

Problem 1

Before we jump into this problem, we will first prove an easy lemma.

Lemma 1. *Let F, B and E be sets with $F \subseteq B \subseteq E$. Then,*

$$(E - F) \cap B = B - F.$$

Proof. We have

$$\begin{aligned} x \in (E - F) \cap B &\iff (x \in E - F) \wedge x \in B \\ &\iff (x \in E) \wedge (x \notin F) \wedge (x \in B) \\ &\iff (x \in E \cap B) \wedge (x \notin F) \\ &\iff (x \in B) \wedge (x \notin F) && \text{Since } B \subseteq E \\ &\iff x \in B - F, \end{aligned}$$

and our proof is complete. \square

Theorem 1. *$E \subseteq \mathbb{R}^n$ is measurable if and only if for every $\epsilon > 0$ there exists a measurable set $B \subseteq E$ such that $|E - B|_e < \epsilon$.*

Solution

Proof. Suppose E is measurable, and let $\epsilon > 0$. By Lemma 3.22 in our textbook, there exists a closed set $B \subseteq E$ such that $|E - B|_e < \epsilon$. Since all closed sets are measurable, B is measurable, and we have proven one direction of the theorem.

Now suppose that for each $\epsilon > 0$, there exists a measurable set $B \subseteq E$ such that $|E - B|_e < \epsilon$. Thus, let $\epsilon > 0$, and fix a measurable $B \subseteq E$ such that $|E - B|_e < \frac{\epsilon}{2}$. Since B is measurable, Lemma 3.22 tells us that there exists a closed set $F \subseteq B$ such that $|B - F|_e < \frac{\epsilon}{2}$. Now, by Caratheory's theorem, we have

$$\begin{aligned} |E - F|_e &= |(E - F) \cap B|_e + |(E - F) - B|_e \\ &= |(E - F) \cap B|_e + |E - B|_e && \text{Since } F \subseteq B \\ &= |B - F|_e + |E - B|_e && \text{By Lemma 1} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus, utilizing Lemma 3.22 one last time, it follows that E is measurable, and our proof is complete. \square

Problem 2

Theorem 2. *Let $\{I_1, I_2, \dots, I_N\}$ be a finite family of closed intervals in \mathbb{R}^1 such that $\mathbb{Q} \cap [0, 1] \subseteq \bigcup_{j=1}^N I_j$. Then $\sum_{j=1}^N |I_j| \geq 1$. Furthermore, if this family of intervals is infinite, this may not be true.*

Solution

Proof. Suppose there exists an $x \in [0, 1] \setminus \mathbb{Q}$ such that $x \notin \bigcup_{j=1}^N I_j$. Then, since each I_j is closed, there exist $\delta_j > 0$ such that

$$(\forall y \in I_j)(|x - y| \geq \delta_j).$$

Define $\delta = \min\{\delta_j | j = 1, 2, \dots, N\}$, which exists because there are only finitely many I_j . By construction, we have that $\delta > 0$. By density of the rational numbers, there exists a rational number $q \in [0, 1]$ such that $|x - q| < \delta$. Thus, $q \notin \bigcup_{j=1}^N I_j$, and we have a contradiction.

Therefore, we have that $[0, 1] \subseteq \bigcup_{j=1}^N I_j$. Now, by definition of Lebesgue measure, we have

$$\begin{aligned} 1 &= |[0, 1]| \\ &\leq \sum_{j=1}^N |I_j|, \end{aligned}$$

and our proof is complete.

For the last remark, let $\{q_1, q_2, \dots\}$ be an enumeration of $\mathbb{Q} \cap [0, 1]$. Define an infinite family of closed intervals $\{I_1, I_2, \dots\}$ such that

$$I_k = [q_k, q_k + \frac{\epsilon}{2}]$$

for some $0 < \epsilon < 1$. Clearly, we have $\mathbb{Q} \cap [0, 1] \subseteq \bigcup_{j=1}^{\infty} I_j$, but we also have

$$\begin{aligned} \sum_{j=1}^{\infty} |I_j| &= \sum_{j=1}^{\infty} |q_k, q_k + \frac{\epsilon}{2^k}| \\ &= \sum_{j=1}^{\infty} \frac{\epsilon}{2^k} \\ &= \epsilon \\ &< 1, \end{aligned}$$

and we have shown that the statement does not hold for an infinite family of closed intervals. \square

Problem 3

Theorem 3. Part (i):

Let $A, B \subseteq \mathbb{R}^n$, with A measurable. If $A \cap B = \emptyset$, then $|A \cup B|_e = |A| + |B|_e$

Part (ii):

If $F \subseteq E$ is closed such that $|E|_e - |F| < \epsilon$ and $|F| < \infty$, then $|E - F|_e < \epsilon$.

