

Estimating Very Large Demand Systems

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Abstract

Whether online, in a supermarket, or elsewhere, people now assemble consumption bundles from an extremely wide variety of goods. We model this as a discrete choice between bundles, to maximize a random utility depending on the attributes of the goods in the bundle. We do not measure attributes directly, but discern them in consumer preferences. Attributes may be orders-of-magnitude fewer than goods - much reducing the effective consumption space.

Under quasi-linear preferences in our model, any given consumer tends to bundle-together goods that are price complements for her; while, on the contrary, she tends to buy substitutes separately.

We estimate consistently, at scale, by using techniques similar to negative-sampling for embedding in machine learning. This involves estimating from a big dataset every purchased good's latent attributes, jointly with every consumer's preferences over attributes.

JEL classification:

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1 Introduction

We present a discrete choice random utility model for analyzing consumer demand for *large* numbers of products. The method allows for the consumer to purchase multiple units of any product and to purchase multiple products at once (one might imagine a consumer selecting a basket of goods in a supermarket). Our model allows for products to be complements or substitutes and allows for demand estimation and welfare evaluation of price changes.

In our model each product has an associated unobservable (to the analyst) vector of attributes. The consumer purchases a basket of products to maximize utility which is derived from the attributes of the products. If the dimensions of the attribute space are much smaller than the number of products we greatly reduce the effective size of the consumption space. While we gain tractability from this smaller consumption space we allow vectors of attributes to be substitutes or complements and so our model retains a considerable degree of flexibility in the types of behavior it can describe.

The parameters of our model can be estimated by maximum likelihood methods but this proves intractable for applications where the number of goods is large (note that the number of baskets which a consumer may purchase is much larger than the number of goods). This intractability motivates our introduction of a new estimation technique suitable for estimating parameters in a discrete choice model with a large choice spaces. The basic idea of the technique is to simulate random variables and then perform maximum likelihood conditioning on the simulated variables. Through carefully designing the distribution of the simulated random variables we are able to obtain a tractable conditional likelihood function even though the original unconditional likelihood function is intractable.

Our model can be explored via a simple example. Imagine a customer in a teashop, selecting a bundle to consume. Three goods are on offer: Tea, Biscuits, and Cake. Each, for the sake of argument, costs £1 and is offered only in unit quantity. A typical bundle, then, might be represented by a quantity-vector q , where $q = (1, 0, 1)'$ would represent the option of ‘tea and cake’; while $q = (0, 0, 0)'$ would represent ‘nothing’.

So the consumer selects from $2^3 = 8$ bundles. Our project is simply to study a multinomial choice model across these eight options. This idea, that these options are themselves

bundles of underlying goods, has been well developed, for example in Gentzkow (2007).

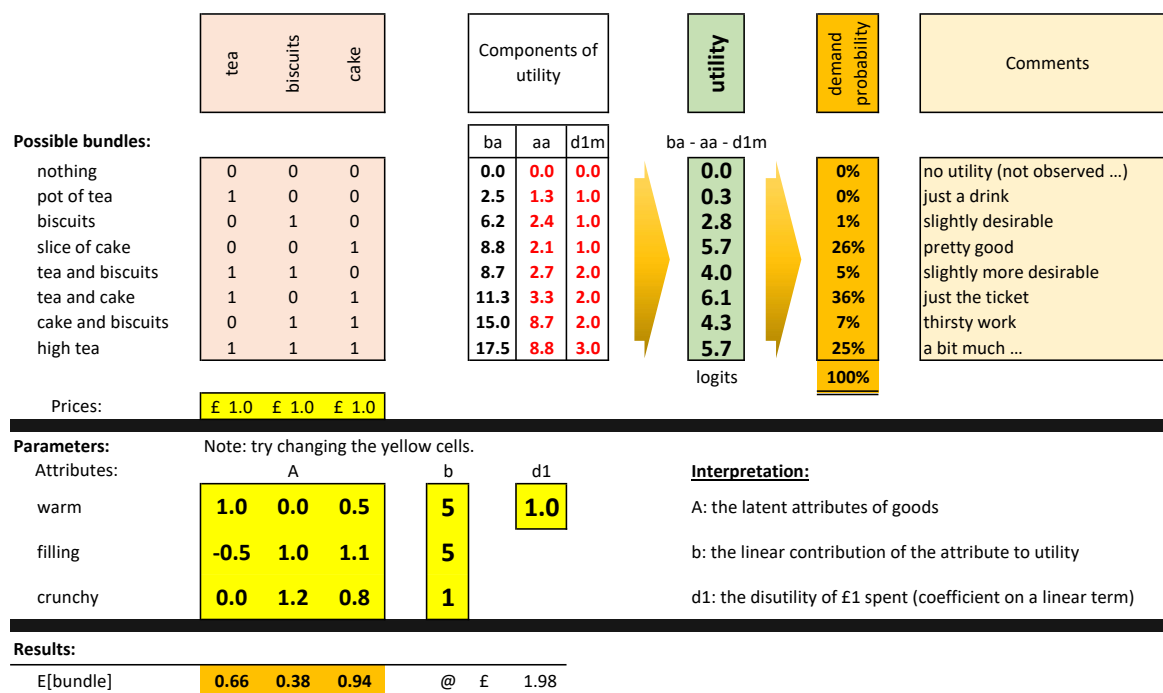


Figure 1: An illustrative example motivated by an individual's trip to a teashop. When three goods can be bought in unit quantity there are 8 possible bundles. Each bundle's composition and expected utility is reported. As the latter serves as the bundle's logit, it gives rise to the demand probabilities. Demand probabilities in turn determine (at the base of the figure) expected demand for each of the three goods – the expected bundle. Against a yellow background some typical values are reported for the parameters of this model, A , b , and d_1 .

However, in the very large demand systems characteristic of a contemporary Western economy, there are thousands of goods, not just three, each of which is at times purchased in a variety of quantities. So, the options, far from numbering only eight, may be very numerous – indeed our model has a multinomial logit structure whose alternatives are overwhelmingly numerous and may be countably infinite in number.

In tackling this profusion, we apply a Gorman-Lancaster linear characteristics model where the characteristics, or attributes, of the goods are a matter of settled fact. However, we do not measure them, but must infer them from consumption patterns.

Let's say our tearoom is offering 'warm', 'filling' and 'crunchy' consumption. Tea, Biscuits and Cake deliver differing quantities of these characteristics. We collect these quantities in a matrix, A , which in this toy example is just (3×3) but in general of

shape $(K \times L)$, where K is the number of attributes and L is the number of goods in the economy. As a settled matter, we take A to be agreed upon by all consumers.

A bundle’s random utility will depend upon the attributes of the goods in the bundle. They are collected as a vector $a \in \mathbb{R}^K$. The bundle ‘tea and cake’ is endowed with the attribute vector $a = (A_1 + A_3)$ (adding the first column of A to the last), while ‘cup of tea’ delivers simply $a = A_1$. For any bundle, q , then, $a = Aq$.

In determining their own preferences, the consumer pays attention to an inner product $b'a$. In the teashop, b (say $(5, 5, 1)'$) contains weightings indicating the desirability to this consumer of, respectively, the bundle’s ‘warm’, ‘filling’ and ‘crunchy’ attributes.

In the direction of parsimony, we allow quadratic terms in a to enter only via a ’s L2 norm, $a'a$. We show later that this is without loss of generality within a broad family of quadratic forms. This is because the latent attributes here are only defined up to a linear transformation.

Finally, the consumer pays attention to the expense of the bundle, $p'q$, applying a coefficient $d_1 > 0$ to it, which serves as a decreasing index of the affluence of the consumer.

If we consider the utility, U , given by:

$$U = b'a - a'a - d_1 p'q, \quad (1)$$

(which we will generalise later) then we stay close to the specification in Lewbel and Nesheim (2019), which points out that this yields quasi-linear preferences in which b_k plays the role of a satiation level or bliss point for attribute k .

The specification also has similarities to the heuristic model of consumer behaviour in Ruiz, Athey, and Blei (2020), although it differs in several respects – including by being quasi-concave in a and in penalizing expense.¹

Our proposal is for independent standard Gumbel errors to be added to (1) giving a random utility. The consumer selects the bundle with highest random utility, so producing a multinomial logistic distribution of outcomes in which (1) functions as the logit. All this is illustrated in Figure 1.

Like Ruiz, Athey, and Blei (2020) we use technology and techniques from machine

¹Ruiz, Athey, and Blei (2020) propose a specification in which each good, ℓ is endowed with both an embedding vector A_ℓ and a second K -vector known as a context vector. At the cost of quasi-concavity this can be captured here by replacing $a'a$ with $a'Da$ where D may be constructed from a square matrix of zeros, 0, and identity matrix, I , of equal size either as $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ or as $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$.

learning in order to estimate such a model at reasonable scale; and we could be said to engage in a `good2vec` exercise which is similar to `word2vec` in natural language processing. Our code, which uses `jax`, at is www.github.com/jeremy-large/RUBE.

In the next Section we situate our ideas in the context of the literature, before moving, in Section 3 to a formal statement of our model. Section 4 presents our comparative statics results regarding price complementarity. In Sections 5 and 6 we provide formal results regarding, respectively, identification and estimation. Section 7 then studies a simulation commensurate with our main empirical application, using data from the market research company, DunnHumby, which application is presented in Section 8.

2 Literature

2.1 Consumption bundles, and continuous consumer choice

Afriat (1967)

Barnett and Serletis (2008)

Banks, Blundell, and Lewbell (1997)

Christensen, Jorgenson, and Lau (1975)

Deaton and Muellbauer (1980)

Gallant (1981)

Gallant and Golub (1984)

Lancaster (1966)

Lewbel and Nesheim (2019) - bundle sparseness

Lewbel and Pendakur (2009)

Stone (1954)

2.2 Discrete choice in the context of random utility

Anderson, Palma, and Thisse (1995)

Ben-Akiva and Lerman (1985)

Berry, Levinsohn, and Pakes (1995, 2004)

Chesher and Santos Silva (2002)

Dubin and McFadden (1984)

Hausman and McFadden (1984)

Hausman and Wise (1978)

McFadden and Train (2000) - mixed MNL

Nevo (1998)

Rosen and Small (1979)

Ruiz, Athey, and Blei (2020) - estimates at scale, machine learning

3 The model

Let \mathbb{N}_0 denote the set of non-negative integers and \mathbb{N} denote the set of positive integers. There are L goods (call them good 1, good 2, ..., good L) which can only be consumed in non-negative integer quantities. Let $\mathbf{q} \in \mathbb{N}_0^L$ denote a consumption bundle. We assume that each good, $\ell \in \{1, \dots, L\}$, has an associated unobservable K -vector of attributes $\boldsymbol{\alpha}_\ell \in \mathbb{R}^K$ where $K \leq L$. Let \mathbf{A} be the $K \times L$ matrix with column ℓ equal to $\boldsymbol{\alpha}_\ell$. That is,

$$\mathbf{A} = [\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_L]$$

Thus, $\sum_{\ell=1}^L \boldsymbol{\alpha}_\ell q_\ell = \mathbf{A}\mathbf{q}$ is the attribute vector which the consumer enjoys. We call \mathbf{A} an *attribute matrix*. For attribute matrix \mathbf{A} let the range of \mathbf{A} , denoted $\text{range}(\mathbf{A})$, be defined by

$$\text{range}(\mathbf{A}) = \{\mathbf{A}\mathbf{q} \in \mathbb{R}^K : \mathbf{q} \in \mathbb{N}_0^L\}$$

We shall assume that the utility of the consumer is a function of the attributes of the consumption bundle purchased as well as the total amount of money spent. That is, the utility of the bundle \mathbf{q} which costs $m \in \mathbb{R}_+$ can be calculated as $U(\mathbf{A}\mathbf{q}, m)$. Such a function U is called an attribute utility function. More specifically, a function $U : \mathcal{A} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is an *attribute utility function* provided that \mathcal{A} is a convex cone in \mathbb{R}^K . The pair (U, \mathbf{A}) is called an *attributes model* when \mathbf{A} is an attribute matrix and U is an attribute utility function whose domain contains $\text{range}(\mathbf{A}) \times \mathbb{R}_+$. We shall work with attributes models which are well-behaved in the following sense.

Definition 1 *An attribute utility function $U : \mathcal{A} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is well-behaved if $U(\mathbf{a}, m)$ is continuously differentiable, strictly concave in \mathbf{a} , and*

$$\frac{\partial U(\mathbf{a}, m)}{\partial m} < 0, \quad \text{for all } (\mathbf{a}, m) \in \mathcal{A} \times \mathbb{R}_+.$$

An attributes model (U, \mathbf{A}) is well-behaved if U is well-behaved.

An attribute utility function $U : \mathcal{A} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is a *Quadratic Attributes utility Function* (Qua function) if U is well-behaved and satisfies

$$U(\mathbf{a}, m) = \tilde{\mathbf{b}}' \begin{bmatrix} \mathbf{a} \\ m \end{bmatrix} - \begin{bmatrix} \mathbf{a}' & m \end{bmatrix} \tilde{\mathbf{B}} \begin{bmatrix} \mathbf{a}' \\ m \end{bmatrix}, \quad \text{for all } (\mathbf{a}, m) \in \mathcal{A} \times \mathbb{R}_+ \quad (2)$$

for some $\tilde{\mathbf{b}} \in \mathbb{R}^{K+1}$ and $(K+1) \times (K+1)$ symmetric matrix $\tilde{\mathbf{B}}$. An attributes model (U, \mathbf{A}) is a *Quadratic attributes model* (Qua model) if U is a Qua function. The following proposition provides a convenient representation of the Qua function.

Proposition 1 *The attribute utility function $U : \mathcal{A} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is a Qua function if and only if for all $(\mathbf{a}, m) \in \mathcal{A} \times \mathbb{R}_+$*

$$U(\mathbf{a}, m) = \mathbf{b}'\mathbf{a} - \mathbf{a}'\mathbf{B}\mathbf{a} - d_1m - d_2m^2 - 2\tilde{\mathbf{d}}'\mathbf{a}m \quad (3)$$

where $\mathbf{b} \in \mathbb{R}^K$, $\tilde{\mathbf{d}} \in \mathbb{R}^K$, \mathbf{B} is a $K \times K$ symmetric positive definite matrix, $\tilde{\mathbf{d}}'\mathbf{a} \geq 0$ for all $\mathbf{a} \in \mathcal{A}$, $d_1 > 0$, and $d_2 \geq 0$.

For convenience of estimation we shall constrain the parameters of the Qua function (specifically, we set \mathbf{B} in (3) to be the identity matrix). We shall show that this restriction is made without loss in our ability to explain behavior. To do this we introduce the notion of equivalent attribute models.

Definition 2 *Two attribute models (U, \mathbf{A}) and $(\tilde{U}, \tilde{\mathbf{A}})$ are equivalent if*

$$U(\mathbf{A}\mathbf{q}, m) = \tilde{U}(\tilde{\mathbf{A}}\mathbf{q}, m), \quad \text{for all } (\mathbf{q}, m) \in \mathbb{N}_0^L \times \mathbb{R}_+ \quad (4)$$

The reason for referring to (U, \mathbf{A}) and $(\tilde{U}, \tilde{\mathbf{A}})$ as equivalent when (4) holds is that neither utility function nor attribute matrix are observable and so (U, \mathbf{A}) and $(\tilde{U}, \tilde{\mathbf{A}})$ have the exact same implications for observables.

If it were so desired one could strengthen the notion of “equivalent attribute models” to requiring that the matrices in (4) have the same dimensions and all results in this paper would still hold. To see that requiring the matrices in (4) to be of the same dimension is indeed a strengthening consider the following example. Let (U, \mathbf{A}) be an attribute model where \mathbf{A} is $K \times L$ and let $\tilde{\mathbf{A}}$ be $(K+1) \times L$ where \mathbf{A} and $\tilde{\mathbf{A}}$ are the same for the first K rows and row $K+1$ of $\tilde{\mathbf{A}}$ is a L vector of all 0s. Then, it is easy to see that (U, \mathbf{A}) and $(U, \tilde{\mathbf{A}})$ are equivalent but involve matrices of different dimensions.

