Problem 1

Let

$$f = \begin{cases} \frac{xy}{(x^2+y^2)^2} & \text{if } (x,y) \in [-1,1]^2 \backslash \{(0,0)\} \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Show that

$$\int_{-1}^{1} \left(\int_{-1}^{1} f(x,y) dy \right) dx = \int_{-1}^{1} \left(\int_{-1}^{1} f(x,y) dx \right) dy,$$

but $f \notin L([-1,1]^2)$.

Solution

We have, for any $x \in [-1,1]$,

$$\begin{aligned} \int_{-1}^{1} f(x,y) dy &= \int_{-1}^{1} \frac{xy}{(x^2+y^2)^2} dy \\ &= 0 && \text{Since we are integrating and odd function over symmetric bounds} \end{aligned}$$

With an identical argument, we have that for any $y \in [0,1]$,

$$\begin{aligned} \int_{-1}^{1} f(x,y) dy &= \int_{-1}^{1} \frac{xy}{(x^2+y^2)^2} dy \\ &= 0. \end{aligned}$$

Thus,

$$0 = \int_{-1}^{1} \left(\int_{-1}^{1} f(x,y) dy \right) dx = \int_{-1}^{1} \left(\int_{-1}^{1} f(x,y) dx \right) dy.$$

Now, to show that $f \notin L([-1,1]^2)$, we will show that $|f| \notin L([-1,1]^2)$. We have

$$|f| = \begin{cases} \frac{|x||y|}{(x^2+y^2)^2} & \text{if } (x,y) \in [-1,1]^2 \backslash \{(0,0)\} \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Since this $|f|$ is continuous almost everywhere, we have that $|f|$ is measurable. Thus, by Tonneli's theorem, we have

$$\begin{aligned} \int\int_{[-1,1]^2} |f(x,y)| dy dx &= \int_{-1}^{1} \left(\int_{-1}^{1} |f(x,y)| dy \right) dx \\ &= \int_{-1}^{1} \left(2 \int_{0}^{1} |f(x,y)| dy \right) dx && \text{Even function over symmetric bounds} \\ &= \int_{-1}^{1} \left(2 \int_{0}^{1} \frac{|x| y}{(x^2+y^2)^2} dy \right) dx \\ &= \int_{-1}^{1} \left(\int_{u(0)}^{u(1)} |x| u^{-2} du \right) dx && \text{u sub with } u = x^2+y^2 \\ &= \int_{-1}^{1} \left(-|x| u^{-1} \right) \Big|_{u(0)}^{u(1)} dx \\ &= \int_{-1}^{1} \left(-\frac{|x|}{x^2+y^2} \right) \Bigg|_{0}^{1} dx \\ &= \int_{-1}^{1} \left(\frac{|x|}{x^2} - \frac{|x|}{x^2+1} \right) dx \\ &= 2 \int_{0}^{1} \left(\frac{x}{x^2} - \frac{x}{x^2+1} \right) dx && \text{Even function over symmetric bounds} \end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^1 \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx \\
&= \infty - \int_1^2 u^{-1} du && \text{u sub with } u = x^2+1 \\
&= \infty - \ln(2) \\
&= \infty.
\end{aligned}
$$

Thus, we have shown that $f \notin L([-1,1]^2)$.

Problem 2

Theorem 1. *Let $A \subseteq \mathbb{R}^p$ be measurable, and $B \subseteq \mathbb{R}^q$ be non-measurable with $|B|_e < \infty$. Then*

Part (i):
If $|A| = 0$, then $A \times B$ is measurable, and

Part (ii):
If $|A| > 0$, then $A \times B$ is not measurable.

