Problem 1

Theorem 1. *Let $f : (a, b) \to \mathbb{R}$ be such that for all $x_0 \in (a, b)$, there is a support line*

$$l_{x_0}(x) = f(x_0) + m(x - x_0)$$

for some $m \in \mathbb{R}$ such that

$$f(x) \geq l_{x_0}(x)$$

for all $x \in (a, b)$. Then f is convex on (a, b).

Solution

Proof. Suppose that f is not convex on (a, b). Then, for some $[x_1, x_2] \subset (a, b)$, there exists an $x_0 \in [x_1, x_2]$ such that $f(x_0) > L(x_0)$, where L is the straight line such that $L(x_1) = f(x_1)$ and $L(x_2) = f(x_2)$. We have $l_{x_0}(x_1) \leq f(x_1) = L(x_1)$ and $l_{x_0}(x_2) \leq f(x_2) = L(x_2)$. Since L and l_{x_0} are both straight lines, we have that $l_{x_0}(x) \leq L(x)$ for all $x \in [x_1, x_2]$. However, we have

$$l_{x_0}(x_0) = f(x_0) + m(x_0 - x_0) = f(x_0) > L(x_0),$$

and we have reached a contradiction. Thus, we have proven that f is convex. □

Problem 2

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous and $f'(x)$ be increasing except on a zero measure subset of $[a, b]$. Then f is convex on $[a, b]$.*

Solution

We will start by proving a lemma which will aid in the proof of this theorem:

Lemma 1. *Let $f : [a, b] \to \mathbb{R}$ be continuous and satisfy the midpoint convexity*

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}.$$

for any $x_1, x_2 \in [a, b]$. Then f is convex on $[a, b]$.

Proof. We will first prove, that for any $n \in \mathbb{N}$, if $x_1, ..., x_{2^n} \in [a, b]$, then

$$f\left(\frac{x_1 + ... + x_{2^n}}{2^n}\right) \leq \frac{f(x_1) + ... + f(x_{2^n})}{2^n}.$$

To prove this, we note that the base case ($n = 1$) is true by midpoint convexity of f. Now, suppose for some $n \in \mathbb{N}$ that if $x_1, ..., x_{2^n} \in [a, b]$, then

$$f\left(\frac{x_1 + ... + x_{2^n}}{2^n}\right) \leq \frac{f(x_1) + ... + f(x_{2^n})}{2^n}.$$

Let $x_1, ..., x_{2^{n+1}} \in [a, b]$. Then, we have

$$\begin{aligned} f\left(\frac{x_1 + ... + x_{2^{n+1}}}{2^{n+1}}\right) &= f\left(\frac{\frac{x_1+...+x_{2^n}}{2^n} + \frac{x_{2^n+1}+...+x_{2^{n+1}}}{2^n}}{2}\right) \\ &\leq \frac{f\left(\frac{x_1+...+x_{2^n}}{2^n}\right) + f\left(\frac{x_{2^n+1}+...+x_{2^{n+1}}}{2^n}\right)}{2} \\ &= \frac{f(x_1) + ... + f(x_{2^{n+1}})}{2^{n+1}}, \end{aligned}$$

and our induction is complete.

From elementary analysis, we have that rational numbers of the form $\frac{m}{2^n}$ with $1 \leq m \leq 2^n$ are dense in $[0, 1]$. Let $[a_1, b_2] \subseteq [a, b]$. Fix some $n \in \mathbb{N}$ and some $1 \leq m \leq 2^n$. Setting $x_i = a_i$ for $1 \leq i \leq m$, and $x_i = b_i$ for $m + 1 \leq i \leq 2^n$, we see from the above argument that

$$\begin{aligned} f\left(\frac{x_1 + ... + x_{2^n}}{2^n}\right) &\leq \frac{f(x_1) + ... + f(x_{2^n})}{2^n} \\ &= \frac{mf(a_i) + (2^n - m)f(b_i)}{2^n} \\ &= \frac{m}{2^n}f(a_i) + (1 - \frac{m}{2^n})f(b_i). \end{aligned}$$

