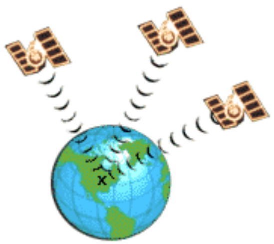
Chapter 11 – Sensor and Navigation Systems



11.1 Low-Pass and Notch Filtering
11.2 Fixed Gain Observer Design
11.3 Kalman Filter Design
11.4 Nonlinear Passive Observer Design
11.5 Integration Filters for IMU and Global Navigation Satellite Systems

Navigation: *is the science of directing a craft by determining its position, course, and distance traveled*

In some cases velocity and acceleration are determined as well. This is usually done by using a satellite navigation system combined with motion sensors such as accelerometers and gyros. The most advanced navigation system for marine applications is the inertial navigation system (INS).

Navigation is derived from the Latin navis, *"ship"*, and agere, *"to drive"*

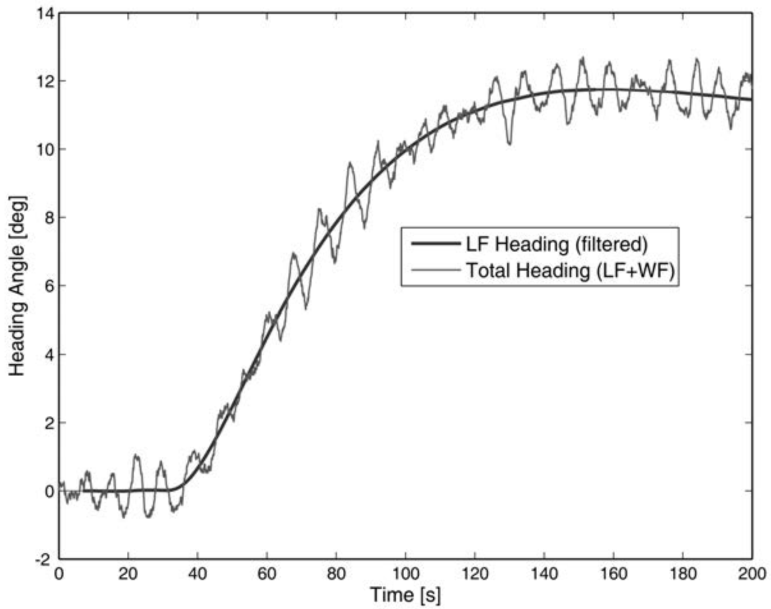
It originally denoted the art of ship driving, including steering and setting the sails.

Chapter 11 – Sensor and Navigation Systems

- *Wave filtering* is one of the most important issues to take into account when designing ship control systems.
- It is important that only the slowly varying disturbances are counteracted by the steering and propulsion systems; the oscillatory wave-frequency (WF) motion due to the waves (1st-order wave-induced forces) should be prevented from entering the feedback loop (Balchen 1976).

Definition 11.1 (Wave Filtering) Wave filtering can be defined as the reconstruction of the low-frequency (LF) motion components from noisy measurements of position, heading and in some cases velocity and acceleration by means of a state observer or a filter.

Remark: If a state observer is applied, estimates of the WF motion components can also be computed.



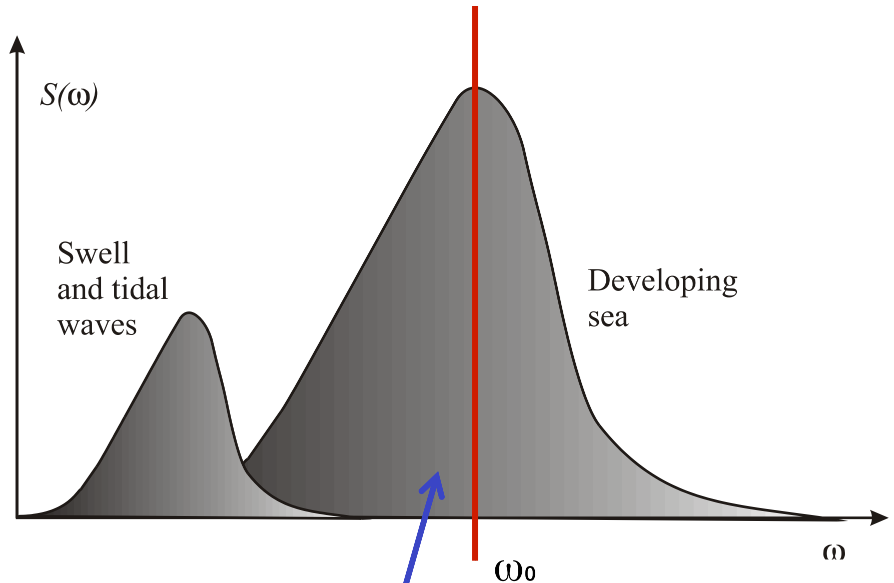
11.1 Low-Pass and Notch Filtering

For wave periods in the interval 5s < T_0 < 20s, the dominating wave frequency (modal frequency) f_0 of a wave spectrum will be in the range:

$$0.05 \text{ Hz} < f_0 < 0.2 \text{ Hz}$$

The circular frequency $\omega_0 = 2\pi f_0$ corresponding to periods T_0 > 5s is:

$$\omega_0 < 1.3 \text{ rad/s}$$



Waves can be accurately described by 1st- and 2nd-order linear wave theory:

- **1st-order wave-induced forces (WF forces)** produce large oscillations about a mean wave force. WF forces are represented as a wave spectrum.
Compensated for by using wave filtering in the state estimator

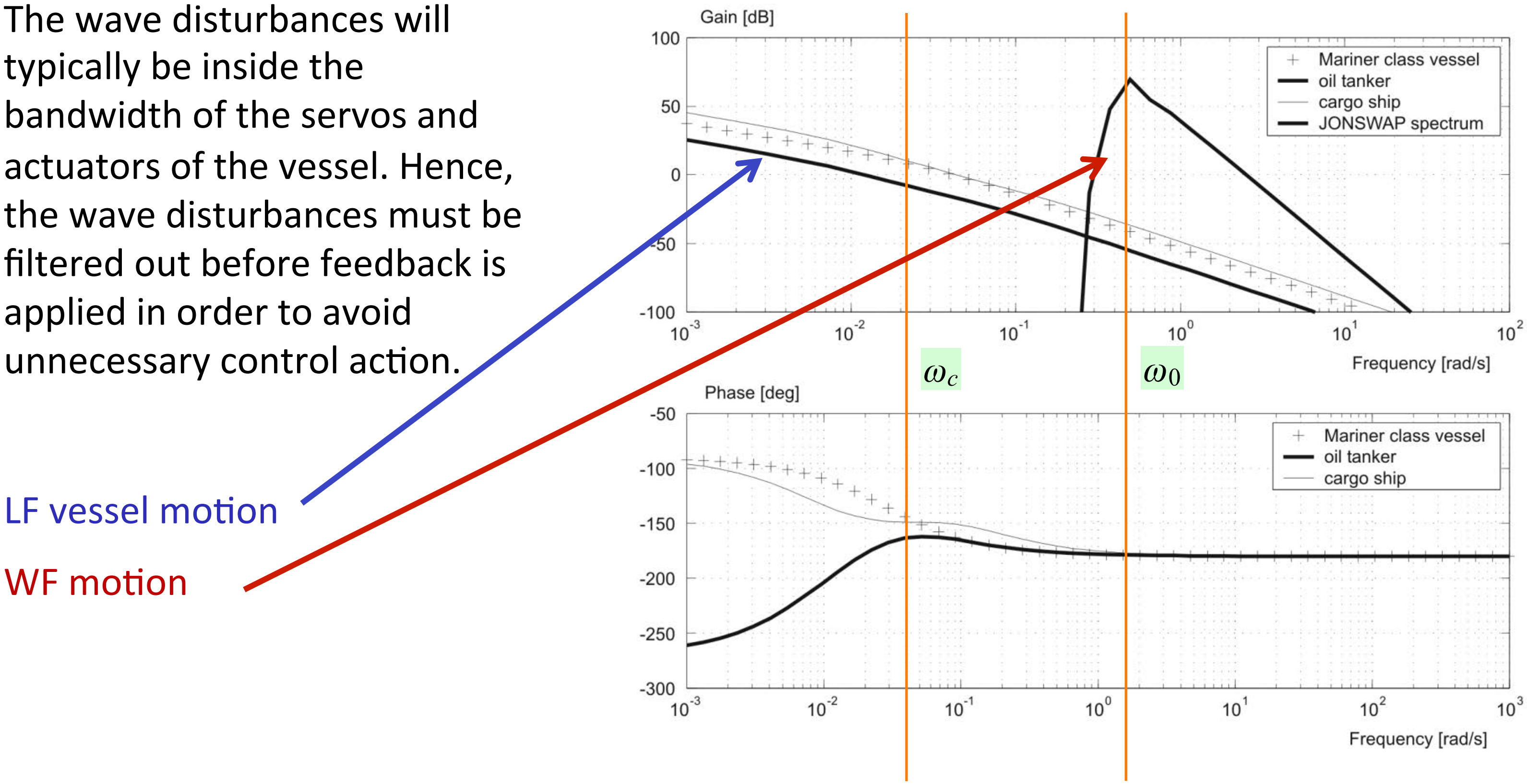
- **2nd-order wave-induced forces** or mean wave (drift) forces are slowly varying forces.
Compensated for by using integral action in the control law

11.1 Low-Pass and Notch Filtering

A feedback control system will typically move the bandwidth ω_b of the vessel up to 0.1 rad/s which still is below the wave spectrum.

The wave disturbances will typically be inside the bandwidth of the servos and actuators of the vessel. Hence, the wave disturbances must be filtered out before feedback is applied in order to avoid unnecessary control action.

For a large oil tanker, the crossover frequency ω_c can be as low as a 0.01 rad/s, while smaller vessels like cargo ships and the *Mariner class vessel*, are close to 0.05 rad/s.

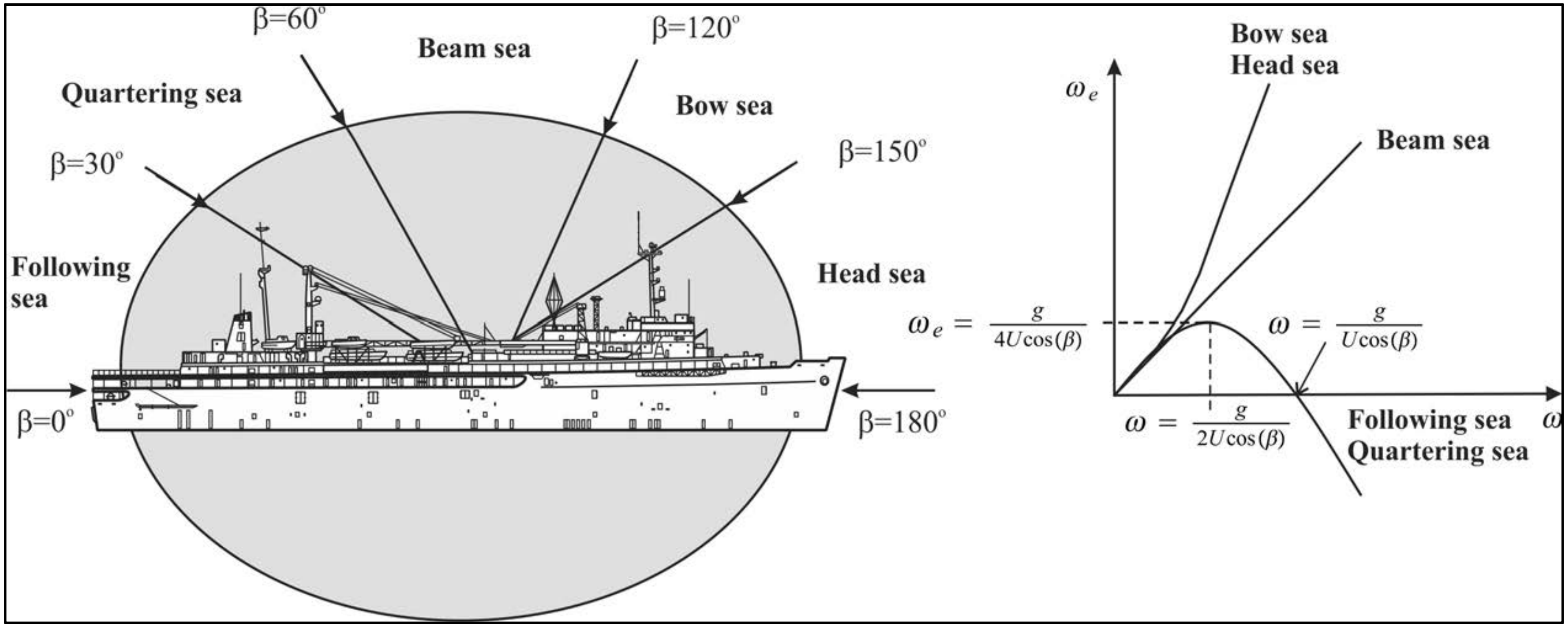


11.1 Low-Pass and Notch Filtering

For a ship moving at forward speed $U > 0$, there will be a shift in the wave spectrum peak frequency ω_0.

The shifted frequency is referred to as the frequency of encounter ω_e and it depends on ship speed U, modal wave frequency ω_0 and wave direction β.

$$\omega_e(U, \omega_0, \beta) = \left|\omega_0 - \frac{\omega_0^2}{g} U \cos\beta\right|$$



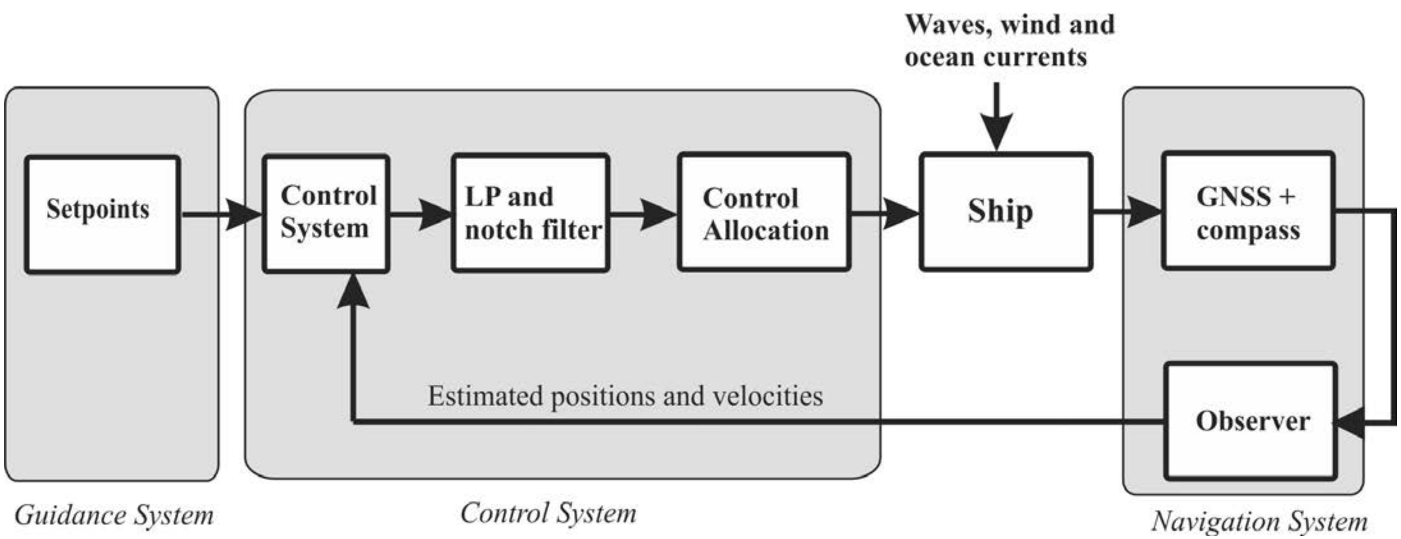
11.1.1 Low-Pass Filtering

For sea states where the encounter frequency ω_e is much higher than the bandwidth ω_b of the control system,

$$\omega_b \ll \omega_e$$

a LP-filter can be used to filter out the 1st-order wave-induced forces. This is typically the case for large vessels such as oil tankers.

For smaller vessels a LP filter in cascade with a notch filter is quite common to use.



LP and notch filters in series with the control system.

11.1.1 Low-Pass Filtering

Autopilot measurement equation:

$$y(s) = \underbrace{h_{\text{ship}}(s)\delta(s)}_{\psi(s)} + \underbrace{h_{\text{wave}}(s)w(s)}_{\psi_w(s)}$$

where

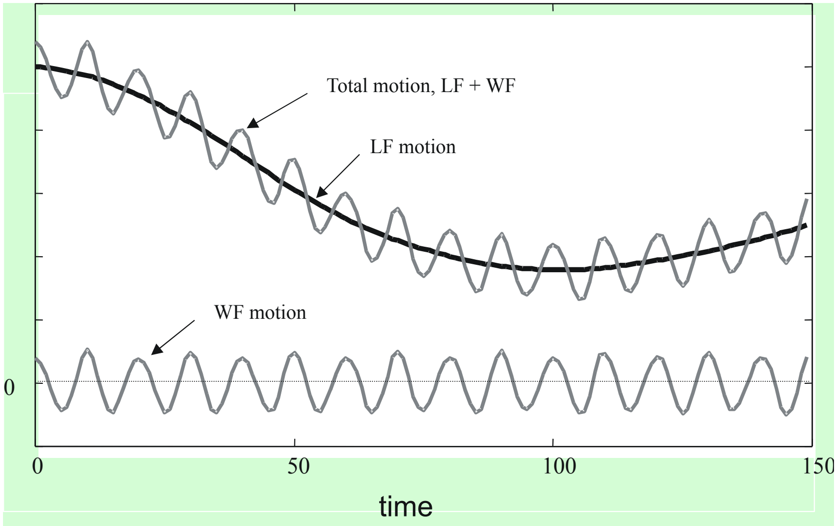
y(s) is the compass measurement

w(s) is zero-mean Gaussian white noise

$\delta(s)$ is the rudder input.

$\psi(s)$ is the LF motion

$\psi_w(s)$ is the WF motion



Linear theory:

Consequently, the feedback control law $\delta(s)$ should be a function of $\psi(s)$ and not y(s) in order to avoid 1st-order wave-induced rudder motions.

$$h_{\text{wave}}(s) = \frac{K_w s}{s^2+2\lambda\omega_0 s+\omega_0^2} \qquad h_{\text{ship}}(s) = \frac{K(1+T_3 s)}{s(1+T_1 s)(1+T_2 s)}$$

11.1.1 Low-Pass Filtering

A first-order low-pass filter with time constant T_f can be designed according to:

$$h_{lp}(s) = \frac{1}{1+T_f s} \qquad \omega_b < \frac{1}{T_f} < \omega_e \qquad \text{(rad/s)}$$

This filter will suppress disturbances over the frequency $1/T_f$.

This criterion is hard to satisfy for smaller vessels.

Higher-order low-pass filters can be designed by using a *Butterworth filter*:

$$h_{lp}(s) = \frac{1}{p(s)}$$

where p(s) is found by solving the Butterworth polynomial:

$$p(s)p(-s) = 1 + (s/j\omega_f)^{2n}$$

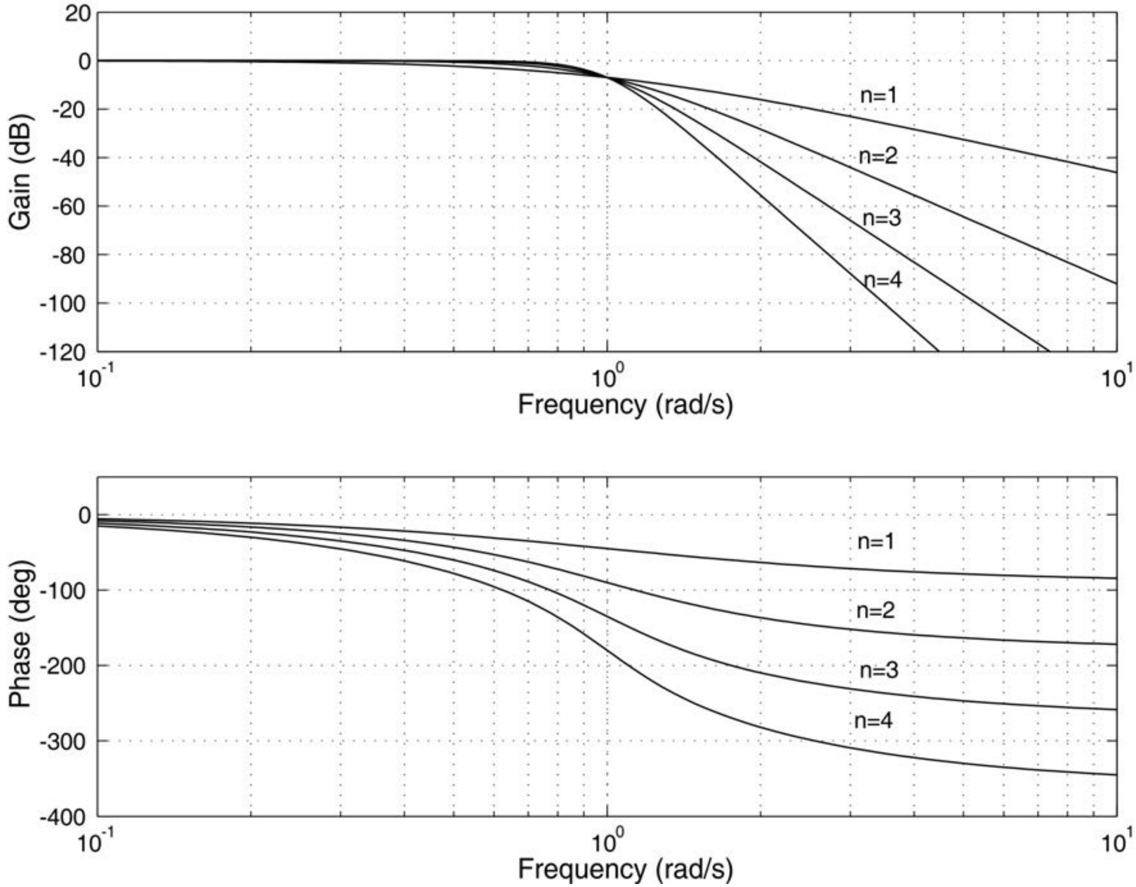
$$(n=1)\, h_{lp}(s) = \frac{1}{1+s/\omega_f}$$

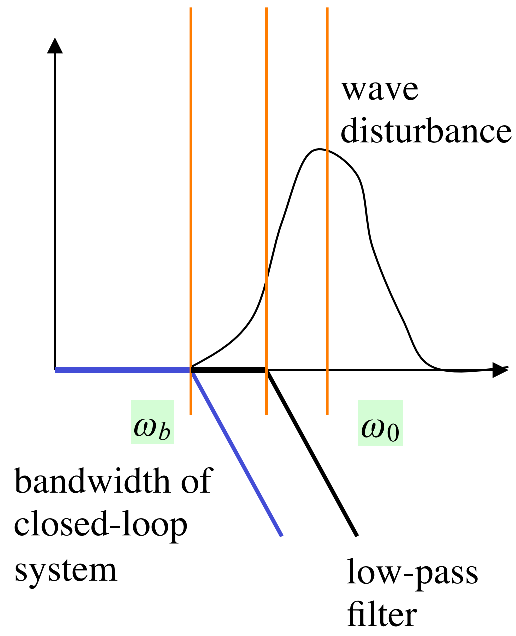
$$(n=2)\, h_{lp}(s) = \frac{\omega_f^2}{s^2 + 2\zeta\omega_f s + \omega_f^2}; \qquad \zeta = \sin(45^o)$$

$$(n=3)\, h_{lp}(s) = \frac{\omega_f^2}{s^2 + 2\zeta\omega_f s + \omega_f^2} \cdot \frac{1}{1+s/\omega_f}; \qquad \zeta = \sin(30^o)$$

$$(n=4)\, h_{lp}(s) = \prod_{i=1}^{2} \frac{\omega_f^2}{s^2 + 2\zeta_i\omega_f s + \omega_f^2}; \qquad \zeta_1 = \sin(22.5^o), \qquad \zeta_2 = \sin(67.5^o)$$

11.1.1 Low-Pass Filtering





A higher-order low-pass filter implies better disturbance suppression to the price of additional phase lag

