \frac{1}{y} - \int_0^1 e^{-xy} dx \right) dy \\
&= \int_1^\infty f(y) \frac{1}{y} dy - \int_1^\infty \left( \int_0^1 f(y) e^{-xy} dx \right) dy \\
&\leq \|f\|_2 \cdot \sqrt{\int_1^\infty y^{-2} dy} - \int_1^\infty \left( \int_0^1 f(y) e^{-xy} dx \right) dy && \text{Cauchy-Schwarz's} \\
&= \|f\|_2 - \int_1^\infty \left( \int_0^1 f(y) e^{-xy} dx \right) dy \\
&= \|f\|_2 \left( 1 - \frac{1}{\|f\|_2} \int_1^\infty \left( \int_0^1 f(y) e^{-xy} dx \right) dy \right).
\end{aligned}$$

Thus, we have shown that there is some  $c < 1$  such that  $\|g\|_1 \leq c\|f\|_2$ .

To estimate  $c$ , let's find a lower bound on  $c$ . Finding a lower bound on  $c$  can be achieved by finding an upper bound on the integral in that last equation. We have

$$\begin{aligned}
\int_1^\infty \left( \int_0^1 f(y) e^{-xy} dx \right) dy &= \int_1^\infty f(y) \frac{1}{y} (1 - e^{-y}) dy \\
&\leq \|f\|_2 \cdot \sqrt{\int_1^\infty \left( \frac{1 - e^{-y}}{y} \right)^2 dy} && \text{Holder's Inequality} \\
&< \|f\|_2 \cdot \sqrt{\frac{3}{4}} && \text{Wolfram alpha for computing integral} \\
&= \|f\|_2 \cdot \frac{\sqrt{3}}{2}.
\end{aligned}$$

Thus, we have

$$\|g\|_1 < \left( 1 - \frac{\sqrt{3}}{2} \right) \cdot \|f\|_2.$$

□

**Problem 4**

**Theorem 3.** Let  $f \in L^1(\mathbb{R})$  and

$$g(x) = \int_{\mathbb{R}} f(y) e^{-(x-y)^2} dy$$

for  $x \in \mathbb{R}$ . Then  $g \in L^p(\mathbb{R})$  for any  $1 \leq p \leq \infty$ , and

$$\|g\|_p \leq \left( \frac{\pi}{p} \right)^{1/p} \cdot \|f\|_1$$

for  $1 \leq p < \infty$ .

**Solution**

*Proof.* From theorem 9.1 in our textbook, we have  $g \in L^p(\mathbb{R})$ , and

$$\begin{aligned}
 \|g\|_p &\leq \|f\|_1 \cdot \|e^{-(x-y)^2}\|_p \\
 &= \|f\|_1 \cdot \left( \int_{\mathbb{R}^2} e^{-p(x-y)^2} \right)^{1/p} \\
 &= \|f\|_1 \cdot \left( \int_{\mathbb{R}^2} e^{-px^2} e^{-py^2} \right)^{1/p} \\
 &= \|f\|_1 \cdot \left( \int_{\mathbb{R}} e^{-px^2} dx \int_{\mathbb{R}} e^{-py^2} dy \right)^{1/p} \\
 &= \|f\|_1 \cdot \left( \sqrt{\frac{\pi}{p}} \sqrt{\frac{\pi}{p}} \right)^{1/p} \\
 &= \left( \frac{\pi}{p} \right)^{1/p} \cdot \|f\|_1.
 \end{aligned}$$

□

**Problem 5**

**Theorem 4.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous and satisfy the midpoint convexity

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}.$$

for any  $x_1, x_2 \in [a, b]$ . Then  $f$  is convex on  $[a, b]$ .

**Solution**

*Proof.* We will first prove, that for any  $n \in \mathbb{N}$ , if  $x_1, \dots, x_{2^n} \in [a, b]$ , then

$$f\left(\frac{x_1 + \dots + x_{2^n}}{2^n}\right) \leq \frac{f(x_1) + \dots + f(x_{2^n})}{2^n}.$$

To prove this, we note that the base case ( $n = 1$ ) is true by midpoint convexity of  $f$ . Now, suppose for some  $n \in \mathbb{N}$  that if  $x_1, \dots, x_{2^n} \in [a, b]$ , then

$$f\left(\frac{x_1 + \dots + x_{2^n}}{2^n}\right) \leq \frac{f(x_1) + \dots + f(x_{2^n})}{2^n}.$$