Returning to our goal of restricting the parameters of the qua function we shall say that attribute utility function U is a *standard qua function* if $U : \mathbb{R}_+ \times \mathbb{R}^{K-1} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ can be expressed as

$$U(\mathbf{a}, m) = \mathbf{b}'\mathbf{a} - \mathbf{a}'\mathbf{a} - d_1 m - d_2 m^2 - 2d_3 a_1 m \quad (5)$$

where a_1 is the first element of \mathbf{a} , $\mathbf{b} \in \mathbb{R}^K$, d_1, d_2, d_3 are non-negative scalars and $d_1 > 0$. Note that a standard Qua function requires attribute 1 to take non-negative values. An attributes model (U, \mathbf{A}) is a *standard Qua model* if U is a standard Qua function.

Proposition 2 *A standard Qua model is a Qua model and every Qua model is equivalent to some standard Qua model.*

Proposition 2 establishes that there is no loss in our ability to explain behavior from assuming that \mathbf{B} in equation (2) is the identity matrix. For this reason we focus our analysis on standard Qua models.

Instead of choosing \mathbf{q} to maximize a deterministic attribute model, given by $U(\mathbf{A}\mathbf{q}, \mathbf{p}'\mathbf{q})$, we assume that the consumer maximizes the random utility function \tilde{U} defined by

$$\tilde{U}(\mathbf{A}\mathbf{q}, \mathbf{p}'\mathbf{q}) = U(\mathbf{A}\mathbf{q}, \mathbf{p}'\mathbf{q}) + \varepsilon_{\mathbf{q}} \quad (6)$$

where $\varepsilon_{\mathbf{q}}$ is a standard Gumbel random variable and $\varepsilon_{\mathbf{q}}, \varepsilon_{\mathbf{q}'}$ are independent when $\mathbf{q} \neq \mathbf{q}'$. Such a model (\tilde{U}, \mathbf{A}) is called a *random attributes model with base utility function* U .

Let (\tilde{U}, \mathbf{A}) be a random attributes model. The function $f : \mathbb{N}_0^L \times \mathbb{R}_{++}^L \rightarrow [0, 1]$ is the *stochastic choice function generated by* (\tilde{U}, \mathbf{A}) if

$$f(\mathbf{q}|\mathbf{p}) = P \left(\tilde{U}(\mathbf{A}\mathbf{q}, \mathbf{p}'\mathbf{q}) \geq \sup_{\tilde{\mathbf{q}} \in \mathbb{N}_0^L} \tilde{U}(\mathbf{A}\tilde{\mathbf{q}}, \mathbf{p}'\tilde{\mathbf{q}}) \right), \quad \text{for all } \mathbf{q} \in \mathbb{N}_0^L \quad (7)$$

Intuitively, $f(\mathbf{q}|\mathbf{p})$ is the probability with which the consumer purchases \mathbf{q} when prices are \mathbf{p} . $f(\cdot|\mathbf{p})$ is a probability mass function provided (i) the supremum in (7) is attained with probability 1 and (ii) the probability of ties are 0. The following proposition gives a convenient expression for f when (\tilde{U}, \mathbf{A}) is a random attributes model with standard Qua base utility function.

Proposition 3 *Let (\tilde{U}, \mathbf{A}) be a random attributes model with a standard Qua base utility function U and let f be the stochastic choice function generated by (\tilde{U}, \mathbf{A}) . Then,*

$$f(\mathbf{q}|\mathbf{p}) = \frac{\exp(U(\mathbf{A}\mathbf{q}, \mathbf{p}'\mathbf{q}))}{\sum_{\tilde{\mathbf{q}} \in \mathbb{N}_0^L} \exp(U(\mathbf{A}\tilde{\mathbf{q}}, \mathbf{p}'\tilde{\mathbf{q}}))} \quad (8)$$

Equation (8) should be understood as saying that (i) the probability mass function of consumption is described by (8) and (ii) the denominator in (8) is finite.

4 Comparative statics

5 Identification

We discuss the identification of U from the distribution of consumption and prices. To do this we introduce notation for the parameters of the Qua model. So, let $\boldsymbol{\theta} = [\mathbf{A}, \mathbf{b}, d_1, d_2, d_3]$ denote a list of the parameters of the standard Qua. Let Θ denote all such lists. Let $U(\cdot; \boldsymbol{\theta})$ denote the utility function defined by

$$U(\mathbf{q}, \mathbf{p}; \boldsymbol{\theta}) = U(\mathbf{A}\mathbf{q}, \mathbf{p}'\mathbf{q}) \quad (9)$$

where $U(\mathbf{A}\mathbf{q}, \mathbf{p}'\mathbf{q})$ is the standard Qua function with parameters \mathbf{b} , d_1 , d_2 , and d_3 . Further, let $f(\mathbf{q}|\mathbf{p}; \boldsymbol{\theta})$ denote the stochastic choice function generated by $U(\cdot; \boldsymbol{\theta})$. For the remainder of the paper we shall use $\mathbf{e}_k = [0, \dots, 0, 1, 0, \dots, 0]$ to denote K -vector whose entries are all 0 except for entry k which is a 1. Recall that a matrix V is *orthogonal* if $V'V = VV' = I$ where I denotes the identity matrix.

Proposition 4 *Let $\boldsymbol{\theta} = [\mathbf{A}, \mathbf{b}, d_1, d_2, d_3] \in \Theta$ and $\tilde{\boldsymbol{\theta}} = [\tilde{\mathbf{A}}, \tilde{\mathbf{b}}, \tilde{d}_1, \tilde{d}_2, \tilde{d}_3] \in \Theta$ where \mathbf{A} and $\tilde{\mathbf{A}}$ are rank K . Let $E \subseteq \mathbb{R}_{++}^L$ be a non-empty open set. The following are equivalent.*

1. $f(\mathbf{q}|\mathbf{p}; \boldsymbol{\theta}) = f(\mathbf{q}|\mathbf{p}; \tilde{\boldsymbol{\theta}})$, for all $\mathbf{q} \in \mathbb{N}_0^L$ and all $\mathbf{p} \in E$.
2. $U(\mathbf{q}, \mathbf{p}; \boldsymbol{\theta}) = U(\mathbf{q}, \mathbf{p}; \tilde{\boldsymbol{\theta}})$, for all $\mathbf{q} \in \mathbb{N}_0^L$ and all $\mathbf{p} \in E$.
3. $[\mathbf{A}'\mathbf{A}, \mathbf{A}'\mathbf{b}, d_1, d_2, d_3\mathbf{A}'\mathbf{e}_1] = [\tilde{\mathbf{A}}'\tilde{\mathbf{A}}, \tilde{\mathbf{A}}'\tilde{\mathbf{b}}, \tilde{d}_1, \tilde{d}_2, \tilde{d}_3\tilde{\mathbf{A}}'\mathbf{e}_1]$.
4. There is a $K \times K$ orthogonal matrix V so that

$$[\mathbf{A}, \mathbf{b}, d_1, d_2, d_3\mathbf{A}'\mathbf{e}_1] = [V\tilde{\mathbf{A}}, V\tilde{\mathbf{b}}, \tilde{d}_1, \tilde{d}_2, \tilde{d}_3\tilde{\mathbf{A}}'\mathbf{e}_1]$$

Define a function ψ by

$$\psi([\mathbf{A}, \mathbf{b}, d_1, d_2, d_3]) = [\mathbf{A}'\mathbf{A}, \mathbf{A}'\mathbf{b}, d_1, d_2, d_3\mathbf{A}'\mathbf{e}_1] \quad (10)$$

Proposition 4 says that two parameter vectors $\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}} \in \Theta$ are empirically indistinguishable if $\psi(\boldsymbol{\theta}) = \psi(\tilde{\boldsymbol{\theta}})$.

6 Estimation

Doing MLE is practically impossible because (8) is intractable. Instead we introduce ideas which still allow the parameters to be consistently estimated. We will let \mathbf{c} be a random L -vector whose probability mass function conditional on price vector \mathbf{p} is $f(\cdot|\mathbf{p})$ defined by (8). We will continue to use \mathbf{q} to denote a deterministic vector in \mathbb{N}_0^L .

The reason MLE is infeasible is because the denominator in (8) has too many terms. What we propose is to construct a random set S which contains a total of J (some finite number of) elements in \mathbb{N}_0^L . One of the elements in S will be the true bundle demanded while the rest will be “imposter” bundles. We may then proceed to estimate the parameters of our model by performing maximum likelihood estimation conditional on S . When properly implemented, this conditional MLE turns the size of the summands which must be evaluated from an infinite series (as in (8)) to the sum of only J numbers.

6.1 Signal Functions

Definition 3 For each $\mathbf{q} \in \mathbb{N}_0^L$ let $S(\mathbf{q})$ be a random subset of \mathbb{N}_0^L . S is a signal function if (i) each realization of $S(\mathbf{q})$ contains \mathbf{q} , (ii) for each $Q \subseteq \mathbb{N}_0^L$ and all $\mathbf{q}, \tilde{\mathbf{q}} \in Q$,

$$P(S(\mathbf{q}) = Q) = P(S(\tilde{\mathbf{q}}) = Q) \quad (11)$$

and (iii) for each $\mathbf{q} \in \mathbb{N}_0^L$ there is a countable collection $\tilde{\mathcal{Q}}$ consisting of subsets of \mathbb{N}_0^L where

$$P(S(\mathbf{q}) \in \tilde{\mathcal{Q}}) = 1$$

The support of $S(\mathbf{q})$ is the collection $\mathcal{Q}(\mathbf{q})$ defined by

$$\mathcal{Q}(\mathbf{q}) = \left\{ Q \subseteq \mathbb{N}_0^L : P(S(\mathbf{q}) = Q) > 0 \right\}$$

The support of S is the collection \mathcal{Q} defined by

$$\mathcal{Q} = \bigcup_{\mathbf{q} \in \mathbb{N}_0^L} \mathcal{Q}(\mathbf{q})$$

We shall specify a specific signal function S which can be computed via a pseudo-random number generator. Now, given such an signal function we shall be interested in estimating the parameters of our model via maximum likelihood estimation conditional on prices and conditional on the signals $S(\mathbf{q})$. There are two important things to note

about the signal function. First, property (i) ensures that any realization of $S(\mathbf{c})$ is a set which contains the true realization of \mathbf{c} as well as some imposter bundles. Property (ii) says that when trying to distinguish between whether the true bundle passed to S is \mathbf{q} or $\tilde{\mathbf{q}}$ one can infer nothing beyond what can be inferred from knowing whether \mathbf{q} and $\tilde{\mathbf{q}}$ are in $S(\mathbf{q})$ or not.

Proposition 5 *Let \mathbf{c} be a random vector on \mathbb{N}_0^L and $\boldsymbol{\rho}$ be random vector on \mathbb{R}_{++}^L . Let S be a signal function with support \mathcal{Q} and assume $S(\mathbf{c})$ and $\boldsymbol{\rho}$ are independent conditional on \mathbf{c} . Let $Q \subseteq \mathbb{N}_0^L$ and let $f(\mathbf{q}|Q, \mathbf{p})$ denote the probability that $\mathbf{c} = \mathbf{q}$ conditional on $S(\mathbf{c}) = Q$ and $\boldsymbol{\rho} = \mathbf{p}$. We have*

$$f(\mathbf{q}|Q, \mathbf{p}) = P(\mathbf{c} = \mathbf{q} | \mathbf{c} \in Q, \boldsymbol{\rho} = \mathbf{p}), \quad \text{for all } Q \in \mathcal{Q} \quad (12)$$

Consequently, if the probability mass function of \mathbf{c} conditional on $\boldsymbol{\rho}$, denoted $f(\mathbf{q}|\mathbf{p})$, satisfies (8), then, for all $Q \in \mathcal{Q}$,

$$f(\mathbf{q}|Q, \mathbf{p}) = \begin{cases} \frac{\exp(U(\mathbf{A}\mathbf{q}, \mathbf{p}'\mathbf{q}))}{\sum_{\tilde{\mathbf{q}} \in Q} \exp(U(\mathbf{A}\tilde{\mathbf{q}}, \mathbf{p}'\tilde{\mathbf{q}}))}, & \text{for } \mathbf{q} \in Q \\ 0, & \text{else.} \end{cases} \quad (13)$$

Our approach will be to estimate the parameters of our model via maximum likelihood conditional on prices and the output of the signal function. We shall need to place some restrictions on the signal function.

Definition 4 *A function $g : \mathbb{N}_0^L \rightarrow \mathbb{R}$ is grounded quadratic if*

$$g(\mathbf{q}) = \mathbf{b}'\mathbf{q} + \mathbf{q}'\mathbf{B}\mathbf{q}, \quad \text{for all } \mathbf{q} \in \mathbb{N}_0^L \quad (14)$$

for some $\mathbf{b} \in \mathbb{R}^L$ and some symmetric $L \times L$ matrix \mathbf{B} . The function g is degenerate if $g(\mathbf{q}) = 0$ for all $\mathbf{q} \in \mathbb{N}_0^L$.

Definition 5 *A signal function S with support set \mathcal{Q} is distinguishing if, for any non-degenerate grounded quadratic function g , there exists a set $Q \in \mathcal{Q}$ and $\mathbf{q}, \tilde{\mathbf{q}} \in Q$ satisfying*

$$g(\mathbf{q}) \neq g(\tilde{\mathbf{q}}) \quad (15)$$

Definition 6 *A signal function S with support \mathcal{Q} is small if*

1. *There exists a number $J \in \mathbb{N}$ so that $|Q| \leq J$ for each $Q \in \mathcal{Q}$.*

2. There exists a number $J' \in \mathbb{N}$ so that

$$\|\mathbf{q}\|_1 \leq J', \quad \text{for all } \mathbf{q} \in \mathbb{N}_0^L \text{ such that } \#\mathcal{Q}(\mathbf{q}) > 1$$

where $\mathcal{Q}(\mathbf{q})$ is the support of $S(\mathbf{q})$.

In Definition 6 item 1. limits the size of sets in the support of S . Item 2. says that all bundles in $\mathcal{Q}(\mathbf{q})$ cannot have many more units of consumption than are in \mathbf{q} itself.

Our consistency result relies on S being small and distinguishing.

6.2 Consistent Estimation

Suppose we have some data on N consumption bundles purchased and the prices of goods $(\mathbf{c}_1, \boldsymbol{\rho}_1), (\mathbf{c}_2, \boldsymbol{\rho}_2), \dots, (\mathbf{c}_N, \boldsymbol{\rho}_N)$. Let $I \in \mathbb{N}$. We also assume that we have the ability to generate NI signal functions $S_{1,1}, S_{1,2}, \dots, S_{1,I}, S_{2,1}, S_{2,2}, \dots, S_{N,I}$. For each n and i let $\mathcal{S}_{n,i} = S_{n,i}(\mathbf{c}_n)$. We shall refer to

$$\mathcal{O}_n = [\mathbf{c}_n, \boldsymbol{\rho}_n, \mathcal{S}_{n,1}, \mathcal{S}_{n,2}, \dots, \mathcal{S}_{n,I}]$$

as observation n . We place the following assumption on the data.

Assumption 1 *The observations $\mathcal{O}_1, \mathcal{O}_2, \dots$ are independent and identically distributed. Further, for each n ,*

$$\boldsymbol{\rho}_n, \mathcal{S}_{n,1}, \mathcal{S}_{n,2}, \dots, \mathcal{S}_{n,I} \tag{16}$$

are independent conditional on \mathbf{c}_n .

We shall assume that the price vectors satisfy the following assumption.

