Problem 1

We know that if f is measurable, then for every $a \in \mathbb{R}^1$, the set $f^{-1}(\{a\})$ is measurable. Use the following function f to show that the converse of this result is not true, where $f : \mathbb{R}^1 \to \mathbb{R}^1$ is defined by

$$f = \begin{cases} e^x & x \in E \\ -e^x & x \in E^c \end{cases}$$

where $E \in \mathbb{R}^1$ is a nonmeasurable set.

Solution

Proof. We have that e^x is a positive function. Thus, $-e^x$ is a negative function. Thus, it follows that

$$\{f > 0\} = E.$$

Since E is nonmeasurable, this tells us that $\{f > 0\}$ is nonmeasurable. With this, we have shown that f is not measurable. □

Problem 2

Theorem 1. *Let $E \in \mathbb{R}^1$ be measurable. Show that if $f : E \to [-\infty, \infty]$ is increasing, then f is measurable.*

Solution

Proof. Let $a \in \mathbb{R}$. We have

$$\{f > a\} = \{x : E | f(x) > a\}.$$

Now, suppose there is no lower bound to $\{f > a\}$. Then,

$$\begin{aligned} (\forall x \in E)(\exists x_a \in \{f > a\})(x_a < x) &\implies (\forall x \in E)(\exists x_a \in \{f > a\})(f(x_a) \leq f(x)) && \text{Since } f \text{ is increasing} \\ &\implies (\forall x \in E)(f(x) > a) \\ &\implies \{f > a\} = E \\ &\implies \{f > a\} \text{ is measurable.} \end{aligned}$$

Suppose then, that $\{f > a\}$ has a lower bound. Then, by a fundamental property of the real numbers, $\{f > a\}$ has a greatest lower bound $l \in \mathbb{R}$. Suppose first that $l \in \{f > a\}$. Then, since f is increasing, we have

$$\begin{aligned} (\forall x \geq l)(f(x) > a) &\implies [l, \infty) \cap E = \{f > a\} \\ &\implies \{f > a\} \text{ Is measurable} && \text{Intersection of two measurable sets.} \end{aligned}$$

Similarly, if $l \notin \{f > a\}$, then

$$\begin{aligned} (\forall x > l)(f(x) > a) &\implies (l, \infty) \cap E = \{f > a\} \\ &\implies \{f > a\} \text{ Is measurable} && \text{Intersection of two measurable sets.} \end{aligned}$$

Therefore, we have shown that f is measurable. □

Problem 3

Theorem 2. *If f is differentiable on $[a, b]$, the f' is measurable.*

Solution

Proof. We first note that

$$f'(t) = \lim_{k\to\infty} k\left(f(t+\frac{1}{k}) - f(t)\right)$$

for $t \in [a,b)$ (for convenience, set $f(t) = f(b)$ when $t > b$, so that $f(t+\frac{1}{k})$ is always defined). By theorem 4.5, it will suffice to show that f' is measurable on $[a,b)$, since it differs from $[a,b]$ only by a set of measure zero.

Thus, we can define a sequence of function which converges to f', $\{f_k\}_{k=1}^{\infty}$, by

$$f_k(t) = \lim_{k\to\infty} k\left(f(t+\frac{1}{k}) - f(t)\right).$$

By theorem 4.12, it will suffice to show that each f_k is measurable.

Since f is differentiable, it is continuous, and therefore measurable. Define $\phi(t) = t + \frac{1}{k}$ for $t \in [a,b)$. Then, since ϕ is continuous, $f(\phi) = f(t+\frac{1}{k})$ is also continuous, and therefore measurable (we could instead have used theorem 4.6 to arrive at the same conclusion).