Let  $x_1, \dots, x_{2^{n+1}} \in [a, b]$ . Then, we have

$$\begin{aligned}
 f\left(\frac{x_1 + \dots + x_{2^{n+1}}}{2^{n+1}}\right) &= f\left(\frac{\frac{x_1 + \dots + x_{2^n}}{2^n} + \frac{x_{2^n+1} + \dots + x_{2^{n+1}}}{2^n}}{2}\right) \\
 &\leq \frac{f\left(\frac{x_1 + \dots + x_{2^n}}{2^n}\right) + f\left(\frac{x_{2^n+1} + \dots + x_{2^{n+1}}}{2^n}\right)}{2} \\
 &= \frac{f(x_1) + \dots + f(x_{2^{n+1}})}{2^{n+1}},
 \end{aligned}$$

and our induction is complete.

From elementary analysis, we have that rational numbers of the form  $\frac{m}{2^n}$  with  $1 \leq m \leq 2^n$  are dense in  $[0, 1]$ . Let  $[a_1, b_2] \subseteq [a, b]$ . Fix some  $n \in \mathbb{N}$  and some  $1 \leq m \leq 2^n$ . Setting  $x_i = a_i$  for  $1 \leq i \leq m$ , and  $x_i = b_i$  for  $m + 1 \leq i \leq 2^n$ , we see from the above argument that

$$\begin{aligned}
 f\left(\frac{x_1 + \dots + x_{2^n}}{2^n}\right) &\leq \frac{f(x_1) + \dots + f(x_{2^n})}{2^n} \\
 &= \frac{mf(a_i) + (2^n - m)f(b_i)}{2^n} \\
 &= \frac{m}{2^n}f(a_i) + \left(1 - \frac{m}{2^n}\right)f(b_i).
 \end{aligned}$$

Finally, let  $\theta \in [0, 1]$  be a real number. For each  $n \in \mathbb{N}$ , define  $1 \leq m_n \leq 2^n$  to be the largest number such that  $\frac{m_n}{2^n} \geq \theta$ . Then,  $\frac{m_n}{2^n} \rightarrow \theta$ , and it follows from the continuity of  $f$  that

$$f(\theta a_i + (1 - \theta)b_i) \leq \theta f(a_i) + (1 - \theta)f(b_i),$$

and we have shown that  $f$  is convex. □

## Problem 6

**Theorem 5.** If  $f_k \rightarrow f$  in  $L^p(\mathbb{R}^n)$ ,  $1 < p < \infty$ ,  $g_k \rightarrow g$  pointwise, and  $\|g_k\|_\infty \leq M$  for all  $k$ , then

$$f_k g_k \rightarrow f g$$

in  $L^p(\mathbb{R}^n)$ .

### Solution

*Proof.* We have

$$\begin{aligned} \|f g - f_k g_k\|_p &= \|f g - g_k f + g_k f - f_k g_k\|_p \\ &\leq \|f g - g_k f\|_p + \|g_k f - f_k g_k\|_p \\ &\leq \|f g - g_k f\|_p + \|g_k(f - f_k)\|_p \\ &\leq \|f g - g_k f\|_p + \|M(f - f_k)\|_p \\ &= \|f g - g_k f\|_p + M\|f - f_k\|_p. \end{aligned}$$

Since  $f_k \rightarrow f$  in  $L^p(\mathbb{R}^n)$ , the second term in the last equation will vanish as  $k \rightarrow \infty$ . Thus, we must show that  $\|f g - g_k f\|_p \rightarrow 0$ . We have

$$\begin{aligned} \|f g - g_k f\|_p^p &= \|f(g - g_k)\|_p^p \\ &= \int_{\mathbb{R}^n} |f(g - g_k)|^p \\ &= \int_{\mathbb{R}^n} |g - g_k|^p |f|^p. \end{aligned}$$

