Problem 1

Use Holder's inequality to show

$$\int_0^1 \sqrt{x}(1-x)^{-1/3}dx \leq \frac{2}{5^{1/3}}.$$

Solution

Using Holder's inequality, let $p = 3$, and let $p' = 3/2$. Then, we have

$$\begin{aligned}
\int_0^1 \sqrt{x}(1-x)^{-1/3}dx &\leq \left(\int_0^1 \sqrt{x}^3 dx\right)^{1/3} \left(\int_0^1 ((1-x)^{-1/3})^{3/2} dx\right)^{2/3} \\
&= \left(\int_0^1 x^{3/2} dx\right)^{1/3} \left(\int_0^1 u^{-1/2} du\right)^{2/3} \\
&= \left(\frac{2}{5}\right)^{1/3} 2^{2/3} \\
&= \frac{2}{5^{1/3}},
\end{aligned}$$

as desired.

Problem 2

Let $E \subseteq \mathbb{R}^n$ be measurable with $|E| = 1$. Let $h \geq 0$ be measurable on E. Let $A = \int_E h dx$. Show that

$$\sqrt{1 + A^2} \leq \int_E \sqrt{1 + h^2} dx \leq 1 + A.$$

Solution

Show the first inequality here!!!!

Now, we have

$$\begin{aligned}
\int_E \sqrt{1+h^2} dx &\leq \int_E \sqrt{1 + h^2 + 2h} dx \\
&= \int_E \sqrt{(1+h)^2} dx \\
&= \int_E 1 + h dx \\
&= 1 + A.
\end{aligned}$$

Problem 3

Find all nonnegative functions $g \in L^3(0,1)$ such that

$$\left(\int_0^1 xg(x)dx\right)^3 = \frac{4}{25}\int_0^1 g^3(x)dx$$

Solution

Using Holders inequality, we can see that if $p = \frac{3}{2}$, and $p' = 3$, then

$$\begin{aligned}
\int_0^1 xg(x)dx &\leq \left(\int_0^1 x^{3/2}dx\right)^{2/3}\left(\int_0^1 g^3(x)dx\right)^{1/3} \\
&= \left(\frac{2}{5}\right)^{2/3}\left(\int_0^1 g^3(x)dx\right)^{1/3} \\
&= \left(\frac{4}{25}\int_0^1 g^3(x)dx\right)^{1/3} \\
\left(\int_0^1 xg(x)dx\right)^3 &= \frac{4}{25}\int_0^1 g^3(x)dx.
\end{aligned}$$

Now, as we proved in class, the equality holds if and only if $\alpha x^{3/2} = g^3(x)$ almost everywhere for some real α. Thus, we have

$$g(x) = \alpha x^{\frac{1}{2}}$$

for some nonnegative real number α and for almost every x.

Problem 4

Let $f \in L^\infty(0,1)$ and $||f||_\infty \leq 1$. Show that

$$\int_0^1 \sqrt{1-f^2(x)}dx \leq \sqrt{1-\left(\int_0^1 f(x)dx\right)^2},$$

and describe the class of functions f for which equality takes place.

Solution

We have

$$\begin{aligned}
\int_0^1 \sqrt{1-f^2(x)}dx &= \int_0^1 \sqrt{(1-f)(1+f)}dx \\
&= \int_0^1 \sqrt{1-f}\sqrt{1+f}dx \\
&\leq \sqrt{\int_0^1 (1-f)dx}\sqrt{\int_0^1 (1+f)dx} && \text{Cauchy-Schwarz's inequality} \\
&= \sqrt{1-\int_0^1 fdx}\sqrt{1+\int_0^1 fdx} \\
&= \sqrt{1-\left(\int_0^1 f(x)dx\right)^2}.
\end{aligned}$$

From the proof of Holder's Inequality, we know that the equality holds iff there exists some $\alpha \in \mathbb{R}$ such that $1 - f = \alpha(1+f)$. Thus,

$$\begin{aligned}
1 - f = \alpha(1+f) &\implies 1 - f = \alpha + \alpha f \\
&\implies 1 - \alpha = (1+\alpha)f \\
&\implies f = \frac{1-\alpha}{1+\alpha}.
\end{aligned}$$

Problem 5

Theorem 1. *Prove that*

$$\int_0^\infty e^{-x}\sqrt{x^4+3x^2+2}dx \leq \sqrt{12},$$

and that the equality does not hold.

Solution

Proof. We have

$$\begin{aligned}
\int_0^\infty e^{-x}\sqrt{x^4+3x^2+2}dx &= \int_0^\infty \sqrt{e^{-2x}(x^4+3x^2+2)}dx \\
&= \int_0^\infty \sqrt{e^{-x}(x^2+2)e^{-x}(x^2+1)}dx \\
&= \int_0^\infty \sqrt{e^{-x}(x^2+2)}\sqrt{e^{-x}(x^2+1)}dx \\
&\leq \sqrt{\int_0^\infty e^{-x}(x^2+2)dx}\sqrt{\int_0^\infty e^{-x}(x^2+1)dx}dx && \text{Cauchy-Schwarz's inequality} \\
&= \sqrt{2+2}\sqrt{2+1} && \text{Apply integration by parts twice} \\
&= \sqrt{12}.
\end{aligned}$$

Assume that the equality holds. Then, there exists some $\alpha \in \mathbb{R}$ such that for almost every $x \in [0, \infty)$, we have

$$e^{-x}(x^2+2) = \alpha e^{-x}(x^2+1).$$

With this, we see

$$\begin{aligned}
x^2+2 = \alpha x^2 + \alpha &\implies x^2(1-\alpha) = 2+\alpha \\
&\implies x^2 = \frac{1-\alpha}{2+\alpha}.
\end{aligned}$$

Thus, for any given α, this can only be true for at most two points in $[0, \infty)$, which contradicts our assumption that this is true almost everywhere. Therefore, we can conclude that the equality does not hold, as desired. □

Problem 6

Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and bounded. Show that

$$||f||_\infty = \sup\{|f(x)| : x \in \mathbb{R}^n\}.$$

Solution

Let $M = \sup\{|f(x)| : x \in \mathbb{R}^n\}$, and let $\alpha < M$. Then, by definition of supremum, there exists an $x \in \mathbb{R}^n$ such that

$$\alpha < |f(x)| \leq M.$$