Problem 1

Let I_1 and I_2 be two intervals in $\mathbb{R}^2$. Show the following elementary fact: $I_1 \cup I_2$ is a union of finitely many non-overlapping intervals J_k for $k = 1, ..., N$, with

$$\sum_{k=1}^{N} v(J_k) \leq v(I_1) + v(I_2).$$

Solution

We will solve this problem by proving the more general case:

Theorem 1. *Let I_1 and I_2 be two intervals in $\mathbb{R}^n$. Then, $I_1 \cup I_2$ is a union of finitely many non-overlapping intervals J_k for $k = 1, ..., N$, with*

$$\sum_{k=1}^{N} v(J_k) \leq v(I_1) + v(I_2).$$

Proof. Let $x \in \mathbb{R}^n$, and let x_i denote the i^{th} component of x. Since I_1 and I_2 are both intervals, we will write them as

$$I_k = \prod_{i=1}^{n} [a_i^k, b_i^k].$$

From elementary set theory, we have

$$I_1 \cup I_2 = (I_1 - I_2) \cup I_2$$

In this way, we have written $I_1 \cup I_2$ as a union of finitely many non-overlapping sets. Thus, if we can write each of these sets as a finite union of non-overlapping intervals, our proof will be complete.

Since I_2 is already an interval, all we have to show is that $I_1 - I_2$ can be written as a finite union of intervals. We have

$$\begin{aligned} x \in I_1 - I_2 &\iff x \in I_1 \wedge x \notin I_2 \\ &\iff (\forall i \in \{1, ..., n\})(a_i^1 \leq x_i \leq b_i^1) \wedge (\exists i \in \{1, ..., n\})(x_i < a_i^2 \wedge x > b_i^2). \end{aligned}$$

This leads us to a very straightforward way of partitioning $I_1 - I_2$ into a finite number of non overlapping intervals: For each $i \in \{1, ..., n\}$, we have at most a choice of three 1-dimensional intervals:

$$\begin{aligned} L_1^i &= [a_i^1 \leq x_i \leq b_i^1] \\ L_2^i &= [a_i^1 \leq x_i \leq \min\{a_i^2, b_i^1\}] \\ L_3^i &= [\max\{a_i^1, b_i^2\} \leq x_i \leq b_i^1]. \end{aligned}$$

We note, that depending on the exact overlap of I_1 and I_2, some of these 1-dimensional intervals may be empty (in which case, we discard). To prevent duplicates, we note that if $a_i^2 \geq b_i^1$, we will remove L_i^2, as this would mean $L_i^2 = L_i^1$. Likewise, in the event that $a_i^1 \geq b_i^2$, remove L_3^i.

Now, by taking cartesian products of the form

$$\prod_{i=1}^{n} L^i \text{ for } L^i \in \{L_1^i, L_2^i, L_3^i\}$$

and removing

$$\prod_{i=1}^{n} L_1^i,$$

we have created a set of at most $3^n - 1$ non-overlapping intervals who's union is equal to $I_1 - I_2$. We will call this number of intervals $N - 1$, and define $J_N = I_2$.

Putting all of this together, we can create a union of finitely many non-overlapping intervals J_k for $k \in \{1, ..., N\}$ such that

$$I_1 \cup I_2 = \sum_{k=1}^{N} J_k.$$

Now all that remains is to show

$$\sum_{k=1}^{N} v(J_k) \leq v(I_1) + v(I_2).$$

We note right away, that since $J_N = I_2$, this amounts to showing

$$\sum_{k=1}^{N-1} v(J_k) \leq v(I_1).$$

Conveniently, this follows directly from the fact that I_2 is a superset of a finite union of non-overlapping intervals:

$$I_2 \supseteq \bigcup_{k=1}^{N-1} J_k.$$

Thus, our proof is complete. $\square$

Problem 2

Part (i):
Show that if $f : [a, b] \to \mathbb{R}$ is continuous where $-\infty < a < b < \infty$, the its graph $E = \{(x, f(x) : x \in [a, b]\}$ as a subset of $\mathbb{R}^2$ has the Lebesgue measure zero.

Part (ii):
Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous then its graph $E = \{(x, f(x) : x \in \mathbb{R}\}$ as a subset of $\mathbb{R}^2$ has the Lebesgue measure zero.

Solution

Before we jump into the proof, we will prove a useful lemma.