Finally, let $\theta \in [0, 1]$ be a real number. For each $n \in \mathbb{N}$, define $1 \leq m_n \leq 2^n$ to be the largest number such that $\frac{m_n}{2^n} \geq \theta$. Then, $\frac{m_n}{2^n} \to \theta$, and it follows from the continuity of f that

$$f(\theta a_i + (1 - \theta)b_i) \leq \theta f(a_i) + (1 - \theta)f(b_i),$$

and we have shown that f is convex. □

Now we are ready to prove the main theorem:

Proof. Let $[x_1, x_2] \subseteq [a, b]$. By the above lemma, it will suffice to show that

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}.$$

Since f is absolutely continuous, we have

$$f\left(\frac{x_1 + x_2}{2}\right) - f(x_1) = \int_{x_1}^{\frac{x_1+x_2}{2}} f'(x)dx,$$

and

$$f(x_2) - f\left(\frac{x_1 + x_2}{2}\right) = \int_{\frac{x_1+x_2}{2}}^{x_2} f'(x)dx.$$

Now, since f' is increasing almost everywhere, we have

$$\int_{x_1}^{\frac{x_1+x_2}{2}} f'(x)dx \leq \int_{\frac{x_1+x_2}{2}}^{x_2} f'(x)dx.$$

With this, we have

$$\begin{aligned} f\left(\frac{x_1 + x_2}{2}\right) - f(x_1) &\leq f(x_2) - f\left(\frac{x_1 + x_2}{2}\right) \\ f\left(\frac{x_1 + x_2}{2}\right) &\leq \frac{f(x_1) + f(x_2)}{2} \end{aligned}$$

and we have proven that f is midpoint-convex, and therefore convex. □

Problem 3

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ and let $E \subseteq [a, b]$. Assume that $f'(x)$ exists with a finite value for any $x \in E$. Then,*

$$|f(E)|_e \leq \int_E |f'(x)|dx.$$

Solution

Proof. We will use the fact that if $f'(x)$ exists for all $x \in A \subseteq [a,b]$, and $|f(x)| \leq p$ for all $x \in A$, then

$$|f(A)|_e \leq p|A|_e.$$

Let $\epsilon > 0$, and define $E_k = \{x \in E : (k-1)\epsilon \leq |f'(x)| < k\epsilon\}$ for each natural k. Then, we have

$$\begin{aligned}
|f(E)|_e &= \left| f(\bigcup_{k\in\mathbb{N}} E_k) \right|_e \\
&= \left| \bigcup_{k\in\mathbb{N}} f(E_k) \right|_e \\
&\leq \sum_{k\in\mathbb{N}} |f(E_k)|_e \\
&\leq \sum_{k\in\mathbb{N}} k\epsilon|E_k| \\
&= \sum_{k\in\mathbb{N}} (k-1)\epsilon|E_k| + \epsilon|E_k| \\
&\leq \sum_{k\in\mathbb{N}} \int_{E_k} |f'(x)|dx + \epsilon \sum_{k\in\mathbb{N}} |E_k| \\
&= \int_E |f'(x)|dx + \epsilon \left| \bigcup_{k\in\mathbb{N}} E_k \right| \\
&= \int_E |f'(x)|dx + \epsilon|E| \\
&\leq \int_E |f'(x)|dx + \epsilon|[a,b]|.
\end{aligned}$$

Thus, taking the limit as $\epsilon \to 0$, we have $|f(E)|_e \leq \int_E |f'(x)|dx$, and our proof is complete. □

Problem 4

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be continuous. If $f'(x)$ exists as a finite number almost on all but a countable subset of $[a,b]$ and $f' \in L([a,b])$, then f is absolutely continuous on $[a,b]$.*

Solution

Proof. By a theorem we recently proved in class, it will suffice to show that f is a null function. Let $N \subseteq [a,b]$ have measure zero. Define $Z \subseteq N$ to be the at most countable set where f' does not have a finite value. Then, we have

$$\begin{aligned}
|f(N)|_e &= |f(N\backslash Z) \cup f(Z)|_e \\
&\leq |f(N\backslash Z)|_e + |f(Z)|_e \\
&\leq \int_{N\backslash Z} |f'(x)|dx + 0 && \text{By Theorem 3 and the fact that } f(Z) \text{ is at most countable} \\
&= 0. && \text{Since } N\backslash Z \text{ has measure zero, and } f' \text{ is finite on } N\backslash Z
\end{aligned}$$

Thus, we have shown that f is absolutely continuous. □