

## Problem 1.6

**Theorem 1.** Let  $\alpha > 0$ . Then  $f(x) = x^\alpha$  is absolutely continuous on every subinterval  $[a, b] \subseteq [0, \infty)$ .

### Solution

*Proof.* We have that  $f$  is differentiable on  $(0, \infty)$  with derivative

$$f'(x) = \alpha x^{\alpha-1}.$$

Thus,  $f$  is differentiable almost everywhere on  $[0, \infty)$ . Now, let  $[a, b] \subseteq [0, \infty)$ . Since  $f'$  is continuous a.e. on  $[a, b]$ ,  $f'$  is integrable on  $[a, b]$ . Furthermore, we have for any  $x \in [a, b]$ ,

$$\begin{aligned} \int_a^x f'(x) dx &= \int_a^x \alpha x^{\alpha-1} dx \\ &= \alpha x^\alpha \Big|_a^x \\ &= b^\alpha - a^\alpha \\ &= f(b) - f(a). \end{aligned}$$

Thus, by theorem 7.29 in our textbook, we have that  $f$  is absolutely continuous on any subinterval of  $[0, \infty)$ .  $\square$

## Problem 1.7

**Theorem 2.** A function  $f$  is absolutely continuous on  $[a, b]$  if and only if given  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $|\sum [f(b_i) - f(a_i)]| < \epsilon$  for any finite collection  $\{[a_i, b_i]\}$  of nonoverlapping subintervals of  $[a, b]$  with  $\sum (b_i - a_i) < \delta$ .

### Solution

*Proof.* Suppose  $f$  is absolutely continuous on  $[a, b]$ , and let  $\epsilon > 0$ . Since  $f$  is absolutely continuous, there exists a  $\delta > 0$  such that  $\sum |f(b_i) - f(a_i)| < \epsilon$  for any finite collection  $\{[a_i, b_i]\}$  of nonoverlapping subintervals of  $[a, b]$  with  $\sum (b_i - a_i) < \delta$ . Thus, if we let  $\{[a_i, b_i]\}$  be a set of nonoverlapping subintervals of  $[a, b]$  with  $\sum (b_i - a_i) < \delta$ , we have

$$\begin{aligned} \epsilon &> \sum |f(b_i) - f(a_i)| \\ &\geq \left| \sum [f(b_i) - f(a_i)] \right|, \end{aligned} \quad \text{Basic property of absolute value}$$

and we have proven the forward direction.

Now, suppose that if we are given  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $|\sum [f(b_i) - f(a_i)]| < \epsilon$  for any finite collection  $\{[a_i, b_i]\}$  of nonoverlapping subintervals of  $[a, b]$  with  $\sum (b_i - a_i) < \delta$ . Let  $\epsilon > 0$ , and choose  $\delta > 0$  such that  $|\sum [f(b_i) - f(a_i)]| < \frac{\epsilon}{2}$  for any finite collection  $\{[a_i, b_i]\}$  of nonoverlapping subintervals of  $[a, b]$  with  $\sum (b_i - a_i) < \delta$ . Let  $\{[a_i, b_i]\}$  be a set of nonoverlapping subintervals of  $[a, b]$  with  $\sum (b_i - a_i) < \delta$ . We have

$$\sum_{i \in \{i: f(b_i) \geq f(a_i)\}} (b_i - a_i) < \delta,$$

which means that

$$\begin{aligned} \frac{\epsilon}{2} &> \left| \sum_{i \in \{i: f(b_i) \geq f(a_i)\}} [f(b_i) - f(a_i)] \right| \\ &= \sum_{i \in \{i: f(b_i) \geq f(a_i)\}} |f(b_i) - f(a_i)|. \end{aligned}$$

Similarly, we have

$$\sum_{i \in \{i: f(b_i) < f(a_i)\}} (b_i - a_i) < \delta,$$

which implies

$$\begin{aligned} \frac{\epsilon}{2} &> \left| \sum_{i \in \{i: f(b_i) < f(a_i)\}} [f(b_i) - f(a_i)] \right| \\ &= \sum_{i \in \{i: f(b_i) < f(a_i)\}} |f(b_i) - f(a_i)|. \end{aligned}$$

