Problem 1.6

Theorem 1. *Let $\alpha > 0$. Then $f(x) = x^\alpha$ is absolutely continuous on every subinterval $[a, b] \subseteq [0, \infty)$.*

Solution

Proof. We have that f is differentiable on $(0, \infty)$ with derivative

$$f'(x) = \alpha x^{\alpha-1}.$$

Thus, f is differentiable almost everywhere on $[0, \infty)$. Now, let $[a, b] \subseteq [0, \infty)$. Since f' is continuous a.e. on $[a, b]$, f' is integrable on $[a, b]$. Furthermore, we have for any $x \in [a, b]$,

$$\begin{aligned}
\int_a^x f'(x)dx &= \int_a^x \alpha x^{\alpha-1} dx \\
&= \alpha x^\alpha \big|_a^x \\
&= b^\alpha - x^\alpha \\
&= f(x) - f(a).
\end{aligned}$$

Thus, by theorem 7.29 in our textbook, we have that f is absolutely continuous on any subinterval of $[0, \infty)$. $\square$

Problem 1.7

Theorem 2. *A function f is absolutely continuous on $[a, b]$ if and only if given $\epsilon > 0$, there exists $\delta > 0$ such that $|\sum[f(b_i) - f(a_i)]| < \epsilon$ for any finite collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$.*

Solution

Proof. Suppose f is absolutely continuous on $[a, b]$, and let $\epsilon > 0$. Since f is absolutely continuous, there exists a $\delta > 0$ such that $\sum |[f(b_i) - f(a_i)]| < \epsilon$ for any finite collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$. Thus, if we let $\{[a_i, b_i]\}$ be a set of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$, we have

$$\begin{aligned}
\epsilon &> \sum |[f(b_i) - f(a_i)]| \\
&\geq |\sum[f(b_i) - f(a_i)]|, && \text{Basic property of absolute value}
\end{aligned}$$

and we have proven the forward direction.

Now, suppose that if we are given $\epsilon > 0$, there exists $\delta > 0$ such that $|\sum[f(b_i) - f(a_i)]| < \epsilon$ for any finite collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$. Let $\epsilon > 0$, and choose $\delta > 0$ such that $|\sum[f(b_i) - f(a_i)]| < \frac{\epsilon}{2}$ for any finite collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$. Let $\{[a_i, b_i]\}$ be a set of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$. We have

$$\sum_{i \in \{i : f(b_i) \geq f(a_i)\}} (b_i - a_i) < \delta,$$

which means that

$$\begin{aligned}
\frac{\epsilon}{2} &> \left| \sum_{i \in \{i : f(b_i) \geq f(a_i)\}} [f(b_i) - f(a_i)] \right| \\
&= \sum_{i \in \{i : f(b_i) \geq f(a_i)\}} |f(b_i) - f(a_i)|.
\end{aligned}$$

Similarly, we have

$$\sum_{i \in \{i : f(b_i) < f(a_i)\}} (b_i - a_i) < \delta,$$

which implies

$$\begin{aligned}
\frac{\epsilon}{2} &> \left| \sum_{i \in \{i : f(b_i) < f(a_i)\}} [f(b_i) - f(a_i)] \right| \\
&= \sum_{i \in \{i : f(b_i) < f(a_i)\}} |f(b_i) - f(a_i)|.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\sum_i |f(b_i) - f(a_i)| &= \sum_{i \in \{i : f(b_i) < f(a_i)\}} |f(b_i) - f(a_i)| + \sum_{i \in \{i : f(b_i) \geq f(a_i)\}} |f(b_i) - f(a_i)| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}$$

and we have shown that f is absolutely continuous. With this, our proof is complete. □

Problem 1.8

Theorem 3. *If f is of bounded variation on $[a, b]$, and if the function $V(x) = V[a, x]$ is absolutely continuous on $[a, b]$, then f is absolutely continuous on $[a, b]$.*

Solution

Proof. Let $\epsilon > 0$. Since $V(x)$ is absolutely continuous, we have that there exists $\delta > 0$ such that $\sum |V(b_i) - V(a_i)| < \epsilon$ for any finite collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$. Let $\{[a_i, b_i]\}$ be a collection of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$. From Theorem 2.2 (part i) in our textbook, since f is of bounded variation, and $V(x)$ is finite for all $x \in [a, b]$. We have

$$\begin{aligned}
\epsilon &> \sum |V(b_i) - V(a_i)| \\
&= \sum (V[a, b_i] - V[a, a_i]) \\
&\geq \sum V[a, b_i] \\
&\geq \sum V[a_i, b_i] && \text{Theorem 2.2 part i} \\
&\geq \sum |f(b_i) - f(a_i)|,
\end{aligned}$$

and we have proven that f is absolutely continuous. □

Problem 1.9

Theorem 4. *If f is of bounded variation on $[a, b]$, then*

$$\int_a^b |f'| \leq V[a, b].$$

Furthermore, if the equality holds in this inequality, then f is absolutely continuous.

