

Problem 1

Theorem 1. Let $f : (a, b) \rightarrow \mathbb{R}$ be such that for all $x_0 \in (a, b)$, there is a support line

$$l_{x_0}(x) = f(x_0) + m(x - x_0)$$

for some $m \in \mathbb{R}$ such that

$$f(x) \geq l_{x_0}(x)$$

for all $x \in (a, b)$. Then f is convex on (a, b) .

Solution

Proof. Suppose that f is not convex on (a, b) . Then, for some $[x_1, x_2] \subset (a, b)$, there exists an $x_0 \in [x_1, x_2]$ such that $f(x_0) > L(x_0)$, where L is the straight line such that $L(x_1) = f(x_1)$ and $L(x_2) = f(x_2)$. We have $l_{x_0}(x_1) \leq f(x_1) = L(x_1)$ and $l_{x_0}(x_2) \leq f(x_2) = L(x_2)$. Since L and l_{x_0} are both straight lines, we have that $l_{x_0}(x) \leq L(x)$ for all $x \in [x_1, x_2]$. However, we have

$$l_{x_0}(x_0) = f(x_0) + m(x_0 - x_0) = f(x_0) > L(x_0),$$

and we have reached a contradiction. Thus, we have proven that f is convex. \square

Problem 2

Theorem 2. Let $f : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous and $f'(x)$ be increasing except on a zero measure subset of $[a, b]$. Then f is convex on $[a, b]$.

Solution

We will start by proving a lemma which will aid in the proof of this theorem:

Lemma 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous and satisfy the midpoint convexity

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}.$$

for any $x_1, x_2 \in [a, b]$. Then f is convex on $[a, b]$.

Proof. We will first prove, that for any $n \in \mathbb{N}$, if $x_1, \dots, x_{2^n} \in [a, b]$, then

$$f\left(\frac{x_1 + \dots + x_{2^n}}{2^n}\right) \leq \frac{f(x_1) + \dots + f(x_{2^n})}{2^n}.$$

To prove this, we note that the base case ($n = 1$) is true by midpoint convexity of f . Now, suppose for some $n \in \mathbb{N}$ that if $x_1, \dots, x_{2^n} \in [a, b]$, then

$$f\left(\frac{x_1 + \dots + x_{2^n}}{2^n}\right) \leq \frac{f(x_1) + \dots + f(x_{2^n})}{2^n}.$$

Let $x_1, \dots, x_{2^{n+1}} \in [a, b]$. Then, we have

$$\begin{aligned} f\left(\frac{x_1 + \dots + x_{2^{n+1}}}{2^{n+1}}\right) &= f\left(\frac{\frac{x_1 + \dots + x_{2^n}}{2^n} + \frac{x_{2^n+1} + \dots + x_{2^{n+1}}}{2^n}}{2}\right) \\ &\leq \frac{f\left(\frac{x_1 + \dots + x_{2^n}}{2^n}\right) + f\left(\frac{x_{2^n+1} + \dots + x_{2^{n+1}}}{2^n}\right)}{2} \\ &= \frac{f(x_1) + \dots + f(x_{2^{n+1}})}{2^{n+1}}, \end{aligned}$$

and our induction is complete.

From elementary analysis, we have that rational numbers of the form $\frac{m}{2^n}$ with $1 \leq m \leq 2^n$ are dense in $[0, 1]$. Let $[a_1, b_2] \subseteq [a, b]$. Fix some $n \in \mathbb{N}$ and some $1 \leq m \leq 2^n$. Setting $x_i = a_i$ for $1 \leq i \leq m$, and $x_i = b_i$ for $m+1 \leq i \leq 2^n$, we see from the above argument that

$$\begin{aligned} f\left(\frac{x_1 + \dots + x_{2^n}}{2^n}\right) &\leq \frac{f(x_1) + \dots + f(x_{2^n})}{2^n} \\ &= \frac{mf(a_i) + (2^n - m)f(b_i)}{2^n} \\ &= \frac{m}{2^n}f(a_i) + \left(1 - \frac{m}{2^n}\right)f(b_i). \end{aligned}$$

Finally, let $\theta \in [0, 1]$ be a real number. For each $n \in \mathbb{N}$, define $1 \leq m_n \leq 2^n$ to be the largest number such that $\frac{m_n}{2^n} \geq \theta$. Then, $\frac{m_n}{2^n} \rightarrow \theta$, and it follows from the continuity of f that

$$f(\theta a_i + (1 - \theta)b_i) \leq \theta f(a_i) + (1 - \theta)f(b_i),$$

and we have shown that f is convex. □

Now we are ready to prove the main theorem:

Proof. Let $[x_1, x_2] \subseteq [a, b]$. By the above lemma, it will suffice to show that

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}.$$

Since f is absolutely continuous, we have

$$f\left(\frac{x_1 + x_2}{2}\right) - f(x_1) = \int_{x_1}^{\frac{x_1 + x_2}{2}} f'(x) dx,$$

and

$$f(x_2) - f\left(\frac{x_1 + x_2}{2}\right) = \int_{\frac{x_1 + x_2}{2}}^{x_2} f'(x) dx.$$

Now, since f' is increasing almost everywhere, we have

$$\int_{x_1}^{\frac{x_1 + x_2}{2}} f'(x) dx \leq \int_{\frac{x_1 + x_2}{2}}^{x_2} f'(x) dx.$$

With this, we have

$$\begin{aligned} f\left(\frac{x_1 + x_2}{2}\right) - f(x_1) &\leq f(x_2) - f\left(\frac{x_1 + x_2}{2}\right) \\ f\left(\frac{x_1 + x_2}{2}\right) &\leq \frac{f(x_1) + f(x_2)}{2} \end{aligned}$$

and we have proven that f is midpoint-convex, and therefore convex. □

Problem 3

Theorem 3. Let $f : [a, b] \rightarrow \mathbb{R}$ and let $E \subseteq [a, b]$. Assume that $f'(x)$ exists with a finite value for any $x \in E$. Then,

$$|f(E)|_e \leq \int_E |f'(x)| dx.$$