

# Math 263 Notes

Jeremiah Vuong  
Los Angeles Pierce College  
[vuongjn5900@student.laccd.edu](mailto:vuongjn5900@student.laccd.edu)

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## 14 Differentiating Functions of Several Variables

### 14.1 The Partial Derivative

#### Question 14.1: Elementary Estimation of Two Variable Functions

Imagine an unevenly heated thin rectangular metal plate lying in the  $xy$ -plane with its lower left corner at the origin and  $x$  and  $y$  measured in meters. The temperature (in  $^{\circ}\text{C}$ ) at the point  $(x, y)$  is  $T(x, y)$ . See Figure 14.2 and Table 14.2. How does  $T$  vary near the point  $(2, 1)$ ?

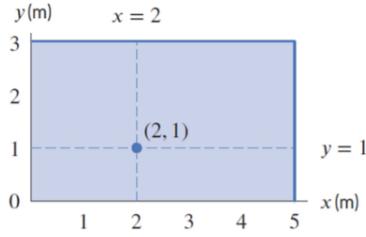


Figure 14.2: Unevenly heated metal plate

Table 14.2 Temperature ( $^{\circ}\text{C}$ ) of a metal plate

3	85	90	<b>110</b>	135	155	180
2	100	110	<b>120</b>	145	190	170
1	<b>125</b>	<b>128</b>	<b>135</b>	<b>160</b>	<b>175</b>	<b>160</b>
0	120	135	<b>155</b>	160	160	150
	0	1	2	3	4	5

**Solution:** We have two directions, such that we go left/right and up/down. Therefore, we define two functions of one variable.

$$u(x) = T(x, 1) \quad \text{also} \quad v(y) = T(2, y)$$

Such that:

$$\begin{aligned} u'(2) &= \lim_{h \rightarrow 0} \frac{u(2+h) - u(2)}{h} = \lim_{h \rightarrow 0} \frac{T(2+h, 1) - T(2, 1)}{h} \\ &\approx \frac{T(3, 1) - T(2, 1)}{3 - 2} = \frac{160 - 135}{1} = 25 \end{aligned}$$

$$\begin{aligned} v'(1) &= \lim_{h \rightarrow 0} \frac{v(1+h) - v(1)}{h} = \lim_{h \rightarrow 0} \frac{T(2, 1+h) - T(2, 1)}{h} \\ &\approx \frac{T(2, 2) - T(2, 1)}{2 - 1} = \frac{120 - 135}{1} = -15 \end{aligned}$$

#### Definition 14.1: Partial Derivatives of $f$ With Respect to $x$ and $y$

For all points at which the limits exist, we define the partial derivatives at the point  $(a, b)$  by

$$\begin{aligned} f_x(a, b) &= \text{Rate of change of } f \text{ with respect to } x \text{ at } (a, b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}, \\ f_y(a, b) &= \text{Rate of change of } f \text{ with respect to } y \text{ at } (a, b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}. \end{aligned}$$

If we let  $a$  and  $b$  vary, we have the partial derivative functions  $f_x(x, y)$  and  $f_y(x, y)$ .

Simply, we find the derivative of the cross section of the function with respect to the variable we are taking the partial derivative of.

If  $z = f(x, y)$ , we can write

$$\begin{aligned} f_x(x, y) &= \frac{\partial z}{\partial x} \quad \text{and} \quad f_y(x, y) = \frac{\partial z}{\partial y}, \\ f_x(a, b) &= \left. \frac{\partial z}{\partial x} \right|_{(a,b)} \quad \text{and} \quad f_y(a, b) = \left. \frac{\partial z}{\partial y} \right|_{(a,b)}. \end{aligned}$$

### Example 14.1 (Partial Derivatives Graphically)

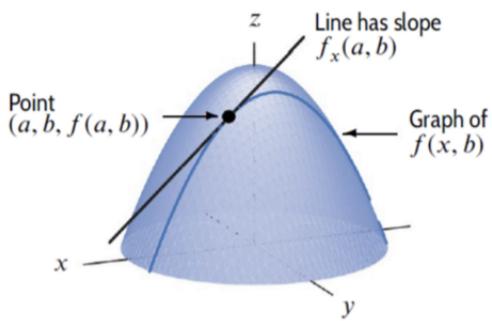


Figure 14.3: The curve  $z = f(x, b)$  on the graph of  $f$  has slope  $f_x(a, b)$  at  $x = a$

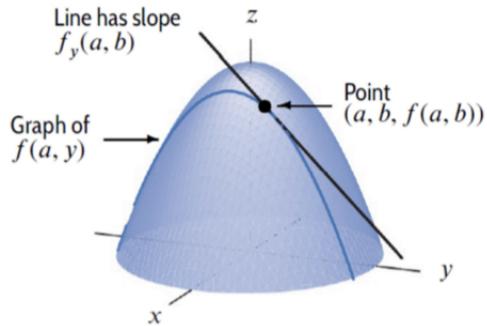


Figure 14.4: The curve  $z = f(a, y)$  on the graph of  $f$  has slope  $f_y(a, b)$  at  $y = b$

### Question 14.2: Applied Analysis of Partial Derivatives

The quantity,  $Q$ , of beef purchased at a store, in kilograms per week, is a function of the price of beef,  $b$ , and the price of chicken,  $c$ , both in dollars per kilogram.

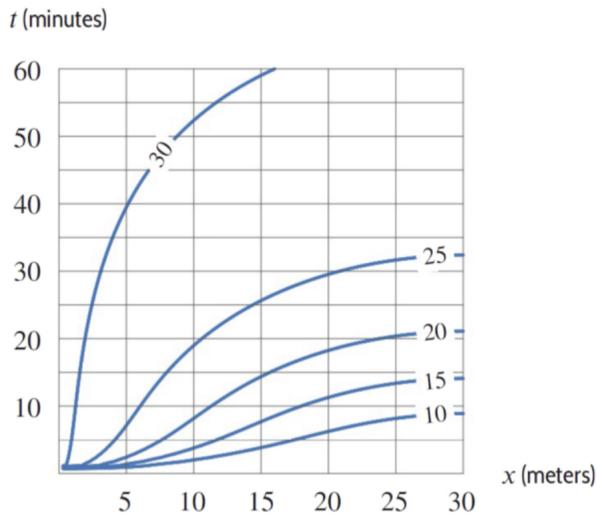
- Do you expect  $\partial Q / \partial b$  to be positive or negative? Explain.
- Do you expect  $\partial Q / \partial c$  to be positive or negative? Explain.
- Interpret the statement  $\partial Q / \partial b = -213$  in terms of quantity of beef purchased.

#### Solution:

- Keep  $c$  constant, increase  $b$ ,  $Q$  increases. The higher the beef prices, the lower purchase quantity. Therefore  $\partial Q / \partial b$  is negative.
- Keep  $b$  constant, increase  $c$ ,  $Q$  increases. The higher the chicken prices, the higher purchase quantity of beef. Therefore  $\partial Q / \partial c$  is positive.
- If the price of beef increases by 1 dollar per kilogram, the quantity of beef purchased decreases by 213 kilograms per week.

### Question 14.3

Figure 14.16 shows a contour diagram for the temperature  $T$  (in  $^{\circ}\text{C}$ ) along a wall in a heated room as a function of distance  $x$  along the wall and time  $t$  in minutes. Estimate  $\partial T/\partial x$  and  $\partial T/\partial t$  at the given points. Give units and interpret your answers. (a)  $x = 15, t = 20$  (b)  $x = 5, t = 12$



**Solution:**

$$\text{a)} \frac{\partial T}{\partial x} \Big|_{(15,20)} \approx \frac{23 - 20}{15 - 24} = \frac{3}{9} \approx -0.3$$

At 15 meters and 20 minutes along the wall the temperature is decreasing by  $0.3 \text{ } ^{\circ}\text{C}$  per minute.

$$\text{b)} \frac{\partial T}{\partial t} \Big|_{(15,20)} \approx \frac{23 - 25}{20 - 25} = \frac{-2}{-5} = 0.4$$

At 15 meters along the wall and 20 minutes the temperature is increasing by  $0.4 \text{ } ^{\circ}\text{C}$  per minute.

Extra)  $T_x(5, 12)$

$$\frac{\partial T}{\partial t} \Big|_{(5,12)} \approx \frac{15 - 26}{20 - 5} = \frac{-11}{15}$$

### Question 14.4

The temperature adjusted for wind chill is a temperature which tells you how cold it feels, as a result of the combination of wind and temperature. Estimate  $f_T(5, 20)$ . What does your answer mean in practical terms?

		Temperature ( $^{\circ}\text{F}$ )							
		35	30	25	20	15	10	5	0
Wind Speed (mph)	5	31	25	19	13	7	1	-5	-11
	10	27	21	15	9	3	-4	-10	-16
	15	25	19	13	6	0	-7	-13	-19
	20	24	17	11	4	-2	-9	-15	-22
	25	23	16	9	3	-4	-11	-17	-24

**Solution:**

$$f_t(5, 20) \approx \frac{f(5, 15) - f(5, 20)}{15 - 20} = \frac{7 - 13}{15 - 20} = \frac{-6}{-5} = 1.2 \text{ F} \text{ per actual degree.}$$

## 14.2 Computing Partial Derivatives Algebraically

### Question 14.5

Given the  $f(x, y) = x^2 + y^2$ , find  $f_x(2, 1)$ .

Find  $f_y(2, 1)$  and note on the following graphs what you found.

**Solution:** Keep  $y$  fixed (because the partial is with respect to  $x$ ):

$$f(x, 1) = x^2 + 1^2 \text{ Such that: } f_x(x, 1) = 2x \implies f_x(2, 1) = 4$$

$$f(x, y) = x^2 + y^2 \implies f(2, y) = (2)^2 + y^2 \implies f(2, y) = 4 + y^2$$

$$\text{Such that: } f_y(2, y) = 2y \implies f_y(2, 1) = 2$$

### Question 14.6

Find the partial derivative:

a)  $f_x$  and  $f_y$  if  $f(x, y) = 5x^2y^3 + 8xy^2 - 3x^2$

b)  $z_x$  if  $z = \frac{1}{2x^2ay} + \frac{3x^5abc}{y}$

c)  $\frac{\partial V}{\partial r}$  and  $\frac{\partial V}{\partial h}$  if  $V = \frac{4}{3}\pi r^2h$

d)  $g_x$  and  $g_y$  if  $g(x, y) = e^{x+3y} \sin(xy)$ .

**Solution:**

a)  $f(x, y) = 5x^2y^3 + 8xy^2 - 3x^2 \implies f_x(x, y) = 10xy^3 + 8y^2 - 6x$

$$f_y(x, y) = 15x^2y^2 + 16xy$$

b) Simplify  $z$  such that anything non  $x$  is constant:  $z = \frac{1}{2ay}x^{-2} + \frac{3abc}{y}x^5$

$$z_x = \frac{1}{2ay} - 2x^{-3} + \frac{3abc}{y}5x^4 = \frac{-1}{x^3ay} + \frac{15abc}{y}x^4$$

c)  $\frac{\partial V}{\partial r} = \frac{8}{3}\pi rh$

$$\frac{\partial V}{\partial h} = \frac{4}{3}\pi r^2$$

d) Note the chain rule:

$$g_x(x, y) = 1 \cdot e^{x+3y} \cdot \sin(xy) + y \cdot \cos(xy)e^{x+3y} = e^{x+3y}[\sin(xy) + y\cos(xy)]$$

$$g_y(x, y) = 3e^{x+3y}\sin(xy) + x\cos(xy)e^{x+3y}$$

### Example 14.2

$f(x, y) = x^2 + y^2$  is (II) since it is the same steepness in both  $x$  and  $y$  direction.



$f(x, y) = e^{x^2} + y^2$  is (III) since as we go to the right the graph is steeper as the contours get denser.



### Question 14.7

Let  $h(x, t) = 5 + \cos(0.5x - t)$  describe a wave. The value of  $h(x, t)$  gives the depth of the water in cm at a distance  $x$  meters from a fixed point and at time  $t$  seconds. Evaluate  $h_x(2, 5)$  and  $h_t(2, 5)$  and interpret each in terms of the wave.

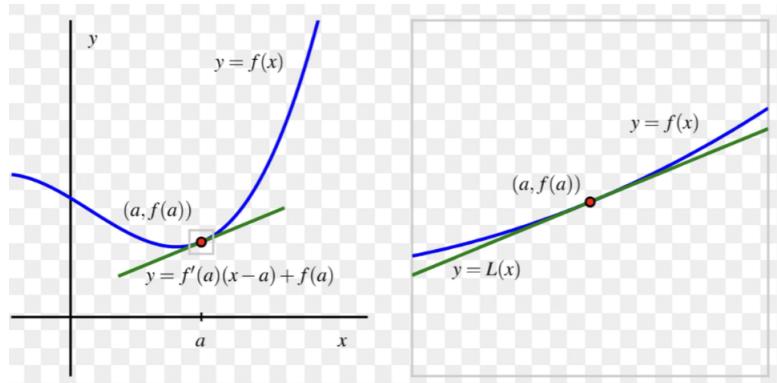
**Solution:**  $h_x(x, y) = -0.5\sin(0.5x - t) \implies h_x(2, 5) = -0.5\sin(1 - 5) \approx -0.38$

At 2 m and 5 s the depth of the water decreases by 0.38 cm per meter.

