Problem 1

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous. Then $|f|$ is absolutely continuous on $[a, b]$, and*

$$\left| \frac{d}{dx} |f(x)| \right| \leq |f'(x)|$$

for almost every $x \in [a, b]$.

Solution

Proof. Let $\epsilon > 0$. Since f is absolutely continuous, there exists a $\delta > 0$ such for any collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$, we have

$$\sum |f(b_i) - f(a_i)| < \epsilon \text{ if } \sum (b_i - a_i) < \delta.$$

Thus, let $\{[a_i, b_i]\}$ be a collection of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$. Then, we have

$$\begin{aligned} \sum ||f(b_i)| - |f(a_i)|| &\leq \sum |f(b_i) - f(a_i)| && \text{The reverse triangle inequality} \\ &< \epsilon. \end{aligned}$$

Thus, $|f|$ is absolutely continuous.

By theorem 7.29 in our textbook, the derivatives of f and $|f|$ exists almost everywhere in $[a, b]$. Let $Z, Z' \subseteq [a, b]$ be the sets of measure zero where the derivatives of f and $|f|$ respectively do not exist. Then, $Z \cup Z'$ has measure zero, and the derivatives of $|f|$ and f exists on $[a, b] \backslash Z \cup Z'$. If we let $x \in [a, b] \backslash Z \cup Z'$, then we have

$$\begin{aligned} \left| \frac{d}{dx} |f(x)| \right| &= \left| \lim_{n \to 0} \frac{|f(x + n)| - |f(x)|}{n} \right| \\ &= \lim_{n \to 0} \frac{||f(x + n)| - |f(x)||}{|n|} \\ &\leq \lim_{n \to 0} \frac{|f(x + n)| - |f(x)|}{|n|} && \text{Reverse triangle inequality} \\ &= \left| \lim_{n \to 0} \frac{f(x + n) - f(x)}{n} \right| \\ &= |f'(x)|, \end{aligned}$$

as desired. □

Problem 2

Use Fubini's theorem and the relation

$$\frac{1}{x} = \int_0^\infty e^{-xy} dy$$

to prove that

$$\lim_{A \to \infty} \int_0^A \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Solution

Proof. Consider the function

$$f(x, y) = \sin(x) e^{-xy}.$$

We have that f is continuous, which implies that f is initegrable on $(0, A) \times (0, \infty)$. Thus, by Fubini's theorem, we have

$$\int_0^A \frac{\sin x}{x} dx = \int_0^\infty \left(\int_0^A \sin(x) e^{-xy} dx \right) dy.$$

By performing integration by parts twice, or by recognizing that

$$\sin(x)e^{-xy} = -\frac{1}{1+y^2}\frac{d}{dx}(y\sin(x)e^{-xy} + \cos(x)e^{-xy}),$$

we have

$$\begin{aligned}\int_0^A \sin(x)e^{-xy}dx &= \int_0^A \left(-\frac{1}{1+y^2}\frac{d}{dx}(y\sin(x)e^{-xy} + \cos(x)e^{-xy})\right)dx \\ &= \frac{1}{1+y^2}[1 - e^{-Ay}(y\sin A + \cos A)].\end{aligned}$$

Plugging this into our original integral, we have

$$\begin{aligned}\int_0^\infty \left(\int_0^A \sin(x)e^{-xy}dx\right)dy &= \int_0^\infty \frac{1}{1+y^2}[1 - e^{-Ay}(y\sin A + \cos A)]dy \\ &= \int_0^\infty \frac{1}{1+y^2}dy - \int_0^\infty \frac{e^{-Ay}(y\sin A + \cos A)}{1+y^2}dy \\ &= \tan^{-1}(y)\big|_0^\infty - \int_0^\infty \frac{e^{-Ay}(y\sin A + \cos A)}{1+y^2}dy \\ &= \frac{\pi}{2} - \int_0^\infty \frac{e^{-Ay}(y\sin A + \cos A)}{1+y^2}dy.\end{aligned}$$

