Problem 1.6

Theorem 1. *Let $\alpha > 0$. Then $f(x) = x^\alpha$ is absolutely continuous on every subinterval $[a, b] \subseteq [0, \infty)$.*

Solution

Proof. We have that f is differentiable on $(0, \infty)$ with derivative

$$f'(x) = \alpha x^{\alpha-1}.$$

Thus, f is differentiable almost everywhere on $[0, \infty)$. Now, let $[a, b] \subseteq [0, \infty)$. Since f' is continuous a.e. on $[a, b]$, f' is integrable on $[a, b]$. Furthermore, we have for any $x \in [a, b]$,

$$\begin{aligned}\int_a^x f'(x)dx &= \int_a^x \alpha x^{\alpha-1}dx \\ &= \alpha x^\alpha\big|_a^x \\ &= b^\alpha - x^\alpha \\ &= f(x) - f(a).\end{aligned}$$

Thus, by theorem 7.29 in our textbook, we have that f is absolutely continuous on any subinterval of $[0, \infty)$. □

Problem 1.7

Theorem 2. *A function f is absolutely continuous on $[a, b]$ if and only if given $\epsilon > 0$, there exists $\delta > 0$ such that $|\sum[f(b_i) - f(a_i)]| < \epsilon$ for any finite collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$.*

Solution

Proof. Suppose f is absolutely continuous on $[a, b]$, and let $\epsilon > 0$. Since f is absolutely continuous, there exists a $\delta > 0$ such that $\sum |[f(b_i) - f(a_i)]| < \epsilon$ for any finite collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$. Thus, if we let $\{[a_i, b_i]\}$ be a set of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$, we have

$$\begin{aligned}\epsilon &> \sum |[f(b_i) - f(a_i)]| \\ &\geq |\sum[f(b_i) - f(a_i)]|, && \text{Basic property of absolute value}\end{aligned}$$

and we have proven the forward direction.

Now, suppose that if we are given $\epsilon > 0$, there exists $\delta > 0$ such that $|\sum[f(b_i) - f(a_i)]| < \epsilon$ for any finite collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$. Let $\epsilon > 0$, and choose $\delta > 0$ such that $|\sum[f(b_i) - f(a_i)]| < \frac{\epsilon}{2}$ for any finite collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$. Let $\{[a_i, b_i]\}$ be a set of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$. We have

$$\sum_{i \in \{i : f(b_i) \geq f(a_i)\}} (b_i - a_i) < \delta,$$

which means that

$$\begin{aligned}\frac{\epsilon}{2} &> \left| \sum_{i \in \{i : f(b_i) \geq f(a_i)\}} [f(b_i) - f(a_i)] \right| \\ &= \sum_{i \in \{i : f(b_i) \geq f(a_i)\}} |f(b_i) - f(a_i)|.\end{aligned}$$

Similarly, we have

$$\sum_{i \in \{i : f(b_i) < f(a_i)\}} (b_i - a_i) < \delta,$$

which implies

$$\begin{aligned}\frac{\epsilon}{2} &> \left| \sum_{i \in \{i : f(b_i) < f(a_i)\}} [f(b_i) - f(a_i)] \right| \\ &= \sum_{i \in \{i : f(b_i) < f(a_i)\}} |f(b_i) - f(a_i)|.\end{aligned}$$

Finally, we have

$$\begin{aligned}\sum_i |f(b_i) - f(a_i)| &= \sum_{i \in \{i : f(b_i) < f(a_i)\}} |f(b_i) - f(a_i)| + \sum_{i \in \{i : f(b_i) \geq f(a_i)\}} |f(b_i) - f(a_i)| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon,\end{aligned}$$

and we have shown that f is absolutely continuous. With this, our proof is complete. □

Problem 1.8

Theorem 3. *If f is of bounded variation on $[a, b]$, and if the function $V(x) = V[a, x]$ is absolutely continuous on $[a, b]$, then f is absolutely continuous on $[a, b]$.*

Solution

Proof. Let $\epsilon > 0$. Since $V(x)$ is absolutely continuous, we have that there exists $\delta > 0$ such that $\sum |V(b_i) - V(a_i)| < \epsilon$ for any finite collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$. Let $\{[a_i, b_i]\}$ be a collection of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$. From Theorem 2.2 (part i) in our textbook, since f is of bounded variation, and $V(x)$ is finite for all $x \in [a, b]$. We have

$$\begin{aligned}\epsilon &> \sum |V(b_i) - V(a_i)| \\ &= \sum (V[a, b_i] - V[a, a_i]) \\ &\geq \sum V[a, b_i] \\ &\geq \sum V[a_i, b_i] && \text{Theorem 2.2 part i} \\ &\geq \sum |f(b_i) - f(a_i)|,\end{aligned}$$

and we have proven that f is absolutely continuous. □

Problem 1.9

Theorem 4. *If f is of bounded variation on $[a, b]$, then*

$$\int_a^b |f'| \leq V[a, b].$$

Furthermore, if the equality holds in this inequality, then f is absolutely continuous.