$h_y(x, y) = \sin(0.5x - t) \implies h_y(2, 5) = \sin(1 - 5) \approx 0.76$

### 14.3 Local Linearity and the Differential

Recall for functions of one variable, if the function is differentiable at a point, "nearby" that point, the curve acts almost like its tangent line. So the curve values can be approximated using the tangent line (the curve's linearization), provided we stay close to that point.

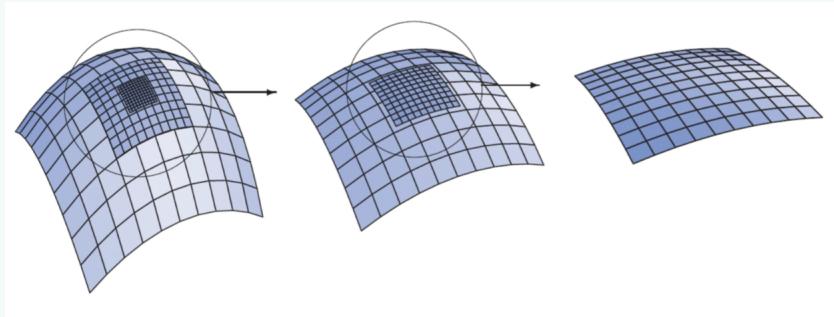


Where the point is  $(a, f(a))$  and the slope  $= m = f'(a)$  such that  $y - y_1 = m(x - x_1) \implies y = f(a) + f'(a)(x - a)$ . For functions of two variables, we have a similar situation.

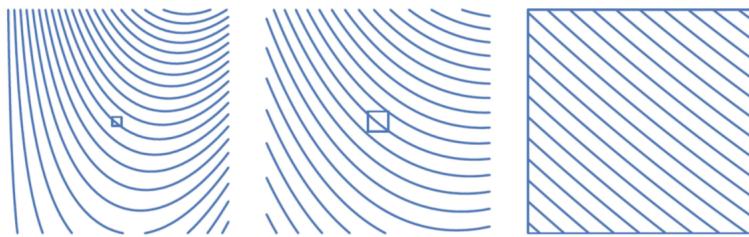
For functions of two variables, we have a similar situation.

#### Example 14.3

Zooming in near a point on the surface makes a surface act "linear" (plane).



Zooming in near a point the contour diagram makes contours lookk like equally spaced parallel lines (plane).



Zooming in near a point on the table of values makes the table of values look linear, that is, each col and row increases/decreases by a consistent amount (plane).

		y		
		0	1	2
x	1	1	2	9
	2	4	5	12
	3	9	10	17

		y		
		0.9	1.0	1.1
x	1.9	4.34	4.61	4.94
	2.0	4.73	5.00	5.33
	2.1	5.14	5.41	5.74

		y		
		0.99	1.00	1.01
x	1.99	4.93	4.96	4.99
	2.00	4.97	5.00	5.03
	2.01	5.01	5.04	5.07

Zooming in on values of  $f(x, y) = x^2 + y^3$  near  $(2, 1)$

When zooming in the  $z$  values act linear.

### Definition 14.2: Tangent Planes & Linear Approximation

Let  $S$  be a surface defined by a differentiable function  $z = f(x, y)$ , and let  $P_0 = (x_0, y_0)$  be a point in the domain of  $f$ . Then, the equation of the tangent plane to  $S$  at  $P_0$  is given by

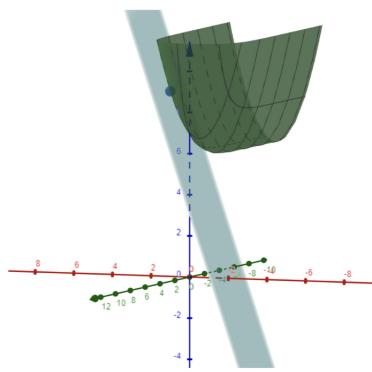
Given a function  $z = f(x, y)$  with continuous partial derivatives that exist at the point  $(x_0, y_0)$ , the linear approximation of  $f$  at the point  $(x_0, y_0)$  is given by

$$z = L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

### Question 14.8

Find the equation of the tangent plane:

$$z = e^y + x + x^2 + 6 \text{ at the point } (1, 0, 9)$$



**Solution:**

$$z_x = 1 + 2x|_{(1,0)} = 3$$

$$z_y = e^y|_{(1,0)} = 1$$

Such that:  $f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$$z = 9 + 3(x-1) + 1(y-0) \implies z = 3x + y + 6$$

### Question 14.9

A student was asked to find the equation of the tangent plane to the surface  $z = x^3 - y^2$  at the point  $(x, y) = (2, 3)$ . The student's answer was

$$z = 3x^2(x-2) - 2y(y-3) - 1.$$

- (a) At a glance, how do you know this is wrong?
- (b) What mistake did the student make?
- (c) Answer the question correctly.

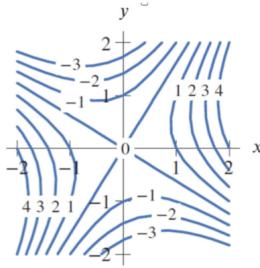
**Solution:** (a) There are multiple x and y's, i.e. it is not a plane.

(b) The student took the general partial derivative, not the partial derivative at the point  $(2, 3)$ .

$$(c) z = 3(2)^2(x-2) - 2(3)(y-3) - 1 \implies L(x, y) = 12(x-2) - 6(y-3) - 1$$

### Question 14.10

Give a linear function approximating  $z = f(x, y)$  near  $(1, -1)$  using its contour diagram in Figure 14.27.



**Solution:**  $z = f(1, -1) + f_x(1, -1)(x-1) + f_y(1, -1)(y+1)$

$$f_x(1, -1) = \frac{\Delta z}{\Delta x} = \frac{-1-0}{1-1.5} = 2$$

$$f_y(1, -1) = \frac{\Delta z}{\Delta y} = \frac{-1-0}{-1-(-.5)} = 2$$

$$z \approx L(x, y) = -1 + 2(x-1) + 2(y+1)$$

### Definition 14.3: Total Differential

Let  $z = f(x, y)$  be a function of two variables with  $(x_0, y_0)$  in the domain of  $f$ , and let  $\Delta x$  and  $\Delta y$  be chosen so that  $(x_0 + \Delta x, y_0 + \Delta y)$  is also in the domain of  $f$ . If  $f$  is differentiable at the point  $(x_0, y_0)$ , then the differentials  $dx$  and  $dy$  are defined as

$$dx = \Delta x \quad \text{and} \quad dy = \Delta y$$

The differential  $dz$ , also called the total differential of  $z = f(x, y)$  at  $(x_0, y_0)$ , is defined as

$$dz = f_x(x_0, y_0) dx + f_y(x_0, y_0) dy$$

We can think of it intuitively via approximation. Where the approximation of change in  $z \rightarrow dz$  is rate of change of x and y times their respective small change.

### Question 14.11

Find the differential for the given function.

$$z = e^{-x} \cos y$$

**Solution:**  $dz = z_x dx + z_y dy$

$$df = f_x dx + f_y dy$$

$$df = -e^{-x} \cos y dx - e^{-x} \sin y dy$$

### Question 14.12

A right circular cylinder has a radius of 50 cm and a height of 100 cm. Use differentials to estimate the change in volume of the cylinder if its height and radius are both increased by 1 cm.

**Solution:** Where  $V = \pi r^2 h = f(r, h)$ , given that the initial radius is 50 cm and height is 100 cm—and the change in both is 1cm.

Such that:  $\frac{\partial V}{\partial r} = V_r = 2\pi r h$      $\frac{\partial V}{\partial h} = V_h = \pi r^2$

$$\begin{aligned} dV &= \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh \\ dV &= 2\pi r h dr + \pi r^2 dh \\ &= 2\pi(50)(100)(1) + \pi(50)^2(1) \\ &\approx 39270 \text{ cm}^3 \end{aligned}$$

### Question 14.13

A small business has \$300,000 worth of equipment and 100 workers. The total monthly production,  $P$  (in thousands of dollars), is a function of the total value of the equipment,  $V$  (in thousands of dollars), and the total number of workers,  $N$ . The differential of  $P$  is given by  $dP = 4.9dN + 0.5dV$ . If the business decides to lay off 3 workers and buy additional equipment worth \$20,000, then what happens to production?

**Solution:**  $P = f(V, N)$  Such that:

$$\begin{aligned} dP &= 4.9dN + 0.5dV \\ &= 4.9(-3) + 0.5(20) \\ &= -4.7 \approx -4700 \text{ in production} \end{aligned}$$

### Question 14.14

Find  $df$  at  $(2, -4)$  if the tangent plane to  $z = f(x, y)$  at  $(2, -4)$  is  $z = -3(x - 2) + 2(y + 4) + 3$ .

**Solution:**  $df = f_x dx + f_y dy$

Recall  $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

Thus,  $df = -3dx + 2dy$

### Question 14.15

- Find the equation of the plane tangent to the graph of  $f(x, y) = x^2 e^{xy}$  at  $(1, 0)$ .
- Find the linear approximation of  $f(x, y)$  for  $(x, y)$  near  $(1, 0)$ .
- Find the differential of  $f$  at the point  $(1, 0)$ .
- Approximate  $f(1.1, 0.1)$  using differentials.

**Solution:**

- The plane tangent to the graph:  $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$   
Thus,  $z = 2(x - 1) + y + 1$

(b) The same thing as (a)

(c)  $df = f_x dx + f_y dy$

Thus,  $df = 2dx + dy$

(d) We defined the original points as  $(1,0)$  such that going to  $(1,1,0.1) \implies \Delta x = \Delta y = 0.1$

$df = 2(0.1) + 0.1 = 0.3$

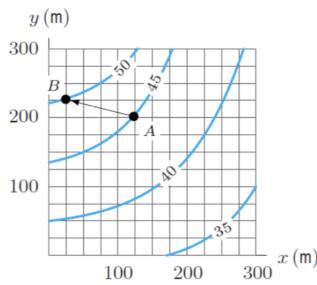
$f(1.1, 0.1) \approx f(1, 0) + df = 1.3$

Where upon checking the exact value,  $f(1, 1, 0.1) = 1.3506\dots$

## 14.4 Gradients and Direction Derivatives (Plane)

### Question 14.16

Figure 14.28 shows the temperature, in  $^{\circ}\text{C}$ , at the point  $(x, y)$ . Estimate the average rate of change of temperature as we walk from point  $A$  to point  $B$ .



**Solution:**  $\frac{\Delta T}{\Delta \text{dist}} = \frac{50-45}{\sqrt{100^2+25^2}} \approx 0.05$  As we move in the direction of  $AB$  the temperature increases by  $0.05 \text{ C}$  per meter.

#### Definition 14.4: Directional Derivative

We can think of the directional derivative as the partial derivative at a certain point with an infinitesimal nudge ( $\partial\vec{v}$ )—of course in the direction of  $\vec{v}$ . Where  $\vec{v}$  unitized (to determine the slope):

$$f_{\vec{v}} = \nabla_{\vec{v}} f(\vec{a}) = \frac{\partial f}{\partial \vec{v}} = \lim_{h \rightarrow 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h}$$

We can say that the directional derivative is in the direction (unitized) of  $\vec{w} = \langle a, b \rangle$ :

$$\nabla_{\vec{w}} f = a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y}$$

Such that:

$$\vec{w} \cdot \nabla f$$

#### Question 14.17

Determine if the directional derivative at the indicated point is positive, negative, or zero, in the direction of the vector  $\vec{V} = \vec{i} + 2\vec{j}$  and in the direction of the vector  $\vec{W} = 2\vec{i} + \vec{j}$ .

**Solution:** If we go in the direction of  $\vec{y}$  we get a negative, since it was at contour 6 and is now looking at 5. If we go in the direction of  $\vec{u}$  we get a positive, since it was at contour 6 and is going towards 7.

#### Question 14.18

If  $f(x, y) = xy$  and  $\vec{v} = 4\vec{i} - 3\vec{j}$ , find the directional derivative at the point  $(2, 6)$  in the direction of  $\vec{v}$

**Solution:**  $f_{\vec{u}}(a, b) = \lim_{h \rightarrow 0} \frac{f(a+hu_1, b+hu_2) - f(a, b)}{h}$

Unitizing  $\vec{v}$ :  $\vec{u} = \langle \frac{4}{5}, -\frac{3}{5} \rangle$

$$\begin{aligned} f_{\vec{u}} &= \lim_{h \rightarrow 0} \frac{f(2 + \frac{4}{5}h, 6 - \frac{3}{5}h) - f(2, 6)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2 + \frac{4}{5}h)(6 - \frac{3}{5}h) - 12}{h} \\ &= \lim_{h \rightarrow 0} \frac{12 + \frac{18}{5}h - \frac{12}{25}h^2 - 12}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(\frac{18}{5} - \frac{12}{25}h)}{h} \\ &= \frac{18}{5} = 3.6 \end{aligned}$$

Standing on  $(2, 6)$  and going in the direction of  $\vec{v}$  the function/z-value increases by 3.6 per unit.