Solution

Proof. Let $N(x)$ and $P(x)$ denote the negative and positive variations of f on $[a, x]$, as in the proof of theorem 2.7 in our textbook. Then, we have

$$f(x) = [P(x) + f(a)] - N(x).$$

We note, that $P(x) + f(a)$ and $N(x)$ are increasing functions. Now, we have

$$\begin{aligned}
\int_a^b |f'| &= \int_a^b |P'(x) - N'(x)| dx \\
&\leq \int_a^b P'(x) + \int_a^b N'(x) \\
&\leq P(b) - P(a) + N(b) - N(a) && \text{By theorem 7.21 in our textbook} \\
&= V(b) - V(a) && \text{By theorem 2.6 in our textbook} \\
&\leq V(b) \\
&= V[a,b].
\end{aligned}$$

Now, suppose the equality holds. That is, suppose

$$\int_a^b |f'| = V[a,b].$$

By theorem 7.24 in our textbook, we have $V'(x) = |f'(x)|$ almost everywhere for $x \in [a,b]$. Thus, we have

$$\begin{aligned}
\int_a^x V'(t)dt &= \int_a^x |f'(t)|dt \\
&= V(x) \\
&= V[a,x] \\
&= V[a,x] + V[a,a] && \text{By theorem 2.2 ii} \\
&= V(x) - V(a).
\end{aligned}$$

Thus, by theorem 7.29, $V(x)$ is absolutely continuous. By the statement we proved in the previous problem, we can conclude that f is absolutely continuous. □

Problem 1.10

Theorem 5. Part (a):
If f is absolutely continuous on $[a,b]$ and Z is a subset of $[a,b]$ with measure zero, then the image set defined by $f(Z) = \{w : w = f(z), z \in Z\}$ also has measure zero. Deduce that the image under f of any measurable subset of $[a,b]$ is measurable. (Compare theorem 3.33)(Hint: use the fact that the image of an interval $[a_i, b_i]$ is an interval of length at most $V(b_i) - V(a_i)$.)

Part (b):
Give an example of a strictly increasing Lipschitz continuous function f and a set Z with measure 0 such that $f^{-1}(Z)$ does not have measure 0 (and cosequently, f^{-1} is not absolutely continuous). (Let $f^{-1}(x) = x + C(x)$ on $[0,1]$, where $C(x)$ is the Cantor-Lebesgue function.

Solution

Before we jump into the proof, we will prove a usefull lemma.

Lemma 1. *If f is an absolutely continuous function on $[a,b]$, and $[a_i, b_i] \subseteq [a,b]$, then the image of $[a_i, b_i]$ under f is an interval with $f([a_i, b_i]) \leq V(b_i) - V(a_i)$.*

Proof. Since f is continuous, it follows immediately from the intermediate value theorem that $f([a_i, b_i])$ is an interval. By Theorem 7.27 in our textbook, we have that f is of bounded variation. Thus, by theorem 2.2, we have

$$\begin{aligned}
V(b_i) - V(a_i) &= V[a, b_i] - V[a, a_i] \\
&= V[a_i, b_i].
\end{aligned}$$

By the extreme value theorem, there exist $c, d \in [a_i, b_i]$ such that the minimum and maximum values of f on $[a_i, b_i]$ are attained at c and d respectively. Define a partition of $[a_i, b_i]$ by

$$T = \{a_i, c, d, b_i\}.$$

Then

$$\begin{aligned}
V[a_i,b_i] &\geq V([a_i,b_i],T) \\
&\geq |f(d)-f(c)| \\
&= |[f(c),f(d)]| \qquad\qquad = |f([a_i,b_i])|,
\end{aligned}$$

and we have proven the lemma. □

Now we are ready for the main proof.