By theorems 4.8 and 4.9, $f(\phi) - f$ is measurable. Finally, by theorem 4.8, $k\left(f(t+\frac{1}{k}) - f(t)\right)$ is measurable, and our proof is complete. □

Problem 4

Theorem 3. *Let f, g be two simple functions on E. Then, $f+g$ and fg are both simple functions on E.*

Solution

Proof. Suppose that f and g assume N and M distinct values respectively. Then, using elementary combinatorics, there are $N \cdot M$ different ordered pairs of the form $(f(t), g(t))$. Thus, $f+g$ and fg have, at most, $N \cdot M$ values that they can assume. Thus, $f+g$ and fg are simple functions, and our proof is complete. □

Problem 5

Theorem 4. *Let $|E| < \infty$ and let f be a measurable function on E which is finite a.e. in E. Then, for every $\epsilon > 0$, there exists a closed set $F \subseteq E$ such that $|E - F| < \epsilon$ and that f is bounded on F.*

Solution

Proof. Let $\epsilon > 0$, and let $G = \{x \in E : f(x) \text{ is infinite}\}$. Then, since E and G are measurable, we have that $E - G$ is also measurable. Define a sequence of sets $\{E_k\}$ such that

$$E_k = \{x \in E - G : -k < f(x) < k\}.$$

Then, since f is finite valued on $E - G$, we have $E_k \nearrow E - G$. Then, by theorem 3.26, we have

$$\lim_{k\to\infty} |E_k| = |E - G| = |E|.$$

Thus, there exists some $k > 0$ such that $|E| < |E_k| + \frac{\epsilon}{2}$. Furthermore, since E_k is measurable, there exists some closed set $F \subseteq E_k$ such that $|E_k - F| < \frac{\epsilon}{2}$. Now, by Caratheodory's theorem, we have

$$\begin{aligned}
|E| &= |E \cap F| + |E - F| && \\
&= |F| + |E - F| && \text{Since } F \subseteq E \\
|E - F| &= |E| - |F| && \text{Since } |E| < \infty.
\end{aligned}$$

In an identical fashion, we have

$$|E_k - F| = |E_k| - |F|.$$

Combining these results, we have

$$\begin{aligned}
|E - F| &= |E| - |F| \\
&< |E_k| + \frac{\epsilon}{2} - |F| \\
&= (|E_k| - |F|) + \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Since $F \subseteq E_k$, we have that $|f(x)| < k$ for all $x \in F$, and is thus bounded on F, and our proof is complete. □

Problem 6

Theorem 5. *Let $|E| < \infty$ and let f be measurable on E. Then there are at most countably many real numbers y such that $|f^{-1}(y)| > 0$.*

Solution

Proof. By definition of a function, we have that $y_1 \neq y_2 \iff f^{-1}(y_1) \neq f^{-1}(y_2)$. Thus, we have that $f^{-1}(y_1)$ are $f^{-1}(y_2)$ are disjoint if and only if $y_1 \neq y_2$.

For each $k \in \mathbb{N}$, define

$$R_k = \left\{ y \in \mathbb{R} := \frac{|E|}{2^k} < |f^{-1}(y)| \leq \frac{|E|}{2^{k-1}} \right\}.$$

Then, we have that each R_k is disjoint, and that

$$\{y \in \mathbb{R} : |f^{-1}(y)| > 0\} = \bigcup_{k=1}^{\infty} R_k.$$

Thus, by addativity of the Lebesgue measure, we must have that the cardinality of R_k is less than 2^k. Thus, each R_k has a finite cardinality, which means that the union over all k is at most countable. Therefore, our proof is complete. □

Problem 7

Theorem 6. *Let f be defined and measurable in $\mathbb{R}^n$. If T is a nonsingular linear transformation of $\mathbb{R}^n$, then $f(Tx)$ is measurable.*

Proof. Let $a \in \mathbb{R}$, and define $E_1 = \{f > a\}$ and $E_2 = \{fT > a\}$. Since f is measurable, it follows that E_1 is measurable. Now, because T is nonsingular, it has an inverse T^{-1}. With this, we have

$$\begin{aligned}
x \in E_2 &\iff f(Tx) > a \\
&\iff Tx \in E_1 \\
&\iff x \in T^{-1}(E_1).
\end{aligned}$$