#### Question 14.19

Find the directional derivative at  $(-1, 1)$  for  $f(x, y) = x^2 + y^2$  in the direction of  $\hat{i} + \hat{j}$

**Solution:** The limit definition in the direction of a point is  $\lim_{h \rightarrow 0} \frac{f(a+hu_1, b+hu_2) - f(a, b)}{h}$

Let  $\vec{v} = \langle 1, 1 \rangle$ . Unitizing it we get  $\vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ . Such that,

$$\begin{aligned} f_{\vec{u}} &= \lim_{h \rightarrow 0} \frac{f(-1 + \frac{1}{\sqrt{2}}h, 1 + \frac{1}{\sqrt{2}}h) - f(-1, 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(-1 + \frac{1}{\sqrt{2}}h)^2 + (1 + \frac{1}{\sqrt{2}}h)^2 - 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 - \frac{2}{\sqrt{2}} + \frac{1}{2}h^2 + 1 + \frac{2}{\sqrt{2}}h + \frac{1}{2}h^2 - 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2}{h} = 0 \end{aligned}$$

#### Question 14.20

If  $f(x, y) = xy$  and  $\vec{v} = 4\vec{i} - 3\vec{j}$ , find the directional derivative at the point  $(2, 6)$  in the direction of  $\vec{v}$  But, using partials this time!

**Solution:**  $f_x = y$  and  $f_y = x$ .

$$f_{\vec{u}} = \langle 6, 2 \rangle \cdot \langle \frac{4}{5}, -\frac{3}{5} \rangle = 3.6$$

#### Definition 14.5: Gradient Vector

$$\text{grad } f(a, b) = \nabla f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle = f_x(a, b)\hat{i} + f_y(a, b)\hat{j}$$

#### Question 14.21

a) Find the gradient vector for the function  $f(x, y) = x^2 - y^2$  at  $(1, 2)$ .

b) Find the directional derivative  $f_{\vec{u}}(1, 2)$  for the function  $f(x, y) = x^2 - y^2$  with  $\vec{u} = \frac{(3\vec{i} - 4\vec{j})}{5}$

**Solution:** to fill in later

**Note:-**

Recalling the definition of the dot product:

$$\begin{aligned} f_{\vec{u}}(a.b) &= \nabla f(a.b) \cdot \vec{u} \\ &= \|\nabla f(a.b)\| \|\vec{u}\| \cos \theta \end{aligned}$$

Max if  $\cos \theta = 1$  or  $\theta = 0$ . Means  $\nabla f$  and  $\vec{u}$  point in the same direction.

Min if  $\cos \theta = -1$  or  $\theta = \pi$ . Means  $\nabla f$  and  $\vec{u}$  point in opposite directions.

0 if  $\cos \theta = 0$  or  $\theta = \frac{\pi}{2}$ . Means  $\nabla f$  and  $\vec{u}$  are perpendicular.

Such that the gradient points in the direction of greatest increase. Such that  $\|\nabla f\|$  is the value of max slope.

The gradient vector is perpendicular to the contour line; always pointing in direction of increase.

**Question 14.22**

Find the gradient vector at given point and explore on its graph.  $f(x, y) = \sin(x^2) + \cos y$ , at  $(\frac{\pi}{2}, 0)$  insert picture later

**Solution:**  $f_x = 2x \cos(x^2)$  and  $f_y = -\sin y$ .

$\nabla f = \langle 2x \cos(x^2), -\sin y \rangle$  Such that  $\nabla f(\frac{\pi}{2}, 0) = \langle -\#, 0 \rangle$

Go left for steepest incline.

**Question 14.23**

The fluid pressure, in atmospheres, at the bottom of a body of water of varying depths is given by  $P(x, y) = 1 + \frac{x^2 y}{100}$ , where  $x$  and  $y$  are measured in meters. Find  $\nabla P(1, 2)$  and  $\|\nabla P(1, 2)\|$  and interpret them in the context of this problem.

**Solution:**  $\nabla P = \langle \frac{2xy}{100}, \frac{x^2}{100} \rangle$

$\nabla P(1, 2) = \langle \frac{4}{100}, \frac{1}{100} \rangle = \langle \frac{1}{25}, \frac{1}{100} \rangle$  ← means go in this direction when you're at (1,2) for maximum increase in fluid pressure.

$\|\nabla P(1, 2)\| = \sqrt{\frac{1}{625} + \frac{1}{10000}} \approx 0.04$  atm per meter.

**Question 14.24**

Which of the following vectors gives the direction of the gradient vector at point A in the contour diagram.  
The scales on the  $x$ - and  $y$ -axes are the same.

**Solution:** A) The gradient vector is perpendicular and towards the greatest increase. Thus,  $\nabla f = -\hat{j}$ .  
B)  $\nabla f = -2\hat{i} - \hat{j}$

**Example 14.4**

In Figure 14.39, which is larger:  $\|\nabla f\|$  at  $P$  or  $\|\nabla f\|$  at  $Q$ ? Explain how you know.  
 $P$ , we have close together contours so function is steeper near there.

**Note:-**

1. The Directional Derivative  $f_{\vec{u}}$  or  $D_{\vec{u}}f$  is simply the slope in a certain direction ( $\vec{u}$ )
2.  $f_{\vec{u}}$  is a rate of change and a scalar value.
3.  $f_{\vec{u}}$  is maximum in the direction of the gradient.
4.  $f_{\vec{u}}$  is minimum in the opposite direction of the gradient vector.
5. The gradient ( $\nabla f$ ) is perpendicular/orthogonal to contours and points in the direction of maximum change.
6. If contours are together closer together  $\|\nabla f\|$  is larger. Means  $f$  is increasing faster.

## 14.5 Gradients and Direction Derivatives in Space

We extend the idea from the Plane (2D) to Space (3D) and define the gradient as  $\text{grad } f(a, b, c) = f_x(a, b, c)\vec{i} + f_y(a, b, c)\vec{j} + f_z(a, b, c)\vec{k}$

As a result the directional derivative becomes

$$f_{\vec{u}}(a, b, c) = \nabla f \cdot \vec{u} = f_x(a, b, c)u_1 + f_y(a, b, c)u_2 + f_z(a, b, c)u_3$$

### Question 14.25

Find the directional derivative using  $f(x, y, z) = xy + z^2$  at  $(1, 2, 3)$  in the direction of  $\vec{i} + \vec{j} + \vec{k}$ .

**Solution:** Let  $\vec{v} = <1, 1, 1>$ , upon unitizing  $\vec{u} = <\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}>$

Getting the directional derivatives and evaluating them at  $(1, 2, 3)$

$$f_x = 2 \quad f_y = 1 \quad f_z = 6$$

Such that

$$f_{\vec{u}}(1, 2, 3) = <2, 1, 6> \cdot <\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}> = \frac{9}{\sqrt{3}}$$

Therefore,  $f$  ( $\mathbb{R}^4$ ) is changing at the rate of change of  $\frac{9}{\sqrt{3}}$  at the point  $(1, 2, 3)$  in the direction of  $\vec{v}$ .

**Note:-**

1.  $\nabla f(a, b, c)$  is in the direction of greatest rate of increase for  $f$ .
2.  $\nabla f(a, b, c)$  is perpendicular to the level surface of  $f$  at  $(a, b, c)$ .
3.  $\|\nabla f(a, b, c)\|$  is the max rate of change of  $f$  at  $(a, b, c)$ .

### Question 14.26

A thin column of ice lies along the  $z$ -axis, and the temperature at any point in space is given by  $T(x, y, z) = 5(x^2 + y^2)$  degrees Celsius, where  $x, y$ , and  $z$  are measured in meters.

- (a) Is the surface shown in Figure 1 the graph of the function  $T$ ? If not, then what is it?
- (b) Suppose that  $P = (0, 2, 2)$  and  $Q = (\sqrt{2}, \sqrt{2}, 0)$  are points on the level surface  $T(x, y, z) = 20$ . Calculate and draw in  $\nabla T(0, 2, 2)$  and  $\nabla T(\sqrt{2}, \sqrt{2}, 0)$ . Then, give a practical interpretation of both of these quantities.
- (c) How fast does the temperature change as you move away from  $P = (0, 2, 2)$  in the direction of  $R = (0, 3, 3)$ ? Include units with your answer.

Figure 1.  $T(x, y, z) = 20$

**Solution:**

- (a) No,  $T$  is a 4D function. The surface is a level surface of  $T$ , where  $T(x, y, z) = 20$ .
- (b)

$$\begin{aligned} \nabla T &= < T_x, T_y, T_z > = < 10x, 10y, 0 > \\ \nabla T(P) &= < 10(0), 10(2), 0 > = < 0, 20, 0 > \\ \nabla T(Q) &= < 10(\sqrt{2}), 10(\sqrt{2}), 0 > = < 10\sqrt{2}, 10\sqrt{2}, 0 > \end{aligned}$$

At P we should go in the direction of  $<0, 20, 0>$  for maximum/fastest increase in temperature.

At Q we should go in the direction of  $<10\sqrt{2}, 10\sqrt{2}, 0>$  for maximum/fastest increase in temperature.

(c) From P to R such that,  $\vec{PR} = \langle 0, 1, 1 \rangle$  and unitizing it we get  $\vec{u} = \langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$$\begin{aligned} f_{\vec{u}}(0, 2, 2) &= \nabla T \cdot \vec{u} \\ &= \langle 0, 20, 0 \rangle \cdot \langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \\ &= \frac{20}{\sqrt{2}} \implies \text{temp is increase by } \frac{20}{\sqrt{2}} \text{ per meter.} \end{aligned}$$

### Question 14.27

Find the directional derivative for the function  $f(x, y, z) = xy + z^2$  at  $(0, 1, 1)$  as you arrive at  $(0, 1, 1)$  from the direction of  $(1, 1, 0)$ .

**Solution:** From  $(1, 1, 0)$  to  $(0, 1, 1)$  we get  $\vec{u} = \langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle$

Taking the partial derivatives and evaluating them at  $(0, 1, 1)$

$$f_x = 1 \quad f_y = 0 \quad f_z = 2$$

Such that

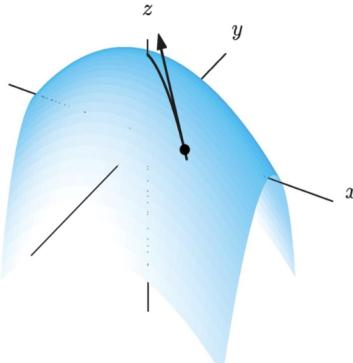
$$f_{\vec{u}}(0, 1, 1) = \langle 1, 0, 2 \rangle \cdot \langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle = \boxed{\frac{1}{\sqrt{2}}}$$

### Question 14.28

A hiker on the surface  $f(x, y) = 4 - x^2 - 2y^2$  at the point  $(1, -1, 1)$  starts to climb along the path of steepest ascent.

A) What is the relation between the vector  $\nabla f(1, -1)$  and a vector tangent to the path at the point  $(1, -1, 1)$  and pointing uphill?

B) At the point  $(1, -1, 1)$  calculate a vector perpendicular to the surface.



**Solution:**

(A) We know that  $\nabla f(1, -1)$  is a  $\mathbb{R}^2$  vector on the xy-plane.

Such that the tangent vector to the surface has the same x and y components as the gradient, but also a z component (which the gradient does not).

(B) We can treat  $f$  (a  $\mathbb{R}^3$  function) as a level surface of a  $\mathbb{R}^4$  function.

$$\begin{aligned} z &= 4 - x^2 - 2y^2 \\ 0 &= 4 - x^2 - 2y^2 - z \\ F(x, y, z) &= 4 - x^2 - 2y^2 - z \end{aligned}$$

If we look at the level surface of  $f = 0$  we get

$$\begin{aligned}\nabla f &= \langle f_x, f_y, f_z \rangle = \langle -2x, -4y, -1 \rangle \\ \nabla f(1, -1, 1) &= \langle -2, 4, -1 \rangle \text{ or } \langle 2, -4, 1 \rangle\end{aligned}$$

**Note:-**

The previous question leads us to revisiting the equation of a tangent plane. Since we know  $\nabla f$  is orthogonal to the level surfaces,  $\nabla f$  can act as a normal to the tangent plane at  $(a, b, c)$ .

We have  $\nabla f(a, b, c) = \langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c) \rangle = \vec{n}$ .

Take any general vector on that plane containing  $(a, b, c)$  we get  $\vec{v} = \langle x - a, y - b, z - c \rangle$

Such that  $\nabla f \cdot \vec{v} = 0$  as the normal to the plane is perpendicular to all vectors on that plane.

$$f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0$$

**Question 14.29**

Find the equation of the tangent plane to the sphere  $x^2 + y^2 + z^2 = 14$  at the point  $(1, 2, 3)$ .

