MA 653-01 HW 4 (10/14/22)

1. Assume that a function f is continuous a.e. in a measurable set $E \in R^n$. Show that f is measurable on E.

   (Hint: For any $a \in \mathbb{R}$, let $A = \{f > a\}$. Show that $A = E \cap (\cup_{x \in A} B(x, \delta_x))$ where δ_x is a positive number that may depend on x and $B(x, \delta_x)$ is the ball centered at x with radius δ_x .)

2. Let f_k $(k = 1, 2, \cdots)$ be measurable functions and finite a.e on E with $|E| < \infty$. Let f be a function on E. Assume that for any $\delta > 0$, there is a measurable set $F \subset E$ such that $|E - F| < \delta$ and $f_k \to f$ on F. Show that $f_k \to f$ a.e. on E and f is measurable on E.

3. Let $f_k = \chi_{|x|<k}$ for $k = 1, 2, \cdots$.

   (i) Show that $f_k(x) \to 1$ for all $x \in \mathbb{R}^n$.

   (ii) Show that f_k does not converge uniformly outside any ball $|x| > m$ for $m > 0$.

   (iii) Show that the conclusion in Egorov's theorem does not hold for (f_k). (This shows that $|E| < \infty$ cannot be removed from Egorov's theorem.)

4. (i) Assume that $f_k \xrightarrow{m} f$ on E and $f_k \leq f_{k+1}$ $(k = 1, 2, \cdots)$. Show that $f_k \to f$ a.e. on E.

   (ii) Assume that $f_k \xrightarrow{m} f$ on E and $|f_k| \leq M$ a.e on E $(k = 1, 2, \cdots)$. Show that $|f| \leq M$ a.e. on E.

5. Let $f_k(x) = \frac{x}{k}$ for $x \in \mathbb{R}$ and $k = 1, 2, \cdots$. Does f_k converge in measure on $\mathbb{R}$? Justify your answer.

6. Show that $f_k \xrightarrow{m} f$ on E if and only if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $k \geq N$,

$$|\{x \in E : |f_k(x) - f(x)| > \varepsilon\}| < \varepsilon.$$

7. Let E be measurable and $|E| < \infty$. Show that $f_k \xrightarrow{m} f$ if and only if any subsequence of (f_{n_k}) has a subsequence $(f_{n_{k_j}})$ that is convervent to f a.e. on E.

8. Assume that $f_k \xrightarrow{m} f$ on E with $|E| < \infty$. Show that $f_k^2 \xrightarrow{m} f^2$ on E.