Problem 1

Theorem 1. *If $E \subseteq [0,1]$ has measure 1, then E is dense in $[0,1]$.*

Solution

Proof. Suppose E is not dense in $[0,1]$. Then,

$$(\exists x \in [0,1])(\exists \epsilon > 0)(B(x;\epsilon) \cap E = \emptyset).$$

Fix this x and ϵ. Then,

$$\begin{aligned} E \subseteq [0,1] \cap B(x;\epsilon)^C &\implies |E|_e \leq |[0,1] \cap B(x;\epsilon)^C|_e \\ &\implies |E|_e \leq 1 - \epsilon \\ &\implies |E|_e < 1, \end{aligned}$$

and we have proven the contrapositive. □

Problem 2

Let F be a proper closed subset in $[0,1]$. Show that $|F| < 1$.

Solution

Proof. Since F is a proper subset in $[0,1]$,

$$(\exists x \in [0,1])(x \notin F).$$

Fix this x. Since F is closed, x is not a boundary point of F, which means

$$(\exists \epsilon > 0)(B(x;\epsilon) \cap F = \emptyset) \implies (\exists \epsilon > 0)(F \subseteq ([0,1] - B(x;\epsilon))).$$

Fixing this ϵ, we have

$$\begin{aligned} |F| &\leq |[0,1] - B(x;\epsilon)| \\ &\leq 1 - \epsilon \\ &< 1, \end{aligned}$$

and our proof is complete. □

Problem 3

Theorem 2. *Let $E_k \subseteq [0,1]$ be measurable with $\sum_{k=1}^N |E_k| > N - 1$. Then*

$$|\bigcap_{k=1}^N E_k| > 0.$$

Proof. We have

$$[0,1] = \bigcap_{k=1}^N E_k + [0,1] \setminus \bigcap_{k=1}^N E_k \implies 1 = \left|\bigcap_{k=1}^N E_k\right| + \left|[0,1] \setminus \bigcap_{k=1}^N E_k\right|.$$

Thus, it will suffice to show that

$$\left|[0,1] \setminus \bigcap_{k=1}^N E_k\right| < 1.$$

We have

$$\begin{aligned}
\left|[0,1]\setminus\bigcap_{k=1}^{N}E_k\right| &= \left|[0,1]\cap\left(\bigcap_{k=1}^{N}E_k\right)^c\right| \\
&= \left|[0,1]\cap\bigcup_{k=1}^{N}E_k^c\right| && \text{De Morgan' Laws} \\
&= \left|\bigcup_{k=1}^{N}[0,1]\cap E_k^c\right| \\
&\leq \sum_{k=1}^{N}|[0,1]\cap E_k^c| && \text{Subadditivity} \\
&= \sum_{k=1}^{N}1-|E_k| \\
&= N-\sum_{k=1}^{N}|E_k| \\
&< N-(N-1) && \text{By initial assumption} \\
&= 1.
\end{aligned}$$

With this, our proof is complete. □

Problem 4

Theorem 3. *If E_1 and E_2 are measurable, then*

$$|E_1\cup E_2|+|E_1\cap E_2|=|E_1|+|E_2|.$$

Proof. We will assume that E_1 and E_2 have finite measure, because otherwise the proof is trivial. We can write E_1 as the union of two disjoint measurable sets

$$E_1=(E_1-E_2)\cup(E_1\cap E_2),$$

and additivity gives us

$$|E_1|=|E_1-E_2|+|E_1\cap E_2|. \tag{1}$$

By symmetry, we also have

$$|E_2|=|E_2-E_1|+|E_1\cap E_2|. \tag{2}$$

Now, in a similar fashion, we can write $E_1\cup E_2$ as a union of disjoint measurable sets

$$E_1\cup E_2=(E_1-E_2)\cup(E_1\cap E_2)\cup(E_2-E_1).$$

Then

$$\begin{aligned}
|E_1\cup E_2| &= |(E_1-E_2)\cup(E_1\cap E_2)\cup(E_2-E_1)| \\
&= |E_1-E_2|+|E_1\cap E_2|+|E_2-E_1| \\
&= |E_1|-|E_1\cap E_2|+|E_1\cap E_2|+|E_2|-|E_1\cap E_2| && \text{by (1) and (2)} \\
&= |E_1|+|E_2|-|E_1\cap E_2|.
\end{aligned}$$

