Eliminar la recurrencia:

En cada caso suponer que $T(1)=1$.

(c) $T(n) = 4T(\lfloor n/2 \rfloor) + n$

Suponemos que n es potencia 2. Es decir, $n = 2^k$.

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&= 4(4T(n/4) + n/2) + n \\
&= 4^2 T(n/4) + 2n + n \\
&= 4^2(4T(n/8) + n/4) + 2n + n \\
&= 4^3 T(n/8) + 4n + 2n + n \\
&= 4^3(4T(n/16) + n/8) + 4n + 2n + n \\
&= 4^4 T(n/16) + 8n + 4n + 2n + n \\
&= 4^4 T(n/2^4) + 2^3 n + 2^2 n + 2^1 n + 2^0 n \\
&\vdots \\
&= 2^i T(n/2^i) + 2^{i-1} n + \ldots + 2^2 n + 2^1 n + 2^0 n \\
&= 2^i T(n/2^i) + n(2^0 + 2^1 + 2^2 + \ldots + 2^{i-1}) \\
&= 2^i T(n/2^i) + n(2^{i-1+1} - 1) \\
&= 2^i T(n/2^i) + n(2^i - 1)
\end{aligned}
$$

$i = \lg n$

$$
\begin{aligned}
&= 2^{\lg n} T(n/2^{\lg n}) + n(2^{\lg n} - 1) \\
&= n T(1) + n(n-1) \\
&= \underline{n + n(n-1)}
\end{aligned}
$$

(d) $T(n) = 2T(\lfloor n/2 \rfloor) + n^3$

Suponemos que n es potencia de 2. Es decir, $n = 2^k$.

$$
\begin{aligned}
T(n) &= 2T(n/2) + n^3 \\
&= 2(2T(n/4) + (n/2)^3) + n^3 \qquad \tfrac{n^3}{2^3} \\
&= 2^2 T(n/4) + \frac{n^3}{2^2} + n^3 \\
&= 2^2(2T(n/8) + (n/4)^3) + \frac{n^3}{2^2} + n^3 \qquad \tfrac{n^3}{2^6} \\
&= 2^3 T(n/8) + \frac{n^3}{2^4} + \frac{n^3}{2^2} + n^3 \\
&= 2^3(2T(n/16) + (n/8)^3) + \frac{n^3}{2^4} + \frac{n^3}{2^2} + n^3 \\
&= 2^4 T(n/16) + \frac{n^3}{2^6} + \frac{n^3}{2^4} + \frac{n^3}{2^2} + \frac{n^3}{2^0} \\
&= 2^4 T(n/2^4) + n^3\left(\frac{1}{2^6} + \frac{1}{2^4} + \frac{1}{2^2} + \frac{1}{2^0}\right) \qquad \left(\left(\frac{1}{2^2}\right)^3 + \left(\frac{1}{2^2}\right)^2 + \left(\frac{1}{2^2}\right)^1 + \left(\frac{1}{2^2}\right)^0\right) \\
&\vdots \\
&= 2^i T(n/2^i) + n^3\left(\frac{(1/4)^{i-1+1} - 1}{1/4 - 1}\right)
\end{aligned}
$$

$i = \lg n$

$$
\begin{aligned}
&= 2^{\lg n} T(n/2^{\lg n}) + n^3\left(\frac{\left(\frac{1}{4}\right)^{\lg n} - 1}{\frac{1}{4} - 1}\right) \qquad 2^{2\lg n} = 2^{\lg n^2} = n^2 \qquad \frac{\frac{1}{n^2} - 1}{-3/4} \\
&= n T(1) + n^3\left(\frac{\frac{1}{n^2} - 1}{-3/4}\right) \\
&= n + \frac{\frac{n^3}{n^2}}{-3/4} - \frac{n^3}{-3/4} \\
&= n - \frac{4n}{3} + \frac{4n^3}{3} \\
&= -\frac{n}{3} + \frac{4n^3}{3} \\
&= \frac{n}{3}(4n^2 - 1) \qquad (2n)^2 \\
&= \frac{n}{3}(2n+1)(2n-1)
\end{aligned}
$$

(a) Suponga que $T(1) = 1$,

$$T(n) = 2T(\lfloor n/2 \rfloor) + n^2$$

$n = 2^k$

$$
\begin{aligned}
T(n) &= 2T(n/2) + n^2 \\
&= 2\left(2T(n/4) + \frac{n^2}{2^2}\right) + n^2 \\
&= 2^2 T(n/2^2) + \frac{n^2}{2} + n^2 \\
&= 2^2\left(2T(n/2^3) + \frac{n^2}{2^4}\right) + \frac{n^2}{2} + n^2 \\
&= 2^3 T(n/2^3) + \frac{n^2}{2^2} + \frac{n^2}{2} + n^2 \\
&\vdots \\
&= 2^i T(n/2^i) + \frac{n^2}{2^0} + \frac{n^2}{2^1} + \cdots + \frac{n^2}{2^{i-1}}
\end{aligned}
$$