Finally, we have

$$\begin{aligned} \sum_i |f(b_i) - f(a_i)| &= \sum_{i \in \{i: f(b_i) < f(a_i)\}} |f(b_i) - f(a_i)| + \sum_{i \in \{i: f(b_i) \geq f(a_i)\}} |f(b_i) - f(a_i)| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$

and we have shown that  $f$  is absolutely continuous. With this, our proof is complete.  $\square$

## Problem 1.8

**Theorem 3.** *If  $f$  is of bounded variation on  $[a, b]$ , and if the function  $V(x) = V[a, x]$  is absolutely continuous on  $[a, b]$ , then  $f$  is absolutely continuous on  $[a, b]$ .*

### Solution

*Proof.* Let  $\epsilon > 0$ . Since  $V(x)$  is absolutely continuous, we have that there exists  $\delta > 0$  such that  $\sum |V(b_i) - V(a_i)| < \epsilon$  for any finite collection  $\{[a_i, b_i]\}$  of nonoverlapping subintervals of  $[a, b]$  with  $\sum (b_i - a_i) < \delta$ . Let  $\{[a_i, b_i]\}$  be a collection of nonoverlapping subintervals of  $[a, b]$  with  $\sum (b_i - a_i) < \delta$ . From Theorem 2.2 (part i) in our textbook, since  $f$  is of bounded variation, and  $V(x)$  is finite for all  $x \in [a, b]$ . We have

$$\begin{aligned} \epsilon &> \sum |V(b_i) - V(a_i)| \\ &= \sum (V[a, b_i] - V[a, a_i]) \\ &\geq \sum V[a, b_i] \\ &\geq \sum V[a_i, b_i] && \text{Theorem 2.2 part i} \\ &\geq \sum |f(b_i) - f(a_i)|, \end{aligned}$$

and we have proven that  $f$  is absolutely continuous.  $\square$

## Problem 1.9

**Theorem 4.** *If  $f$  is of bounded variation on  $[a, b]$ , then*

$$\int_a^b |f'| \leq V[a, b].$$

*Furthermore, if the equality holds in this inequality, then  $f$  is absolutely continuous.*

### Solution

*Proof.* Let  $N(x)$  and  $P(x)$  denote the negative and positive variations of  $f$  on  $[a, x]$ , as in the proof of theorem 2.7 in our textbook. Then, we have

$$f(x) = [P(x) + f(a)] - N(x).$$

We note, that  $P(x) + f(a)$  and  $N(x)$  are increasing functions. Now, we have

$$\begin{aligned}
 \int_a^b |f'| &= \int_a^b |P'(x) - N'(x)| dx \\
 &\leq \int_a^b P'(x) + \int_a^b N'(x) \\
 &\leq P(b) - P(a) + N(b) - N(a) && \text{By theorem 7.21 in our textbook} \\
 &= V(b) - V(a) && \text{By theorem 2.6 in our textbook} \\
 &\leq V(b) \\
 &= V[a, b].
 \end{aligned}$$

Now, suppose the equality holds. That is, suppose

$$\int_a^b |f'| = V[a, b].$$

By theorem 7.24 in our textbook, we have  $V'(x) = |f'(x)|$  almost everywhere for  $x \in [a, b]$ . Thus, we have

$$\begin{aligned}
 \int_a^x V'(t) dt &= \int_a^x |f'(t)| dt \\
 &= V(x) \\
 &= V[a, x] \\
 &= V[a, x] + V[a, a] && \text{By theorem 2.2 ii} \\
 &= V(x) - V(a).
 \end{aligned}$$

Thus, by theorem 7.29,  $V(x)$  is absolutely continuous. By the statement we proved in the previous problem, we can conclude that  $f$  is absolutely continuous.  $\square$

## Problem 1.10

**Theorem 5.** Part (a):

If  $f$  is absolutely continuous on  $[a, b]$  and  $Z$  is a subset of  $[a, b]$  with measure zero, then the image set defined by  $f(Z) = \{w : w = f(z), z \in Z\}$  also has measure zero. Deduce that the image under  $f$  of any measurable subset of  $[a, b]$  is measurable. (Compare theorem 3.33) (Hint: use the fact that the image of an interval  $[a_i, b_i]$  is an interval of length at most  $V(b_i) - V(a_i)$ .)

