Math 275 Notes

Jeremiah Vuong
Los Angeles Valley College
vuongjn5900@student.laccd.edu

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1.1 Slope (Direction) Fields

Equations containing derivatives are differential equations.
A differential equation that describes some physical process is often called a mathematical model of the process.

Definition 1.1: Slope (direction) field

Given the differential equation $dy/dt = f(t, y)$. If we systematically evaluate f over a rectangular grid of points in the ty-plane and draw a line element at each point (t, y) of the grid with slope $f(t, y)$, then the collection of all these line elements is called a slope (direction) field of the differential equation $dy/dt = f(t, y)$

We can graph undefined slopes in a slope field by using a vertical line; horizontal for 0.

Question 1.1: pg. 8: #6

Write down a differential equation of the form dy/dt = ay + b whose solutions diverge from y = 2.

Solution:
Given that the equation is dependent on just y, we can say that it is an autonomous differential equation (DE). We can subsequently solve for the DE by setting $\frac{dy}{dt} = f(y) = 0$, such that its solution is its equilibrium solution. Since $y = 2$, $\boxed{\dfrac{dy}{dt} = y - 2}$. Thus, any number that is less than 2 will result in the solution's phase portrait diverging (-) from $y = 2$ — vice versa for numbers greater than 2.

1.2 tfi

1.3 Classification on DEs

The order of a DE is the order of the highest derivative that appears in the equation.

Definition 1.2: Linear ODE

The ODE $F\left(t, y, y', \ldots, y^{(n)}\right) = 0$ is said to be linear if F is a linear function of the variables $y, y', \ldots, y^{(n)}$. Thus the general linear ODE of order n is $a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_n(t)y = g(t)$.

Question 1.2: pg. 22

Determine the order of the given DE; also state whether the equation is linear or nonlinear:
1) $t^2y'' + ty' + 2y = \sin t$
2) $\left(1 + y^2\right)y'' + ty' + y = e^t$

Solution:
1) The order of the DE is 2, and it is linear: noticed how all the coefficents of n derivatives of y are functions of t.
2) The order of the DE is 2, and it is nonlinear: noticed how the coefficents of n derivatives of y are functions of t and y.

Definition 1.3: Nth-order IDE solutions

Any function h, defined on an interval and possessing at least n derivatives that are continuous on this interval, which substituted into an nth-order ODE reduces the equation to an identity, is said to be a solution of the equation on the interval.

Question 1.3: pg. 22: #9)

Verify that the functions $y_1(t) = t^{-2}$ and $y_2(t) = t^{-2} \ln t$ are solutions of the DE

$$t^2y'' + 5ty' + 4y = 0$$

Solution:
We can verify that $y_1(t)$ and $y_2(t)$ are solutions of the DE by substituting them into the DE and verifying that the equation holds true.

$$\begin{aligned} y_1 &= t^{-2} \\ y_1' &= -2t^{-3} \\ y_1'' &= 6t^{-4} \end{aligned}$$

Substituting: $t^2y'' + 5ty' + 4y = t^2(6t^{-4}) + 5t(-2t^{-3}) + 4t^{-2} = 0$
For homework 1, we can verify that $y_2(t)$ is a solution of the DE.

Question 1.4: #12

Determine the values of r for which $y'' + y' - 6y = 0$ has solutions of the form $y = e^{rt}$

Solution:
We can solve for the values of r by substituting $y = e^{rt}$ into the DE and solving for r. $y = e^{rt}$, $y' = re^{rt}$, and $y'' = r^2e^{rt}$. Substituting: $r^2e^{rt} + re^{rt} - 6e^{rt} = 0$. Factoring out e^{rt}, we get $e^{rt}(r^2 + r - 6) = 0$. Thus, $r^2 + r - 6 = 0$, and $\boxed{r = 2, -3}$.

