Problem 1

Theorem 1. *Let $|E| < \infty$ and E_i $(i = 1, 2, ..., m)$ be measurable subsets of E. Let $k \in \{1, 2, ..., m\}$. Show that if every point of E belongs to at least k of E_i, then there is i such that $|E_i| \geq \frac{k}{m}|E|$.*

Solution

Proof. Since each $k \leq \sum_{i=1}^m \chi_{E_i}$ for each $k \in E$, theorem 5.5 tells us that

$$\int_E k \leq \int_E \sum_{i=1}^m \chi_{E_i}.$$

The left hand side of this inequality is easily evaluated to be $k|E|$ by Corollary 5.4. For the right hand side, we can use theorem 5.14 to see

$$\begin{aligned}\int_E \sum_{i=1}^m \chi_{E_i} &= \sum_{i=1}^m \int_E \chi_{E_i} \\ &= \sum_{i=1}^m |E_i| \\ &\leq \sum_{j=1}^m \max\{|E_i| : i = 1, 2, ..., m\} \\ &= m \cdot \max\{|E_i| : i = 1, 2, ..., m\}.\end{aligned}$$

Thus, we have that

$$k|E| \leq m \cdot \max\{|E_i| : i = 1, 2, ..., m\} \implies \frac{k}{m} = \max\{|E_i| : i = 1, 2, ..., m\},$$

and our proof is complete. □

Problem 2

Let f be continuous and nonnegative on $[a, b]$ where $-\infty < a < b < \infty$. Define a nondecreasing sequence of step functions $\{\phi_k\}$ on $[a, b]$ such that $\phi_k \to f$ on $[a, b]$. Then, use the monotone convergence theorem to show that

$$(L) \int_{[a,b]} f(x)dx = (R) \int_a^b f(x)dx.$$

Solution

Since f is continuous, we have that it is Reimann integrable and Lebesgue integrable. For each $k \in \mathbb{N}$, lets create a partition of $[a, b]$, $\Gamma_k = \{I_1^k, I_2^k, ..., I_{N_k}^k\}$ with norm $|\Gamma_k| < \frac{1}{k}$. For each partition, define a function $\phi_k : [a, b] \to \mathbb{R}$ such that for $x \in I_n^k$, we have

$$\phi_k(x) = \inf\{f(x_n) : x_n \in I_n^k\}.$$

Then, we clearly have that $\phi_k \nearrow f$. By the monotone convergence theorem, we have

$$(L) \int_{[a,b]} \phi_k(x)dx \to (L) \int_a^b f(x)dx.$$

Also, we have by corollary 5.4 that

$$(L) \int_{[a,b]} \phi_k(x)dx = \sum_{n=1}^{N_k} \inf\{f(x_n) : x_n \in I_n^k\} \cdot |k|$$

which is nothing more than a lower Reimann sum of f. Thus, we can conclude that

$$(L)\int_{[a,b]}\phi_k(x)dx \to (R)\int_a^b f(x)dx,$$

and ultimately that

$$(L)\int_{[a,b]} f(x)dx = (R)\int_a^b f(x)dx.$$

Problem 3

Theorem 2. *Let $f \geq 0$ be measurable in $\mathbb{R}^n$. For $k = 1, 2, ...,$, define the cut-off functions*

$$f = \begin{cases} f(x) & \text{if } f(x) < k \\ 0 & \text{if } f(x) \geq k \end{cases}$$

Then, (i) each f_k is measurable on $\mathbb{R}^n$ and (ii)

$$\int_{\mathbb{R}^n} f_k(x)dx \to \int_{\mathbb{R}^n} f(x)dx$$

as $k \to \infty$.

Solution

Proof. Part (i):

Let $a \in \mathbb{R}$. Suppose first that $a \geq k$. Then

$$\begin{aligned}\{f_k > a\} &= \{x \in \mathbb{R} : f_k(x) > a\} \\ &\subseteq \{x \in \mathbb{R} : f_k(x) > k\} \\ &= \emptyset,\end{aligned}$$

and we can conclude that $\{f_k > a\} = \emptyset$, which is measurable.

Now suppose that $0 < a < k$. Then, we have

$$\begin{aligned}\{f_k > a\} &= \{a < f_k < k\} \\ &= \{a < f < k\},\end{aligned}$$

which is measurable, since f is measurable.

