Homework 5

1. Use the Holder's inequality to show

$$\int_0^1 \sqrt{x}\,(1-x)^{-\frac{1}{3}}\,dx \le \frac{2}{\sqrt[3]{5}}.$$

2. Let $E \subseteq \mathbb{R}^n$ be m'ble with $|E| = 1$. Let $h \ge 0$ be m'ble on E. ~~Let~~ $A = \int_E h\,dx$. Show that

$$\sqrt{1+A^2} \le \int_E \sqrt{1+h^2}\,dx \le 1 + A.$$

3. Find all nonnegative functions $g \in L^3(0,1)$ such that

$$\left(\int_0^1 x\,g(x)\,dx \right)^3 = \frac{4}{25} \int_0^1 g^3(x)\,dx.$$

4. Let $f \in L^\infty(0,1)$ and $\|f\|_\infty \le 1$.

(a) Show that

$$\int_0^1 \sqrt{1 - f^2(x)}\,dx \le \sqrt{1 - \left(\int_0^1 f(x)\,dx\right)^2}.$$

(b) Describe the class of functions f for which the equality take place.

5. Prove that $\int_0^\infty e^{-x}\sqrt{x^4+3x^2+2}\,dx \le \sqrt{12}$.

Also show that the equality does not hold.

6. Let $f: \mathbb{R}^n \to \mathbb{R}$ be continuous and bounded. Show that

$$\|f\|_\infty = \sup\{|f(x)| : x \in \mathbb{R}^n\}.$$