Solution

Proof. Part (i):

Since A is measurable, we have

$$\begin{aligned} |A \cup B|_e &= |(A \cup B) \cap A|_e + |(A \cup B) - A|_e \\ &= |A| + |(A \cup B) - A|_e \\ &= |A| + |B|_e \end{aligned}$$

By Caratheory's theorem

Since $A \cap B = \emptyset$

Since $(A \cup B) - A = B$

and we have completed the proof of this part.

Part (ii):

Since F is closed (and therefore measurable), we can use Caratheodory's theorem once more, to see

$$\begin{aligned} |E|_e &= |E \cap F|_e + |E - F|_e \\ &= |F| + |E - F|_e. \end{aligned} \quad \text{Since } F \subseteq E$$

Since $|F|$ is finite, we can subtract it from both sides to yield

$$\begin{aligned} |E - F|_e &= |E|_e - |F| \\ &< \epsilon, \end{aligned}$$

as desired. \square

Problem 4

Theorem 4. Let A and B be subsets of \mathbb{R}^n . Then

$$|A \cup B|_e + |A \cap B|_e \leq |A|_e + |B|_e.$$

Solution

Proof. Let $\epsilon > 0$. By theorem 3.6 in our textbooks, there exist open sets G_A, G_B such that $A \subseteq G_A$, $B \subseteq G_B$, $|G_A| \leq |A|_e + \frac{\epsilon}{2}$, and $|G_B| \leq |B|_e + \frac{\epsilon}{2}$. With this, we have

$$\begin{aligned} |A \cup B|_e + |A \cap B|_e &\leq |G_A \cup G_B| + |G_A \cap G_B| && \text{Since } A \cup B \subseteq G_A \cup G_B \text{ and } A \cap B \subseteq G_A \cap G_B \\ &= |G_A| + |G_B| && \text{From HW2 Problem 4} \\ &\leq |A|_e + \frac{\epsilon}{2} + |B|_e + \frac{\epsilon}{2} \\ &= |A|_e + |B|_e + \epsilon. \end{aligned}$$

Since this is true for any $\epsilon > 0$, we have

$$|A \cup B|_e + |A \cap B|_e \leq |A|_e + |B|_e,$$

and our proof is complete. \square

Problem 5

Theorem 5. *If $E \subseteq \mathbb{R}$ is measurable and $|E| > 0$, then there are $x, y \in E$, with $x \neq y$, such that $x - y$ is a rational number.*

Solution

Proof. Since E has nonzero measure, there exists a bounded subset $F \subseteq E$ such that $|F| > 0$. Since F is bounded, there exists integers $a < b$ such that $F \subseteq [a, b]$. Thus, it will suffice to show that there are $x, y \in F$ such that $x - y$ is a rational number.

Let $\{q_1, q_2, \dots\}$ be an enumeration of the rational numbers contained in $[0, 1]$. Define a family of sets $\{F_k\}$ by

$$F_k = F + q_1,$$

where we have just translated F some rational number in $[0, 1]$. Now, suppose that $F_j \cap F_k = \emptyset$ if $j \neq k$. Using the fact that $\bigcup_{k=1}^{\infty} F_k \subseteq [a, b+1]$, we have

$$\begin{aligned} |[a, b+1]| &\geq \left| \bigcup_{k=1}^{\infty} F_k \right| \\ &= \sum_{k=1}^{\infty} |F_k| && \text{Since the sets are disjoint, by assumption} \\ &= \sum_{k=1}^{\infty} |F| && \text{By translation invariance of Lebesgue measure} \\ &= \infty, \end{aligned}$$

which is a contradiction, since $|[a, b+1]|$ is finite. Thus, there exist numbers $j, k \in \mathbb{N}$, with $j \neq k$, such that $F_j \cap F_k \neq \emptyset$. Thus, there exist $f_j, f_k \in F$ such that

$$f_j + q_j = f_k + q_k.$$

Then, we have

$$f_j - f_k = q_k - q_j.$$

Since the rational numbers are closed under subtraction, we have that $f_j - f_k$ is rational, and our proof is complete. \square

Problem 6

Theorem 6. Let E be the nonmeasurable set in $I = [0, 1]$ constructed in the lecture. Then

Part (i):

$$|E|_i = 0.$$

Part (ii):

$$|I| < |E|_e + |I - E|_e.$$

Part (iii):

$$|I| > |E|_i + |I - E|_i.$$

Solution

Proof. Part (i):

Let $F \subseteq E$ be closed. □

Problem 7

Theorem 7. Let $f(x) = x^3$. If $E \subseteq \mathbb{R}$ is a zero measure set, then $|f(E)| = 0$.

