Quick Sensitivity Curves for Pulsar Timing Arrays

Making unrealistic sensitivity curves more realistic

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October 3, 2024

Outline

Pulsars

- Pulsars are rapidly rotating neutron stars that emit beams of radiation.
- "millisecond" pulsars are highly stable
- Can track changes in relative motion between Earth and pulsars

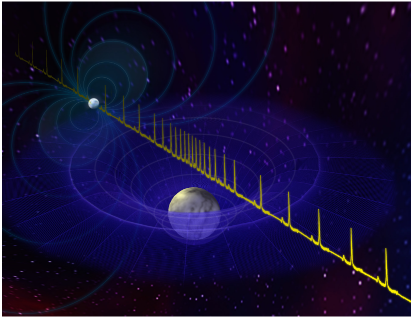

Figure: Relative motion of the earth and pulsar can be seen in the arrival times of pulses.

Pulsars Timing Arrays (PTAs)

- an array of millisecond pulsars
- track relative changes in Earth-pulsar distance over decades
- correlations between pulsars across the sky reveal gravitational waves

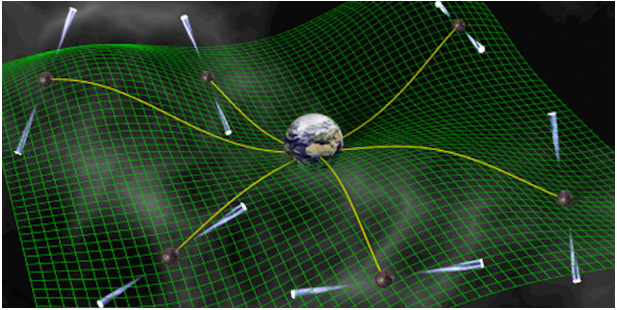

Figure: Artist rendering of a PTA

Sensitivity Curves

In general, sensitivity curves are a way to characterize the sensitivity of a detector to a given signal.

- Encode the noise properties of the detector.
- Can be used to assess to the "detectability" of something.
- Useful figures of merit for detector characterization.
- tools exist to calculate the sensitivity curve for gravitational wave detectors ...

Sensitivity Curves

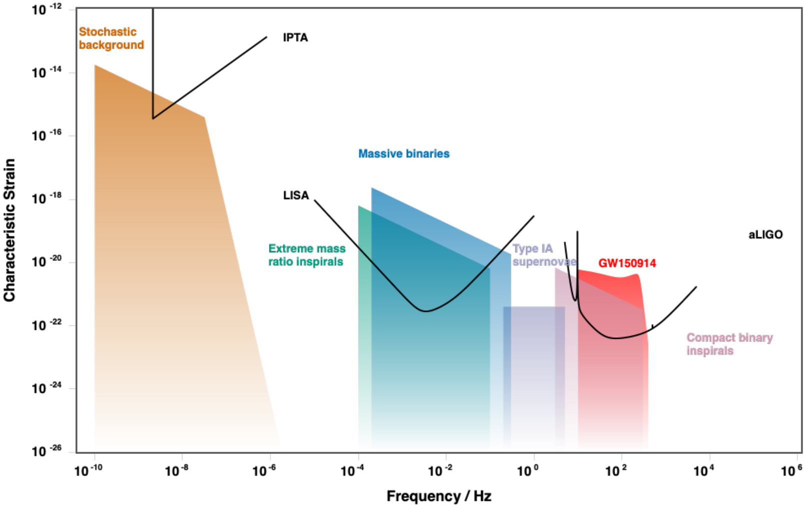

Figure: Community resource for approximate GW sensitivity curves
`https://gwplotter.com/`

Sensitivity Curves

But PTA sensitivity curves are not pizza slices !!