11.1.2 Cascaded Low-Pass and Notch Filtering

For smaller craft the bandwidth of the controller can be close to or within the range of the wave spectrum. This problem can be handled by using a *low-pass filter in cascade with a notch filter*:

$$\hat{\psi}(s) = h_{lp}(s)h_n(s)y(s)$$

where

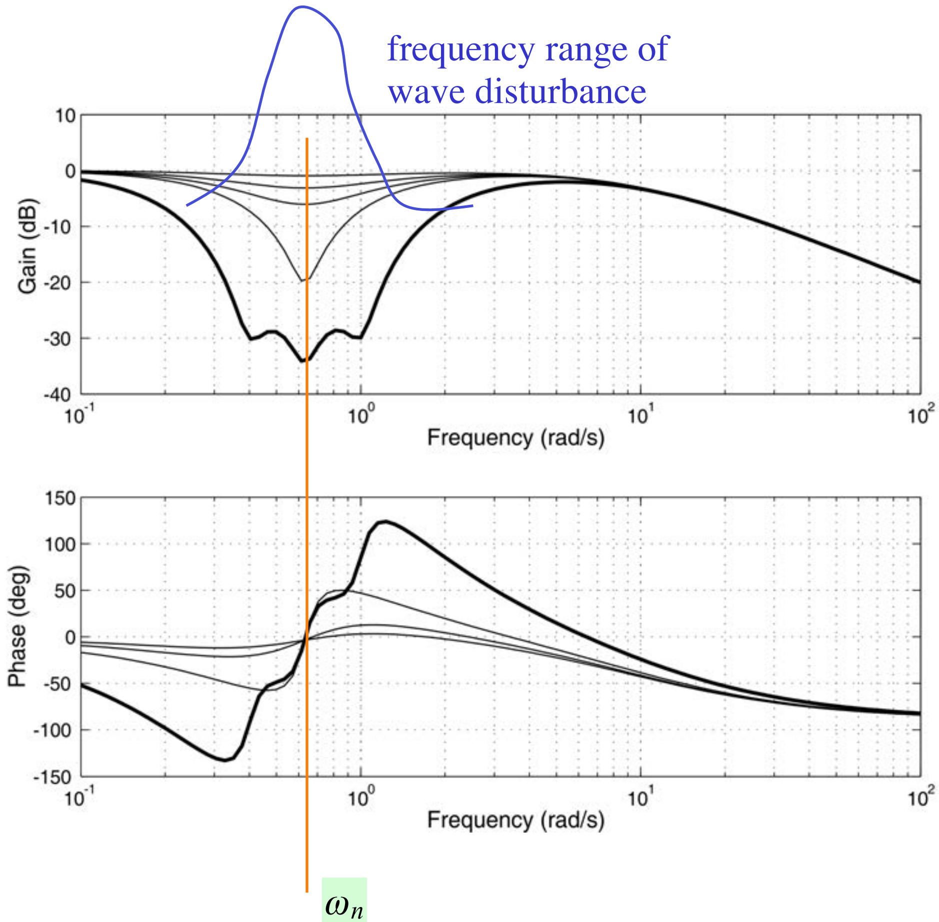
$$h_n(s) = \frac{s^2+2\zeta\omega_n s+\omega_n^2}{(s+\omega_n)^2}$$

For a vessel moving at forward speed U the optimal notch frequency will be:

$$\omega_n = \omega_e$$

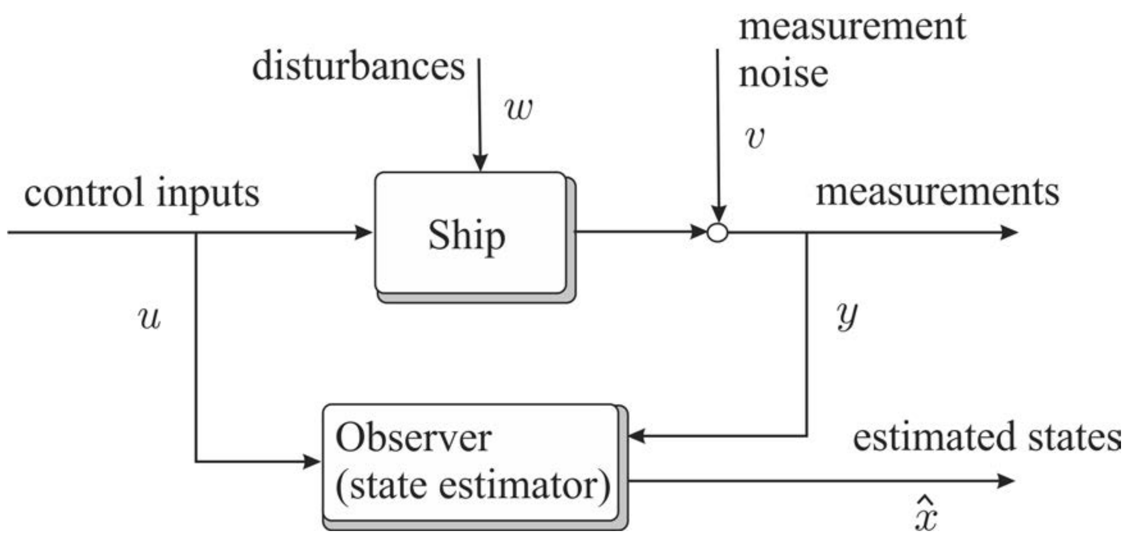
but… notch filtering also introduces additional phase lag!

therefore… use Kalman filtering or a linear/nonlinear observer



11.2 Fixed Gain Observer Design

The simplest state estimator is designed as a fixed gain observer where the ultimate goal of the observer is to reconstruct the unmeasured state vector **x** from the measurements **u** and **y** of a dynamical system.



11.2.1 Observability

Definition 11.2 (Observability) Consider the linear time-invariant system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{H}\mathbf{x}$$

The state and output matrices (**A**, **H**) must satisfy the *observability condition* to ensure that the state **x** can be reconstructed from the output **y** and the input **u**. The observability condition requires that the matrix:

$$\mathcal{O} = [\mathbf{H}^\top \mid \mathbf{A}^\top\mathbf{H}^\top \mid \ldots \mid (\mathbf{A}^\top)^{n-1}\mathbf{H}^\top]$$

must be of full column rank such that (at least) a left inverse exists.

If the *observability matrix* $\mathcal{O}$ is nonsingular, the poles of the error dynamics can be placed in the left half-plane by using the Matlab function:

k = place(A',h,p)'

where *p = [p_1,...,p_n]* is a vector describing the desired locations of the observer error poles (must be distinct).

11.2.2 Luenberger Observer

- An alternative to conventional filtering of wave disturbances is to apply a state estimator (observer).
- A state estimator can be designed to separate the LF components of the motion from the noisy measurements by using a model of the ship and the wave disturbances (WF model).

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w} \\ \mathbf{y} &= \mathbf{H}\mathbf{x} + \mathbf{v} \end{aligned}$$

plant

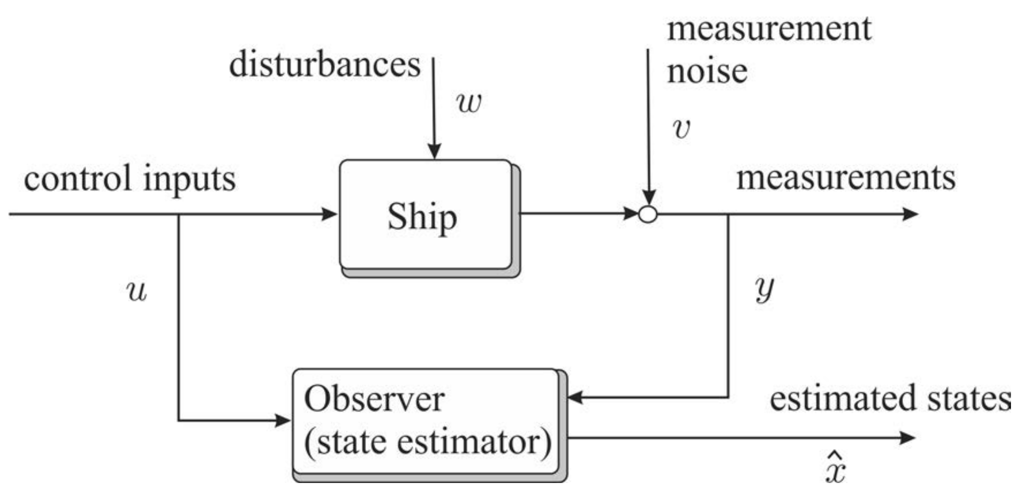
Observer goal: reconstruct the *unmeasured* state vector **x** from the measurements **u** and **y**

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \boldsymbol{\gamma}(\mathbf{y}, \hat{\mathbf{y}}) \\ \hat{\mathbf{y}} &= \mathbf{H}\hat{\mathbf{x}} \end{aligned}$$

observer (copy of dynamics)

where $\boldsymbol{\gamma} = \boldsymbol{\gamma}(y, y)$ is an *injection term* to be constructed such that:

$$\hat{\mathbf{x}} \rightarrow \mathbf{x} \text{ as } t \rightarrow \infty$$



11.2.2 Luenberger Observer

Luenberger Observer (Luenberger, 1964)

Assume that **w = v = 0**. Defining the estimation error as:

$$\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$$

then error dynamics takes the form:

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} - \boldsymbol{\gamma}(\mathbf{y}, \hat{\mathbf{y}})$$

A fixed-gain (Luenberger) observer is found by choosing the injection term as:

$$\boldsymbol{\gamma}(\mathbf{y}, \hat{\mathbf{y}}) = \mathbf{K}\boldsymbol{\varepsilon}, \qquad \boldsymbol{\varepsilon} = \mathbf{y} - \hat{\mathbf{y}} = \mathbf{H}\tilde{\mathbf{x}} \qquad \mathbf{K} = \text{constant}$$

Hence, the error dynamics become:

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}\mathbf{H})\tilde{\mathbf{x}}$$

Asymptotical convergence of **x** to zero can be obtained for a constant **K** if the system (**A**,**H**) is *observable*.

11.2.3 Case Study: Luenberger Observer for Heading Autopilots using only Compass Measurements

Example 11.1 (Nomoto ship model exposed to wind, waves and ocean currents)
Let a 1st-order Nomoto model describe the LF motion of a ship:

$$\begin{aligned}\dot{\psi} &= r\\ \dot{r} &= -\frac{1}{T}r + \frac{K}{T}(\delta - b) + w_r\\ \dot{b} &= -\frac{1}{T_b}b + w_b\end{aligned}$$

where b is the rudder offset (counteracts slowly varying moments on the ship due to wave drift forces, LF wind and ocean currents).

A linear wave model can be used to model the wave response:

$$\begin{aligned}\dot{\xi}_w &= \psi_w\\ \dot{\psi}_w &= -\omega_0^2\xi_w - 2\lambda\omega_0\psi_w + K_w w_w\end{aligned}$$

$$h(s) = \frac{K_w s}{s^2+2\lambda\omega_0 s+\omega_0^2}$$

The process noise terms, w_r, w_b, and w_w are modeled as white noise processes.

The compass measurement equation can be expressed by the sum:

$$y = \psi + \psi_w + v$$

where v represents zero-mean Gaussian measurement noise.

Notice that the yaw rate r nor the wave states ξ_w *and* ψ_w are measured.

11.2.3 Case Study: Luenberger Observer for Heading Autopilots using only Compass Measurements

Example 11.1 (Nomoto ship model exposed to wind, waves and ocean currents, cont.)

State-space model:

$$\mathbf{x} = [\xi_w, \psi_w, \psi, r, b]^\top$$

$$\mathbf{w} = [w_w, w_r, w_b]^\top$$

$$u = \delta$$

$$\mathbf{A} = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ -\omega_0^2 & -2\lambda\omega_0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{T} & -\frac{K}{T} \\ 0 & 0 & 0 & 0 & -\frac{1}{T_b} \end{array}\right], \quad \mathbf{b} = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ \frac{K}{T} \\ 0 \end{array}\right]$$

$$\mathbf{E} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \underbrace{2\lambda\omega_0\sigma}_{K_w} & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad \mathbf{h}^\top = [0, 1, 1, 0, 0]$$

Is the ship model exposed to environmental forces observable? YES

This implies that *yaw rate, bias* and *waves states* can be estimated using a single compass measurement! → see example on next page

11.2.3 Case Study: Luenberger Observer for Heading Autopilots using only Compass Measurements

Example 11.2 (Luenberger observer gains) It is straightforward to see that the autopilot model with wave frequency, wind and current models is observable from the input to the compass measurement *y*. Matlab →

```
K = 1; T=50; lambda = 0.1; wo =1; Tb = 1000;

A = [ 0 1 0 0 0
     -wo*wo -2*lambda*wo 0 0 0
      0 0 0 1 0
      0 0 0 -1/T -K/T
      0 0 0 0 -1/Tb       ]

h = [0,1,1,0,0]'

n = rank(obsv(A,h'))
```

results in *n = 5* corresponding to $\mathrm{rank}(\mathcal{O}) = 5$. Hence, the system is observable.
The *Luenberger filter gains* can be computed by using:

k = place(A',h,[p1,p2,p3,p4,p5])'

where *p1,p2,p3,p4,p5* are the desired closed-loop poles of $\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{k}\mathbf{h}^{\top})\tilde{\mathbf{x}}$

11.3 Kalman Filter Design

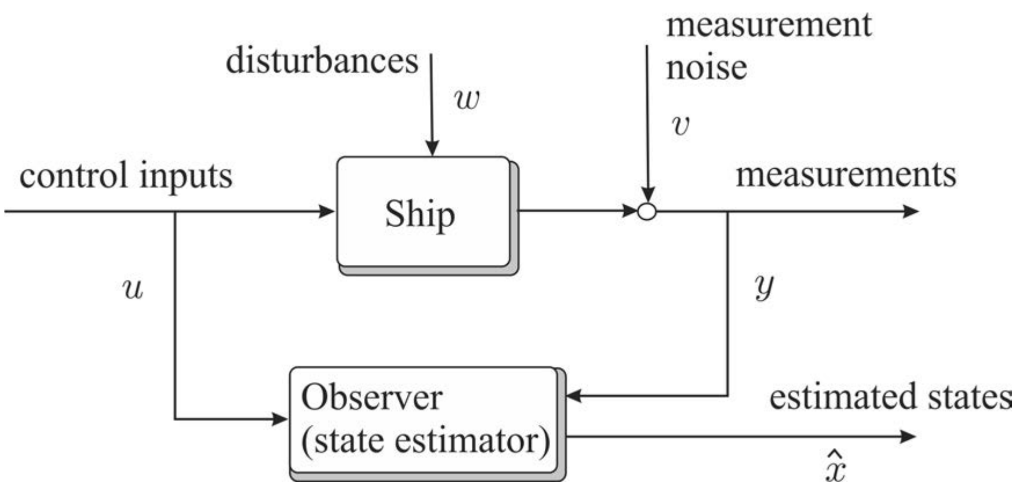
- An alternative solution to the pole-placement technique is to apply a Kalman filter (Kalman 1960) to compute the estimator gain matrix **K**.
- Kalman filtering (or *optimal state estimation* in sense of minimum variance) allows the user to estimate the state **x** of a dynamic system from a noise-contaminated input-output pair (**u**, **y**).

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w}$$

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v}$$

w is a zero-mean Gaussian white noise process with covariance matrix $\mathbf{Q} = \mathbf{Q}^{\mathrm{T}} > 0$

v is a zero-mean Gaussian white noise process with covariance matrix $\mathbf{R} = \mathbf{R}^{\mathrm{T}} > 0$



If the system is observable, the state vector **x** can be reconstructed recursively through the measurement vector **y** and the control input vector **u**

11.3.1 Discrete-Time Kalman Filter

Discrete-Time Kalman Filter Design

The discrete-time KF is defined in terms of the discretized system model:

$$
\begin{aligned}
\mathbf{x}(k+1) &= \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Delta}\mathbf{u}(k) + \mathbf{\Gamma}\mathbf{w}(k) \\
\mathbf{y}(k) &= \mathbf{H}\mathbf{x}(k) + \mathbf{v}(k)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{\Phi} &= \exp(\mathbf{A}h) \approx \mathbf{I} + \mathbf{A}h + \frac{1}{2}(\mathbf{A}h)^2 + \ldots + \frac{1}{N!}(\mathbf{A}h)^N \\
\mathbf{\Delta} &= \mathbf{A}^{-1}(\mathbf{\Phi} - \mathbf{I})\mathbf{B}, \qquad \mathbf{\Gamma} = \mathbf{A}^{-1}(\mathbf{\Phi} - \mathbf{I})\mathbf{E}
\end{aligned}
$$

The discretized system matrices can be computed in Matlab as:

```
[PHI,DELTA]=c2d(A,B,h)
[PHI,GAMMA]=c2d(A,E,h)
```

where `PHI`=$\mathbf{\Phi}$, `DELTA`=$\mathbf{\Delta}$ and `GAMMA`=$\mathbf{\Gamma}$.
Notice that Euler integration implies choosing $N = 1$, that is $\mathbf{\Phi}(k) = \mathbf{I} + \mathbf{A}h$

11.3.1 Discrete-Time Kalman Filter

Discrete-Time Kalman Filter Algorithm

The algorithm, requires that the state estimation error covariance matrix **P**(k) = **P**(k)$^\top$> 0 is computed on-line.

Design matrices	$\mathbf{Q}(k) = \mathbf{Q}^\top(k) > 0,\ \mathbf{R}(k) = \mathbf{R}^\top(k) > 0$ (usually constant)
Initial conditions	$\overline{\mathbf{x}}(0) = \mathbf{x}_0$ $\overline{\mathbf{P}}(0) = E[(\mathbf{x}(0) - \hat{\mathbf{x}}(0))(\mathbf{x}(0) - \hat{\mathbf{x}}(0))^\top] = \mathbf{P}_0$
Kalman gain matrix	$\mathbf{K}(k) = \overline{\mathbf{P}}(k)\mathbf{H}^\top(k)\ [\mathbf{H}(k)\overline{\mathbf{P}}(k)\mathbf{H}^\top(k) + \mathbf{R}(k)]^{-1}$
State estimate update	$\hat{\mathbf{x}}(k) = \overline{\mathbf{x}}(k) + \mathbf{K}(k)\ [\mathbf{y}(k) - \mathbf{H}(k)\overline{\mathbf{x}}(k)]$
Error covariance update	$\hat{\mathbf{P}}(k) = [\mathbf{I} - \mathbf{K}(k)\mathbf{H}(k)]\ \overline{\mathbf{P}}(k)\ [\mathbf{I} - \mathbf{K}(k)\mathbf{H}(k)]^\top + \mathbf{K}(k)\mathbf{R}(k)\mathbf{K}^\top(k), \quad \hat{\mathbf{P}}(k) = \hat{\mathbf{P}}(k)^\top > 0$
State estimate propagation	$\overline{\mathbf{x}}(k+1) = \mathbf{\Phi}(k)\hat{\mathbf{x}}(k) + \mathbf{\Delta}(k)\mathbf{u}(k)$
Error covariance propagation	$\overline{\mathbf{P}}(k+1) = \mathbf{\Phi}(k)\hat{\mathbf{P}}(k)\mathbf{\Phi}^\top(k) + \mathbf{\Gamma}(k)\mathbf{Q}(k)\mathbf{\Gamma}^\top(k)$

11.3.2 Continuous-Time Kalman Filter

Continuous-Time KF Algorithm

Design matrices	$\mathbf{Q}(t) = \mathbf{Q}^\top(t) > 0, \quad \mathbf{R}(t) = \mathbf{R}^\top(t) > 0$ (usually constant)
Initial conditions	$\hat{\mathbf{x}}(0) = \mathbf{x}_0$ $\mathbf{P}(0) = E[(\mathbf{x}(0) - \hat{\mathbf{x}}(0))(\mathbf{x}(0) - \hat{\mathbf{x}}(0))^\top] = \mathbf{P}_0$
Kalman gain matrix	$\mathbf{K}(t) = \mathbf{P}(t)\mathbf{H}^\top(t)\mathbf{R}^{-1}(t)$
State estimate propagation Error covariance propagation	$\boxed{\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}(t)\hat{\mathbf{x}}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{K}(t)[\mathbf{y}(t) - \mathbf{H}(t)\hat{\mathbf{x}}(t)]}$ $\dot{\mathbf{P}}(t) = \mathbf{A}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}^\top(t) + \mathbf{E}(t)\mathbf{Q}(t)\mathbf{E}^\top(t)$ $\quad - \mathbf{P}(t)\mathbf{H}^\top(t)\mathbf{R}^{-1}(t)\mathbf{H}(t)\mathbf{P}(t), \quad \mathbf{P}(t) = \mathbf{P}^\top(t) > 0$

Linear system model

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w} \\
\mathbf{y} &= \mathbf{H}\mathbf{x} + \mathbf{v}
\end{aligned}
$$

injection term

copy of dynamics

11.3.2 Continuous-Time Kalman Filter

Steady-State Kalman Filter

In the linear case it is computationally advantageous to use the *steady-state solution* of the KF. This filter has the same structure as the fixed-gain observers. The only difference is the method for computation of the filter gain matrix.

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w} \\ \mathbf{y} &= \mathbf{H}\mathbf{x} + \mathbf{v}\end{aligned}$$

System equations

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{K}_\infty(\mathbf{y} - \mathbf{H}\hat{\mathbf{x}})$$

$$\mathbf{K}_\infty = \mathbf{P}_\infty \mathbf{H}^\top \mathbf{R}^{-1}$$

Kalman filter with fixed gain

Matlab: *[k,P] = lqe(A,E,h,Q,R)*

where $\mathbf{P}_\infty = \mathbf{P}_\infty^\top > 0$ is the positive definite solution of the *algebraic matrix Riccati equation:*

$$\mathbf{A}\mathbf{P}_\infty + \mathbf{P}_\infty\mathbf{A}^\top + \mathbf{E}\mathbf{Q}\mathbf{E}^\top - \mathbf{P}_\infty\mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}\mathbf{P}_\infty = \mathbf{0}$$

This solution can only be applied for linear time-invariant (LTI) systems.

11.3.3 Extended Kalman Filter

Design matrices	$\mathbf{Q}(k)=\mathbf{Q}^\top(k) > 0,\ \mathbf{R}(k)=\mathbf{R}^\top(k) > 0$ (usually constant)
Initial conditions	$\bar{\mathbf{x}}(0) = \mathbf{x}_0$ $\bar{\mathbf{P}}(0) = E[(\mathbf{x}(0)-\hat{\mathbf{x}}(0))(\mathbf{x}(0)-\hat{\mathbf{x}}(0))^\top] = \mathbf{P}_0$
Kalman gain matrix State estimate update Error covariance update	$\mathbf{K}(k) = \bar{\mathbf{P}}(k)\mathbf{H}^\top(k)\,[\mathbf{H}(k)\bar{\mathbf{P}}(k)\mathbf{H}^\top(k)+\mathbf{R}(k)]^{-1}$ $\hat{\mathbf{x}}(k) = \bar{\mathbf{x}}(k)+\mathbf{K}(k)\boxed{[\mathbf{y}(k)-\mathbf{H}(k)\bar{\mathbf{x}}(k)]}$ $\hat{\mathbf{P}}(k) = [\mathbf{I}-\mathbf{K}(k)\mathbf{H}(k)]\,\bar{\mathbf{P}}(k)\,[\mathbf{I}-\mathbf{K}(k)\mathbf{H}(k)]^\top$ $\quad +\mathbf{K}(k)\mathbf{R}(k)\mathbf{K}^\top(k),\quad \hat{\mathbf{P}}(k) = \hat{\mathbf{P}}(k)^\top > 0$
State estimate propagation Error covariance propagation	$\bar{\mathbf{x}}(k+1) = \mathcal{F}(\hat{\mathbf{x}}(k),\mathbf{u}(k))$ $\bar{\mathbf{P}}(k+1) = \mathbf{\Phi}(k)\hat{\mathbf{P}}(k)\mathbf{\Phi}^\top(k)+\mathbf{\Gamma}(k)\mathbf{Q}(k)\mathbf{\Gamma}^\top(k)$

Nonlinear system model

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})+\mathbf{B}\mathbf{u}+\mathbf{E}\mathbf{w}$$
$$\mathbf{y} = \mathbf{H}\mathbf{x}+\mathbf{v}$$

injection term

Discretized approximate model

$$\mathcal{F}(\hat{\mathbf{x}}(k),\mathbf{u}(k)) = \hat{\mathbf{x}}(k)+h[\mathbf{f}(\hat{\mathbf{x}}(k))+\mathbf{B}\mathbf{u}(k)]$$

$$\mathbf{\Phi}(k) = \mathbf{I}+h\left.\frac{\partial \mathbf{f}(\mathbf{x}(k),\mathbf{u}(k))}{\partial \mathbf{x}(k)}\right|_{\mathbf{x}(k)=\hat{\mathbf{x}}(k)}$$
$$\mathbf{\Gamma}(k) = h\mathbf{E}$$

11.3.4 Corrector-Predictor Representation for Nonlinear Observers

Example 11.3 (Corrector-Predictor for Ship Navigation using Two Measurement Rates)

When implementing nonlinear observers, the corrector-predictor representation of the discrete time EKF can be used to effectively handle slow measurement rates, multiple measurement rates and dead-reckoning.

$$\dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}}, \mathbf{u}) + \boldsymbol{\gamma}(\mathbf{y}, \hat{\mathbf{y}})$$

$$\boldsymbol{\gamma}(\mathbf{y}, \hat{\mathbf{y}}) = \mathbf{K}(\mathbf{y} - \hat{\mathbf{y}})$$

$$\mathbf{K}_d = h\mathbf{K}$$

$$\begin{array}{ll} \text{Corrector} & \hat{\mathbf{x}}(k) = \bar{\mathbf{x}}(k) + \mathbf{K}_d\,[\mathbf{y}(k) - \bar{\mathbf{y}}(k)] \\ \text{Predictor} & \bar{\mathbf{x}}(k+1) = \hat{\mathbf{x}}(k) + h\mathbf{f}(\hat{\mathbf{x}}(k), \mathbf{u}(k)) \end{array}$$

$h = 0.01$ *sampling time*
$\text{GPS} = 10$ *counter for GPS measurements*
$\bar{x} = x_0$ *initial state vector*

while estimating

Method A	**Method B**
$y_{\text{IMU}} = \textit{measurement}$	$y_{\text{IMU}} = \textit{measurement}$
$y_{\text{GPS}} = \textit{measurement}$	
$K_d = [hk_{\text{IMU}}, 0]^\top$	$K_d = [hk_{\text{IMU}}, 10hk_{\text{GPS}}]^\top$
if $\text{GPS} = 10$	*if* $\text{GPS} = 10$
$\quad K_d = [hk_{\text{IMU}}, 10hk_{\text{GPS}}]^\top$	$\quad y_{\text{GPS}} = \textit{measurement}$
$\quad \text{GPS} = 0$	$\quad \text{GPS} = 0$
end	*end*
if dead-reckoning (no updates)	*if* dead-reckoning (no updates)
$\quad K_d = [0, 0]^\top$	$\quad K_d = [0, 0]^\top$
end	*end*
$y = [y_{\text{IMU}}^\top, y_{\text{GPS}}^\top]^\top$	$y = [y_{\text{IMU}}^\top, y_{\text{GPS}}^\top]^\top$
$\hat{x} = \bar{x} + K_d\,[y - H\bar{x}]$	$\hat{x} = \bar{x} + K_d\,[y - H\bar{x}]$
$u = \textit{control system (optionally)}$	$u = \textit{control system (optionally)}$
$\bar{x} = \hat{x} + hf(\hat{x}, u)$	$\bar{x} = \hat{x} + hf(\hat{x}, u)$
$\text{GPS} = \text{GPS} + 1$	$\text{GPS} = \text{GPS} + 1$

end

11.3.5 Case Study: Kalman Filter for Heading Autopilots using only Compass Measurements

The main sensor components for a heading controlled marine craft are:

- **Magnetic and/or gyroscopic compasses measuring ψ**
- **Yaw rate gyro measuring r**

In many commercial systems only the compass is used for feedback control since the yaw rate can be estimated quite well by a state estimator.