Assumption 2 *There is a non-empty open set $\mathcal{P} \subseteq \mathbb{R}_{++}^L$ so that for any non-empty open set $\mathcal{P}' \subseteq \mathcal{P}$ we have $P(\boldsymbol{\rho}_n \in \mathcal{P}') > 0$ for all n .*

Assumption 3 *The signal function $S_{n,i}$ is small and distinguishing for all n and i .*

Assumption 4 *There exists parameters $\boldsymbol{\theta}^* = [\mathbf{A}^*, \mathbf{b}^*, d_1^*, d_2^*, d_3^*] \in \Theta$ so that, for each n , the consumption bundle \mathbf{c}_n has conditional probability mass function $f(\mathbf{q}|\mathbf{p}; \boldsymbol{\theta}^*)$ defined by (8). That is,*

$$f(\mathbf{q}|\mathbf{p}; \boldsymbol{\theta}^*) = P(\mathbf{c}_n = \mathbf{q} | \boldsymbol{\rho}_n = \mathbf{p}) \tag{17}$$

Further, \mathbf{A}^ has rank K .*

Assumption 5 For each n , the following moments exist and are finite

$$\mathbf{E}[\mathbf{c}_n \mathbf{c}'_n], \quad \mathbf{E}[\boldsymbol{\rho}_n \boldsymbol{\rho}'_n], \quad \mathbf{E}[\mathbf{c}'_n \mathbf{c}_n \boldsymbol{\rho}_n], \quad \text{and} \quad \mathbf{E}[\mathbf{c}'_n \mathbf{c}_n \boldsymbol{\rho}'_n \boldsymbol{\rho}_n] \quad (18)$$

We shall estimate $\boldsymbol{\theta}$ by maximizing the following log likelihood function.

$$\mathcal{L}_N(\boldsymbol{\theta}) = \frac{1}{NI} \sum_{n=1}^N \sum_{i=1}^I \ln \left(f(\mathbf{c}_n | \mathcal{S}_{i,n}, \boldsymbol{\rho}_n; \boldsymbol{\theta}) \right) \quad (19)$$

where $f(\mathbf{c}_n | \mathcal{S}_{i,n}, \boldsymbol{\rho}_n; \boldsymbol{\theta})$ is defined by (13). Let $\hat{\boldsymbol{\theta}}_N \in \Theta$ satisfy

$$\hat{\boldsymbol{\theta}}_N \in \underset{\boldsymbol{\theta} \in \Theta}{\arg \max} \mathcal{L}_N(\boldsymbol{\theta}) \quad (20)$$

when this argmax exists.

Theorem 1 Suppose assumptions 1-5 hold. Let $\hat{\boldsymbol{\theta}}_N$ satisfy (20) when the argmax is non-empty. Let ψ be defined by (10). Then,

$$\psi(\hat{\boldsymbol{\theta}}_N) \xrightarrow{a.s.} \psi(\boldsymbol{\theta}^*) \quad (21)$$

6.3 Asymptotic Distribution

We refine Proposition 4 as follows.

Proposition 6 Let $\boldsymbol{\theta} = [\mathbf{A}, \mathbf{b}, d_1, d_2, d_3] \in \Theta$ and $\tilde{\boldsymbol{\theta}} = [\tilde{\mathbf{A}}, \tilde{\mathbf{b}}, \tilde{d}_1, \tilde{d}_2, \tilde{d}_3] \in \Theta$ where \mathbf{A} and $\tilde{\mathbf{A}}$ are rank K and $d_3 \neq 0$. Let $E \subseteq \mathbb{R}_{++}^L$ be a non-empty open set. Then, items 1.-4. in Proposition 4 are true if and only if either of the following are true

$$5. [\mathbf{A}' \mathbf{A}, \mathbf{A}' \mathbf{b}, \mathbf{A}' \mathbf{e}_1, d_1, d_2, d_3] = [\tilde{\mathbf{A}}' \tilde{\mathbf{A}}, \tilde{\mathbf{A}}' \tilde{\mathbf{b}}, \tilde{\mathbf{A}}' \mathbf{e}_1, \tilde{d}_1, \tilde{d}_2, \tilde{d}_3].$$

6. There is a $K \times K$ orthogonal matrix V so that $V \mathbf{e}_1 = \mathbf{e}_1$ and

$$[\mathbf{A}, \mathbf{b}, d_1, d_2, d_3] = [V \tilde{\mathbf{A}}, V \tilde{\mathbf{b}}, \tilde{d}_1, \tilde{d}_2, \tilde{d}_3]$$

Definition 7 A parameter vector $\boldsymbol{\theta} = [\mathbf{A}, \mathbf{b}, d_1, d_2, d_3] \in \Theta$ is in T_1 -form if

$$[\mathbf{e}_1 \ \mathbf{A}] = [B \ \tilde{B}] \quad (22)$$

where B is a $K \times K$ lower triangle matrix and \tilde{B} is a $K \times (L - K + 1)$ matrix.

A parameter vector $\boldsymbol{\theta} = [\mathbf{A}, \mathbf{b}, d_1, d_2, d_3] \in \Theta$ is in T_2 -form if

$$\mathbf{A} = [B \ \tilde{B}] \quad (23)$$

where B is a $K \times K$ lower triangle matrix and \tilde{B} is a $K \times (L - K)$ matrix.

The significance of T_1 and T_2 forms is presented in the following Proposition.

Proposition 7 *Let ψ be defined by (10) and let $\boldsymbol{\theta} = [\mathbf{A}, \mathbf{b}, d_1, d_2, d_3] \in \Theta$. Suppose that each set of K distinct columns of $[\mathbf{e}_1 \ \mathbf{A}]$ are linearly independent. The following holds.*

1. *If $d_3 > 0$ then there exists a unique $\boldsymbol{\theta}^* \in \Theta$ in T_1 -form so that $\psi(\boldsymbol{\theta}) = \psi(\boldsymbol{\theta}^*)$.*
2. *If $d_3 = 0$ then there exists a unique $\boldsymbol{\theta}^* \in \Theta$ in T_2 -form so that $\psi(\boldsymbol{\theta}) = \psi(\boldsymbol{\theta}^*)$.*

[Proof of Proposition 7.] First, suppose $d_3 > 0$. Let \mathbf{A}_{K-1} denote the $K \times K - 1$ matrix whose columns are the first $K - 1$ columns of \mathbf{A} . Let $C = [\mathbf{e}_1, \mathbf{A}_{K-1}]$. By assumption we know that C is full-rank. Thus, from Lemma 1 there is a unique orthogonal matrix V and a unique upper triangle matrix B whose diagonals are positive numbers so that $C = V'B$.

Define $\mathbf{A}^* = V\mathbf{A}$ and $\mathbf{b}^* = V\mathbf{b}$. We claim that $V\mathbf{e}_1 = \mathbf{e}_1$. Let B_1 denote the first column of B . As B is a upper triangle matrix with positive numbers on the diagonal it must be the case that $B_1 = k\mathbf{e}_1$ where $k > 0$. Next, we know $VC = B$ (using the fact that V is orthogonal) which implies $V\mathbf{e}_1 = k\mathbf{e}_1$. Now,

$$1 = \mathbf{e}_1' \mathbf{e}_1 = \mathbf{e}_1' V' V \mathbf{e}_1 = k^2 \mathbf{e}_1' \mathbf{e}_1 = k^2$$

which shows that indeed $k = 1$ (recall that $k > 0$). So, we have shown that $V\mathbf{e}_1 = \mathbf{e}_1$.

Let $\boldsymbol{\theta}^* = [\mathbf{A}^*, \mathbf{b}^*, d_1, d_2, d_3]$. Clearly, $\boldsymbol{\theta}^* \in \Theta$, \mathbf{A}^* is rank K , and $\boldsymbol{\theta}^*$ is in T_1 -form. Thus, using the fact that $V\mathbf{e}_1 = \mathbf{e}_1$, the definition of $\boldsymbol{\theta}^*$, and Proposition 6 (specifically, we use the part of Proposition 6 which says that item 6. implies 4.) we see that $\psi(\boldsymbol{\theta}) = \psi(\boldsymbol{\theta}^*)$.

Next, suppose $\tilde{\boldsymbol{\theta}} = [\tilde{\mathbf{A}}, \tilde{\mathbf{b}}, \tilde{d}_1, \tilde{d}_2, \tilde{d}_3] \in \Theta$ is in T_1 -form and $\psi(\boldsymbol{\theta}) = \psi(\tilde{\boldsymbol{\theta}})$. We show that $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}^*$. As $\psi(\tilde{\boldsymbol{\theta}}) = \psi(\boldsymbol{\theta})$ Proposition 6 shows that there is an orthogonal matrix \tilde{V} so that $\tilde{V}\mathbf{e}_1 = \mathbf{e}_1$ and

$$[\tilde{V}\mathbf{A}, \tilde{V}\mathbf{b}, d_1, d_2, d_3] = [\tilde{\mathbf{A}}, \tilde{\mathbf{b}}, \tilde{d}_1, \tilde{d}_2, \tilde{d}_3] \quad (24)$$

Let $\tilde{\mathbf{A}}_{K-1}$ denote the first $K - 1$ columns of $\tilde{\mathbf{A}}$. As $\tilde{\boldsymbol{\theta}}$ is in T_1 -form we know that $[\mathbf{e}_1 \ \tilde{\mathbf{A}}]$ is an upper triangle matrix. Thus, (note that $\tilde{V}\mathbf{e}_1 = \mathbf{e}_1$ implies $\tilde{V}'\mathbf{e}_1 = \mathbf{e}_1$) we see

$$[\mathbf{e}_1 \ \mathbf{A}_{K-1}] = C = V'B = \tilde{V}'[\mathbf{e}_1 \ \tilde{\mathbf{A}}_0] \quad (25)$$

But, Lemma 1 guarantees that the factorization in (25) is unique and so $V = \tilde{V}$. Thus, from (24) we see that $\boldsymbol{\theta}^* = \tilde{\boldsymbol{\theta}}$.

Next, consider the case where $d_3 = 0$. Let \mathbf{A}_K denote the $K \times K$ matrix whose columns are the first K columns of \mathbf{A} . Let $C = \mathbf{A}_K$. The proof is now a simpler version of the $d_3 > 0$ case and so we omit it.

Lemma 1 *Suppose C is some $K \times K$ matrix. There exists $K \times K$ orthogonal matrix V and $K \times K$ upper triangle matrix B where the diagonals of B are positive numbers so that*

$$C = V'B \quad (26)$$

If C is full-rank then the factorization in (26) is unique.

The existence of the factorization in (26) is well-known (it's called QR factorization). Further, its uniqueness when C is full-rank is also well-known.

Lemma 2 *Let $J \in \mathbb{N}$ where $K \leq J$ and let B_n be a sequence of $K \times J$ triangle matrices. Let $B_{n,i,j}$ denote row i column j of matrix B_n . Suppose that $B_{n,i,i} > 0$ for all n and all $i \in \{1, \dots, K\}$. Further, suppose that there exists a $K \times J$ triangle matrix B where (i) $B_{i,i} > 0$ for all $i \in \{1, \dots, K\}$ and (ii) $B'_n B_n \rightarrow B'B$. Then, $B_n \rightarrow B$.*

We shall show first that

$$B_{n,1,j} \rightarrow B_{1,j} \quad \text{for all } j \in \{1, \dots, J\} \quad (27)$$

After that we show that if, for some $\bar{i} \in \{1, \dots, K-1\}$,

$$B_{n,i,j} \rightarrow B_{i,j}, \quad \text{for all } i \in \{1, 2, \dots, \bar{i}\} \text{ and all } j \in \{1, \dots, J\} \quad (28)$$

then also

$$B_{n,\bar{i}+1,j} \rightarrow B_{\bar{i}+1,j}, \quad \text{for all } j \in \{1, \dots, J\} \quad (29)$$

This will establish the result as a consequence of mathematical induction.

So, let us first show that $B_{n,1,j} \rightarrow B_{1,j}$ for all $j \in \{1, \dots, J\}$. To proceed, note that $B'_n B_n \rightarrow B'B$ implies that

$$\sum_{j=1}^K B_{n,j,k} B_{n,j,i} \longrightarrow \sum_{j=1}^K B_{j,k} B_{j,i}, \quad \text{for all } k, i \in \{1, 2, \dots, J\} \quad (30)$$

Evaluating (30) with $i = k = 1$ and using the fact that B_n and B are triangle matrices gives

$$B_{n,1,1}^2 \rightarrow B_{1,1}^2$$

Using the fact that $B_{n,1,1} > 0$ we see $B_{n,1,1} \rightarrow B_{1,1}$. Now, evaluate (30) with $k = 1$, $i \in \{2, \dots, J\}$ and using the fact that B_n and B are triangle matrices gives

$$B_{n,1,1}B_{n,1,i} \rightarrow B_{1,1}B_{1,i}$$

Now, given that we have shown $B_{n,1,1} \rightarrow B_{1,1} > 0$ it must be the case that $B_{n,1,i} \rightarrow B_{1,i}$. Thus, we have shown (27). Next, suppose (28) holds for some $\bar{i} \in \{1, \dots, K-1\}$. Evaluating (30) with $k = i = \bar{i} + 1$ and using the fact that B_n and B are triangle matrices we see

$$\sum_{j=1}^{\bar{i}+1} B_{n,j,\bar{i}+1}^2 \rightarrow \sum_{j=1}^{\bar{i}+1} B_{j,\bar{i}+1}^2$$

which shows $B_{n,\bar{i}+1,\bar{i}+1} \rightarrow B_{\bar{i}+1,\bar{i}+1}$ after appealing to (28) and the fact that $B_{n,\bar{i}+1,\bar{i}+1} > 0$. Now, evaluate (30) with $k = \bar{i} + 1$, $i \in \{\bar{i} + 2, \dots, J\}$ and using the fact that B_n and B are triangle matrices gives

$$\sum_{j=1}^{\bar{i}+1} B_{n,j,\bar{i}+1}B_{n,j,i} \rightarrow \sum_{j=1}^{\bar{i}+1} B_{j,\bar{i}+1}B_{j,i}$$

Now, given that we have shown $B_{n,\bar{i}+1,\bar{i}+1} \rightarrow B_{\bar{i}+1,\bar{i}+1} > 0$ and we have assumed (28) it must be the case that $B_{n,\bar{i}+1,i} \rightarrow B_{\bar{i}+1,i}$. Thus, (29) is shown and the proof is complete.

Let $\mathcal{T}_1 : \Theta \rightarrow \Theta$ satisfy the following. When, $\boldsymbol{\theta} = [\mathbf{A}, \mathbf{b}, d_1, d_2, d_3]$ is such that each set of K distinct columns of $[\mathbf{e}_1 \ \mathbf{A}]$ are linearly independent then $\mathcal{T}_1(\boldsymbol{\theta})$ returns the unique parameter vector guaranteed to exist by Proposition 7 which satisfies $\psi(\mathcal{T}_1(\boldsymbol{\theta})) = \psi(\boldsymbol{\theta})$. If it is not the case that each set of K distinct columns of $[\mathbf{e}_1 \ \mathbf{A}]$ are linearly independent then let $\mathcal{T}_1(\boldsymbol{\theta}) = \boldsymbol{\theta}$.

Lemma 3 *Let $\boldsymbol{\theta}_n$ be a sequence in Θ , let $\boldsymbol{\theta}^* = [\mathbf{A}^*, \mathbf{b}^*, d_1^*, d_2^*, d_3^*] \in \Theta$ where $d_3 > 0$, and suppose $\psi(\boldsymbol{\theta}_n) \rightarrow \psi(\boldsymbol{\theta}^*)$. Then, $\mathcal{T}_1(\boldsymbol{\theta}_n) \rightarrow \mathcal{T}_1(\boldsymbol{\theta}^*)$.*

Let $\boldsymbol{\theta}_n = [\mathbf{A}_n, \mathbf{b}_n, d_{1,n}, d_{2,n}, d_{3,n}]$. Let $\mathbf{A}_{n,K-1}$ denote the first $K-1$ columns of \mathbf{A}_n . Let $C_n = [\mathbf{e}_1, \mathbf{A}_{n,K-1}]$. Let $C_n = V_n' B_n$ denote the factorization given by Lemma 1. From Lemma 7 we know

$$\mathcal{T}_1(\boldsymbol{\theta}_n) = [V_n \mathbf{A}_n, V_n \mathbf{b}_n, d_1, d_2, d_3]$$

7 Simulation study

8 Empirical application

We apply our model to scanner data made available by DunnHumby, a customer data science company. The dataset, named ‘The Complete Journey’ by DunnHumby, describes approximately 2.6m purchases by 2,500 households in the US over a two-year period. The households are described as frequent shoppers at a retailer.

