Problem 1

Theorem 1. *Let $f : (a, b) \to \mathbb{R}$ be such that for all $x_0 \in (a, b)$, there is a support line*

$$l_{x_0}(x) = f(x_0) + m(x - x_0)$$

for some $m \in \mathbb{R}$ such that

$$f(x) \geq l_{x_0}(x)$$

for all $x \in (a, b)$. Then f is convex on (a, b).

Solution

Proof. Suppose that f is not convex on (a, b). Then, for some $[x_1, x_2] \subset (a, b)$, there exists an $x_0 \in [x_1, x_2]$ such that $f(x_0) > L(x_0)$, where L is the straight line such that $L(x_1) = f(x_1)$ and $L(x_2) = f(x_2)$. We have $l_{x_0}(x_1) \leq f(x_1) = L(x_1)$ and $l_{x_0}(x_2) \leq f(x_2) = L(x_2)$. Since L and l_{x_0} are both straight lines, we have that $l_{x_0}(x) \leq L(x)$ for all $x \in [x_1, x_2]$. However, we have

$$l_{x_0}(x_0) = f(x_0) + m(x_0 - x_0) = f(x_0) > L(x_0),$$

and we have reached a contradiction. Thus, we have proven that f is convex. □

Problem 2

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous and $f'(x)$ be increasing except on a zero measure subset of $[a, b]$. Then f is convex on $[a, b]$.*

Solution

We will start by proving a lemma which will aid in the proof of this theorem:

Lemma 1. *Let $f : [a, b] \to \mathbb{R}$ be continuous and satisfy the midpoint convexity*

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}.$$

for any $x_1, x_2 \in [a, b]$. Then f is convex on $[a, b]$.

Proof. We will first prove, that for any $n \in \mathbb{N}$, if $x_1, ..., x_{2^n} \in [a, b]$, then

$$f\left(\frac{x_1 + ... + x_{2^n}}{2^n}\right) \leq \frac{f(x_1) + ... + f(x_{2^n})}{2^n}.$$

To prove this, we note that the base case ($n = 1$) is true by midpoint convexity of f. Now, suppose for some $n \in \mathbb{N}$ that if $x_1, ..., x_{2^n} \in [a, b]$, then

$$f\left(\frac{x_1 + ... + x_{2^n}}{2^n}\right) \leq \frac{f(x_1) + ... + f(x_{2^n})}{2^n}.$$

Let $x_1, ..., x_{2^{n+1}} \in [a, b]$. Then, we have

$$\begin{aligned} f\left(\frac{x_1 + ... + x_{2^{n+1}}}{2^{n+1}}\right) &= f\left(\frac{\frac{x_1+...+x_{2^n}}{2^n} + \frac{x_{2^n+1}+...+x_{2^{n+1}}}{2^n}}{2}\right) \\ &\leq \frac{f\left(\frac{x_1+...+x_{2^n}}{2^n}\right) + f\left(\frac{x_{2^n+1}+...+x_{2^{n+1}}}{2^n}\right)}{2} \\ &= \frac{f(x_1) + ... + f(x_{2^{n+1}})}{2^{n+1}}, \end{aligned}$$

and our induction is complete.

From elementary analysis, we have that rational numbers of the form $\frac{m}{2^n}$ with $1 \leq m \leq 2^n$ are dense in $[0,1]$. Let $[a_1, b_2] \subseteq [a,b]$. Fix some $n \in \mathbb{N}$ and some $1 \leq m \leq 2^n$. Setting $x_i = a_i$ for $1 \leq i \leq m$, and $x_i = b_i$ for $m+1 \leq i \leq 2^n$, we see from the above argument that

$$\begin{aligned} f\left(\frac{x_1 + ... + x_{2^n}}{2^n}\right) &\leq \frac{f(x_1) + ... + f(x_{2^n})}{2^n} \\ &= \frac{mf(a_i) + (2^n - m)f(b_i)}{2^n} \\ &= \frac{m}{2^n} f(a_i) + (1 - \frac{m}{2^n}) f(b_i). \end{aligned}$$