Now,  $|g - g_k|^p |f|^p \leq (2M)^p |f|^p \in L(\mathbb{R}^n)$ , thus, by the lebesgue dominated convergence theorem, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} |g - g_k|^p |f|^p &= \int_{\mathbb{R}^n} \lim_{k \rightarrow \infty} |g - g_k|^p |f|^p \\ &= 0, \end{aligned}$$

and our proof is complete. □

## Problem 7

**Theorem 6.** Suppose  $f_k \rightarrow f$  a.e. on  $\mathbb{R}^n$  and that  $f_k, f \in L^p(\mathbb{R}^n)$ , for  $1 < p < \infty$ . If  $\|f_k\|_p \leq M < \infty$ , then

$$f_k g \rightarrow f g$$

in  $L^1(\mathbb{R}^n)$  for all  $g \in L^{p'}(\mathbb{R}^n)$ , with

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

### Solution

*Proof.* We have

$$\begin{aligned} \|f_k g - f g\|_1 &= \|g(f_k - f)\|_1 \\ &= \|g\|_{p'} \cdot \|f_k - f\|_p \end{aligned} \quad \text{Holder's inequality.}$$

Thus, it will suffice to show that  $\|f_k - f\|_p \rightarrow 0$ . We have

$$\begin{aligned} |f_k - f|^p &\leq (2 \max\{|f|, |f_k|\})^p \\ &= 2^p \max\{|f|^p, |f_k|^p\} \\ &\leq 2^p(|f|^p + |f_k|^p). \end{aligned}$$

Now,

$$\begin{aligned} \int_{\mathbb{R}^n} 2^p(|f|^p + |f_k|^p) &= 2^p \int_{\mathbb{R}^n} |f|^p + 2^p \int_{\mathbb{R}^n} |f_k|^p \\ &= 2^p \|f\|_p^p + 2^p \|f_k\|_p^p \\ &\leq 2^p \|f\|_p^p + 2^p M^p. \end{aligned}$$

Thus, we can use the Lebesgue dominated convergence theorem to yield

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} |f_k - f|^p &= \int_{\mathbb{R}^n} \lim_{k \rightarrow \infty} |f_k - f|^p \\ &= 0, \end{aligned}$$

and our proof is complete. □

## Problem 8

**Theorem 7.** Let  $f \in L^2(\mathbb{R})$ , and  $g(x) = xf(x) \in L^2(\mathbb{R})$ . Then  $f \in L^1(\mathbb{R})$  and

$$\|f\|_1 \leq \sqrt{2}(\|f\|_2 + \|g\|_2).$$

### Solution

*Proof.* We have

$$\begin{aligned} \|f\|_1 &= \int_{\mathbb{R}} |f| \\ &= \int_{|x| < 1} |f| + \int_{|x| \geq 1} |f| \\ &= \int_{|x| < 1} |(1)f| + \int_{|x| \geq 1} \left| \frac{1}{x} g \right| \\ &\leq \left( \int_{|x| < 1} 1 \right)^{1/2} \cdot \left( \int_{|x| < 1} |f|^2 \right)^{1/2} + \left( \int_{|x| \geq 1} \left| \frac{1}{x} \right|^2 \right)^{1/2} \cdot \left( \int_{|x| \geq 1} |g|^2 \right)^{1/2} \quad \text{Cauchy Schwarz's} \\ &= \sqrt{2} \left( \int_{|x| < 1} |f|^2 \right)^{1/2} + \sqrt{2} \left( \int_{|x| \geq 1} |g|^2 \right)^{1/2} \\ &= \sqrt{2} \left( \int_{\mathbb{R}} |f|^2 \right)^{1/2} + \sqrt{2} \left( \int_{\mathbb{R}} |g|^2 \right)^{1/2} \\ &= \sqrt{2}(\|f\|_2 + \|g\|_2), \end{aligned}$$

as desired. □

## Problem 9

**Theorem 8.** Let

$$f(x) = \begin{cases} x^{3/2} \sin \frac{1}{x} & x \in (0, 1] \\ 0 & \text{if } x = 0 \end{cases}$$

Then  $f$  is absolutely continuous on  $[0, 1]$ .

**Solution**

*Proof.* For  $x \in (0, 1]$ , we have

$$f'(x) = \frac{3}{2}x^{1/2} \sin \frac{1}{x} + x^{1/2} \cos \frac{1}{x},$$

and  $f'(0) = 0$ . Thus, we have for all  $x \in [0, 1]$ ,

$$|f'(x)| \leq \frac{5}{2}.$$

Since every function on a compact interval with a bounded derivative is Lipschitz continuous, we can conclude that  $f$  is Lipschitz continuous. Since Lipschitz continuity implies absolute continuity, our proof is complete.  $\square$