Proof. Part (a):
Let $\epsilon > 0$. Since f is absolutely continuous on $[a,b]$, theorem 7.31 tells us that $V(x)$ is absolutely continuous on $[a,b]$. Thus, there exists a $\delta > 0$ such that $\sum |V(b_i) - V(a_i)| < \epsilon$ for any countable collection $\{[a_i,b_i]\}$ of nonoverlapping subintervals of $[a,b]$ with $\sum(b_i - a_i) < \delta$. Define an open set G such that $[a,b] \subseteq G$, with $|G| < \delta$. Since G is open, by theorem 1.11 in our textbook, there exists a countable collection $\{[a_i,b_i]\}$ of nonoverlapping subintervals of $[a,b]$ whose union is $[a_i,b_i]$. Thus, since $\sum(b_i - a_i) < \delta$, we have

$$\begin{aligned}
\epsilon &> \sum |V(b_i) - V(a_i)| \\
&\geq \sum |f([a_i, bi])| && \text{By Lemma 1} \\
&\geq \left|\bigcup f([a_i,b_i])\right| \\
&\geq |f(G)| \\
&\geq |f(Z)|.
\end{aligned}$$

Since this is true for all $\epsilon < 0$, we have that $|f(Z)| = 0$, and we have shown that f maps sets of measure zero to sets of measure zero.

Now, let E be any measurable set. Then, we use theorem 3.28 in our textbook write $E = H \cup Z$, where H is a set of type F_σ and Z is of measure zero. Since $f(E) = f(H) \cup f(Z)$, we have that TE is the the union of two measurable sets, and is therefore measurable.

Part (b):
Let $g : [0,1] \to [0,2]$ be defined by

$$g(x) = x + C(x),$$

where $C : [0,1] \to [0,1]$ is the Cantor Lebesgue function. This function is injective, thus it's inverse function $f : [0,2] \to [0,1]$ is well defined. Since g is strictly increasing, we have that f is strictly increasing. We will first show that f is Lipschitz continuous. Let $x, y \in [0,2]$ with $x < y$. Then, we have

$$\begin{aligned}
\left|\frac{y-x}{f(y)-f(x)}\right| &= \frac{y-x}{f(y)-f(x)} && \text{Since } f \text{ is increasing} \\
&= \frac{g(f(y)) - g(f(x))}{f(y)-f(x)} \\
&= \frac{f(y)-f(x)}{f(y)-f(x)} + \frac{C(f(y)) - C(f(x))}{f(y)-f(x)} \\
&= 1 + \frac{C(f(y)) - C(f(x))}{f(y)-f(x)} \\
&\geq 1 && \text{Since the Cantor Lebesgue function is non decreasing.}
\end{aligned}$$

Thus, we have

$$|f(y) - f(x)| \leq |y - x|,$$

and we can conclude f is Lipschitz continuous.

Now, let $\mathcal{C}$ be the Cantor set, and consider the set $[0,1]\backslash\mathcal{C}$. We have that $[0,1]\backslash\mathcal{C}$ is the sum of countably many disjoint intervals $\{I_i\}$, and C is constant on each one of these intervals. Let $C(x) = c_i$ for $x \in I_i$. Then, we have

$$g(I_i) = \{x + c_i : x \in I_i\}.$$

Thus, by translation invariance of the lebesgue outer measure, we have

$$|g(I_i)|_e = |I_i|.$$

With this, we have

$$\begin{aligned} |g([0,1]\backslash\mathcal{C})|_e &= \left| g\left(\bigcup I_i\right)\right| \\ &= \left|\bigcup g(I_i)\right|_e \\ &= \sum_{i=1}^{\infty} |g(I_i)|_e \\ &= 1. \end{aligned}$$

Now, we know that $|\mathcal{C}| = 0$. Furthermore, we have $g([0,1]) = [0,2]$. Finally, we have

$$g([0,1]) = g(\mathcal{C}) \cup g([0,1]\backslash\mathcal{C}),$$

and by theorem 3.34 in our textbook, we can conclude that $|g(\mathcal{C})| = 1$. □

Problem 2

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be monotone, and suppose $f'(x)$ exists and is finite at every $x \in [a,b]$. Then f is absolutely continuous.*

Solution

Proof. Proof is under development. □

Problem 3

Theorem 7. *A function $f : [a,b] \to \mathbb{R}$ is Lipschitz continuous on $[a,b]$ if and only if f is an indefinite integral of a bounded measurable function on $[a,b]$.*

Solution

We will start with a useful lemma.

Lemma 2. *Let $f : [a,b] \to \mathbb{R}$ be Lipschitz continuous on $[a,b]$. Then, the derivative of f is bounded.*

Proof. Since f is Lipschitz continuous on $[a,b]$, there exists an $M > 0$ such that for all $x, h \in [a,b]$, where $f'(x)$ exists and $x + h \in [a,b]$, we have

$$|f(x+h) - f(x)| \leq M|h| \implies \left|\frac{f(x+h) - f(x)}{h}\right| \leq M.$$

Taking the limit as $h \to 0$, we have that $f'(x) \leq M$, and we have proven the lemma. □

Now we are ready to prove the main theorem.