LF vessel model

$$\begin{aligned} \dot{\psi} &= r \\ \dot{r} &= -\frac{1}{T}r + \frac{1}{m}(\tau_{wind} + \tau_{\mathrm{N}}) + b + w_2 \\ \dot{b} &= w_3 \end{aligned}$$

WF model

$$\begin{aligned} \dot{\xi}_w &= \psi_w \\ \dot{\psi}_w &= -\omega_0^2\, \xi_w - 2\lambda\omega_0\, \psi_w + w_1 \end{aligned}$$

Compass measurement

$$y = \psi + \psi_w + v$$

Yaw moment (control input)

$$\tau_{\mathrm{N}} = m\frac{K}{T}\delta = N_\delta \delta \qquad m = I_z - N_{\dot{r}}$$

11.3.5 Case Study: Kalman Filter for Heading Autopilots using only Compass Measurements

The LF and WF models must be written in state-space form in order to use the Kalman filter algorithm.

State-space model

$$\boldsymbol{x} = [\xi_w, \psi_w, \psi, r, b]^\top$$
$$u = \tau_{wind} + \tau_{\mathrm{N}}$$
$$\boldsymbol{w} = [w_1, w_2, w_3]^\top$$

$$\dot{\boldsymbol{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u + \mathbf{w}$$
$$y = \mathbf{h}^\top \mathbf{x} + v$$

$$\boldsymbol{A} = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ -\omega_0^2 & -2\lambda\omega_0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1/T & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \boldsymbol{b} = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 1/m \\ 0 \end{array}\right]$$

$$\boldsymbol{E} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \qquad \boldsymbol{h}^\top = [0, 1, 1, 0, 0]$$

11.3.5 Case Study: Kalman Filter for Heading Autopilots using only Compass Measurements

Example 11.4 (Continuous-Time Steady-State KF for Ship Autopilots)
For the ship-wave system (11.66)–(11.67), the SISO continuous-time state estimator takes the form:

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{b}u + \boldsymbol{k}_\infty(y - \boldsymbol{h}^\top \boldsymbol{x})$$

where the Kalman filter gain is:

$$\boldsymbol{k}_\infty = \frac{1}{r}\boldsymbol{P}_\infty \boldsymbol{h}$$

The covariance matrix $\boldsymbol{P}_\infty = \boldsymbol{P}_\infty^\top > 0$ *is given by the ARE:*

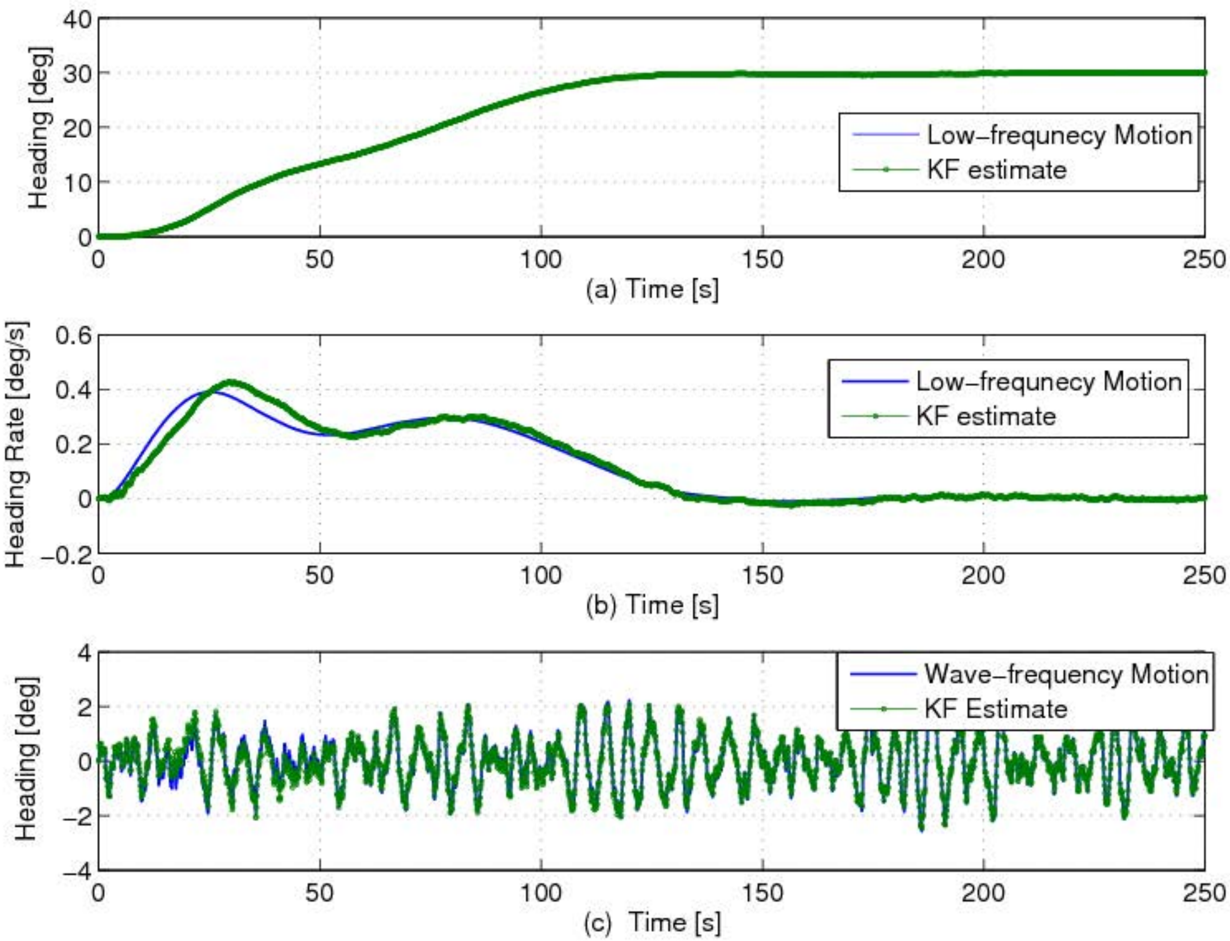
$$\boldsymbol{A}\boldsymbol{P}_\infty + \boldsymbol{P}_\infty \boldsymbol{A}^\top + \boldsymbol{E}\boldsymbol{Q}\boldsymbol{E}^\top - \frac{1}{r}\boldsymbol{P}_\infty \boldsymbol{h}\boldsymbol{h}^\top \boldsymbol{P}_\infty = \boldsymbol{0}$$

The KF gain $\boldsymbol{k}_\infty$ *is computed in Matlab as:*

```
R = r
Q = diag{q11,q22,q33}

[k,P] = lqe(A,E,h,Q,R)
```

11.3.5 Case Study: Kalman Filter for Heading Autopilots using only Compass Measurements



True LF heading ψ and estimate.

True LF heading rate r and estimate.

True WF component of the heading ψ_w and estimate.

11.3.6 Case Study: Kalman Filter for Dynamic Positioning Systems using GNSS and Compass Measurements

History:

Dynamic positioning (DP) systems have been commercially available for marine craft since the 1960's. The first DP systems were designed using conventional PID controllers in cascade with low pass and/or notch filters to suppress the wave-induced motion.

From the middle of the 1970's more advanced filtering techniques were available thanks to the *Kalman filter* (Kalman 1960). This motivated *Balchen* and coauthors to define wave filtering in terms of linear optimal estimation theory; see *Balchen et al.* (1976, 1980a, 1980b). A similar design technique has been proposed by *Grimble et al*. (1979, 1980a).

The Kalman Filter as a DP Observer:

In many cases, measurements of the vessel velocities are not available. Hence, estimates of the velocities must be computed from noisy position and heading measurements through a state observer. Unfortunately, the position and heading measurements are corrupted with colored noise due to wind, waves and ocean currents as well as sensor noise. Only the slowly varying disturbances should be counteracted by the propulsion system, whereas the oscillatory motion due to the waves (1st-order wave-induced forces) should not enter the feedback loop. This require wave filtering.

11.3.6 Case Study: Kalman Filter for Dynamic Positioning Systems using GNSS and Compass Measurements

Navigation Systems

Several position measurement systems are commercially available, for instance:

- **Local Navigation Systems**
  - Hydro-acoustic positioning reference (HPR)
  - Taut wire
- **Global Navigation Satellite Systems (GNSS)**
  - Galileo – EU
  - Navstar GPS – USA
  - Global Orbiting Navigation Satellite System (GLONASS) - Russian

11.3.6 Case Study: Kalman Filter for Dynamic Positioning Systems using GNSS and Compass Measurements

In many DP systems the wave filtering and state estimation problems are solved by using the linear or Extended Kalman Filter (EKF). The major drawback of the EKF is that the kinematic and kinetic equations of motions must be *linearized about varying velocities and yaw angle,* for instance by using:

$$\begin{aligned}\dot{\boldsymbol{\eta}} &= \mathbf{R}(\psi)\boldsymbol{\nu} \\ \mathbf{M}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{D}\exp(-\alpha V_{rc})\boldsymbol{\nu}_r + \mathbf{d}(V_{rc},\gamma_{rc}) &= \boldsymbol{\tau} + \boldsymbol{\tau}_{wind}\end{aligned}$$

$$\mathbf{d}(V_{rc},\gamma_{cr}) = \begin{bmatrix} -\frac{1}{2}\rho A_{Fc}C_X(\gamma_{rc})V_{rc}^2 \\ -\frac{1}{2}\rho A_{Lc}C_Y(\gamma_{rc})V_{rc}^2 \\ -\frac{1}{2}\rho A_{Lc}L_{oa}C_N(\gamma_{rc})V_{rc}^2 - N_{|r|r}|r|r \end{bmatrix}$$

Nonlinear damping

An alternative approach is to linearize the nonlinear damping model (small velocity assumption) such that vessel parallel coordinates can be used (no need for look-up tables):

$$\mathbf{D}\exp(-\alpha V_{rc})\boldsymbol{\nu}_r + \boldsymbol{d}(V_{rc},\gamma_{rc}) \approx \mathbf{D}\boldsymbol{\nu} - \mathbf{R}^\top(\psi)\mathbf{b}$$

11.3.6 Case Study: Kalman Filter for Dynamic Positioning Systems using GNSS and Compass Measurements

Vessel kinematics and kinetics

$$
\begin{aligned}
\dot{\boldsymbol{\eta}} &= \mathbf{R}(\psi)\boldsymbol{\nu} \\
\mathbf{M}\dot{\boldsymbol{\nu}} + \mathbf{D}\boldsymbol{\nu} &= \boldsymbol{\tau} + \mathbf{R}^{\top}(\psi)\mathbf{b} + \mathbf{w}_3
\end{aligned}
$$

LF model with linear damping and nonlinear rotation matrix

where $\boldsymbol{\eta} = [N, E, \psi]^{\top}, \boldsymbol{\nu} = [u, v, r]$

1st-order wave response model

1st-order wave forces

$$
\begin{aligned}
\dot{\boldsymbol{\xi}} &= \mathbf{A}_w \boldsymbol{\xi} + \mathbf{E}_w \mathbf{w}_1 \\
\boldsymbol{\eta}_w &= \mathbf{C}_w \boldsymbol{\xi}
\end{aligned}
$$

where $\boldsymbol{\xi} \in \mathbb{R}^6$ is the state vector, $\mathbf{w}_1 \in \mathbb{R}^3$ is a vector of zero-mean Gaussian white noise, and $\mathbf{A}_w \in \mathbb{R}^{6\times 6}$, $\mathbf{E}_w \in \mathbb{R}^{6\times 3}$ and $\mathbf{C}_w \in \mathbb{R}^{3\times 6}$ are constant matrices of appropriate dimensions

Bias modeling (slowly varying disturbances)

2nd-order wave forces, ocean currents and wind forces

$$
\dot{\mathbf{b}} = \mathbf{w}_2
$$

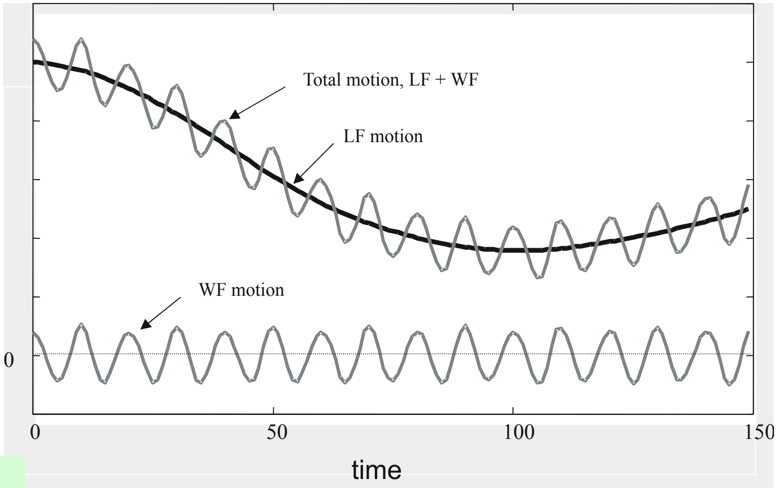
where $\mathbf{w}_2$ is a vector of zero-mean Gaussian white noise.

11.3.6 Case Study: Kalman Filter for Dynamic Positioning Systems using GNSS and Compass Measurements

Measurement model

The position and yaw angle measurements are generated by using the principle of linear superposition:

$$\mathbf{y} = \boldsymbol{\eta} + \boldsymbol{\eta}_w + \mathbf{v}$$



Resulting DP observer model

$$\begin{aligned}
\dot{\boldsymbol{\xi}} &= \mathbf{A}_w \boldsymbol{\xi} + \mathbf{E}_w \mathbf{w}_1 \\
\dot{\boldsymbol{\eta}} &= \mathbf{R}(\psi)\boldsymbol{\nu} \\
\dot{\mathbf{b}} &= \mathbf{w}_2 \qquad (\text{alternatively } \dot{\mathbf{b}} = -\mathbf{T}^{-1}\mathbf{b} + \mathbf{w}_2) \\
\mathbf{M}\dot{\boldsymbol{\nu}} &= -\mathbf{D}\boldsymbol{\nu} + \mathbf{R}^\top(\psi)\mathbf{b} + \boldsymbol{\tau} + \mathbf{w}_3 \\
\mathbf{y} &= \boldsymbol{\eta} + \mathbf{C}_w \boldsymbol{\xi} + \mathbf{v}
\end{aligned}$$

The nonlinear rotation matrices can be removed by using vessel parallel coordinates

11.3.6 Case Study: Kalman Filter for Dynamic Positioning Systems using GNSS and Compass Measurements

Linear vessel parallel (VP) Kalman filter design

Since the only nonlinear term in the system model is the rotation matrix **R** it is convenient to use vessel parallel coordinates (assumes that the heading angle ψ is constant):

$$
\begin{aligned}
\boldsymbol{\eta}_p &= \mathbf{R}^\top(\psi)\boldsymbol{\eta} \\
\mathbf{b}_p &= \mathbf{R}^\top(\psi)\mathbf{b}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\boldsymbol{\eta}}_p &= \mathbf{R}^\top(\psi)\dot{\boldsymbol{\eta}} + \dot{\mathbf{R}}^\top(\psi)\boldsymbol{\eta} \\
&= \mathbf{R}^\top(\psi)\mathbf{R}(\psi)\boldsymbol{\nu} + \dot{\mathbf{R}}^\top(\psi)\mathbf{R}(\psi)\boldsymbol{\eta}_p \\
&= \boldsymbol{\nu} + \dot{\mathbf{R}}^\top(\psi)\boldsymbol{\eta}
\end{aligned}
$$

This gives the linear model:

$$
\begin{aligned}
\dot{\boldsymbol{\xi}} &= \mathbf{A}_w\boldsymbol{\xi} + \mathbf{E}_w\mathbf{w}_1 \\
\dot{\boldsymbol{\eta}}_p &= \boldsymbol{\nu} \\
\dot{\mathbf{b}}_p &= \mathbf{w}_2 \qquad (\text{alternatively } \dot{\mathbf{b}}_p = -\mathbf{T}^{-1}\mathbf{b}_p + \mathbf{w}_2) \\
\mathbf{M}\dot{\boldsymbol{\nu}} &= -\mathbf{D}\boldsymbol{\nu} + \mathbf{b}_p + \boldsymbol{\tau} + \boldsymbol{\tau}_{wind} + \mathbf{w}_3 \\
\mathbf{y} &= \boldsymbol{\eta}_p + \mathbf{C}_w\boldsymbol{\xi} + \mathbf{v}
\end{aligned}
$$

zero for constant heading, and negligible for vessels with large inertia—that is, slow rotational velocity

When using this model and the linear KF, the heading angle ψ must change slowly. If not use the EKF!

11.3.6 Case Study: Kalman Filter for Dynamic Positioning Systems using GNSS and Compass Measurements

State-space model using VP coordinates

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w} \\
\mathbf{y} &= \mathbf{H}\mathbf{x} + \mathbf{v}
\end{aligned}
$$

$$
\mathbf{A} = \left[\begin{array}{c|ccc}
\mathbf{A}_w & \mathbf{0}_{6\times 3} & \mathbf{0}_{6\times 3} & \mathbf{0}_{6\times 3} \\
\hline
\mathbf{0}_{3\times 6} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{I}_{3\times 3} \\
\mathbf{0}_{3\times 6} & \mathbf{0}_{3\times 3} & -\mathbf{T}^{-1} & \mathbf{0}_{3\times 3} \\
\mathbf{0}_{3\times 6} & \mathbf{0}_{3\times 3} & \mathbf{M}^{-1} & -\mathbf{M}^{-1}\mathbf{D}
\end{array}\right], \quad
\mathbf{B} = \left[\begin{array}{c}
\mathbf{0}_{6\times p} \\
\hline
\mathbf{0}_{3\times p} \\
\mathbf{0}_{3\times p} \\
\mathbf{M}^{-1}\mathbf{B}_u
\end{array}\right]
$$

$$
\mathbf{E} = \left[\begin{array}{c|cc}
\mathbf{E}_w & \mathbf{0}_{6\times 3} & \mathbf{0}_{6\times 3} \\
\hline
\mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\
\mathbf{0}_{3\times 3} & \mathbf{I}_{3\times 3} & \mathbf{0}_{3\times 3} \\
\mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{M}^{-1}
\end{array}\right], \quad
\mathbf{H} = \left[\begin{array}{c|ccc}
\mathbf{C}_w & \mathbf{I}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3}
\end{array}\right]
$$

11.3.6 Case Study: Kalman Filter for Dynamic Positioning Systems using GNSS and Compass Measurements

Linear Kalman filter using VP coordinates

The continuous-time filter equations for this system is given by:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \underbrace{\mathbf{P}\mathbf{H}^\top\mathbf{R}^{-1}}_{\mathbf{K}}(\mathbf{y} - \mathbf{H}\hat{\mathbf{x}})$$

$$\dot{\mathbf{P}} = \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^\top + \mathbf{E}\mathbf{Q}\mathbf{E}^\top - \mathbf{P}\mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}\mathbf{P}$$

The covariance matrices $\mathbf{Q} = \mathbf{Q}^\top$ and $\mathbf{R} = \mathbf{R}^\top$ are design matrices:

- The measurement covariance matrix can be chosen as $\mathbf{R} = \mathrm{diag}\{r_1, r_2, r_3\}$ where the elements r_1 and r_2 are the covariance of the GNSS position measurements and r_3 is the compass covariance.
- The matrix $\mathbf{Q}$ is usually chosen to be diagonal with positive tunable parameters. These are usually found by trial and error.

Notice that only local exponential stability can be proven for VP coordinates.

Global exponential stability can, however, be obtained using a nonlinear passive observer →

11.4 Nonlinear Passive Observer Designs

Extended Kalman Filtering has been applied in a large number of ship applications, see Fossen (1994) and references therein, Balchen, Jenssen and Sælid (1976, 1980), Grimble, Patton and Wise (1980), Katebi and Grimble (1989) to mention some.

Drawback: If the EKF is combined with a state feedback controller using the estimates of the states global asymptotic/exponential stability <u>cannot</u> be guaranteed.

The EKF is only locally exponentially stable (LES) since the Ricatti equations are based on the linearized model of the plant using the transition matrix.

Global exponential stability (GES) of an observer-controller system requires:

- The existence of a nonlinear separation principle
- A nonlinear globally stable observer

Conclusion: There is no guarantee for global stability when applying the EKF in cascade with your favorite full state feedback controller.

11.4 Nonlinear Passive Observer Designs

The drawbacks of the Kalman filter are:

It is difficult and time-consuming to tune the state estimator (stochastic system with 15 states and 120 covariance equations).

The main reason for this is that the numerous covariance tuning parameters may be difficult to relate to physical quantities. This results in an ad hoc tuning procedure for the process covariance matrix **Q** while the measurement covariance matrix **R** usually is well defined in terms of sensor specifications.

only local results

This motivates the search for a fixed gain observer covering the whole state space (global exponential stability)

Alternative Solution:

Fossen, T. I. and J. P. Strand (1999). Passive Nonlinear Observer Design for Ships Using Lyapunov Methods: Experimental Results with a Supply Vessel, *Automatica,* Vol. 35, No. 1, pp. 3-16, January 1999.

11.4.1 Case Study: Passive Observer for Dynamic Positioning using GNSS and Compass Measurements

Dynamic positioning (DP) Model

$$
\begin{aligned}
\dot{\xi} &= A_w \xi \\
\dot{\eta} &= R(y_3)\nu \\
\dot{b} &= -T^{-1} b \qquad \text{(alternatively } \dot{b} = 0\text{)} \\
M\dot{\nu} &= -D\nu + R^\top(y_3) b + \tau + \tau_{\text{wind}} \\
y &= \eta + C_w \xi
\end{aligned}
$$

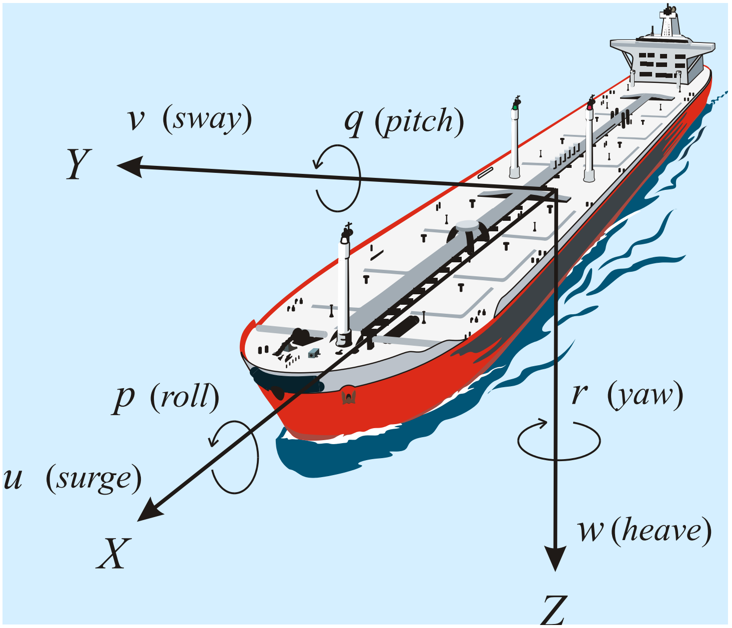
Position/heading (Earth) $\eta = [N, E, \psi]^\top$

Velocity (Body) $\nu = [u, v, r]^\top$

$$
R(\psi) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
M = \begin{bmatrix} m - X_{\dot{u}} & 0 & 0 \\ 0 & m - Y_{\dot{v}} & m x_g - Y_{\dot{r}} \\ 0 & m x_g - Y_{\dot{r}} & I_z - N_{\dot{r}} \end{bmatrix}
$$

$$
D = \begin{bmatrix} -X_u & 0 & 0 \\ 0 & -Y_v & -Y_r \\ 0 & -N_v & -N_r \end{bmatrix}
$$



11.4.1 Case Study: Passive Observer for Dynamic Positioning using GNSS and Compass Measurements

Environmental models (wind, waves and ocean currents):

Stochastic bias models:

$$\dot{\mathbf{b}} = \mathbf{T}^{-1}\mathbf{b} + \mathbf{w}$$

$$\dot{\mathbf{b}} = \mathbf{w}$$

white noise

Deterministic bias models:

$$\xrightarrow{\mathbf{w} = \mathbf{0}}$$

$$\dot{\mathbf{b}} = \mathbf{T}^{-1}\mathbf{b}$$

$$\dot{\mathbf{b}} = \mathbf{0}$$

Stochastic wave model:

$$\dot{\boldsymbol{\xi}} = \mathbf{A}_w\boldsymbol{\xi} + \mathbf{B}_w\mathbf{w}$$

$$\boldsymbol{\eta}_w = \mathbf{C}_w\boldsymbol{\xi}$$

white noise

Deterministic wave model:

$$\xrightarrow{\mathbf{w} = \mathbf{0}}$$

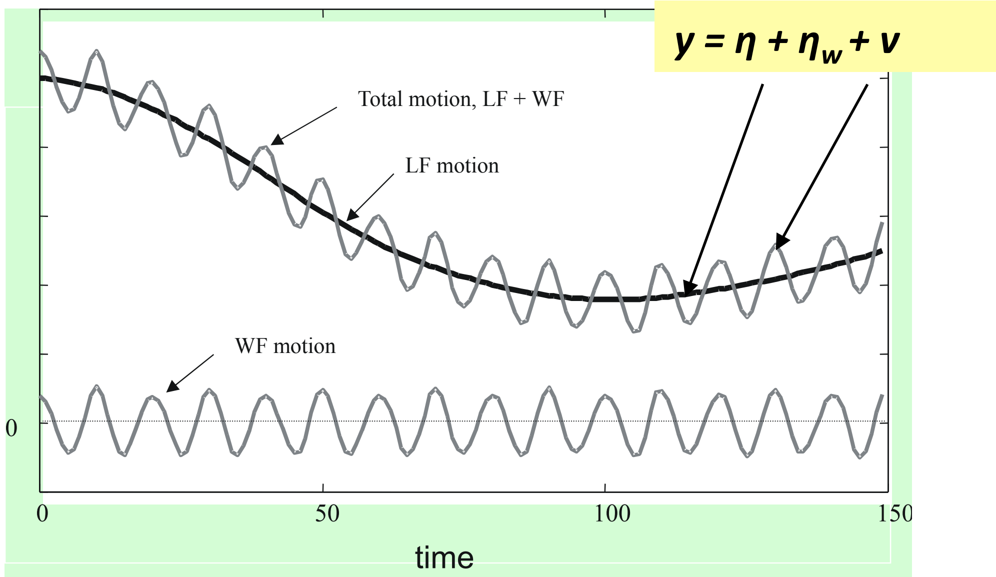
$$\dot{\boldsymbol{\xi}} = \mathbf{A}_w\boldsymbol{\xi}$$

$$\boldsymbol{\eta}_w = \mathbf{C}_w\boldsymbol{\xi}$$

$$\frac{\eta_{w_i}}{w_i}(s) = \frac{K_{w_i}s}{s^2 + 2\lambda\omega_{0_i}s + \omega_{0_i}^2}$$ 1st-order wave-induced forces

The passive nonlinear observer is derived in a deterministic setting but it can be used in stochastic systems.