We exclude from our study the final 600,000 purchases for a future out-of-sample validation exercise. We study the remaining 2m purchases and find that they were made on 216,251 separate check-out occasions.

We view each such check-out at the till as recording a single consumption bundle. On average, bundles cost \$28.47 and contained 8 items. The most expensive bundle cost \$961.49; and the biggest bundle contained 141 items.

For each good, we observe an SKU code, store sub-department information, and anonymous manufacturer and brand codes. We also observe a product description such as for example `CONDIMENTS/SAUCES|STEAK & WORCHESTER SAUCE`, as well as product size in, for example Ounces or Gallons.

When we distinguish products by their descriptions, manufacturers, brands and departments, as well as by their sizes bucketed into 20% bins, we find 26,633 goods. On average, each of these goods is mentioned 68 times in our data. However, we observe a long tail of rarely-purchased goods. Many goods are seen, for example, fewer than 25 times. We track the 7,500 goods which are purchased more than 25 times. The remaining goods relate to 6% of purchases: in calculating bundle utility we account for their expense.

Some of the 2,500 households participate considerably more than others. We estimate household-specific preference parameters, $(b_u, d_{1u}, d_{2u}, d_{3u})$, for the 1,250 most active households, u . DunnHumby provides self-reported demographic information for 701 of these households. We pool all other households, u^* , such that $(b_{u^*}, d_{1u^*}, d_{2u^*}, d_{3u^*})$ is constant across them.

We use the code at www.github.com/jeremy-large/RUBE to estimate our model. This uses a stochastic gradient descent method. Stochasticity arises both from signal set generation, and from the selective batching of our dataset into groups of 1024 consump-

tion bundles. These batches are not independently drawn from our dataset, but rather are simply accessed sequentially. When the whole dataset has been accessed once, we consider that an *epoch* is completed, and we begin again to pass through our data, generating fresh signal sets for each revisited batch. We use signal sets of size 100. Section 7 simulates this arrangement and discusses its behaviour in more detail.

To prepare our estimation, we broadly seed all our unconstrained real parameters with independent random normals of variance $1/K$, and all positive parameters with exponentially-distributed random variables of mean $1/K$.

We hold-out a small sliver, two per cent, of our data for validation testing. Each bundle in this validation dataset is equipped with two fixed signal sets. During estimation, we consider an ancillary classification exercise where after each epoch, we identify the bundle in the signal set with the greatest estimated Base Utility; and we count how often this is the true bundle. This may be thought of as Bayes classification using interim, plug-in, parameter values.

When all parameters of the model are set to zero, this classification exercise yields a hold-out accuracy of 1%. With our randomly-seeded parameters, this accuracy stands at 7%. But, towards the end after estimation, accuracy plateaus at around 35%. Over a third of the time, therefore, our fitted classifier is able to identify the truth from among a set of 100 possible bundles.

We set $K = 12$, so we estimate 12 latent characteristics, or embedding coordinates. The fitted \hat{A} can be viewed as L observations of a multivariate random variable, whose empirical sample covariance is $\hat{A}\hat{A}'$. Applying principal component analysis to \hat{A} rotates the embedding coordinates such that its empirical covariance is diagonal. The contribution of the smallest variance in this diagonalization would appear to be fairly low, measuring only 2.2% of the sum of the variances. Although this may be a reassuring indication (that there is not much work left, for our final degree of freedom to do), further work is needed to understand how best to select the tuning parameter, K , for various purposes.

As a simple sanity-check of our fit, Figure 2 compares our estimates of d_{1u} , which captures the disutility of expenditure, with self-reported household income for the 701 households where these overlap.

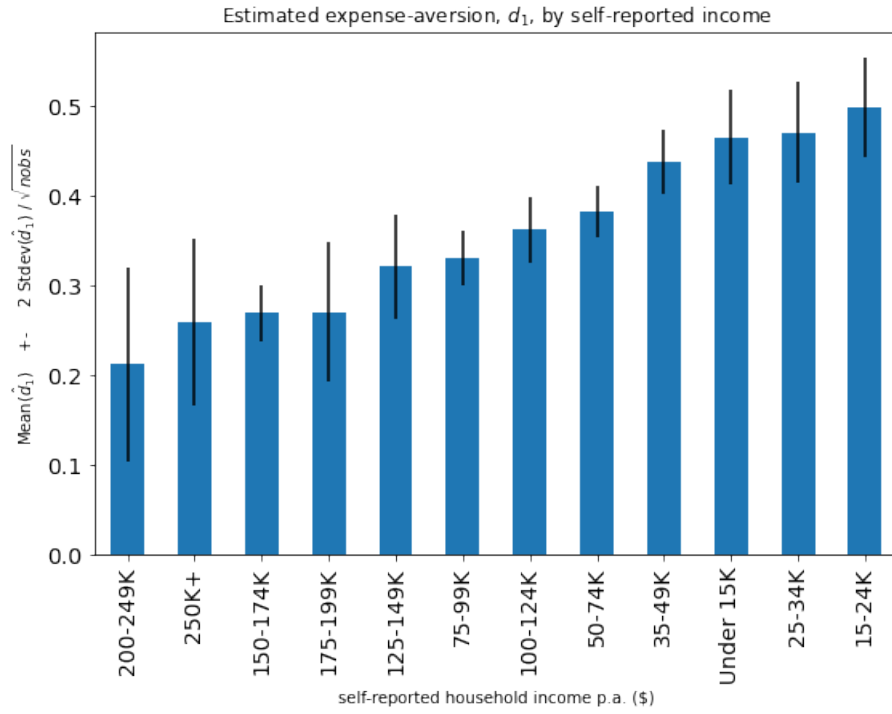


Figure 2: We estimate household-specific preference parameters, $(b_u, d_{1u}, d_{2u}, d_{3u})$, for the 1,250 most active households, u . DunnHumby provides self-reported demographic information for 701 of these households. These households are partitioned by income band, and the mean and standard deviation of d_{1u} in each band are displayed in this bar chart.

9 Conclusion

Well well.

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10 Appendix

[Proof of Proposition 1.] Suppose U satisfies (2). Decompose $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{B}}$ as

$$\tilde{\mathbf{b}} = \begin{bmatrix} \mathbf{b} \\ -d_1 \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & \tilde{\mathbf{d}} \\ \tilde{\mathbf{d}}' & d_2 \end{bmatrix} \quad (31)$$

where $\mathbf{b} \in \mathbb{R}^K$, $d_1 \in \mathbb{R}$, \mathbf{B} is a $K \times K$ symmetric matrix (as $\tilde{\mathbf{B}}$ is symmetric), $\tilde{\mathbf{d}} \in \mathbb{R}^K$, $d_2 \in \mathbb{R}$. Plugging these definitions into (2) yields (3). On the other hand, if we start by assuming that U satisfies (3) with symmetric \mathbf{B} then we may define $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{B}}$ via (31) to show that U can be expressed as (2) with symmetric $\tilde{\mathbf{B}}$.

So, we must show that U , which satisfies (3) with symmetric \mathbf{B} , is well-behaved if and only if \mathbf{B} is positive definite and

$$\tilde{\mathbf{d}}' \mathbf{a} \geq 0 \quad \text{for all } \mathbf{a} \in \mathcal{A}, \quad d_1 > 0, \quad \text{and} \quad d_2 \geq 0. \quad (32)$$

First, note that U is strictly concave in \mathbf{a} if and only if \mathbf{B} is positive definite. Next, note that if (32) holds then clearly $\partial U/\partial m < 0$. So we must show that if U is strictly decreasing in m then (32) holds.

The fact that $\mathbf{0} \in \mathcal{A}$ ensures that if $\partial U/\partial m < 0$ then $d_1 > 0$ and $d_2 \geq 0$. Next, suppose that there exists some $\tilde{\mathbf{a}} \in \mathcal{A}$ so that $\tilde{\mathbf{d}}'\tilde{\mathbf{a}} < 0$. As \mathcal{A} is a cone we know that $k\tilde{\mathbf{a}} \in \mathcal{A}$ for all $k \geq 0$. Thus, we may find some large enough k so that $\partial U(k\tilde{\mathbf{a}}, m)/\partial m > 0$ (for some m close to 0). This establishes that $\tilde{\mathbf{d}}'\mathbf{a} \geq 0$ for all $\mathbf{a} \in \mathcal{A}$ when $\partial U/\partial m < 0$.

[Proof of Proposition 2.] First we prove that a standard Qua model is a Qua model. So, let (U, A) be a standard Qua (i.e. U satisfies (5) where d_1, d_2, d_3 are non-negative and $d_1 > 0$). U clearly satisfies (3) where \mathbf{B} is the $K \times K$ identity matrix and $\tilde{\mathbf{d}} = [d_3, 0, 0, \dots]$ (that is, $\tilde{\mathbf{d}}$ is a vector with d_3 on the first element and 0 on the remaining elements). So, by Proposition 1 we see that a standard Qua model is a Qua model.

Next, we show that every Qua model is equivalent to some standard Qua model. Let (U, \mathbf{A}) be a Qua model. By Proposition 1 we know that U satisfies (3) for some positive definite matrix \mathbf{B} and also equation (32) hold.

As \mathbf{B} is positive definite we know that both $\mathbf{B}^{1/2}$ and $\mathbf{B}^{-1/2}$ exist. Let $\mathbf{v}_1 \in \mathbb{R}^K$ be defined by

$$\mathbf{v}_1 = \frac{\mathbf{B}^{-1/2}\tilde{\mathbf{d}}}{\|\mathbf{B}^{-1/2}\tilde{\mathbf{d}}\|}$$

Next, let $\mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_K$ be K -vectors so that the $K \times K$ matrix $\mathbf{V} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_K]$ is orthogonal (i.e. $\mathbf{V}'\mathbf{V} = \mathbf{V}\mathbf{V}' = \mathbf{I}$ where \mathbf{I} is the $K \times K$ identity matrix).

Define $d_3 = \|\mathbf{B}^{-1/2}\tilde{\mathbf{d}}\| \geq 0$. From the definition of \mathbf{V} we have

$$\tilde{\mathbf{d}}'\mathbf{B}^{-1/2}\mathbf{V}\mathbf{a} = d_3 a_1, \quad \text{for all } \mathbf{a} = [a_1, a_2, \dots, a_K] \in \mathbb{R}^K \quad (33)$$

Define an attribute utility function $\tilde{U} : \mathbb{R}_+ \times \mathbb{R}^{K-1} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$\tilde{U}(\mathbf{a}, m) = \mathbf{b}'\mathbf{B}^{-1/2}\mathbf{V}\mathbf{a} - \mathbf{a}'\mathbf{a} - d_1 m - d_2 m^2 - 2d_3 a_1 m \quad (34)$$

Define an attribute matrix $\tilde{\mathbf{A}}$ by

$$\tilde{\mathbf{A}} = \mathbf{V}'\mathbf{B}^{1/2}\mathbf{A}$$

Given $d_3 \geq 0$ it is clear that $(\tilde{U}, \tilde{\mathbf{A}})$ is a standard Qua model. We claim that (U, \mathbf{A}) and

$(\tilde{U}, \tilde{\mathbf{A}})$ are equivalent. Letting $\mathbf{e}_1 = [1, 0, 0, \dots, 0] \in \mathbb{R}^K$, $\mathbf{q} \in \mathbb{N}_0^L$, and using (33) we see

$$\begin{aligned}
\tilde{U}(\tilde{\mathbf{A}}\mathbf{q}, m) &= \mathbf{b}'\mathbf{B}^{-1/2}\mathbf{V}\tilde{\mathbf{A}}\mathbf{q} - \mathbf{q}'\tilde{\mathbf{A}}'\tilde{\mathbf{A}}\mathbf{q} - d_1m - d_2m^2 - 2\tilde{\mathbf{d}}'\mathbf{B}^{-1/2}\mathbf{V}\tilde{\mathbf{A}}\mathbf{q}m \\
&= \mathbf{b}'\mathbf{B}^{-1/2}\mathbf{V}\mathbf{V}'\mathbf{B}^{1/2}\mathbf{A}\mathbf{q} - \mathbf{q}'\tilde{\mathbf{A}}'\tilde{\mathbf{A}}\mathbf{q} - d_1m - d_2m^2 - 2\tilde{\mathbf{d}}'\mathbf{B}^{-1/2}\mathbf{V}\tilde{\mathbf{A}}\mathbf{q}m \\
&= \mathbf{b}'\mathbf{A}\mathbf{q} - \mathbf{q}'\tilde{\mathbf{A}}'\tilde{\mathbf{A}}\mathbf{q} - d_1m - d_2m^2 - 2\tilde{\mathbf{d}}'\mathbf{B}^{-1/2}\mathbf{V}\tilde{\mathbf{A}}\mathbf{q}m \\
&= \mathbf{b}'\mathbf{A}\mathbf{q} - \mathbf{q}'\mathbf{A}'\mathbf{B}^{1/2}\mathbf{V}\mathbf{V}'\mathbf{B}^{1/2}\mathbf{A}\mathbf{q} - d_1m - d_2m^2 - 2\tilde{\mathbf{d}}'\mathbf{B}^{-1/2}\mathbf{V}\tilde{\mathbf{A}}\mathbf{q}m \\
&= \mathbf{b}'\mathbf{A}\mathbf{q} - \mathbf{q}'\mathbf{A}'\mathbf{B}\mathbf{A}\mathbf{q} - d_1m - d_2m^2 - 2\tilde{\mathbf{d}}'\mathbf{B}^{-1/2}\mathbf{V}\tilde{\mathbf{A}}\mathbf{q}m \\
&= \mathbf{b}'\mathbf{A}\mathbf{q} - \mathbf{q}'\mathbf{A}'\mathbf{B}\mathbf{A}\mathbf{q} - d_1m - d_2m^2 - 2\tilde{\mathbf{d}}'\mathbf{B}^{-1/2}\mathbf{V}\mathbf{V}'\mathbf{B}^{1/2}\mathbf{A}\mathbf{q}m \\
&= \mathbf{b}'\mathbf{A}\mathbf{q} - \mathbf{q}'\mathbf{A}'\mathbf{B}\mathbf{A}\mathbf{q} - d_1m - d_2m^2 - 2\tilde{\mathbf{d}}'\mathbf{A}\mathbf{q}m \\
&= U(\mathbf{A}\mathbf{q}, m)
\end{aligned}$$

which shows that (U, \mathbf{A}) and $(\tilde{U}, \tilde{\mathbf{A}})$ are equivalent.

10.1 Proof of Proposition 3

The proof of Proposition 3 relies on several lemmas. The first two lemmas are well-known and so we state them without proof. They concern the Gumbel and logistic distributions.

For $b \in \mathbb{R}$ write $X \sim \text{Gumbel}(b)$ if X is a random variable with CDF

$$F_X(x) = \exp(-\exp(b-x))$$

The following lemma shows that the max of two independently distributed Gumbel random variables is itself a Gumbel random variable.

Lemma 4 *Suppose X and Y are independent and $X \sim \text{Gumbel}(b_X)$ and $Y \sim \text{Gumbel}(b_Y)$. Then*

$$\max(X, Y) \sim \text{Gumbel}\left(\ln(\exp(b_X) + \exp(b_Y))\right)$$

Next, for $b \in \mathbb{R}$ write $X \sim \text{Logistic}(b)$ if X is a random variable with CDF

$$F_X(x) = \frac{1}{1 + \exp(b-x)}$$

The next lemma claims that the difference of two independent Gumbel random variables is a logistic random variable.