Now suppose that $a = 0$. Then,

$$\begin{aligned}\{f_k > a\} &= \{f_k > 0\} \\ &= \{0 < f < k\}\end{aligned}$$

which is measurable, since f is measurable.

Finally, suppose that $a < 0$. Then,

$$\begin{aligned}\{f_k > a\} &= \{f_k \geq 0\} \\ &= \mathbb{R}^n && \text{Since } f_k \text{ is a nonnegative function.}\end{aligned}$$

Thus, in every case, $\{f_k > a\}$ is measurable, and we have shown that each f_k is measurable.

Part (ii):

Since f is finite a.e., theorem 5.10 in our text tells us that it will suffice to show that

$$\int_E f_k(x)dx \to \int_E f(x)dx,$$

where $E \subseteq \mathbb{R}^n$ is the set of all $x \in \mathbb{R}^n$ such that $f(x)$ is finite. Since each f_k is measurable and nonnegative, the monotone convergence theorem tells us that if $f_k \nearrow f$ on E, then we will have the desired result. Thus, we will show that $f_k \nearrow f$ on E.

Let $x \in E$. Since f is finite on E, there exists some least $K \in \mathbb{N}$ such that $f(x) < K$. Thus, for all $k \geq K$, we have that $f_k(x) = f(x)$. Now suppose that $k < K$. Then, we have $f_k(x) = 0 < f(x)$. Since this is true for any x, we have shown that $f_k \leq f$ for all k and that $f_k \to f$. Thus, $f_k \nearrow f$, and our proof is complete. ☐

Problem 4

Theorem 3. *Suppose that $f \geq 0$ is continuous on $(0, 1]$ and the improper Riemann integral*

$$(R)\int_0^1 f(x)dx = \lim_{a \to 0^+} (R)\int_a^1 f(x)dx$$

exists (finite or $+\infty$). Then,

$$(L)\int_{(0,1]} f(x)dx = (R)\int_0^1 f(x)dx.$$

Solution

Proof. Define the sequence of functions $\{f_k\}$ by

$$f_k = \chi_{[\frac{1}{k},1]} f.$$

Clearly, we have that $f_k \leq f$ and $f_k \to f$, and therefore that $f_k \nearrow f$. Furthermore, since indicator functions are measurable, and f is measurable, we have by theorem 4.10 that each f_k is measurable. Thus, we have by the monotone convergence theorem that

$$(L)\int_{(0,1]} f_k dx \to (L)\int_{(0,1]} f dx.$$

In addition to this, we have that for each k,

$$\begin{aligned}
(L)\int_{(0,1]} f_k dx &= (L)\int_{(0,\frac{1}{k})} f_k dx + (L)\int_{[\frac{1}{k},1]} f_k dx && \text{By theorem 5.7} \\
&= 0 + (L)\int_{[\frac{1}{k},1]} f_k dx && \text{Since } f_k = 0 \text{ on } (0, \frac{1}{k}) \\
&= (L)\int_{[\frac{1}{k},1]} f_k dx \\
&= (R)\int_{\frac{1}{k}}^1 f(x)dx && \text{By problem 2}
\end{aligned}$$

Thus, putting all of this together, we have

$$\begin{aligned}
(R)\int_0^1 f(x)dx &= \lim_{k\to\infty} (R)\int_{\frac{1}{k}}^1 f(x)dx \\
&= \lim_{k\to\infty} (L)\int_{[\frac{1}{k},1]} f_k dx \\
&= \lim_{k\to\infty} (L)\int_{(0,1]} f_k dx \\
&= (L)\int_{(0,1]} f dx,
\end{aligned}$$

and our proof is complete. □

Problem 5

Theorem 4. *Let f be nonnegative and measurable on a measurable set $E \subseteq \mathbb{R}^n$ with $|E| < \infty$.*

Part (i):
If $f \leq M$ a.e. on E where $M > 0$ is constant, then

$$\int_E f = \inf \sum_j [\sup_{x \in E_j} f(x)]|E_j|,$$

where the infimum is taken over all decompositions $E = \bigcup_j E_j$ of E into the union of a finite number of disjoint measurable sets E_j.

Part (ii):

The statement of (i) can fail if we allow f to be unbounded.

