Minicurso de Verão da FGV/2025: Fundamentos da Otimização Multiobjetivo: Métodos, Teoria e Aplicações

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Problema multiobjetivo ou vetorial

- Estamos interessados no seguinte problema de otimização multiobjetivo:

$$\min_{x \in \mathbb{R}^n} F(x), \tag{1}$$

  sendo $F : \mathbb{R}^n \to \mathbb{R}^m$ um função multiobjetiva diferenciável.

- Um ponto $x^* \in \mathbb{R}^n$ é uma solução Pareto fraca (fracamente Pareto eficiente) para o problema (1) se **não existe** $x \in \mathbb{R}^n$ tal que $F(x) \prec F(x^*)$

Observação, essa ordem pode ser dada por um cone, isto é um conjunto $K \subset \mathbb{R}^m$ pontudo com interior não vazio. Neste caso, diz-se que $u \preceq v$ se $v - u \in int(K)$. Neste sentido, x^* é uma solução Pareto fraca de F se não existe $x \in\in \mathbb{R}^n$ tal que $F(x) \preceq F(x^*)$. Consideramos um conjunto compacto G^* como sendo o gerador de K^*, dual de K.

Método gradiente vetorial

0) (Passo Inicial) Escolha $x^0 \in \mathbb{R}^n$, $\delta \in (0,1)$ e faça $k = 0$;

1) (Direção de Descida) Compute v^k como sendo a única solução de

$$\min_{v \in \mathbb{R}^n} \left\{ \max_{s \in G^*} \langle s, F'(x^k)v \rangle + \frac{1}{2}|v|^2 \right\}, \tag{2}$$

2) (Critério de Parada) Se $v^k = 0$, pare;

3) (Escolha do Passo) Determine o primeiro inteiro não negativo $\ell_k := \ell$ satisfazendo

$$F(x^k + 2^{-\ell} v^k) \preceq F(x^k) + 2^{-\ell} \delta J_F(x^k) v^k; \tag{3}$$

4) (Atualização), Faça $\alpha_k = 2^{-\ell_k}$, $x^{k+1} := x^k + \alpha_k v^k$, $k \leftarrow k+1$ e retorne ao Passo 1.

Direção do método gradiente vetorial projetado

Se o problema for obter $min_{x \in C} F(x)$ onde C é um conjunto convexo, fechado e não vazio, então o método gradiente multiobjetivo se estende a este caso considerando o seguinte subproblema para obter a direção de descida:

$$y^k = \arg \min_{y \in C - x^k} \left\{ \max_{s \in G^*} \langle s, \nabla F(x^k) v \rangle + \frac{1}{2} |v|^2 \right\}, \tag{4}$$

De onde atualizamos $x^{k+1} = x^k + \alpha_k (y^k - x^k)$, para algum comprimento de passo α_k.

Método gradiente proximal multiobjetivo

O método gradiente proximal multiobjetivo produz uma sequência de pontos $\{x^k\}$ por meio do seguinte procedimento iterativo:

Compute x^{k+1} como sendo uma solução do seguinte subproblema problema

$$x^{k+1} = \arg\min_{x \in \in R^n} \left\{ \max_{j=1\ldots,m} \langle \nabla F_j(x^k), x - x^k \rangle + g_j(x) - g_j(x^k) + \frac{1}{2}\|x - x^k\|^2 \right\}, \tag{5}$$

Multiobjective Frank-Wolfe(conditional gradient) method

Passo 0. Inicialização Escolha $x_0 \in C$ e inicialize $k \leftarrow 0$.
Passo 1. Calcular a direção de busca Calcule uma solução ótima $p(x_k)$ e o valor ótimo $\theta(x_k)$ como:

$$p(x_k) \in \arg\min_{u \in C} \max_{j \in J} \langle \nabla f_j(x_k), u - x_k \rangle$$

e

$$\theta(x_k) := \max_{j \in J} \langle \nabla f_j(x_k), p(x_k) - x_k \rangle.$$

Defina a direção de busca como $d(x_k) := p(x_k) - x_k$.
Passo 2. Critério de parada Se $\theta(x_k) = 0$, então pare.
Passo 3. Calcular o tamanho do passo e iterar Calcule $\lambda_k \in (0, 1]$ e defina:

$$x_{k+1} := x_k + \lambda_k d(x_k).$$

Passo 4. Iniciar uma nova iteração Atualize $k \leftarrow k + 1$ e volte ao Passo 1.

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