Thus, we have

$$\lim_{A\to\infty}\int_0^A \frac{\sin x}{x}dx = \frac{\pi}{2} - \lim_{A\to\infty}\int_0^\infty \frac{e^{-Ay}(y\sin A + \cos A)}{1+y^2}dy.$$

We can see that

$$\left|\frac{e^{-Ay}(y\sin A + \cos A)}{1+y^2}\right| \le e^{-Ay}(y+1).$$

Thus, since

$$\int_0^\infty e^{-Ay}(y+1)dy = \frac{1}{A} + \frac{1}{A^2},$$

we have that $e^{-Ay}(y+1)$ is integrable. By the Lebesgue dominated convergence theorem,

$$\begin{aligned}\lim_{A\to\infty}\int_0^\infty \frac{e^{-Ay}(y\sin A + \cos A)}{1+y^2}dy &= \int_0^\infty \lim_{A\to\infty}\frac{e^{-Ay}(y\sin A + \cos A)}{1+y^2}dy \\ &= \int_0^\infty (0)dy \\ &= 0.\end{aligned}$$

With this, our proof is complete. □

Problem 3

Theorem 2. *Let $E = [1,\infty)$ and $f \in L^2(E)$. Assume that $f \ge 0$ almost every where on E. Let*

$$g(x) = \int_E f(y)e^{-xy}dy$$

for all $x \in E$. Then $g \in L^1(E)$ and

$$||g||_1 \le c||f||_2$$

for some $c < 1$. (Also Estimate c).

Solution

Proof. We have

$$\begin{aligned}
||g||_1 &= \int_E |g| dx \\
&= \int_1^\infty \left| \int_E f(y) e^{-xy} dy \right| dx \\
&= \int_1^\infty \int_1^\infty f(y) e^{-xy} dy dx \\
&= \int_1^\infty f(y) \left(\int_0^\infty e^{-xy} dx - \int_0^1 e^{-xy} dx \right) dy \\
&= \int_1^\infty f(y) \left(\frac{1}{y} - \int_0^1 e^{-xy} dx \right) dy \\
&= \int_1^\infty f(y) \frac{1}{y} dy - \int_1^\infty \left(\int_0^1 f(y) e^{-xy} dx \right) dy \\
&\leq ||f||_2 \cdot \sqrt{\int_1^\infty y^{-2} dy} - \int_1^\infty \left(\int_0^1 f(y) e^{-xy} dx \right) dy && \text{Cauchy-Schwarz's} \\
&= ||f||_2 - \int_1^\infty \left(\int_0^1 f(y) e^{-xy} dx \right) \\
&= ||f||_2 \left(1 - \frac{1}{||f||_2} \int_1^\infty \left(\int_0^1 f(y) e^{-xy} dx \right) dy \right).
\end{aligned}$$

Thus, we have shown that there is some $c < 1$ such that $|g||_1 \leq c||f||_2$.

To estimate c, let's find a lower bound on c. Finding a lower bound on c can be achieved by finding an upper bound on the integral in that last equation. We have

$$\begin{aligned}
\int_1^\infty \left(\int_0^1 f(y) e^{-xy} dx \right) dy &= \int_1^\infty f(y) \frac{1}{y} (1 - e^{-y}) dy \\
&\leq ||f||_2 \cdot \sqrt{\int_1^\infty \left(\frac{1 - e^{-y}}{y} \right)^2 dy} && \text{Holder's Inequality} \\
&< ||f||_2 \cdot \sqrt{\frac{3}{4}} && \text{Wolfram alpha for computing integral} \\
&= ||f||_2 \cdot \frac{\sqrt{3}}{2}.
\end{aligned}$$

Thus, we have

$$||g||_1 < \left(1 - \frac{\sqrt{3}}{2} \right) \cdot ||f||_2.$$

□

Problem 4

Theorem 3. *Let $f \in L^1(\mathbb{R})$ and*

$$g(x) = \int_{\mathbb{R}} f(y) e^{-(x-y)^2} dy$$

for $x \in \mathbb{R}$. Then $g \in L^p(\mathbb{R})$ for any $1 \leq p \leq \infty$, and

$$||g||_p \leq \left(\frac{\pi}{p} \right)^{1/p} \cdot ||f||_1$$

for $1 \leq p < \infty$.

