Problem 1

Let $I = [a_1, b_1] \times [a_2, b_2]$, a closed interval in $\mathbb{R}^2$. Given $\epsilon > 0$, construct an interval I_ϵ such that the following conditions hold:

1. $I \subset \mathring{I}_\epsilon$
2. $v(I_\epsilon) - v(I) < \epsilon$.

Solution

Define a new interval

$$I_\epsilon = [a_1 - \delta, b_1 + \delta] \times [a_2 - \delta, b_2 + \delta]$$

for some $\delta > 0$. This clearly satisfies condition (1). Now, we must find an upper bound on δ such that condition 2 holds. Thus, we must find the value of δ such that

$$v(I_\epsilon) = v(I) + \epsilon. \tag{1}$$

By definition, we have

$$v(I) = (b_1 - a_1)(b_2 - a_2),$$

and

$$\begin{aligned} v(I_\epsilon) &= ((b_1 + \delta) - (a_1 - \delta))((b_2 + \delta) - (a_2 - \delta)) \\ &= (b_1 - a_1 + 2\delta)(b_2 - a_2 + 2\delta) \\ &= 4\delta^2 + 2\delta(b_1 + b_2 - a_1 - a_2) + v(I). \end{aligned}$$

Combining this with equation (1), we have

$$\begin{aligned} 4\delta^2 + 2\delta(b_1 + b_2 - a_1 - a_2) + v(I) = v(I) + \epsilon \\ 4\delta^2 + 2\delta(b_1 + b_2 - a_1 - a_2) - \epsilon = 0. \end{aligned}$$

For brevity, we define

$$B = 2(b_1 + b_2 - a_1 - a_2),$$

so that we have

$$4\delta^2 + B\delta - \epsilon = 0.$$

Using the quadratic formula, we have

$$\delta = \frac{-B \pm \sqrt{B^2 + 16\epsilon}}{8}.$$

Thus, choosing δ such that

$$\delta < \frac{-B \pm \sqrt{B^2 + 16\epsilon}}{8},$$

we have satisfied condition 2.

Problem 2

Use the result in Problem 1 to show that given a sequence $\{I_k\}_{k=1}^{\infty}$ in $\mathbb{R}^2$ and $\epsilon > 0$, there exists a sequence of intervals $\{I_k^\epsilon\}_{k=1}^{\infty}$ such that

1. $I_k \subset \mathring{I}_k^\epsilon$, $k = 1, 2, ...$
2. $\sum_{k=1}^{\infty} v(I_k^\epsilon) < \sum_{k=1}^{\infty} v(I_k) + \epsilon$

Solution

From our solution to Problem 1, we can create an interval I_k^ϵ such that condition 1 holds and

$$v(I_k^\epsilon) < v(I_k) + 2^{-k}\epsilon$$

for all $k \in \mathbb{N}$. Thus, summing over all k, we have

$$\begin{aligned}\sum_{k=1}^{\infty} v(I_k^\epsilon) &< \sum_{k=1}^{\infty}(v(I_k) + 2^{-k}\epsilon)\\ &= \sum_{k=1}^{\infty} v(I_k) + \sum_{k=1}^{\infty} 2^{-k}\epsilon\\ &= \sum_{k=1}^{\infty} v(I_k) + \epsilon\sum_{k=1}^{\infty} 2^{-k}\\ &= \sum_{k=1}^{\infty} v(I_k) + \epsilon,\end{aligned}$$

and we have shown that $\{I_k^\epsilon\}_{k=1}^\infty$ satisfies condition 2.