Question 1.5: #14

Determine the values of r for which $t^2y'' + 4ty' + 2y = 0$ has solutions of the form $y = t^r$ for $t > 0$

Solution:
$y = t^r$, $y' = rt^{r-1}$, and $y'' = r(r-1)t^{r-2}$. Substituting: $t^2(r(r-1)t^{r-2}) + 4t(rt^{r-1}) + 2t^r = 0$. Factoring out t^r, we get $t^r[r(r-1) + 4r + 2] = 0$. $t > 0$ such that $t^r \neq 0$, and $r(r-1) + 4r + 2 = 0$. Thus, $r^2 + 3r + 2 = 0$, and $\boxed{r = -1, -2}$.

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2.1 Linear DEs

Definition 2.1: First-order linear DE

An ODE of the form $dy/dt + p(t)y = g(t)$ is called a first-order linear differential equation in standard form

Example 2.1 (Solving a Linear First-Order DE)

i) Put the linear equation in standard form.
ii) From the standard form of the equation identify p(t) and then find the integrating factor $e^{\int p(t)dt}$. No constant need be used in evaluating the indefinite integral in the exponent.
iii) Multiply both sides of the standard form equation by the integrating factor. The left-hand side of the resulting equation is automatically the derivative of the product of the integrating factor and y : $\frac{d}{dt}\left[e^{\int p(t)dt}y\right] = e^{\int p(t)dt}g(t)$.
iv) Integrate both sides of the last equation and solve for y

2.2 Separable DEs

Definition 2.2: Seperable DE

An ODE written in the differential form $M(x)dx + N(y)dy = 0$ is said to be separable since terms involving each variable may be placed on opposite sides of the equation. A separable equation can be solved by integrating the functions M and N.

Question 2.1: pg. 38: #4)

Solve the DE: $\quad xy' = \left(1 - y^2\right)^{1/2}$

Solution:
We can rewrite the DE as $x\frac{dy}{dx} = \sqrt{1-y^2}$, we can then seperate the variables and integrate both sides. $xdy = \sqrt{1-y^2}dx \implies \int dy \frac{1}{\sqrt{1-y^2}} = \int dx \frac{1}{x}$. Such that, $\arcsin(y) = \ln|x| + C$. Thus, $\boxed{y = \sin(\ln|x| + C)}$, we call this the general solution.

Definition 2.3: General Order n-th ODE

Trying to solve the ODE $\frac{d^n y}{dx^n} = f\left(x, y, y', \ldots, y^{(n-1)}\right)$ subject to the conditions $\quad y\left(x_0\right) = y_0, y'\left(x_0\right) = y_1, \ldots, y^{(n-1)}\left(x_0\right) = y_{n-1}$, where $\mathrm{y_0, y_1, \ldots, y_{n-1}}$ are arbitrary real constants, is called an nth-order initial-value problem (IVP). The values of y(x) and its first $\mathrm{n-1}$ derivatives at $\mathrm{x_0}$ are called initial conditions (IC).

Question 2.2: pg. 38: #16)

Find the solution of the initial value problem $\sin(2x)dx + \cos(3y)dy = 0, \quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{3}$ in explicit form.

Solution:
We can solve the DE by seperating the variables and integrating both sides. $\sin(2x)dx = -\cos(3y)dy \implies \int \sin(2x)dx = -\int \cos(3y)dy$. Such that, $-\frac{1}{2}\cos(2x) = -\frac{1}{3}\sin(3y) + C$. We can solve for C by using the initial condition $y(\frac{\pi}{2}) = \frac{\pi}{3}$. $-\frac{1}{2}\cos(\pi) = -\frac{1}{3}\sin(\pi) + C \implies C = \frac{1}{2}$. Thus, $-\frac{1}{2}\cos(2x) = \frac{1}{3}\sin(3y) + \frac{1}{2}$.

$$\begin{aligned}
6[-\frac{1}{2}\cos(2x)] &= 6[-\frac{1}{3}\sin(3y) + \frac{1}{2}] \\
-3\cos(2x) &= 2\sin(3y) + 3 \\
-3 - 3cos(2x) &= -2\sin(3y) \\
(3\cos(2x) + 3) \cdot \frac{1}{2} &= \sin(3y) \\
&\boxed{y = \frac{1}{3} \cdot \arcsin(\frac{3 + 3\cos(2x)}{2})}
\end{aligned}$$

Definition 2.4: Homogeneous DEs

A first-order DE in differential form $M(x, y)dx + N(x, y)dy = 0$ is said to be homogeneous if both coefficient functions M and N have the same degree.