$i = \lg n$

$$
\begin{aligned}
&= n T(1) + \left(\frac{1}{2^0} + \frac{1}{2^1} + \cdots + \frac{1}{2^{\lg n - 1}}\right) n^2 \\
&= n + \sum_{k=0}^{\lg n - 1} \left(\frac{1}{2}\right)^k n^2 \\
&= n + \left(\frac{(1/2)^{\lg n - 1 + 1} - 1}{1/2 - 1}\right) n^2 \qquad \frac{(1/2)^{\lg n} - 1}{-1/2} = \frac{1/n - 1}{-1/2} = \frac{\frac{1-n}{n}}{\frac{-1}{2}} = \frac{2(1-n)}{-n} = \frac{2(n-1)}{n} \\
&= n + \frac{2(n-1)}{n} n^2
\end{aligned}
$$

$$
\begin{aligned}
T(n) &= n + 2(n-1)n \\
&= n + 2n^2 - 2n \\
&= 2n^2 - n \\
&= \underline{n(2n-1)}
\end{aligned}
$$

(b)

$$
\begin{aligned}
T(n) &= 2T(n-1) + 3n - 2 \\
&= 2(2T(n-2) + 3(n-1) - 2) + 3n - 2 \\
&= 2^2 T(n-2) + 2 \cdot 3(n-1) - 2^2 + 3n - 2 \\
&= 2^2(2T(n-3) + 3(n-2) - 2) + 2 \cdot 3(n-1) - 2^2 + 3n - 2 \\
&= 2^3 T(n-3) + 3 \cdot 2^2(n-2) - 2^3 + 3 \cdot 2^1(n-1) - 2^2 + \underbrace{3n - 2}_{3 \cdot 2^0(n-0) - 2^1} \\
&\vdots \\
&= 2^i T(n-i) + 3(2^{i-1}(n-(i-1)) + 2^{i-2}(n-(i-2)) + \ldots + 2^0(n-0)) - (2^1 + 2^2 + 2^3 + \ldots + 2^i)
\end{aligned}
$$

$6n - 6 - 2^2 + 3n - 2$

$6n + 3n - 6 - 2^2 - 2$

$i = n-1$

$$
\begin{aligned}
&= 2^{n-1} T(1) + 3\underbrace{(2^{n-2}(2) + 2^{n-3}(3) + \ldots + 2^{n-n}(n))}_{S} - (2^1 + 2^2 + 2^3 + \ldots + 2^{n-1})
\end{aligned}
$$

$$
\begin{aligned}
S &= 2 \cdot 2^{n-2} + 3 \cdot 2^{n-3} + \ldots + n \cdot 2^0 \\
2S &= 2 \cdot 2^{n-1} + 3 \cdot 2^{n-2} + \ldots + n \cdot 2^1 \\
\hline
S &= 1 \cdot 2^{n-2} + 1 \cdot 2^{n-3} + \ldots + 1 \cdot 2^1 + 2 \cdot 2^{n-1} - n \cdot 2^0 \\
&= \sum_{k=1}^{n-2} 2^k + 2^n - n \\
&= \frac{2^{n-2+1} - 1}{2-1} - 1 + 2^n - n \\
&= 2^{n-1} - 2 + 2^n - n \\
&= 2^{n-1}(1+2) - 2 - n \\
&= 2^{n-1} \cdot 3 - 2 - n
\end{aligned}
$$

$$
\begin{aligned}
&= 2^{n-1} \cdot 1 + 3(2^{n-1} \cdot 3 - 2 - n) - \left(\frac{2^{n-1+1} - 1}{2-1} - 1\right) \\
&= 2^{n-1} + 2^{n-1} \cdot 3^2 - 3 \cdot 2 - 3n - 2^n + 2 \\
&= 2^{n-1} + 2^{n-1} \cdot 3^2 - 2^n - 3n - 4 \\
&= 2^{n-1}(\underbrace{1 + 3^2 - 2}_{8}) - 3n - 4 \\
&= 2^{n-1} \cdot 2^3 - 3n - 4 \\
&= \underline{2^{n+2} - 3n - 4}
\end{aligned}
$$

Exámenes Pasados

Ejercicio 3 (5 ptos). Resuelva la recurrencia **de manera explícita**, usando los métodos vistos en clase. Debe dar **la solución exacta** cuando n es potencia de 3. Cuando no lo es, debe acotar adecuadamente.

$$T(n) = \begin{cases} 1 & n = 1 \\ 3T(\lfloor n/3 \rfloor) + n^2 & \text{caso contrário} \end{cases}$$

Puede suponer que es conocido que $T(n)$ es una función creciente.

