Number Theory

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- This implies that we can do all the computation with integers *modulo n*.
- The integers, modulo some n form a structure called a *ring*.
- Special rules apply, also loads of interesting properties.

Modular arithmetic

Some of the allowed operations:

- Addition and subtraction modulo n

$$(a \bmod n) + (b \bmod n) = (a + b \bmod n)$$
$$(a \bmod n) - (b \bmod n) = (a - b \bmod n)$$

- Multiplication

$$(a \bmod n)(b \bmod n) = (ab \bmod n)$$

- Exponentiation

$$(a \bmod n)^b = (a^b \bmod n)$$

- *Note:* We are only working with integers.

Modular arithmetic

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- We could end up with a fraction!
- Division with k equals multiplication with the *multiplicative inverse* of k.
- The *multiplicative inverse* of an integer a, is the element a^{-1} such that

$$a \cdot a^{-1} = 1 \pmod{n}$$

Modular arithmetic

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- What about logarithm? YES!
  - But difficult.
  - Basis for some cryptography such as elliptic curve, Diffie-Hellmann.
- Google "Discrete Logarithm" if you want to know more.

Modular arithmetic

- Prime number is a positive integer greater than 1 that has no positive divisor other than 1 and itself.
- Greatest Common Divisor of two integers a and b is the largest number that divides both a and b.
- Least Common Multiple of two integers a and b is the smallest integer that both a and b divide.
- Prime factor of an positive integer is a prime number that divides it.
- Prime factorization is the decomposition of an integer into its prime factors. By the fundamental theorem of arithmetics, every integer greater than 1 has a unique prime factorization.

Extended Euclidean algorithm

- The Euclidean algorithm is a recursive algorithm that computes the GCD of two numbers.

```
int gcd(int a, int b){
    return b == 0 ? a : gcd(b, a % b);
}
```

- Runs in $O(\log^2 N)$.

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- Runs in $O(\log^2 N)$.
- Notice that this can also compute LCM

$$\text{lcm}(a, b) = \frac{a \cdot b}{\gcd(a, b)}$$

- See Wikipedia to see how it works and for proofs.

Extended Euclidean algorithm

- Reversing the steps of the Euclidean algorithm we get the Bézout's identity

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  which simply states that there always exist x and y such that the equation above holds.
- The extended Euclidean algorithm computes the GCD and the coefficients x and y.
- Each iteration it add up how much of b we subtracted from a and vice versa.

Extended Euclidean algorithm

```
int egcd(int a, int b, int& x, int& y) {
    if (b == 0) {
        x = 1;
        y = 0;
        return a;
    } else {
        int d = egcd(b, a % b, x, y);
        x -= a / b * y;
        swap(x, y);
        return d;
    }
}
```

Applications

- Essential step in the RSA algorithm.
- Essential step in many factorization algorithms.
- Can be generalized to other algebraic structures.
- Fundamental tool for proofs in number theory.
- Many other algorithms for GCD

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  - Iterate up to $\sqrt{n}$ instead.
  - $O(\sqrt{N})$
- Even better: If n is not a prime, it has a prime divisor $\leq \sqrt{n}$
  - Iterate over the prime numbers up to $\sqrt{n}$.
  - There are $\sim N/\ln(N)$ primes less N, therefore $O(\sqrt{N}/\log N)$.

Primality testing

- Trial division is a deterministic primality test.
- Many algorithms that are probabilistic or randomized.
- Fermat test; uses Fermat's little theorem.
- Probabilistic algorithms that can only prove that a number is composite such as Miller-Rabin.
- AKS primality test, the one that proved that primality testing is in P.

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Sieve of Eratosthenes

```
vector<int> eratosthenes(int n){
    bool *isMarked = new bool[n+1];
    memset(isMarked, 0, n+1);
    vector<int> primes;
    int i = 2;
    for(; i*i <= n; ++i)
        if (!isMarked[i]) {
            primes.push_back(i);
            for(int j = i; j <= n; j += i)
                isMarked[j] = true;
        }
    for (; i <= n; i++)
        if (!isMarked[i])
            primes.push_back(i);
    return primes;
}
```

Integer factorization

The fundamental theorem of arithmetic states that

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We can therefore store integers as lists of their prime powers.
To factor an integer n:

- Use the sieve of Eratosthenes to generate all the primes up $\sqrt{n}$
- Iterate over all the primes generated and check if they divide n, and determine the largest power that divides n.

```cpp
map<int, int> factor(int N) {
    vector<int> primes;
    primes = eratosthenes(static_cast<int>(sqrt(N+1)));
    map<int, int> factors;
    for(int i = 0; i < primes.size(); ++i){
        int prime = primes[i], power = 0;
        while(N % prime == 0){
            power++;
            N /= prime;
        }
        if(power > 0){
            factors[prime] = power;
        }
    }
    if (N > 1) {
        factors[N] = 1;
    }
    return factors;
}
```

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The prime factors can be quite useful.