Proof. Suppose that f is Lipschitz continuous on $[a,b]$. Then, f is absolutely continuous, and by theorem 7.29 in our textbook that

$$\begin{aligned} f(x) - f(a) = \int_a^x f' &\implies f(x) = \int_a^x f' + f(a) \\ &\implies f(x) = \int_a^x \left(f' + \frac{f(a)}{x-a}\right). \end{aligned}$$

Thus, combining this with the results of Lemma 2, we have shown that f is an indefinite integral of a bounded measurable function on $[a,b]$.

Now, suppose that f is the indefinite integral of a bounded measurable function F. That is, suppose

$$f(x) = \int_a^x F,$$

for some function where $|F| \leq M$ for some $M > 0$. Then, for any $a_i < b_i \in [a, b]$, we have

$$\begin{aligned} |f(b_i) - f(a_i)| &= \left| \int_a^{b_i} F - \int_a^{a_i} F \right| \\ &= \left| \int_{a_i}^{b_i} F \right| \\ &\leq \int_a^{b_i} |F| \\ &= M(b_i - a_i). \end{aligned}$$

Thus, f is Lipschitz continuous, and our proof is complete. ☐

Problem 4

Theorem 8. *Show that if $f[a, b] \to \mathbb{R}$ is continuous, and $|D^+ f(x)| \leq M$ for all $x \in [a, b]$, where*

$$D^+ f(x) = \lim_{h \searrow 0} \frac{f(x + h) - f(x)}{h},$$

then f satisfies a Lipschitz condition on $[a, b]$.

Solution

Proof. Proof is still under development. ☐

Problem 5

Theorem 9. *A function $f : [a, b] \to \mathbb{R}$ satisfies a Lipschitz condition if and only if for all $\epsilon > 0$, there exists a $\delta > 0$, such that any finite collection of intervals $\{[a_i, b_i]\}_{i=1}^n$ in $[a, b]$ (which are not necessarily nonoverlapping) satisfying*

$$\sum_{i=1}^n (b_i - a_i) < \delta,$$

it holds that

$$\left| \sum_{i=1}^n (f(b_i) - f(a_i)) \right| < \epsilon.$$

Solution

Proof. Suppose that f is Lipschitz continuous with Lipschitz constant M. Let $\epsilon > 0$, and define $\delta = \epsilon/M$. Then, for any finite collection of intervals $\{[a_i, b_i]\}_{i=1}^n$ in $[a, b]$ satisfying

$$\sum_{i=1}^n (b_i - a_i) < \delta,$$

we have

$$\begin{aligned}\left|\sum_{i=1}^{n}(f(b_i)-f(a_i))\right| &\leq \sum_{i=1}^{n}|f(b_i)-f(a_i)| \\ &\leq \sum_{i=1}^{n} M|b_i-a_i| \\ &= M\sum_{i=1}^{n}|b_i-a_i| \\ &< M\delta \\ &= M\frac{\epsilon}{M} \\ &= \epsilon.\end{aligned}$$

Thus, we have proven the forward direction.

Now we will prove the other direction by contrapositive. Suppose f is not Lipschitz continuous. Then, for any $\epsilon > 0$, there exists two points $x < y \in [a, b]$ such that

$$|f(y)-f(x)| > \frac{1}{\epsilon}|x-y|.$$

Fix an $\epsilon > 0$ and the corresponding x, y. Suppose first that $|y-x| < \epsilon$. Then, choose $n \in \mathbb{N}$ such that

$$\frac{\epsilon}{2} \leq n|y-x| < \epsilon.$$

With this, we have

$$\frac{\epsilon}{2} \leq \sum_{i=1}^{n}(y-x) < \epsilon,$$

and

$$\begin{aligned}\sum_{i=1}^{n}|f(y)-f(x)| &\geq \sum_{i=1}^{n}\frac{1}{\epsilon}|y-x| \\ &= \frac{1}{\epsilon}\sum_{i=1}^{n}(y-x) \\ &\geq \frac{1}{\epsilon}\cdot\frac{\epsilon}{2} \\ &= \frac{1}{2}.\end{aligned}$$

