Problem 1

Theorem 1. *Let f be measurable on a measurable set E in $\mathbb{R}^n$. Define an extension of f on $\mathbb{R}$ by*

$$\bar{f}(x) = \begin{cases} f(x) & x \in E \\ 0 & \text{if } x \in \mathbb{R}^n - E \end{cases}$$

Then

Part (i):
$\bar{f}$ is measurable on $\mathbb{R}^n$, and

Part (i):
If $f \in L(E)$, then $\bar{f} \in L(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \bar{f} = \int_E f.$$

Solution

Proof. Part (i):
Let $a \in \mathbb{R}$. We have

$$\begin{aligned} \{\bar{f} > a\} &= \{x \in \mathbb{R}^n | \bar{f}(x) > a\} \\ &= \{x \in E | f(x) > a\} \cup \{x \in E^c | 0 > a\}. \end{aligned}$$

The fist set in the union above is measurable, since f is measurable. The second set is either the empty set (if $a \geq 0$), or is E^c if $a < 0$. Since the empty set and the complement of a measurable set are both measurable, and the union of two measurable sets is measurable, we have shown that $\{\bar{f} > a\}$ is measurable.

Part (ii):
Suppose $f \in L(E)$. By theorem 5.24, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \bar{f} &= \int_{\mathbb{R}^n - E} \bar{f} + \int_E \bar{f} \\ &= \int_{\mathbb{R}^n - E} (0) + \int_E f \\ &= \int_E f. \end{aligned}$$

Thus, since $f \in L(E)$, $\int_E f$ is finite, and we can conclude that $\int_{\mathbb{R}^n} \bar{f}$ is finite. Thus, we have shown that $f \in L(\mathbb{R}^n)$. ☐

Problem 2

Theorem 2. *Let f_k $(k = 1, 2, ...)$ and f be measurable and finite a.e. on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} |f_k - f| dx \to 0$ as $k \to \infty$. Then f_k converges to f in measure.*

Solution

Proof. Suppose that f_k does not converge to f in measure. Then, there exists some $\epsilon > 0$ such that

$$\lim_{k \to \infty} |\{x \in \mathbb{R}^n : |f(x) - f_k(x)| > \epsilon\}| \neq 0.$$

Thus, there exists some $\delta > 0$, such that for all $K \in \mathbb{N}$, there exists a $k \geq K$ such that

$$\lim_{k \to \infty} |\{x \in \mathbb{R}^n : |f(x) - f_k(x)| > \epsilon\}| > \delta.$$

Define $E = \{x \in \mathbb{R}^n : |f(x) - f_k(x)| > \epsilon\}$. Then, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |f_k - f| dx &= \int_{\mathbb{R}^n - E} |f_k - f| dx + \int_E |f_k - f| dx && \text{By theorem 5.7} \\
&\geq \int_E |f_k - f| dx && \text{Since } |f_k - f| \geq 0 \\
&\geq \int_E \epsilon dx && \text{By theorem 5.5} \\
&\geq |E|\epsilon && \text{By corollary 5.5} \\
&\geq \delta\epsilon
\end{aligned}$$

Thus, for every $K \in \mathbb{N}$, there exists a $k \geq K$ such that $\int_{\mathbb{R}^n} |f_k - f| dx \geq \delta\epsilon$, and we have shown that $\int_{\mathbb{R}^n} |f_k - f| dx$ does not go to 0. With this, we have proven the contrapositive, and our proof is complete. □

Problem 3

Theorem 3. *Let $f \in L(\mathbb{R}^n)$. Then*

$$\lim_{k\to\infty} \int_{|x|>k} |f(x)| dx = 0, \qquad \textit{and} \qquad \lim_{k\to\infty} \int_{|f(x)|>k} |f(x)| dx = 0.$$

Solution

Proof. Let $\epsilon > 0$. By theorem 5.21, we have that $|f| \in L(\mathbb{R}^n)$. By theorem 5.24, we have

$$\int_{\mathbb{R}^n} |f(x)| dx = \sum_{k=1}^{\infty} \int_{k-1 \leq |x| < k} |f(x)| dx.$$