**Solution:** We can think of the equation as a level surface of a  $\mathbb{R}^4$  function. Such that

$$x^2 + y^2 + z^2 - 14 = F(x, y, z)$$

One level surface is 0:

$$F(x, y, z) = x^2 + y^2 + z^2 - 14 = 0$$

$\nabla F$  will be normal to the sphere

$$\begin{aligned}\nabla F &= \langle F_x, F_y, F_z \rangle = \langle 2x, 2y, 2z \rangle \\ \nabla F(1, 2, 3) &= \langle 2, 4, 6 \rangle\end{aligned}$$

Such that

$$2(x - 1) + 4(y - 2) + 6(z - 3) = 0$$

**Question 14.30**

Find the equation of the tangent plane to the surface  $2e^z = \cos z + x^3yz + xy^3$  at the point  $(1, 1, 0)$ .

**Solution:** in notebook. fill in later.

**Question 14.31**

The temperature of a gas at the point  $(x, y, z)$  is given by  $G(x, y, z) = x^2 - 5xy + y^2z$ .

- (a) What is the rate of change in the temperature at the point  $(1, 2, 3)$  in the direction  $\vec{v} = 2\vec{i} + \vec{j} - 4\vec{k}$  ?
- (b) What is the direction of maximum rate of change of temperature at the point  $(1, 2, 3)$  ?
- (c) What is the maximum rate of change at the point  $(1, 2, 3)$  ?

**Solution:**

(a)

$$\begin{aligned}\nabla G(x, y, z) &= \langle 2x - 5y, -5x + 2yz, y^2 \rangle \\ \nabla(1, 2, 3) &= \langle -8, 7, 4 \rangle\end{aligned}$$

Unitizing  $\vec{v}$  we get  $\vec{u} = \langle \frac{2}{\sqrt{21}}, \frac{1}{\sqrt{21}}, -\frac{4}{\sqrt{21}} \rangle$

$$G_{\vec{u}}(1, 2, 3) = \nabla G \cdot \vec{u} = \langle -8, 7, 4 \rangle \cdot \langle \frac{2}{\sqrt{21}}, \frac{1}{\sqrt{21}}, -\frac{4}{\sqrt{21}} \rangle = \boxed{\#}$$

- (b) The direction of maximum rate of change is the direction of the gradient vector.  $\langle -8, 7, 4 \rangle$   
(c) Find the magnitude of the gradient vector at  $(1, 2, 3)$ .

$$\|\nabla G(1, 2, 3)\| = \sqrt{(-8)^2 + 7^2 + 4^2} = \boxed{\#}$$

### Question 14.32

At what point on the surface  $z = 1+x^2+y^2$  is its tangent plane parallel to the following plane?  $z = 4+6x-4y$

**Note:-**

Summary of Gradients  
To fill in

## 14.6 The Chain Rule

Recall from single variable Calculus  $y = f(x)$  and  $x = g(t)$ , we have  $y = f(g(t))$

So  $y$  is a function of  $x$  and  $x$  is a function of  $t$ , so ultimately  $y$  is a function of  $t$ . When there is a change in  $t$  that will cause a change in  $x$  and the change in  $x$  will cause a change in  $y$ .

And  $\frac{dy}{dt} = \frac{dy}{dx} = \frac{dx}{dt}$

Now we have  $z = f(x, y)$  and  $x = g(t), y = h(t)$ , so we get  $z = f((t), h(t))$  we want to find  $\frac{dz}{dt}$  that means How does the change in  $t$  cause a change in  $x$  and a change in  $y$ , that ultimately causes a change in  $z$ ? We use local linearity to figure this out.

We know

$$\begin{aligned}\frac{\Delta x}{\Delta t} &= \frac{dx}{dt} \implies \Delta x = \frac{dx}{dt} \Delta t \\ \frac{\Delta y}{\Delta t} &= \frac{dy}{dt} \implies \Delta y = \frac{dy}{dt} \Delta t\end{aligned}$$

Such that

$$\begin{aligned}\Delta z &= f_x dx + f_y dy \\ &= \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y \\ &= \frac{\partial f}{\partial y} \cdot \frac{dx}{dt} \Delta t + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \Delta t \\ &= \Delta t \left( \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \right) \\ \frac{\Delta z}{\Delta t} &= \frac{\Delta t \left( \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \right)}{\Delta t}\end{aligned}$$

**Definition 14.6: The Chain Rule for  $z = f(x, y), x = g(t), y = h(t)$**

If  $f, g$ , and  $h$  are differentiable and if  $z = f(x, y), x = g(t), y = h(t)$ , then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}.$$

Insert derivative tree

**Note:-**

$z$  is a function of two variables  $x$  and  $y$ . So use partials.  
 $x$  and  $y$  are functions of  $t$ . So use regular derivatives.

**Question 14.33**

Find  $\frac{dz}{dt}$ . Assume the variables are restricted to the domains on which the functions are defined.

$$z = x \sin y + y \sin x, x = t^2, y = \ln t$$

**Solution:** We know that  $z$  depends on  $x$  and  $y$  and  $x$  and  $y$  depend on  $t$ .

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= (\sin y + y \cos x)(2t) + (x \cos y + \sin x)(1/t) \\ \frac{dz}{dt} &= (\sin(\ln t) + \ln t \cos(t^2))(2t) + (t^2 \cos(\ln t) + \sin(t^2))(1/t)\end{aligned}$$

**Question 14.34**

Find  $\frac{dz}{dt}$ . Assume the variables are restricted to the domains on which the functions are defined.

$$z = (x + y)e^y, x = 2t, y = 1 - t^2$$

**Solution:**

**Definition 14.7**

If  $f, g, h$  are differentiable and if  $z = f(x, y)$ , with  $x = g(u, v)$  and  $y = h(u, v)$ . Then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \text{ and } \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$$

**Question 14.35**

Find  $\frac{dz}{du}$  and  $\frac{dz}{dv}$ . Assume the variables are restricted to the domains on which the functions are defined.

$$z = \ln(xy), x = (u^2 + v^2)^2, y = (u^3 + v^3)^2$$

- A) For  $\frac{dz}{du}$  use the chain rule.
- B) For  $\frac{dz}{dv}$  do a direct substitution to get  $z = f(u, v)$ , then differentiate.

**Solution:**

A)

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \\ &= \left(\frac{1}{xy} \cdot y\right)(4u^3 + 4uv^2) + \left(\frac{1}{xy} \cdot x\right)(6u^5 + 6u^2v^3) \\ &= \frac{1}{(u^2 + v^2)^2} \cdot 4u(u^2 + v^2) + \frac{1}{(u^3 + v^3)^2} \cdot 6u^2(u^3 + v^3) \\ \frac{\partial z}{\partial u} &= \frac{4u}{u^2 + v^2} + \frac{6u^2}{u^3 + v^3}\end{aligned}$$

B)

$$\begin{aligned} z &= \ln(xy) = \ln((u^2 + v^2)^2(u^3 + v^3)^2) \\ &= \ln(u^2 + v^2)^2 + \ln(u^3 + v^3)^2 \\ &= 2\ln(u^2 + v^2) + 2\ln(u^3 + v^3) \\ \frac{\partial z}{\partial v} &= 2 \cdot \frac{1}{y^2 + v^2} \cdot 2v + 2 \cdot \frac{1}{u^3 + v^3} \cdot 3v^2 \\ Z_v &= \frac{4v}{u^2 + v^2} + \frac{6v^2}{u^3 + v^3} \end{aligned}$$

### Question 14.36

- A) Suppose that  $w = f(x, y, z)$  is a differentiable function of  $x, y$ , and  $z$ , where  $x = g(t), y = h(t)$ , and  $z = k(t)$  are all differentiable functions of  $t$ . Draw a dependence diagram and write out the corresponding formula(s) for the Chain Rule.
- B) Suppose that  $z = f(x, y)$  is a differentiable function of  $x$  and  $y$ , where  $x = g(r, s, t)$  and  $y = h(r, s, t)$  are both differentiable functions of  $r, s$ , and  $t$ . Draw a dependence diagram and then write out the corresponding formula(s) for the Chain Rule.

*Solution:*

### Question 14.37: Test Likely

If  $z = f(x, y)$ , where  $f$  is differentiable and

$$\begin{array}{ll} x = g(t) & y = h(t) \\ g(3) = 2 & h(3) = 7 \\ g'(3) = 5 & h'(3) = -4 \\ f_x(2, 7) = 6 & f_y(2, 7) = -8 \end{array}$$

Find  $\frac{dz}{dt}$  when  $t = 3$

*Solution:*

### Question 14.38

Wheat production  $W$  in a given year depends on the average temperature  $T$  and the annual rainfall  $R$ . Scientists estimate that the average temperature is rising at a rate of 0.15 degrees Celsius/year and rainfall is decreasing at a rate of 0.1 cm/year. They also estimate that, at current production levels,  $W_T = -2$  and  $W_R = 8$ .

- a) What is the significance of the signs of these partial derivatives?  
b) Estimate the current rate of change of wheat production,  $\frac{dW}{dt}$ .

**Solution:** We think of  $\frac{dT}{dt} = 0.15$  C/year and  $\frac{dR}{dt} = -0.1$  cm/year.

$W_T = -2 = \frac{\partial W}{\partial T}$  wheat prod/temp and  $W_R = 8 = \frac{\partial W}{\partial R}$  wheat prod/rainfall.

- a)  $W_T$  as the temperature  $T$  increases, wheat production  $W$  decreases.  
 $W_R$  as the rainfall  $R$  increases, wheat production  $W$  increases.

b)

$$\begin{aligned} \frac{dW}{dt} &= \frac{\partial W}{\partial T} \cdot \frac{dT}{dt} + \frac{\partial W}{\partial R} \cdot \frac{dR}{dt} \\ &= (-2)(0.15) + (8)(-0.1) \\ &= -1.1 \text{ wheat production/year} \end{aligned}$$

## 14.7 Second-Order Partial Derivatives

### Note:-

Recall from single variable Calculus the first derivative of  $f$  gave us some information about  $f$ . If  $f'(x) > 0$  then  $f$  is increasing and if  $f'(x) < 0$  then  $f$  is decreasing.

And the second derivative of  $f$  also gave us some information, it told us how  $f$  increases or decreases. If  $f''(x) > 0$  then  $f$  is concave up and if  $f''(x) < 0$  then  $f$  is concave down.

If the concave is up the instantaneous rate of changes are increasing, vice versa.

### Definition 14.8: Second-Order Partial Derivatives

$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}$$

$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}$$

### Question 14.39

Calculate all four second-order partial derivatives for  $z = e^{xy}$

*Solution:*

$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} (ye^{xy}) = y^2 e^{xy}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} (xe^{xy}) = x^2 e^{xy}$$

$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} (xe^{xy}) = xy e^{xy} + e^{xy}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (ye^{xy}) = xy e^{xy} + e^{xy}$$

### Question 14.40

Calculate all four second-order partial derivatives for  $z = f(x, y) = x^2 + x \ln y$

*Solution:*

$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} (2x + \ln y) = 2$$

$$\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{x}{y} \right) = -\frac{x}{y^2}$$

$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left( \frac{x}{y} \right) = \frac{1}{y}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (2x + \ln y) = \frac{1}{y}$$

In our examples,  $\frac{\partial^2 z}{\partial x \partial y}$  and  $\frac{\partial^2 z}{\partial y \partial x}$  are the same. This is not a coincidence. These so-called mixed partials are equal whenever they are both continuous. And most functions we can imagine are not only continuous but also differentiable, so most of the function we deal with here will have equal mixed partial derivatives.

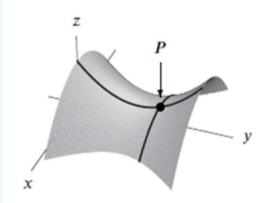
### Theorem 14.1 Equality of Mixed Partial Derivatives

If  $f_{xy}$  and  $f_{yx}$  are continuous at  $(a, b)$ , an interior point of their domain, then  $f_{xy}(a, b) = f_{yx}(a, b)$ .

### Example 14.5

$f_{xx}(P) < 0$  as the curve parallel to the z-axis is concave down.

$f_{yy}(P) > 0$  as the curve parallel to the y-axis is concave up.



### Example 14.6

A)

$f_x(P) < 0$  Such that  $x$  is increasing and  $z$  is decreasing.

$f_{xx}(P) > 0$  Such that  $f_x(P) < 0$  and is getting less steep.

$f_y(P) = 0$  as there is no change in the  $\hat{j}$  direction.

$$f_{yy}(P) = 0 \implies f_{xy}(P) = 0$$

B)

$f_x(P) < 0$

$f_{xx}(P) > 0$  such that is is negative and getting less steep.

$f_y(P) > 0$

$f_{yy}(P) > 0$  such that is is positive and getting more steep.

$f_{xy}(P) < 0$  such that  $f_x(P) < 0, -1, -2, -3$ .