Lemma 5 *Let X and Y be independent and $X \sim \text{Gumbel}(b_X)$ and $Y \sim \text{Gumbel}(b_Y)$. Then*

$$Y - X \sim \text{Logistic}(b_Y - b_X)$$

The next lemma establishes that a certain series is finite.

Lemma 6 Suppose (U, \mathbf{A}) is a standard Qua model. Then, for all $\mathbf{p} \in \mathbb{R}_{++}^L$,

$$\sum_{\mathbf{q} \in \mathbb{N}_0^L} \exp \left(U(\mathbf{A}\mathbf{q}, \mathbf{p}'\mathbf{q}) \right) < \infty \quad (35)$$

The proof of Lemma 6 appears in subsection 10.1.1. We can now prove Proposition 3. [Proof of Proposition 3.] Fix some consumption bundle $\mathbf{q} \in \mathbb{N}_0^L$ and some price vector $\mathbf{p} \in \mathbb{R}_{++}^L$. Let $\{\tilde{\mathbf{q}}^i\}_{i \in \mathbb{N}}$ be a non-repeating enumeration of the elements in $\mathbb{N}_0^L \setminus \{\mathbf{q}\}$. For $I \in \mathbb{N} \cup \{\infty\}$ let v_I be defined by

$$v_I = \ln \left(\sum_{i=1}^I \exp \left(U(\mathbf{A}\tilde{\mathbf{q}}^i, \mathbf{p}'\tilde{\mathbf{q}}^i) \right) \right) \quad (36)$$

and let \tilde{v}_I be defined by

$$\tilde{v}_I = \sup_{i \leq I} \tilde{U}(\mathbf{A}\tilde{\mathbf{q}}^i, \mathbf{p}'\tilde{\mathbf{q}}^i) \quad (37)$$

Repeated application of Lemma 4 gives

$$\tilde{v}_I \sim \text{Gumbel}(v_I), \quad \text{for } I \in \mathbb{N} \quad (38)$$

By Theorem 13.4(i) in billingsley86 \tilde{v}_∞ is a random variable taking values in the extended real line. Let $a \in \mathbb{R}$. For $I \in \mathbb{N} \cup \{\infty\}$ let E_I be the event that \tilde{v}_I is less than or equal to a . We see that $E_I \downarrow E_\infty$ and using the continuity property of probability measures (see billingsley86 Theorem 2.1) we see

$$P(E_\infty) = \lim_{I \rightarrow \infty} P(E_I) = \lim_{I \rightarrow \infty} \exp(-\exp(v_I - a)) = \exp(-\exp(v_\infty - a))$$

From Lemma 6 we know that $v_\infty < \infty$ and thus we see that

$$\tilde{v}_\infty \sim \text{Gumbel}(v_\infty) \quad (39)$$

Applying Lemma 5 we see

$$\begin{aligned} f(\mathbf{q}; \mathbf{p}) &= P \left(\tilde{U}(\mathbf{q}, \mathbf{p}'\mathbf{q}) \geq \max_{\tilde{\mathbf{q}} \in \mathbb{N}_0^L} \tilde{U}(\tilde{\mathbf{q}}, \mathbf{p}'\tilde{\mathbf{q}}) \right) \\ &= P \left(\tilde{U}(\mathbf{q}, \mathbf{p}'\mathbf{q}) \geq \tilde{v}_\infty \right) \\ &= P \left(0 \geq \tilde{v}_\infty - \tilde{U}(\mathbf{q}, \mathbf{p}'\mathbf{q}) \right) \\ &= \frac{1}{1 + \exp(v_\infty - U(\mathbf{A}\mathbf{q}, \mathbf{p}'\mathbf{q}))} \\ &= \frac{\exp(U(\mathbf{A}\mathbf{q}, \mathbf{p}'\mathbf{q}))}{\exp(U(\mathbf{A}\mathbf{q}, \mathbf{p}'\mathbf{q})) + \exp(v_\infty)} \\ &= \frac{\exp(U(\mathbf{A}\mathbf{q}, \mathbf{p}'\mathbf{q}))}{\sum_{\tilde{\mathbf{q}} \in \mathbb{N}_0^L} \exp(U(\mathbf{A}\tilde{\mathbf{q}}, \mathbf{p}'\tilde{\mathbf{q}}))} \end{aligned}$$

which completes the proof.

10.1.1 Proof of Lemma 6

The proof of Lemma 6 rests on a lemma which provides an upper bound on the number of elements in a certain set. So, for some set B let $|B|$ denote the number of elements in B . For some vector $\mathbf{v} = [v_1, \dots, v_L] \in \mathbb{R}^L$ let $\|\mathbf{v}\|_1 = \sum_{\ell=1}^L |v_\ell|$. For $Q \in \mathbb{N}_0$, define the set $B(Q)$ by

$$B(Q) = \left\{ \mathbf{q} \in \mathbb{N}_0^L : \|\mathbf{q}\|_1 = Q \right\} \quad (40)$$

The following lemma gives a upper bound on $|B(Q)|$.

Lemma 7 For $Q \in \mathbb{N}_0^L$,

$$|B(Q)| \leq (Q + 1)^{L-1} \quad (41)$$

[Proof of Lemma 7.] The number $|B(Q)|$ is the number of vectors $\mathbf{q} \in \mathbb{N}_0^L$ whose elements sum up to Q . This coincides with the number of ways in which Q identical balls can be placed into L distinct boxes. We may thus find an expression for $|B(Q)|$ using the well-known formula for counting the number of ways of placing identical balls into distinct boxes and thus,

$$|B(Q)| = \binom{Q + L - 1}{L - 1}$$

Applying this formula we have

$$|B(Q)| = \frac{(Q + L - 1)!}{(L - 1)!Q!} = \prod_{\ell}^{L-1} \left(\frac{Q + \ell}{\ell} \right) \leq \prod_{\ell}^{L-1} (Q + 1) = (Q + 1)^{L-1}$$

which establishes (41). We now prove Lemma 6. [Proof of Lemma 6.] Given the quadratic form of U , the fact that U is strictly concave in \mathbf{a} and $\partial U / \partial m < 0$, it is clear that there is some $\mathbf{a}^* \in \mathbb{R}_+ \times \mathbb{R}^{K-1}$ and $u^* \in \mathbb{R}$ so that

$$u^* = U(\mathbf{a}^*, 0) \geq U(\mathbf{a}, m), \quad \text{for all } \mathbf{a} \in \mathbb{R}_+ \times \mathbb{R}^{K-1} \text{ and } m \in \mathbb{R}_+ \quad (42)$$

Fix some $\mathbf{p} \in \mathbb{R}_{++}^L$ and let $\rho > 0$ be the smallest element in \mathbf{p} . Clearly,

$$\mathbf{p}'\mathbf{q} \leq \rho \|\mathbf{q}\|_1 \quad (43)$$

Let $\mathbf{e}_1 = [1, 0, \dots, 0]$. Now, applying (5), (42), the fact that $d_3 \geq 0$, and (43), we see

$$\begin{aligned} U(\mathbf{A}\mathbf{q}, \mathbf{p}'\mathbf{q}) &= \mathbf{b}'\mathbf{A}\mathbf{q} - \mathbf{q}'\mathbf{A}'\mathbf{A}\mathbf{q} - d_1\mathbf{p}'\mathbf{q} - d_2(\mathbf{p}'\mathbf{q})^2 - 2d_3\mathbf{e}_1'\mathbf{A}\mathbf{q}\mathbf{p}'\mathbf{q} \\ &\leq u^* - d_1\mathbf{p}'\mathbf{q} - d_2(\mathbf{p}'\mathbf{q})^2 - 2d_3\mathbf{e}_1'\mathbf{A}\mathbf{q}\mathbf{p}'\mathbf{q} \\ &\leq u^* - d_1\mathbf{p}'\mathbf{q} - d_2(\mathbf{p}'\mathbf{q})^2 \\ &\leq u^* - d_1\rho\|\mathbf{q}\|_1 - d_2\rho^2\|\mathbf{q}\|_1^2 \end{aligned} \quad (44)$$

Define $B(Q)$ by (40). Applying (44) and Lemma 7 we see

$$\begin{aligned}
\sum_{\mathbf{q} \in \mathbb{N}_0^L} \exp\left(U(\mathbf{A}\mathbf{q}, \mathbf{p}'\mathbf{q})\right) &\leq \sum_{\mathbf{q} \in \mathbb{N}_0^L} \exp\left(u^* - d_1\rho\|\mathbf{q}\|_1 - d_2\rho^2\|\mathbf{q}\|_1^2\right) \\
&= \exp(u^*) \sum_{\mathbf{q} \in \mathbb{N}_0^L} \exp\left(-d_1\rho\|\mathbf{q}\|_1 - d_2\rho^2\|\mathbf{q}\|_1^2\right) \\
&= \exp(u^*) \sum_{Q=0}^{\infty} \sum_{\mathbf{q} \in B(Q)} \exp\left(-d_1\rho\|\mathbf{q}\|_1 - d_2\rho^2\|\mathbf{q}\|_1^2\right) \\
&= \exp(u^*) \sum_{Q=0}^{\infty} \sum_{\mathbf{q} \in B(Q)} \exp\left(-d_1\rho Q - d_2\rho^2 Q^2\right) \\
&\leq \exp(u^*) \sum_{Q=0}^{\infty} (Q+1)^{L-1} \exp\left(-d_1\rho Q - d_2\rho^2 Q^2\right) < \infty
\end{aligned}$$

where the final inequality can be shown to follow using the “ratio test” for the absolute convergence of series.

10.2 Identification and Estimation Proofs

The following class of utility functions will prove useful.

Definition 8 *A utility function $V : \mathbb{N}_0^L \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is a quadratic expenditure modified utility function (Quem utility function) if*

$$V(\mathbf{q}, m) = \boldsymbol{\gamma}'_0 \mathbf{q} - \mathbf{q}' \mathbf{G} \mathbf{q} - \gamma_1 m - \gamma_2 m^2 - 2\boldsymbol{\gamma}'_3 \mathbf{q} m \quad (45)$$

where $\boldsymbol{\gamma}_0 \in \mathbb{R}^L$, \mathbf{G} is a $L \times L$ symmetric matrix, γ_1, γ_2 are real numbers, and $\boldsymbol{\gamma}_3 \in \mathbb{R}^L$. A Quem V is degenerate if $V(\mathbf{q}, m) = 0$ for all $\mathbf{q} \in \mathbb{N}_0^L$ and all $m \in \mathbb{R}$.

Let $\boldsymbol{\gamma} = [\mathbf{G}, \boldsymbol{\gamma}_0, \gamma_1, \gamma_2, \boldsymbol{\gamma}_3]$ be a vector of parameters of the Quem utility function and let Γ be all such vectors. For $\boldsymbol{\gamma} \in \Gamma$ let $V(\cdot, \cdot; \boldsymbol{\gamma}) : \mathbb{N}_0^L \times \mathbb{R}_{++}^L \rightarrow \mathbb{R}$ be defined by

$$V(\mathbf{q}, \mathbf{p}; \boldsymbol{\gamma}) = V(\mathbf{q}, \mathbf{p}'\mathbf{q}) \quad (46)$$

where $V(\mathbf{q}, \mathbf{p}'\mathbf{q})$ refers to the Quem utility function, defined by (45), with parameters $\boldsymbol{\gamma}$. For $\boldsymbol{\gamma} \in \Gamma$, non-empty $Q \subseteq \mathbb{N}_0^L$, and $\mathbf{p} \in \mathbb{R}_{++}^L$ let $h(\cdot|Q, \mathbf{p}; \boldsymbol{\gamma}) : \mathbb{N}_0^L \rightarrow [0, 1]$ be defined by

$$h(\mathbf{q}|Q, \mathbf{p}; \boldsymbol{\gamma}) = \frac{\exp(V(\mathbf{q}, \mathbf{p}; \boldsymbol{\gamma}))}{\sum_{\tilde{\mathbf{q}} \in Q} \exp(V(\tilde{\mathbf{q}}, \mathbf{p}; \boldsymbol{\gamma}))}, \quad \text{for all } \mathbf{q} \in \mathbb{N}_0^L \quad (47)$$

where $V(\mathbf{q}, \mathbf{p}; \boldsymbol{\gamma})$ is defined by (46). Also, define $h(\mathbf{q}|\mathbf{p}; \boldsymbol{\gamma})$ by

$$h(\mathbf{q}|\mathbf{p}; \boldsymbol{\gamma}) = h(\mathbf{q}|\mathbb{N}_0^L, \mathbf{p}; \boldsymbol{\gamma}), \quad \text{for all } \mathbf{q} \in \mathbb{N}_0^L \quad (48)$$

where $h(\mathbf{q}|\mathbb{N}_0^L, \mathbf{p}; \boldsymbol{\gamma})$ is defined by (47). The following lemma connects the Quem utility function to the standard Qua utility function.

Lemma 8 *Let $\psi : \Theta \rightarrow \Gamma$ be defined by (10). Then,*

$$U(\mathbf{q}, \mathbf{p}; \boldsymbol{\theta}) = V(\mathbf{q}, \mathbf{p}; \psi(\boldsymbol{\theta})), \quad \text{for all } \mathbf{q} \in \mathbb{N}_0^L, \mathbf{p} \in \mathbb{R}_{++}^L, \boldsymbol{\theta} \in \Theta \quad (49)$$

Consequently, if $f(\mathbf{q}|\mathbf{p}; \boldsymbol{\theta})$ satisfies (8) then

$$f(\mathbf{q}|\mathbf{p}; \boldsymbol{\theta}) = h(\mathbf{q}|\mathbf{p}; \psi(\boldsymbol{\theta})), \quad \text{for all } \mathbf{q} \in \mathbb{N}_0^L \quad (50)$$

Further, if $f(\cdot; Q, \mathbf{p}; \boldsymbol{\theta})$ satisfies (13) for some $Q \subseteq \mathbb{N}_0^L$ then

$$f(\mathbf{q}|Q, \mathbf{p}; \boldsymbol{\theta}) = h(\mathbf{q}|Q, \mathbf{p}; \psi(\boldsymbol{\theta})), \quad \text{for all } \mathbf{q} \in \mathbb{N}_0^L \quad (51)$$

The proof is just a matter of plugging in definitions.

The following is crucial for the “identification” proof.