Since the integral on the left is finite, and all of the terms in the summation are nonnegative, we have that there exists some $K \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^n} |f(x)| dx - \sum_{k=1}^{K} \int_{k-1 \leq |x| < k} |f(x)| dx < \epsilon.$$

Using theorem 5.24 once more, we have

$$\int_{\mathbb{R}^n} |f(x)| dx - \sum_{k=1}^{K} \int_{k-1 \leq |x| < k} |f(x)| dx = \int_{|x| \geq K} |f(x)| dx.$$

Using theorem 5.5 part (iii), we have

$$\begin{aligned}
\int_{|x|<K} |f(x)| dx &< \int_{|x| \geq K} |f(x)| dx \\
&< \epsilon,
\end{aligned}$$

and we have shown that

$$\lim_{k\to\infty} \int_{|x|>k} |f(x)| dx = 0.$$

For the next part, we will proceed in a nearly identical fashion. Using theorem 5.24, we have

$$\int_{\mathbb{R}^n} |f(x)| dx = \sum_{k=1}^{\infty} \int_{k-1 \leq |f(x)| < k} |f(x)| dx.$$

To see that each of the sets we are integrating over in the summation, recall that $|f|$ is measurable by theorem 5.1. Thus, since f is measurable, we have that each of these sets are measurable by corollary 4.2.

Using the same argument as before, we have that there exists a $K \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^n} |f(x)|dx - \sum_{k=1}^{K} \int_{k-1 \le |f(x)| < k} |f(x)|dx < \epsilon.$$

From theorem 5.24, we have

$$\int_{\mathbb{R}^n} |f(x)|dx - \sum_{k=1}^{K} \int_{k-1 \le |f(x)| < k} |f(x)|dx = \int_{|f(x)| \ge K} |f(x)|dx.$$

Finally, we have

$$\begin{aligned}\int_{|f(x)| > K} |f(x)|dx &\le \int_{|f(x)| \ge K} |f(x)|dx \\ &< \epsilon.\end{aligned}$$

With this, we have shown that

$$\lim_{k\to\infty} \int_{|f(x)| > k} |f(x)|dx = 0,$$

completing the proof. □

Problem 4

Give an example of a bounded continuous function f on $(0, \infty)$ such that $\lim_{x\to\infty} f(x) = 0$ but $\int_{(0,\infty)} |f|^p dx = \infty$ for any $p > 0$.

Solution

Consider the function f on $(0, \infty)$ defined by

$$f(x) = \begin{cases} \sum_{n=1}^{\infty} 2^{-n} x^{-1/n} & \text{for } x \ge 1 \\ 1 & \text{otherwise} \end{cases}$$

Since $x^{-1/n} \le 1$ for all n and all $x \in (1, \infty)$, we have for all $x \in (1, \infty)$,

$$\begin{aligned}\sum_{n=1}^{\infty} 2^{-n} x^{-1/n} &\le \sum_{n=1}^{\infty} 2^{-n} \\ &= 1.\end{aligned}$$

Therefore, this function is bounded. Furthermore, since this is a sum of continuous functions, f is also continuous. Finally, since each term goes to zero as $x \to \infty$, we have that $f \to 0$ as $x \to \infty$.

Now it just remains to show that the integral of $|f|^p$ diverges. We have

$$\begin{aligned}\int_{(0,\infty)} |f|^p dx &\le \int_{(1,\infty)} |f|^p dx && \text{By theorem 5.5} \\ &= \int_{(1,\infty)} \left| \sum_{n=1}^{\infty} 2^{-n} x^{-1/n} \right|^p dx \\ &\le \int_{(1,\infty)} \sum_{n=1}^{\infty} \left| 2^{-n} x^{-1/n} \right|^p dx \\ &= \int_{(1,\infty)} \sum_{n=1}^{\infty} 2^{-np} x^{-p/n} dx \\ &= \sum_{n=1}^{\infty} \int_{(1,\infty)} 2^{-np} x^{-p/n} dx && \text{By 5.16}\end{aligned}$$

Now, lets examine the integral of a single term, and compute it as the limit of a Reimann integral:

$$\int_{(1,\infty)} 2^{-np} x^{-p/n} dx = \left. \frac{n x^{\frac{n-p}{n}}}{n-p} \right|_1^\infty .$$

Thus, once $n > p$, this integral diverges, and we have shown that $\int_{(0,\infty)} |f|^p dx = \infty$ for any $p > 0$.

