Poisson Blending

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1   Introduction

The mathematical tool at the heart of the approach is the Poisson partial differential equation with Dirichlet boundary conditions which specifies the Laplacian of an unknown function over the domain of interest, along with the unknown function values over the boundary of the domain. The motivation is twofold.

Poisson partial differential equations

Dirichlet boundary conditions

Laplacian operator

2   Poisson solution to guided interpolation

Guided interpolation

- Image interpolation using a guidance vector field

- Solve the interpolation problem for each color component separately

- Let S a closed subset of $\mathbb{R}^2$, be the image definition domain.

- Let Ω be a closed subset of S with boundary $\partial\Omega$.

- Let $f*$ be a known scalar function defined over S minus the interior of Ω.

- Let f be a unknown scalar function defined over the interior of Ω.

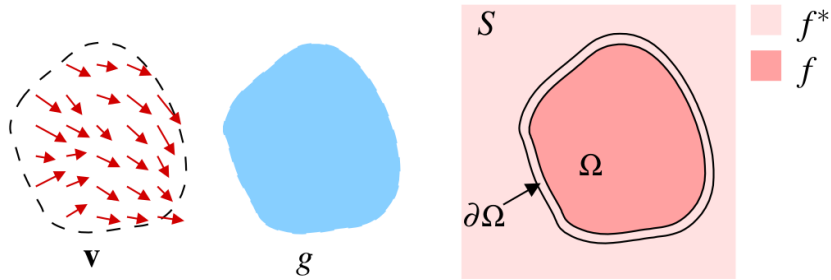- Let v be a vector field defined over Ω

Figure 1: Guided interpolation notations. Unknown function f interpolates in domain Ω the destination function f^*, under guidance of vector field v, which might be or not the gradient field of a source function g.

The simplest interpolant f of $f*$ over Ω is the membrane interpolant defined as the solution of the minimization problem:

$$\min_f \iint_\Omega |\nabla f|^2 \text{ with } f|_{\partial\Omega} = f^*|_{\partial\Omega} \tag{1}$$

where $\nabla. = (\frac{\partial}{\partial_x}, \frac{\partial}{\partial_y})$ is the gradient operator. The minimizer must satisfy the associated Euler-Lagrange equation:

$$\Delta f = 0 \text{ over } \Omega \text{ with } f|_{\partial\Omega} = f^*|_{\partial\Omega} \tag{2}$$

where $\Delta. = (\frac{\partial^2}{\partial_{x^2}}, \frac{\partial^2}{\partial_{y^2}})$ is the Laplacian operator. Equation 2 is a Laplace equation with Dirichlet boundary conditions.

The route proposed in this work is to modify the problem by introducing further constraints in the form of a guidance field as follows:

A **guidance field** is a vector field $\mathbf{v}$ used in an extended version of the minimization problem (1) above:

$$\min_f \iint_\Omega |\nabla f - \mathbf{v}|^2 \text{ with } f|_{\partial\Omega} = f^*|_{\partial\Omega} \tag{3}$$

whose solution is the unique solution of the following Poisson equation with Dirichlet boundary conditions:

$$\Delta f = \text{div}\mathbf{v} \text{ over } \Omega \text{ with } f|_{\partial\Omega} = f^*|_{\partial\Omega} \tag{4}$$

where $\text{div.} = \frac{\partial.}{\partial_x} + \frac{\partial.}{\partial_y}$ is the divergence operator of $\mathbf{v} = (u, v)$.

Three Poisson equations of the form (4) are solved independently in the three color channels of the chosen color space.

When the guidance field $\mathbf{v}$ is conservative, i.e., it is the gradient of some function g, a helpful alternative way of understanding what Poisson interpolation does is to define the correction function $\tilde{f}$ on Ω such that $f = \tilde{f} + g$. The Poisson equation (4) then becomes:

$$\Delta \tilde{f} = 0 \text{ over } \Omega \text{ with } \tilde{f}|_{\partial\Omega} = (f^* - g)|_{\partial\Omega} \tag{5}$$

Discrete Poisson solver

Seamless cloning