Part (b):

Give an example of a strictly increasing Lipschitz continuous function  $f$  and a set  $Z$  with measure 0 such that  $f^{-1}(Z)$  does not have measure 0 (and consequently,  $f^{-1}$  is not absolutely continuous). (Let  $f^{-1}(x) = x + C(x)$  on  $[0, 1]$ , where  $C(x)$  is the Cantor-Lebesgue function.

## Solution

Before we jump into the proof, we will prove a useful lemma.

**Lemma 1.** If  $f$  is an absolutely continuous function on  $[a, b]$ , and  $[a_i, b_i] \subseteq [a, b]$ , then the image of  $[a_i, b_i]$  under  $f$  is an interval with  $f([a_i, b_i]) \subseteq V(b_i) - V(a_i)$ .

*Proof.* Since  $f$  is continuous, it follows immediately from the intermediate value theorem that  $f([a_i, b_i])$  is an interval. By Theorem 7.27 in our textbook, we have that  $f$  is of bounded variation. Thus, by theorem 2.2, we have

$$\begin{aligned}
 V(b_i) - V(a_i) &= V[a, b_i] - V[a, a_i] \\
 &= V[a_i, b_i].
 \end{aligned}$$

By the extreme value theorem, there exist  $c, d \in [a_i, b_i]$  such that the minimum and maximum values of  $f$  on  $[a_i, b_i]$  are attained at  $c$  and  $d$  respectively. Define a partition of  $[a_i, b_i]$  by

$$T = \{a_i, c, d, b_i\}.$$

Then

$$\begin{aligned} V[a_i, b_i] &\geq V([a_i, b_i], T) \\ &\geq |f(d) - f(c)| \\ &= |[f(c), f(d)]| = |f([a_i, b_i])|, \end{aligned}$$

and we have proven the lemma.  $\square$

Now we are ready for the main proof.

*Proof. Part (a):*

Let  $\epsilon > 0$ . Since  $f$  is absolutely continuous on  $[a, b]$ , theorem 7.31 tells us that  $V(x)$  is absolutely continuous on  $[a, b]$ . Thus, there exists a  $\delta > 0$  such that  $\sum |V(b_i) - V(a_i)| < \epsilon$  for any countable collection  $\{[a_i, b_i]\}$  of nonoverlapping subintervals of  $[a, b]$  with  $\sum (b_i - a_i) < \delta$ . Define an open set  $G$  such that  $[a, b] \subseteq G$ , with  $|G| < \delta$ . Since  $G$  is open, by theorem 1.11 in our textbook, there exists a countable collection  $\{[a_i, b_i]\}$  of nonoverlapping subintervals of  $[a, b]$  whose union is  $[a, b]$ . Thus, since  $\sum (b_i - a_i) < \delta$ , we have

$$\begin{aligned} \epsilon &> \sum |V(b_i) - V(a_i)| \\ &\geq \sum |f([a_i, b_i])| && \text{By Lemma 1} \\ &\geq \left| \bigcup f([a_i, b_i]) \right| \\ &\geq |f(G)| \\ &\geq |f(Z)|. \end{aligned}$$

Since this is true for all  $\epsilon > 0$ , we have that  $|f(Z)| = 0$ , and we have shown that  $f$  maps sets of measure zero to sets of measure zero.

Now, let  $E$  be any measurable set. Then, we use theorem 3.28 in our textbook write  $E = H \cup Z$ , where  $H$  is a set of type  $F_\sigma$  and  $Z$  is of measure zero. Since  $f(E) = f(H) \cup f(Z)$ , we have that  $f(E)$  is the union of two measurable sets, and is therefore measurable.