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- The number of divisors

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- The number of divisors

$$\sigma_0(n) = \prod_{i=1}^{k}(e_1 + 1)$$

- The sum of all divisors in x-th power

$$\sigma_m(n) = \prod_{i=1}^{k} \frac{(p^{(e_i+1)x} - 1)}{(p_i - 1)}$$

Integer factorization

- The Euler's totient function

$$\phi(n) = n \cdot \prod_{i=1}^{k}(1 - p)$$

Integer factorization

- The Euler's totient function

$$\phi(n) = n \cdot \prod_{i=1}^{k}(1 - p)$$

- Euler's theorem, if a and n are coprime

$$a^{\phi(n)} = 1 \pmod{n}$$

  Fermat's theorem is a special case when n is a prime.

Combinatorics

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Combinatorics is study of countable discrete structures.

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Generic enumeration problem: We are given an infinite sequence of sets $A_1, A_2, \ldots A_n, \ldots$ which contain objects satisfying a set of properties. Determine

$$a_n := |A_n|$$

for general n.

Basic counting

- Factorial

$$n! = 1 \cdot 2 \cdot 3 \cdots n$$

- Binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

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  Number of ways to choose k objects from a set of n objects, ignoring order.

Basic counting

Properties

- $$\binom{n}{k} = \binom{n}{n-k}$$
- $$\binom{n}{0} = \binom{n}{n} = 1$$
- $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Basic counting

Pascal triangle!

$$
\begin{array}{ccccccccccccccccccccc}
 & & & & & & & & \binom{0}{0} & & & & & & & & & & \\
 & & & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & & & & & \\
 & & & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & & & & & \\
 & & & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & & & & & \\
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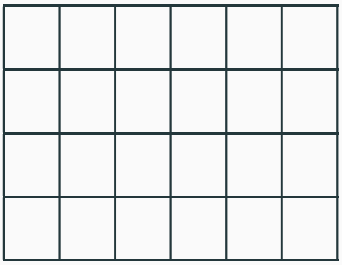
\end{array}
$$

Example

How many rectangles can be formed on a $m \times n$ grid?



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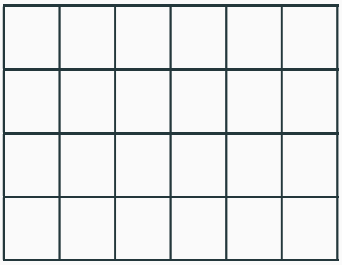

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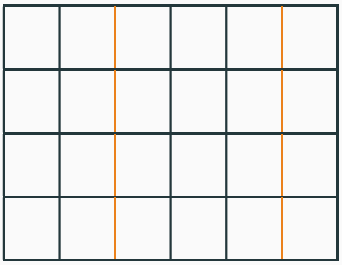

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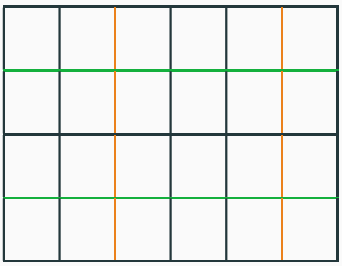

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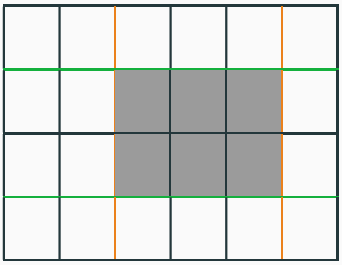

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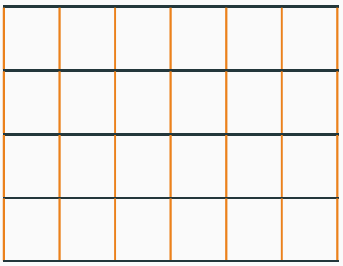

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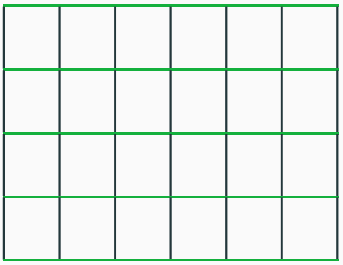

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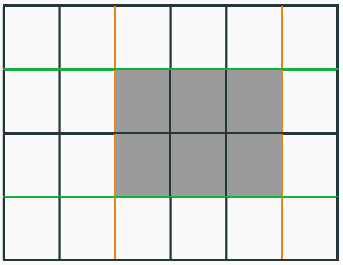
Example

How many rectangles can be formed on a $m \times n$ grid?