Adding $|E_1\cap E_2|$ to both sides gives us the desired result. □

Problem 5

Theorem 4. *Suppose that the measurable sets $A_1, A_2, \ldots$ are "almost disjoint" in the sense that $|A_i\cap A_j|=0$ if $i\neq j$. Then,*

$$\left|\bigcup_{k=1}^{\infty}A_k\right|=\sum_{k=1}^{\infty}|A_k|.$$

Solution

Proof. We will proceed by induction. As the result is trivial if any of these sets have infinite measure, we will assume they all have finite measure. Using the results of the previous problem, we have

$$\begin{aligned} A_1 \cup A_2 &= |A_1| + |A_2| - |A_1 \cap A_2| \\ &= |A_1| + |A_2|, \end{aligned}$$

and the base case holds.

Now suppose that for some $N > 1$ we have

$$\left| \bigcup_{k=1}^{N} A_k \right| = \sum_{k=1}^{N} |A_k|. \tag{3}$$

We have

$$\begin{aligned} \left| \bigcup_{k=1}^{N+1} A_k \right| &= \left| A_{N+1} \cup \left(\bigcup_{k=1}^{N} A_k \right) \right| \\ &= |A_{N+1}| + \left| \bigcup_{k=1}^{N} A_k \right| - \left| A_{N+1} \cap \left(\bigcup_{k=1}^{N} A_k \right) \right| && \text{By results of previous problem} \\ &= |A_{N+1}| + \sum_{k=1}^{N} |A_k| - \left| \bigcup_{k=1}^{N} A_k \cap A_{N+1} \right| && \text{By (3)} \\ &= \sum_{k=1}^{N+1} |A_k| - \left| \bigcup_{k=1}^{N} A_k \cap A_{N+1} \right| \\ &= \sum_{k=1}^{N+1} |A_k|. && \text{Countable union of sets of measure zero} \end{aligned}$$

Since this is true for all $N > 1$, we can conclude

$$\left| \bigcup_{k=1}^{\infty} A_k \right| = \sum_{k=1}^{\infty} |A_k|,$$

and our proof is complete. □

Problem 6

Define the inner measure of E by $|E|_i = \sup |F|$, where the supremum is taken over all closed subsets F of E. Show that (i) $|E|_i \leq |E|_e$, and (ii) if $|E|_e < +\infty$, then E is measurable if and only if $|E|_i = |E|_e$.

Proof. Part (i):

Suppose, for sake of contradiction, that $|E|_i > |E|_e$. Define $\delta > 0$ by $\delta = |E|_i - |E|_e$. By definition of supremum, there exists a closed $F \subseteq E$ such that

$$|F| > |E|_i - \delta.$$

Then, we have $|F| > |E|_e$, which contradicts the monotonicity of the outer measure. Thus, we have proven (i).

Part (ii):

Suppose that $|E|_e < +\infty$. By Lemma 3.22, we have that E is measurable if and only if for every $\epsilon > 0$, there exists a closed $F \subseteq E$ such that $|E - F| < \epsilon$.

Suppose first that E is measurable. Let $\epsilon > 0$. Then, by Lemma 3.22, there exists a closed $F \subseteq E$ such that $|E - F| < \epsilon$. Furthermore, since E and F are measurable, we know that $E - F$ is also measurable, and

$$\begin{aligned} |E| &= |E - F| + |F| \\ &< \epsilon + |F|. \end{aligned}$$

Since ϵ was arbitrary, we can conclude that $|E| = |E|_e \leq F$ for every closed $F \subseteq E$. Thus, by definition of supremum, we can conclude that $|E|_e \leq |E|_i$. Combining this with the result of part (i), we have $|E|_e = |E|_i$.

Now suppose that $|E|_e = |E|_i$, and let $\epsilon > 0$. Then, by definition of supremum, there exists a closed $F \subseteq E$ such that $|E|_e - |F| < \epsilon$. Now, using the fact that F is measurable, the Caratheodory property tells us that

$$\begin{aligned}
|E|_e &= |E \cap F|_e + |E - F|_e \\
&= |F| + |E - F|_e && \text{Since } F \subseteq E \\
|E - F|_e &= |F| - |E|_e && \text{We can do this because all three terms are finite.} \\
&< \epsilon.
\end{aligned}$$

Thus, by Lemma 3.22, E is measurable. $\square$