11.4.1 Case Study: Passive Observer for Dynamic Positioning using GNSS and Compass Measurements



Observer requirements:

- Observer must reconstruct η, η_w and ν from y
- Only η and ν are used for feedback

Position/heading (Earth)

$\eta = [x, y, z]^T$

Velocity (Body)

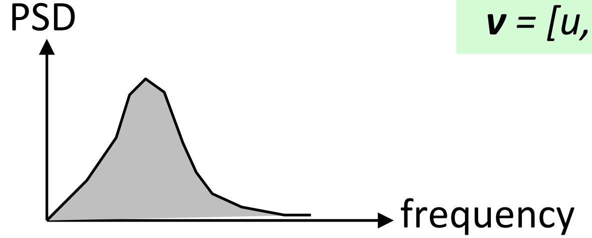
$\nu = [u, v, r]^T$

Measurement Equation (GNSS+Compass)

$y = \eta + \eta_w + v$ ← measurement noise

Low-frequency motion from the ship model

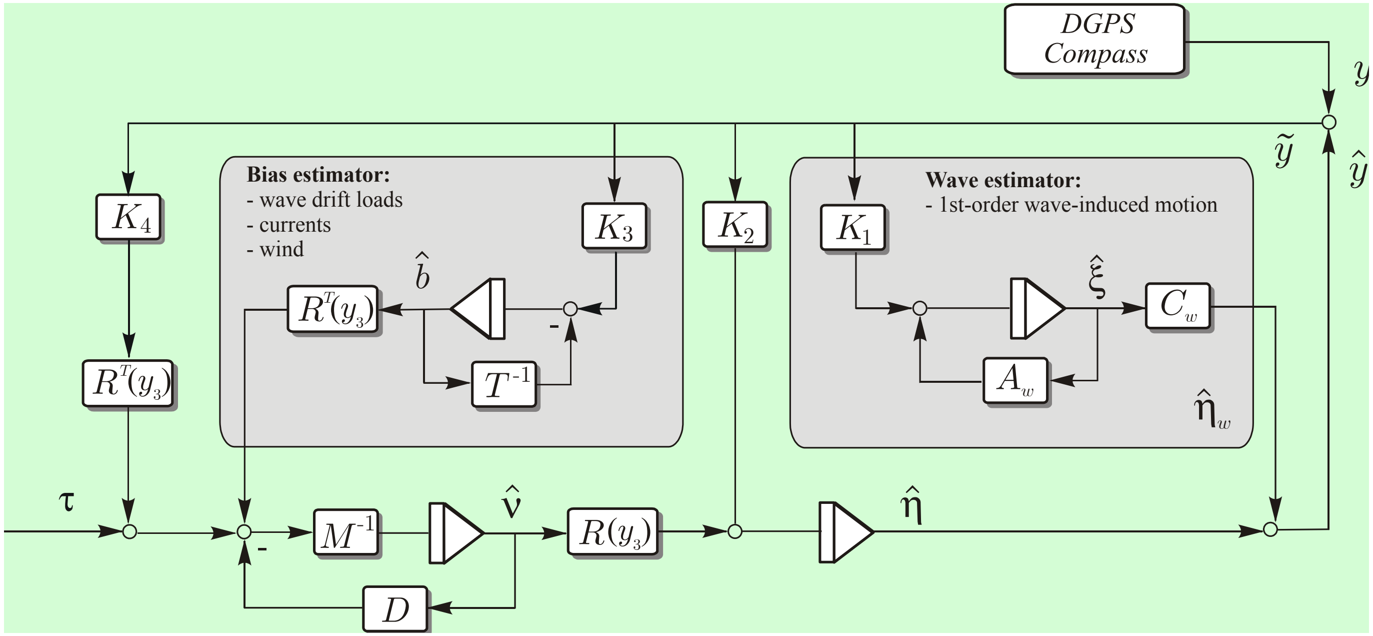
Wave frequency motion generated by a wave spectrum



11.4.1 Case Study: Passive Observer for Dynamic Positioning using GNSS and Compass Measurements

$$\begin{aligned}
\dot{\hat{\boldsymbol{\xi}}} &= \mathbf{A}_w\hat{\boldsymbol{\xi}} + \mathbf{K}_1(\boldsymbol{\omega}_o)\tilde{\mathbf{y}} && \text{wave model}\\
\dot{\hat{\boldsymbol{\eta}}} &= \mathbf{R}(y_3)\hat{\boldsymbol{\nu}} + \mathbf{K}_2\tilde{\mathbf{y}} && \text{kinematics}\\
\dot{\hat{\mathbf{b}}} &= -\mathbf{T}^{-1}\hat{\mathbf{b}} + \mathbf{K}_3\tilde{\mathbf{y}} \qquad (\text{alternatively } \dot{\hat{\mathbf{b}}} = \mathbf{K}_3\tilde{\mathbf{y}}) && \text{bias}\\
\mathbf{M}\dot{\hat{\boldsymbol{\nu}}} &= -\mathbf{D}\hat{\boldsymbol{\nu}} + \mathbf{R}^\top(y_3)\hat{\mathbf{b}} + \boldsymbol{\tau} + \mathbf{R}^\top(y_3)\mathbf{K}_4\tilde{\mathbf{y}} && \text{dynamics}\\
\mathbf{y} &= \hat{\boldsymbol{\eta}} + \mathbf{C}_w\hat{\boldsymbol{\xi}} && \text{measurements}
\end{aligned}$$

Goal:
choose the *gains* $\mathbf{K}_i$ such that the error dynamics is *passive* and *GES*



11.4.1 Case Study: Passive Observer for Dynamic Positioning using GNSS and Compass Measurements

Observer error dynamics (position and WF models):

$$\begin{aligned} \dot{\hat{\boldsymbol{\eta}}}_0 &= \mathbf{A}_0\hat{\boldsymbol{\eta}}_0 + \mathbf{B}_0\mathbf{R}(y_3)\hat{\boldsymbol{\nu}} + \mathbf{K}_0(\omega_o)\tilde{\mathbf{y}} \\ \mathbf{y} &= \mathbf{C}_0\hat{\boldsymbol{\eta}}_0 \end{aligned}$$

$$\mathbf{K}_0(\omega_o) = \begin{bmatrix} \mathbf{K}_1(\omega_o) \\ \mathbf{K}_2 \end{bmatrix}$$

$$\hat{\boldsymbol{\eta}}_0 = [\hat{\boldsymbol{\xi}}^\top, \hat{\boldsymbol{\eta}}^\top]^\top$$

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{A}_w & \mathbf{0}_{6\times3} \\ \mathbf{0}_{3\times6} & \mathbf{0}_{3\times3} \end{bmatrix}, \quad \mathbf{B}_0 = \begin{bmatrix} \mathbf{0}_{6\times3} \\ \mathbf{I}_{3\times3} \end{bmatrix}, \quad \mathbf{C}_0 = \begin{bmatrix} \mathbf{C}_w & \mathbf{I}_{3\times3} \end{bmatrix}$$

Observer error dynamics including velocity/bias:

$$\begin{aligned} \dot{\tilde{\boldsymbol{\eta}}}_0 &= [\mathbf{A}_0 - \mathbf{K}_0(\omega_o)\mathbf{C}_0]\tilde{\boldsymbol{\eta}}_0 + \mathbf{B}_0\mathbf{R}(y_3)\tilde{\boldsymbol{\nu}} \\ \dot{\tilde{\mathbf{b}}} &= -\mathbf{T}^{-1}\tilde{\mathbf{b}} - \mathbf{K}_3\tilde{\mathbf{y}} \qquad (\text{alternatively } \dot{\tilde{\mathbf{b}}} = -\mathbf{K}_3\tilde{\mathbf{y}}) \\ \mathbf{M}\dot{\tilde{\boldsymbol{\nu}}} &= -\mathbf{D}\tilde{\boldsymbol{\nu}} + \mathbf{R}^\top(y_3)\tilde{\mathbf{b}} - \mathbf{R}^\top(y_3)\mathbf{K}_4\tilde{\mathbf{y}} \end{aligned}$$

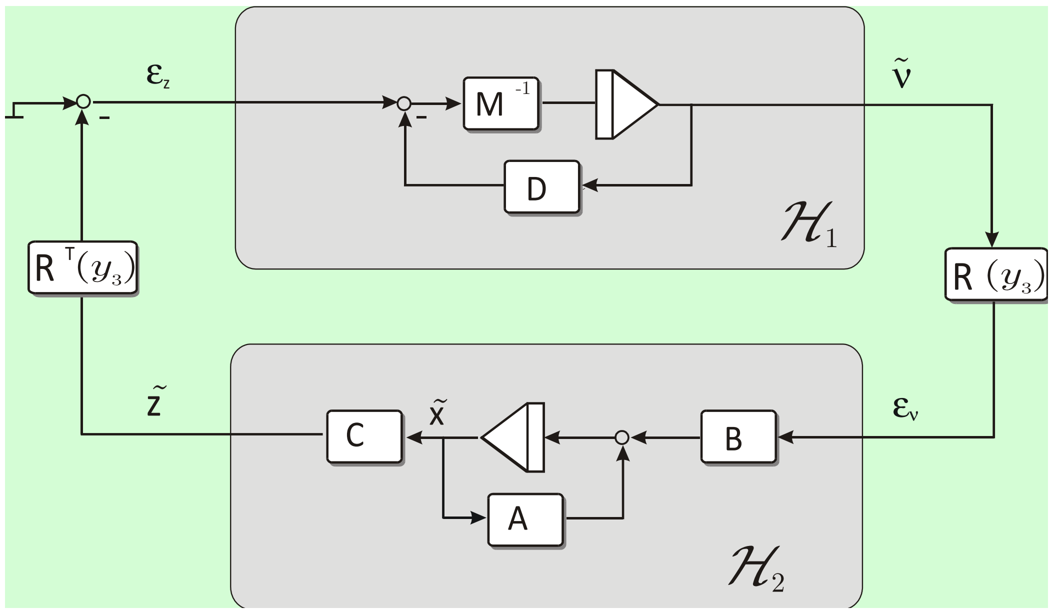
$$\begin{aligned} \tilde{\boldsymbol{\nu}} &= \boldsymbol{\nu} - \hat{\boldsymbol{\nu}} \\ \tilde{\mathbf{b}} &= \mathbf{b} - \hat{\mathbf{b}} \\ \tilde{\boldsymbol{\eta}}_0 &= \boldsymbol{\eta}_0 - \hat{\boldsymbol{\eta}}_0 \end{aligned}$$

11.4.1 Case Study: Passive Observer for Dynamic Positioning using GNSS and Compass Measurements

Forming two passive blocks $\mathcal{H}_1$ and $\mathcal{H}_2$:

$$\mathbf{M}\dot{\tilde{\mathbf{v}}} = -\mathbf{D}\tilde{\mathbf{v}} - \mathbf{R}^\top(y_3)\tilde{\mathbf{z}}$$

$$\boldsymbol{\varepsilon}_z = -\mathbf{R}^\top(y_3)\tilde{\mathbf{z}}$$



$$\tilde{\mathbf{z}} = \mathbf{K}_4\tilde{\mathbf{y}} - \tilde{\mathbf{b}}$$

$$\boldsymbol{\varepsilon}_v = \mathbf{R}(y_3)\tilde{\mathbf{v}}$$

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}\mathbf{R}(y_3)\tilde{\mathbf{v}}$$
$$\tilde{\mathbf{z}} = \mathbf{C}\tilde{\mathbf{x}}$$

$$\tilde{\mathbf{x}} = \begin{bmatrix} \tilde{\boldsymbol{\eta}}_0 \\ \tilde{\mathbf{b}} \end{bmatrix}$$

11.4 Nonlinear Passive Observer Designs

Passivity is a property of engineering systems, most commonly used in electronic engineering and control systems.

A **passive component**, may be either a component that consumes (but does not produce) energy, or a component that is incapable of power gain. A component that is not passive is called an **active component**.

An electronic circuit consisting entirely of passive components is called a passive circuit (and has the same properties as a passive component).

A transfer functions $h(s)$ must have phase greater than -90° in order to be passive.

Passivity is related to stability and Lyapunov analysis can be used to prove passivity/stability in nonlinear systems while for linear systems the Kalman-Yakubovich-Popov (KYP) Lemma can be used to prove stability.

11.4.1 Case Study: Passive Observer for Dynamic Positioning using GNSS and Compass Measurements

Definition 6.3 (Khalil 2002) A nonlinear system is said to be passive if there exists a continuously differentiable positive definite function *V(x)* (called storage function) such that:

$$u^T y \geq \dot{V}$$

Moreover, it is said to be

- *Lossless if* $u^T y = \dot{V}$
- *Input-feedforward passive if* $u^T y \geq \dot{V} + u^T \varphi(u)$ *for some function* $j(u)$
- *Input strictly passive if* $u^T y \geq \dot{V} + u^T \varphi(u)$ *and* $u^T \phi(u) > 0$*, for all* $u \neq 0$
- *Output-feedback passive if* $u^T y \geq \dot{V} + y^T \rho(y)$ *for some function* $r(y)$
- *Output strictly passive if* $u^T y \geq \dot{V} + y^T \rho(y)$ *and* $y^T \rho(y) > 0$ *, for all* $y \neq 0$

- *Strictly passive if* $u^T y \geq \dot{V} + \psi(x)$ *for some positive definite function* $y(x)$

11.4.1 Case Study: Passive Observer for Dynamic Positioning using GNSS and Compass Measurements

Proposition 11.1 (Strictly Passive Velocity Error Dynamics)

The mapping $\mathcal{H}_1$ is *strictly passive*.

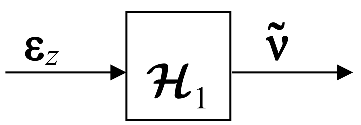
Proof: Let,

$$S_1 = \frac{1}{2}\tilde{\mathbf{v}}^\top \mathbf{M}\tilde{\mathbf{v}}$$

be a positive definite storage function. Time differentiation of S_1, yields:

$$\dot{S}_1 = -\frac{1}{2}\tilde{\mathbf{v}}^\top(\mathbf{D}+\mathbf{D}^\top)\tilde{\mathbf{v}} - \tilde{\mathbf{z}}^\top \mathbf{R}(y_3)\tilde{\mathbf{v}}$$

Using the fact that $\boldsymbol{\varepsilon}_z = -\mathbf{R}^\top(y_3)\tilde{\mathbf{z}}$, yields:



Hence:

$$\boldsymbol{\varepsilon}_z^\top \tilde{\mathbf{v}} = \dot{S}_1 + \frac{1}{2}\tilde{\mathbf{v}}^\top(\mathbf{D}+\mathbf{D}^\top)\tilde{\mathbf{v}}$$

This proves that $\mathcal{H}_1$ is strictly passive.

11.4.1 Case Study: Passive Observer for Dynamic Positioning using GNSS and Compass Measurements

Theorem 6.3 (Khalil 2002) *The feedback connection of two time-invariant dynamical systems is GAS if the origin of the nominal system (u = 0) is asymptotically stable and*

- *both feedback components are strictly passive*
- *both feedback components are output strictly passive and zero-state observable, or*
- *one component is strictly passive and the other is output strictly passive and zero-state observable*

In addition the storage function for each component must be radially unbounded

---

1. The mapping $\boldsymbol{\varepsilon}_z \mapsto \tilde{\mathbf{v}}$ is strictly passive (block $\mathcal{H}_1$)
2. Post-multiplication with the bounded transformation matrix $\mathbf{R}(y_3)$ and pre-multiplication by it's transpose will not affect the passivity properties.
3. Hence, it only remains to show that the mapping $\boldsymbol{\varepsilon}_v \mapsto \tilde{\mathbf{z}}$ (block $\mathcal{H}_2$) is strictly passive

---

For linear systems passivity can easily be checked by applying the

Kalman-Yakubovich-Popov (KYP) Lemma.

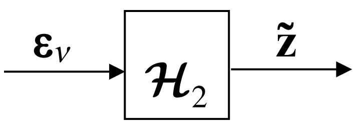
11.4.1 Case Study: Passive Observer for Dynamic Positioning using GNSS and Compass Measurements

Emma 11.1 (Kalman-Yakubovich-Popov)
Let $\mathbf{Z}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$ be an m×m transfer function matrix, where $\mathbf{A}$ is Hurwitz, $(\mathbf{A},\mathbf{B})$ is controllable, and $(\mathbf{A},\mathbf{C})$ is observable. Then $\mathbf{Z}(s)$ is strictly positive real (SPR) if and only if there exist positive definite matrices $\mathbf{P} = \mathbf{P}^{\top}$ and $\mathbf{Q} = \mathbf{Q}^{\top}$ such that:

$$\begin{aligned} \mathbf{P}\mathbf{A} + \mathbf{A}^{\top}\mathbf{P} &= -\mathbf{Q} \\ \mathbf{B}^{\top}\mathbf{P} &= \mathbf{C} \end{aligned}$$

Since $\mathcal{H}_1$ is strictly passive and $\mathcal{H}_2$, given by three matrices ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$) according to

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}\boldsymbol{\varepsilon}_v \\ \tilde{\mathbf{z}} &= \mathbf{C}\tilde{\mathbf{x}} \end{aligned}$$



can be made SPR by choosing the gain matrices $\mathbf{K}_i$ ($i=1,\ldots,4$) according to the KYP Lemma. Hence, according to Lemma 6.4 (Khalil 2002), $\mathcal{H}_2$ is strictly passive since $\mathcal{H}_2$ is SPR

Interconnected system $\mathcal{H}_1$ and $\mathcal{H}_2$ is GAS

11.4.1 Case Study: Passive Observer for Dynamic Positioning using GNSS and Compass Measurements

Determination of the Observer Gains

The mapping $\mathcal{H}_2$ describes three decoupled systems in *surge, sway*, and *yaw.*

This suggests that the observer gain matrices should have a diagonal structure:

$$
\begin{aligned}
\mathbf{K}_1(\boldsymbol{\omega}_o) &= \begin{bmatrix} \mathrm{diag}\{K_{11}(\omega_{o1}), K_{12}(\omega_{o2}), K_{13}(\omega_{o3})\} \\ \mathrm{diag}\{K_{14}(\omega_{o1}), K_{15}(\omega_{o3}), K_{16}(\omega_{o3})\} \end{bmatrix} \\
\mathbf{K}_2 &= \mathrm{diag}\{K_{21}, K_{22}, K_{23}\} \\
\mathbf{K}_3 &= \mathrm{diag}\{K_{31}, K_{32}, K_{33}\} \\
\mathbf{K}_4 &= \mathrm{diag}\{K_{41}, K_{42}, K_{43}\}
\end{aligned}
$$

function of wave frequencies in surge, sway, and yaw.
Opens for:

-gain scheduling
-adaptive observer

11.4.1 Case Study: Passive Observer for Dynamic Positioning using GNSS and Compass Measurements

Three decoupled transfer functions:

$$\tilde{\mathbf{z}}(s) = \mathbf{H}(s)\boldsymbol{\varepsilon}_y(s) = \mathbf{H}_0(s)\mathbf{H}_B(s)\boldsymbol{\varepsilon}_y(s)$$

$$\begin{aligned}
\mathbf{H}_0(s) &= \mathbf{C}_0[s\mathbf{I} + \mathbf{A}_0 - \mathbf{K}_0(\boldsymbol{\omega}_0)\mathbf{C}_0]^{-1}\mathbf{B}_0 \\
\mathbf{H}_B(s) &= \mathbf{K}_4 + (s\mathbf{I} + \mathbf{T}^{-1})^{-1}\mathbf{K}_3
\end{aligned}$$

H_o is determined by using pole placement:

$$h_{oi}(s) = \frac{s^2 + 2\lambda_i\omega_{oi}s + \omega_{oi}^2}{s^3 + (K_{1(i+3)} + K_{2i} + 2\lambda_i\omega_{oi})s^2 + (\omega_{oi}^2 + 2\lambda_i\omega_{oi}K_{2i} - K_{1i}\omega_{oi}^2)s + \omega_{oi}^2 K_{2i}}$$

$$h_{di}(s) = \frac{s^2 + 2\lambda_i\omega_{oi}s + \omega_{oi}^2}{\left(s^2 + 2\zeta_{ni}\omega_{oi}s + \omega_{oi}^2\right)(s + \omega_{ci})}$$

The desired structure is low-pass + notch

$$\begin{aligned}
K_{1i}(\omega_{oi}) &= -2(\zeta_{ni} - \lambda_i)\frac{\omega_{ci}}{\omega_{oi}} \\
K_{1(i+3)}(\omega_{oi}) &= 2\omega_{oi}(\zeta_{ni} - \lambda_i) \\
K_{2i} &= \omega_{ci}
\end{aligned}$$

11.4.1 Case Study: Passive Observer for Dynamic Positioning using GNSS and Compass Measurements

The remaining gains K_3 and K_4 in $\mathbf{H}_B$ is found by frequency shaping. The transfer functions $h_i(s)$ must all have phase greater than -90° in order to be passive.