Solution

Proof. Part (i):
Suppose $|A| = 0$. Define $H \subseteq \mathbb{R}^q$ such that $H \subseteq B$, H is of type G_δ, and $|H| = |B|_e$. Then, we have

$$
\begin{aligned}
|A \times B|_e &\leq |A \times H| && \text{Since } A \times B \subseteq A \times H \\
&= |A| \cdot |H| \\
&= |A| \cdot |B|_e \\
&= 0 \cdot |B|_e \\
&= 0 && \text{Since } |B|_e < \infty.
\end{aligned}
$$

Thus, $A \times B$ is measurable. □

Part (ii):
We will prove the contrapositive. Thus, suppose $A \times B$ is measurable. Then, by the first theorem we proved in class this semester, we have that for almost every $x \in \mathbb{R}^p$, $(A \times B)_x$ is measurable. Now, by definition, we have that for $x \in A$,

$$(A \times B)_x = B,$$

and for $x \notin A$,

$$(A \times B)_x = \emptyset.$$

Thus, for almost every $x \in A$, we have that B is measurable. However, since we know B is not measurable, we can conclude that A is a set of measure zero, and our proof of the contrapostive is complete.

Problem 3

Show that

$$\int\int_{[0,\infty]^2} \frac{dxdy}{(1+x)(1+xy^2)} = \frac{\pi^2}{2}.$$

Solution

Since our integrand is continuous, it is measurable. Thus, since we also have that our integrand is nonnegative, Tonneli's theorem allows us to compute this as an itterated integral:

$$\begin{aligned}
\int\int_{[0,\infty]^2} \frac{dxdy}{(1+x)(1+xy^2)} &= \int_0^\infty \frac{1}{(1+x)} \left(\int_0^\infty \frac{1}{1+xy^2} dy \right) dx \\
&= \int_0^\infty \frac{1}{x(1+x)} \left(\int_0^\infty \frac{1}{\frac{1}{x}+y^2} dy \right) dx \\
&= \int_0^\infty \frac{1}{x(1+x)} \left(\frac{\pi\sqrt{x}}{2} \right) dx && \text{Integral table} \\
&= \frac{\pi}{2} \int_0^\infty \frac{1}{\sqrt{x}(1+x)} dx \\
&= \frac{\pi^2}{2} && \text{Integral table.}
\end{aligned}$$

Problem 4

Let $f(x,y) = e^{-y}\sin(2xy)$ for $(x,y) \in E = [0,1] \times [0,\infty)$. Apply Fubini's theorem to show

$$\int_0^\infty e^{-y} \frac{sin^2 y}{y} dy = \frac{1}{4}\ln 5.$$

Solution

Since f is continuous, f is measurable, and the classical formula presented at the beginning of chapter 6 tells us that the integral of f will be equal to it's iterated integral.

Now, we note that for $y \in [0,\infty)$, we have

$$\int_0^1 e^{-y}\sin(2xy)dx = e^{-y}\frac{sin^2 y}{y}.$$

Thus, using Fubini's theorem, we have

$$\begin{aligned}
\int_0^\infty e^{-y} \frac{sin^2 y}{y} dy &= \int_0^\infty \left(\int_0^1 e^{-y}\sin(2xy)dx \right) dy \\
&= \int_0^1 \left(\int_0^\infty e^{-y}\sin(2xy)dy \right) dx \\
&= \int_0^1 \frac{2x}{4x^2+1} dx && \text{After two applications of integration by parts} \\
&= \int_{u(0)}^{u(1)} \frac{du}{4u} && \text{u sub with } u = 4x^2+1 \\
&= \frac{1}{4}\ln(4x^2+1)\Big|_0^1 \\
&= \frac{1}{4}\ln 5.
\end{aligned}$$

If you would like to see the work for integration by parts, let me know, and I can send you a picture of my white board.

Problem 5

Theorem 2. Part (i):
Let f be measurable on $\mathbb{R}^p$, and g be measurable on $\mathbb{R}^q$. Then $h(x,y) = f(x)g(x)$ is measurable on $\mathbb{R}^{p+q}$.