- A rectangle needs 4 edges, 2 vertical and 2 horizontal.
  - 2 vertical
  - 2 horizontal
- Total number of ways we can form a rectangle

$$\binom{n}{2}\binom{m}{2} = \frac{n!m!}{(n-2)!(m-2)!2!2!}$$
$$= \frac{n(n-1)m(m-1)}{4}$$

Multinomial

What if we have many objects with the same value?

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- Number of permutations on n objects, where n_i is the number of objects with the i-th value.(Multinomial)

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- Number of way to choose k objects from a set of n objects with, where each value can be chosen more than once.

$$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}$$

Example

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- Let's imagine we have a bit string consisting only of 1 of length $n + k - 1$

$$\underbrace{1\,1\,1\,1\,1\,1\,1\ldots1}_{n+k-1}$$

Example

- Choose $n-1$ bits to be swapped for 0

$$1\ldots101\ldots10\ldots01\ldots1$$

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$$\underbrace{1\ldots1}_{x_1}0\underbrace{1\ldots1}_{x_2}0\ldots0\underbrace{1\ldots1}_{x_n}$$

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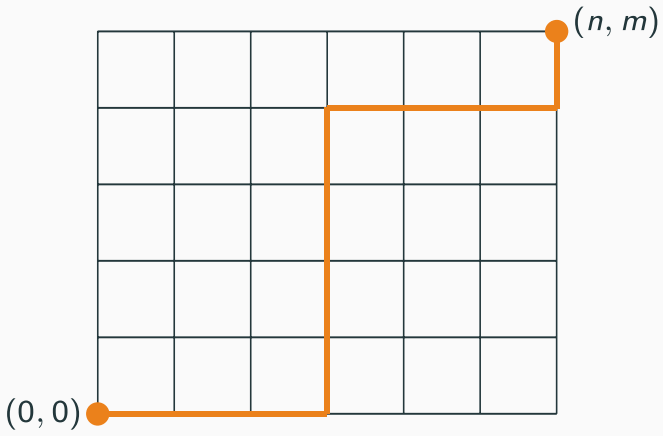
- Number of ways to choose the bits to swap

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Example

How many different lattice paths are there from $(0, 0)$ to (n, m)?



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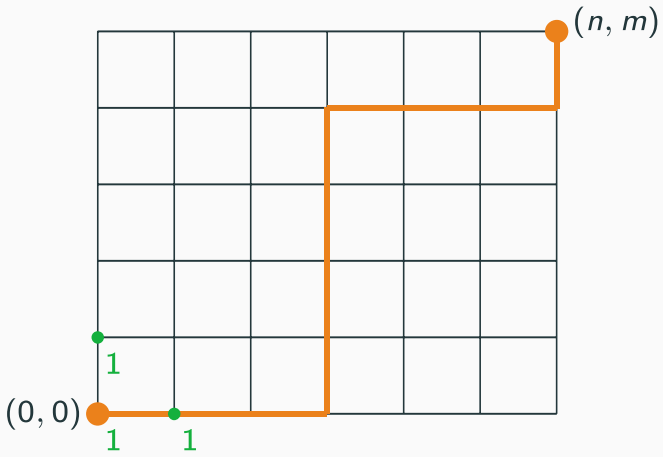

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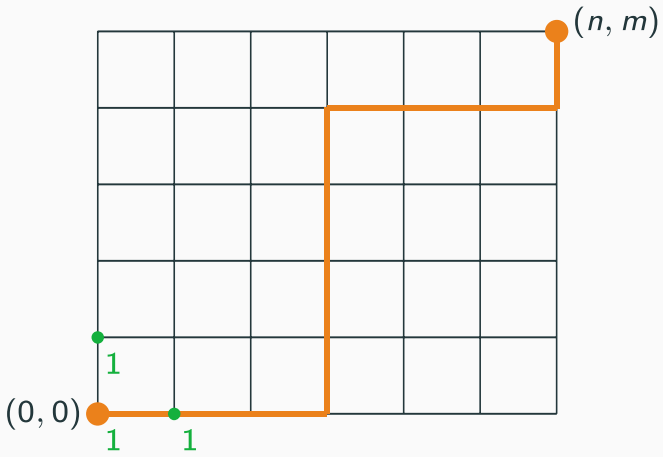