*Part (b):*

Let  $g : [0, 1] \rightarrow [0, 2]$  be defined by

$$g(x) = x + C(x),$$

where  $C : [0, 1] \rightarrow [0, 1]$  is the Cantor Lebesgue function. This function is injective, thus its inverse function  $f : [0, 2] \rightarrow [0, 1]$  is well defined. Since  $g$  is strictly increasing, we have that  $f$  is strictly increasing. We will first show that  $f$  is Lipschitz continuous. Let  $x, y \in [0, 2]$  with  $x < y$ . Then, we have

$$\begin{aligned} \left| \frac{y - x}{f(y) - f(x)} \right| &= \frac{y - x}{f(y) - f(x)} && \text{Since } f \text{ is increasing} \\ &= \frac{g(f(y)) - g(f(x))}{f(y) - f(x)} \\ &= \frac{f(y) - f(x)}{f(y) - f(x)} + \frac{C(f(y)) - C(f(x))}{f(y) - f(x)} \\ &= 1 + \frac{C(f(y)) - C(f(x))}{f(y) - f(x)} \\ &\geq 1 && \text{Since the Cantor Lebesgue function is non decreasing.} \end{aligned}$$

Thus, we have

$$|f(y) - f(x)| \leq |y - x|,$$

and we can conclude  $f$  is Lipschitz continuous.

Now, let  $\mathcal{C}$  be the Cantor set, and consider the set  $[0, 1] \setminus \mathcal{C}$ . We have that  $[0, 1] \setminus \mathcal{C}$  is the sum of countably many disjoint intervals  $\{I_i\}$ , and  $C$  is constant on each one of these intervals. Let  $C(x) = c_i$  for  $x \in I_i$ . Then, we have

$$g(I_i) = \{x + c_i : x \in I_i\}.$$

Thus, by translation invariance of the lebesgue outer measure, we have

$$|g(I_i)|_e = |I_i|.$$

With this, we have

$$\begin{aligned} |g([0, 1] \setminus \mathcal{C})|_e &= \left| g \left( \bigcup I_i \right) \right| \\ &= \left| \bigcup g(I_i) \right|_e \\ &= \sum_{i=1}^{\infty} |g(I_i)|_e \\ &= 1. \end{aligned}$$

Now, we know that  $|\mathcal{C}| = 0$ . Furthermore, we have  $g([0, 1]) = [0, 2]$ . Finally, we have

$$g([0, 1]) = g(\mathcal{C}) \cup g([0, 1] \setminus \mathcal{C}),$$

and by theorem 3.34 in our textbook, we can conclude that  $|g(\mathcal{C})| = 1$ .  $\square$

## Problem 2

**Theorem 6.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be monotone, and suppose  $f'(x)$  exists and is finite at every  $x \in [a, b]$ . Then  $f$  is absolutely continuous.

### Solution

*Proof.* Proof is under development.  $\square$

## Problem 3

**Theorem 7.** A function  $f : [a, b] \rightarrow \mathbb{R}$  is Lipschitz continuous on  $[a, b]$  if and only if  $f$  is an indefinite integral of a bounded measurable function on  $[a, b]$ .

### Solution

We will start with a useful lemma.

**Lemma 2.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be Lipschitz continuous on  $[a, b]$ . Then, the derivative of  $f$  is bounded.

*Proof.* Since  $f$  is Lipschitz continuous on  $[a, b]$ , there exists an  $M > 0$  such that for all  $x, h \in [a, b]$ , where  $f'(x)$  exists and  $x + h \in [a, b]$ , we have

$$|f(x + h) - f(x)| \leq M|h| \implies \left| \frac{f(x + h) - f(x)}{h} \right| \leq M.$$

Taking the limit as  $h \rightarrow 0$ , we have that  $f'(x) \leq M$ , and we have proven the lemma.  $\square$

Now we are ready to prove the main theorem.

*Proof.* Suppose that  $f$  is Lipschitz continuous on  $[a, b]$ . Then,  $f$  is absolutely continuous, and by theorem 7.29 in our textbook that

$$\begin{aligned} f(x) - f(a) &= \int_a^x f' \implies f(x) = \int_a^x f' + f(a) \\ &\implies f(x) = \int_a^x \left( f' + \frac{f(a)}{x - a} \right). \end{aligned}$$

Thus, combining this with the results of Lemma 2, we have shown that  $f$  is an indefinite integral of a bounded measurable function on  $[a, b]$ .

