MA 653-01 Homework 5 (10/28/2022)

1. Let $|E| < \infty$ and E_i $(i = 1, 2, \cdots, m)$ be measurable subsets of E. Let $k \in \{1, 2, \cdots, m\}$. Show that if every point of E belongs to at least k of E_i, then there is i such that $|E_i| \geq \frac{k}{m}|E|$. (Hint: Consider the integral $\int_E \sum_{i=1}^{m} \chi_{E_i}$.)

2. Let f be continuous and nonnegative on $[a, b]$ where $-\infty < a < b < \infty$. Define a nondecreasing sequence of step functions $\{\phi_k\}$ on $[a, b]$ such that $\phi_k \to f$ on $[a, b]$, and then using the MCT theorem to show that
$$(L) \int_{[a,b]} f(x)\, dx = (R) \int_a^b f(x)\, dx.$$

3. Let $f \geq 0$ be measurable in $\mathbb{R}^n$. For $k = 1, 2, \cdots$, define the cut-off functions
$$f_k(x) = \begin{cases} f(x) & \text{if } f(x) < k, \\ 0 & \text{if } f(x) \geq k. \end{cases}$$
Show that (i) each f_k is measurable on $\mathbb{R}^n$; (ii) $\int_{\mathbb{R}^n} f_k(x)\, dx \to \int_{\mathbb{R}^n} f(x)\, dx$ as $k \to \infty$.

4. Suppose that $f \geq 0$ is continuous on $(0, 1]$ and the improper Riemann integral $(R) \int_0^1 f(x)\, dx = \lim_{a \to 0^+} (R) \int_a^1 f(x)\, dx$ exists (finite or $+\infty$). Use the results in Problems 2 and 3 to show that
$$(L) \int_{(0,1]} f(x)\, dx = (R) \int_0^1 f(x)\, dx.$$

5. Let f be nonnegative and measurable on a measurable set $E \subset \mathbb{R}^n$ with $|E| < \infty$.

   (i) If $f \leq M$ a.e. on E where $M > 0$ is constant, then
$$\int_E f = \inf \sum_j [\sup_{x \in E_j} f(x)] |E_j|,$$
   where the infimum is taken over all decompositions $E = \sum_j E_j$ of E into the union of a finite number of disjoint measurable sets E_j.

   Hint: Consider the nonnegative function $M - f$.

   (ii) Use a counter-example (e.g., $f(x) = 1/\sqrt{x}$ with $E = (0, 1]$) to show that (i) is false if the boundedness of f is not held.