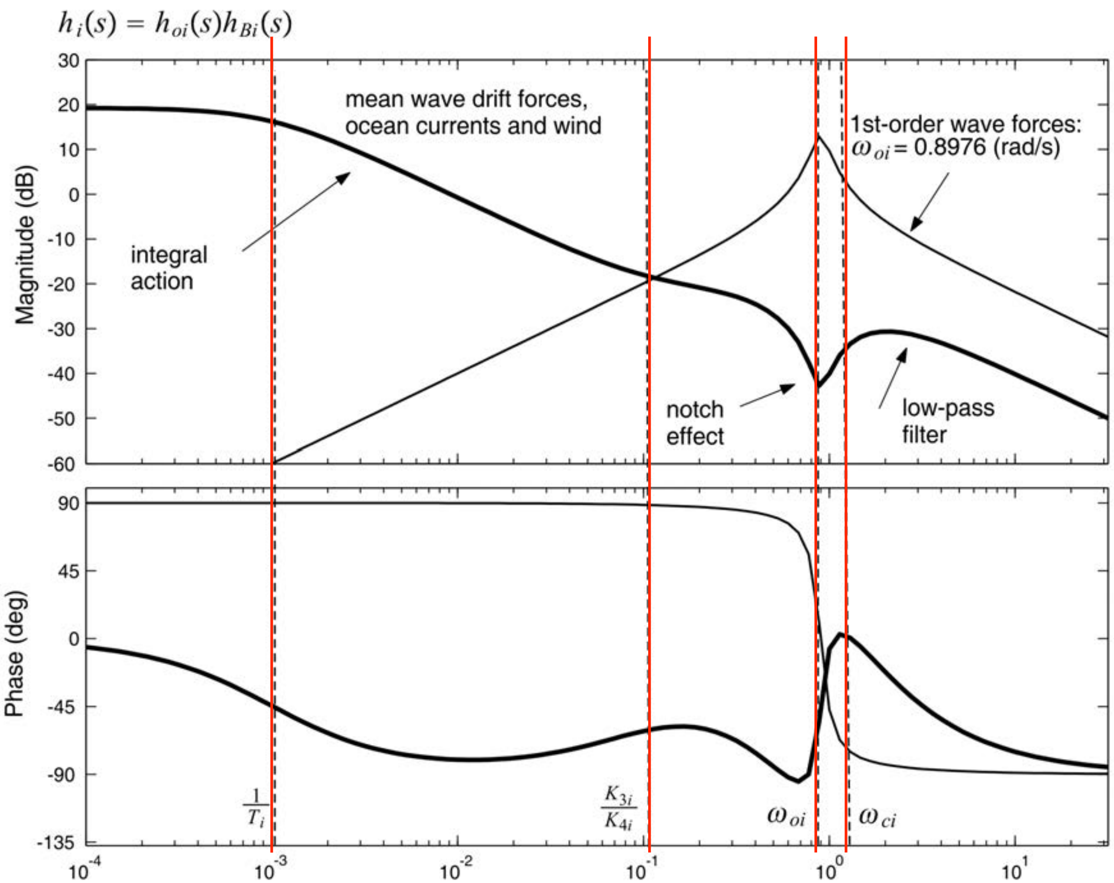
This is satisfied for:

$$1/T_i \ll K_{3i}/K_{4i} < \omega_{oi} < \omega_{ci}$$

$$\mathbf{H}(s) = \mathbf{H}_0(s)\mathbf{H}_B(s)$$

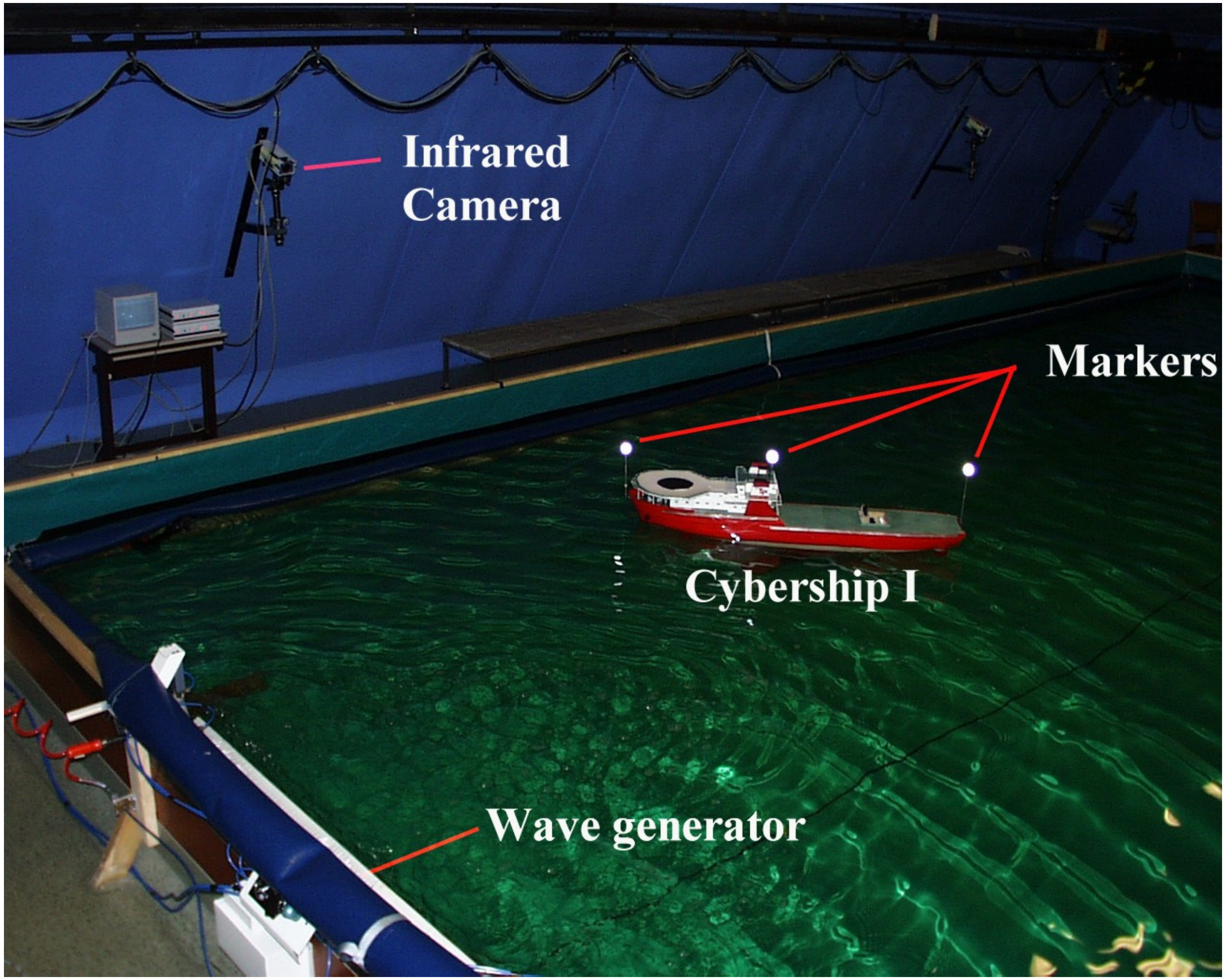
$$h_{Bi}(s) = K_{4i}\frac{s+\left(\frac{1}{T_i}+\frac{K_{3i}}{K_{4i}}\right)}{s+\frac{1}{T_i}} \overset{T_i \gg 1}{\approx} K_{4i}\frac{s+\frac{K_{3i}}{K_{4i}}}{s+\frac{1}{T_i}}$$

$$h_{di}(s) = \frac{s^2+2\lambda_i\omega_{oi}s+\omega_{oi}^2}{\left(s^2+2\zeta_{ni}\omega_{oi}s+\omega_{oi}^2\right)(s+\omega_{ci})}$$



11.4.1 Case Study: Passive Observer for Dynamic Positioning using GNSS and Compass Measurements

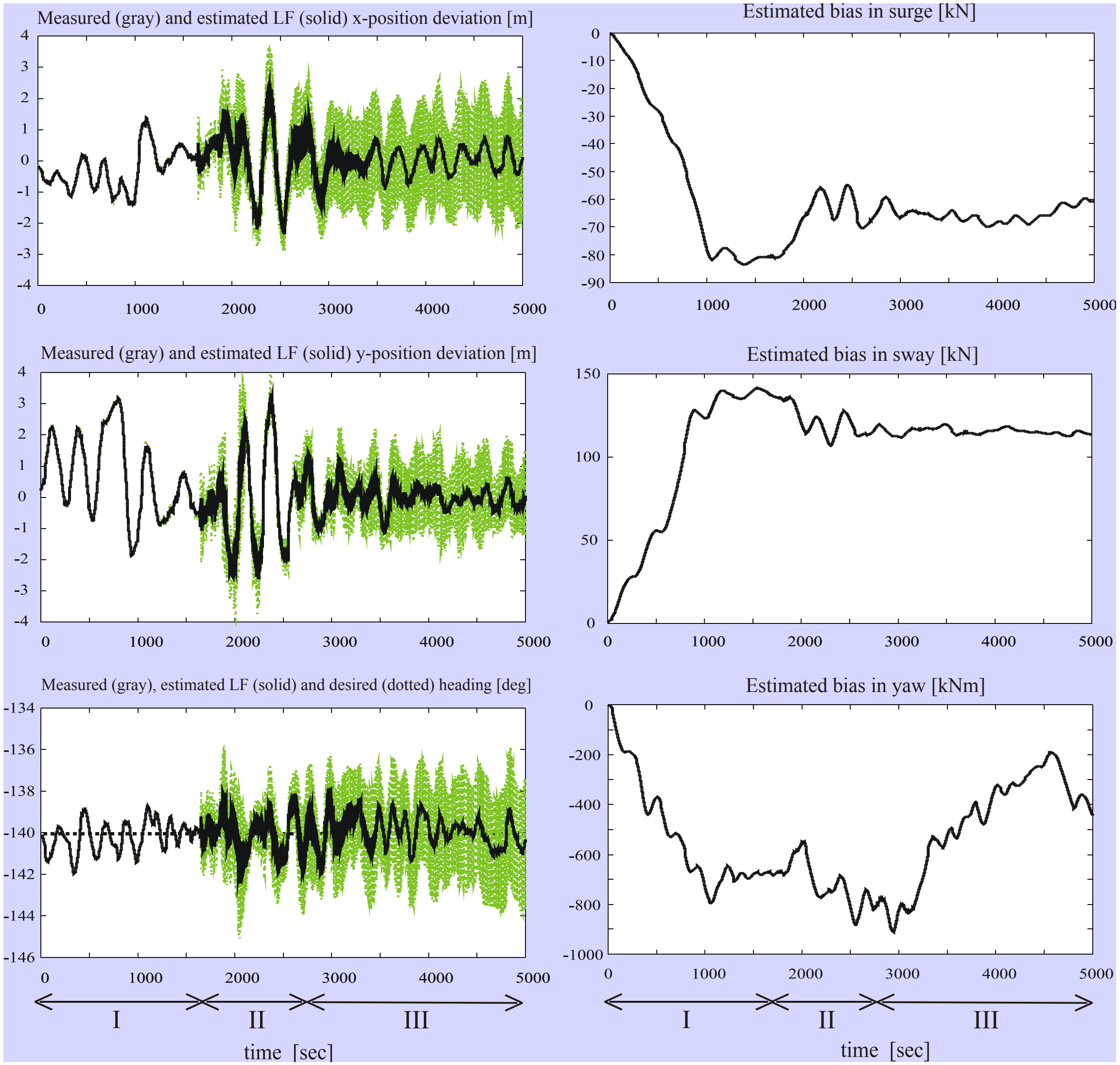
The first experiments were performed in the GNC laboratory at the Department of Engineering Cybernetics, NTNU using CyberShip I which is offshore supply vessel scale 1:70.



11.4.1 Case Study: Passive Observer for Dynamic Positioning using GNSS and Compass Measurements

Experimental results: implemented and tested onboard several ships and rigs offshore.

Reduced commissioning time: easy to tune compared to the Extended Kalman Filter.



11.4.2 Case Study: Passive Observer for Heading Autopilots using only Compass Measurements

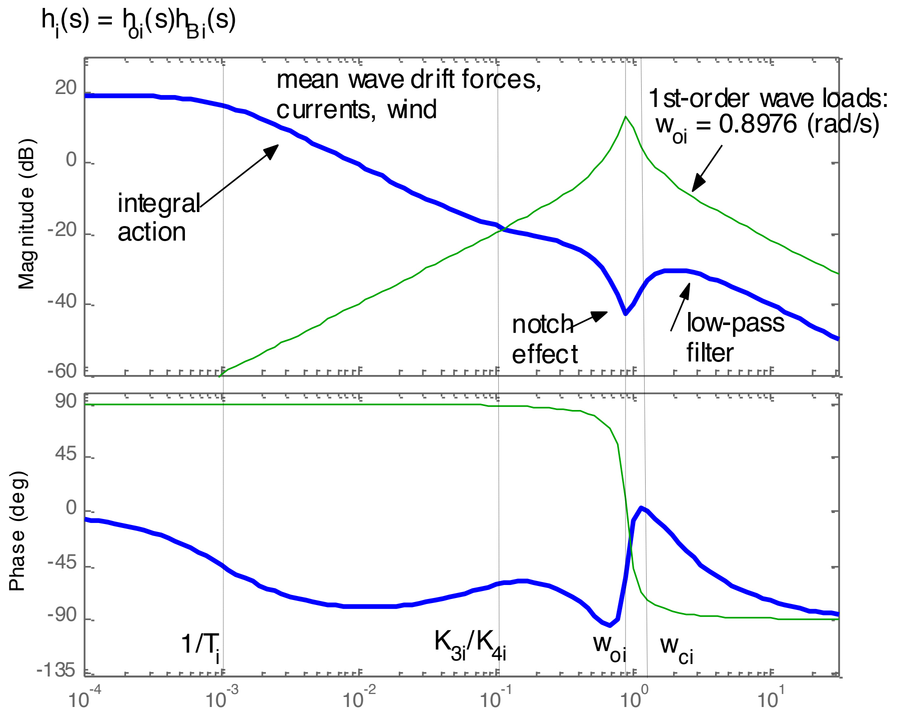
Passivity-Based Pole Placement

The observer error dynamics can be reformulated as two subsystems for yaw angle/rudder bias, and yaw rate. Fossen and Strand (1999) have shown that these systems forms a *passive interconnection* if the observer gains are chosen according to:

$$\mathbf{k} = \begin{bmatrix} -2\omega_0(1-\lambda)/\omega_c \\ 2\omega_0(1-\lambda) \\ \omega_c \\ K_4 \\ K_5 \end{bmatrix}$$

where $\omega_c > \omega_0$ is the filter cut-off frequency and:

$$0 < 1/T_b < K_5/K_4 < \omega_0 < \omega_c$$



11.4.2 Case Study: Passive Observer for Heading Autopilots using only Compass Measurements

Example 11.7 (Passive Wave Filtering) Consider the *Mariner class cargo ship* with $K = 0.185\ \mathrm{s}^{-1}$ and $T = T_1 + T_2 - T_3 = 107.3$ s (Strøm-Tejsen 1965). The bias time constant is chosen to be rather large, that is $T_b = 100$ s. The wave response model is modeled by a linear approximation to the JONSWAP spectrum with $\lambda = 0.1$ and $\omega_0 = 1.2$ rad/s.

State-space model:

wave-frequency model

ship + bias models

$$\mathbf{A} = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ -1.96 & -0.26 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -0.0093 & -0.0017 \\ 0 & 0 & 0 & 0 & -0.001 \end{array}\right], \quad \mathbf{b} = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 0.0017 \\ 0 \end{array}\right]$$

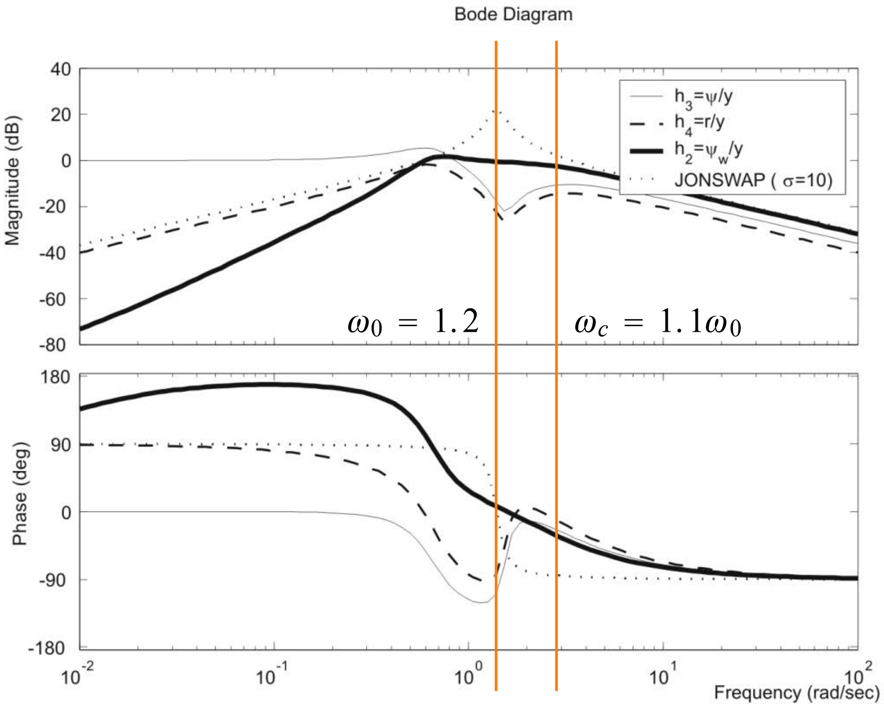
$$\mathbf{E} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ 0.26\,\sigma & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \qquad \mathbf{h}^\top = [0, 1, 1, 0, 0]$$

11.4.2 Case Study: Passive Observer for Heading Autopilots using only Compass Measurements

Example 11.7 (Passive Wave Filtering, cont.) Using passivity as a tool for filter design with cut-off frequency $\omega_c = 1.1\omega_0$, yields:

$$\mathbf{k} = \begin{bmatrix} K_1 \\ \frac{K_2}{K_3} \\ K_4 \\ K_5 \end{bmatrix} = \begin{bmatrix} -2\omega_0(1-\lambda)/\omega_c \\ \frac{2\omega_0(1-\lambda)}{\omega_c} \\ K_4 \\ K_5 \end{bmatrix} = \begin{bmatrix} -1.64 \\ \frac{1.80\ \omega_0}{1.10\ \omega_0} \\ K_4 \\ K_5 \end{bmatrix}$$

Notice that the notch effect at ω_0 is more than -20 dB for h_3(s) and h_4(s) representing the state estimates $\hat{\psi}$ *and* $\hat{r}$. We also see that high-frequency motion components above ω_c is low-pass filtered. Finally, the transfer function h_2(s) representing reconstruction of the WF motion $\hat{\psi}_w$ filters out signals on the outside of the wave response spectrum.

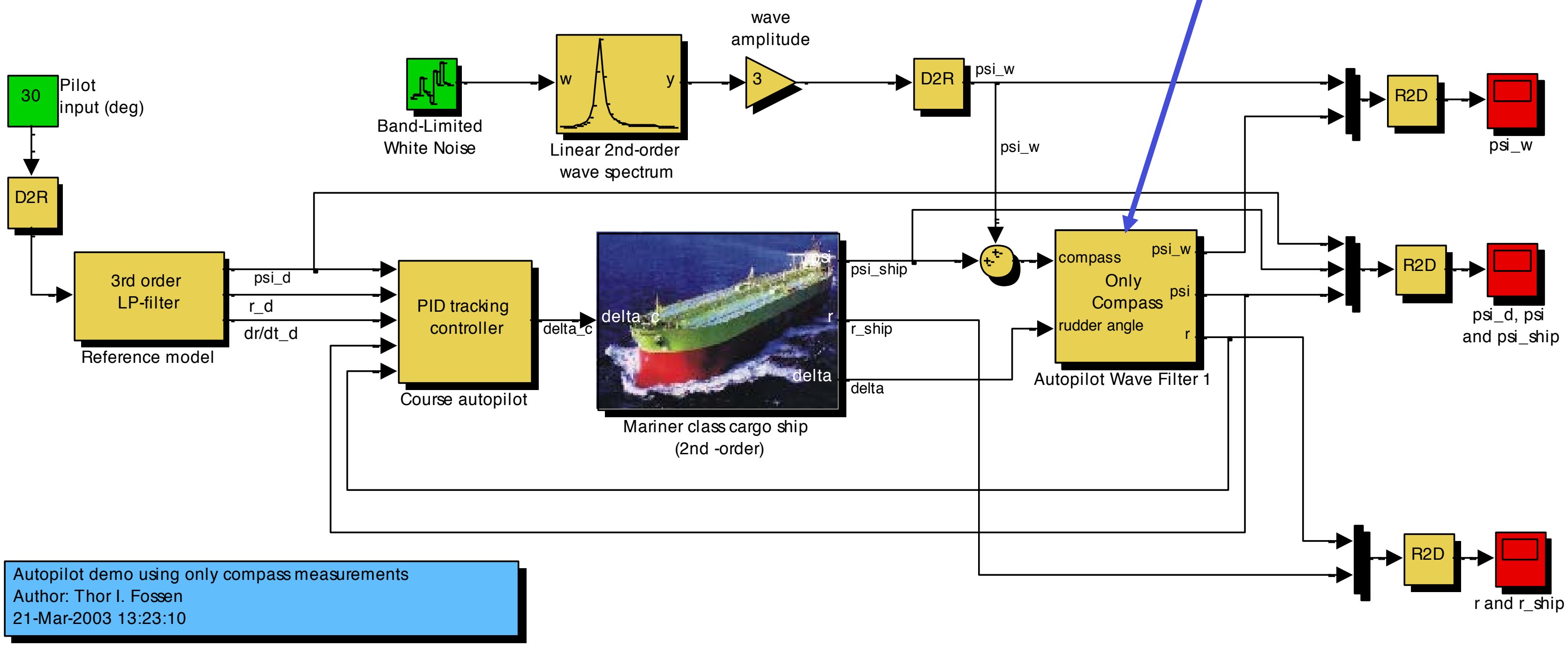


Bode plot showing the wave filter transfer functions and the JONSWAP spectrum.

11.4.2 Case Study: Passive Observer for Heading Autopilots using only Compass Measurements

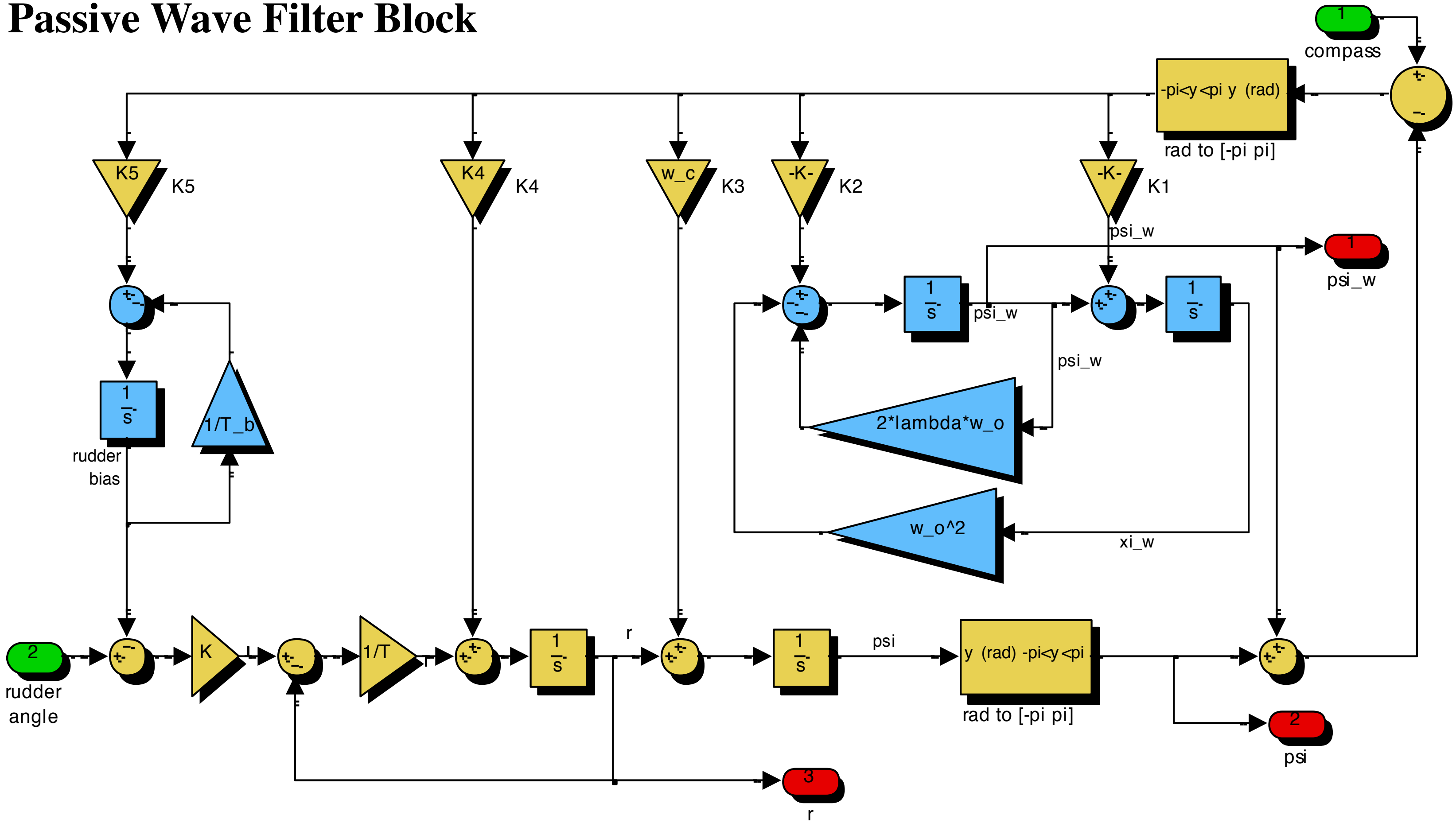
MSS Toolbox Simulink example

passive wave filter

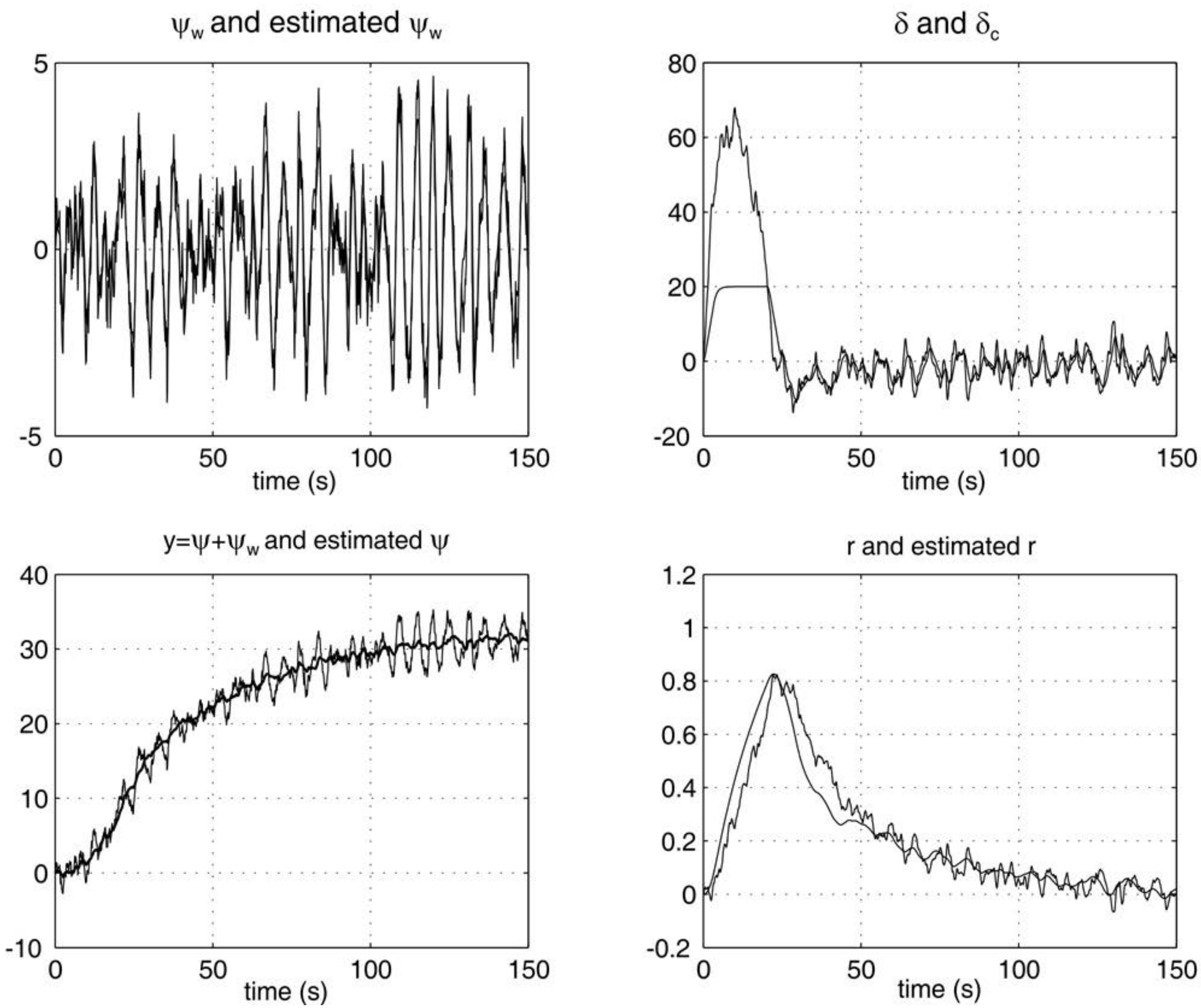


11.4.2 Case Study: Passive Observer for Heading Autopilots using only Compass Measurements

Passive Wave Filter Block



11.4.2 Case Study: Passive Observer for Heading Autopilots using only Compass Measurements



11.4.3 Case Study: Passive Observer for Heading Autopilots using both Compass and Rate Measurements

It is advantageous to integrate gyro and compass measurements in the observer. This results in less variance and better accuracy of the state estimates.