Lemma 9 *Let h be defined by (47). Let S be a distinguishing signal function with support \mathcal{Q} , let $E \subseteq \mathbb{R}_{++}^L$ be a non-empty open set, and let $\boldsymbol{\gamma}, \tilde{\boldsymbol{\gamma}} \in \Gamma$. Then, $\boldsymbol{\gamma} \neq \tilde{\boldsymbol{\gamma}}$ if and only if there exists a non-empty open set $E' \subseteq E$ so that*

$$h(\mathbf{q}|Q, \mathbf{p}; \boldsymbol{\gamma}) \neq h(\mathbf{q}|Q, \mathbf{p}; \tilde{\boldsymbol{\gamma}}), \quad \text{for some } Q \in \mathcal{Q}, \mathbf{q} \in Q, \text{ and all } \mathbf{p} \in E' \quad (52)$$

Lemma 10 *Let \mathbf{A} and $\tilde{\mathbf{A}}$ be two full row rank $K \times L$ matrices which satisfy*

$$\mathbf{A}'\mathbf{A} = \tilde{\mathbf{A}}'\tilde{\mathbf{A}} \quad (53)$$

Define a $K \times K$ matrix V by

$$V = (\mathbf{A}\mathbf{A}')^{-1}\mathbf{A}\tilde{\mathbf{A}}' \quad (54)$$

Then, V is orthogonal and

$$V\tilde{\mathbf{A}} = \mathbf{A} \quad (55)$$

We first show that V is orthogonal. We have

$$VV' = (\mathbf{A}\mathbf{A}')^{-1}\mathbf{A}\tilde{\mathbf{A}}'\tilde{\mathbf{A}}\mathbf{A}(\mathbf{A}\mathbf{A}')^{-1} = (\mathbf{A}\mathbf{A}')^{-1}\mathbf{A}\mathbf{A}'\mathbf{A}(\mathbf{A}\mathbf{A}')^{-1} = I$$

which made use of (53). Next, note that

$$\begin{aligned} V &= V(\tilde{\mathbf{A}}\tilde{\mathbf{A}}')(\tilde{\mathbf{A}}\tilde{\mathbf{A}}')^{-1} = (\mathbf{A}\mathbf{A}')^{-1}\mathbf{A}\tilde{\mathbf{A}}'(\tilde{\mathbf{A}}\tilde{\mathbf{A}}')^{-1} = (\mathbf{A}\mathbf{A}')^{-1}\mathbf{A}\mathbf{A}'\mathbf{A}\tilde{\mathbf{A}}'(\tilde{\mathbf{A}}\tilde{\mathbf{A}}')^{-1} \\ &= \mathbf{A}\tilde{\mathbf{A}}'(\tilde{\mathbf{A}}\tilde{\mathbf{A}}')^{-1} \end{aligned} \quad (56)$$

which made use of (53). Using (56) we see

$$V'V = (\tilde{\mathbf{A}}\tilde{\mathbf{A}}')^{-1}\tilde{\mathbf{A}}\mathbf{A}'\mathbf{A}\tilde{\mathbf{A}}'(\tilde{\mathbf{A}}\tilde{\mathbf{A}}')^{-1} = (\tilde{\mathbf{A}}\tilde{\mathbf{A}}')^{-1}\tilde{\mathbf{A}}\tilde{\mathbf{A}}'\tilde{\mathbf{A}}\tilde{\mathbf{A}}'(\tilde{\mathbf{A}}\tilde{\mathbf{A}}')^{-1} = I$$

which again used (53). So, we have shown that V is orthogonal. We now show that (55) holds. So,

$$V\tilde{\mathbf{A}} = (\mathbf{A}\mathbf{A}')^{-1}\mathbf{A}\tilde{\mathbf{A}}'\tilde{\mathbf{A}} = (\mathbf{A}\mathbf{A}')^{-1}\mathbf{A}\mathbf{A}'\mathbf{A} = \mathbf{A}$$

which shows (55).

[Proof of Proposition 4.] It is trivial to show that $4. \implies 3. \implies 2. \implies 1.$ So, we focus on the other direction. We first show that $1. \implies 3.$ or more precisely we show that not $3.$ implies not $1.$ So, assume that item $3.$ does not hold.

Let ψ be defined by (10). As $3.$ does not hold we see that $\psi(\boldsymbol{\theta}) \neq \psi(\tilde{\boldsymbol{\theta}})$. Next, let S be the signal function which satisfies $S(\mathbf{q}) = \mathbb{N}_0$ for all \mathbf{q} . It is easily shown that S is distinguishing. Thus, by setting $\boldsymbol{\gamma} = \psi(\boldsymbol{\theta})$ and $\tilde{\boldsymbol{\gamma}} = \psi(\tilde{\boldsymbol{\theta}})$ we may apply Lemma 9 to see that there is some non-empty open set $E' \subseteq E$ so that (52) holds. From Lemma 8 equation (49) we see that item $1.$ does not hold.

Now, we show $3. \implies 4.$ So, suppose $3.$ holds. Define V by (54). From Lemma 10 we know that V is orthogonal and further (55). Thus, to complete the proof that $3. \implies 4.$ we must show that $\mathbf{b} = V\tilde{\mathbf{b}}$. Using the fact that $\mathbf{A}'\mathbf{b} = \tilde{\mathbf{A}}'\tilde{\mathbf{b}}$ we see

$$V\tilde{\mathbf{b}} = (\mathbf{A}\mathbf{A}')^{-1}\mathbf{A}\tilde{\mathbf{A}}'\tilde{\mathbf{b}} = (\mathbf{A}\mathbf{A}')^{-1}\mathbf{A}\mathbf{A}'\mathbf{b} = \mathbf{b}$$

and so $\mathbf{b} = V\tilde{\mathbf{b}}$ as required.

[Proof of Proposition 6.] It is easily seen that $6.$ implies $5.$ implies $3.$ (this is true even when $d_3 = 0$.) To complete the proof we show that when $d_3 \neq 0$ we have $4.$ implies $6.$. So, suppose that $4.$ holds and let V be the orthogonal matrix from item $4.$. Note that $d_3\mathbf{A}'\mathbf{e}_1 = \tilde{d}_3\tilde{\mathbf{A}}'\mathbf{e}_1$ implies

$$\tilde{d}_3\mathbf{A}'V'\mathbf{e}_1 = d_3\mathbf{A}'\mathbf{e}_1$$

doing left multiplication on both sides by $(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}$ gives

$$\tilde{d}_3V'\mathbf{e}_1 = d_3\mathbf{e}_1 \tag{57}$$

Using the fact that V is orthogonal we see $\tilde{d}_3^2 = d_3^2$. This implies $\tilde{d}_3 = d_3$ as \tilde{d}_3 and d_3 are both greater than or equal to 0. Going back to (57) and using the fact that V is orthogonal and the assumption that $d_3 > 0$ we see

$$\mathbf{e}_1 = V\mathbf{e}_1$$

Thus, we have shown that item 6. holds.

10.2.1 Proof of Lemma 9

The proof of Lemma 9 relies on 3 lemmas.

Lemma 11 *Let V be a non-degenerate Quem utility function and let $E \subseteq \mathbb{R}_+^L$ be a non-empty open set. There exists a $\mathbf{p} \in E$ so that $V(\mathbf{q}, \mathbf{p}'\mathbf{q})$ is a non-degenerate grounded quadratic function of \mathbf{q} .*

Let V be a Quem utility function. Suppose that $V(\mathbf{q}, \mathbf{p}'\mathbf{q})$ is a degenerate grounded quadratic function of \mathbf{q} for all $\mathbf{p} \in E$. We shall show that V must be degenerate. First, note that

$$V(\mathbf{q}, \mathbf{p}'\mathbf{q}) = 0 \quad \text{for all } \mathbf{q} \in \mathbb{N}_0^L \text{ and all } \mathbf{p} \in E \quad (58)$$

Let us differentiate $V(\mathbf{q}, \mathbf{p}'\mathbf{q})$ twice with respect to \mathbf{p} . This gives

$$\partial_{\mathbf{p}} V(\mathbf{q}, \mathbf{p}'\mathbf{q}) = -\gamma_1 \mathbf{q} - 2\gamma_2 \mathbf{p}'\mathbf{q}\mathbf{q} - 2\gamma_3' \mathbf{q}\mathbf{q} \quad (59)$$

$$\partial_{\mathbf{p}}^2 V(\mathbf{q}, \mathbf{p}'\mathbf{q}) = -2\gamma_2 \mathbf{q}\mathbf{q}' \quad (60)$$

Equation (58) implies that the expressions in both (59) and (60) are 0. From equation (60) we see that $\gamma_2 = 0$ (fix \mathbf{q} and vary \mathbf{p} to see this). Now, from (59) and the fact that $\gamma_2 = 0$ we see

$$-\gamma_1 - 2\gamma_3' \mathbf{q} = 0, \quad \text{for all } \mathbf{q} \neq 0$$

But, this is easily shown to imply $\gamma_1 = 0$ and $\gamma_3 = 0$. So, we may now express $V(\mathbf{q}, \mathbf{p}'\mathbf{q})$ as

$$V(\mathbf{q}, \mathbf{p}'\mathbf{q}) = \gamma_0' \mathbf{q} - \mathbf{q}' \mathbf{G} \mathbf{q}$$

It is easy to show that $\gamma_0 = 0$ and $\mathbf{G} = 0$ from (58). Thus, V is indeed degenerate.

Lemma 12 *Let S be a distinguishing signal function with support \mathcal{Q} . Let $E \subseteq \mathbb{R}_{++}^L$ be a non-empty open set. For each non-degenerate Quem utility function V there exists a $Q \in \mathcal{Q}$, $\mathbf{q}, \tilde{\mathbf{q}} \in Q$, and a non-empty open set $E' \subseteq E$ satisfying*

$$V(\mathbf{q}, \mathbf{p}'\mathbf{q}) \neq V(\tilde{\mathbf{q}}, \mathbf{p}'\tilde{\mathbf{q}}), \quad \text{for all } \mathbf{p} \in E' \quad (61)$$

Let V be a non-degenerate Quem utility function. From Lemma 11 there exists a $\bar{\mathbf{p}} \in E$ so that $V(\mathbf{q}, \bar{\mathbf{p}}'\mathbf{q})$ is a non-degenerate grounded quadratic function of \mathbf{q} . As S is distinguishing there exists a set $Q \in \mathcal{Q}$ and $\mathbf{q}, \tilde{\mathbf{q}} \in Q$ satisfying

$$V(\mathbf{q}, \bar{\mathbf{p}}'\mathbf{q}) \neq V(\tilde{\mathbf{q}}, \bar{\mathbf{p}}'\tilde{\mathbf{q}}) \quad (62)$$

But, as V is a continuous function we may find some neighborhood of $\bar{\mathbf{p}}$ so that (62) holds for any \mathbf{p} in this set. This establishes (61).

Lemma 13 *Let (b_n) and (\tilde{b}_n) be two sequences in \mathbb{R} where $\sum_{n=1}^{\infty} \exp(b_n) < \infty$ and $\sum_{n=1}^{\infty} \exp(\tilde{b}_n) < \infty$. There exists a number $k \in \mathbb{R}$ so that $b_N = \tilde{b}_N + k$ for all $N \in \mathbb{N}$ if and only if*

$$\frac{\exp(b_N)}{\sum_{n=1}^{\infty} \exp(b_n)} = \frac{\exp(\tilde{b}_N)}{\sum_{n=1}^{\infty} \exp(\tilde{b}_n)}, \quad \text{for each } N \in \mathbb{N} \quad (63)$$

The “only if” part is obvious so let’s prove the “if” part. Let k be

$$k = \ln \left(\frac{\sum_{n=1}^{\infty} \exp(b_n)}{\sum_{n=1}^{\infty} \exp(\tilde{b}_n)} \right)$$

Rearranging (63) we see

$$\exp(b_N - \tilde{b}_N) = \frac{\sum_{n=1}^{\infty} \exp(b_n)}{\sum_{n=1}^{\infty} \exp(\tilde{b}_n)}$$

Taking logs of both sides and rearranging yields the result.

We may now prove Lemma 9. [Proof of Lemma 9.] The “if” part is obvious so we focus on the “only if” part. Suppose $\gamma \neq \tilde{\gamma}$. Let $W : \mathbb{N}_0^L \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$W(\mathbf{q}, m) = V(\mathbf{q}, m; \tilde{\gamma}) - V(\mathbf{q}, m; \gamma)$$

It is easily shown that W is a non-degenerate Quem utility function. Thus, by Lemma 12 there exists $Q \in \mathcal{Q}$, $\mathbf{q}, \tilde{\mathbf{q}} \in Q$, and a non-empty open set $E' \subseteq E$ so that

$$W(\mathbf{q}, \mathbf{p}'\mathbf{q}) \neq W(\tilde{\mathbf{q}}, \mathbf{p}'\tilde{\mathbf{q}}), \quad \text{for all } \mathbf{p} \in E'$$

Plugging in the definition of W and rearranging we see

$$V(\tilde{\mathbf{q}}, \mathbf{p}; \tilde{\gamma}) - V(\mathbf{q}, \mathbf{p}; \tilde{\gamma}) \neq V(\tilde{\mathbf{q}}, \mathbf{p}; \gamma) - V(\mathbf{q}, \mathbf{p}; \gamma)$$

This implies that for each $\mathbf{p} \in E'$ there is no constant $k \in \mathbb{R}$ so that both $V(\mathbf{q}, \mathbf{p}; \tilde{\gamma}) = V(\mathbf{q}, \mathbf{p}; \gamma) + k$ and $V(\tilde{\mathbf{q}}, \mathbf{p}; \tilde{\gamma}) = V(\tilde{\mathbf{q}}, \mathbf{p}; \gamma) + k$. Thus, Lemma 13 establishes (52).

10.2.2 Proof of Proposition 5

[Proof of Proposition 5] Fix some $\mathbf{q} \in \mathbb{N}_0^L$ and let $Q \in \mathcal{Q}$. Consider two cases: Case 1: $\mathbf{q} \notin Q$ and Case 2: $\mathbf{q} \in Q$. Suppose Case 1 holds. As the event $\mathbf{c} \in Q$ clearly implies $\mathbf{c} \neq \mathbf{q}$ we see that the right hand side of (12) is 0. On the other hand, by property (i)

in the definition of a signal function, we see that the event $S(\mathbf{c}) = Q$ implies the event $\mathbf{c} \in Q$. But, we have already noted that the event $\mathbf{c} \in Q$ implies $\mathbf{c} \neq \mathbf{q}$ and so the left hand side of (12) is also 0. Thus, (12) holds under Case 1.

Now, consider Case 2. We write \mathcal{S} for $S(\mathbf{c})$. By equation (11) and the assumption that $S(\mathbf{c})$ and $\boldsymbol{\rho}$ are independent conditional on \mathbf{c}

$$P(\mathcal{S} = Q | \mathbf{c} = \mathbf{q}, \boldsymbol{\rho}) = P(\mathcal{S} = Q | \mathbf{c} = \tilde{\mathbf{q}}, \boldsymbol{\rho}) \quad (64)$$

Now,

$$\begin{aligned} P(\mathbf{c} = \mathbf{q} | \mathcal{S} = Q, \boldsymbol{\rho}) &= \frac{P(\mathbf{c} = \mathbf{q} \text{ and } \mathcal{S} = Q | \boldsymbol{\rho})}{P(\mathcal{S} = Q | \boldsymbol{\rho})} \\ &= \frac{P(\mathbf{c} = \mathbf{q} \text{ and } \mathcal{S} = Q | \boldsymbol{\rho})}{\sum_{\tilde{\mathbf{q}} \in Q} P(\mathbf{c} = \tilde{\mathbf{q}} \text{ and } \mathcal{S} = Q | \boldsymbol{\rho})}, && \text{by prop (i) of def 3} \\ &= \frac{P(\mathbf{c} = \mathbf{q} | \boldsymbol{\rho}) P(\mathcal{S} = Q | \mathbf{c} = \mathbf{q}, \boldsymbol{\rho})}{\sum_{\tilde{\mathbf{q}} \in Q} P(\mathbf{c} = \tilde{\mathbf{q}} | \boldsymbol{\rho}) P(\mathcal{S} = Q | \mathbf{c} = \tilde{\mathbf{q}}, \boldsymbol{\rho})} \\ &= \frac{P(\mathbf{c} = \mathbf{q} | \boldsymbol{\rho})}{\sum_{\tilde{\mathbf{q}} \in Q} P(\mathbf{c} = \tilde{\mathbf{q}} | \boldsymbol{\rho})}, && \text{by (64)} \\ &= P(\mathbf{c} = \mathbf{q} | \mathbf{c} \in Q, \boldsymbol{\rho}) \end{aligned}$$

which establishes (12) for Case 2. Finally, (13) is a easy consequence of (12).

10.2.3 Proof of Theorem 1

The proof requires several lemmas.

Lemma 14 *Let $\tilde{\boldsymbol{\theta}} = [\tilde{\mathbf{A}}, \tilde{\mathbf{b}}, d_1, d_2, d_3] \in \Theta$ and suppose that $\tilde{\mathbf{A}}$ has rank K . For $\delta > 0$ let B_δ denote a closed ball in Γ centered at $\psi(\tilde{\boldsymbol{\theta}})$. There is $\bar{\delta} > 0$ such that $\psi^{-1}(B_\delta)$ is compact for all $\delta \in (0, \bar{\delta}]$.*

Lemma 15 *Suppose Assumptions 3 and 5 hold. Then,*

$$\mathbf{E} \left[\left| \ln(h(\mathbf{c}_n | \mathcal{S}_{n,i}, \boldsymbol{\rho}_n; \boldsymbol{\gamma})) \right| \right] < \infty, \quad \text{for all } \boldsymbol{\gamma} \in \Gamma, \text{ and all } n, i \quad (65)$$

Lemma 16 *The function $\ln h(\mathbf{q} | Q, \mathbf{p}; \boldsymbol{\gamma})$ defined in (47) is concave in $\boldsymbol{\gamma} \in \Gamma$.*

Lemma 17 *Suppose $F_N(\boldsymbol{\gamma})$ is a random variable for each $N \in \mathbb{N}$ and $\boldsymbol{\gamma} \in \Gamma$. Suppose that F_N satisfies the following.*

1. $F_N : \Gamma \rightarrow \mathbb{R}$ is concave for each $N \in \mathbb{N}$.

