Problem 1

Let

$$f = \begin{cases} \frac{xy}{(x^2+y^2)^2} & \text{if } (x,y) \in [-1,1]^2 \backslash \{(0,0)\} \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Show that

$$\int_{-1}^{1} \left(\int_{-1}^{1} f(x,y)dy \right) dx = \int_{-1}^{1} \left(\int_{-1}^{1} f(x,y)dx \right) dy,$$

but $f \notin L([-1,1]^2)$.

Solution

We have, for any $x \in [-1,1]$,

$$\begin{aligned} \int_{-1}^{1} f(x,y)dy &= \int_{-1}^{1} \frac{xy}{(x^2+y^2)^2} dy \\ &= 0 && \text{Since we are integrating and odd function over symmetric bounds} \end{aligned}$$

With an identical argument, we have that for any $y \in [0,1]$,

$$\begin{aligned} \int_{-1}^{1} f(x,y)dy &= \int_{-1}^{1} \frac{xy}{(x^2+y^2)^2} dy \\ &= 0. \end{aligned}$$

Thus,

$$0 = \int_{-1}^{1} \left(\int_{-1}^{1} f(x,y)dy \right) dx = \int_{-1}^{1} \left(\int_{-1}^{1} f(x,y)dx \right) dy.$$

Now, to show that $f \notin L([-1,1]^2)$, we will show that $|f| \notin L([-1,1]^2)$. We have

$$|f| = \begin{cases} \frac{|x||y|}{(x^2+y^2)^2} & \text{if } (x,y) \in [-1,1]^2 \backslash \{(0,0)\} \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Since this $|f|$ is continuous almost everywhere, we have that $|f|$ is measurable. Thus, by Tonneli's theorem, we have

$$\begin{aligned} \int\int_{[-1,1]^2} |f(x,y)|dydx &= \int_{-1}^{1} \left(\int_{-1}^{1} |f(x,y)|dy \right) dx \\ &= \int_{-1}^{1} \left(2\int_{0}^{1} |f(x,y)|dy \right) dx && \text{Even function over symmetric bounds} \\ &= \int_{-1}^{1} \left(2\int_{0}^{1} \frac{|x|y}{(x^2+y^2)^2} dy \right) dx \\ &= \int_{-1}^{1} \left(\int_{u(0)}^{u(1)} |x|u^{-2}du \right) dx && \text{u sub with } u = x^2+y^2 \\ &= \int_{-1}^{1} \left(-|x|u^{-1} \right)\Big|_{u(0)}^{u(1)} dx \\ &= \int_{-1}^{1} \left(-\frac{|x|}{x^2+y^2} \right)\Bigg|_{0}^{1} dx \\ &= \int_{-1}^{1} \left(\frac{|x|}{x^2} - \frac{|x|}{x^2+1} \right) dx \\ &= 2\int_{0}^{1} \left(\frac{x}{x^2} - \frac{x}{x^2+1} \right) dx && \text{Even function over symmetric bounds} \end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^1 \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx \\
&= \infty - \int_1^2 u^{-1} du && \text{u sub with } u = x^2+1 \\
&= \infty - \ln(2) \\
&= \infty.
\end{aligned}
$$

Thus, we have shown that $f \notin L([-1,1]^2)$.

Problem 2

Theorem 1. *Let $A \subseteq \mathbb{R}^p$ be measurable, and $B \subseteq \mathbb{R}^q$ be non-measurable with $|B|_e < \infty$. Then*

Part (i):
If $|A| = 0$, then $A \times B$ is measurable, and

Part (ii):
If $|A| > 0$, then $A \times B$ is not measurable.

Solution

Proof. Part (i):
Suppose $|A| = 0$. Define $H \subseteq \mathbb{R}^q$ such that $H \subseteq B$, H is of type G_δ, and $|H| = |B|_e$. Then, we have

$$
\begin{aligned}
|A \times B|_e &\leq |A \times H| && \text{Since } A \times B \subseteq A \times H \\
&= |A| \cdot |H| \\
&= |A| \cdot |B|_e \\
&= 0 \cdot |B|_e \\
&= 0 && \text{Since } |B|_e < \infty.
\end{aligned}
$$

Thus, $A \times B$ is measurable. □

Part (ii):
We will prove the contrapositive. Thus, suppose $A \times B$ is measurable. Then, by the first theorem we proved in class this semester, we have that for almost every $x \in \mathbb{R}^p$, $(A \times B)_x$ is measurable. Now, by definition, we have that for $x \in A$,

$$(A \times B)_x = B,$$

and for $x \notin A$,

$$(A \times B)_x = \emptyset.$$

Thus, for almost every $x \in A$, we have that B is measurable. However, since we know B is not measurable, we can conclude that A is a set of measure zero, and our proof of the contrapostive is complete.