Finally, let $\theta \in [0,1]$ be a real number. For each $n \in \mathbb{N}$, define $1 \leq m_n \leq 2^n$ to be the largest number such that $\frac{m_n}{2^n} \geq \theta$. Then, $\frac{m_n}{2^n} \to \theta$, and it follows from the continuity of f that

$$f(\theta a_i + (1-\theta) b_i) \leq \theta f(a_i) + (1-\theta) f(b_i),$$

and we have shown that f is convex. $\square$

Now we are ready to prove the main theorem:

Proof. Let $[x_1, x_2] \subseteq [a,b]$. By the above lemma, it will suffice to show that

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}.$$

Since f is absolutely continuous, we have

$$f\left(\frac{x_1 + x_2}{2}\right) - f(x_1) = \int_{x_1}^{\frac{x_1+x_2}{2}} f'(x)dx,$$

and

$$f(x_2) - f\left(\frac{x_1 + x_2}{2}\right) = \int_{\frac{x_1+x_2}{2}}^{x_2} f'(x)dx.$$

Now, since f' is increasing almost everywhere, we have

$$\int_{x_1}^{\frac{x_1+x_2}{2}} f'(x)dx \leq \int_{\frac{x_1+x_2}{2}}^{x_2} f'(x)dx.$$

With this, we have

$$\begin{aligned} f\left(\frac{x_1 + x_2}{2}\right) - f(x_1) &\leq f(x_2) - f\left(\frac{x_1 + x_2}{2}\right) \\ f\left(\frac{x_1 + x_2}{2}\right) &\leq \frac{f(x_1) + f(x_2)}{2} \end{aligned}$$

and we have proven that f is midpoint-convex, and therefore convex. $\square$

Problem 3

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ and let $E \subseteq [a,b]$. Assume that $f'(x)$ exists with a finite value for any $x \in E$. Then,*

$$|f(E)|_e \leq \int_E |f'(x)|dx.$$

Proof. We start by defining the set of values of f on E as $f(E) = \{f(x) : x \in E\}$. We then define the outer measure of this set as $|f(E)|_e = \inf \sum_{i=1}^{\infty} |I_i| : f(E) \subseteq \bigcup_{i=1}^{\infty} I_i$, where the I_i are intervals.

Now, let $A \subseteq E$ be a subset of E. By the mean value theorem, for any $x, y \in A$, there exists a point $c \in A$ such that $f(x) - f(y) = f'(c)(x-y)$. Therefore, taking absolute values and using the triangle inequality, we have $|f(x) - f(y)| = |f'(c)||x-y| \leq \sup_{x \in A} |f'(x)||x-y|$.

We can now cover $f(A)$ with intervals of length at most $\sup_{x\in A}|f'(x)|$ as follows: for each $f(x) \in f(A)$, choose an interval I_x centered at $f(x)$ with length ϵ_x, where ϵ_x is such that $2\epsilon_x \sup_{x\in A}|f'(x)| \geq \epsilon_x|A|e$ (such an ϵ_x exists since $\sup x \in A|f'(x)|$ is finite and $|A|e$ is also finite). Then, we have $f(A) \subseteq \bigcup x \in AI_x$ and $\sum_{x\in A} l(I_x) \leq 2\sup_{x\in A}|f'(x)| \sum_{x\in A} \epsilon_x \leq 2\sup_{x\in A}|f'(x)| \cdot |A|_e$.

By the definition of the outer measure of $f(E)$, we have $|f(E)|e \leq \sum i = 1^\infty l(I_i)$ for any covering of $f(E)$ by intervals I_i, including the one above. Therefore, we obtain $|f(A)|e \leq \sum x \in Al(I_x) \leq 2\sup_{x\in A}|f'(x)| \cdot |A|_e$. Taking the infimum over all coverings of $f(E)$ by intervals, we have $|f(A)|e \leq \inf 2 \sup x \in A|f'(x)| \cdot |A|_e$. $\square$