One simple way to do this is to treat the *gyro measurements* as an *input* to the system model:

$$\begin{aligned} \dot{\psi} &= u_{\text{gyro}} + b \\ \dot{b} &= w_b \end{aligned}$$

where b denotes the gyro bias, w_b is Gaussian white noise and u_{gyro} is the rate gyro measurement.

This model will give proper wave filtering of the state ψ. However, the estimate of r is not wave filtered, since this signal is taken directly from the gyro measurement u_{gyro}. This can be solved by filtering u_{gyro} with a notch filter $h_{\text{notch}}(s)$ and a low-pass filter to the cost of some phase lag:

$$u_f = h_{\text{notch}}(s)\, h_{\text{lp}}(s)\, u_{\text{gyro}}$$

11.4.3 Case Study: Passive Observer for Heading Autopilots using both Compass and Rate Measurements

The resulting model becomes:

$$\left.\begin{aligned} \dot{\hat{\xi}}_w &= \hat{\psi}_w + K_1\,\varepsilon \\ \dot{\hat{\psi}}_w &= -\omega_0^2\,\hat{\xi}_w - 2\lambda\omega_0\,\hat{\psi}_w + K_2\,\varepsilon \end{aligned}\right\} \quad \text{WF model}$$

$$\dot{\hat{\psi}} = u_f + \hat{b} + K_3\,\varepsilon \qquad \text{yaw kinematics with gyro input}$$

$$\dot{\hat{b}} = -\frac{1}{T_b}\hat{b} + K_4\,\varepsilon \qquad \text{gyro bias}$$

where $\varepsilon = y - \hat{\psi} - \hat{\psi}_w$ and $T_b \gg 0$

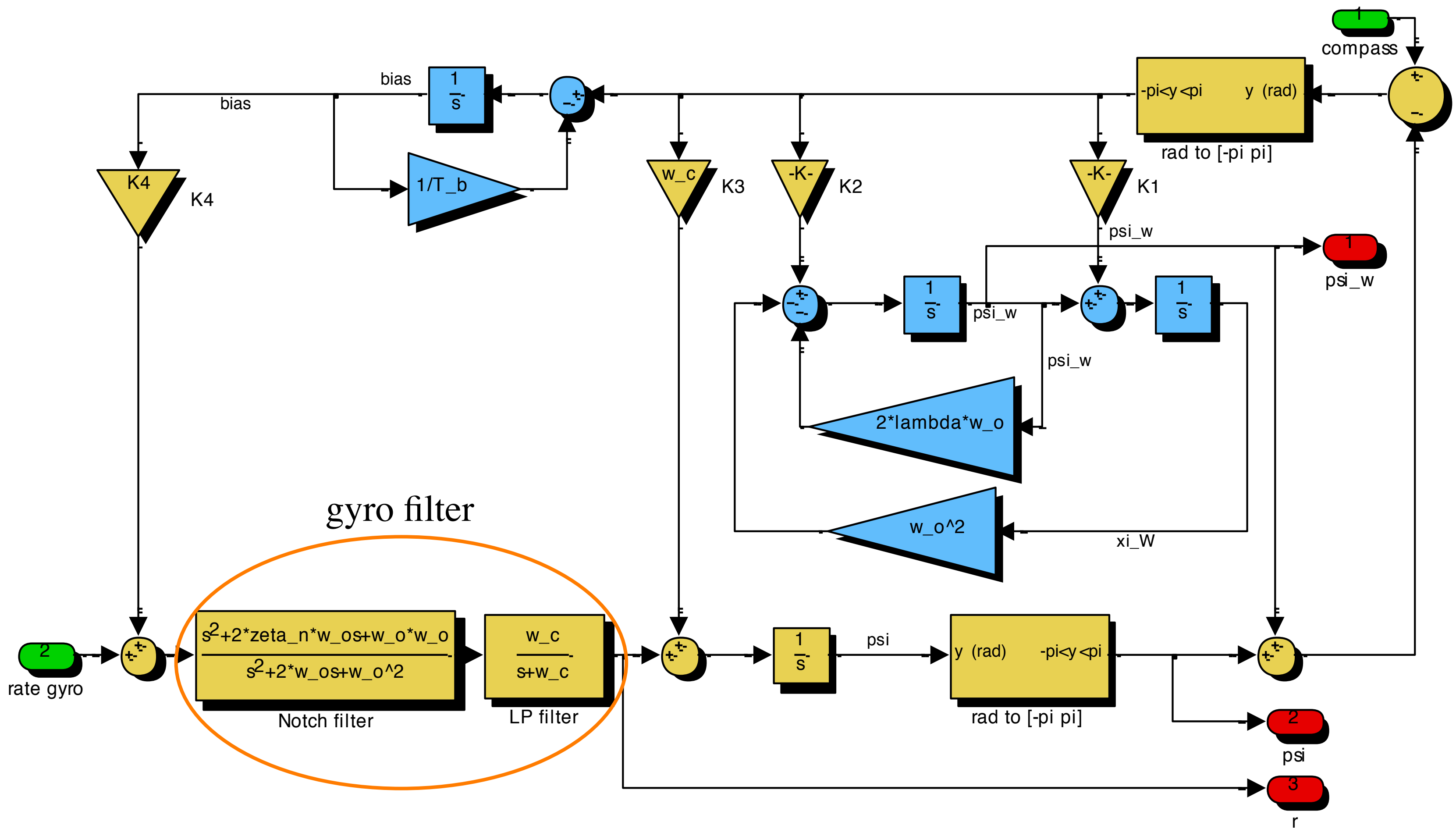
Notice that the gyro bias must be estimated on-line since it will vary with temperature and possible scale factor/misalignment errors when mounted onboard the ship. This is a slowly varying process so the gain K_4 can be chosen quite small reflecting a large bias time constant.

If *passivity-based pole placement* is used, K_1,K_2 and K_3 become:

$$K_1 = -2\omega_0(1-\lambda)/\omega_c, \quad K_2 = 2\omega_0(1-\lambda), \quad K_3 = \omega_c$$

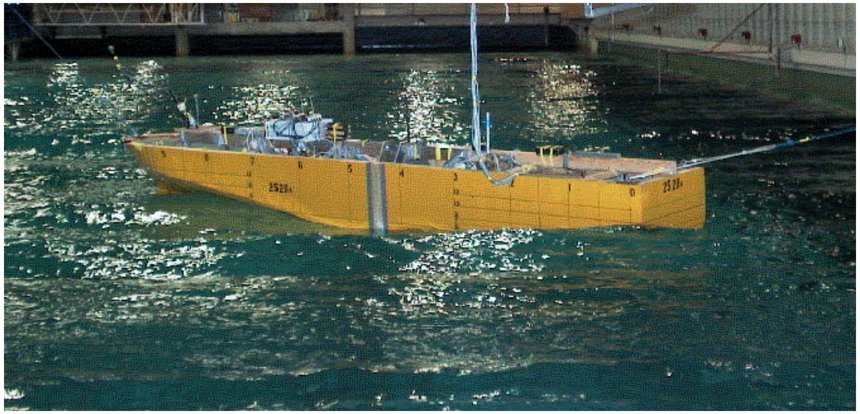
11.4.3 Case Study: Passive Observer for Heading Autopilots using both Compass and Rate Measurements

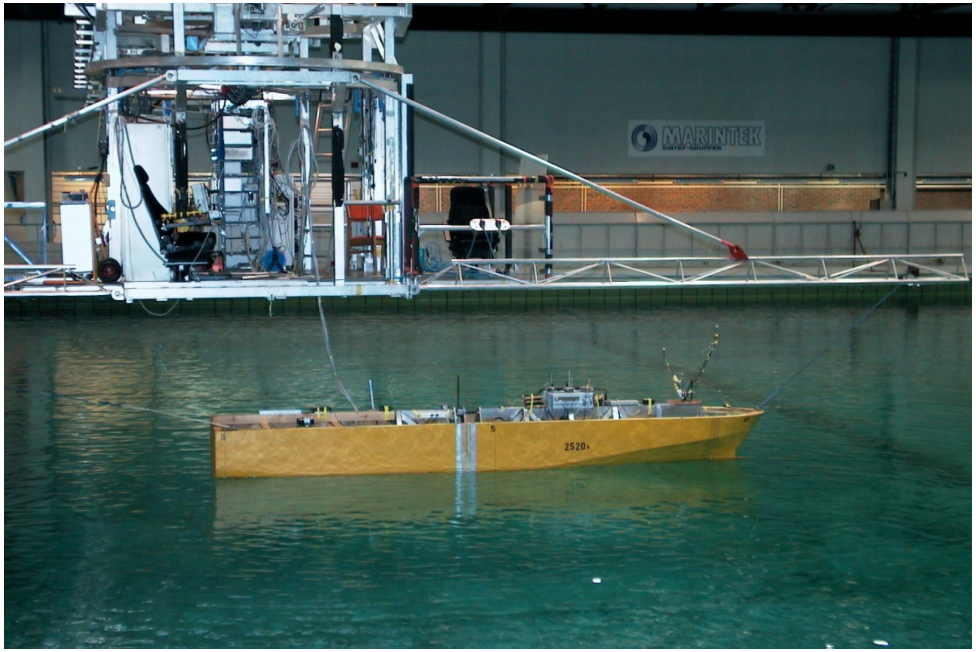
MSS Toolbox Simulink Example



11.4.3 Case Study: Passive Observer for Heading Autopilots using both Compass and Rate Measurements

The wave filter has been tested on a scale model of *MV Autoprestige* of the United European Car Carriers (UECC)





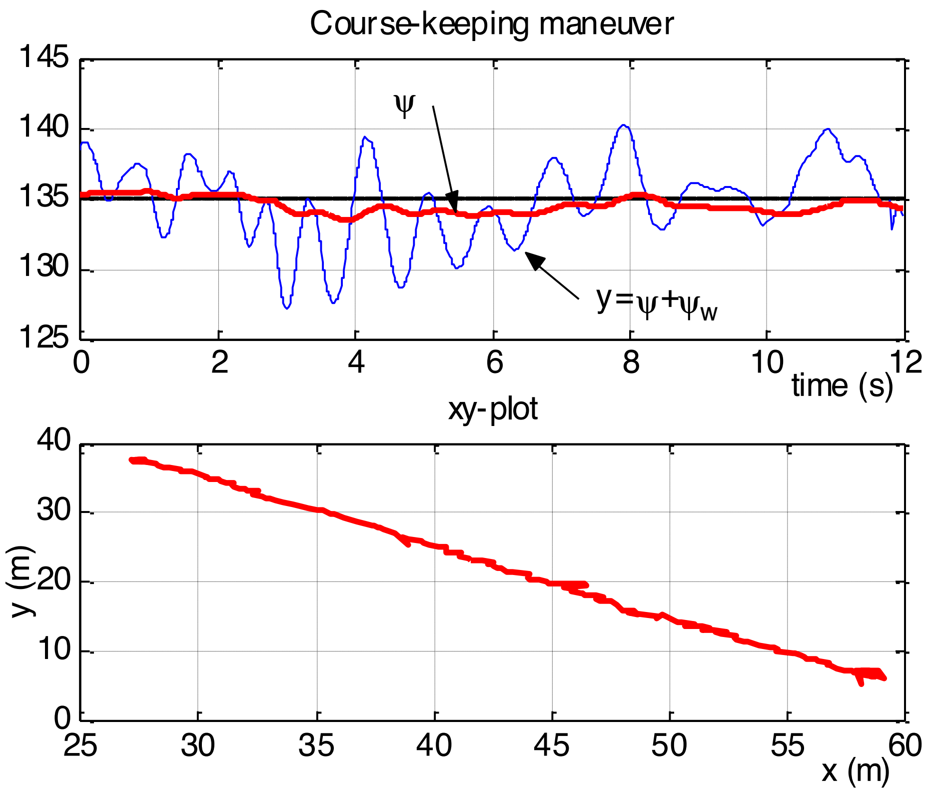
The maneuvering test were performed in the Ocean Basin at MARINTEK in Trondheim April 2001.

11.4.3 Case Study: Passive Observer for Heading Autopilots using both Compass and Rate Measurements

It is seen that the WF motion components are quite well removed from the estimate of resulting in good course-keeping capabilities.

We also notice that the estimate of ψ_w is quite good, while r could be slightly improved by changing the observer gains.

Significant wave height: H_s= 1.3 m (full scale)
Frequency of encounter: ω_e= 1.07 rad/s
Cruise speed: U = 2.3 m/s (model scale)



11.5 Integration Filter for IMU and Global Navigation Satellite Systems

Inertial Navigation Systems

An inertial navigation system (INS) consists of:

- Inertial measurement unit (IMU)
- Global reference system (GNSS)
- Software (state observer) that computes *position*, *velocity* and *attitude* from the measurements

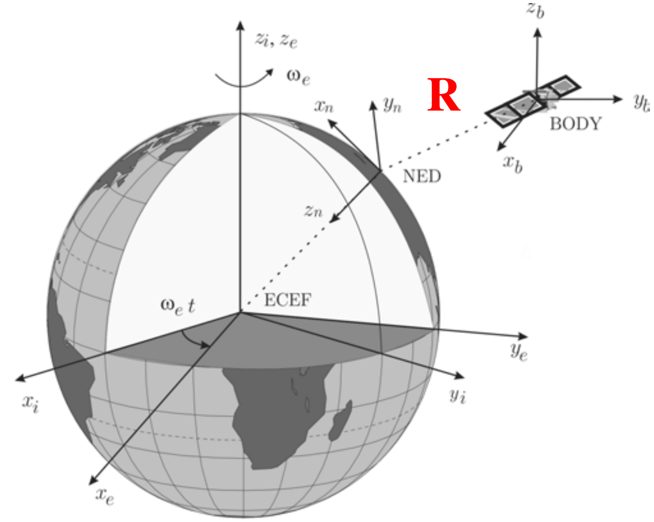
When integrating the angular velocity (gyro) and linear acceleration (accelerometers) using the *strapdown equations*, drift must be prevented. This is obtained by using a GNSS as reference for position and the resulting system is known a *strapdown inertial navigation system*.

In case of GNSS drop-outs the strapdown equations are integrated without corrections of the observer (zero observer gains). Hence, the position and velocity will drift off until the GNSS signals returns. The bias estimates are frozen during GNSS drop-outs since they require position updates.

11.5 Integration Filter for IMU and Global Navigation Satellite Systems

Coordinate systems for local navigation:

BODY – body-fixed frame
NED – North-East-Down frame is approximated as the inertial frame by neglecting the Earth rotation and movement of the Earth in the solar system



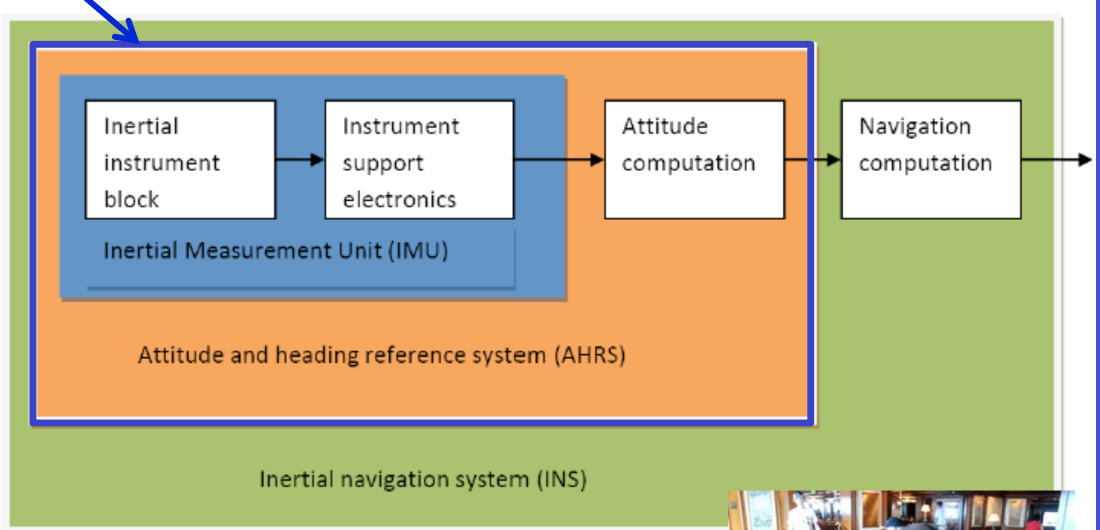
Attitude can be described by a 3 x 3 matrix (9-elements) relating a vector in BODY to a vector in NED

Consequently, a NED vector $\mathbf{v}^i$ is related to a BODY vector $\mathbf{v}^b$ by the rotation matrix according to

$$\mathbf{v}^n = \mathbf{R}\mathbf{v}^b \quad \text{where} \quad \mathbf{R} \in SO(3)$$

O(3) is the group of all orthogonal matrices, i.e. $\mathbf{R}\mathbf{R}^T = \mathbf{I}_3$ and $\mathbf{R}^\mathrm{T} = \mathbf{R}^{-1}$

SO(3) special orthogonal group. The subgroup of the O(3) satisfying: $\det(\mathbf{R}) = 1$



GNSS/INS integration

11.5 Integration Filter for IMU and Global Navigation Satellite Systems

Inertial Measurement Systems

Today inertial measurement technology is available for commercial users thanks to a significant reduction in price the last decade. As a consequence of this, low cost inertial sensors can be integrated with satellite navigation system using a conventional Kalman filter or a nonlinear state observer.

LN-200 (*Courtesy Litton)*

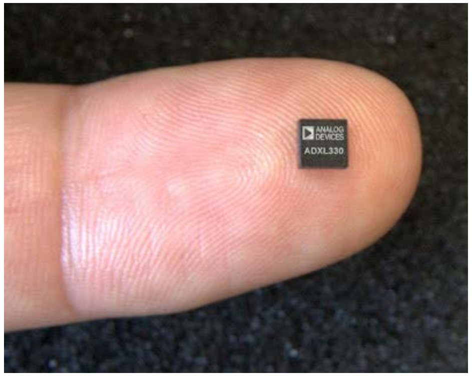
ISA (Inertial Sensor Assembly) - cluster of three gyros and three accelerometers that measure <u>angular velocity</u> and <u>linear acceleration</u>, respectively.

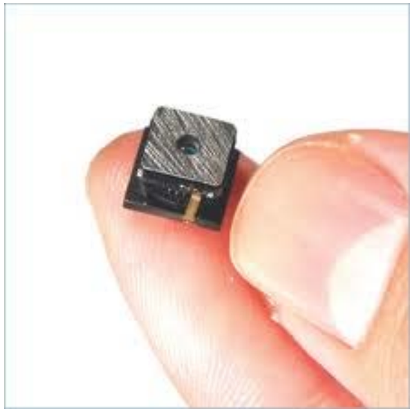
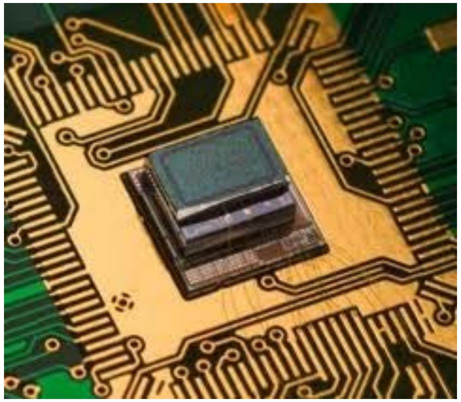
IMU (Inertial Measurement Unit) - consists of an ISA, hardware to interface the ISA, and low level software that performs down-sampling, temperature calibration, and vibration (sculling and coning) compensation

MEMS

Microelectromechanical systems (MEMS) is the technology of very small mechanical devices driven by electricity.

It merges at the nano-scale into nanoelectromechanical systems (NEMS) and nanotechnology.



Today inertial measurement technology is available for commercial users thanks to a MEMS technology and a significant reduction in price the last decades.

Low-cost inertial sensors can be integrated with a satellite navigation system or other positioning systems using a conventional Kalman filter or a nonlinear state observer.

Inertial Measurement Units (IMUs)

MicorStrain 3DM-GX3® -45 Attitude and Heading Reference System (IMU) with build-in EKF

~ $2,000.

Accelerometer range	± 5 *g* standard; ± 1.7 *g*, ± 16 *g*, and ± 50 *g* also available
Accelerometer bias stability	± 0.005 *g* for ± 5 *g* range ± 0.003 *g* for ± 1.7 *g* range ± 0.010 *g* for ± 16 *g* range ± 0.050 *g* for ± 50 *g* range
Accelerometer nonlinearity	0.2 %
Gyro range	± 300°/sec standard, ± 1200°/sec, ± 600°/sec, ± 50°/sec also available
Gyro bias stability	± 0.2°/sec for ± 300°/sec
Gyro nonlinearity	0.2 %
Magnetometer range	± 2.5 Gauss
Magnetometer nonlinearity	0.4 %
Magnetometer bias stability	0.01 Gauss

Low-Cost Motion Sensors

MTi Attitude and Heading Reference System (IMU) with build-in EKF

~ $2,000

Dimensions: 58mm x 58mm

Attitude and Heading	
Static accuracy (roll/pitch)	<0.5 deg
Static accuracy (heading)[1]	<1 deg
Dynamic accuracy[2]	2 deg RMS
Angular resolution[3]	0.05 deg
Dynamic range:	
- Pitch	± 90 deg
- Roll/Heading	± 180 deg
Maximum update rate:	
- Onboard processing	256 Hz
- External processing	512 Hz
Specified performance operating range[4]	0...+55 °C

Sensor performance	Rate of turn	Acceleration	Magnetic field
Dimensions	3 axes	3 axes	3 axes
Full Scale (standard)	± 300 deg/s	± 50 m/s²	± 750 mGauss
Linearity	0.1% of FS	0.2% of FS	0.2% of FS
Bias stability[5]	1 deg/s	0.02 m/s²	0.1 mGauss
Scale Factor stability[5]	-	0.03%	0.5%
Noise	0.05 deg/s/√Hz	0.002 m/s²/√Hz	0.5 mGauss
Alignment error	0.1 deg	0.1 deg	0.1 deg
Bandwidth	40 Hz	30 Hz	10 Hz
Max update rate	512 Hz	512 Hz	512 Hz

Inertial Measurement Units (IMUs)

ADIS16488 Tri-Axis Inertial Sensor with Magnetometer and Pressure sensor by Analog Devices

~ $2000

10 DOF IMU Mounted in a Cube:
3-axis, digital gyroscope
Tri-axis, ±18 *g* digital accelerometer
Tri-axis, ±2.5 Gauss digital magnetometer
Digital pressure sensor, 300 mbar to 1100 mbar
550 ms start-up time
Gyro drift 0.0014 deg/s

Factory-calibrated sensitivity, bias, and axial alignment
Operating temperature range: −40°C to +85°C

Gyro bias stability for some inertial sensors

MEMS Units	drift deg/s	drift deg/h
Sensonor STIM 300	0.0001 deg/s	0.5 deg/h
Analog Devices ADIS 16488	0.0014 deg/s	5.1 deg/h
Analog Devices ADIS 16405	0.007 deg/s	25.2 deg/h
MicroStrain 3DM-GX3® -45	0.2 deg/s	720.0 deg/h
XSENS MTi	1.0 deg/s	3600.0 deg/h

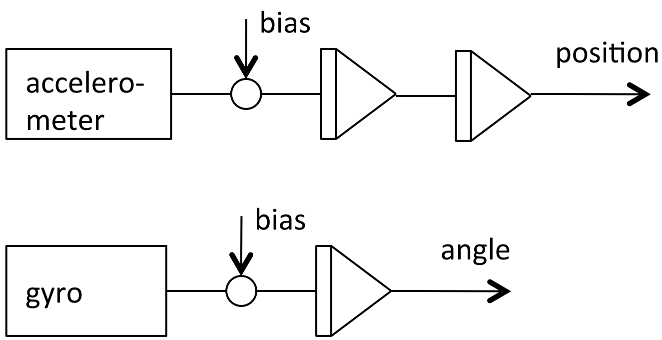
Earth rotation 0.0042 deg/s

~ $10000

11.5 Integration Filter for IMU and Global Navigation Satellite Systems

- An IMU can be integrated with a satellite navigation system in a state observer to obtain estimates of position and velocity in 6 DOF.
- A stand-alone IMU solution, where
  - acceleration measurements are integrated twice to obtain positions
  - gyro measurements (angular rates) are integrated once to obtain attitude

  will drift due to sensors biases, misalignments, temperature variations etc.