Lemma 1. *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is uniformly continuous.*

Proof. Suppose that f is not uniformly continuous. Then,

$$(\exists \epsilon > 0)(\forall \delta > 0)(\exists x, y \in [a, b])(|x - y| < \delta \wedge |f(x) - f(y)| \geq \epsilon).$$

Fix this ϵ. Using the axiom of choice, we can construct two sequences $\{x_k\}$ and $\{y_k\}$ such that

$$|x_k - y_k| < \frac{1}{k} \wedge |f(x_k) - f(y_k)| \geq \epsilon$$

for all k. By sequential compactness of $[a, b]$, there exists a convergent subsequence $\{x_{k_j}\}$ of $\{x_k\}$ that converges to some $x \in [a, b]$. Let $\epsilon_0 > 0$. Since $\{x_{k_j}\}$ converges to x, there exists some $N \in \mathbb{N}$ such that

$$|x_{k_j} - x| < \frac{\epsilon_0}{2}$$

for all $j \geq N$. Choose K such that $1/K < \frac{\epsilon_0}{2}$ and $K \geq N$. Then, we have for all $j \geq K$

$$\begin{aligned} |y_{k_j} - x| &\leq |y_{k_j} - x_{k_j}| + |x_{k_j} - x| \\ &< \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} \\ &< \epsilon_0. \end{aligned}$$

Thus, we have shown that $\{y_{k_j}\}$ converges to x as well.

Now we assume, for sake of contradiction, that $\{f(y_{k_j})\}$ and $\{f(x_{k_j})\}$ converge to $f(x)$. Then, there exists a $J \in \mathbb{N}$ such that $|f(x_{k_J}) - f(x)| < \frac{\epsilon}{2}$ and $|f(y_{k_J}) - f(x)| < \frac{\epsilon}{2}$. Finally, utilizing the triangle inequality, we have

$$\begin{aligned}|f(x_{k_J}) - f(y_{k_J})| &\leq |f(x_{k_J}) - f(x)| + |f(y_{k_J}) - f(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon,\end{aligned}$$

and we have reached a contradiction. Thus, f is not continuous, and we have proven the contrapositive. □

We will now begin the main proof.

Proof. Part (i):

Suppose $f : [a, b] \to \mathbb{R}$ is continuous where $-\infty < a < b < \infty$. Let $\epsilon > 0$. By Lemma 1, f is uniformly continuous on $[a, b]$, and there exists a $\delta > 0$ such that for all $x, y \in [a, b]$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{b - a}.$$

Let $\{x_0, ..., x_N\}$ be a partition of $[a, b]$ such that $|x_k - x_{k-1}| < \delta$ for all $k \in \mathbb{N}$. Define

$$\begin{aligned}m_k &= \min\{f(x) | x_{k-1} \leq x < x\}, \text{ and} \\ M_k &= \max\{f(x) | x_{k-1} \leq x < x\}.\end{aligned}$$

Then, we have

$$\{[x_{k-1}, x_k] \times [m_k, M_k] | k \in \{1, ..., N\}\}$$

forms a covering (made of closed intervals) of the graph, G, of f. By the definition of outer measure, we have

$$\begin{aligned}|G|_e &\leq \sum_{k=1}^{N} |[x_{k-1}, x_k] \times [m_k, M_k]|_e \\ &= \sum_{k=1}^{N} (x_k - x_{k-1})(M_k - m_k) \\ &< \sum_{k=1}^{N} (x_k - x_{k-1}) \frac{\epsilon}{b - a} \\ &= \frac{\epsilon}{b - a} \sum_{k=1}^{N} (x_k - x_{k-1}) \\ &= \frac{\epsilon}{b - a}(b - a) && \text{By definition of a partition} \\ &= \epsilon.\end{aligned}$$

Taking the limit as ϵ goes to 0, we have the desired result.