Now suppose $|y-x| \geq \epsilon$. We will break $[x, y]$ into n subintervals $[x_i, y_i]$ of equal length, satisfying

$$\frac{\epsilon}{2} \leq \frac{y-x}{n} < \epsilon.$$

Suppose, for sake of contradiction, that for all i, we have $|f(y_i)-f(x_i)| \leq \frac{1}{\epsilon}|y_i-x_i|$. Then, by the triangle inequality, we have

$$\begin{aligned}|f(y)-f(x)| &\leq \sum_{i=1}^{n}|f(y_i)-f(x_i)| \\ &\leq \sum_{i=1}^{n}\frac{1}{\epsilon}|y_i-x_i| \\ &\leq \frac{1}{\epsilon}|y-x|,\end{aligned}$$

which is a contradiction. Thus, for some $I \in \{1, ..., n\}$, we have

$$|f(y_I)-f(x_I)| > \frac{1}{\epsilon}|y_I-x_I|.$$

Then, we have

$$\frac{\epsilon}{2} \leq \sum_{i=1}^{n}(y_I - x_I) < \epsilon,$$

and

$$\begin{aligned}
\sum_{i=1}^{n}|f(y_I) - f(x_I)| &\geq \sum_{i=1}^{n}\frac{1}{\epsilon}|y_I - x_I| \\
&= \frac{1}{\epsilon}\sum_{i=1}^{n}(y_I - x_I) \\
&\geq \frac{1}{\epsilon}\cdot\frac{\epsilon}{2} \\
&= \frac{1}{2}.
\end{aligned}$$

With this, our proof is complete. □

Problem 6

Use the function

$$f(x) = \begin{cases} x\sin\frac{1}{x} & \text{if } x \in (0, \pi] \\ 0 & \text{if } x = 0 \end{cases}$$

to show that the following statement is not true: If $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a + \epsilon, b]$ for any small $\epsilon > 0$, and f is continuous on $[a, b]$, then f is absolutely continuous on $[a, b]$.

Solution

For any small $\epsilon > 0$, we have for all $x \in [\epsilon, \pi]$,

$$\begin{aligned}
|f'(x)| &= \left|\sin\frac{1}{x} - \frac{\cos\frac{1}{x}}{x}\right| \\
&\leq \left|\sin\frac{1}{x}\right| + \left|\frac{\cos\frac{1}{x}}{x}\right| \\
&\leq 1 + \frac{1}{\epsilon}.
\end{aligned}$$

Thus, since $f'(x)$ is bounded on $x \in [\epsilon, \pi]$, an easy argument using the mean value theorem allows us to conclude that f is Lipschitz continuous, and therefore absolutely continuous.

Now, let $\delta > 0$. Choose $n \in \mathbb{N}$ large enough that $\frac{2}{\pi(1+2n)} < \delta$, and define $x_i = \frac{2}{\pi(1+2i)}$ for $i \geq n$. Then, we have $\{[x_i, x_{i-1}]\}_{i=n+1}^{\infty}$ is a set of nonoverlapping subintervals of $[0, \pi]$ such that $\sum_{i=n+1}^{\infty}(x_i - x_{i-1}) < \delta$. Furthermore, since

$$\frac{1}{x_i} = \frac{\pi}{2} + i\pi,$$

we have that $f(x_i)$ alternates between 1 and -1. Assume, without loss of generality, that $f(x_n) = -1$. Then, we have

$$\begin{aligned}
\sum_{i=n+1}^{\infty} |f(x_i) - f(x_{i-1})| &= \sum_{i=n+1}^{\infty} |x_i + x_{i-1}| \\
&= \sum_{i=n+1}^{\infty} \left(\frac{2}{\pi(1+2i)} + \frac{2}{\pi(1+2(i-1))} \right) \\
&= \frac{2}{\pi} \sum_{i=n+1}^{\infty} \left(\frac{1}{1+2i} + \frac{1}{2i-1} \right) \\
&= \frac{2}{\pi} \sum_{i=n+1}^{\infty} \frac{2i-1+2i+1}{(1+2i)(2i-1)} \\
&= \frac{2}{\pi} \sum_{i=n+1}^{\infty} \frac{4i}{4i^2-1} \\
&\geq \frac{2}{\pi} \sum_{i=n+1}^{\infty} \frac{4i}{4i^2} \\
&= \frac{2}{\pi} \sum_{i=n+1}^{\infty} \frac{1}{i} \\
&= \infty && \text{Harmonic series does not converge.}
\end{aligned}$$

Since this is true for any $\delta > 0$, we can conclude that f is not absolutely continuous (or even of bounded variation) on $[0, \pi]$.

Problem 7

Theorem 10. *If $f : [a, b] \to \mathbb{R}$ is continuous, $f \in BV[a, b]$, and $f \in AC[a + \epsilon, b]$ for any small $\epsilon > 0$, then $f \in AC[a, b]$.*