

Problem 1

Let I_1 and I_2 be two intervals in \mathbb{R}^2 . Show the following elementary fact: $I_1 \cup I_2$ is a union of finitely many non-overlapping intervals J_k for $k = 1, \dots, N$, with

$$\sum_{k=1}^N v(J_k) \leq v(I_1) + v(I_2).$$

Solution

We will solve this problem by proving the more general case:

Theorem 1. *Let I_1 and I_2 be two intervals in \mathbb{R}^n . Then, $I_1 \cup I_2$ is a union of finitely many non-overlapping intervals J_k for $k = 1, \dots, N$, with*

$$\sum_{k=1}^N v(J_k) \leq v(I_1) + v(I_2).$$

Proof. Let $x \in \mathbb{R}^n$, and let x_i denote the i^{th} component of x . Since I_1 and I_2 are both intervals, we will write them as

$$I_k = \prod_{i=1}^n [a_i^k, b_i^k].$$

From elementary set theory, we have

$$I_1 \cup I_2 = (I_1 - I_2) \cup I_2$$

In this way, we have written $I_1 \cup I_2$ as a union of finitely many non-overlapping sets. Thus, if we can write each of these sets as a finite union of non-overlapping intervals, our proof will be complete.

Since I_2 is already an interval, all we have to show is that $I_1 - I_2$ can be written as a finite union of intervals. We have

$$\begin{aligned} x \in I_1 - I_2 &\iff x \in I_1 \wedge x \notin I_2 \\ &\iff (\forall i \in \{1, \dots, n\})(a_i^1 \leq x_i \leq b_i^1) \wedge (\exists i \in \{1, \dots, n\})(x_i < a_i^2 \wedge x > b_i^2). \end{aligned}$$

This leads us to a very straightforward way of partitioning $I_1 - I_2$ into a finite number of non overlapping intervals: For each $i \in \{1, \dots, n\}$, we have at most a choice of three 1-dimensional intervals:

$$\begin{aligned} L_1^i &= [a_i^1 \leq x_i \leq b_i^1] \\ L_2^i &= [a_i^1 \leq x_i \leq \min\{a_i^2, b_i^1\}] \\ L_3^i &= [\max\{a_i^1, b_i^2\} \leq x_i \leq b_i^1]. \end{aligned}$$

We note, that depending on the exact overlap of I_1 and I_2 , some of these 1-dimensional intervals may be empty (in which case, we discard). To prevent duplicates, we note that if $a_i^2 \geq b_i^1$, we will remove L_2^i , as this would mean $L_2^i = L_1^i$. Likewise, in the event that $a_i^1 \geq b_i^2$, remove L_3^i .

Now, by taking cartesian products of the form

$$\prod_{i=1}^n L^i \text{ for } L^i \in \{L_1^i, L_2^i, L_3^i\}$$

and removing

$$\prod_{i=1}^n L_1^i,$$

we have created a set of at most $3^n - 1$ non-overlapping intervals whose union is equal to $I_1 - I_2$. We will call this number of intervals $N - 1$, and define $J_N = I_2$.

Putting all of this together, we can create a union of finitely many non-overlapping intervals J_k for $k \in \{1, \dots, N\}$ such that

$$I_1 \cup I_2 = \sum_{k=1}^N J_k.$$

Now all that remains is to show

$$\sum_{k=1}^N v(J_k) \leq v(I_1) + v(I_2).$$

We note right away, that since $J_N = I_2$, this amounts to showing

$$\sum_{k=1}^{N-1} v(J_k) \leq v(I_1).$$

Conveniently, this follows directly from the fact that I_2 is a superset of a finite union of non-overlapping intervals:

$$I_2 \supseteq \bigcup_{k=1}^{N-1} J_k.$$

Thus, our proof is complete. □

Problem 2

Part (i):

Show that if $f : [a, b] \rightarrow \mathbb{R}$ is continuous where $-\infty < a < b < \infty$, then its graph $E = \{(x, f(x)) : x \in [a, b]\}$ as a subset of \mathbb{R}^2 has the Lebesgue measure zero.

Part (ii):

Show that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly continuous then its graph $E = \{(x, f(x)) : x \in \mathbb{R}\}$ as a subset of \mathbb{R}^2 has the Lebesgue measure zero.

Solution

Before we jump into the proof, we will prove a useful lemma.

Lemma 1. *If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then f is uniformly continuous.*

Proof. Suppose that f is not uniformly continuous. Then,

$$(\exists \epsilon > 0)(\forall \delta > 0)(\exists x, y \in [a, b])(|x - y| < \delta \wedge |f(x) - f(y)| \geq \epsilon).$$

Fix this ϵ . Using the axiom of choice, we can construct two sequences $\{x_k\}$ and $\{y_k\}$ such that

$$|x_k - y_k| < \frac{1}{k} \wedge |f(x_k) - f(y_k)| \geq \epsilon$$

for all k . By sequential compactness of $[a, b]$, there exists a convergent subsequence $\{x_{k_j}\}$ of $\{x_k\}$ that converges to some $x \in [a, b]$. Let $\epsilon_0 > 0$. Since $\{x_{k_j}\}$ converges to x , there exists some $N \in \mathbb{N}$ such that

$$|x_{k_j} - x| < \frac{\epsilon_0}{2}$$

for all $j \geq N$. Choose K such that $1/K < \frac{\epsilon_0}{2}$ and $K \geq N$. Then, we have for all $j \geq K$

$$\begin{aligned} |y_{k_j} - x| &\leq |y_{k_j} - x_{k_j}| + |x_{k_j} - x| \\ &< \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} \\ &< \epsilon_0. \end{aligned}$$

Thus, we have shown that $\{y_{k_j}\}$ converges to x as well.