2. There exists a function $F : \Gamma \rightarrow \mathbb{R}$ so that $F_N(\gamma) \xrightarrow{a.s.} F(\gamma)$ for all $\gamma \in \Gamma$.

Then, F is concave and for any compact set $C \subseteq \Gamma$

$$\sup_{\gamma \in C} \left| F_N(\gamma) - F(\gamma) \right| \xrightarrow{a.s.} 0 \quad (66)$$

Lemma 18 *Let $F : \Gamma \rightarrow \mathbb{R}$ be a continuous function. Suppose there is a unique $\bar{\gamma}$ which maximizes F . That is, there exists a $\bar{\gamma} \in \Gamma$ so that*

$$F(\bar{\gamma}) > F(\gamma), \quad \text{for all } \gamma \neq \bar{\gamma} \quad (67)$$

For $\delta > 0$ let B_δ be a closed ball in Γ centered at $\bar{\gamma}$ with radius $\delta > 0$. For every compact set $C \subseteq \Gamma$ and every $\delta > 0$ there exists a $\varepsilon > 0$ so that

$$F(\bar{\gamma}) - F(\gamma) > \varepsilon, \quad \text{for all } \gamma \in C \setminus B_\delta \quad (68)$$

We now prove Theorem 1.

[Proof of Theorem 1.] For convenience let $\gamma^* = \psi(\theta^*)$. Let B_δ denote a closed ball in Γ of radius δ centered at γ^* . By Assumption 4 and Lemma 14 there is a $\bar{\delta} > 0$ so that $\psi^{-1}(B_\delta)$ is compact for all $\delta \in (0, \bar{\delta}]$.

Next, define $Q_N : \Gamma \rightarrow \mathbb{R}$ by

$$Q_N(\gamma) = \frac{1}{NI} \sum_{n=1}^N \sum_{i=1}^I \ln \left(h(\mathbf{c}_n | \mathcal{S}_{i,n}, \boldsymbol{\rho}_n; \gamma) \right)$$

From Lemma 8 equation (51) we see

$$Q_N(\psi(\theta)) = \mathcal{L}_N(\theta), \quad \text{for all } \theta \in \Theta \quad (69)$$

where \mathcal{L}_N is defined by (19). Define $Q : \Gamma \rightarrow \mathbb{R}$ by

$$Q(\gamma) = \mathbf{E} \left[\ln \left(h(\mathbf{c}_1 | \mathcal{S}_{1,1}, \boldsymbol{\rho}_1; \gamma) \right) \right] \quad (70)$$

From Assumptions 3 and 5 and Lemma 15 we know $Q(\gamma)$ is finite for all $\gamma \in \Gamma$. Thus, by Assumption 1 and the strong law of large numbers we know

$$Q_N(\gamma) \xrightarrow{a.s.} Q(\gamma), \quad \text{for all } \gamma \in \Gamma \quad (71)$$

From Lemma 16 it is clear that Q_N is concave and so we may use (71) and Lemma 17 to see

$$\sup_{\gamma \in B_{\bar{\delta}}} \left| Q_N(\gamma) - Q(\gamma) \right| \xrightarrow{a.s.} 0 \quad (72)$$

From now on we conditional our analysis on a point in the sample space on which

$$\sup_{\gamma \in B_{\bar{\delta}}} |Q_N(\gamma) - Q(\gamma)| \rightarrow 0 \quad (73)$$

We shall show that $\psi(\hat{\theta}_N) \rightarrow \gamma^*$. This will prove the theorem as the set of all points in the sample space which satisfy (73) contain a probability 1 event (by equation (72)).

Let $\delta \in (0, \bar{\delta})$ and for convenience let $\hat{\gamma}_N = \psi(\hat{\theta}_N)$. We shall show that there exists a $\bar{N} \in \mathbb{N}$ so that

$$\|\hat{\gamma}_N - \gamma^*\| < \delta, \quad \text{for all } N \geq \bar{N} \quad (74)$$

Of course (74) implies $\psi(\hat{\theta}_N) \equiv \hat{\gamma}_N \rightarrow \gamma^*$ and so the proof is complete if we can show (74).

From Assumption 2, and Lemma 9, and the conditional Kullback-Leibler information inequality we see

$$Q(\gamma^*) > Q(\gamma), \quad \text{for all } \gamma \neq \psi(\theta^*)$$

Thus, we may apply Lemma 18 to see that there exists an $\varepsilon > 0$ so that

$$Q(\gamma^*) - Q(\gamma) > \varepsilon, \quad \text{for all } \gamma \in B_{\bar{\delta}} \setminus B_{\delta} \quad (75)$$

From (73) there exists a $\bar{N} \in \mathbb{N}$ so that

$$\sup_{\gamma \in B_{\bar{\delta}}} |Q_N(\gamma) - Q(\gamma)| < \frac{\varepsilon}{2}, \quad \text{for all } N \geq \bar{N} \quad (76)$$

We claim that

$$Q_N(\gamma^*) > Q_N(\gamma), \quad \text{for all } \gamma \in B_{\bar{\delta}} \setminus B_{\delta} \quad \text{and all } N \geq \bar{N} \quad (77)$$

To see (77) use equations (75) and (76) and note that for all $\gamma \in B_{\bar{\delta}} \setminus B_{\delta}$ and all $N \geq \bar{N}$

$$\begin{aligned} Q_N(\gamma^*) - Q_N(\gamma) &= Q_N(\gamma^*) - Q(\gamma^*) + Q(\gamma^*) - Q(\gamma) + Q(\gamma) - Q_N(\gamma) \\ &\geq Q(\gamma^*) - Q_N(\gamma) - |Q_N(\gamma^*) - Q(\gamma^*)| - |Q(\gamma) - Q_N(\gamma)| \\ &> \varepsilon - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 0 \end{aligned}$$

Thus, (77) holds. We next claim that

$$Q_N(\gamma^*) > Q_N(\gamma), \quad \text{for all } \gamma \notin B_{\delta} \quad \text{and all } N \geq \bar{N} \quad (78)$$

So, let $\gamma \notin B_{\delta}$. If $\gamma \in B_{\bar{\delta}}$ then (78) follows from (77) so suppose that $\gamma \notin B_{\bar{\delta}}$. Let $t \in (0, 1)$ be chosen so that $t\gamma + (1-t)\gamma^*$ is on the boundary of $B_{\bar{\delta}}$. Now, we use (77) and the concavity of Q_N to see

$$Q_N(\gamma^*) > Q_N(t\gamma + (1-t)\gamma^*) \geq tQ_N(\gamma) + (1-t)Q_N(\gamma^*)$$

which gives (78) after rearranging.

B_δ is compact and so

$$\mathcal{L}_N(\boldsymbol{\theta}) \quad \boldsymbol{\theta} \in B_\delta \quad (79)$$

is non-empty. Further, from (78) we see that the argmax in (20) and (79) coincide. Thus, $\hat{\boldsymbol{\theta}}_N$ is an element of the argmax in (79) (for all $N \geq \bar{N}$) and consequently $\|\hat{\boldsymbol{\gamma}}_N - \boldsymbol{\gamma}^*\| \leq \delta$ for all $N \geq \bar{N}$. Thus, (74) holds and so the theorem has been proved.

10.3 Proofs of lemmas used to prove Theorem 1

[Proof of Lemma 14.] For $\boldsymbol{\theta} = [\mathbf{A}, \mathbf{b}, d_1, d_2, d_3] \in \Theta$ let $\boldsymbol{\theta}^1 = \mathbf{A}$, $\boldsymbol{\theta}^2 = \mathbf{b}$, $\boldsymbol{\theta}^3 = d_1$, $\boldsymbol{\theta}^4 = d_2$, and $\boldsymbol{\theta}^5 = d_3$. That is, $\boldsymbol{\theta}^1$ denotes the first entry in $\boldsymbol{\theta}$, $\boldsymbol{\theta}^2$ denotes the second entry, and so forth. Similarly, for $\boldsymbol{\gamma} = [\mathbf{G}, \gamma_0, \gamma_1, \gamma_2, \gamma_3] \in \Gamma$ let $\boldsymbol{\gamma}^1 = \mathbf{G}$, $\boldsymbol{\gamma}^2 = \gamma_0$, $\boldsymbol{\gamma}^3 = \gamma_1$, $\boldsymbol{\gamma}^4 = \gamma_2$, and $\boldsymbol{\gamma}^5 = \gamma_3$.

Now, as $\tilde{\mathbf{A}}$ has rank K it is clear that $\tilde{\mathbf{A}}' \tilde{\mathbf{A}}$ also has rank K . Thus, there exists a closed ball C around $\tilde{\mathbf{A}}' \tilde{\mathbf{A}}$ which only contains matrices of rank K . Let $\bar{\delta} > 0$ be a number small enough so that $\boldsymbol{\gamma} \in B_{\bar{\delta}}$ implies $\boldsymbol{\gamma}^0 \in C$. Let $\delta \in (0, \bar{\delta}]$. We shall show that $\psi^{-1}(B_\delta)$ is compact.

As B_δ is closed and ψ is continuous we know that $\psi^{-1}(B_\delta)$ is closed. So, the desired conclusion holds if we can show that $\psi^{-1}(B_\delta)$ is bounded. Now, if $\psi^{-1}(B_\delta)$ were unbounded then there would be a sequence in $\psi^{-1}(B_\delta)$ denoted $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \dots$ where one of the vectors or matrices in $\boldsymbol{\theta}_n = [\mathbf{A}_n, \mathbf{b}_n, d_{1,n}, d_{2,n}, d_{3,n}]$ has an element whose absolute value goes to ∞ . We shall show that this cannot be the case for any of the vectors nor the matrix in $[\mathbf{A}_n, \mathbf{b}_n, d_{1,n}, d_{2,n}, d_{3,n}]$.

First, it is clear that the sequence of matrices \mathbf{A}_n could not have an element whose absolute value tends to infinity as we know $\mathbf{A}_n' \mathbf{A}_n \in C$. Second, as \mathbf{A}_n is bounded we may assume, without loss of generality, that \mathbf{A}_n converges to some matrix \mathbf{A} (for if not just consider sub-sequences). From the definition of C , the fact that $\mathbf{A}_n' \mathbf{A}_n \in C$, and C is closed we see that each matrix \mathbf{A}_n is rank K and additionally, \mathbf{A} is rank K . Let $\lambda_n > 0$ be the smallest eigenvalue of $\mathbf{A}_n \mathbf{A}_n'$. As \mathbf{A} is rank K (and so $\mathbf{A} \mathbf{A}'$ is also rank K) we know λ_n does not tend to 0. Now, we have

$$\|\mathbf{A}_n' \mathbf{b}_n\| = \sqrt{\mathbf{b}_n' \mathbf{A}_n \mathbf{A}_n' \mathbf{b}_n} \geq \lambda_n \|\mathbf{b}_n\|$$

Thus, if \mathbf{b}_n has an entry which tends to infinity in absolute value then also $\|\mathbf{A}_n' \mathbf{b}_n\| \rightarrow \infty$ which contradicts $\boldsymbol{\theta}_n \in \psi^{-1}(B_\delta)$.

Next, note that neither $|d_{1,n}|$ nor $|d_{2,n}|$ tend to infinity as a direct result of $\gamma_n \in \psi^{-1}(B_\delta)$. Finally, we have

$$||d_{3,n}\mathbf{A}'_n\mathbf{e}_1|| = |d_{3,n}|\sqrt{\mathbf{e}'_1\mathbf{A}_n\mathbf{A}'_n\mathbf{e}_1} \geq |d_{3,n}|\lambda_n$$

So, if $|d_{3,n}| \rightarrow \infty$ then $||d_{3,n}\mathbf{A}'_n\mathbf{e}_1||$ also goes to infinity which contradicts $\theta_n \in \psi^{-1}(B_\delta)$.

[Proof of Lemma 16.] It is clear that $V(\mathbf{q}, \boldsymbol{\rho}; \boldsymbol{\gamma})$, defined by (46), is linear in parameters and so it can be represented as

$$V(\mathbf{q}, \boldsymbol{\rho}; \boldsymbol{\gamma}) = w(\mathbf{q}, \boldsymbol{\rho})'\boldsymbol{\gamma}$$

where $w(\mathbf{q}, \boldsymbol{\rho})$ is some transformation of the data. Thus, $\ln h(\mathbf{q}|Q, \boldsymbol{\rho}; \boldsymbol{\gamma})$ can be expressed as

$$\ln h(\mathbf{q}|Q, \boldsymbol{\rho}; \boldsymbol{\gamma}) = \ln \left(\frac{\exp(w(\mathbf{q}, \boldsymbol{\rho})'\boldsymbol{\gamma})}{\sum_{\tilde{\mathbf{q}} \in Q} \exp(w(\tilde{\mathbf{q}}, \boldsymbol{\rho})'\boldsymbol{\gamma})} \right)$$

But, this is the form of the log likelihood of the conditional logit and it is well-known that this function is concave in $\boldsymbol{\gamma}$. See for example mcfadden74.

[Proof of Lemma 17.] Let $\tilde{\Gamma} = \{\gamma_1, \gamma_2, \dots\}$ be some countable dense subset of Γ . By item 2 of the lemma there is some probability 1 event E so that $F_N(\gamma_k) \rightarrow F(\gamma_k)$ for all $k \in \mathbb{N}$ as $N \rightarrow \infty$ on E . Let $C \subseteq \Gamma$ be compact. By rockafellar70 Theorem 10.8 and item 1 of the lemma, the function F is concave and the function F_N converges uniformly on C to F for any sample space point in E . But, E is a probability 1 event and so (66) holds.

[Proof of Lemma 18.] Suppose $C \subseteq \Gamma$ is compact and suppose, for a contradiction, that there is some $\delta > 0$ so that there is no $\varepsilon > 0$ so that (68) holds. Then, we can find a convergent sequence γ_n in C where $F(\gamma_n) \rightarrow F(\bar{\gamma})$ and each element of the sequence satisfies $||\bar{\gamma} - \gamma_n|| > \delta$. From the continuity of F the limit of the sequence would contradict the condition in (67).

[Proof of Lemma 15.] Let $\boldsymbol{\gamma} \in \Gamma$. Using Assumption 3 it is straightforward to show that there is some $M \in \mathbb{R}$ large enough so that for all $\mathbf{q} \in \mathbb{N}_0^L$, $\tilde{\mathbf{q}} \in S(\mathbf{q})$, and all $\mathbf{p} \in \mathbb{R}_{++}^L$

$$|V(\tilde{\mathbf{q}}, \mathbf{p}; \boldsymbol{\gamma})| \leq \left[\sum_{k=0}^2 \sum_{j=0}^2 ||\mathbf{q}||_k^k ||\mathbf{p}||_j^j \right] M$$

Again applying Assumption 3 we see

$$\begin{aligned}
0 \leq -\ln(h(\mathbf{c}_n | \mathcal{S}_{n,i}, \boldsymbol{\rho}_n; \gamma)) &= -\ln\left(\frac{\exp(V(\mathbf{c}_n, \boldsymbol{\rho}_n; \gamma))}{\sum_{\tilde{\mathbf{c}} \in \mathcal{S}_{n,i}} \exp(V(\tilde{\mathbf{c}}, \boldsymbol{\rho}_n; \gamma))}\right) \\
&\leq -\ln\left(\frac{\exp\left(-\left[\sum_{k=0}^2 \sum_{j=0}^2 \|\mathbf{c}_n\|_k^k \|\boldsymbol{\rho}_n\|_j^j\right] M\right)}{J \exp\left(\left[\sum_{k=0}^2 \sum_{j=0}^2 \|\mathbf{c}_n\|_k^k \|\boldsymbol{\rho}_n\|_j^j\right] M\right)}\right) \\
&= 2 \left[\sum_{k=0}^2 \sum_{j=0}^2 \|\mathbf{c}_n\|_k^k \|\boldsymbol{\rho}_n\|_j^j \right] M + \ln J
\end{aligned}$$

Using this inequality and Assumption 5 we have

$$\mathbf{E}\left[|\ln(h(\mathbf{c}_n | \mathcal{S}_{n,i}, \boldsymbol{\rho}_n; \gamma))|\right] \leq 2 \left[\sum_{k=0}^2 \sum_{j=0}^2 \mathbf{E}\left[\|\mathbf{c}_n\|_k^k \|\boldsymbol{\rho}_n\|_j^j\right] \right] M + \ln J < \infty$$

which is what we needed to show.