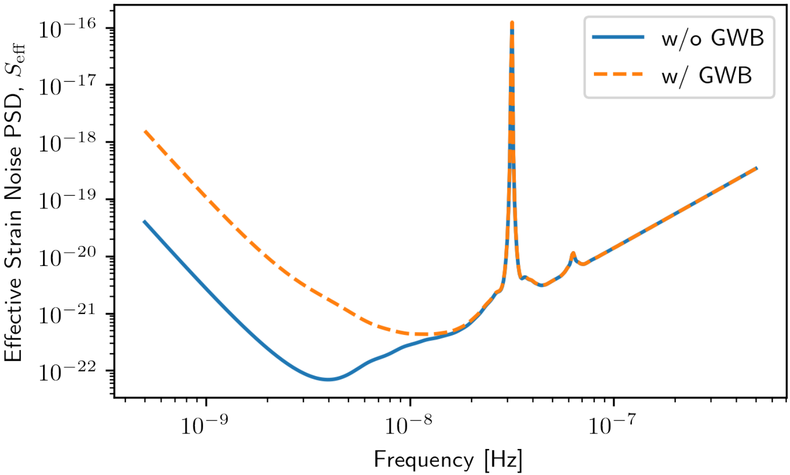

Figure: Example "realistic" sensitivity curve for a PTA
https://hasasia.readthedocs.io

Sensitivity Curves

Project goals:

- **derive a quick sensitivity curve for PTAs**
- implement quick sensitivity curve into *GW Plotter*
- share with the rest of the GW astronomy community

Quick Sensitivity Curve Derivation

We will start from the realistic sensitivity curves presented in eq. 92 of Hazboun et al. (2019),

$$S_{\mathrm{eff}}(f) = \left(\sum_{I} \sum_{J>I} \frac{T_{IJ}}{T_{\mathrm{obs}}} \frac{\chi_{IJ}^2}{S_I(f) S_J(f)} \right)^{-1/2}$$

- S_{eff} is the effective background sensitivity
- T_{obs}: Total observational time span
- T_{IJ}: Overlapping observational time span between pulsars I and J
- χ_{IJ}: correlation coefficients
- $S_I(f)$: Individual pulsar sensitivity I

Derivation of Quick Sensitivity Curves

Simplifying assumptions from GW Plotter:

- Uniform distribution of pulsars across the sky
- Identical noise properties for all pulsars
- $T_I = T_{\mathrm{obs}}$ (all pulsars observed for full timespan)
- Same observational cadence for all pulsars

Simplified Sensitivity Equation

Assuming uniform pulsar distribution across the sky,

$$\chi^2_{IJ} \approx 1/48$$

$$S_{\mathrm{eff}}(f) = \left(\sum_I \sum_{J>I} \frac{T_{IJ}}{T_{\mathrm{obs}}} \frac{1}{48 S_I(f) S_J(f)} \right)^{-1/2}$$

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Assuming that all pulsars have been observed for the total timespan, $T_{\mathrm{obs}} = T_{IJ}$,

$$S_{\mathrm{eff}}(f) = \left(\sum_I \sum_{J>I} \frac{1}{48 S_I(f) S_J(f)} \right)^{-1/2}$$

Simplified Sensitivity Equation

Assuming that all pulsars have the same noise properties, $S_I \approx S_J$. So the sum can be simplified...

$$\sum_I \sum_{J>I} \frac{1}{48 S_I(f) S_J(f)} \approx \frac{N_{\mathrm{psr}}(N_{\mathrm{psr}}-1)}{2} \frac{1}{48 S_I(f)^2}$$

And our expression for sensitivity becomes...

$$S_{\mathrm{eff}}(f) = \left(\frac{N_{\mathrm{psr}}(N_{\mathrm{psr}}-1)}{2} \frac{1}{48 S_I(f)^2} \right)^{-1/2}$$

So we only need to calculate the sensitivity for one pulsar!

Final Sensitivity Equation

Single pulsar sensitivity is given by:

$$S_I(f) = \frac{1}{\mathcal{N}_I^{-1}(f)\mathcal{R}(f)} = \frac{12\pi^2 f^2}{\mathcal{N}_I^{-1}(f)}$$

where $\mathcal{N}_I^{-1}(f)$ is the noise-weighted inverse transmission function for pulsar I.