Now we assume, for sake of contradiction, that $\{f(y_{k_j})\}$ and $\{f(x_{k_j})\}$ converge to $f(x)$. Then, there exists a $J \in \mathbb{N}$ such that $|f(x_{k_J}) - f(x)| < \frac{\epsilon}{2}$ and $|f(y_{k_J}) - f(x)| < \frac{\epsilon}{2}$. Finally, utilizing the triangle inequality, we have

$$\begin{aligned} |f(x_{k_J}) - f(y_{k_J})| &\leq |f(x_{k_J}) - f(x)| + |f(y_{k_J}) - f(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$

and we have reached a contradiction. Thus, f is not continuous, and we have proven the contrapositive. \square

We will now begin the main proof.

Proof. Part (i):

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous where $-\infty < a < b < \infty$. Let $\epsilon > 0$. By Lemma 1, f is uniformly continuous on $[a, b]$, and there exists a $\delta > 0$ such that for all $x, y \in [a, b]$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{b - a}.$$

Let $\{x_0, \dots, x_N\}$ be a partition of $[a, b]$ such that $|x_k - x_{k-1}| < \delta$ for all $k \in \mathbb{N}$. Define

$$\begin{aligned} m_k &= \min\{f(x) | x_{k-1} \leq x < x_k\}, \text{ and} \\ M_k &= \max\{f(x) | x_{k-1} \leq x < x_k\}. \end{aligned}$$

Then, we have

$$\{[x_{k-1}, x_k] \times [m_k, M_k] | k \in \{1, \dots, N\}\}$$

forms a covering (made of closed intervals) of the graph, G , of f . By the definition of outer measure, we have

$$\begin{aligned} |G|_e &\leq \sum_{k=1}^N |[x_{k-1}, x_k] \times [m_k, M_k]|_e \\ &= \sum_{k=1}^N (x_k - x_{k-1})(M_k - m_k) \\ &< \sum_{k=1}^N (x_k - x_{k-1}) \frac{\epsilon}{b - a} \\ &= \frac{\epsilon}{b - a} \sum_{k=1}^N (x_k - x_{k-1}) \\ &= \frac{\epsilon}{b - a} (b - a) && \text{By definition of a partition} \\ &= \epsilon. \end{aligned}$$

Taking the limit as ϵ goes to 0, we have the desired result.

Part (ii):

Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly continuous. For every integer a , define

$$p_a = \{(x, f(x)) | a - 1 \leq x \leq a\}.$$

By part (i), each of these sets has measure zero. Then, we can write the graph of f as a countable union of measurable sets with measure zero:

$$\{(x, f(x)) | x \in \mathbb{R}\} = \bigcup_{a \in \mathbb{Z}} p_a.$$

Thus, by Theorem 3.12 in our textbook,

$$\begin{aligned} |\{(x, f(x)) | x \in \mathbb{R}\}| &\leq \sum_{a \in \mathbb{Z}} |p_a| \\ &= 0, \end{aligned}$$

and our proof is complete. \square

Problem 3

Let $A = \{r_1, r_2, \dots\}$ be the set of rational numbers in $(0, 1)$. Given $\epsilon \in (0, 1)$, let

$$A_k = (r_k - \frac{\epsilon}{2^k}, r_k + \frac{\epsilon}{2^k}) \cap (0, 1), \quad k = 1, 2, \dots,$$

and

$$E = \bigcup_{k=1}^{\infty} A_k.$$

Compute $|E|$ and $|\partial E|$.

Solution

We make two assumptions: $\epsilon \in (0, \frac{1}{2})$, and we are just supposed to show that $|\partial E| > 0$.

Part 1: Finding $|\bar{E}|$

Let $x \in [0, 1]$. Let $\delta > 0$. Since the rational numbers are dense in the real numbers, we have

$$\begin{aligned} (\exists r_k \in A)(|x - r_k| < \delta) &\iff B(x; \delta) \cap E \neq \emptyset \\ &\iff x \in \bar{E} \end{aligned} \quad \text{Since } \delta \text{ was arbitrary.}$$

Thus, we have

$$\bar{E} = [0, 1] \implies |\bar{E}| = 1.$$

Part 2: Finding the upper bound of $|E|$

We have

$$\begin{aligned} |E| &\leq \sum_{k=1}^{\infty} |[r_k - \frac{\epsilon}{2^k}, r_k + \frac{\epsilon}{2^k}]| \\ &= \sum_{k=1}^{\infty} \frac{2\epsilon}{2^k} \\ &= 2\epsilon \\ &< 2 \left(\frac{1}{2} \right) \\ &= 1. \end{aligned}$$

Part 3: Showing that $|\partial E| > 0$

Since E is a countable union of open sets, E is open, and is thus measurable and disjoint from its boundary. Since \bar{E} and E are both measurable, we have from theorem 3.19 in our textbook that ∂E is measurable. Furthermore, from lemma 3.16 in our textbook,

$$\begin{aligned} |\bar{E}| &= |E| + |\partial E| \\ |\partial E| &= |\bar{E}| - |E| \\ &> 0 \end{aligned} \quad \text{Since } |\bar{E}| = 1 > |E|.$$