Solution

Proof. Part (i):

In light of theorem 5.10, we will assume that $f \le M$ everywhere on E. Then, we have that the measurable function $M - f$ is nonnegative on E. Utilizing theorem 5.8, we have

$$\begin{aligned}
\int_E (M-f) &= \sup \sum_j [\inf_{x\in E_j}(M - f(x))]|E_j| \\
&= -\inf\left(-\sum_j [\inf_{x\in E_j}(M - f(x))]|E_j|\right) && \text{Using the relation between inf and sup we used to prove 4.11} \\
&= -\inf \sum_j [\sup_{x\in E_j}(f(x) - M)]|E_j| && \text{Using the same relation again} \\
&= -\inf \sum_j ([\sup_{x\in E_j} f(x)] - M)|E_j| \\
&= \inf \sum_j M|E_j| - \inf \sum_j [\sup_{x\in E_j} f(x)]|E_j| \\
&= \sum_j M|E_j| - \inf \sum_j [\sup_{x\in E_j} f(x)]|E_j| \\
&= \int_E M - \inf \sum_j [\sup_{x\in E_j} f(x)]|E_j| && \text{By theorem 5.7 and corollary 5.4.}
\end{aligned}$$

We have by theorem 5.5 part (i) that $\int_E f \le \int_E M$. Combining this with that fact $\int_E M = M|E|$, we have that the integral of f is finite. Finally, using corollary 5.15, we have that

$$\begin{aligned}
\int_E M - \int_E f &= \int_E (M-f) \\
&= \int_E M - \inf \sum_j [\sup_{x\in E_j} f(x)]|E_j| \\
-\int_E f &= -\inf \sum_j [\sup_{x\in E_j} f(x)]|E_j| \\
\int_E f &= \inf \sum_j [\sup_{x\in E_j} f(x)]|E_j|,
\end{aligned}$$

and our proof of (i) is complete.

Part (ii):

Consider the function $f(x) = \frac{1}{\sqrt{x}}$ with $E = (0, 1]$. Taking the limit of Reimann integrals in the same manner as problem 4 leads us easily to the fact that

$$\int_{(0,1]} f(x)dx = 2.$$

Now, let $\{E_j\}$ be some finite decomposition of E into disjoint measurable sets. Since this decomposition is finite, there must be some E_j that contains an interval $(0, \epsilon)$ for some $\epsilon > 0$. Thus, we have

$$\begin{aligned}
\sum_j [\sup_{x\in E_j} f(x)]|E_j| &\ge [\sup_{x\in(0,\epsilon)} f(x)]|(0,\epsilon)| \\
&= \infty \cdot \epsilon \\
&= \infty.
\end{aligned}$$

Since this is true for any such decomposition, we have that

$$\begin{aligned}\inf \sum_j [\sup_{x\in E_j} f(x)]|E_j| &= \infty \\ &\neq \int_{(0,1]} f(x)dx,\end{aligned}$$

and our proof is complete. □

Problem 6

Theorem 5. *Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on E. If $f_k \to f$ and $f_k \le f$ a.e. on E, then*

$$\int_E f_k \to \int_E f.$$

Solution

Proof. Since $f_k \le f$ a.e., we can define for each k a set of measure zero

$$Z_k = \{f_k > f\}.$$

With this, define $Z = \bigcup_{k=1}^{\infty} Z_k$. Then, Z has zero measure, and we have that for each k, $f_k \le f$ and $f_k \to f$ on $E - Z$. Thus,

$$\begin{aligned}
\lim_{k\to\infty} \int_E f_k &= \lim_{k\to\infty} \left(\int_{E-Z} f_k + \int_Z f_k \right) && \text{By theorem 5.7} \\
&= \lim_{k\to\infty} \int_{E-Z} f_k && \text{By theorem 5.9, since } |Z| = 0 \\
&= \int_{E-Z} f && \text{By the monotone convergence theorem} \\
&= \int_{E-Z} f + \int_Z f \\
&= \int_E f,
\end{aligned}$$

and our proof is complete. □

Problem 7

Theorem 6. *If $f \in L(0,1)$, then $x^k f(x) \in L(0,1)$ for $k = 1, 2, ...,$, and*

$$\int_0^1 x^k f(x)dx \to 0.$$

Solution

Proof. Since x^k is continuous, we have that x^k is measurable. Furthermore, we have that $|x^k| = x^k \le 1$ for all $x \in (0,1)$. Thus, by theorem 5.30, $x^k f(x) \in L(0,1)$ for each $k \in \mathbb{N}$.

