Multiplicación – DC

Dados dos números naturales a y b con n dígitos, podemos expresarlos.

$$a = a_1 \times 10^m + a_2$$
$$b = b_1 \times 10^m + b_2$$

Donde, $m = \lceil n/2 \rceil$.

Entonces tenemos,

$$a \times b = (a_1 \times 10^m + a_2)(b_1 \times 10^m + b_2)$$
$$= (a_1 b_1) \times 10^{2m} + a_1 \times b_2 \times 10^m + a_2 \cdot b_1 \times 10^m + a_2 b_2$$
$$= \underbrace{(a_1 b_1) 10^{2m} + (a_1 b_2 + a_2 b_1) 10^m + a_2 b_2}_{\text{4 sub problemas}}$$

Teorema Maestro

$T(n) = aT(n/b) + cn^k$

$\lg a/\lg b > k \quad \Theta(n^{\lg a/\lg b})$

$\lg a/\lg b = k \quad \Theta(n^k \lg n)$

$\lg a/\lg b < k \quad \Theta(n^k)$

Siempre va de exp el mayor.
Cuando son iguales se añade lgn.

MULTIPLICACION-DC(a,b,n)	costo	veces
if n==1:	1	1
return a.b	T(n)	1
$a_1 = \lfloor a/10^{n/2} \rfloor$	T(n)	1
$a_2 = a \bmod 10^{n/2}$	T(n)	1
$b_1 = \lfloor b/10^{n/2} \rfloor$	T(n)	1
$b_2 = b \bmod 10^{n/2}$	T(n)	1
a_1b_1 = MULTIPLICACION-DC(a_1, b_1, n/2)	T(n/2)	1
a_1b_2 = MULTIPLICACION-DC(a_1, b_2, n/2)	T(n/2)	1
a_2b_1 = MULTIPLICACION-DC(a_2, b_1, n/2)	T(n/2)	1
a_2b_2 = MULTIPLICACION-DC(a_2, b_2, n/2)	T(n/2)	1
return $a_1b_1 \cdot 10^{n/2} + (a_1b_2 + a_2b_1) \cdot 10^{n/2} + a_2b_2$	T(n)	1

$$T(n) = \begin{cases} \Theta(n) & n=1 \\ 4T(n/2) + \Theta(n) & \text{c.c.} \end{cases} \quad \lg 4/\lg 2 = 2 > k=1$$

$a=4 \; b=2 \; k=1$

$\Theta(n^2)$

Algoritmo de Karatsuba

Tomamos:

$$a = a_1 \times 10^m + a_2$$
$$b = b_1 \cdot 10^m + b_2$$

donde, $m = \lceil n/2 \rceil$

Entonces, tenemos:

$$a \cdot b = (a_1 \times 10^m + a_2)(b_1 \cdot 10^m + b_2)$$
$$= a_1 b_1 \cdot 10^{2m} + a_1 b_2 \cdot 10^m + a_2 b_1 \cdot 10^m + a_2 b_2$$
$$= a_1 b_1 \cdot 10^{2m} + (a_1 b_2 + a_2 b_1) 10^m + a_2 b_2$$
$$= \underline{a_1 b_1} \cdot 10^{2m} + (\underline{(a_1+a_2)(b_1+b_2)} - a_1 b_1 - a_2 b_2) 10^m + \underline{a_2 b_2}$$

Solo 3 llamadas recursivas

$a_1 b_2 + a_2 b_1$

$(a_1+a_2)(b_1+b_2) - a_1 b_1 - a_2 b_2$

$$T(n) = \begin{cases} \Theta(n) & n=1 \\ 3T(n/2) + \Theta(n) & \text{c.c.} \end{cases} \quad a=3 \; b=2 \; k=1 \quad \lg 3 > 1 \quad T(n) = \Theta(n^{\lg 3}) \approx \Theta(n^{1.58\ldots})$$

Multiplicación de Matrices

Supongamos por facilidad que n es potencia de 2.

MULTIPLICA_DC

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

Por lo tanto:

$$\left.\begin{array}{ll} C_{11} = (A_{11}B_{11} + A_{12}B_{21}) & 2T(n/2) \\ C_{12} = (A_{11}B_{12} + A_{12}B_{22}) & 2T(n/2) \\ C_{21} = (A_{21}B_{11} + A_{22}B_{21}) & 2T(n/2) \\ C_{22} = (A_{21}B_{12} + A_{22}B_{22}) & 2T(n/2) \end{array}\right\rangle$$

Strassen reduce las llamadas a 7.

$$T(n) = \begin{cases} \theta(1) & n = 1 \\ 7T(n/2) + \theta(n^2) & c.c. \end{cases}$$

$T(n) = \theta(n^{\lg 7})$ por teorema maestro.