Problem 3

Show that

$$\int\int_{[0,\infty]^2} \frac{dxdy}{(1+x)(1+xy^2)} = \frac{\pi^2}{2}.$$

Solution

Since our integrand is continuous, it is measurable. Thus, since we also have that our integrand is nonnegative, Tonneli's theorem allows us to compute this as an itterated integral:

$$\begin{aligned}
\int\int_{[0,\infty]^2} \frac{dxdy}{(1+x)(1+xy^2)} &= \int_0^\infty \frac{1}{(1+x)} \left(\int_0^\infty \frac{1}{1+xy^2} dy \right) dx \\
&= \int_0^\infty \frac{1}{x(1+x)} \left(\int_0^\infty \frac{1}{\frac{1}{x}+y^2} dy \right) dx \\
&= \int_0^\infty \frac{1}{x(1+x)} \left(\frac{\pi\sqrt{x}}{2} \right) dx && \text{Integral table} \\
&= \frac{\pi}{2} \int_0^\infty \frac{1}{\sqrt{x}(1+x)} dx \\
&= \frac{\pi^2}{2} && \text{Integral table.}
\end{aligned}$$

Problem 4

Let $f(x,y) = e^{-y}\sin(2xy)$ for $(x,y) \in E = [0,1] \times [0,\infty)$. Apply Fubini's theorem to show

$$\int_0^\infty e^{-y} \frac{sin^2 y}{y} dy = \frac{1}{4} \ln 5.$$

Solution

Since f is continuous, f is measurable, and the classical formula presented at the beginning of chapter 6 tells us that the integral of f will be equal to it's iterated integral.

Now, we note that for $y \in [0,\infty)$, we have

$$\int_0^1 e^{-y} \sin(2xy) dx = e^{-y} \frac{sin^2 y}{y}.$$

Thus, using Fubini's theorem, we have

$$\begin{aligned}
\int_0^\infty e^{-y} \frac{sin^2 y}{y} dy &= \int_0^\infty \left(\int_0^1 e^{-y} \sin(2xy) dx \right) dy \\
&= \int_0^1 \left(\int_0^\infty e^{-y} \sin(2xy) dy \right) dx \\
&= \int_0^1 \frac{2x}{4x^2+1} dx && \text{After two applications of integration by parts} \\
&= \int_{u(0)}^{u(1)} \frac{du}{4u} && \text{u sub with } u = 4x^2+1 \\
&= \frac{1}{4} \ln(4x^2+1) \Big|_0^1 \\
&= \frac{1}{4} \ln 5.
\end{aligned}$$

If you would like to see the work for integration by parts, let me know, and I can send you a picture of my white board.

Problem 5

Theorem 2. Part (i):
Let f be measurable on $\mathbb{R}^p$, and g be measurable on $\mathbb{R}^q$. Then $h(x,y) = f(x)g(x)$ is measurable on $\mathbb{R}^{p+q}$.

Part (ii):
Let $f \in L(\mathbb{R}^p)$, *and* $g \in L(\mathbb{R}^q)$. *Then* $h(x,y) = f(x)g(y) \in L(\mathbb{R}^{p+q})$ *and*

$$\int\int_{\mathbb{R}^{p+q}} f(x)g(y)dxdy = \left(\int_{\mathbb{R}^p} f(x)dx\right)\left(\int_{\mathbb{R}^q} g(y)dy\right).$$

Proof. Part (i):
Define $\bar{f}(x,y) = f(x)$ and $\bar{g}(x,y) = g(y)$ for all $(x,y) \in \mathbb{R}^{p+q}$. Then, for any $a \in \mathbb{R}$, we have

$$\{\bar{f} > a\} = \{f > a\} \times \mathbb{R}^q, \text{ and } \{\bar{g} > a\} = \mathbb{R}^p \times \{g > a\}.$$

Now, since f and g are measurable, and (as we proved last semester) the cartesian product of measurable sets is measurable, we can conclude that $\bar{f}$ and $\bar{g}$ are measurable. Finally, since $h = \bar{f} \cdot \bar{g}$, theorem 4.10 in our textbook tells us that h is measurable.

Part (ii):
Define $h(x,y)$ as we did in part (i). By theorem 4.6, we have that $|h(x,y)| = |f(x)| \cdot |g(y)|$ is measurable. Thus, by Tonneli's theorem, we have

$$\begin{aligned}
\int\int_{\mathbb{R}^{p+q}} |h(x,y)|dxdy &= \int_{\mathbb{R}^p}\left(\int_{\mathbb{R}^q} |h(x,y)|dy\right)dx \\
&= \int_{\mathbb{R}^p}\left(\int_{\mathbb{R}^q} |f(x)||g(y)|dy\right)dx \\
&= \left(\int_{\mathbb{R}^p} |f(x)|dx\right)\left(\int_{\mathbb{R}^q} |g(y)|dy\right) \quad \text{Since there is no } x \text{ dependence in } \int_{\mathbb{R}^q} |g(y)|dy.
\end{aligned}$$

Thus, since f and g are integrable, the two integrals in the last expression are finite, and we can conclude that h is integrable. Finally, since h is integrable, Fubini's theorem tells us

$$\begin{aligned}
\int\int_{\mathbb{R}^{p+q}} f(x)g(y)dxdy &= \int_{\mathbb{R}^p}\left(\int_{\mathbb{R}^q} f(x)g(y)dy\right)dx \\
&= \left(\int_{\mathbb{R}^p} f(x)dx\right)\left(\int_{\mathbb{R}^q} g(y)dy\right) \quad \text{Since there is no } x \text{ dependence in } \int_{\mathbb{R}^q} g(y)dy.
\end{aligned}$$

With this, our proof is complete. □