Problem 1

Let $I = [a_1, b_1] \times [a_2, b_2]$, a closed interval in $\mathbb{R}^2$. Given $\epsilon > 0$, construct an interval I_ϵ such that the following conditions hold:

1. $I \subset \mathring{I}_\epsilon$
2. $v(I_\epsilon) - v(I) < \epsilon$.

Solution

Define a new interval

$$I_\epsilon = [a_1 - \delta, b_1 + \delta] \times [a_2 - \delta, b_2 + \delta]$$

for some $\delta > 0$. This clearly satisfies condition (1). Now, we must find an upper bound on δ such that condition 2 holds. Thus, we must find the value of δ such that

$$v(I_\epsilon) = v(I) + \epsilon. \tag{1}$$

By definition, we have

$$v(I) = (b_1 - a_1)(b_2 - a_2),$$

and

$$\begin{aligned} v(I_\epsilon) &= ((b_1 + \delta) - (a_1 - \delta))((b_2 + \delta) - (a_2 - \delta)) \\ &= (b_1 - a_1 + 2\delta)(b_2 - a_2 + 2\delta) \\ &= 4\delta^2 + 2\delta(b_1 + b_2 - a_1 - a_2) + v(I). \end{aligned}$$

Combining this with equation (1), we have

$$\begin{aligned} 4\delta^2 + 2\delta(b_1 + b_2 - a_1 - a_2) + v(I) &= v(I) + \epsilon \\ 4\delta^2 + 2\delta(b_1 + b_2 - a_1 - a_2) - \epsilon &= 0. \end{aligned}$$

For brevity, we define

$$B = 2(b_1 + b_2 - a_1 - a_2),$$

so that we have

$$4\delta^2 + B\delta - \epsilon = 0.$$

Using the quadratic formula, we have

$$\delta = \frac{-B \pm \sqrt{B^2 + 16\epsilon}}{8}.$$

Thus, choosing δ such that

$$\delta < \frac{-B \pm \sqrt{B^2 + 16\epsilon}}{8},$$

we have satisfied condition 2.

Problem 2

Use the result in Problem 1 to show that given a sequence $\{I_k\}_{k=1}^\infty$ in $\mathbb{R}^2$ and $\epsilon > 0$, there exists a sequence of intervals $\{I_k^\epsilon\}_{k=1}^\infty$ such that

1. $I_k \subset \mathring{I}_k^\epsilon$, $k = 1, 2, ...$
2. $\sum_{k=1}^\infty v(I_k^\epsilon) < \sum_{k=1}^\infty v(I_k) + \epsilon$

Solution

From our solution to Problem 1, we can create an interval I_k^ϵ such that condition 1 holds and

$$v(I_k^\epsilon) < v(I_k) + 2^{-k}\epsilon$$

for all $k \in \mathbb{N}$. Thus, summing over all k, we have

$$\begin{aligned}\sum_{k=1}^{\infty} v(I_k^\epsilon) &\leq \sum_{k=1}^{\infty}(v(I_k) + 2^{-k}\epsilon)\\ &= \sum_{k=1}^{\infty} v(I_k) + \sum_{k=1}^{\infty} 2^{-k}\epsilon\\ &= \sum_{k=1}^{\infty} v(I_k) + \epsilon\sum_{k=1}^{\infty} 2^{-k}\\ &= \sum_{k=1}^{\infty} v(I_k) + \epsilon,\end{aligned}$$

and we have shown that $\{I_k^\epsilon\}_{k=1}^\infty$ satisfies condition 2.

Problem 3

Use the definition of the outer measure to show:

1. Let $E \subseteq \mathbb{R}^2$ be countable. Then
$$|E|_e = 0.$$
2. Let E be the edge (4 boundaries) of an interval $I \subseteq \mathbb{R}^2$. Then
$$|E|_e = 0.$$

Solution

Proof. Part 1:
Since E is countable, it can be denoted as

$$E = \{x_k : k \in \mathbb{N}\}.$$

By definition of outer measure, we have $|E|_e \geq 0$. Let $\epsilon > 0$. Define a cover $\{I_k\}$ of E such that I_k is an interval centered at x_k with $v(I_k) = \epsilon 2^{-k}$. Then, we have

$$\begin{aligned}|E|_e &\leq \sum_{k=1}^{\infty} v(I_k)\\ &= \sum_{k=1}^{\infty} \epsilon 2^{-k}\\ &= \epsilon.\end{aligned}$$

Thus, we have proven 1.

Part 2:
Let $I = [a_1, b_1] \times [a_2, b_2]$, and let $\epsilon > 0$. The boundary of I is given by

$$\partial I = ([a_1, b_1] \times \{a_2\}) \cup ([a_1, b_1] \times \{b_2\}) \cup (\{a_1\} \times [a_2, b_2]) \cup (\{b_1\} \times [a_2, b_2]).$$

Define

$$\begin{aligned}\delta_v &= \frac{\epsilon}{4(b_1 - a_1)}\\ \delta_h &= \frac{\epsilon}{4(b_2 - a_2)}.\end{aligned}$$

Then, we can define a finite cover for I:

$$C = \{[a_1, b_1] \times [a_2 - \frac{\delta_v}{2}, a_2 + \frac{\delta_v}{2}], [a_1, b_1] \times [b_2 - \frac{\delta_v}{2}, b_2 + \frac{\delta_v}{2}], [a_1 - \frac{\delta_h}{2}, a_1 + \frac{\delta_h}{2}] \times [a_2, b_2],$$
$$[b_1 - \frac{\delta_h}{2}, b_1 + \frac{\delta_h}{2}] \times [a_2, b_2]\}.$$

By design, we have $v(i) = \frac{\epsilon}{4}$ for all $i \in C$. Thus, we have

$$\begin{aligned} 0 &\leq |\partial I|_e \\ &\leq \sum_{i \in C} v(i) \\ &= \epsilon. \end{aligned}$$

Since ϵ was arbitrary, our proof is complete. □