Thus, $E_2 = T^{-1}(E_1)$. Since the inverse of a linear transformation is itself a linear transformation, Theorem 3.35 tells us that $T^{-1}(E_1)$ is measurable, and our proof is complete. □

Problem 8

Theorem 7. *Let $f(x,t)$ be a function on $E \times \mathbb{R}$ where E is a measurable subset in $\mathbb{R}^n$. Assume that (i) for almost every $x \in E$, $f(x,t)$ is continuous as a function of $t \in \mathbb{R}$; (ii) for every $t \in \mathbb{R}$, $f(x,t)$ is measurable as a function of $x \in E$. Such conditions (i) and (ii) are called Caratheory's conditions.*

Let $g : E \to \mathbb{R}$ be a measurable function. Show that $F(x) = f(x, g(x))$ is measurable on E.

Proof. Since g is measurable, there exists a sequence of measurable simple functions $\{g_k\}_{k=1}^{\infty}$ that converges to g. With each of these g_k being measurable, we can write

$$g_k(x) = \sum_{n=1}^{N_k} a_n^k \chi_{E_n^k}(x),$$

where each of the a_n^k are distinct for every n, and each of the E_n^k are disjoint and measurable for every n. Now, we define

$$F_k(x) = f(x, g_k(x)).$$

We have

$$\{F_k > a\} = \bigcup_{n=1}^{N_k} \{x \in E_n^k : f(x, a_n^k) > a\}. \tag{1}$$

Since $f(x, a_n^k)$ is continuous for almost every x, it is measurable. Thus, each set in the union of (1) is measurable, and we can conclude that F_k is measurable.

Now, all that remains is to show that $F_k \to F$. Let $x \in E$ be such that $f(x,t)$ is continuous as function of t, and let $\epsilon > 0$. Then, there exists some $\delta > 0$ such that $|f(x, g(x)) - f(y, g(y))| < \epsilon$ whenever $y \in E$ is such that $|y - x| < \delta$. Now, since $g_k \to g$, there exists a $K \in \mathbb{N}$ such that for all $k \geq K$, $|g(x) - g_k(x)| < \delta$. Thus, it follows that for all $k \geq K$, we have $|f(x, g(x)) - f(y, g_k(x))| < \epsilon$. Thus, by theorem 4.12, F is measurable, and our proof is complete. $\square$

Problem 9

Theorem 8. *Let $f : [0, \infty) \to \mathbb{R}$ be continuous. For each $k \in \mathbb{N}$, divide the interval $[0, k)$ into k^2 disjoint subintervals*

$$\left[\frac{j-1}{k}, \frac{j}{k}\right), \qquad j = 1, 2, ..., k^2,$$

and define the step function

$$f_k = \begin{cases} f\left(\frac{j-1}{k}\right) & x \in \left[\frac{j-1}{k}, \frac{j}{k}\right),\ j = 1, 2, ..., k^2 \\ f(k) & x \geq k \end{cases}$$

Then, $f_k(x) \to f(x)$ for every $x \in [0, \infty)$.

Proof. Let $x \in [0, \infty)$, and let $\epsilon > 0$. Since f is continuous, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $y \in [0, \infty)$ such that $|x - y| < \delta$. Now, choose $K \in \mathbb{N}$ to be large enough that $x < K$ and $\frac{1}{K} < \epsilon$. Then, for all $k \geq K$, there exists some $j \in \{1, 2, ..., k^2\}$ such that

$$f_k(x) = f\left(\frac{j-1}{k}\right).$$

Thus, for all $k \geq K$, we have

$$\begin{aligned} |f(x) - f_k(x)| &= \left| f(x) - f\left(\frac{j-1}{k}\right) \right| \\ &< \epsilon && \text{Since } x \in \left[\frac{j-1}{k}, \frac{j}{k}\right) \implies \left| x - \frac{j-1}{k} \right| < \frac{1}{k} < \frac{1}{K} < \delta. \end{aligned}$$

Thus, we have shown that $f_k(x) \to f(x)$, and our proof is complete. $\square$