Either of the substitutions $y = ux$ or $x = vy$, where u and v are new dependent variables, will reduce a homogeneous equation to a separable first-order differential equation.

Although either of the indicated substitutions can be used for every homogeneous differential equation, in practice we try $x = vy$ whenever the function M is simpler than N, and $y = ux$ whenever N is simpler than M.

Question 2.3: pg. 39: #26)

Solve:

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$$

Solution:
Seperating the differential, $x^2 dy = (x^2 + xy + y^2)dx \implies 0 = (x^2 + xy + y^2)dx - x^2 dy$. Let $y = ux$, such that $dy = udx + xdu$. Where $0 = (x^2 + x(ux) + (ux)^2)dx - x^2(udx + xdu)$. Distributing,

$$\begin{aligned}
0 &= x^2dx + ux^2dx + u^2x^2dx - ux^2dx - x^3du \\
x^3du &= x^2dx + u^2x^2dx \\
\int \frac{du}{u^2+1} &= \int \frac{dx}{x} \\
\arctan(u) &= \ln|x| + C \\
\arctan(\frac{y}{x}) &= \ln|x| + C \\
&\boxed{y = x\tan(\ln|x| + C)}
\end{aligned}$$

Question 2.4: #28)

Solve:

$$\frac{dy}{dx} = \frac{4y - 3x}{2x - y}$$

Solution:
Cross multplying, $(2x-y)dy = (4y-3x)dx \implies 0 = (4y-3x)dx + (y-2x)dy$. Let $x = vy$, such that $dx = vdy + ydv$. Where $0 = (4y - 3vy)(ydy + ydv) + (y - 2vy)dy$. Distributing,

$$\begin{aligned}
0 &= 4vydy + 4y^2dv - 3v^2ydy - 3vy^2dv + ydy - 2vydy \\
0 &= 2vydy + 4y^2dv - 3v^2ydy - 3vy^2dv + ydy \\
3vy^2dv - 4y^2dv &= 2vydy - 3vy^2dy - ydy \\
y^2(3v-4)dv &= y(2v - 3v^2 + 1)dy \\
y^2(3v-4)dv &= -y(3v^2 - 2v - 1)dy \\
\int \frac{3v-4}{(v-1)(3v+1)}dv &= -\int \frac{1}{y}dy
\end{aligned}$$

Breaking up the L.H.S. into partial fractions,

$$\begin{aligned}
\frac{3v-4}{(v-1)(3v+1)} &= \frac{A}{v-1} + \frac{B}{3v+1} \\
3v - 4 &= A(3v+1) + B(v-1)
\end{aligned}$$

Let $v = -\frac{1}{3}$, such that $3(-\frac{1}{3}) - 4 = A(0) + B(-\frac{1}{3} - 1) \implies B = \frac{15}{4}$.
Let $v = 1$, such that $3(1) - 4 = A(3+1) + B(0) \implies A = -\frac{1}{4}$.

$$\int \frac{-\frac{1}{4}}{v-1} + \frac{\frac{15}{4}}{3v+1} dv = -\int \frac{1}{y} dy$$

$$-\frac{1}{4}\ln|v-1| + \frac{15}{12}\ln|3v+1| = -\ln|y| + C$$

$$-\frac{1}{3}\ln|\frac{x}{y} - 1| + \frac{5}{4}\ln|\frac{3x}{y} + 1| = -\ln|y| + C$$

$$\boxed{-\ln|\frac{x-y}{y}| + 5\ln|\frac{3x+y}{y}| = -4\ln|y| + C}$$