Part (ii):
Let $f \in L(\mathbb{R}^p)$, and $g \in L(\mathbb{R}^q)$. Then $h(x, y) = f(x)g(y) \in L(\mathbb{R}^{p+q})$ and

$$\int\int_{\mathbb{R}^{p+q}} f(x)g(y)dxdy = \left(\int_{\mathbb{R}^p} f(x)dx\right)\left(\int_{\mathbb{R}^q} g(y)dy\right).$$

Proof. Part (i):
Define $\bar{f}(x, y) = f(x)$ and $\bar{g}(x, y) = g(y)$ for all $(x, y) \in \mathbb{R}^{p+q}$. Then, for any $a \in \mathbb{R}$, we have

$$\{\bar{f} > a\} = \{f > a\} \times \mathbb{R}^q, \text{ and } \{\bar{g} > a\} = \mathbb{R}^p \times \{g > a\}.$$

Now, since f and g are measurable, and (as we proved last semester) the cartesian product of measurable sets is measurable, we can conclude that $\bar{f}$ and $\bar{g}$ are measurable. Finally, since $h = \bar{f} \cdot \bar{g}$, theorem 4.10 in our textbook tells us that h is measurable.

Part (ii):
Define $h(x, y)$ as we did in part (i). By theorem 4.6, we have that $|h(x, y)| = |f(x)| \cdot |g(y)|$ is measurable. Thus, by Tonneli's theorem, we have

$$\begin{aligned}
\int\int_{\mathbb{R}^{p+q}} |h(x, y)|dxdy &= \int_{\mathbb{R}^p}\left(\int_{\mathbb{R}^q} |h(x, y)|dy\right)dx \\
&= \int_{\mathbb{R}^p}\left(\int_{\mathbb{R}^q} |f(x)||g(y)|dy\right)dx \\
&= \left(\int_{\mathbb{R}^p} |f(x)|dx\right)\left(\int_{\mathbb{R}^q} |g(y)|dy\right) \qquad \text{Since there is no } x \text{ dependence in } \int_{\mathbb{R}^q} |g(y)|dy.
\end{aligned}$$

Thus, since f and g are integrable, the two integrals in the last expression are finite, and we can conclude that h is integrable. Finally, since h is integrable, Fubini's theorem tells us

$$\begin{aligned}
\int\int_{\mathbb{R}^{p+q}} f(x)g(y)dxdy &= \int_{\mathbb{R}^p}\left(\int_{\mathbb{R}^q} f(x)g(y)dy\right)dx \\
&= \left(\int_{\mathbb{R}^p} f(x)dx\right)\left(\int_{\mathbb{R}^q} g(y)dy\right) \qquad \text{Since there is no } x \text{ dependence in } \int_{\mathbb{R}^q} g(y)dy.
\end{aligned}$$

With this, our proof is complete. □

Problem 6

Let $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$ for $(x, y) \neq (0, 0)$. Show that

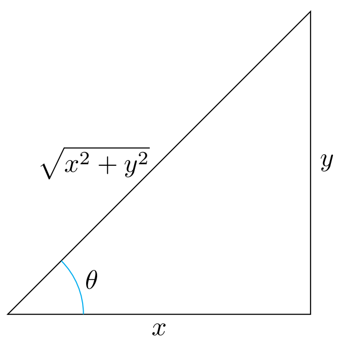
$$\int_0^1\left(\int_0^1 f(x, y)dy\right)dx = 0.$$

Solution

First, let us examine the inner most integral

$$\int_0^1 \frac{x^2 - y^2}{x^2 + y^2}dy.$$

As we normally do for integrals of this form, we will draw a triangle, and use this to make some handy variable changes.



Thus, using this, we let

$$y = x \tan \theta$$
$$dy = x \sec^2 \theta d\theta$$
$$\cos \theta = \frac{x}{\sqrt{x^2 + y^2}},$$

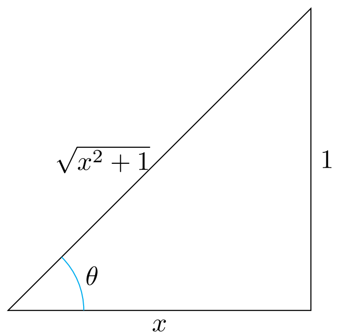
With this change of variables, our integral becomes

$$\begin{aligned}\int_0^1 \frac{x^2 - y^2}{x^2 + y^2} dy &= \int_{\theta(0)}^{\theta(1)} x(2 - \sec^2 \theta) dy \\ &= (2x\theta - x \tan \theta)\Big|_{\theta(0)}^{\theta(1)} \\ &= \left(2x \tan^{-1}\left(\frac{y}{x}\right) - x\frac{y}{x}\right)\Big|_0^1 \\ &= 2x \tan^{-1}\left(\frac{1}{x}\right) - 1.\end{aligned}$$