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From the above equation, it follows that:

$$S_{\text{eff}}(f) = \left(\frac{96}{N_{\mathrm{psr}}(N_{\mathrm{psr}} - 1)}\right)^{1/2} \frac{12\pi^2 f^2}{\mathcal{N}_I^{-1}(f)}$$

Invserse Noise-Weighted Transmission

$$\mathcal{N}^{-1}(f) \equiv \frac{1}{2T_{\mathsf{span}}} \sum_{I,J} \left[G \left(G^T C G \right)^{-1} G^T \right]_{I,J} e^{i2\pi f \left(t_i - t_j \right)},$$

Most expensive part of the computation because C is an $N_{\rm toa} \times N_{\rm toa}$ covariance matrix. (memory intensive)

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So we approximate...

$$\mathcal{N}_I^{-1}(f) \approx \frac{\mathcal{T}_I(f)}{P_{\mathrm{N}}(f)}$$

where $\mathcal{T}_I(f) \approx \left(1 + \frac{1}{T_{\mathrm{obs}} f}\right)^{-6}$ is an approximation to the transmission function and $P_{\mathrm{N}}(f)$ is the power in the noise.

Transmission Function

Here we use a first order polynomial approximation to the transmission function:

$$\mathcal{T}_I(f) \approx \left(1 + \frac{1}{T_{\mathrm{obs}} f}\right)^{-6}$$

The transmission function describes how much power we lose at each frequency due to our fit to a timing model.

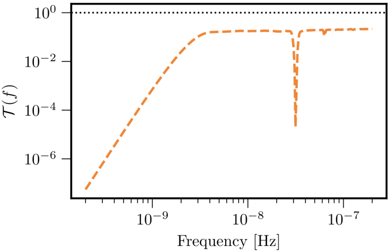

Figure: Example of an unapproximated transmission function.

Noise Power

The power in the noise is given by:

$$P_N = P_{\mathrm{WN}} + P_{\mathrm{RN}} = 2\Delta t\sigma^2 + A_{\mathrm{RN}} f^{-\gamma_{\mathrm{RN}}}$$

where

- σ is the time of arrival uncertainty
- Δt is the observational cadence
- A_{RN} is the red-noise amplitude
- and γ_{RN} is the red-noise spectral index

Final Expression

Putting it all together, we have the final expression for the simplified sensitivity curve...

$$S_{\text{eff}}(f) = \left(\frac{96}{N_{\mathrm{psr}}(N_{\mathrm{psr}} - 1)} \right)^{1/2} \frac{12\pi^2 f^2}{\mathcal{T}_I(f) P_{\mathrm{N}}(f)}$$

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... and now we just have to learn JavaScript ...

GW Plotter Update

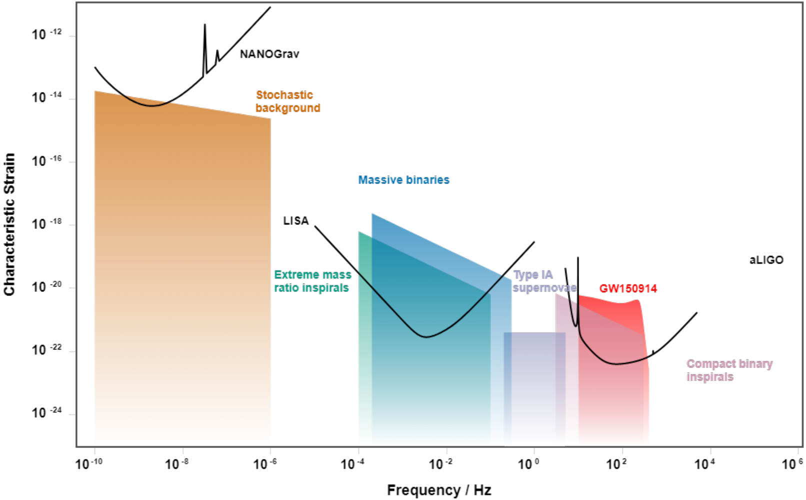

Figure: My more realistic sensitivity curve approximation in GW Plotter !!
`https://jeremy-baier/github.io/gwplotter`

Conclusions

- Sensitivity curves are a way to characterize the sensitivity of a detector to a given signal.
- PTA sensitivity curves are not pizza slices !!
- With some approximations, I derive approximate but quick sensitivity curve for PTAs.
- I implement quick background sensitivity curves into *GW Plotter* to better represent the PTA community.

References

- Hazboun, J. S., Romano, J. D., Smith, T. L. (2019). Realistic sensitivity curves for pulsar timing arrays. *Phys. Rev. D*, 100, 104028.
- Moore, C. J., Cole, R. H., Berry, C. P. L. (2015). Gravitational-wave sensitivity curves. *Class. Quantum Grav.*, 32, 015014.
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