Now, suppose that  $f$  is the indefinite integral of a bounded measurable function  $F$ . That is, suppose

$$f(x) = \int_a^x F,$$

for some function where  $|F| \leq M$  for some  $M > 0$ . Then, for any  $a_i < b_i \in [a, b]$ , we have

$$\begin{aligned} |f(b_i) - f(a_i)| &= \left| \int_a^{b_i} F - \int_a^{a_i} F \right| \\ &= \left| \int_{a_i}^{b_i} F \right| \\ &\leq \int_a^{b_i} |F| \\ &= M(b_i - a_i). \end{aligned}$$

Thus,  $f$  is Lipschitz continuous, and our proof is complete.  $\square$

## Problem 4

**Theorem 8.** Show that if  $f[a, b] \rightarrow \mathbb{R}$  is continuous, and  $|D^+f(x)| \leq M$  for all  $x \in [a, b]$ , where

$$D^+f(x) = \lim_{h \searrow 0} \frac{f(x+h) - f(x)}{h},$$

then  $f$  satisfies a Lipschitz condition on  $[a, b]$ .

## Solution

*Proof.* Proof is still under development.  $\square$

## Problem 5

**Theorem 9.** A function  $f : [a, b] \rightarrow \mathbb{R}$  satisfies a Lipschitz condition if and only if for all  $\epsilon > 0$ , there exists a  $\delta > 0$ , such that any finite collection of intervals  $\{[a_i, b_i]\}_{i=1}^n$  in  $[a, b]$  (which are not necessarily nonoverlapping) satisfying

$$\sum_{i=1}^n (b_i - a_i) < \delta,$$

it holds that

$$\left| \sum_{i=1}^n (f(b_i) - f(a_i)) \right| < \epsilon.$$

## Solution

*Proof.* Suppose that  $f$  is Lipschitz continuous with Lipschitz constant  $M$ . Let  $\epsilon > 0$ , and define  $\delta = \epsilon/M$ . Then, for any finite collection of intervals  $\{[a_i, b_i]\}_{i=1}^n$  in  $[a, b]$  satisfying

$$\sum_{i=1}^n (b_i - a_i) < \delta,$$

we have

$$\begin{aligned}
 \left| \sum_{i=1}^n (f(b_i) - f(a_i)) \right| &\leq \sum_{i=1}^n |f(b_i) - f(a_i)| \\
 &\leq \sum_{i=1}^n M |b_i - a_i| \\
 &= M \sum_{i=1}^n |b_i - a_i| \\
 &< M\delta \\
 &= M \frac{\epsilon}{M} \\
 &= \epsilon.
 \end{aligned}$$

Thus, we have proven the forward direction.

Now we will prove the other direction by contrapositive. Suppose  $f$  is not Lipschitz continuous. Then, for any  $\epsilon > 0$ , there exists two points  $x < y \in [a, b]$  such that

$$|f(y) - f(x)| > \frac{1}{\epsilon} |y - x|.$$

Fix an  $\epsilon > 0$  and the corresponding  $x, y$ . Suppose first that  $|y - x| < \epsilon$ . Then, choose  $n \in \mathbb{N}$  such that

$$\frac{\epsilon}{2} \leq n|y - x| < \epsilon.$$

With this, we have

$$\frac{\epsilon}{2} \leq \sum_{i=1}^n (y - x) < \epsilon,$$

and

$$\begin{aligned}
 \sum_{i=1}^n |f(y) - f(x)| &\geq \sum_{i=1}^n \frac{1}{\epsilon} |y - x| \\
 &= \frac{1}{\epsilon} \sum_{i=1}^n (y - x) \\
 &\geq \frac{1}{\epsilon} \cdot \frac{\epsilon}{2} \\
 &= \frac{1}{2}.
 \end{aligned}$$

Now suppose  $|y - x| \geq \epsilon$ . We will break  $[x, y]$  into  $n$  subintervals  $[x_i, y_i]$  of equal length, satisfying

$$\frac{\epsilon}{2} \leq \frac{y - x}{n} < \epsilon.$$

Suppose, for sake of contradiction, that for all  $i$ , we have  $|f(y_i) - f(x_i)| \leq \frac{1}{\epsilon} |y_i - x_i|$ . Then, by the triangle inequality, we have

$$\begin{aligned}
 |f(y) - f(x)| &\leq \sum_{i=1}^n |f(y_i) - f(x_i)| \\
 &\leq \sum_{i=1}^n \frac{1}{\epsilon} |y_i - x_i| \\
 &\leq \frac{1}{\epsilon} |y - x|,
 \end{aligned}$$