Problem 5

Evaluate the limit

$$\lim_{n\to\infty} \int_0^\infty \frac{n x^{n-3}}{1+x^n} \sin\frac{x}{n} dx,$$

and justify your answer.

Solution

To the solve this problem, we will use the Lebesgue Dominated Convergence Theorem. We will break this integral up over two sets, as the integrand has different behavior on $(0,1)$ than it does on $[1,\infty)$. Thus, let us first consider

$$\lim_{n\to\infty} \int_0^1 \frac{n x^{n-3}}{1+x^n} \sin\frac{x}{n} dx.$$

We need to show that there exists a function $\phi \in L(0,1)$ such that for some $K \in \mathbb{N}$, $|f_n| \leq \phi$ for all $n \geq K$. By simple evaluation, we see

$$\begin{aligned}
\left| \frac{n x^{n-3}}{1+x^n} \sin\frac{x}{n} \right| &= \frac{n x^{n-3}}{1+x^n} \left| \sin\frac{x}{n} \right| \\
&\leq \frac{n x^{n-3}}{1+x^n} \frac{x}{n} && \text{A result from the mean value theorem} \\
&\leq \frac{x^{n-2}}{1+x^n} \\
&\leq \frac{x^{n-2}}{1} \\
&\leq 1 && \text{For } n > 2.
\end{aligned}$$

With this, we are free to use the LDCT on this integral. Evalutating the limit,

$$\begin{aligned}
\lim_{n\to\infty} \frac{n x^{n-3}}{1+x^n} \sin\frac{x}{n} &= \lim_{n\to\infty} n x^{n-3} \sin\frac{x}{n} \\
&= x^{-3} \lim_{n\to\infty} n x^n \sin\frac{x}{n} \\
&= x^{-3} (\lim_{n\to\infty} n x^n)(\lim_{n\to\infty} \sin\frac{x}{n})
\end{aligned}$$

Since $-1 \leq \sin\frac{x}{n} \leq 1$, it will suffice to show that $\lim_{n\to\infty} n x^n = 0$. We have

$$\begin{aligned}
\lim_{n\to\infty} n x^n &= \lim_{n\to\infty} \frac{n}{1/x^n} \\
&= \lim_{n\to\infty} \frac{n}{x^{-n}} \\
&= \lim_{n\to\infty} \frac{1}{-x^{-n}\ln(x)} && \text{L'Hopital's Rule} \\
&= 0.
\end{aligned}$$

Thus, we have shown that

$$\lim_{n\to\infty} \frac{n x^{n-3}}{1+x^n} \sin\frac{x}{n} = 0,$$

and the LDCT tells us that

$$\lim_{n\to\infty}\int_0^1 \frac{nx^{n-3}}{1+x^n}\sin\frac{x}{n}dx = 0.$$

Now we consider the integral

$$\lim_{n\to\infty}\int_1^\infty \frac{nx^{n-3}}{1+x^n}\sin\frac{x}{n}dx.$$

Again, we must first verify that this is a candidate for the LDCT,:

$$\begin{aligned}
\left|\frac{nx^{n-3}}{1+x^n}\sin\frac{x}{n}\right| &= \frac{nx^{n-3}}{1+x^n}\left|\sin\frac{x}{n}\right| \\
&\le \frac{nx^{n-3}}{1+x^n}\frac{x}{n} && \text{A result from the mean value theorem} \\
&= \frac{x^{n-2}}{1+x^n} \\
&\le \frac{x^{n-2}}{x^n} \\
&= x^{-2}.
\end{aligned}$$