Now, we can solve the final integral:

$$\begin{aligned}\int_0^1 \left(\int_0^1 f(x, y) dy\right) dx &= \int_0^1 \left(2x \tan^{-1}\left(\frac{1}{x}\right) - 1\right) dx \\ &= -1 + \int_0^1 2x \tan^{-1}\left(\frac{1}{x}\right) dx\end{aligned}$$

Once again, we will draw a triangle to help determine an appropriate change of variables:



With this, we have

$$\tan^{-1}\left(\frac{1}{x}\right) = \theta$$
$$\sin^{-2} \theta = x^2 + 1$$
$$x = \cot \theta$$
$$dx = -(\csc \theta)^2$$

Thus, plugging these subsitutions in and making some simplifications, we have

$$-1 + \int_0^1 2x \tan^{-1}\left(\frac{1}{x}\right) dx = -1 - 2\int_{\theta(0)}^{\theta(1)} \theta \frac{\cos \theta}{\sin^3 \theta} d\theta.$$

Now, recognizing that

$$\frac{d}{d\theta}\left(\theta(\sin \theta)^{-2}\right) = csc^2(\theta) - 2\theta \frac{\cos \theta}{\sin^3 \theta}$$

we can use integration by parts to reduce our integral to

$$\begin{aligned}
-1-2\int_{\theta(0)}^{\theta(1)}\theta\frac{\cos\theta}{\sin^3\theta}d\theta &= -1+\theta\sin^{-2}\theta\Big|_{\theta(0)}^{\theta(1)}-\int_{\theta(0)}^{\theta(1)}\csc^2\theta d\theta \\
&= -1+\theta\sin^{-2}\theta\Big|_{\theta(0)}^{\theta(1)}+\cot\theta\Big|_{\theta(0)}^{\theta(1)} \\
&= -1+(x^2+1)\tan^{-1}\left(\frac{1}{x}\right)\Bigg|_0^1+x\Big|_0^1 \\
&= -1+2\frac{\pi}{4}-\frac{\pi}{2}+1 \\
&= 0,
\end{aligned}$$

as desired.

1 Problem 7

Theorem 3. Part (a):
Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every $x\in\mathbb{R}^1$, $\{y:(x,y)\in E\}$ has measure zero. Then E has measure 0, and for almost every $y\in\mathbb{R}^1$, $\{x:(x,y)\in E\}$ has measure zero.

Part (b):
Let $f(x,y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x\in\mathbb{R}$, $f(x,y)$ is finite for almost every y. Then, for almost every $y\in\mathbb{R}$, $f(x,y)$ is finite for almost every x.

Solution

Proof. Part (a):
Define $A,B\subseteq\mathbb{R}$ such that $E=A\times B$. We have that for $x\in A$,

$$\{y:(x,y)\in E\}=B,$$

and for $x\notin A$,

$$\{y:(x,y)\in E\}=\emptyset.$$

Thus, either $|B|=0$, or $|A|=0$. Armed with this information, let's examine the set $\{x:(x,y)\in E\}$. We have that for $y\in B$,

$$\{x:(x,y)\in E\}=A,$$

and for $y\notin B$,

$$\{x:(x,y)\in E\}=\emptyset.$$

If $|A|=0$, it follows that this set has zero measure for all y. Suppose now that $|B|=0$. Then, for almost every y (that is, the ones not in the measure zero set B), we have that $\{x:(x,y)\in E\}=\emptyset$, which implies it has measure zero.