C)

$f_x(P) = 0$  as there is no change in the  $\hat{i}$  direction.

$$f_{xx}(P) = 0 \implies f_{xy}(P) = 0$$

$f_y(P) > 0$  as the z-value is increasing

$f_{yy}(P) < 0$  as  $f_y > 0$  and is getting less steep.

### 14.7.1 Taylor Polynomials

Taylor Polynomial of Degree 1 Approximating  $f(x, y)$  for  $(x, y)$  near  $(0, 0)$  If  $f$  has continuous first-order partial derivatives, then

$$f(x, y) \approx L(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y$$

We can improve this approximation, using a Taylor polynomial of degree 2 approximating  $f(x, y)$  at  $(0, 0)$  with a quadratic approximation of  $f$ , basically a quadratic function of the form:

$$f(x, y) \approx Q(x, y) = A + Bx + Cy + Dx^2 + Exy + Fy^2$$

$f(0, 0) = A$  as all the  $x$  die out due to 0.

$$f_x(x, y) = B + 2Dx + Ey|_{(0,0)} \implies f_x(0, 0) = B$$

$$f_{xx}(x, y) = 2D|_{(0,0)} \implies f_{xx}(0, 0) = 2D \implies D = \frac{1}{2}f_{xx}(0, 0)$$

$$\begin{aligned} f_y(x, y) &= C + Ex + 2Fy|_{(0,0)} \implies f_y(0, 0) = C \\ f_{yy}(x, y) &= 2F|_{(0,0)} \implies f_{yy}(0, 0) = 2F \implies F = \frac{1}{2}f_{yy}(0, 0) \\ f_{xy}(x, y) &= E|_{(0,0)} \implies f_{xy}(0, 0) = E \end{aligned}$$

$$Q(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2}f_{xx}(0, 0)x^2 + \frac{1}{2}f_{yy}(0, 0)y^2 + f_{xy}(0, 0)xy$$

### Question 14.41

- A) Find the first degree Taylor polynomial about  $(0, 0)$  for  $e^x \cos y$ .  
 B) Find the quadratic Taylor polynomial about  $(0, 0)$  for  $e^x \cos y$ .

**Solution:**

A)

$$\begin{aligned} L(x, y) &= f(0, 0) + f_x(0, 0)x + f_y(0, 0)y \\ &= 1 + x \end{aligned}$$

B)

$$\begin{aligned} Q(x, y) &= f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2}f_{xx}(0, 0)x^2 + \frac{1}{2}f_{yy}(0, 0)y^2 + f_{xy}(0, 0)xy \\ &= 1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2 \end{aligned}$$

## 14.8 Differentiability (Condensed)

Recall from single variable calculus,

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

Simply, if the limit exists, you have a tangent line at that point.

When you zoom on the curve, of the differentiable point, the curve acts like its tangent line.

A function was NOT differentiable:

- If the limit does not exist.
- Not continuous
- vertical line (not defined)

### Definition 14.9: Differentiability for Functions of Two Variables

A function  $f(x, y)$  is differentiable at the point  $(a, b)$  if there is a linear function  $L(x, y) = f(a, b) + m(x - a) + n(y - b)$  such that if the error  $E(x, y)$  is defined by

$$f(x, y) = L(x, y) + E(x, y)$$

and if  $h = x - a, k = y - b$ , then the relative error

$$\frac{E(a+h, b+k)}{\sqrt{h^2 + k^2}} \text{ satisfies } \frac{E(a+h, b+k)}{\sqrt{h^2 + k^2}} = 0$$

The function  $f$  is differentiable on a region  $R$  if it is differentiable at each point of  $R$ . The function  $L(x, y)$  is called the local linearization of  $f(x, y)$  near  $(a, b)$ .

Simply, a function  $f$  is differentiable at  $(a, b)$  if it can be "well-approximated" by its tangent plane at  $(a, b)$  for nearby values.

#### Question 14.42

Consider the function  $f(x, y) = \sqrt{x^2 + y^2}$ . Is  $f$  differentiable at the origin?

Insert image.

**Solution:** No, it has a sharp corner at the origin. It is not smooth.

#### Question 14.43

List the points in the  $xy$ -plane, if any, at which the function  $z = |x| + |y|$  is not differentiable.

**Solution:** No, it's not differentiable along the x or y axis.

#### Question 14.44

Is  $x^2 + y^2 + z^2 = 9$  differentiable at a)  $(0, 0, 3)$  b)  $(0, 3, 0)$

**Solution:** a)  $\implies z = 3$  such that it is a tangent plane, yes.

b) no, the tangent plane becomes vertical such that there is no  $c$  that exists such that  $z = c$ .

## 15 Optimization: Local and Global Extrema

### 15.1 Local and Global Extrema

Recall from single variable calculus,

The local extrema, the local max/min, the function's highest/lowest value in a "nearby region."

The absolute/global extrema, the global max/min, the function's highest/lowest value in its domain.

Also recall the critical points of  $f(x)$ , where  $f'(x) = 0$  or  $f'(x)$  DNE.

However, a critical point does not guarantee a local extrema.

#### Definition 15.1

$f$  has a local maximum at the point  $P_0$  if  $f(P_0) \geq f(P) \quad \forall P$  near  $P_0$   
to fill in

Since we know a function has the greatest increase in the direction of the gradient and the greatest decrease in the direction opposite to the gradient, if we do have a local min/max then

$$\nabla f(P_0) = \vec{0} \text{ or } \nabla f(P_0) = \text{undefined.}$$

#### Example 15.1

Once we find the Critical Points, we have to analyze and see if indeed a local max/min occur there. For our visual purposes, we can have three things happen at the critical points (though possibly none occur):

- A) min
- B) max
- C) neither max/min.

#### Question 15.1

Find and analyze the critical points of  $f(x, y) = -\sqrt{x^2 + y^2}$

**Solution:** We know that this is a cone.

$$f(x, y) = -(x^2 + y^2)^{\frac{1}{2}}$$
$$f_x = \frac{-x}{\sqrt{x^2 + y^2}}$$
$$f_y = \frac{-y}{\sqrt{x^2 + y^2}}$$

$$f_x = 0 \implies -x = 0 \quad x = 0 \quad y \neq 0$$

$$f_y = 0 \implies -y = 0 \quad y = 0 \quad x \neq 0$$

#### Question 15.2

Find and analyze the critical points of  $f(x, y) = x^2 - y^2$

**Solution:**  $\nabla f = < 2x, 2y >$

$$f_x = 0 \implies x = 0 \quad f_y = 0 \implies y = 0$$

Our critical point is  $(0, 0)$ .

### Definition 15.2: Second Derivative Test

Suppose  $f_x(x_0, y_0) = 0$  and  $f_y(x_0, y_0) = 0$ , and let

$$D = D(x_0, y_0) = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

Such that  $D$  is the determinant, a determinant of  $\begin{vmatrix} f_{xy} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$

- If  $D > 0$  and  $f_{xx}(x_0, y_0) > 0$ , then  $f$  has a local minimum at  $(x_0, y_0)$ .
- If  $D > 0$  and  $f_{xx}(x_0, y_0) < 0$ , then  $f$  has a local maximum at  $(x_0, y_0)$ .
- If  $D < 0$ , then  $f$  has a saddle point at  $(x_0, y_0)$ .
- If  $D = 0$ , then anything can happen:  $f$  could have a local maximum, or a local minimum, or a saddle point at  $(x_0, y_0)$ .

### Question 15.3

Find the critical points and classify as local maxima, local minima, saddle point or neither.

A)

$$f(x, y) = 5 + 6x - x^2 + xy - y^2$$

B)

$$f(x, y) = x^2y + 2y^2 - 2xy + 6$$

**Solution:**

A)  $\nabla f = \vec{0}$

$$f_x = 0 \implies 6 - 2x + y = 0$$

$$f_y = 0 \implies x - 2y = 0$$

Such that,  $y = 2$ ,  $x = 4$ , we have a critical point at  $(4, 2)$

$$f_{xx} = -2 \quad f_{yy} = -2 \quad f_{xy} = 1$$

$$\begin{aligned} D(x_0, y_0) &= f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 \\ &= (-2)(-2) - (1)^2 \\ &= 3 \end{aligned}$$

Such that  $D > 0$  and  $f_{xx} < 0$  we have a local maxima at  $(4, 2)$

B)  $\nabla f = \vec{0}$

$$f_x = 0 \implies 2xy - 2y = 0 \implies 2y(x - 1) = 0$$

Such that  $y = 0$  or  $x = 1$  is a critpoint

$$f_y = 0 \implies x^2 + 4y - 2x = 0$$

$$f_y : y = 0 \implies x = 2$$

$$f_y : x = 1 \implies y = \frac{1}{4}$$

Our critical points are at  $(0, 0)$ ,  $(2, 0)$ ,  $(1, \frac{1}{4})$

$$f_{xx} = 2y \quad f_{yy} = 4 \quad f_{xy} = 2x - 2$$

$$D(x_0, y_0) = 8y - (2x - 2)^2$$

Classifying:

$$D(0, 0) = -4 < 0 \text{ saddle point at } (0, 0)$$

$$D(2, 0) = -\# < 0 \text{ saddle point at } (2, 0)$$

$$D(1, \frac{1}{4}) = 2 > 0 \text{ and } f_{xx} = \frac{1}{2} > 0 \text{ local min at } (1, \frac{1}{4})$$

### Theorem 15.1 Extreme Value Theorem

If  $z = f(x, y)$  is a continuous function on a closed and bounded region  $R$  defined synonymously  $[a, b]$ , then  $f$  has a global maximum at some point in  $R$  and a global minimum at some point in  $R$ .

#### Note:-

If  $f$  has a global min/max it will occur at the critical point of  $f$  or the boundary of it.

Also note that this theorem doesn't tell us where the global min/max occurs, just that it exists.

### Example 15.2

Given below are contour plots for four different functions  $f$ . Find the global maximum and the global minimum of each function on the closed region  $R = \{(x, y) : -2 \leq x \leq 2, -2 \leq y \leq 2\}$ . Do you notice anything special about the points that you chose?

- The global min at  $(0, 2)$  and  $(0, -2)$  and the global max at  $(2, 0)$  and  $(-2, 0)$
- The global min at  $(0, 0)$  and the global max at  $(-2, 2)$ ,  $(2, 2)$ ,  $(-2, -2)$ , and  $(2, -2)$

### Question 15.4

Find the global maximum and minimum of the function on  $-1 \leq x \leq 1, -1 \leq y \leq 1$ , and say whether it occurs on the boundary of the square.

A)

$$z = x^2 + y^2$$

B)

$$z = -e^{x^2+y^2}$$

C)

$$f(x, y) = x^2 - y^2$$

#### Solution:

- A) Critical point at  $(0, 0)$ . Short way (for the purpose of notes):  $z = x^2 + y^2$  is a paraboloid, such that the min is at the origin and the max is at the corners of the square.  
B) Critical point at  $(0, 0)$ .  $z = -e^{x^2+y^2}$  is a paraboloid, such that the min is at the corners of the square and the max is at the origin.  
C) Critical point at  $(0, 0)$ .

### Question 15.5

Does the function have a global maxima or minima?  $G(x, y) = -2x^2 - 5y^2$

Critical point at  $(0, 0)$  such that  $f(0, 0) = 0$ , such that it has to be the global max as all other value are negative. We can not have a defined global min as it is not bounded.

### Question 15.6: Distance Optimization

What is the shortest distance from the surface  $xy + 3x + z^2 = 9$  to the origin?

**Solution:** We want to minimize the distance from the origin to the surface.

Such that we use the distance formula towards  $(0, 0, 0) \implies d = \sqrt{x^2 + y^2 + z^2}$

$$z^2 = 9 - xy - 3x \implies d = \sqrt{x^2 + y^2 + 9 - xy - 3x}$$

We form the function  $f(x, y) = d^2 = x^2 + y^2 + 9 - xy - 3x$

We take the gradient to find the critical points  $\nabla f = \langle 2x - y - 3, 2y - x \rangle = \vec{0} \implies (2, 1)$

We perform the second derivative test,  $f_{xx} = 2 > 0 \quad f_{yy} = 2 \quad f_{xy} = -1 \quad D = 3 > 0$

Therefore, since  $f$  is a paraboloid as  $x, y \rightarrow \infty \implies f \rightarrow \infty$  we have a global minimum at  $(2, 1)$

Finally, we plug in  $(2, 1)$  into  $d = \sqrt{x^2 + y^2 + 9 - xy - 3x}$  to get  $d = \sqrt{14}$

### Question 15.7: Volume Optimization

An international airline has a regulation that each passenger can carry a suitcase having the sum of its width, length and height less than or equal to 135 cm. Find the dimensions of the suitcase of maximum volume that a passenger may carry under this regulation.