which is a contradiction. Thus, for some  $I \in \{1, \dots, n\}$ , we have

$$|f(y_I) - f(x_I)| > \frac{1}{\epsilon} |y_I - x_I|.$$

Then, we have

$$\frac{\epsilon}{2} \leq \sum_{i=1}^n (y_I - x_I) < \epsilon,$$

and

$$\begin{aligned} \sum_{i=1}^n |f(y_I) - f(x_I)| &\geq \sum_{i=1}^n \frac{1}{\epsilon} |y_I - x_I| \\ &= \frac{1}{\epsilon} \sum_{i=1}^n (y_I - x_I) \\ &\geq \frac{1}{\epsilon} \cdot \frac{\epsilon}{2} \\ &= \frac{1}{2}. \end{aligned}$$

With this, our proof is complete. □

## Problem 6

Use the function

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \in (0, \pi] \\ 0 & \text{if } x = 0 \end{cases}$$

to show that the following statement is not true: If  $f : [a, b] \rightarrow \mathbb{R}$  is absolutely continuous on  $[a + \epsilon, b]$  for any small  $\epsilon > 0$ , and  $f$  is continuous on  $[a, b]$ , then  $f$  is absolutely continuous on  $[a, b]$ .

## Solution

For any small  $\epsilon > 0$ , we have for all  $x \in [\epsilon, \pi]$ ,

$$\begin{aligned} |f'(x)| &= \left| \sin \frac{1}{x} - \frac{\cos \frac{1}{x}}{x} \right| \\ &\leq \left| \sin \frac{1}{x} \right| + \left| \frac{\cos \frac{1}{x}}{x} \right| \\ &\leq 1 + \frac{1}{\epsilon}. \end{aligned}$$

Thus, since  $f'(x)$  is bounded on  $x \in [\epsilon, \pi]$ , an easy argument using the mean value theorem allows us to conclude that  $f$  is Lipschitz continuous, and therefore absolutely continuous.

Now, let  $\delta > 0$ . Choose  $n \in \mathbb{N}$  large enough that  $\frac{2}{\pi(1+2n)} < \delta$ , and define  $x_i = \frac{2}{\pi(1+2i)}$  for  $i \geq n$ . Then, we have  $\{[x_i, x_{i-1}]\}_{i=n+1}^\infty$  is a set of nonoverlapping subintervals of  $[0, \pi]$  such that  $\sum_{i=n+1}^\infty (x_i - x_{i-1}) < \delta$ . Furthermore, since

$$\frac{1}{x_i} = \frac{\pi}{2} + i\pi,$$

we have that  $f(x_i)$  alternates between 1 and  $-1$ . Assume, without loss of generality, that  $f(x_n) = -1$ . Then, we



have

$$\begin{aligned}
 \sum_{i=n+1}^{\infty} |f(x_i) - f(x_{i-1})| &= \sum_{i=n+1}^{\infty} |x_i + x_{i-1}| \\
 &= \sum_{i=n+1}^{\infty} \left( \frac{2}{\pi(1+2i)} + \frac{2}{\pi(1+2(i-1))} \right) \\
 &= \frac{2}{\pi} \sum_{i=n+1}^{\infty} \left( \frac{1}{1+2i} + \frac{1}{2i-1} \right) \\
 &= \frac{2}{\pi} \sum_{i=n+1}^{\infty} \frac{2i-1+2i+1}{(1+2i)(2i-1)} \\
 &= \frac{2}{\pi} \sum_{i=n+1}^{\infty} \frac{4i}{4i^2-1} \\
 &\geq \frac{2}{\pi} \sum_{i=n+1}^{\infty} \frac{4i}{4i^2} \\
 &= \frac{2}{\pi} \sum_{i=n+1}^{\infty} \frac{1}{i} \\
 &= \infty
 \end{aligned}$$

Harmonic series does not converge.

Since this is true for any  $\delta > 0$ , we can conclude that  $f$  is not absolutely continuous (or even of bounded variation) on  $[0, \pi]$ .

