Problem 1

Theorem 1. *Let $|E| < \infty$ and E_i $(i = 1, 2, ..., m)$ be measurable subsets of E. Let $k \in \{1, 2, ..., m\}$. Show that if every point of E belongs to at least k of E_i, then there is i such that $|E_i| \geq \frac{k}{m}|E|$.*

Solution

Proof. Since each $k \leq \sum_{i=1}^{m} \chi_{E_i}$ for each $k \in E$, theorem 5.5 tells us that

$$\int_E k \leq \int_E \sum_{i=1}^{m} \chi_{E_i}.$$

The left hand side of this inequality is easily evaluated to be $k|E|$ by Corollary 5.4. For the right hand side, we can use theorem 5.14 to see

$$\begin{aligned} \int_E \sum_{i=1}^{m} \chi_{E_i} &= \sum_{i=1}^{m} \int_E \chi_{E_i} \\ &= \sum_{i=1}^{m} |E_i| \\ &\leq \sum_{j=1}^{m} \max\{|E_i| : i = 1, 2, ..., m\} \\ &= m \cdot \max\{|E_i| : i = 1, 2, ..., m\}. \end{aligned}$$

Thus, we have that

$$k|E| \leq m \cdot \max\{|E_i| : i = 1, 2, ..., m\} \implies \frac{k}{m} = \max\{|E_i| : i = 1, 2, ..., m\},$$

and our proof is complete. □

Problem 2

Let f be continuous and nonnegative on $[a, b]$ where $-\infty < a < b < \infty$. Define a nondecreasing sequence of step functions $\{\phi_k\}$ on $[a, b]$ such that $\phi_k \to f$ on $[a, b]$. Then, use the monotone convergence theorem to show that

$$(L) \int_{[a,b]} f(x)dx = (R) \int_a^b f(x)dx.$$

Solution

Since f is continuous, we have that it is Reimann integrable and Lebesgue integrable. For each $k \in \mathbb{N}$, lets create a partition of $[a, b]$, $\Gamma_k = \{I_1^k, I_2^k, ..., I_{N_k}^k\}$ with norm $|\Gamma_k| < \frac{1}{k}$. For each partition, define a function $\phi_k : [a, b] \to \mathbb{R}$ such that for $x \in I_n^k$, we have

$$\phi_k(x) = \inf\{f(x_n) : x_n \in I_n^k\}.$$

Then, we clearly have that $\phi_k \nearrow f$. By the monotone convergence theorem, we have

$$(L) \int_{[a,b]} \phi_k(x)dx \to (L) \int_a^b f(x)dx.$$

Also, we have by corollary 5.4 that

$$(L) \int_{[a,b]} \phi_k(x)dx = \sum_{n=1}^{N_k} \inf\{f(x_n) : x_n \in I_n^k\} \cdot |k|$$

which is nothing more than a lower Reimann sum of f. Thus, we can conclude that

$$(L)\int_{[a,b]}\phi_k(x)dx \to (R)\int_a^b f(x)dx,$$

and ultimately that

$$(L)\int_{[a,b]} f(x)dx = (R)\int_a^b f(x)dx.$$

Problem 3

Theorem 2. *Let $f \geq 0$ be measurable in $\mathbb{R}^n$. For $k = 1, 2, ...,$, define the cut-off functions*

$$f = \begin{cases} f(x) & \text{if } f(x) < k \\ 0 & \text{if } f(x) \geq k \end{cases}$$

Then, (i) each f_k is measurable on $\mathbb{R}^n$ and (ii)

$$\int_{\mathbb{R}^n} f_k(x)dx \to \int_{\mathbb{R}^n} f(x)dx$$

as $k \to \infty$.

Solution

Proof. Part (i):
Let $a \in \mathbb{R}$. Suppose first that $a \geq k$. Then

$$\begin{aligned}\{f_k > a\} &= \{x \in \mathbb{R} : f_k(x) > a\} \\ &\subseteq \{x \in \mathbb{R} : f_k(x) > k\} \\ &= \emptyset,\end{aligned}$$

and we can conclude that $\{f_k > a\} = \emptyset$, which is measurable.

Now suppose that $0 < a < k$. Then, we have

$$\begin{aligned}\{f_k > a\} &= \{a < f_k < k\} \\ &= \{a < f < k\},\end{aligned}$$

which is measurable, since f is measurable.

Now suppose that $a = 0$. Then,

$$\begin{aligned}\{f_k > a\} &= \{f_k > 0\} \\ &= \{0 < f < k\}\end{aligned}$$

which is measurable, since f is measurable.

Finally, suppose that $a < 0$. Then,

$$\begin{aligned}\{f_k > a\} &= \{f_k \geq 0\} \\ &= \mathbb{R}^n && \text{Since } f_k \text{ is a nonnegative function.}\end{aligned}$$

Thus, in every case, $\{f_k > a\}$ is measurable, and we have shown that each f_k is measurable.

Part (ii):
Since f is finite a.e., theorem 5.10 in our text tells us that it will suffice to show that

$$\int_E f_k(x)dx \to \int_E f(x)dx,$$

where $E \subseteq \mathbb{R}^n$ is the set of all $x \in \mathbb{R}^n$ such that $f(x)$ is finite. Since each f_k is measurable and nonnegative, the monotone convergence theorem tells us that if $f_k \nearrow f$ on E, then we will have the desired result. Thus, we will show that $f_k \nearrow f$ on E.

Let $x \in E$. Since f is finite on E, there exists some least $K \in \mathbb{N}$ such that $f(x) < K$. Thus, for all $k \geq K$, we have that $f_k(x) = f(x)$. Now suppose that $k < K$. Then, we have $f_k(x) = 0 < f(x)$. Since this is true for any x, we have shown that $f_k \leq f$ for all k and that $f_k \to f$. Thus, $f_k \nearrow f$, and our proof is complete. □