**Solution:** We want to maximize the volume of the suitcase. We know  $l + w + h \leq 135$  and  $V = lwh$

We subsequently want the max and want to reduce the function to 2 variables,  $l + w + h = 135 \implies l = 135 - w - h$ . We get the function  $V(w, h) = (135 - w - h)wh = 135wh - w^2h - wh^2$

Finding the critical point(s),  $\nabla V = \langle 135h - 2wh - h^2, 135w - 2wh - w^2 \rangle = \vec{0}$ . Such that  $(45, 45)$

Taking the second derivative test we find that  $D > 0$  and  $f_{ww} < 0$  such that we have a local max at  $(45, 45)$

As  $f$  has a quadratic opening, we have a global max at  $(45, 45)$  such that the dimensions should be  $45 \times 45 \times 45$

## 15.2 Lagrange Multipliers

### Definition 15.3: Lagrange Multipliers

To maximize an objective function  $f(x, y)$  ("multiple" level curves) subject to constraint  $g(x, y) = c$  (one level curve). Both  $f$  and  $g$  are satisfied when their subsequent level curve(s) intersect. Visually, we can see the max of  $f$  subject to constraint  $g$  occurs when the level curves are tangent. Such that moving along  $g$  on any other intersection point  $f$  can get higher/lower values for  $f$ . As  $g$  and  $f$  are tangent to each other, their gradients are parallel.

#### Method of Lagrange Multipliers:

- 1) Solve the following system of equations

$$\begin{aligned}\nabla f(x, y) &= \lambda \nabla g(x, y) \\ g(x, y, z) &= c\end{aligned}$$

- 2) Plug in all solutions,  $(x, y, z)$ , from the first step into  $f(x, y, z)$  and identify the minimum and maximum values, provided they exist and  $\nabla g \neq \vec{0}$  at the point.

### Theorem 15.2

If  $f(x, y)$  has a lagrange solution at  $(a, b)$  subject to the constraint  $g(x, y) = c$ , then  $g(a, b) = c$ .

### Question 15.8

Find the maximum and minimum values of  $f(x, y) = 2x + y$  with the constraint  $x^2 + y^2 = 5$

**Solution:**

- 1)  $\nabla f = \langle 2, 1 \rangle \quad \nabla g = \langle 2x, 2y \rangle$

$$\langle 2, 1 \rangle = \lambda \langle 2x, 2y \rangle \quad 1 = \lambda x \quad 1 = \lambda 2y$$
$$\lambda x = \lambda 2y \implies x = 2y \text{ and } x^2 + y^2 = 5 \implies y = \pm 1.$$

Our points are  $(2, 1)$  and  $(-2, -1)$

$$2) f(2, 1) = 5 \quad f(-2, -1) = -5$$

Therefore, the max is at  $(2, 1)$  and the min is at  $(-2, -1)$

### Question 15.9

Find the extreme values of the function  $f(x, y) = x^2 + 2y^2$  on the circle  $x^2 + y^2 = 1$ .

**Solution:**

$$1) \nabla f = \langle 2x, 4y \rangle \quad \nabla g = \langle 2x, 2y \rangle$$
$$\langle 2x, 4y \rangle = \lambda \langle 2x, 2y \rangle \quad 4y = \lambda 2y \quad 2x = \lambda 2x.$$

Such that  $(0, 1)$ ,  $(0, -1)$ ,  $(1, 0)$ , and  $(-1, 0)$  are our points.

$$2) f(0, 1) = 2 \quad f(0, -1) = 2 \quad f(1, 0) = 1 \quad f(-1, 0) = 1$$

Therefore, the max is at  $(0, 1)$  and  $(0, -1)$  and the min is at  $(1, 0)$  and  $(-1, 0)$

## 16 Integrating Functions of Several Variables

### 16.1 The Definite Integral of a Function of Two Variables

#### Definition 16.1: Double Integral

Suppose the function  $f$  is continuous on  $R$ , the rectangle  $a \leq x \leq b, c \leq y \leq d$ . If  $(u_{ij}, v_{ij})$  is any point in the  $j$ -th subrectangle, we define the definite integral of  $f$  over  $R$

$$\int_R f dA = \lim_{\Delta x, \Delta y \rightarrow 0} \sum_{i,j} f(u_{ij}, v_{ij}) \Delta x \Delta y.$$

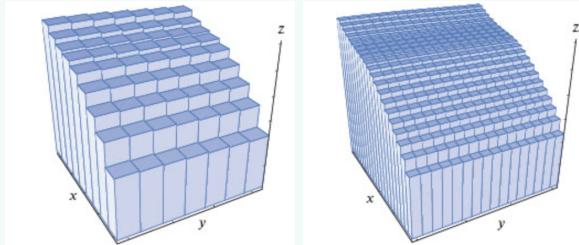
The case when  $R$  is not rectangular is considered. Sometimes we think of  $dA$  as being the area of an infinitesimal rectangle of length  $dx$  and height  $dy$ , so that  $dA = dx dy$ . Then we use the notation

$$\int_R f dA = \int_R f(x, y) dx dy = \iint_R f(x, y) dx dy$$

For this definition, we used a particular type of Riemann sum with equal-sized rectangular subdivisions. In a general Riemann sum, the subdivisions do not all have to be the same size.

#### Example 16.1 (Interpretation of the Double Integral as Volume)

As the number of subdivisions grows, the tops of the bars approximate the surface better, and the volume of the bars gets closer to the volume under the graph of the function



Such that if  $x, y, z$  represent length and  $f$  is positive, then

$$\text{Volume under graph of } f \text{ above region } R = \int_R f dA.$$

#### Example 16.2 (Interpretation of the Double Integral as Area)

In the special case that  $f(x, y) = 1$  for all points  $(x, y)$  in the region  $R$ , each term in the Riemann sum is of the form  $1 \cdot \Delta A = \Delta A$  and the double integral gives the area of the region  $R$  :

$$\text{Area}(R) = \int_R 1 dA = \int_R dA$$

**Example 16.3** (Interpretation of the Double Integral as Average Value)

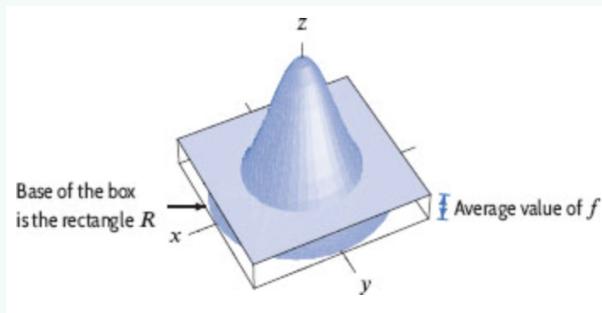
As in the one-variable case, the definite integral can be used to define the average value of a function:

$$\text{Average value of } f = \frac{1}{\text{Area of } R} \int_R f dA$$

We can rewrite this as

$$\text{Average value} \times \text{Area of } R = \int_R f dA$$

If we interpret the integral as the volume under the graph of  $f$ , then we can think of the average value of  $f$  as the height of the box with the same volume that is on the same base.

**Question 16.1: Over/Under Estimates**

Values of  $f(x, y)$  are in Table 16.5. Let  $R$  be the rectangle  $1 \leq x \leq 1.2, 2 \leq y \leq 2.4$ . Find Riemann sums which are reasonable over and underestimates for  $\int_R f(x, y) dA$  with  $\Delta x = 0.1$  and  $\Delta y = 0.2$ .

Table 16.5

		x		
		1.0	1.1	1.2
y	2.0	5	7	10
	2.2	4	6	8
	2.4	3	5	4

**Solution:** Inside Notebook

**Question 16.2**

Figure 16.8 shows a contour plot of population density, people per square kilometer, in a rectangle of land 3 km by 2 km. Estimate the population in the region represented by Figure 16.8.

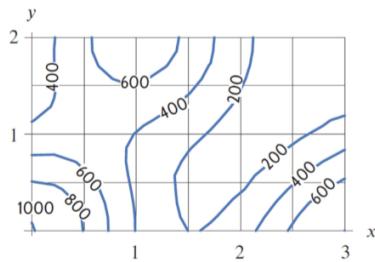


Figure 16.8

**Solution:**

We are given people per square kilometer, such that  $\int_R f dA$  is the population as  $f$  is the population density,  $A$  is the area and  $\frac{f}{A} \cdot A$  is people. We let  $\Delta x = \Delta y = 1$  such that we have 6 rectangles.

$$\begin{aligned}\text{population} &\approx \Delta x \Delta y [650 + 180 + 400 + 500 + 450 + 100] \\ &= 1\text{km}^2 (2280 \frac{\text{ppl}}{\text{km}^2}) = 2280 \text{ people}\end{aligned}$$

### Question 16.3

Decide (without calculation) whether the integrals are positive, negative, or zero. Let  $D$  be the region inside the unit circle centered at the origin, let  $R$  be the right half of  $D$ , and let  $B$  be the bottom half of  $D$ .

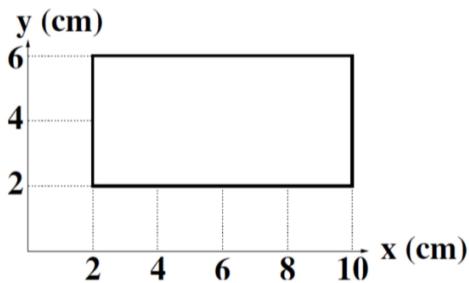
1.  $\int_D 1 dA$
2.  $\int_R 5x dA$
3.  $\int_B 5x dA$
4.  $\int_D (y^3 + y^5) dA$

### Solution:

1. Positive, as it is the area of the unit circle.
2. Positive, as 5,  $x$ ,  $dA$  are all positive.
3. Zero, as  $x$  is symmetric.
4. Zero, as  $y^3$  and  $y^5$  is odd.

### Question 16.4

Let  $R$  denote the rectangle  $[2, 10] \times [2, 6]$  in the  $xy$ -plane, and suppose that a thin metal plate having density  $\rho(x, y) = x + y$  grams per  $\text{cm}^2$  occupies this rectangular region (see diagram to the right). Using  $\Delta x = \Delta y = 2$ , estimate the mass of the metal plate. Estimate the average density of the metal plate



**Solution:** We can split the rectangle into 8 squares. Such if that  $m = \int_R p(x, y) dA$  we can estimate  $m \approx \Delta x \Delta y [\sum_N B_N]$ . Such that  $m \approx (2)(2)[6 + 8 + 10 + 12 + 8 + 10 + 12 + 14] = 320 \text{ g}$ . We can define  $f_{\text{avg}} = \rho_{\text{avg}} = \frac{1}{A} m \implies \frac{1}{(4)(8)} 320 = 10$

## 16.2 Iterated Integrals

### Theorem 16.1 Double integrals as Iterated Integrals

If  $R$  is the rectangle  $a \leq x \leq b, c \leq y \leq d$  and  $f$  is a continuous function on  $R$ , then the integral of  $f$  over  $R$  exists and is equal to the iterated integral

$$\int_R f dA = \int_{y=c}^{y=d} \left( \int_{x=a}^{x=b} f(x, y) dx \right) dy$$

The expression  $\int_{y=c}^{y=d} \left( \int_{x=a}^{x=b} f(x, y) dx \right) dy$  can be written  $\int_c^d \int_a^b f(x, y) dx dy$ .

Such that over  $R$  we could have added the columns (fixed  $x$ ) first. This leads to an iterated integral where  $x$  is constant in the inner integral instead of  $y$ ,  $\int_R f(x, y) dA = \int_a^b \left( \int_c^d f(x, y) dy \right) dx$ .

It does not matter in which order we integrate over a rectangular region  $R$ ; in other words, we can add up the columns first or rows first.

$$\int_R f dA = \int_c^d \left( \int_a^b f(x, y) dx \right) dy = \int_a^b \left( \int_c^d f(x, y) dy \right) dx$$

### Question 16.5

Evaluate the integral

A)

$$\int_0^3 \int_0^4 (4x + 3y) dx dy$$

B) Switch the order of (A).

C)

$$\int_0^1 \int_0^1 y e^{xy} dx dy$$

#### **Solution:**

We define our  $R$  as  $0 \leq y \leq 3, 0 \leq x \leq 4$

A)

$$\int_0^3 \int_0^4 (4x + 3y) dx dy = \int_0^3 (32 + 12y) dy = \boxed{150}$$

B)

$$\int_0^4 \int_0^3 (4x + 3y) dy dx = \int_0^4 (12x + \frac{27}{2}) dx = \boxed{150}$$

C) We define our  $R$  as  $0 \leq x \leq 1, 0 \leq y \leq 1$

$$\int_0^1 \int_0^1 y e^{xy} dx dy = \int_0^1 (e^y - 1) dy = \boxed{e - 2}$$

### 16.2.1 Double Integrals over General Regions

#### Question 16.6

Evaluate the integral

$$\int_0^2 \int_0^y y dx dy$$

**Solution:**

We define our  $R$  as  $0 \leq x \leq y, 0 \leq y \leq 2$ , such that we define  $R$  as a triangle.

$$\int_0^2 \int_0^y y dx dy = \int_0^2 (y^2) dy = \left[ \frac{8}{3} \right]$$

We can switch our limits of integration.

$$\int_0^2 \int_x^2 y dy dx = \int_0^2 (2 - \frac{x^2}{2}) dx = \left[ \frac{8}{3} \right]$$

#### Example 16.4 (Integrating a triangle horizontally and vertically)

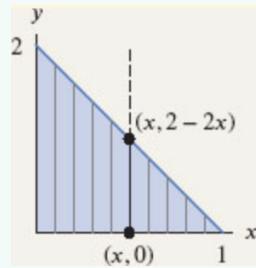
The density at the point  $(x, y)$  of a triangular metal plate, is  $\delta(x, y)$ . Express its mass as an iterated integral.