Solution

Proof. From theorem 9.1 in our textbook, we have $g \in L^p(\mathbb{R})$, and

$$\begin{aligned} ||g||_p &\leq ||f||_1 \cdot ||e^{-(x-y)^2}||_p \\ &= ||f||_1 \cdot \left(\int_{\mathbb{R}^2} e^{-p(x-y)^2} \right)^{1/p} \\ &= ||f||_1 \cdot \left(\int_{\mathbb{R}^2} e^{-px^2} e^{-py^2} \right)^{1/p} \\ &= ||f||_1 \cdot \left(\int_{\mathbb{R}} e^{-px^2} dx \int_{\mathbb{R}} e^{-py^2} dy \right)^{1/p} \\ &= ||f||_1 \cdot \left(\sqrt{\frac{\pi}{p}} \sqrt{\frac{\pi}{p}} \right)^{1/p} \\ &= \left(\frac{\pi}{p} \right)^{1/p} \cdot ||f||_1. \end{aligned}$$

□

Problem 5

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be continuous and satisfy the midpoint convexity*

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}.$$

for any $x_1, x_2 \in [a,b]$. Then f is convex on $[a,b]$.

Solution

Proof. We will first prove, that for any $n \in \mathbb{N}$, if $x_1, ..., x_{2^n} \in [a,b]$, then

$$f\left(\frac{x_1 + ... + x_{2^n}}{2^n}\right) \leq \frac{f(x_1) + ... + f(x_{2^n})}{2^n}.$$

To prove this, we note that the base case ($n = 1$) is true by midpoint convexity of f. Now, suppose for some $n \in \mathbb{N}$ that if $x_1, ..., x_{2^n} \in [a,b]$, then

$$f\left(\frac{x_1 + ... + x_{2^n}}{2^n}\right) \leq \frac{f(x_1) + ... + f(x_{2^n})}{2^n}.$$

Let $x_1, ..., x_{2^{n+1}} \in [a,b]$. Then, we have

$$\begin{aligned} f\left(\frac{x_1 + ... + x_{2^{n+1}}}{2^{n+1}}\right) &= f\left(\frac{\frac{x_1 + ... + x_{2^n}}{2^n} + \frac{x_{2^n+1} + ... + x_{2^{n+1}}}{2^n}}{2}\right) \\ &\leq \frac{f\left(\frac{x_1 + ... + x_{2^n}}{2^n}\right) + f\left(\frac{x_{2^n+1} + ... + x_{2^{n+1}}}{2^n}\right)}{2} \\ &= \frac{f(x_1) + ... + f(x_{2^{n+1}})}{2^{n+1}}, \end{aligned}$$

and our induction is complete.

From elementary analysis, we have that rational numbers of the form $\frac{m}{2^n}$ with $1 \leq m \leq 2^n$ are dense in $[0,1]$. Let $[a_1, b_2] \subseteq [a,b]$. Fix some $n \in \mathbb{N}$ and some $1 \leq m \leq 2^n$. Setting $x_i = a_i$ for $1 \leq i \leq m$, and $x_i = b_i$ for $m + 1 \leq i \leq 2^n$, we see from the above argument that

$$\begin{aligned} f\left(\frac{x_1 + ... + x_{2^n}}{2^n}\right) &\leq \frac{f(x_1) + ... + f(x_{2^n})}{2^n} \\ &= \frac{mf(a_i) + (2^n - m)f(b_i)}{2^n} \\ &= \frac{m}{2^n} f(a_i) + (1 - \frac{m}{2^n}) f(b_i). \end{aligned}$$