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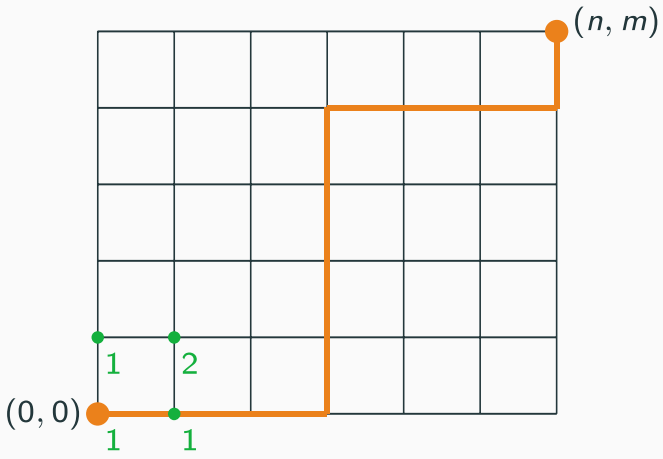

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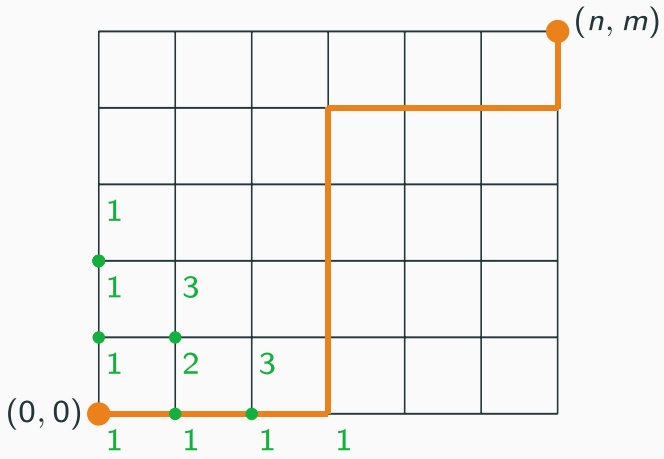

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- Paths to $(1, 1)$ is the sum of number of paths to $(0, 1)$ and $(1, 0)$.
- Number of paths to (i, j) is the sum of the number of paths to $(i - 1, j)$ and $(i, j - 1)$.

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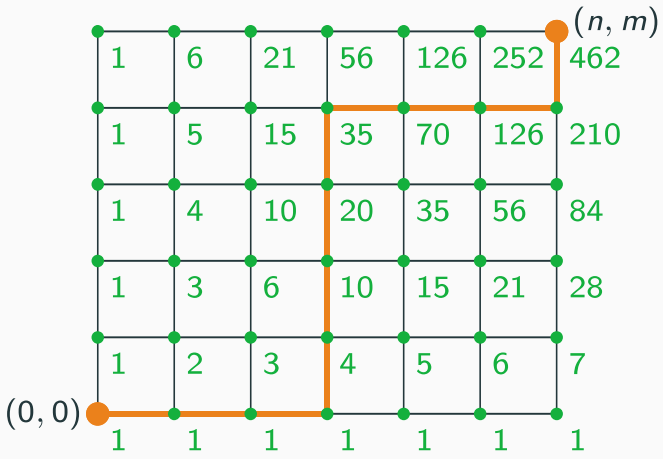

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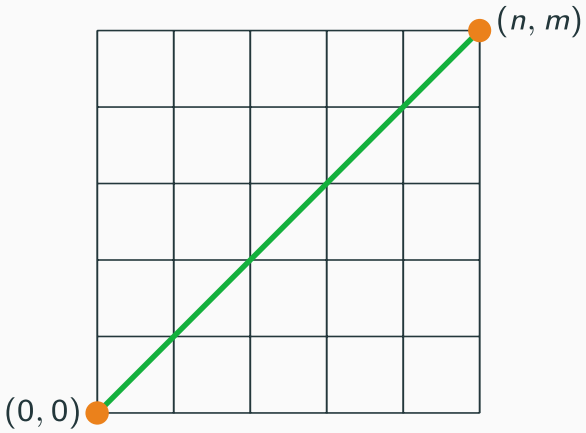
$$\binom{i+j}{i}$$

Catalan

What if we are not allowed to cross the main diagonal?