Solution

Proof. Suppose first that E is bounded. Then, as I proved on homework 1, f is uniformly continuous. Thus, there exists some $M > 0$ such that for all $x, y \in E$, we have

$$|f(x) - f(y)| \leq M|x - y|.$$

Furthermore, it follows that for any interval I , we have

$$|f(I)| \leq M|I|.$$

Now let $\epsilon > 0$. Since E is of measure zero, there exists a cover of $\{I_1, I_2, \dots\}$ of E consisting of closed intervals such that

$$\sum_{j=1}^{\infty} |I_j| < \frac{\epsilon}{M}.$$

We have

$$f(E) = \bigcup_{j=1}^{\infty} f(I_j),$$

therefore

$$\begin{aligned} |f(E)| &\leq \sum_{j=1}^{\infty} |f(I_j)| \\ &\leq \sum_{j=1}^{\infty} M|I_j| \\ &< M \frac{\epsilon}{M} \\ &= \epsilon. \end{aligned}$$

Since this is true for any ϵ , we can conclude that $|f(E)| = 0$.

Now, suppose that E is not bounded. We have that

$$E = \bigcup_{k=1}^{\infty} E \cap B(0; k),$$

where $B(0; k)$ is the open ball centered at the origin with radius k . Furthermore, we have

$$f(E) = \bigcup_{k=1} f(E \cap B(0; k)).$$

Now, since $E \cap B(0; k)$ is bounded for each k and has measure zero, the previous result allows us to conclude that $|f(E \cap B(0; k))| = 0$, and therefore that $|f(E)| = 0$, as desired. \square

Problem 8

Theorem 8. *If f and g are continuous functions on \mathbb{R}^n and are equal a.e. in \mathbb{R}^n , then $f \equiv g$ on \mathbb{R}^n .*

Solution

Proof. Suppose that g is continuous, and that $f \not\equiv g$. Then, there exists some point x_0 such that $f(x_0) \neq g(x_0)$. Since f and g are equal almost everywhere, we can construct a sequence $\{x_k\}$ such that $f(x_k) = g(x_k)$ and $|x_0 - x_k| < \frac{1}{k}$ for all $k \in \mathbb{N}$. By construction, we have that $x_k \rightarrow x_0$. By continuity of g , we have that $g(x_k) \rightarrow g(x_0)$. Furthermore, since $f(x_k) = g(x_k)$ for all k , we have that $f(x_k) \rightarrow g(x_0)$, implying that $f(x_k) \not\rightarrow f(x_0)$. Thus, we have shown that f is not continuous, and we have proven the contrapositive. \square

Problem 9

Theorem 9. *Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set E with $|E| < \infty$. If $|f_k(x)| \leq M_x < \infty$ for all k for each $x \in E$, then given $\epsilon > 0$, there is a closed $F \subseteq E$ and a finite $M > 0$ such that $|E - F| < \epsilon$ and $|f_k(x)| \leq M$ for all k and all $x \in F$.*

Solution

Proof. By the remark after theorem 4.6 in our textbook, we have that $|f_k|$ is measurable for each k . By theorem 4.11 in our text book, $\sup_k |f_k(x)|$ is measurable. Furthermore, we have by assumption that $\sup_k |f_k(x)|$ is finite everywhere. With this, we can construct a sequence of measurable sets $\{E_k\}$ such that $E_k \nearrow E$ defined by

$$E_k = \{x \in E \mid \sup_k |f_k(x)| \leq k\}.$$

By theorem 3.26 in the book, we have

$$\lim_{k \rightarrow \infty} |E_k| = |E|.$$

Thus, there exists an $M > 0$ such that $|E| - |E_M| < \frac{\epsilon}{2}$. Using Lemma 3.22, we can find a closed $F \subseteq E_M$ such that $|E_M - F| < \frac{\epsilon}{2}$. Using Caratheory's theorem, we have

$$\begin{aligned} |E_n| &= |E_n \cap F| + |E_n - F| \\ &= |F| + |E_n - F| && \text{Since } F \subseteq E_n \\ |F| &= |E_n| - |E_n - F| && \text{Since } |E| < \infty \implies |E_n| < \infty \implies |F| < \infty, \end{aligned}$$

and, using the exact same approach,

$$\begin{aligned} |E - F| &= |E| - |E \cap F| \\ &= |E| - |F| \\ &= |E| - (|E_n| - |E_n - F|) \\ &= (|E| - |E_n|) + |E_n - F| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Finally, since $F \subseteq E_M$, we have that

$$(\forall x \in F)(\sup_k |f_k(x)| \leq M) \implies (\forall x \in F)(\forall k)(|f_k(x)| \leq M),$$

and our proof is complete. \square

Problem 10

Theorem 10. Let f be a measurable function and finite a.e. on $E \subseteq \mathbb{R}^n$. Show that there is a sequence of continuous functions g_k on \mathbb{R}^n such that $g_k \rightarrow f$ a.e. on E .