Thus, since $\int_1^\infty x^-2dx = 1$, we have found our $\phi \in L(1,\infty)$ such that $|f_n| \le \phi$. Therefore, we are free to use the LDCT. Evaluating the limit, we see

$$\begin{aligned}
\lim_{n\to\infty}\frac{nx^{n-3}}{1+x^n}\sin\frac{x}{n} &= \lim_{n\to\infty}\frac{nx^{n-3}}{x^n}\sin\frac{x}{n} && x^n \text{ dominates 1 in the limit} \\
&= \lim_{n\to\infty}\frac{n}{x^3}\sin\frac{x}{n} \\
&= x^{-3}\lim_{n\to\infty}\frac{\sin\frac{x}{n}}{n^{-1}} \\
&= x^{-3}\lim_{n\to\infty}\frac{x\ln(n)\cos\frac{x}{n}}{\ln(n)} && \text{L'Hopital's Rule} \\
&= x^{-2}\lim_{n\to\infty}\cos\frac{x}{n} \\
&= x^{-2}.
\end{aligned}$$

Putting it all together, we have

$$\begin{aligned}
\lim_{n\to\infty}\int_0^\infty \frac{nx^{n-3}}{1+x^n}\sin\frac{x}{n}dx &= \lim_{n\to\infty}\int_0^1 \frac{nx^{n-3}}{1+x^n}\sin\frac{x}{n}dx + \lim_{n\to\infty}\int_1^\infty \frac{nx^{n-3}}{1+x^n}\sin\frac{x}{n}dx \\
&= \int_0^1 \lim_{n\to\infty}\frac{nx^{n-3}}{1+x^n}\sin\frac{x}{n}dx + \int_1^\infty \lim_{n\to\infty}\frac{nx^{n-3}}{1+x^n}\sin\frac{x}{n}dx \\
&= \int_0^1 0dx + \int_1^\infty x^-2dx \\
&= 0+1 \\
&= 1.
\end{aligned}$$

Problem 6

Let $f : \mathbb{R}^n \to [0,\infty)$ be measurable and let $0 < \alpha < 1$. Evaluate the limit

$$\lim_{n\to\infty}\int_{\mathbb{R}^n} n\left(\left(1+\frac{f(x)}{n}\right)^\alpha - 1\right)dx,$$

and justify your answer.

Solution

For this problem, we will use the Monotone Convergence Theorem for Nonnegative functions. Define the sequence $\{a_n\}$ by

$$a_n = n\left(\left(1+\frac{f(x)}{n}\right)^{\alpha} - 1\right).$$

In order to use the monotone convergence theorem, we will need to prove that this sequence is increasing. To do this, we will treat n as as a continuous variable, and take the derivative of a_n:

$$\begin{aligned}
\frac{d}{dn}a_n &= \left(1+\frac{f(x)}{n}\right)^{\alpha} - 1 + n\left(1+\frac{f(x)}{n}\right)^{\alpha-1} f(x)\ln(n) \\
&\geq n\left(1+\frac{f(x)}{n}\right)^{\alpha-1} f(x)\ln(n) && \text{Since } \left(1+\frac{f(x)}{n}\right)^{\alpha} \geq 1 \\
&\geq 0 && \text{Since all terms are nonnegative}
\end{aligned}$$

Thus, a_n is an increasing sequence. Now, we just need to find what this sequence converges to, and we can compute the integral. We have

$$\begin{aligned}
\lim_{n\to\infty} n\left(\left(1+\frac{f(x)}{n}\right)^{\alpha} - 1\right) &= \lim_{n\to\infty} \frac{\left(1+\frac{f(x)}{n}\right)^{\alpha} - 1}{n^{-1}} \\
&= \lim_{n\to\infty} \frac{\left(1+\frac{f(x)}{n}\right)^{\alpha-1} \alpha f(x)\ln(n)}{\ln(n)} && \text{L'Hopital's Rule} \\
&= \lim_{n\to\infty} \left(1+\frac{f(x)}{n}\right)^{\alpha-1} \alpha f(x) \\
&= \alpha f(x).
\end{aligned}$$