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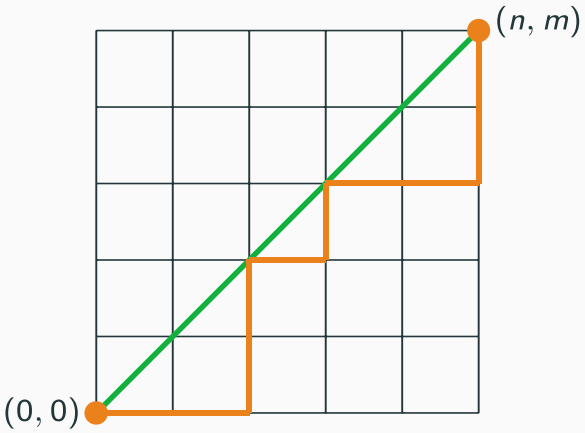

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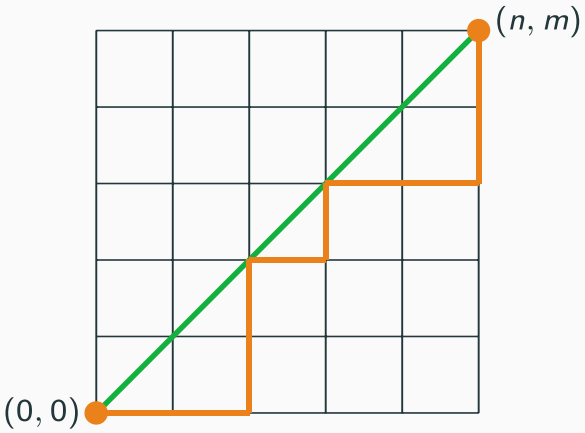

- The number of paths from $(0, 0)$ to (n, m)

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

- C_n are known as Catalan numbers.
- Many problems involve solutions given by the Catalan numbers.

Catalan

- Number of different ways $n+1$ factors can be completely parenthesized.

$$((ab)c)d \quad (a(bc))d \quad (ab)(cd) \quad a((bc)d) \quad a(b(cd))$$

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- And aloot more.

Fibonacci

The Fibonacci sequence is defined recursively as

$$f_1 = 1$$
$$f_2 = 1$$
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But we can do even better.

Fibonacci as matrix

The Fibonacci sequence can be represented by a vectors

$$\begin{pmatrix} f_{n+2} \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$$

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Or simply as a matrix

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Using fast exponentiaton, we can calculate f_n in $O(\log N)$ time.

Fibonacci as matrix

Any linear recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \dots c_k a_{n-k}$$

can be expressed in the same way

$$\begin{pmatrix} a_{n+1} \\ a_n \\ \vdots \\ a_{n-k} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & \dots & c_k \\ 1 & 0 & \dots & 0 \\ \vdots & & \vdots & \\ 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_{n-k-1} \end{pmatrix}$$

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With a recurrence relation defined as a linear function of the k preceding terms the running time will be $O(k^3 \log N)$.

Game Theory

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Game theory is the study of strategic decision making, but in competitive programming we are mostly interested in combinatorial games.

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Example:

- There is a pile of k matches.
- Player can remove 1, 2 or 3 from the pile and alternate on moves.
- The player who removes the last match wins.
- There are two players, and the first player starts.
- Assuming that both players play perfectly, who wins?

Example

We can analyse these types of games with *backward induction*.

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We can analyse these types of games with *backward induction*.
We call a state N-position if it is a winning state for the next player to move, and P-position if it is a winning position for the previous player.

- All terminal positions are P-positions.
- If every reachable state from the current one is a N-position then the current state is a P-position.
- If at least one P-position can be reached from the current one, then the current state is a N-position.
- A position is a P-position if all reachable states form the current one are N position.

Example

Let's analyse our previous game.

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- The terminal position is a P-position.

0	1	2	3	4	5	6	7	8	9	10	11	12	...
P													

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Let's analyse our previous game.

- The terminal position is a P-position.
- The positions reachable from the terminal positions are N-positions.
- Position 4 can only reach N-positions, therefore a P position.

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- The positions reachable from the terminal positions are N-positions.
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- The next 3 positions can reach the P-position 4, therefore they are N-positions.

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P	N	N	N	P	N	N	N						

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- And so on.

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P	N	N	N	P	N	N	N	P	N	N	N	P	...

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We can see a clear pattern of the N and P positions in the previous game. – Easy to prove that a position is P if $x \equiv 0 \pmod 4$.

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- Many games can be analyzed this way.
- Not only one dimensional games.
- What if there are n piles instead of 1?
- What if we can remove 1, 3 or 4?

The game called Nim

- There are n piles, each containing x_i chips.
- Player can remove from exactly one pile, and can remove any number of chips.
- The player who removes the last match wins.
- There are two players, and the first player starts and they alternate on moves.
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The game called Nim

Nim can be analyzed with N and P position.