Finally, let $\theta \in [0,1]$ be a real number. For each $n \in \mathbb{N}$, define $1 \le m_n \le 2^n$ to be the largest number such that $\frac{m_n}{2^n} \ge \theta$. Then, $\frac{m_n}{2^n} \to \theta$, and it follows from the continuity of f that

$$f(\theta a_i + (1-\theta)b_i) \le \theta f(a_i) + (1-\theta)f(b_i),$$

and we have shown that f is convex. $\square$

Problem 6

Theorem 5. *If $f_k \to f$ in $L^p(\mathbb{R}^n), 1 < p < \infty$, $g_k \to g$ pointwise, and $||g_k||_\infty \le M$ for all k, then*

$$f_k g_k \to fg$$

in $L^p(\mathbb{R}^n)$.

Solution

Proof. We have

$$\begin{aligned} ||fg - f_k g_k||_p &= ||fg - g_k f + g_k f - f_k g_k||_p \\ &\le ||fg - g_k f||_p + ||g_k f - f_k g_k||_p \\ &\le ||fg - g_k f||_p + ||g_k(f - f_k)||_p \\ &\le ||fg - g_k f||_p + ||M(f - f_k)||_p \\ &= ||fg - g_k f||_p + M||f - f_k||_p. \end{aligned}$$

Since $f_k \to f$ in $L^p(\mathbb{R}^n)$, the second term in the last equation will vanish as $k \to \infty$. Thus, we must show that $||fg - g_k f||_p \to 0$. We have

$$\begin{aligned} ||fg - g_k f||_p^p &= ||f(g - g_k)||_p^p \\ &= \int_{\mathbb{R}} |f(g - g_k)|^p \\ &= \int_{\mathbb{R}} |g - g_k|^p |f|^p. \end{aligned}$$

Now, $|g - g_k|^p |f|^p \le (2M)^p |f|^p \in L(\mathbb{R}^n)$, thus, by the lebesgue dominated convergence theorem, we have

$$\begin{aligned} \lim_{k\to\infty} \int_{\mathbb{R}^n} |g - g_k|^p |f|^p &= \int_{\mathbb{R}^n} \lim_{k\to\infty} |g - g_k|^p |f|^p \\ &= 0, \end{aligned}$$

and our proof is complete. $\square$

Problem 7

Theorem 6. *Suppose $f_k \to f$ a.e. on $\mathbb{R}^n$ and that $f_k, f \in L^p(\mathbb{R}^n)$, for $1 < p < \infty$. If $||f_k||_p \le M < \infty$, then*

$$f_k g \to fg$$

in $L^1(\mathbb{R}^n)$ for all $g \in L^{p'}(\mathbb{R}^n)$, with

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Solution

Proof. We have

$$\begin{aligned} ||f_k g - fg||_1 &= ||g(f_k - f)||_p \\ &= ||g||_{p'} \cdot ||f_k - f||_p && \text{Holder's inequality.} \end{aligned}$$

Thus, it will suffice to show that $||f_k - f||_p \to 0$. We have

$$\begin{aligned} |f_k - f|^p &\leq (2\max\{|f|, |f_k|\})^p \\ &= 2^p \max\{|f|^p, |f_k|^p\} \\ &\leq 2^p(|f|^p + |f_k|^p). \end{aligned}$$

Now,

$$\begin{aligned} \int_{\mathbb{R}^n} 2^p(|f|^p + |f_k|^p) &= 2^p \int_{\mathbb{R}^n} |f|^p + 2^p \int_{\mathbb{R}^n} |f_k|^p \\ &= 2^p||f||_p^p + 2^p||f_k||_p^p \\ &\leq 2^p||f||_p^p + 2^p M^p. \end{aligned}$$

Thus, we can use the Lebesgue dominated convergence theorem to yield

$$\begin{aligned} \lim_{k\to\infty} \int_{\mathbb{R}^n} |f_k - f|^p &= \int_{\mathbb{R}^n} \lim_{k\to\infty} |f_k - f|^p \\ &= 0, \end{aligned}$$

and our proof is complete. $\square$