Thus, the Monotone Convergence Theorem for Nonnegative Functions leads us to conclude that

$$\lim_{n\to\infty} \int_{\mathbb{R}^n} n\left(\left(1+\frac{f(x)}{n}\right)^{\alpha} - 1\right)dx = \alpha\int_{\mathbb{R}^n} f(x)dx$$

Problem 7

Theorem 4. *Let $E \in \mathbb{R}^n$ be measurable and $|E| < \infty$. Let f be a measurable function on E that is finite a.e. on E. Let $\{a_k\}$ be a positive and strictly increasing sequence such that $\lim_{k\to\infty} a_k = \infty$. Then f is integrable if and only if*

$$\sum_{k=1}^{\infty}(a_{k+1} - a_k)|\{|f| \geq a_k\}| < \infty.$$

Solution

This problem has stumped me.

Problem 8

Apply the result in problem 7 with $a_k = k$ to show the following:

Assume that f is a measurable function on $[0,1]$ and is finite a.e. on $[0,1]$ satisfying

$$|\{x \in [0,1] : |f(x)| \geq y\}| \leq \frac{1}{y^2}$$

for all $y > 0$. Show that f is integrable on $[0,1]$.

Solution

We have

$$\begin{aligned}\sum_{k=1}^{\infty}(a_{k+1}-a_k)|\{|f|\geq a_k\}| &\leq \sum_{k=1}^{\infty}(k+1-k)\frac{1}{k^2} \\ &= \sum_{k=1}^{\infty}\frac{1}{k^2} \\ &= \frac{\pi^2}{6} && \text{Result from elementary analysis} \\ &< \infty.\end{aligned}$$

Thus, by the result of problem 7, we can conclude that f is integrable on $[0,1]$.

Problem 9

Let $f(x,y)$ be a function defined on the unit square $[0,1]\times[0,1]$ which is continuous in each variable separately. Show that f is a measurable function of (x,y). Is the same true if f is only assumed to be continuous in x for each y?

Solution

Define a sequence of functions $f_n(x,y)$ for $n\in\mathbb{N}$ such that

$$f_n(x,y)=f(m/n,x)$$

where $m\in\mathbb{Z}$ is such that $\frac{m-1}{n}<x\leq\frac{m}{n}$. We claim that each f_n is measurable and that $\{f_n\}$ converges to f.

To see that f_n is measurable, let $a\in\mathbb{R}$. We have

$$\{f_n>a\}=\bigcup_{m=0}^{n}\left(\frac{m-1}{n},\frac{m}{n}\right]\times\{y\in[0,1]:f(m/n,y)>a\}.$$

Since f is a continuous function of y for fixed x, and continuous functions are measurable, we have that $\{y\in[0,1]:f(m/n,y)>a\}$ is measurable. Finally, since intervals are measurable, and the cartesian product of two measurable sets is measurable, we have that every set in the sum is measurable. Finally, by theorem 3.12, we can conclude that $\{f_n>a\}$ is a measurable set, and f_n is measurable.

Now we must show that $f_n\to f$. Let $x_0,y_0\in[0,1]$, and let $\epsilon>0$. Since $f(x,y_0)$ is a continuous function of x, there exists a $\delta>0$ such that for all $x\in B(x_0;\delta)$, we have

$$|f(x,y_0)-f(x_0,y_0)|<\epsilon.$$

Choose $N\in\mathbb{N}$ large enough that $\frac{1}{N}<\delta$. Then we have that for all $n\geq N$,

$$\begin{aligned}|\frac{m}{n}-x_0| &\leq |\frac{m}{n}-\frac{m-1}{n}| \\ &< \delta,\end{aligned}$$

which means

$$\begin{aligned}|f_n(x_0,y_0)-f(x_0,y_0)| &= |f(m/n,y_0)-f(x_0,y_0)| \\ &< \epsilon.\end{aligned}$$

Thus, we have shown that $f_n\to f$. Since $f_n\to f$ and each f_n is measurable, we can conclude from theorem 4.12 that f is measurable.

It is not necessary for f to be continuous in y for each x, but it is necessary that f be measurable in y for each x. Continuity is sufficient, but not necessary for measurability in y for each x.