$n = 3^k$

$$T(n) = 3T(n/3) + n^2$$
$$= 3\left(3T(n/9) + \frac{n^2}{3^2}\right) + n^2$$
$$= 9T(n/9) + \frac{n^2}{3} + n^2$$
$$= 9\left(3T(n/27) + \frac{n^2}{9^2}\right) + \frac{n^2}{3} + n^2$$
$$= 27T(n/27) + \frac{n^2}{9} + \frac{n^2}{3} + n^2$$
$$\vdots$$
$$= 3^i T(n/3^i) + \left(\frac{n^2}{3^{i-1}} + \frac{n^2}{3^{i-2}} + \dots + \frac{n^2}{3} + n^2\right)$$
$$= 3^i T(n/3^i) + \left(\frac{1}{3^{i-1}} + \frac{1}{3^{i-2}} + \dots + \frac{1}{3} + 1\right) n^2$$

$i = \log_3 n$

$$= n \cdot T(1) + \left(\frac{1}{3^0} + \frac{1}{3^1} + \dots + \frac{1}{3^{i-2}} + \frac{1}{3^{i-1}}\right) n^2$$
$$= n \cdot 1 + \left(1 + \left(\frac13\right)^1 + \dots + \left(\frac13\right)^{i-2} + \left(\frac13\right)^{i-1}\right) n^2$$
$$= n + \sum_{i=0}^{\log_3 n - 1} \left(\frac13\right)^i n^2$$
$$= n + \frac{\left(\frac13\right)^{\log_3 n - 1 + 1} - 1}{\frac13 - 1} n^2$$

$$\frac{\frac{1}{3^{\log_3 n}} - 1}{\frac13 - 1} = \frac{\frac1n - 1}{\frac13 - 1} = \frac{\frac{1-n}{n}}{-\frac23}$$
$$\frac{3(1-n)}{-2n}$$

$$= n - \frac{3(1-n)}{2n} n^2$$
$$= n - \left(\frac{3-3n}{2n}\right) n^2$$
$$= n + \left(\frac{3n-3}{2n}\right) n^2$$
$$= n + \left(\frac{3n-3}{2}\right) n$$
$$= n + \frac{3n^2 - n}{2}$$
$$= \frac{3n^2 - n}{2}$$
$$= n\left(\frac{3n-1}{2}\right)$$

PRUEBAS POR INDUCCIÓN

Ejercicio 3 (3 ptos). Considere la siguiente recurrencia sobre los números naturales

$$T(n) = \begin{cases} 0 & n = 0 \\ 0 & n = 1 \\ T(n) = T(n-1) + T(n-2) + 1 & \text{caso contrário.} \end{cases}$$

Pruebe por inducción que $T(n) = \Omega((3/2)^n)$. $0 \le c(3/2)^n \le T(n)$, c positivo, $n \ge n_0 > 0$

Caso base, $n = 2$

$$0 \le c\left(\frac32\right)^2 = c\left(\frac94\right) \le T(2) = 1$$

$n = 2$
$$c\left(\frac32\right)^2 = c\frac94 \le T(2) = 1$$
$$\frac{9c}{4} \le 1$$
$$c \le \frac49$$

Paso inductivo, $n > 2$

$$T(n) = T(n-2) + T(n-1) + 1$$
$$\le c\left(\frac32\right)^{n-2} + c\left(\frac32\right)^{n-1} + 1$$
$$= c\left(\frac32\right)^{n-2}\left(1 + \frac32\right) + 1$$
$$= c\left(\frac32\right)^{n-2}\left(\frac52\right) + 1$$

$$c\left(\frac32\right)^n \left(\frac32\right)^{-2} \left(\frac52\right)$$
$$c\left(\frac32\right)^n \left(\frac49\right)\left(\frac52\right)$$

$$= c\left(\frac{10}{9}\right)\left(\frac32\right)^n + 1$$
$$\le c\left(\frac{10}{9}\right)\left(\frac32\right)^n$$

Reemplazamos $c = 4/9$

$$= \left(\frac49\right)\left(\frac{10}{9}\right)\left(\frac32\right)^n$$
$$\le \frac49\left(\frac32\right)^n.$$

Concluimos que $T(n) = O(n^2)$ con $n_0 = 2$ y $C = \frac49$.

Ejemplo 3.4. *Sea $F : \mathbb{N} \to \mathbb{R}^>$ definido por*

$$T(n) = \begin{cases} 1 & n = 1 \\ 2T(\lfloor \frac{n}{2} \rfloor) + n & \text{caso contrário} \end{cases}$$

Compruebe por inducción que $T(n) = O(n^2)$. Compruebe por inducción que $T(n) = \Omega(n \lg n)$.

Probaremos por inducción en n que $T(n) \le 2n^2$ para $n \ge 1$

Si $n = 1$

$$T(1) = 1 \le 2 \le 2n^2.$$

Si $n > 1$, tenemos que

$$T(n) = 2T(\lfloor n/2 \rfloor) + n$$
$$\le 2T(n/2) + n$$
$$\overset{(h.i.)}{\le} 2\left(\frac n2\right)^2 + n$$
$$\le 4\left(\frac n2\right)^2 + n$$
$$\le n^2 + n^2$$
$$\le 2n^2 \blacksquare$$

$0 \le T(n) \le cn^2$
$T(n/2) \le c\left(\frac n2\right)^2$
$2T(n/2) \le 2c\left(\frac n2\right)^2$
$2T(n/2) + n \le 2c\left(\frac n2\right)^2 + n$