## Problem 7

**Theorem 10.** If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous,  $f \in BV[a, b]$ , and  $f \in AC[a + \epsilon, b]$  for any small  $\epsilon > 0$ , then  $f \in AC[a, b]$ .

### Solution

To prove this theorem, we will invoke a rather useful lemma.

**Lemma 3.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous, and of bounded variation. Then,  $V(x)$  is continuous.

*Proof.* Let  $x_0 \in [a, b]$ , and let  $\epsilon > 0$ . Define a partition  $T = \{x_0 < x_1 < \dots < x_n = b\}$  of  $[x_0, b]$  such that

$$V[x_0, b] \leq V(f, T) + \epsilon/2.$$

Since  $f$  is continuous, there exists some  $\delta_1 > 0$  such that for all  $|f(x_0) - f(x)| < \epsilon/2$  if  $|x - x_0| < \delta_1$ . Define  $\delta = \min\{\delta_1, x_1 - x_0\}$ . Then,

$$\begin{aligned}
 V(b) - V(x_0) &= V[x_0, b] && \text{By theorem 2.2 in our textbook} \\
 &\leq V(f, T) + \epsilon/2 \\
 &= \sum_{i=1}^n |f(x_i) - f(x_{i-1})| + \epsilon/2 \\
 &= |f(x_1) - f(x_0)| + \sum_{i=2}^n |f(x_i) - f(x_{i-1})| + \epsilon/2 \\
 &\leq |f(x_0) - f(x)| + |f(x_1) - f(x)| + \sum_{i=2}^n |f(x_i) - f(x_{i-1})| + \epsilon/2 \\
 &< \epsilon/2 + V[x, b] + \epsilon/2 \\
 &= \epsilon + V(b) - V(x).
 \end{aligned}$$

Thus, we can conclude that  $V(x) - V(x_0) < \epsilon$ . Thus, it follows that  $V(x)$  is continuous.  $\square$

Now we are ready to prove the main theorem.

*Proof.* Let  $\epsilon > 0$ . Since  $f$  is continuous,  $V(x) = V[a, x]$  is continuous for all  $x \in [a, b]$ . Thus, there exists a  $\delta_1 > 0$  such that  $V(a + \delta_1) < \epsilon/2$ . Furthermore, since  $f$  is of bounded variation, on  $[a + \delta_1, b]$ , there exists a  $\delta_2 > 0$  such that  $\sum |f(b_i) - f(a_i)| < \epsilon/2$  for any countable collection  $\{[a_i, b_i]\}$  of nonoverlapping subintervals of  $[a + \delta_1, b]$  with  $\sum (b_i - a_i) < \delta_2$ .

Now, let  $\{[a_i, b_i]\}$  be a set of nonoverlapping subintervals of  $[a, b]$  with  $\sum (b_i - a_i) < \delta_2$ . Assume, without loss of generality, that there is a single interval  $[a_n, b_n]$  such that  $a_n < a + \delta_1 < b_n$ . Let  $L = \{i : [a_i, b_i] \subseteq [a, a + \delta_1]\}$  and let  $R = \{i : [a_i, b_i] \subseteq [a + \delta_1, b]\}$ . Then, we have

$$\begin{aligned}
 \sum_i |f(b_i) - f(a_i)| &= \sum_{i \in L} |f(b_i) - f(a_i)| + \sum_{i \in R} |f(b_i) - f(a_i)| + |f(b_n) - f(a_n)| \\
 &\leq \sum_{i \in L} |f(b_i) - f(a_i)| + \sum_{i \in R} |f(b_i) - f(a_i)| + |f(b_n) - f(a + \delta_1)| + |f(a + \delta_1) - f(a_n)| \\
 &= \left( \sum_{i \in L} |f(b_i) - f(a_i)| + |f(a + \delta_1) - f(a_n)| \right) + \left( \sum_{i \in R} |f(b_i) - f(a_i)| + |f(b_n) - f(a + \delta_1)| \right) \\
 &< V(a + \delta_1) + \frac{\epsilon}{2}, \text{ Left hand points can extend to a partition of } [a, a + \delta_1] \\
 &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
 &= \epsilon.
 \end{aligned}$$

Thus, we have proven that  $f$  is absolutely continuous on  $[a, b]$ . □