Solution

Proof. By the alternative version of Lusin's theorem, we can construct a sequence of closed sets $\{F_k^*\}$ such that $F_k^* \subseteq E$ and $|E - F_k^*| < \frac{1}{k}$, and a sequence of continuous functions $\{g_k^*\}$ such that $g_k^* : \mathbb{R}^n \rightarrow \mathbb{R}$ and $g_k^*(x) = f(x)$ for all $x \in F_k^*$.

Now, this sequence of continuous functions does not necessarily converge, as $\{F_k^*\}$ is not necessarily an increasing sequence. One way to remedy this, is to define a new sequence, $\{F_k\}$ such that $F_1 = F_1^*$, and $F_k = F_k^* \cup F_{k-1}$ for $k > 1$. By design, we have that $F_k \nearrow E$, and each F_k is closed. Using this, we will define a new sequence of functions $\{g_k\}$, where $g_1 = g_1^*$, and $g_{k+1}(x) = g_{k+1}^*(x)$ when $x \in F_{k+1}^*$, and $g_{k+1}(x) = g_k(x)$ when $x \notin F_{k+1}^*$. All that remains is to show that each g_k is continuous, and our proof will be complete.

Since g_1^* is continuous, it follows that g_1 is continuous. Now suppose that for some $k \geq 1$ that g_k is continuous. Let $\{x_j\}$ be any convergent sequence in \mathbb{R}^n , and call it's limit x . Now, either $x \in F_{k+1}^*$, or $x \notin F_{k+1}^*$. Suppose first that $x \in F_{k+1}^*$. Since F_{k+1}^* is closed, there exists some $J \in \mathbb{N}$ such that $x_j \in F_{k+1}^*$ for all $j \geq J$. Thus, $g_{k+1}(x_j) = g_{k+1}^*(x_j)$ for all $j \geq J$. Then, by continuity of g_{k+1}^* , $g_{k+1}(x_j) \rightarrow g_{k+1}(x)$, and we have that g_{k+1} is continuous in F_{k+1}^* .

Now, suppose that $x \notin F_{k+1}^*$. Then, since F_{k+1}^* is closed, there exists some $J \in \mathbb{N}$ such that $x_j \notin F_{k+1}^*$ for all $j \geq J$. Thus, $g_{k+1}(x_j) = g_k(x_j)$ for all $j \geq J$. By continuity of g_k , we have that $g_{k+1}(x_j) \rightarrow g_{k+1}(x)$, and we have shown that g_{k+1} is continuous everywhere. Since we have already shown the base case of $k = 1$, the principle of mathematical induction allows us to conclude that g_k is continuous for all k , and our proof is complete. \square

Problem 11

Theorem 11. Let f be a measurable function and finite a.e. on $[a, b]$. Then, there is a sequence of polynomials p_k such that $p_k \rightarrow f$ a.e. on $[a, b]$.

Solution

Proof. As we proved in the previous problem, there exists a sequence of continuous functions g_k on \mathbb{R}^n such that $g_k \rightarrow f$ a.e. on $[a, b]$. By the Stone-Weierstrass theorem, we can construct a sequence of polynomials p_k such that for all $x \in [a, b]$, $|p_k(x) - g_k(x)| < \frac{1}{k}$. Now, let $x \in [a, b]$ be such that $g_k(x) \rightarrow f(x)$, and let $\epsilon > 0$. Choose $K_1 \in \mathbb{N}$ to be such that $\frac{1}{K_1} < \frac{\epsilon}{2}$. Additionally, choose $K_2 \in \mathbb{N}$ large enough that for all $k \geq K_2$, we have $|g_k(x) - f(x)| < \frac{\epsilon}{2}$. Then, if we define K as the maximum of K_1 and K_2 , we have that for all $k \geq K$,

$$\begin{aligned} |p_k(x) - f(x)| &\leq |p_k(x) - g_k(x)| + |g_k(x) - f(x)| && \text{By the triangle inequality} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus, since ϵ was arbitrary, we have shown that $p_k \rightarrow f$ a.e. on $[a, b]$, and our proof is complete. \square