Now, we must show that E has measure zero. Suppose first that $|A|=0$. Let $H\subseteq\mathbb{R}$ be of type G_δ with $B\subseteq H$, and $|H|=|B|_e$. Then, we have

$$\begin{aligned}
|A\times B|_e &\le |A\times H| && \text{Since } A\times B\subseteq A\times H \\
&= |A|\cdot|H| && \text{We proved this in class for measurable sets} \\
&= 0\cdot|H| \\
&= 0.
\end{aligned}$$

Thus, $|E|=0$. The case of $|B|=0$ can be handled in a nearly identical fashion.

Part (b):
Since for almost every $x\in\mathbb{R}$, $f(x,y)$ is finite for almost every y, there exist zero measure sets $A,B\subseteq\mathbb{R}$ such that for every $x\in\mathbb{R}\backslash A$, and for every $y\in\mathbb{R}\backslash B$, $f(x,y)$ is finite. Thus, for every $y\in\mathbb{R}\backslash B$, $f(x,y)$ is finite for every $x\in\mathbb{R}\backslash A$. Thus, by direct definition of "almost everywhere", we have that for almost every $y\in\mathbb{R}$, $f(x,y)$ is finite for almost every x. ☐

Problem 8

If f and g are measurable in $\mathbb{R}^n$ show that $h(x,y) = f(x)g(y)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$. Deduce that if E_1 and E_2 are measurable subsets of $\mathbb{R}^n$, then their cartesian product is measurable.

Solution

We proved the first part of this in part (i) of problem 5. We proved the second part of this problem in class last semester at the start of chapter 5 (you proved a more general version of Lemma 5.2 in the book).

Problem 9

Theorem 4. *Let f be measurable on $(0,1)$. If $f(x) - f(y)$ is integrable over the square $[0,1]^2$, show that $f \in L(0,1)$.*

Solution

Proof. Since $f(x) - f(y)$ is integrable over the square $[0,1]^2$, Fubini's theorem tells us

$$\begin{aligned}
\int\int_{[0,1]^2} (f(x) - f(y))dxdy &= \int_0^1 \left(\int_0^1 (f(x) - f(y))dy\right) dx \\
&= \int_0^1 \left(f(x) - \int_0^1 f(y)dy\right) dx \qquad f(x) \text{ acts as a constant when integrating over } y \\
&= \int_0^1 f(x)dx - \int_0^1 f(y)dy \qquad \int_0^1 f(y) \text{ acts as a constant when integrating over } x.
\end{aligned}$$

Thus, since $f(x)-f(y)$ is integrable over the square $[0,1]^2$, the integral must be finite. From this, we can conclude that the last two integrals on the write side must me finite, and we have shown that $f \in L(0,1)$. ☐

Problem 10

Theorem 5. *Let $A \subseteq \mathbb{R}^p$ be measurable, and let $B \subseteq \mathbb{R}^q$ be an arbitrary set. Then*

$$|A \times B|_e = |A| \cdot |B|_e.$$

Solution

Proof. Let $H \subseteq \mathbb{R}^q$ be of type G_δ such that $B \subseteq H$ and $|B|_e = |H|$. Then,

$$\begin{aligned}
|A \times B|_e &\leq |A \times H| && \text{Since } A \times B \subseteq A \times H \\
&= |A| \cdot |H| \\
&= |A| \cdot |B|_e.
\end{aligned}$$

Now, to show inequality in the reverse direction, let $U \subseteq \mathbb{R}^{p+q}$ be of type G_δ such that $|U| = |A \times B|_e$. We have

$$\begin{aligned}
|A \times B|_e &= |U| \\
&= \int\int_U \chi_U dxdy \\
&= \int_{\mathbb{R}^p} \left(\int_{U_x} \chi_U dy\right) dx && \text{By Fubini's theorem} \\
&\geq \int_A \left(\int_{U_x} \chi_U dy\right) dx && \text{Since } A \subseteq \mathbb{R}^p \\
&= \int_A |U_x| dx \\
&\geq \int_A |B|_e && \text{Since } B \subseteq U_x \\
&= |A| \cdot |B|_e.
\end{aligned}$$

Thus, we have shown that $|A \times B|_e = |A| \cdot |B|_e$, and our proof is complete. $\square$