The vertical strip that starts at the point  $(x, 0)$  ends at the point  $(x, 2 - 2x)$ , because the top edge of the triangle is the line  $y = 2 - 2x$ . See Figure 16.16. On this vertical strip,  $y$  goes from 0 to  $2 - 2x$ . Hence, the inside integral is

$$\int_0^{2-2x} \delta(x, y) dy$$

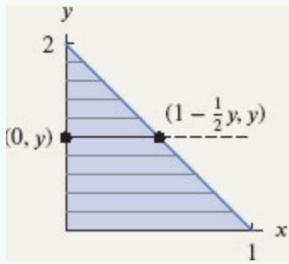
Finally, since there is a vertical strip for each  $x$  between 0 and 1, the outside integral goes from  $x = 0$  to  $x = 1$ . Thus, the iterated integral we want is

$$\text{Mass} = \int_0^1 \int_0^{2-2x} \delta(x, y) dy dx$$



We could have chosen to integrate in the opposite order, keeping  $y$  fixed in the inner integral instead of  $x$ . The limits are formed by looking at horizontal strips instead of vertical ones, and expressing the  $x$ -values at the end points in terms of  $y$ . To find the right endpoint of the strip, we use the equation of the top edge of the triangle in the form  $x = 1 - \frac{1}{2}y$ . Thus, a horizontal strip goes from  $x = 0$  to  $x = 1 - \frac{1}{2}y$ . Since there is a strip for every  $y$  from 0 to 2, the iterated integral is

$$\text{Mass} = \int_0^2 \int_0^{1-\frac{1}{2}y} \delta(x, y) dx dy$$



**Note:-**

**Limits on Iterated Integrals**

- The limits on the outer integral must be constants.
- The limits on the inner integral can involve only the variable in the outer integral. For example, if the inner integral is with respect to  $x$ , its limits can be functions of  $y$ .

**Steps on how to get limits of integration**

1. Draw the region,  $R$  function is being integration over.
2. If outer limits are  $x$ 's, walk left to right and see what  $y$  hits.
3. If outer limits are  $y$ 's, walk bottom to top and see what  $x$  hits.

**Question 16.7**

Sketch the region and integrate

$$\int_0^6 \int_{x/3}^2 x \sqrt{y^3 + 1} dy dx$$

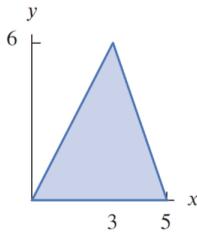
**Solution:** Our region is  $0 \leq x \leq 6$ ,  $\frac{x}{3} \leq y \leq 2$

$\int_{x/3}^2 x \sqrt{y^3 + 1} dy$  is not a trivial integral, such that we can switch the order of integration. Hint: for u-sub.

$$\int_0^2 \int_0^{3y} x \sqrt{y^3 + 1} dx dy = \int_0^2 \left( \frac{9}{2} y^2 \sqrt{y^3 + 1} \right) dy = [26]$$

**Question 16.8**

Write  $\int_R f dA$  as an iterated integral For the shaded region,  $R$ .



**Solution:** In notebook.

**Question 16.9**

Evaluate the integral by reversing the order of integration.

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \frac{1}{1+x^4} dx dy$$

**Solution:** We define  $R$  as  $0 \leq y \leq 8$ ,  $\sqrt[3]{y} \leq x \leq 2$ . In notebook.

**Question 16.10**

Evaluate the integral by reversing the order of integration.

$$\int_0^3 \int_{y^2}^9 y \sin(x^2) dx dy$$

**Solution:** In notebook

**Question 16.11**

Integrate  $f(x,y) = xy$  over the region  $R$ .

### 16.3 Triple Integrals

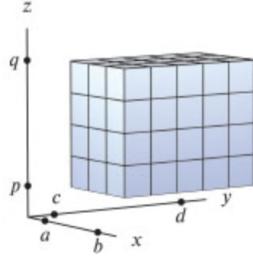
**Definition 16.2: Triple Integral as an Iterated Integral**

$$\int_W f dV = \int_p^q \left( \int_c^d \left( \int_a^b f(x, y, z) dx \right) dy \right) dz$$

where  $y$  and  $z$  are treated as constants in the innermost ( $dx$ ) integral, and  $z$  is treated as a constant in the middle ( $dy$ ) integral. Other orders of integration are possible.

**Example 16.5 (Interpretation of Triple Integrals as Infentesimal Boxes times  $f$ )**

First we subdivide  $W$  into smaller regions, then we multiply the volume of each region by a value of the function in that region, and then we add the results. For example, if  $W$  is the box  $a \leq x \leq b, c \leq y \leq d, p \leq z \leq q$ , then we subdivide each side into  $n, m$ , and  $l$  pieces, thereby chopping  $W$  into  $nml$  smaller boxes.



The volume of each smaller box is

$$\Delta V = \Delta x \Delta y \Delta z$$

where  $\Delta x = (b - a)/n$ , and  $\Delta y = (d - c)/m$ , and  $\Delta z = (q - p)/l$ . Using this subdivision, we pick a point  $(u_{ijk}, v_{ijk}, w_{ijk})$  in the  $ijk$ -th small box and construct a Riemann sum

$$\sum_{i,j,k} f(u_{ijk}, v_{ijk}, w_{ijk}) \Delta V$$

If  $f$  is continuous, as  $\Delta x, \Delta y$ , and  $\Delta z$  approach 0, this Riemann sum approaches the definite integral,  $\int_W f dV$ , called a triple integral, which is defined

$$\int_W f dV = \lim_{\Delta x, \Delta y, \Delta z \rightarrow 0} \sum_{i,j,k} f(u_{ijk}, v_{ijk}, w_{ijk}) \Delta x \Delta y \Delta z$$

For example, mass is density times volume, where we define the density at any point using  $f$ . Such that  $\int_W f dV$  is the mass of the solid/ $\mathbb{R}^3$  region  $W$ .

**Note:-**

**Limits on Triple Integrals**

- The limits for the outer integral are constants.
  - The limits for the middle integral can involve only one variable (that in the outer integral).
  - The limits for the inner integral can involve two variables (those on the two outer integrals).
- Same as the Double Integral** - If  $\rho(x, y, z)$  is density, then  $\int_W \rho dV$  is the total quantity in the solid region  $W$ .
- $\int_W 1 dV$  is the volume of the solid region  $W$ .

**Question 16.12**

A cube C has sides of length 4 cm and is made of a material of variable density. If one corner is at the origin and the adjacent corners are on the positive  $x$ ,  $y$ , and  $z$  axes, then the density at the point  $(x, y, z)$  is  $\delta(x, y, z) = 1 + xyz$  g/cm<sup>3</sup>. Find the mass ( $m$ ) of the cube.

**Solution:**

$$\begin{aligned} m &= \int_C \rho dV = \int_0^4 \int_0^4 \int_0^4 (1 + xyz) dx dy dz \\ &= \int_0^4 \int_0^4 (4 + 8yz) dy dz \\ &= \int_0^4 (16 + 64z) dz \\ &= \boxed{576 \text{ g}} \end{aligned}$$

**Question 16.13**

Set up an iterated integral to compute the mass of the solid cone bounded by  $z = \sqrt{x^2 + y^2}$  and  $z = 3$ , if the density is given by  $\delta(x, y, z) = z$ .

**Solution:** In notebook.

**Question 16.14**

Sketch the region of integration

A)

$$\int_{-1}^1 \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} f(x, y, z) dy dz dx$$

B)

$$\int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_0^{\sqrt{1-x^2-z^2}} f(x, y, z) dy dx dz$$

**Question 16.15**

Write a triple integral, including limits, that gives the specified volume.

- A) Between  $2x + 2y + z = 6$  and  $3x + 4y + z = 6$  and above  $x + y \leq 1$ ,  $x \geq 0$ ,  $y \geq 0$ .
- B) Between the top portion of the sphere  $x^2 + y^2 + z^2 = 9$  and the plane  $z = 2$ .

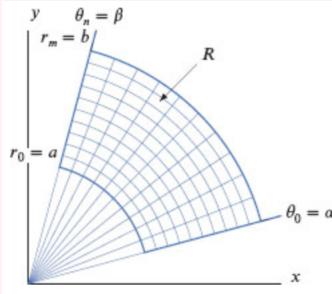
**Question 16.16**

Find the volume of the region bounded by  $z = x^2$ ,  $0 \leq x \leq 5$ , and the planes  $y = 0$ ,  $y = 3$ , and  $z = 0$ .

## 16.4 Double Integrals in Polar Coordinates

### Definition 16.3: Double Integrals in Polar Coordinates

A rectangular grid is constructed from vertical and horizontal lines of the form  $x = k$  (a constant) and  $y = l$  (another constant). In polar coordinates,  $r = k$  gives a circle of radius  $k$  centered at the origin and  $\theta = l$  gives a ray emanating from the origin (at angle  $l$  with the  $x$ -axis). A polar grid is built out of these circles and rays. Suppose we want to integrate  $f(r, \theta)$  over the region  $R$  below



Choosing  $(r_{ij}, \theta_{ij})$  in the  $j$ -th bent rectangle above gives a Riemann sum:

$$\sum_{i,j} f(r_{ij}, \theta_{ij}) \Delta A.$$

If  $\Delta r$  and  $\Delta\theta$  are small, the shaded region is approximately a rectangle with sides  $r\Delta\theta$  and  $\Delta r$ , so

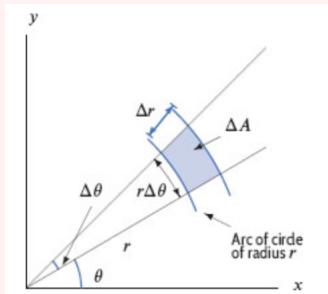
$$\Delta A \approx r\Delta\theta\Delta r$$

Thus, the Riemann sum is approximately

$$\sum_{i,j} f(r_{ij}, \theta_{ij}) r_{ij} \Delta\theta \Delta r$$

If we take the limit as  $\Delta r$  and  $\Delta\theta$  approach 0, we obtain

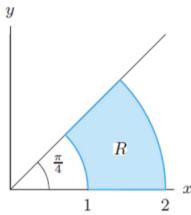
$$\int_R f dA = \int_\alpha^\beta \int_a^b f(r, \theta) r dr d\theta$$



When computing integrals in polar coordinates, use  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $x^2 + y^2 = r^2$ . Put  $dA = r dr d\theta$  or  $dA = r d\theta dr$ .

### Question 16.17

Computer the integral of  $f(x, y) = 1/(x^2 + y^2)^{3/2}$  over the region  $R$  shown below

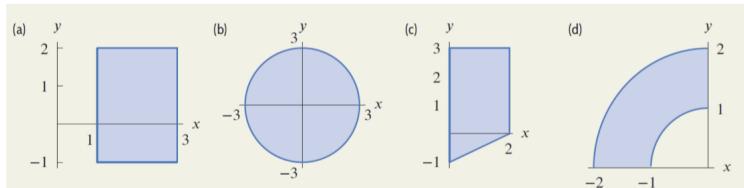


**Solution:** We can define  $f(x, y)$  as  $f(x, \theta) = \frac{1}{r^3}$ .

$$\int_0^{\frac{\pi}{4}} \int_1^2 \frac{1}{r^3} r dr d\theta = \frac{\pi}{8}$$

### Question 16.18

For each region set up an iterated integral of an arbitrary function  $f(x, y)$  over the region, decide which is best to use, polar or rectangular coordinates.



**Solution:**

A)

$$\int_1^3 \int_{-1}^2 f(x, y) dy dx$$

B)

$$\int_0^{2\pi} \int_0^3 f(r, \theta) r dr d\theta$$

C)

$$\int_0^2 \int_{\frac{1}{2}x-1}^3 f(x, y) dy dx$$

D)

$$\int_{\frac{\pi}{2}}^{\pi} \int_1^2 f(r, \theta) r dr d\theta$$

### Question 16.19

Sketch the region of integration shown:

A)

$$\int_3^4 \int_{3\pi/4}^{3\pi/2} f(r, \theta) r d\theta dr$$

B)

$$\int_0^{\pi/4} \int_0^{1/\cos \theta} f(r, \theta) r dr d\theta$$

**Solution:** In notebook.

**Question 16.20**

Evaluate the integral,  $\int_R (x^2 - y^2) dA$ , where  $R$  is the first quadrant region between the circles of radius 1 and radius 2 .

*Solution:*

**Question 16.21**

- (a) Sketch the region of integration of

$$\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} x dy dx + \int_1^2 \int_0^{\sqrt{4-x^2}} x dy dx$$

- (b) Evaluate the quantity in part (a).

*Solution:* In notebook.