We will now use the monotone convergence theorem (part (ii)) to show that $\int_0^1 x^k f(x)dx \to 0$. Define $f_k = x^k f(x)$, and $\phi = |f|$. Clearly, we have that $f_k \le \phi$ on $(0,1)$. By theorem 5.21, we have that $\phi \in L(0,1)$. Thus, it just remains to show that $f_k \to 0$ a.e. on $(0,1)$.

By theorem 5.22, we have that $f(x)$ is finite for almost all $x \in (0,1)$. Let $x \in (0,1)$ be such that $f(x)$ is finite, and let $\epsilon > 0$. We have that $x^k \to 0$, thus, there exists some $K \in \mathbb{N}$ such that for all $k \ge K$, $x^k < \frac{\epsilon}{|f(x)|}$. Then, we have for all $k \geq K$,

$$\begin{aligned}|f_k(x)| &= |x^k f(x)| \\ &= x^k |f(x)| \\ &< \frac{\epsilon}{|f(x)|}|f(x)| \\ &= \epsilon.\end{aligned}$$

Thus, $f_k \to 0$ a.e. on $(0,1)$, and the monotone convergence theorem leads us to conclude that

$$\int_0^1 x^k f(x)dx \to 0,$$

and our proof is complete. □

Problem 8

Theorem 7. Part (a):
Let $\{f_k\}$ be a sequence of measurable functions on E. Then $\sum f_k$ converges absolutely a.e. in E if $\sum \int_E |f_k| < +\infty$.

Part (b):
If $\{r_k\}$ denotes the rational numbers in $[0,1]$ and $\{a_k\}$ satisfies $\sum |a_k| < \infty$, then $\sum a_k|x - r_k|^{-1/2}$ converges absolutely a.e. in $[0,1]$.

Solution

Proof. Part (a):
Suppose that $\sum \int_E |f_k| < +\infty$. Since each f_k is measurable, we have that the nonnegative functions $|f_k|$ are measurable. Thus, by theorem 5.16, we have

$$\sum \int_E |f_k| = \int_E \sum |f_k|.$$

By theorem 5.22, we have $\int_E \sum |f_k| < \infty$ implies that $\sum |f_k|$ is finite a.e. in E. Thus, we have shown that $\sum |f_k|$ converges absolutely a.e. in E.

Part (b):
We have that $|x - r_k|^{-1/2}$ is continuous and nonnegative on $[0, r_k)$ and $(r_k, 1]$, and is thus Riemann integrable on these sets. Furthermore, by the results of Problem 4, the Riemann integral coincides with the Lebesgue integral on these sets. Thus, we have

$$\begin{aligned}\int_{[0,1]} a_k|x - r_k|^{-1/2}dx &= \int_{[0,r_k)} a_k|x - r_k|^{-1/2} + \int_{(r_k,1]} a_k|x - r_k|^{-1/2}dx \\ &= \int_{[0,r_k)} a_k(r_k - x)^{-1/2} + \int_{(r_k,1]} a_k(x - r_k)^{-1/2}dx \\ &= (R)\int_0^{r_k} a_k(r_k - x)^{-1/2} + (R)\int_{r_k}^1 a_k(x - r_k)^{-1/2}dx.\end{aligned}$$

Using the standard techniques from elementary real analysis (u substitution), we have

$$(R)\int_0^{r_k} a_k(r_k - x)^{-1/2} = 2a_k r^{1/2},$$

and

$$(R)\int_{r_k}^1 a_k(x - r_k)^{-1/2}dx = 2a_k(1 - r_k)^{1/2}.$$

With this, we have

$$\begin{aligned}
\int_{[0,1]} a_k|x-r_k|^{-1/2}dx &= 2a_k r_k^{1/2} + 2a_k(1-r_k)^{1/2} \\
&= 2a_k(r_k^{1/2} + (1-r_k)^{1/2}) \\
&\leq 4a_k.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\sum_k \int_{[0,1]} |a_k||x-r_k|^{-1/2}dx &\leq \sum_k 4|a_k| \\
&= 4\sum_k |a_k| \\
&< \infty.
\end{aligned}$$

By part (a), we have shown that $\sum a_k|x-r_k|^{-1/2}$ converges absolutely a.e. in $[0,1]$. $\square$