Problems from the textbook in HW 2

10. If E_1 and E_2 are measurable, show that $|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$.

13. Motivated by (3.7), define the *inner measure* of E by $|E|_i = \sup |F|$, where the supremum is taken over all closed subsets F of E. Show that (i) $|E|_i \leq |E|_e$, and (ii) if $|E|_e < +\infty$, then E is measurable if and only if $|E|_i = |E|_e$. (Use Lemma 3.22.)

15. If E is measurable and A is any subset of E, show that $|E| = |A|_i + |E - A|_e$. (See Exercise 13 for the definition of $|A|_i$.) As a consequence, using Exercise 13, show that if $A \subset [0,1]$ and $|A|_e + |[0,1] - A|_e = 1$, then A is measurable.

30. Let $f : \mathbf{R^n} \to \mathbf{R^1}$ be continuous. Show that the inverse image $f^{-1}(B)$ of a Borel set B is a Borel set; see p. 64 in Section 4.1 for the definition of the inverse image of a set. (The collection of sets $\{E : f^{-1}(E) \text{ is a Borel set}\}$ is a σ-algebra and contains all open sets in $\mathbf{R^1}$; cf. Exercise 10 of Chapter 4 and Corollary 4.15.) See also Exercise 22 of Chapter 4.

32. Let E be a set in $\mathbf{R^n}$ with $|E|_e > 0$ and let θ satisfy $0 < \theta < 1$. Show that there is a set $E_\theta \subset E$ with $|E_\theta|_e = \theta |E|_e$ and that E_θ can be chosen to be measurable if E is measurable. (If $Q(r)$ denotes the cube with edge length r centered at the origin, $0 < r < \infty$, then $|E \cap Q(r)|_e$ is a continuous monotone function of r.)