An INS state estimator aided by GNSS can be used to estimate the bias terms and thus give accurate estimates of position and attitude

A low-cost IMU/GNSS strapdown integration technique can be used for local navigation by neglecting the Earth rotation and assuming that the GNSS signals are available all the time.

For vehicles and robots, the North-East-Down reference frame can be assumed to be inertial even though the Earth is moving relatively to the star-fixed reference frame.

11.5 Integration Filter for IMU and Global Navigation Satellite Systems

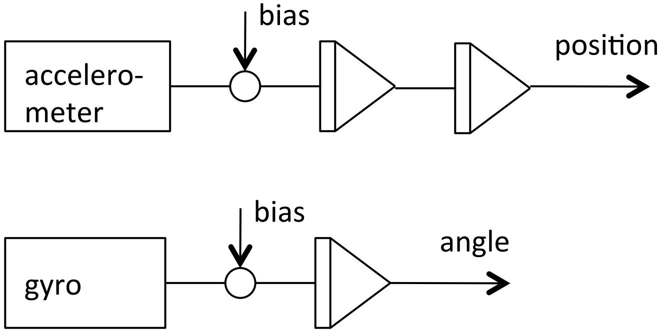
Strapdown INS Principle

An inertial navigation system (INS) consists of:

- Inertial measurement unit (IMU)
- Global reference system (GNSS or other positioning systems)
- Software (state observer) that computes *position*, *velocity* and *attitude* (PVA) from the measurements

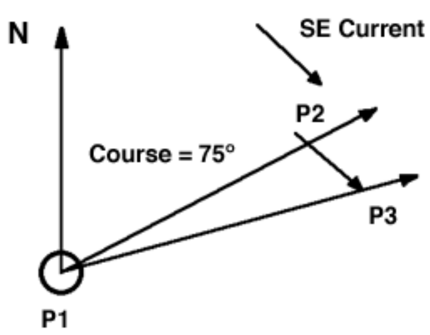
When integrating the angular velocity (gyro) and linear acceleration (accelerometers) using the strapdown equations, drift must be prevented.

This is obtained by using a GNSS as reference for position and the resulting system is known a strapdown inertial navigation system.

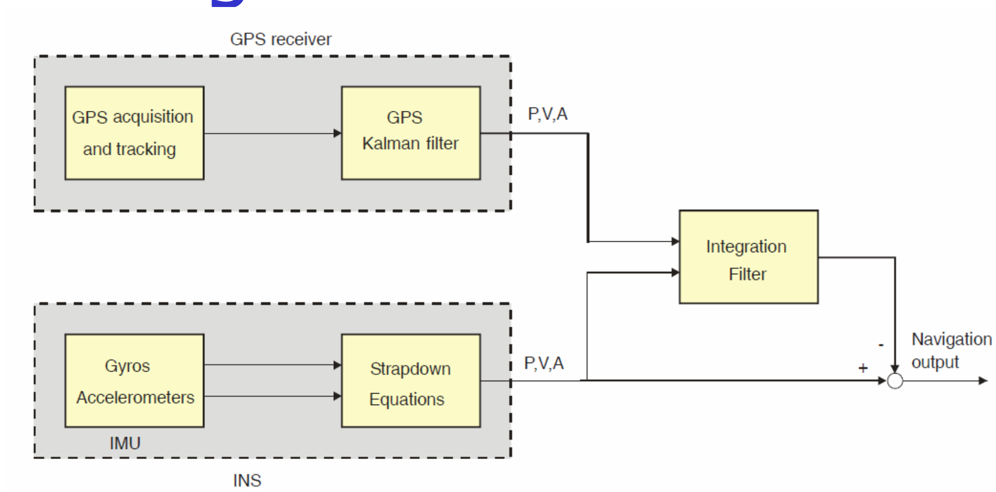


In case of GNSS drop-outs the strapdown equations are integrated without corrections of the observer (zero observer gains). This is referred to as *dead-reckoning*.

Hence, the position and velocity will drift off until the GNSS signals returns. The bias estimates are frozen during GNSS drop-outs since they require position updates.

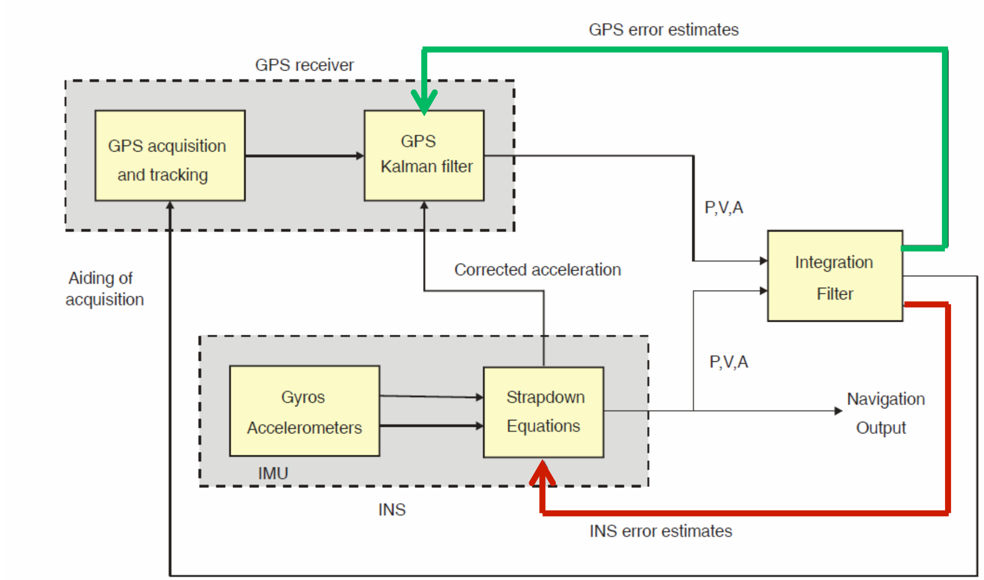


11.5 Integration Filter for IMU and Global Navigation Satellite



Uncoupled Filter:
No interaction between GNSS and INS

- The two PVA estimates are combined in an integration filter, usually an EKF to minimize the variance of the estimates



Loosely Coupled Filter:
Feedback to GNSS and INS.
1) **GNNS feedback**: Can improve GNSS tracking if implemented.
2) **INS feedback:** Can remove acceleration and gyro biases by feedback and improve performance and stability

11.5 Integration Filter for IMU and Global Navigation Satellite Systems

State-of-the-art methods:
- Extended Kalman Filter (EKF)
- Unscented Kalman Filter (UKF)

- Only local stability and convergence due to linearization
- n(n+1)/2 differential equations (*Ricatti equations*) in an n-state EKF system
- Difficult to avoid that the state covariance matrix is singular/ numerical ill-conditioned

Rudolf E. Kalman (1930-)

1993 Crassidis et al. – Survey paper
2003 Markley – special version for quaternions (Multiplicative EKF)

J. L. Crassidis, F. L. Markley and Y. Cheng (2007). Survey of nonlinear attitude estimation methods, *Journal of Guidance, Control and Dynamics*, vol. 30, no. 1, pp. 12–28.
F. L. Markley (2003). Attitude error representations for Kalman filtering, *Journal of Guidance, Control and Dynamics*, vol. 26, no. 2, pp. 311–317.

11.5 Integration Filter for IMU and Global Navigation Satellite Systems

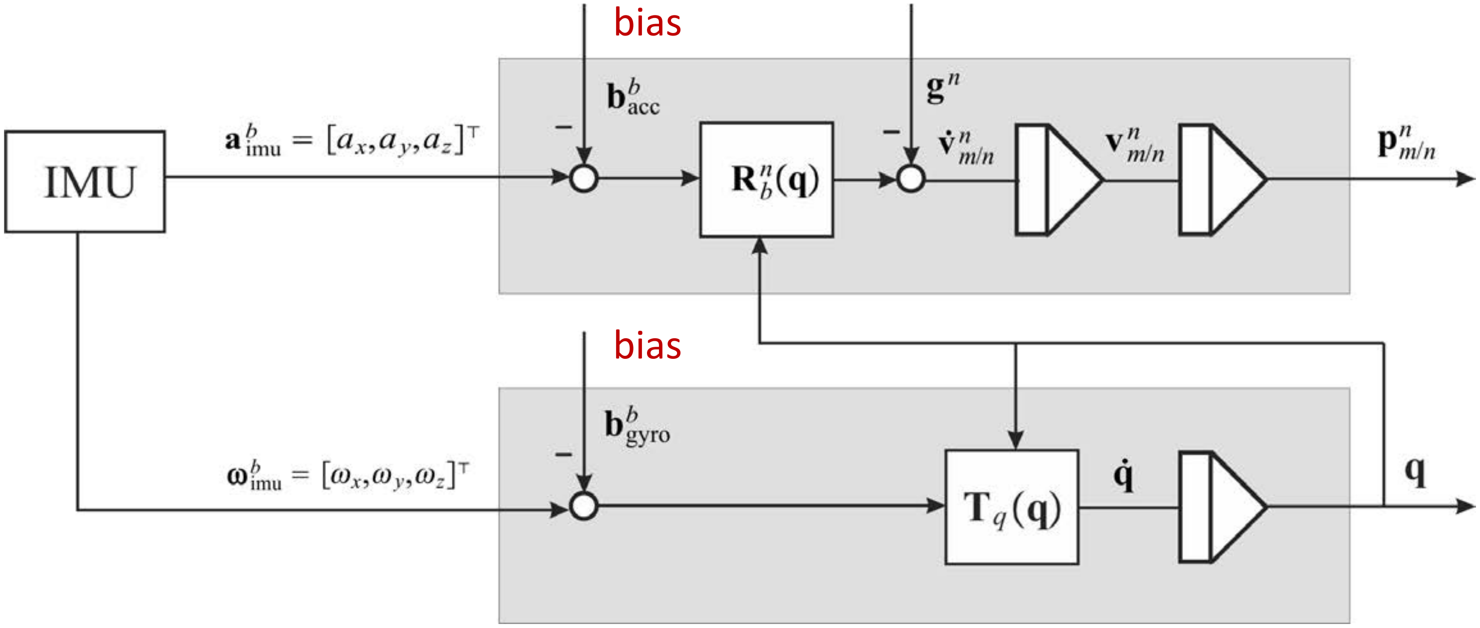
IMU Measurements:

$$
\begin{aligned}
\mathbf{a}^b_{\mathrm{imu}} &= \mathbf{R}^b_n(\boldsymbol{\Theta})(\dot{\mathbf{v}}^n_{m/n} + \mathbf{g}^n) + \mathbf{b}^b_{\mathrm{acc}} + \mathbf{w}^b_{\mathrm{acc}} \\
\boldsymbol{\omega}^b_{\mathrm{imu}} &= \boldsymbol{\omega}^b_{m/n} + \mathbf{b}^b_{\mathrm{gyro}} + \mathbf{w}^b_{\mathrm{gyro}} \\
\mathbf{m}^b_{\mathrm{imu}} &= \mathbf{R}^b_n(\boldsymbol{\Theta})\mathbf{m}^n + \mathbf{b}^b_{\mathrm{mag}} + \mathbf{w}^b_{\mathrm{mag}}
\end{aligned}
$$

Linear acceleration

Angular velocity

Magnetic field



11.5.1 Integration Filter for Position and Linear Velocity

Integration of IMU and GNSS Position Measurements:

$$\dot{\mathbf{v}}^n_{m/n} = \mathbf{R}^b_n(\boldsymbol{\Theta})^\top[\mathbf{a}^b_{\mathrm{imu}} - \mathbf{b}^b_{\mathrm{acc}} - \mathbf{w}^b_{\mathrm{acc}}] - \mathbf{g}^n$$

$$\boldsymbol{\Theta} = [\phi, \theta, \psi]^\top$$

Dynamics:

$$\begin{aligned}
\dot{\mathbf{p}}^n_{m/n} &= \mathbf{v}^n_{m/n} \\
\dot{\mathbf{v}}^n_{m/n} &= \mathbf{R}^n_b(\boldsymbol{\Theta})[\mathbf{a}^b_{\mathrm{imu}} - \mathbf{b}^b_{\mathrm{acc}}] - \mathbf{g}^n \\
\dot{\mathbf{b}}^b_{\mathrm{acc}} &= \mathbf{0} \\
\mathbf{y}_1 &= \mathbf{p}^n_{m/n}
\end{aligned}$$

Attitude measurements

IMU measurements

$$\mathbf{p}^n_{\mathrm{gnss}} = [N_{\mathrm{gnss}}, E_{\mathrm{gnss}}, D_{\mathrm{gnss}}]^\top$$

GNSS measurements

Observer (copy of dynamics):

$$\begin{aligned}
\dot{\hat{\mathbf{p}}}^n_{m/n} &= \hat{\mathbf{v}}^n_{m/n} + \mathbf{K}_1\tilde{\mathbf{y}}_1 \\
\dot{\hat{\mathbf{v}}}^n_{m/n} &= \mathbf{R}^n_b(\boldsymbol{\Theta})[\mathbf{a}^b_{\mathrm{imu}} - \hat{\mathbf{b}}^b_{\mathrm{acc}}] - \mathbf{g}^n + \mathbf{K}_2\tilde{\mathbf{y}}_1 \\
\dot{\hat{\mathbf{b}}}^b_{\mathrm{acc}} &= \mathbf{K}_3\mathbf{R}^n_b(\boldsymbol{\Theta})^\top\tilde{\mathbf{y}}_1 \\
\hat{\mathbf{y}}_1 &= \hat{\mathbf{p}}^n_{m/n}
\end{aligned}$$

injection terms

$$\tilde{\mathbf{y}}_1 = \mathbf{y}_1 - \mathbf{y}_1 = \mathbf{p}^n - \hat{\mathbf{p}}^n$$

$$\hat{\mathbf{v}}^b = \mathbf{R}^b_n(\boldsymbol{\Theta})\hat{\mathbf{v}}^n$$

11.5.1 Integration Filter for Position and Linear Velocity

Observer error dynamics:

$$\begin{bmatrix} \dot{\tilde{\mathbf{p}}}^n_{m/n} \\ \dot{\tilde{\mathbf{v}}}^n_{m/n} \\ \dot{\tilde{\mathbf{b}}}^b_{\mathrm{acc}} \end{bmatrix} = \begin{bmatrix} -\mathbf{K}_1 & \mathbf{I} & \mathbf{0} \\ -\mathbf{K}_2 & \mathbf{0} & -\mathbf{R}^n_b(\mathbf{\Theta}) \\ -\mathbf{K}_3\mathbf{R}^n_b(\mathbf{\Theta})^\top & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{p}}^n_{m/n} \\ \tilde{\mathbf{v}}^n_{m/n} \\ \tilde{\mathbf{b}}^b_{\mathrm{acc}} \end{bmatrix}$$

$$\Updownarrow$$

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{\Theta})\mathbf{x}$$

Property: A matrix **K** is said to commute with the rotation matrix **R**(Θ) if:

$$\mathbf{K}\mathbf{R}(\mathbf{\Theta}) = \mathbf{R}(\mathbf{\Theta})\mathbf{K}$$

Examples of **K** matrices satisfying this property are linear combinations:

$$\mathbf{K} = a_1\mathbf{R}(\mathbf{\Theta}) + a_2\mathbf{I} + a_3\mathbf{k}^\top\mathbf{k}$$

$$\mathbf{k} = [0, 0, 1]^\top$$

11.5.1 Integration Filter for Position and Linear Velocity

Define the transformation:

$$\mathbf{x} = \mathbf{T}(\boldsymbol{\Theta})\mathbf{z} \qquad \mathbf{T}(\boldsymbol{\Theta}) = \mathrm{diag}\{\mathbf{R}_b^n(\boldsymbol{\Theta}),\ \mathbf{R}_b^n(\boldsymbol{\Theta}),\ \mathbf{I}\}$$

$$\dot{\mathbf{T}}(\boldsymbol{\Theta}) = \mathrm{diag}\{\mathbf{R}_b^n(\boldsymbol{\Theta})\mathbf{S}(\boldsymbol{\omega}_{m/n}^b),\ \mathbf{R}_b^n(\boldsymbol{\Theta})\mathbf{S}(\boldsymbol{\omega}_{m/n}^b),\ \mathbf{0}\} \approx \mathbf{0}$$

This assumption is OK for marine craft

Then

$$\dot{\mathbf{z}} = \mathbf{T}(\boldsymbol{\Theta})^\top \mathbf{A}(\boldsymbol{\Theta})\mathbf{T}(\boldsymbol{\Theta})\mathbf{z}$$

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} \qquad \mathbf{A} = \mathbf{T}(\boldsymbol{\Theta})^\top \mathbf{A}(\boldsymbol{\Theta})\mathbf{T}(\boldsymbol{\Theta}) \qquad \mathbf{A} = \begin{bmatrix} -\mathbf{K}_1 & \mathbf{I} & \mathbf{0} \\ -\mathbf{K}_2 & \mathbf{0} & \mathbf{I} \\ -\mathbf{K}_3 & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

if the observer gain matrices $\mathbf{K}_i$ (i=1,…,3) commute with the rotation matrix $\mathbf{R}(\Theta)$

How to choose the observer gains using pole placement?

If the matrices $\mathbf{K}_i$ are chosen diagonal

$$\mathbf{K}_i = \mathrm{diag}\{k_i, k_i, l_i\}, \qquad i = 1,2,3$$

stability can be checked by computing the *eigenvalues* of **A**. Notice that the eigenvalues of **A** and **A**(Θ) are equal.

11.5.1 Integration of IMU and GNSS Position and Velocity Measurements

The observer can be extended to include velocity measurements.

The gains are again found by using commutation of matrices.

$$
\begin{aligned}
\dot{\hat{\mathbf{p}}}^n_{m/n} &= \hat{\mathbf{v}}^n_{m/n} + \mathbf{K}_{11}\tilde{\mathbf{y}}_1 + \mathbf{K}_{21}\tilde{\mathbf{y}}_2 \\
\dot{\hat{\mathbf{v}}}^n_{m/n} &= \mathbf{R}^n_b(\boldsymbol{\Theta})[\mathbf{a}^b_{\text{imu}} - \hat{\mathbf{b}}^b_{\text{acc}}] - \mathbf{g}^n + \mathbf{K}_{12}\tilde{\mathbf{y}}_1 + \mathbf{K}_{22}\tilde{\mathbf{y}}_2 \\
\dot{\hat{\mathbf{b}}}^b_{\text{acc}} &= \mathbf{K}_{13}\mathbf{R}^n_b(\boldsymbol{\Theta})^\top\tilde{\mathbf{y}}_1 + \mathbf{K}_{23}\mathbf{R}^n_b(\boldsymbol{\Theta})^\top\tilde{\mathbf{y}}_2 \\
\hat{\mathbf{y}}_1 &= \hat{\mathbf{p}}^n_{m/n} \\
\hat{\mathbf{y}}_2 &= \hat{\mathbf{v}}^n_{m/n}
\end{aligned}
$$

Error dynamics:

$$
\begin{bmatrix} \dot{\tilde{\mathbf{p}}}^n_{m/n} \\ \dot{\tilde{\mathbf{v}}}^n_{m/n} \\ \dot{\tilde{\mathbf{b}}}^b_{\text{acc}} \end{bmatrix}
=
\begin{bmatrix}
-\mathbf{K}_{11} & \mathbf{I} - \mathbf{K}_{21} & \mathbf{0} \\
-\mathbf{K}_{12} & -\mathbf{K}_{22} & -\mathbf{R}^n_b(\boldsymbol{\Theta}) \\
-\mathbf{K}_{13}\mathbf{R}^n_b(\boldsymbol{\Theta})^\top & -\mathbf{K}_{23}\mathbf{R}^n_b(\boldsymbol{\Theta})^\top & \mathbf{0}
\end{bmatrix}
\begin{bmatrix} \tilde{\mathbf{p}}^n_{m/n} \\ \tilde{\mathbf{v}}^n_{m/n} \\ \tilde{\mathbf{b}}^b_{\text{acc}} \end{bmatrix}
$$

$$\Updownarrow$$

$$\dot{\mathbf{x}} = \mathbf{A}(\boldsymbol{\Theta})\mathbf{x}$$

Choosing $\mathbf{K}_{ij}$ diagonal so they commute with $\mathbf{R}(\Theta)$, yields:

$$
\begin{aligned}
\dot{\mathbf{z}} &= \mathbf{T}^\top(\boldsymbol{\Theta})\mathbf{A}\mathbf{T}(\boldsymbol{\Theta})\mathbf{z} \\
&= \mathbf{A}\mathbf{z}
\end{aligned}
$$

$\longleftarrow$ check eigenvalues of **A**

11.5.2 Accelerometer and Compass Aided Attitude Observer

The observers based on *Salcudean's* work assume that the attitude is available as an explicit quaternion measurement. This typically requires a separate algebraic resolution of the attitude usually a static accelerometer-to-attitude mapping.

The *static acceleration solution* gives inaccurate results for accelerated vehicles exhibiting Coriolis and centripetal accelerations.

- **1991 Salcuedan** – First nonlinear Lyapunov-based attitude observer where the real part of the quaternion estimation error is used as injection term. Exponential stability is proven.
- **1999 Vik, Shiriaev and Fossen** – The *Salcuedan* observer is extended to include gyro bias. Convergence is proven by using *Barbalat's lemma*. Local stability by linearization.
- **2003 Thienel and Sanner** – A gyro persistency-of-excitation (PE) condition for the Vik et al. (1999) observer is given and exponential stability of the quaternion and bias estimation errors follows.

S. Salcudean (1991). A Globally Convergent Angular Velocity Observer for Rigid Body Motion, *IEEE Transaction on Automatic Control* TAC-36(12).
B. Vik and T. I. Fossen (2001). A nonlinear observer for GPS and INS integration, *Proc. IEEE Conference on Decision and Control*, Orlando, FL, pp. 2956–2961.
J. K. Thienel and R. M. Sanner (2003). A coupled nonlinear spacecraft attitude controller and observer with an unknown constant gyro bias and gyro noise, *IEEE Transactions on Automatic Control*, vol. 48, no. 11, pp. 2011–2015.

11.5.2 Accelerometer and Compass Aided Attitude Observer

Static Mapping from Linear Accelerations to Roll and Pitch Angles:

$$\overset{0}{\dot{\mathbf{v}}^n_{m/n}} = \mathbf{R}^b_n(\boldsymbol{\Theta})^\top [\mathbf{a}^b_{\text{imu}} - \overset{0}{\mathbf{b}^b_{\text{acc}}} - \overset{0}{\mathbf{w}^b_{\text{acc}}}] - \mathbf{g}^n$$

$$\mathbf{a}^b_{\text{imu}} \approx \mathbf{R}^b_n(\boldsymbol{\Theta})\mathbf{g}^n$$

$$\Updownarrow$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \approx \mathbf{R}^b_n(\boldsymbol{\Theta}) \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} = \begin{bmatrix} -g\sin(\theta) \\ g\cos(\theta)\sin(\phi) \\ g\cos(\theta)\cos(\phi) \end{bmatrix}$$

Taking the ratios:

$$\frac{a_y}{a_z} \approx \tan(\phi), \quad \frac{a_x}{g} \approx -\sin(\theta), \quad \frac{a_y^2 + a_z^2}{g^2} \approx \cos^2(\theta)$$

implies that:

$$\phi \approx \operatorname{atan}\left(\frac{a_y}{a_z}\right)$$

$$\theta \approx -\operatorname{atan}\left(\frac{a_x}{\sqrt{a_y^2 + a_z^2}}\right)$$

This is a method for computation of the static roll and pitch angles using 3 linear accelerometers.

- OK for marine craft but not high-acceleration maneuvers (need Coriolis-centripetal corrections).
- The mapping needs bias calibration/removal

The special solution of the attitude observer when only the roll and pitch angles are estimated (no compass measurement) is referred to as a vertical reference unit (VRU).