Solution

Proof. Let $N(x)$ and $P(x)$ denote the negative and positive variations of f on $[a, x]$, as in the proof of theorem 2.7 in our textbook. Then, we have

$$f(x) = [P(x) + f(a)] - N(x).$$

We note, that $P(x) + f(a)$ and $N(x)$ are increasing functions. Now, we have

$$\begin{aligned}
\int_a^b |f'| &= \int_a^b |P'(x) - N'(x)| dx \\
&\leq \int_a^b P'(x) + \int_a^b N'(x) \\
&\leq P(b) - P(a) + N(b) - N(a) && \text{By theorem 7.21 in our textbook} \\
&= V(b) - V(a) && \text{By theorem 2.6 in our textbook} \\
&\leq V(b) \\
&= V[a, b].
\end{aligned}$$

Now, suppose the equality holds. That is, suppose

$$\int_a^b |f'| = V[a, b].$$

By theorem 7.24 in our textbook, we have $V'(x) = |f'(x)|$ almost everywhere for $x \in [a, b]$. Thus, we have

$$\begin{aligned}
\int_a^x V'(t) dt &= \int_a^x |f'(t)| dt \\
&= V(x) \\
&= V[a, x] \\
&= V[a, x] + V[a, a] && \text{By theorem 2.2 ii} \\
&= V(x) - V(a).
\end{aligned}$$

Thus, by theorem 7.29, $V(x)$ is absolutely continuous. By the statement we proved in the previous problem, we can conclude that f is absolutely continuous. ☐

Problem 1.10

Theorem 5. Part (a):
If f is absolutely continuous on $[a, b]$ and Z is a subset of $[a, b]$ with measure zero, then the image set defined by $f(Z) = \{w : w = f(z), z \in Z\}$ also has measure zero. Deduce that the image under f of any measurable subset of $[a, b]$ is measurable. (Compare theorem 3.33)(Hint: use the fact that the image of an interval $[a_i, b_i]$ is an interval of length at most $V(b_i) - V(a_i)$.)

Part (b):
Give an example of a strictly increasing Lipschitz continuous function f and a set Z with measure 0 such that $f^{-1}(Z)$ does not have measure 0 (and cosequently, f^{-1} is not absolutely continuous). (Let $f^{-1}(x) = x + C(x)$ on $[0, 1]$, where $C(x)$ is the Cantor-Lebesgue function.

Solution

Before we jump into the proof, we will prove a usefull lemma.

Lemma 1. *If f is an absolutely continuous function on $[a, b]$, and $[a_i, b_i] \subseteq [a, b]$, then the image of $[a_i, b_i]$ under f is an interval with $f([a_i, b_i]) \leq V(b_i) - V(a_i)$.*

Proof. Since f is continuous, it follows immediately from the intermediate value theorem that $f([a_i, b_i])$ is an interval. By Theorem 7.27 in our textbook, we have that f is of bounded variation. Thus, by theorem 2.2, we have

$$\begin{aligned}
V(b_i) - V(a_i) &= V[a, b_i] - V[a, a_i] \\
&= V[a_i, b_i].
\end{aligned}$$

By the extreme value theorem, there exist $c, d \in [a_i, b_i]$ such that the minimum and maximum values of f on $[a_i, b_i]$ are attained at c and d respectively. Define a partition of $[a_i, b_i]$ by

$$T = \{a_i, c, d, b_i\}.$$

Then

$$\begin{aligned} V[a_i, b_i] &\geq V([a_i, b_i], T) \\ &\geq |f(d) - f(c)| \\ &= |[f(c), f(d)]| && = |f([a_i, b_i])|, \end{aligned}$$

and we have proven the lemma. □

Now we are ready for the main proof.

Proof. Part (a):
Let $\epsilon > 0$. Since f is absolutely continuous on $[a, b]$, theorem 7.31 tells us that $V(x)$ is absolutely continuous on $[a, b]$. Thus, there exists a $\delta > 0$ such that $\sum |V(b_i) - V(a_i)| < \epsilon$ for any countable collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$. Define an open set G such that $[a, b] \subseteq G$, with $|G| < \delta$. Since G is open, by theorem 1.11 in our textbook, there exists a countable collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$ whose union is $[a_i, b_i]$. Thus, since $\sum(b_i - a_i) < \delta$, we have

$$\begin{aligned} \epsilon &> \sum |V(b_i) - V(a_i)| \\ &\geq \sum |f([a_i, bi])| && \text{By Lemma 1} \\ &\geq \left| \bigcup f([a_i, b_i]) \right| \\ &\geq |f(G)| \\ &\geq |f(Z)|. \end{aligned}$$

Since this is true for all $\epsilon < 0$, we have that $|f(Z)| = 0$, and we have shown that f maps sets of measure zero to sets of measure zero.

Now, let E be any measurable set. Then, we use theorem 3.28 in our textbook write $E = H \cup Z$, where H is a set of type F_σ and Z is of measure zero. Since $f(E) = f(H) \cup f(Z)$, we have that TE is the the union of two measurable sets, and is therefore measurable. □