**Question 16.22**

Find the volume of a solid bounded by the plane  $z = 0$  and paraboloid  $z = 1 - x^2 - y^2$

*Solution:*

## 16.5 Integrals in 3D Shapes

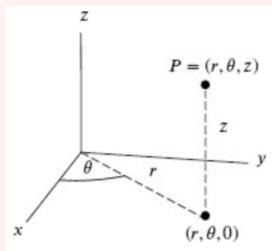
### 16.5.1 Integrals in Cylindrical Coordinates

#### Definition 16.4: Relation Between Cartesian and Cylindrical Coordinates

Each point in 3-space is represented using  $0 \leq r < \infty, 0 \leq \theta \leq 2\pi, -\infty < z < \infty$ .

$$\begin{aligned}x &= r \cos \theta, \\y &= r \sin \theta, \\z &= z\end{aligned}$$

As with polar coordinates in the plane, note that  $x^2 + y^2 = r^2$ .



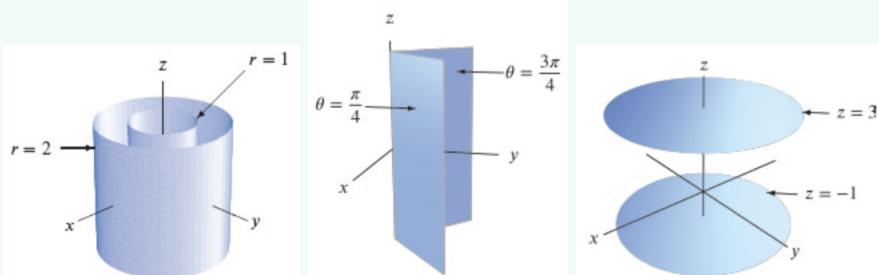
When computing integrals in cylindrical coordinates, put  $dV = r dr d\theta dz$ . Other orders of integration are also possible.

#### Example 16.6 (Fundamental Cylindrical Surfaces Visualized)

The surfaces  $r = 1$  and  $r = 2$

The surfaces  $\theta = \pi/4$  and  $\theta = 3\pi/4$

The surfaces  $z = -1$  and  $z = 3$



#### Question 16.23: From Example 16.7

Find the mass of a wedge of cheese cut from a cylinder 4 cm high and 6 cm in radius; this wedge subtends an angle of  $\pi/6$  at the center. Its density is 1.2 grams/cm<sup>3</sup>.

**Solution:**

We know that the mass is density times volume. Such that  $m = \int_0^4 \int_0^{\pi/2} \int_0^6 1.2 r dr d\theta dz = 45.24$  g

#### Question 16.24

Choose a coordinate system and set up a triple integral, including limits of integration, for a density function  $f$  over the region given.

A) B) Insert from lecture

**Solution:**

A)

$$\int_0^5 \int_0^3 \int_0^1 f dx dy dz$$

B)

$$\int_0^1 \int_0^{2\pi} \int_0^4 f r dr d\theta dz$$

**Question 16.25**

Evaluate

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2 + y^2) dz dy dx$$

by analyzing in cylindrical coordinates.

**Solution:** In Notebook.

**Question 16.26**

Write a triple integral representing the volume above the cone  $z = \sqrt{x^2 + y^2}$  and below the sphere of radius 2 centered at the origin. Include limits of integration but do not evaluate. Use: Cylindrical coordinates

**Solution:** In Notebook.

**Question 16.27**

Write a triple integral representing the volume of the region between spheres of radius 1 and 2, both centered at the origin. Include limits of integration but do not evaluate. Use: Cylindrical coordinates.  
Write answer as difference of two integrals.

**Solution:** In Notebook.

**Question 16.28**

A sphere has density at each point proportional to the square of the distance of the point from the z-axis. The density is  $2 \text{ gm/cm}^3$  at a distance of 2 cm from the axis. What is the mass of the sphere if it is centered at the origin and has radius 3 cm ?

**Solution:** In Notebook.

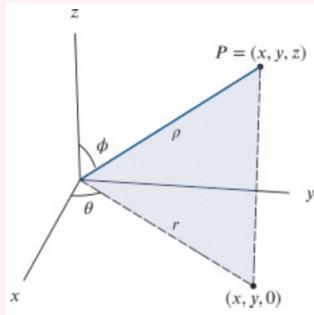
## 16.5.2 Integrals in Spherical Coordinates

### Definition 16.5: Relation Between Cartesian and Spherical Coordinates

Each point in 3-space is represented using  $0 \leq \rho < \infty$ ,  $0 \leq \phi \leq \pi$ , and  $0 \leq \theta \leq 2\pi$ .

$$\begin{aligned}x &= \rho \sin \phi \cos \theta \\y &= \rho \sin \phi \sin \theta \\z &= \rho \cos \phi \\r &= \rho \sin \phi\end{aligned}$$

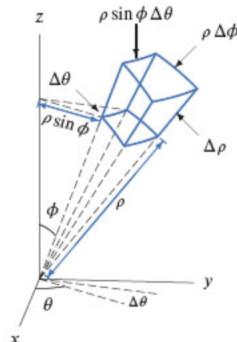
Also,  $\rho^2 = x^2 + y^2 + z^2$ .



### Theorem 16.2

When computing integrals in spherical coordinates, put  $dV = \rho^2 \sin \phi d\rho d\phi d\theta$ . Other orders of integration are also possible.

**Proof:** We have infinitesimal bits,  $\Delta\rho$ ,  $\rho\Delta\phi$ , and  $r\Delta\theta = \rho \sin \phi \Delta\theta$  Such that  $dV = \rho^2 \sin \phi d\rho d\phi d\theta$ .

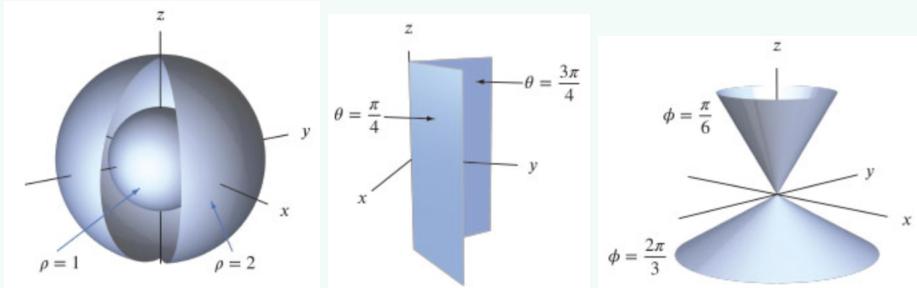


### Example 16.7 (Fundamental Spherical Surfaces Visualized)

The surfaces  $\rho = 1$  and  $\rho = 2$

The surfaces  $\theta = \pi/4$  and  $\theta = 3\pi/4$

The surfaces  $\phi = \pi/6$  and  $\phi = 2\pi/3$



Extra: we can say  $\rho = 2$  with  $0 \leq \theta \leq 3\pi/2$

### Question 16.29

Use spherical coordinates to derive the formula for the volume of a sphere of radius  $r$ .

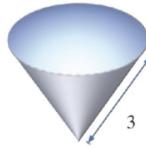
**Solution:**

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^r \rho^2 \sin \phi d\rho d\phi d\theta = \frac{4}{3}\pi r^3$$

### Question 16.30

Choose a coordinate system and set up a triple integral, including limits of integration, for a density function  $f$  over the region given.

A piece of a sphere; angle at the center is  $\pi/3$ .



**Solution:**

$$\int_0^{2\pi} \int_0^{\pi/6} \int_0^3 \rho^2 \sin \phi d\rho d\phi d\theta$$

### Question 16.31

Write a triple integral representing the volume of the region between spheres of radius 1 and 2, both centered at the origin. Include limits of integration but do not evaluate. Use spherical coordinates.

**Solution:**

$$\int_0^{2\pi} \int_0^{\pi} \int_1^2 \rho^2 \sin \phi d\rho d\phi d\theta$$

### Question 16.32

Write a triple integral representing the volume above the cone  $z = \sqrt{x^2 + y^2}$  and below the sphere of radius 2 centered at the origin. Include limits of integration but do not evaluate. Use spherical coordinates.

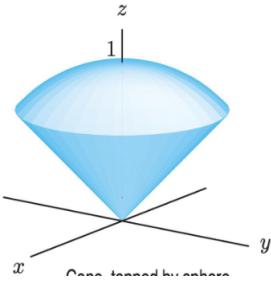
**Solution:**

We have a cone, such that  $z = \sqrt{x^2 + y^2} = r \implies \rho \cos \phi = \rho \sin \phi \implies \tan \phi = 1 \implies \phi = \frac{\pi}{4}$

$$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^2 \rho^2 \sin \phi d\rho d\phi d\theta$$

### Question 16.33

Setup the volume of the region shown in each coordinate system



**Solution:**

FInding where the sphere intersects the cone, we have  $z^2 = x^2 + y^2$  and  $z^2 = 1 - x^2 - y^2$ , such that  $x^2 + y^2 = 1 - x^2 - y^2 \implies r = \sqrt{\frac{1}{2}}$  Rectangular:

$$\int_{-\sqrt{\frac{1}{2}}}^{\sqrt{\frac{1}{2}}} \int_{-\sqrt{\frac{1-x^2}{2}}}^{\sqrt{\frac{1-x^2}{2}}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dz dy dx$$

Cylindrical:

$$\int_0^{2\pi} \int_0^{\sqrt{\frac{1}{2}}} \int_r^{\sqrt{1-r^2}} r dz dr d\theta$$

Spherical:

$$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \rho^2 \sin \phi d\rho d\phi d\theta$$

# 17 Parameters, Coordinates, and Integrals

## 17.1 Parameterized Curves

Recall "lazy parameterization"; where  $y = x^2$ , let  $x = t$ ,  $y = t^2$ . However,  $x = \sqrt{t} \implies y = t$ , only picks up the right side.

### Example 17.1 (Parameterized Curves (Circles))

A)  $x = \cos t$   $y = \sin t$   $0 \leq t \leq 2\pi$

Visually the parameterization is a circle, traced once, such that it is moving counter clockwise.

To eliminate a parameter we can square both  $x$  and  $y$ , such that  $\sin^2(t) + \cos^2(t) = 1 \implies x^2 + y^2 = 1$

We can takeaway that parameterized curves shows us direction of motion as  $t$  increases.

B)  $x = \sin 2t$   $y = \cos 2t$   $0 \leq t \leq 2\pi$

Subsequently the same, however, we start at  $(0,1)$  and goes around twice.

$$x = 2 \cos t + 1 \quad y = 2 \sin t + 3 \quad 0 \leq t \leq 2\pi$$

In the example above, 2 is the radius and  $+c$  is the shift.

### Question 17.1

Give the parameterization of

A) a unit circle in the  $xy$ -plane with center at the origin.

B) a circle of radius 5 in the  $xz$ -plane with center at the origin.

C) a circle of radius 3 parallel to the  $yz$ -plane with center at  $(6, 7, 8)$ .

D) a circle of radius 2 parallel to the  $xy$ -plane with center  $(0, 0, 1)$  traversed counterclockwise when viewed from below.

### Solution:

A) Let  $x = \cos t$ ,  $y = \sin t$ , and  $z = 0$ .

B) Let  $x = 5 \cos t$ ,  $y = 0$ , and  $z = 5 \sin t$ .

C) Let  $x = 6$ ,  $y = 3 \cos t + 7$ , and  $z = 3 \sin t + 8$

D) Let  $x = 2 \sin t$ ,  $y = 2 \cos t$ , and  $z = 1$ .

### Definition 17.1: Parametric Equations of a Line

Through the point  $(x_0, y_0, z_0)$  and parallel to the vector  $\vec{ai} + \vec{bj} + \vec{ck}$  are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

### Definition 17.2: Parametric Equation of a Line in Vector Form

The line through the point with position vector  $\vec{r}_0 = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$  in the direction of the vector  $\vec{v} = \vec{ai} + \vec{bj} + \vec{ck}$  has parametric equation

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

### Question 17.2

- A) Find parametric equations for the line in the direction of  $3\vec{i} - 3\vec{j} + \vec{k}$  through  $(1, 2, 3)$   
B) Find parametric equations for the line parallel to the  $z$ -axis through  $(1, 0, 0)$

### Solution:

- A) We can say the direction vector is the rate of change per  $t$ . Such that  $x = 1 + 3t$ ,  $y = 2 - 3t$ , and  $z = 3 + t$ .  
B)  $x = 1 + 0t$ ,  $y = 0 + 0t$ , and  $z = 0 + 1t$ .

## 17.2 Change of Coordinates in a Multiple Integral

### Definition 17.3: Change of Variables and Jacobian

The Jacobian of the transformation  $x = g(u, v)$ ,  $y = h(u, v)$  is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

Suppose that we want to integrate  $f(x, y)$  over the region  $R$ . Under the transformation  $x = g(u, v)$ ,  $y = h(u, v)$  the region becomes  $S$  and the integral becomes,

$$\iint_R f(x, y) dA = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| d\bar{A}$$

where  $d\bar{A}$  is the area element in the  $uv$ -plane.