11.5.2 Accelerometer and Compass Aided Attitude Observer

Quaternion-Based Attitude Observer:

$$\dot{\mathbf{q}} = \mathbf{T}_q(\mathbf{q})\boldsymbol{\omega}^b_{b/n} \qquad \mathbf{T}_q(\mathbf{q}) = \frac{1}{2}\begin{bmatrix} -\boldsymbol{\varepsilon}^\top \\ \eta\mathbf{I} + \mathbf{S}(\boldsymbol{\varepsilon}) \end{bmatrix}$$

Gyro measurement equation:

$$\boldsymbol{\omega}^b_{m/n} = \boldsymbol{\omega}^b_{\text{imu}} - \mathbf{b}^b_{\text{gyro}} - \mathbf{w}^b_{\text{gyro}}$$

Motion Reference Unit
(Courtesy Kongsberg Seatex)

Attitude Dynamics:

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{T}_q(\mathbf{q})\left[\boldsymbol{\omega}^b_{\text{imu}} - \mathbf{b}^b_{\text{gyro}} - \mathbf{w}^b_{\text{gyro}}\right] \\ \dot{\mathbf{b}}^b_{\text{gyro}} &= \mathbf{0} \end{aligned}$$

Observer:

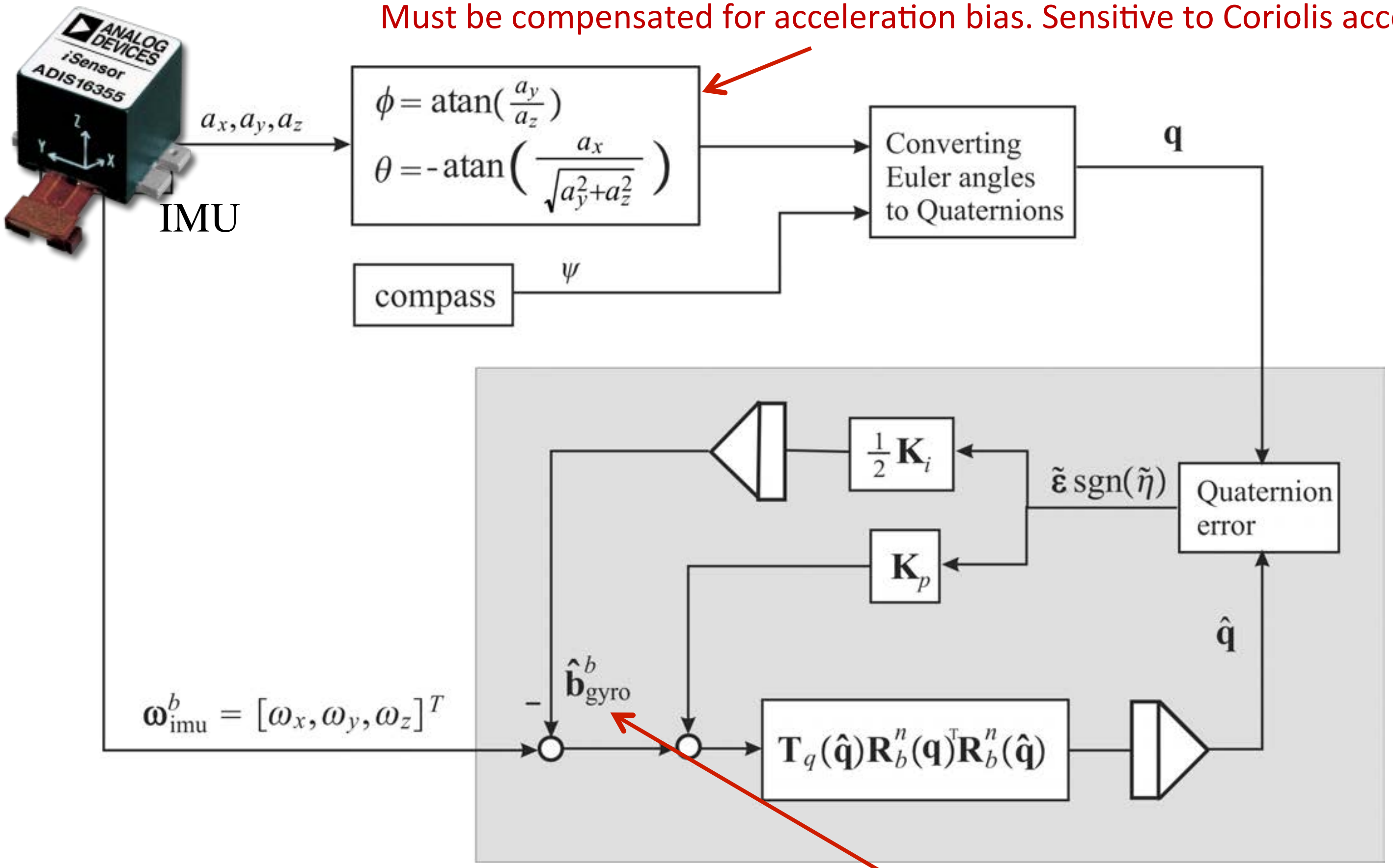
$$\begin{aligned} \dot{\hat{\mathbf{q}}} &= \mathbf{T}_q(\hat{\mathbf{q}})\underbrace{\mathbf{R}^n_b(\mathbf{q})^\top\mathbf{R}^n_b(\hat{\mathbf{q}})}_{\mathbf{R}^\top(\tilde{\mathbf{q}})}\left[\boldsymbol{\omega}^b_{\text{imu}} - \hat{\mathbf{b}}^b_{\text{gyro}} + \mathbf{K}_p\tilde{\boldsymbol{\varepsilon}}\,\text{sgn}(\tilde{\eta})\right] \\ \dot{\hat{\mathbf{b}}}^b_{\text{gyro}} &= -\frac{1}{2}\mathbf{K}_i\tilde{\boldsymbol{\varepsilon}}\,\text{sgn}(\tilde{\eta}) \end{aligned}$$

This observer is an extension of the Salcudean (1991) observer. Bias estimation was first included by Vik, Shiriaev and Fossen (1999). The proof is also found in Vik and Fossen (2001) using Lyapunov analysis and quaternion error dynamics.

Next: how to implement this?

11.5.2 Accelerometer and Compass Aided Attitude Observer

Key assumption: Roll and pitch angle mappings based on static acceleration. Must be compensated for acceleration bias. Sensitive to Coriolis acceleration



S. Salcudean (1991). A Globally Convergent Angular Velocity Observer for Rigid Body Motion, IEEE Transaction on Automatic Control TAC-36(12). (no gyro bias estimation)

B. Vik, A. Shiriaev and T. I. Fossen (1999). Nonlinear Observer Design for Integration of DGPS and INS, In "*New Directions in Nonlinear Observer Design*" (H. Nijmeijer and T. I. Fossen, Eds.), Springer (with gyro bias estimation).

11.5.3 Attitude Observer using Acceleration and Magnetic Field Directions

Normalized acceleration and magnetometer vector measurements in BODY

$$\mathbf{v}_1^b = \frac{\mathbf{a}_{\mathrm{imu}}^b}{\left\| \mathbf{a}_{\mathrm{imu}}^b \right\|}, \qquad \mathbf{v}_2^b = \frac{\mathbf{m}_{\mathrm{mag}}^b}{\left\| \mathbf{m}_{\mathrm{mag}}^b \right\|}$$

Estimate of vector measurements in NED (reference vectors)

$$\hat{\mathbf{v}}_1^b = \mathbf{R}_n^b(\hat{\mathbf{q}})\mathbf{v}_1^n \qquad \hat{\mathbf{v}}_2^b = \mathbf{R}_n^b(\hat{\mathbf{q}})\mathbf{v}_2^n$$

Inertial acceleration $v_1^n = a^n$ must be estimated while the Inertial magnetic field $v_2^n = m^n$ is known

Hamel and Mahony (2006)
Mahony, Hamel and Pflimlin (2008)

$$\boldsymbol{\omega}_{\mathrm{mes}}^b = -\mathrm{vex}\left(\sum_{i=1}^{n} \frac{k_i}{2} \left(\mathbf{v}_i^b (\hat{\mathbf{v}}_i^b)^\top - \hat{\mathbf{v}}_i^b (\mathbf{v}_i^b)^\top \right) \right)$$

$$\dot{\hat{\mathbf{q}}} = \mathbf{T}_q(\hat{\mathbf{q}}) \left[\boldsymbol{\omega}_{\mathrm{imu}}^b - \hat{\mathbf{b}}_{\mathrm{gyro}}^b + \mathbf{K}_p \boldsymbol{\omega}_{\mathrm{mes}}^b \right]$$

$$\dot{\hat{\mathbf{b}}}_{\mathrm{gyro}}^b = -\frac{1}{2} \mathbf{K}_i \boldsymbol{\omega}_{\mathrm{mes}}^b$$

Grip, Fossen, Johansen and Saberi (2012)

$$\dot{\hat{\mathbf{b}}}_{\mathrm{gyro}}^b = \mathrm{Proj}(\hat{\mathbf{b}}_{\mathrm{gyro}}^b, \mathbf{K}_i \boldsymbol{\omega}_{\mathrm{mes}}^b)$$

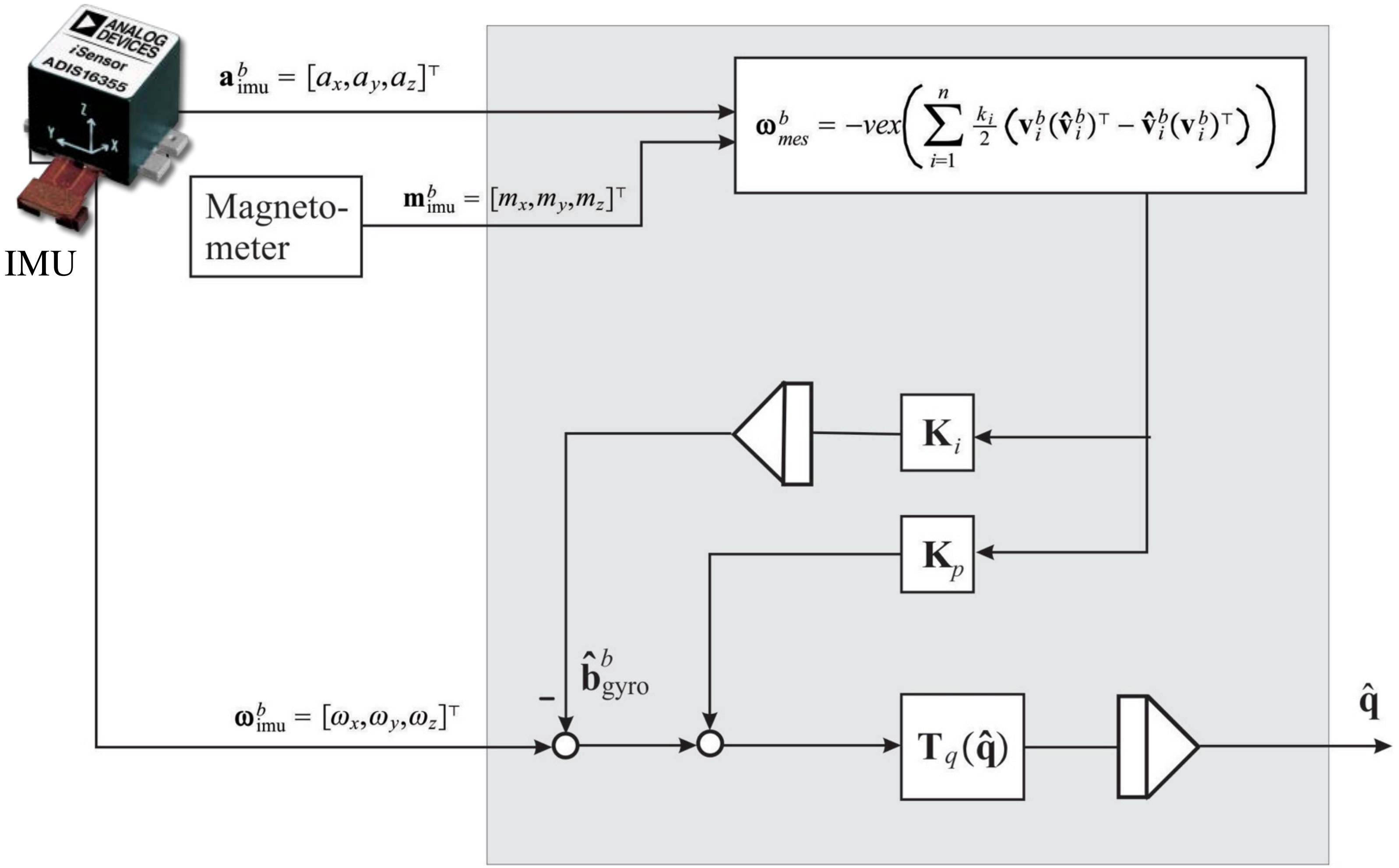
- **H. F. Grip, T. I. Fossen, T. A. Johansen and A. Saberi (2012).** Attitude estimation using biased gyro and vector measurements with time-varying reference vectors. *IEEE Transactions Automatic Control*.
- **T. Hamel and R. Mahony (2006)**. Attitude estimation on SO(3) based on direct inertial measurements, Proc. IEEE Int. Conf. Robotics Automation, Orlando, FL, pp. 2170–2175.
- **R. Mahony, T. Hamel, and J.-M. Pflimlin (2008).** Nonlinear complementary filters on the Special Orthogonal Group, IEEE Trans. Automat. Contr., vol. 53, no. 5, pp. 1203–1218.

Next: how to implement this?

11.5.3 Attitude Observer using Acceleration and Magnetic Field Directions

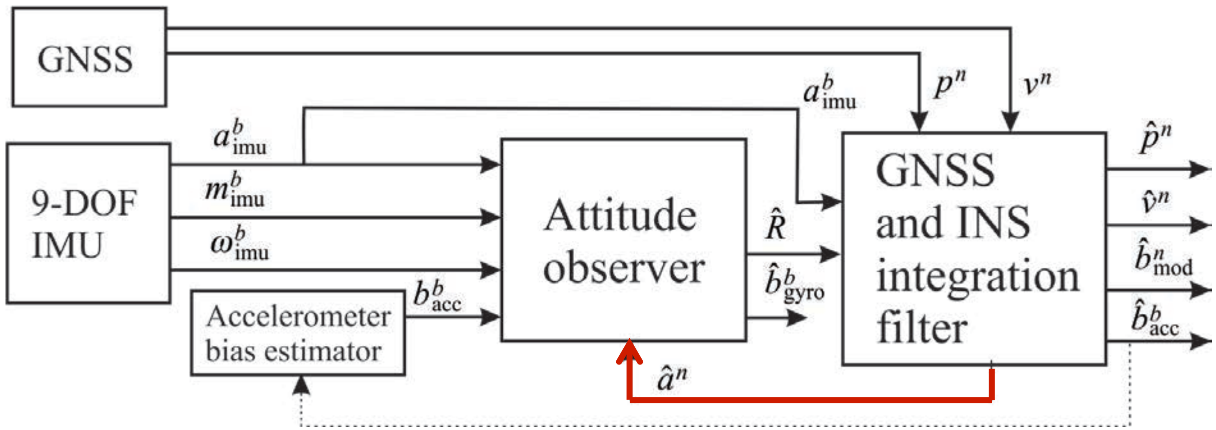
Uses direct measurements to update the observer but....

needs an inertial reference vector $\mathbf{a}^n$



11.5.3 Attitude Observer using Acceleration and Magnetic Field Directions

GNSS and INS Integration



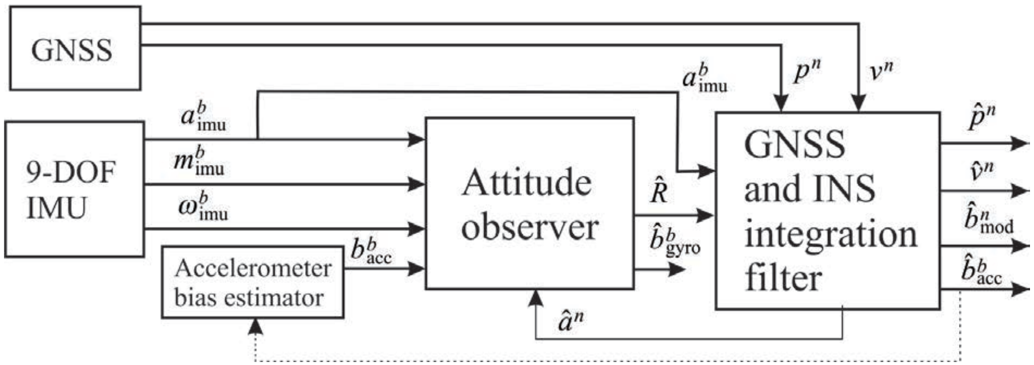
Loosely coupled cascaded filter

The NED acceleration is estimated using a kinematic model and fed back to the attitude observer.

Removes the assumption that the NED acceleration is static (usually computed from the gravity vector). This improves the performance when considering accelerated vehicles.

Recent results (2011-2014) on deterministic nonlinear observers design give explicit stability guarantees (global asymptotic and exponential stability)

11.5.3 Attitude Observer using Acceleration and Magnetic Field Directions



Cascade for <u>accelerated vehicles</u>. Estimates inertial acceleration and avoids assumptions such as $\mathbf{a}^n = \ddot{\mathbf{p}}^n$, alternatively $\mathbf{a}^n \approx -\mathbf{g}^n$

Estimation of Inertial Acceleration $\mathbf{a}^n$, Position $\mathbf{p}^n$ and Velocity $\mathbf{v}^n$

- Nonlinear cascaded observer based on vector measurements. Local exponentially stable under the assumption of constant reference vector $\mathbf{a}^n$

  M.-D. Hua (2010). Attitude estimation for accelerated vehicles using GPS/INS measurements. *Control Engineering Practice, 18(2010):723-732.*

- Nonlinear feedback connection where $\mathbf{a}^n$ is estimated and fed back to the attitude observer. GES nonlinear observer for time-varying $\mathbf{a}^n$ based on the rotation matrix representation.

  H. F. Grip, T. I. Fossen, T. A. Johansen and A. Saberi (2012). Integration of GNSS and IMU measurements with gyro bias estimation and explicit stability guarantees, *Proc. American Control Conference,* June 27-29, Montréal, Canada

- GES nonlinear observer for estimation time-varying $\mathbf{a}^n$ based on the rotation matrix representation <u>with exponential convergence to SO(3)</u>

  H. F. Grip, T. I. Fossen, T. A. Johansen and A. Saberi (2014). Globally Exponentially Stable Attitude and Gyro Bias Estimation with Application to GNSS/INS Integration, *Automatica, to appear.*

Experimental Testing using UAV

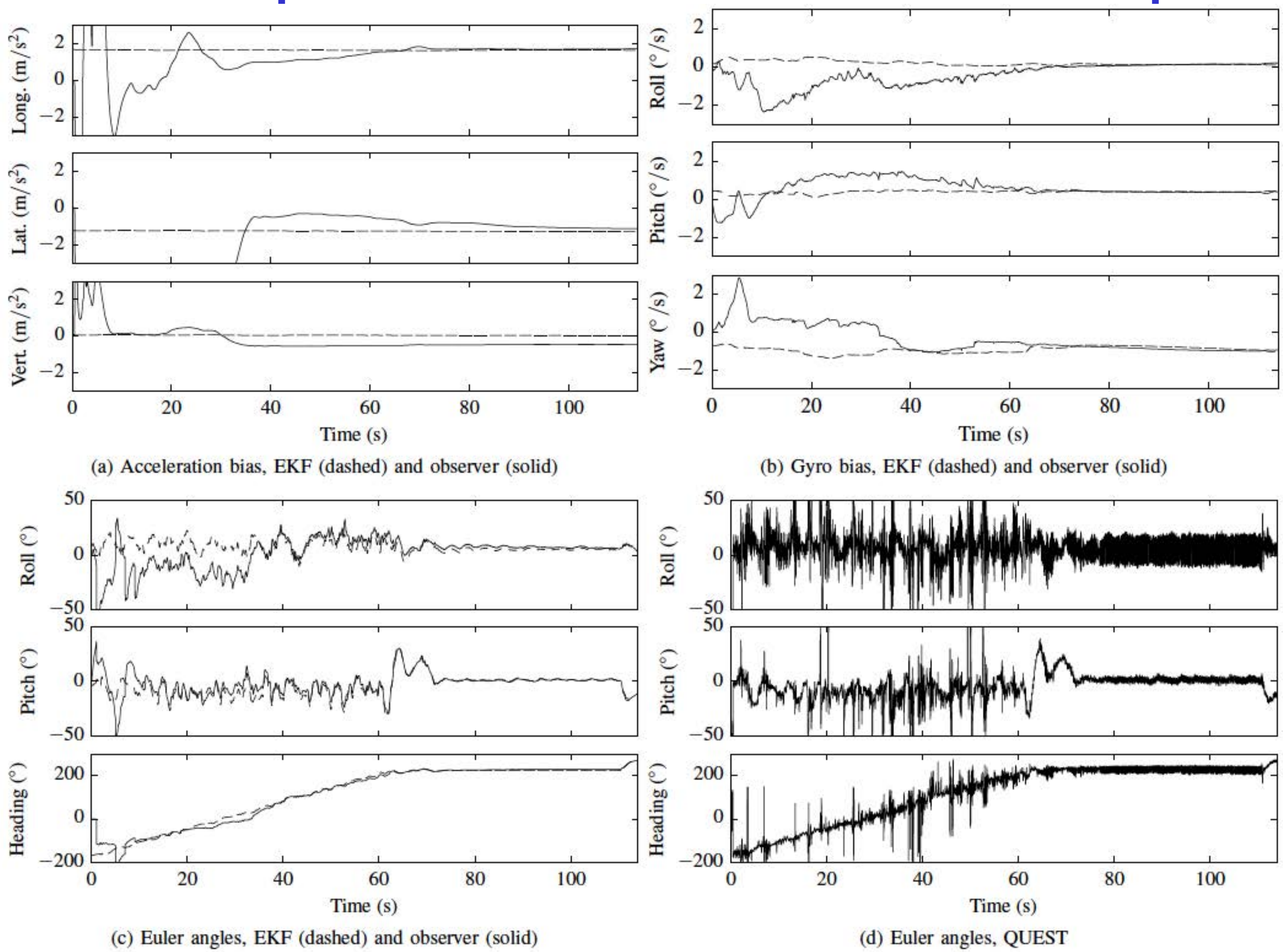
Adaptive Flight Inc, Atlanta, USA

The adaptive Flight Hornet mini helicopter equipped with the FCS-20 autopilot system. Autopilot system includes IMU, GPS and EKF for navigation.

IMU: accelerometers and gyros (100 Hz)
magnetometers (10 Hz)
GPS: position and velocity (5 Hz)

Acceleration reference vector is found by differentiating the GPS position twice (only for demonstration)

UAV Example: The Hornet Mini Helicopter



(a) Acceleration bias, EKF (dashed) and observer (solid)

(b) Gyro bias, EKF (dashed) and observer (solid)

(c) Euler angles, EKF (dashed) and observer (solid)

(d) Euler angles, QUEST

- **H. F. Grip, T. I. Fossen, T. A. Johansen and A. Saberi (2012).** Attitude estimation using biased gyro and vector measurements with time-varying reference vectors. *IEEE Transactions Automatic Control*.

Experimental Testing using Aircraft

Piper Cherokee 140 light fixed-wing aircraft

Håvard – the pilot

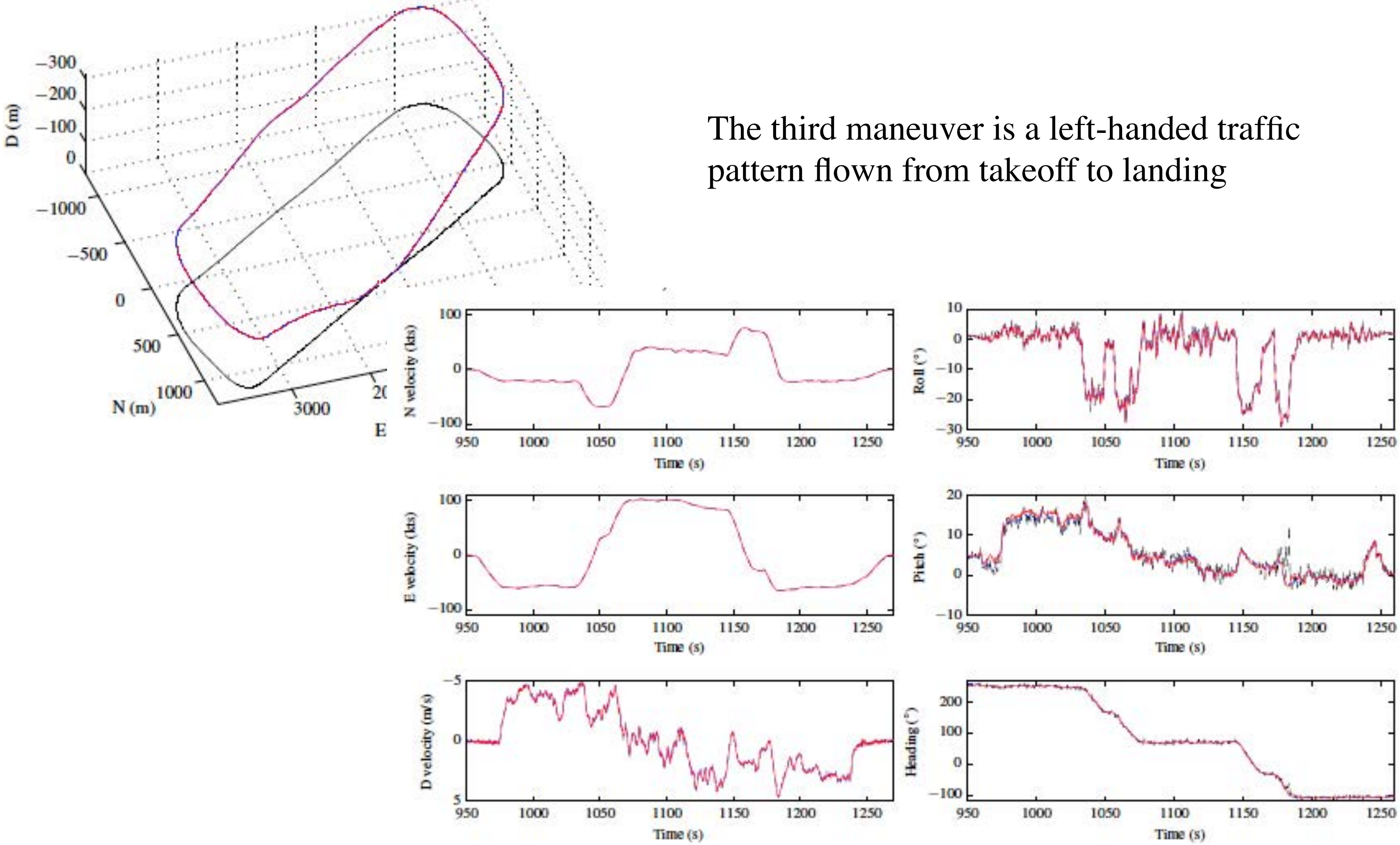
The Piper Cherokee is equipped with:

- **IMU:** XSens Mti, accelrometers, gyros and magnetometers (100 Hz)
- **GNSS:** u-Blox LEA-6H, position and Doppler-based velocity measurements (5 Hz**)**

Mounted on a bulkhead within the tail of the aircraft. The GNSS antenna is mounted on top of the instrument panel.

Traffic Pattern Maneuver

The third maneuver is a left-handed traffic pattern flown from takeoff to landing



(b) Measured (blue, dashed) and estimated (red, solid) velocity (c) QUEST (black, dashed), MEKF (blue, dashed), and observer (red, solid) attitude estimates

H. F. Grip, T. I. Fossen, T. A. Johansen and A. Saberi (2014). Globally Exponentially Stable Attitude and Gyro Bias Estimation with Application to GNSS/INS Integration, *Automatica, to appear.*