Part (ii):

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. For every integer a, define

$$p_a = \{(x, f(x)) | a - 1 \leq x \leq a\}.$$

By part (i), each of these sets has measure zero. Then, we can write the graph of f as a countable union of measurable sets with measure zero:

$$\{(x, f(x)) | x \in \mathbb{R}\} = \bigcup_{a \in \mathbb{Z}} p_a.$$

Thus, by Theorem 3.12 in our textbook,

$$\begin{aligned}|\{(x, f(x)) | x \in \mathbb{R}\}| &\leq \sum_{a \in \mathbb{Z}} |p_a| \\ &= 0,\end{aligned}$$

and our proof is complete. □

Problem 3

Let $A = \{r_1, r_2, ...\}$ be the set of rational numbers in $(0,1)$. Given $\epsilon \in (0, \frac{1}{2})$, let

$$A_k = (r_k - \frac{\epsilon}{2^k}, r_k + \frac{\epsilon}{2^k}) \cap (0,1), \qquad k = 1, 2, \dots,$$

and

$$E = \bigcup_{k=1}^{\infty} A_k.$$

Show that $0 < |E| < 2\epsilon$ and $|\partial E| > 0$.

Solution

Part 1: Finding $|\bar{E}|$

Let $x \in [0,1]$. Let $\delta > 0$. Since the rational numbers are dense in the real numbers, we have

$$\begin{aligned}
(\exists r_k \in A)(|x - r_k| < \delta) &\iff B(x;\delta) \cap E \neq \emptyset \\
&\iff x \in \bar{E} && \text{Since } \delta \text{ was arbitrary.}
\end{aligned}$$

Thus, we have

$$\bar{E} = [0,1] \implies |\bar{E}| = 1.$$

Part 2: Finding the upper and lower bounds of $|E|$

We have

$$\begin{aligned}
|E| &\leq \sum_{k=1}^{\infty} |[r_k - \frac{\epsilon}{2^k}, r_k + \frac{\epsilon}{2^k}]| \\
&= \sum_{k=1}^{\infty} \frac{2\epsilon}{2^k} \\
&= 2\epsilon \\
&< 2\left(\frac{1}{2}\right) \\
&= 1.
\end{aligned}$$

Furthermore, we have

$$A_1 \subseteq E \implies |A_1| \leq |E|.$$

Now, the exact measure of $|A_1|$ depends on the enumeration of the rationals chosen, but we have an easy lower bound:

$$\frac{\epsilon}{2} \leq |A_1| \leq |E|.$$

Thus, since $\epsilon > 0$, we have shown $|E| > 0$.

Part 3: Showing that $|\partial E| > 0$

Since E is a countable union of open sets, E is open, and is thus measurable and disjoint from it's boundary. Since $\bar{E}$ and E are both measurable, we have from theorem 3.19 in our textbook that ∂E is measurable. Furthermore, from lemma 3.16 in our textbook,

$$\begin{aligned}
|\bar{E}| &= |E| + |\partial E| \\
|\partial E| &= |\bar{E}| - |E| \\
&> 0 && \text{Since } |\bar{E}| = 1 > |E|.
\end{aligned}$$

Problem 4

Part (i):

Theorem 2. *If $E \subset \mathbb{R}^n$ is such that $|E| = 0$, then $\mathring{E} = \emptyset$.*

Part (ii):

Theorem 3. *If $E \subset [0,1]$ is Lebesgue measurable with $|E| = 1$, then E is dense in $[0,1]$.*

Solution

Part (i):

Proof. Suppose that $\mathring{E} \neq \emptyset$. Let $x \in \mathring{E}$. Then, since the interior is an open set, we have

$$(\exists \epsilon > 0)(B(x;\epsilon) \subseteq E).$$

Define the open interval centered at x:

$$I = \prod_{k=1}^{n} (x_k - \frac{\epsilon}{\sqrt{n}}, x_k + \frac{\epsilon}{\sqrt{n}}).$$

By design, we have

$$I \subseteq B(x;\epsilon) \implies I \subseteq E.$$

Thus, by sub-additivity of the Lebesgue measure, we have

$$\begin{aligned} 0 &< |I| \\ &= \frac{2n\epsilon}{\sqrt{n}} \\ &\leq |E|. \end{aligned}$$

With this, we have proven the contrapositive, and our proof is complete. □

Part (ii):

Proof. Suppose E is not dense in $[0,1]$. Then,

$$(\exists x \in [0,1])(\exists \epsilon > 0)(B(x;\epsilon) \cap E = \emptyset).$$

Fix this x and ϵ. Then,

$$\begin{aligned} E \subseteq [0,1] \cap B(x;\epsilon)^C &\implies |E|_e \leq |[0,1] \cap B(x;\epsilon)^C|_e \\ &\implies |E|_e \leq 1 - \epsilon \\ &\implies |E|_e < 1, \end{aligned}$$

and we have proven the contrapositive. □