10.4 Signal Function Lemma

Let B be an $M \times M$ symmetric matrix and let $b_{i,j}$ denote the entry in row i , column j . Let $(B) = [b_{1,1}, \dots, b_{M,M}] \in \mathbb{R}^M$. That is, (B) is the vector composed of the elements on the diagonal of B . Similarly, let (B) denote the vector

$$(B) = [b_{i,j}]_{i < j} = [b_{1,2}, b_{1,3}, \dots, b_{M-1,M}]$$

In other words, $(B) \in \mathbb{R}^{M(M-1)/2}$ denotes the off-diagonal elements of the bottom-left triangle of matrix B .

Let S be a signal function. Let $\mathcal{Q}_1(\mathbf{q}) \equiv \mathcal{Q}(\mathbf{q})$ denote the support of $S(\mathbf{q})$. Let $\mathcal{Q}_2(\mathbf{q})$ be defined by

$$\mathcal{Q}_2(\mathbf{q}) = \bigcup_{\tilde{\mathbf{q}} \in \mathcal{Q}(\mathbf{q})} \mathcal{Q}_1(\tilde{\mathbf{q}})$$

In other words, $\mathcal{Q}_2(\mathbf{q})$ is the union of the supports of $S(\tilde{\mathbf{q}})$ where the union is taken over all $\tilde{\mathbf{q}} \in \mathcal{Q}(\mathbf{q})$. Note that because $\mathbf{q} \in \mathcal{Q}(\mathbf{q})$ we have $\mathcal{Q}_1(\mathbf{q}) \subseteq \mathcal{Q}_2(\mathbf{q})$. Similarly, for all $n \in \mathbb{N}$ define

$$\mathcal{Q}_{n+1}(\mathbf{q}) = \bigcup_{\tilde{\mathbf{q}} \in \mathcal{Q}(\mathbf{q})} \mathcal{Q}_n(\tilde{\mathbf{q}})$$

Finally, let $\mathcal{Q}_\infty(\mathbf{q}) = \bigcup_{n \in \mathbb{N}} \mathcal{Q}_n(\mathbf{q})$. It is easy to verify that $\tilde{\mathbf{q}} \in \mathcal{Q}_\infty(\mathbf{q})$ if and only if there exists a sequence $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N$ so that

$$\mathbf{q}_1 \in \mathcal{Q}(\mathbf{q}), \quad \mathbf{q}_2 \in \mathcal{Q}(\mathbf{q}_1), \quad \mathbf{q}_3 \in \mathcal{Q}(\mathbf{q}_2), \quad \dots, \quad \mathbf{q}_N \in \mathcal{Q}(\mathbf{q}_{N-1}), \quad \text{and} \quad \tilde{\mathbf{q}} \in \mathcal{Q}(\mathbf{q}_N)$$

For a set $Q \subseteq \mathbb{N}_0^L$ let $1_Q(\mathbf{q})$ be defined by

$$1_Q(\mathbf{q}) = \begin{cases} 1, & \text{if } \mathbf{q} \in Q \\ 0, & \text{else} \end{cases}$$

Lemma 19 *Let S be a signal function and let $\tilde{Q} = \{\mathbf{q}_1, \dots, \mathbf{q}_N\}$ be a finite subset of \mathbb{N}_0^L . Let \tilde{Q} be defined by*

$$\tilde{Q} = \bigcup_{\mathbf{q} \in \tilde{Q}} \mathcal{Q}_\infty(\mathbf{q})$$

Let $\{Q_1, \dots, Q_M\}$ be an arbitrary enumeration of the elements in \tilde{Q} . If the matrix

$$D = \begin{bmatrix} \mathbf{q}'_1 & (\mathbf{q}_1 \mathbf{q}'_1) & (\mathbf{q}_1 \mathbf{q}'_1) & 1_{Q_1}(\mathbf{q}_1) & \dots & 1_{Q_M}(\mathbf{q}_1) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{q}'_N & (\mathbf{q}_N \mathbf{q}'_N) & (\mathbf{q}_N \mathbf{q}'_N) & 1_{Q_1}(\mathbf{q}_N) & \dots & 1_{Q_M}(\mathbf{q}_N) \end{bmatrix}$$

has full column rank then S is distinguishing.

We prove the lemma by supposing that S is not distinguishing and then we show that the matrix D does not have full column rank. So, suppose S is not distinguishing. This means that there is a $\mathbf{b} \in \mathbb{R}^L$ and a symmetric $L \times L$ matrix B where, for all $Q \in \tilde{Q}$

$$\mathbf{b}'\mathbf{q} + \mathbf{q}'B\mathbf{q} = \mathbf{b}'\tilde{\mathbf{q}} + \tilde{\mathbf{q}}'B\tilde{\mathbf{q}}, \quad \text{for all } \mathbf{q}, \tilde{\mathbf{q}} \in Q \quad (80)$$

and it is not the case that both $\mathbf{b} = 0$ and $B = 0$. Note that (80) implies that for each $m \in \{1, \dots, M\}$ there exists a number β_m so that

$$\beta_m = \mathbf{b}'\mathbf{q} + \mathbf{q}'B\mathbf{q}, \quad \text{for all } \mathbf{q} \in Q_m$$

This clearly implies

$$0 = \mathbf{b}'\mathbf{q}_n + \mathbf{q}'_n B \mathbf{q}_n - \sum_{m=1}^M \beta_m 1_{Q_m}(\mathbf{q}_n), \quad \text{for all } n \in \{1, \dots, N\} \quad (81)$$

Let \mathbf{v} be the vector defined by

$$\mathbf{v} = [\mathbf{b}', (B)', 2(B)', -\beta_1, \dots, -\beta_M]$$

Applying (81) we see

$$\begin{aligned} D\mathbf{v} &= \begin{bmatrix} \mathbf{b}'\mathbf{q}_1 + (B)'(\mathbf{q}_1 \mathbf{q}'_1) + 2(B)'(\mathbf{q}_1 \mathbf{q}'_1) - \sum_{m=1}^M \beta_m 1_{Q_m}(\mathbf{q}_1) \\ \vdots \\ \mathbf{b}'\mathbf{q}_N + (B)'(\mathbf{q}_N \mathbf{q}'_N) + 2(B)'(\mathbf{q}_N \mathbf{q}'_N) - \sum_{m=1}^M \beta_m 1_{Q_m}(\mathbf{q}_N) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{b}'\mathbf{q}_1 + \mathbf{q}'_1 B \mathbf{q}_1 - \sum_{m=1}^M \beta_m 1_{Q_m}(\mathbf{q}_1) \\ \vdots \\ \mathbf{b}'\mathbf{q}_N + \mathbf{q}'_N B \mathbf{q}_N - \sum_{m=1}^M \beta_m 1_{Q_m}(\mathbf{q}_N) \end{bmatrix} = \mathbf{0} \end{aligned}$$

Thus, D cannot have full column rank.

Lemma 20 *If S defined by (??) is a signal function then it is distinguishing.*

Let $v_{1,\ell}, v_{2,\ell}, \dots, v_{J,\ell}$ enumerate the values in I_ℓ . Let $\tilde{L} = L(L-1)/2$. Let \tilde{Q}_1 denote the finite subset of \mathbb{N}_0^L which satisfies

$$\tilde{Q}_1 = \left\{ v_{1,\ell} \mathbf{e}_\ell \in \mathbb{N}_0^L : \ell \in \{1, \dots, L\} \right\} \equiv \{\mathbf{q}_1, \dots, \mathbf{q}_L\}$$

Let \tilde{Q}_2 be defined by

$$\tilde{Q}_2 = \left\{ v_{2,\ell} \mathbf{e}_\ell \in \mathbb{N}_0^L : \ell \in \{1, \dots, L\} \right\} \equiv \{\mathbf{q}_{L+1}, \dots, \mathbf{q}_{2L}\}$$

Let \tilde{Q}_{11} be defined by

$$\tilde{Q}_{11} = \left\{ v_{1,\ell} \mathbf{e}_\ell + v_{1,k} \mathbf{e}_k \in \mathbb{N}_0^L : \ell \neq k \right\} \equiv \{\mathbf{q}_{2L+1}, \dots, \mathbf{q}_{\tilde{L}}\}$$

Let \tilde{Q}_3 be the singleton set defined by

$$\tilde{Q}_3 = \{v_{3,1} \mathbf{e}_1\} \equiv \{\mathbf{q}_{\tilde{L}+1}\}$$

Let \tilde{Q}_{12} be the singleton set defined by

$$\tilde{Q}_{12} = v_{1,1} \mathbf{e}_1 + v_{2,2} \mathbf{e}_2 \equiv \{\mathbf{q}_{\tilde{L}+2}\}$$

Let \tilde{Q} be the union of the sets just defined. That is,

$$\tilde{Q} = \tilde{Q}_1 \cup \tilde{Q}_2 \cup \tilde{Q}_3 \cup \tilde{Q}_{11}$$

It is clear that for all $\mathbf{q}, \tilde{\mathbf{q}} \in \tilde{Q}_1 \cup \tilde{Q}_2 \cup \tilde{Q}_3$ we have $\mathcal{Q}_\infty(\mathbf{q}) = \mathcal{Q}_\infty(\tilde{\mathbf{q}})$. Let Q_1 denote this subset of \mathbb{N}_0^L (so, $Q_1 = \mathcal{Q}_\infty(\mathbf{q})$ for any $\mathbf{q} \in \tilde{Q}_1 \cup \tilde{Q}_2 \cup \tilde{Q}_3$). Similarly, it is clear that for all $\mathbf{q}, \tilde{\mathbf{q}} \in \tilde{Q}_{11} \cup \tilde{Q}_{12}$ we have $\mathcal{Q}_\infty(\mathbf{q}) = \mathcal{Q}_\infty(\tilde{\mathbf{q}})$. Let Q_2 denote this subset of \mathbb{N}_0^L (so, $Q_2 = \mathcal{Q}_\infty(\mathbf{q})$ for any $\mathbf{q} \in \tilde{Q}_{11} \cup \tilde{Q}_{12}$).

We shall apply Lemma 19. To do this, let D denote the matrix in Lemma 19. Let V_1 denote the $L \times L$ diagonal matrix whose diagonal entries are $v_{1,1}, v_{1,2}, \dots, v_{1,L}$. Let V_2 be the $L \times L$ diagonal matrix whose diagonal entries are $v_{2,1}, v_{2,2}, \dots, v_{2,L}$. Let V_3 denote the $\tilde{L} \times \tilde{L}$ diagonal matrix whose diagonal entries are $v_{1,\ell} v_{1,k}$ for $\ell < k$ where the entries may be enumerated as $v_{1,1} v_{1,2}, \dots, v_{1,1} v_{1,L}, v_{1,2} v_{1,3}, \dots, v_{1,2} v_{1,L}, v_{1,3} v_{1,4}, \dots, v_{1,L-1} v_{1,L}$. If D and D' are two matrices with the same rank then write $D \sim D'$. We will alter D using

elementary row and column operations (which preserve the rank). It can be verified that D can be expressed as

$$D = \begin{bmatrix} V_1 & V_1^2 & 0 & \iota & 0 \\ V_2 & V_2^2 & 0 & \iota & 0 \\ X_1 & X_2 & V_3 & 0 & \iota \\ v_{3,1}\mathbf{e}'_1 & v_{3,1}^2\mathbf{e}'_1 & 0 & 1 & 0 \\ \mathbf{x}'_1 & \mathbf{x}'_2 & v_{1,1}v_{2,2}\mathbf{e}'_1 & 0 & 1 \end{bmatrix}$$

First, we perform a column and row swap which yields

$$D \sim \begin{bmatrix} V_1 & V_1^2 & \iota & 0 & 0 \\ V_2 & V_2^2 & \iota & 0 & 0 \\ v_{3,1}\mathbf{e}'_1 & v_{3,1}^2\mathbf{e}'_1 & 1 & 0 & 0 \\ X_1 & X_2 & 0 & V_3 & \iota \\ \mathbf{x}'_1 & \mathbf{x}'_2 & 0 & v_{1,1}v_{2,2}\mathbf{e}'_1 & 1 \end{bmatrix} \equiv D'$$

Note that the matrix D' has the same rank as D and because of the presence of the zeros in the upper right of D' we see that D' has full rank if the following two matrices are each full rank

$$D_1 = \begin{bmatrix} V_1 & V_1^2 & \iota \\ V_2 & V_2^2 & \iota \\ v_{3,1}\mathbf{e}'_1 & v_{3,1}^2\mathbf{e}'_1 & 1 \end{bmatrix} \quad \text{and} \quad D_2 = \begin{bmatrix} V_3 & \iota \\ v_{1,1}v_{2,2}\mathbf{e}'_1 & 1 \end{bmatrix}$$

Now, we have

$$\begin{aligned} D_1 &\sim \begin{bmatrix} I & V_1 & V_1^{-1}\iota \\ V_2 & V_2^2 & \iota \\ v_{3,1}\mathbf{e}'_1 & v_{3,1}^2\mathbf{e}'_1 & 1 \end{bmatrix} \sim \begin{bmatrix} I & V_1 & V_1^{-1}\iota \\ 0 & V_2^2 - V_2V_1 & \iota - V_2V_1^{-1}\iota \\ 0 & (v_{3,1}^2 - v_{3,1}v_{1,1})\mathbf{e}'_1 & 1 - \frac{v_{3,1}}{v_{1,1}} \end{bmatrix} \\ &\sim \begin{bmatrix} I & V_1 & V_1^{-1}\iota \\ 0 & I & (V_2 - V_1)^{-1}(V_2^{-1} - V_1^{-1})\iota \\ 0 & (v_{3,1}^2 - v_{3,1}v_{1,1})\mathbf{e}'_1 & 1 - \frac{v_{3,1}}{v_{1,1}} \end{bmatrix} \\ &= \begin{bmatrix} I & V_1 & V_1^{-1}\iota \\ 0 & I & -V_2^{-1}V_1^{-1}\iota \\ 0 & (v_{3,1}^2 - v_{3,1}v_{1,1})\mathbf{e}'_1 & 1 - \frac{v_{3,1}}{v_{1,1}} \end{bmatrix} \\ &\sim \begin{bmatrix} I & V_1 & V_1^{-1}\iota \\ 0 & I & V_2^{-1}V_1^{-1}\iota \\ 0 & 0 & 1 - \frac{v_{3,1}}{v_{1,1}} + \frac{v_{3,1}^2 - v_{3,1}v_{1,1}}{v_{1,1}v_{2,1}} \end{bmatrix} \end{aligned}$$

So, D_1 is full rank if $1 - \frac{v_{3,1}}{v_{1,1}} + \frac{v_{3,1}^2 - v_{3,1}v_{1,1}}{v_{1,1}v_{2,1}} \neq 0$. This is equivalent to

$$v_{1,1}v_{2,1} - v_{2,1}v_{3,1} + v_{3,1}^2 - v_{1,1}v_{3,1} = 0$$

. We have

$$v_{1,1}v_{2,1} - v_{2,1}v_{3,1} + v_{3,1}^2 - v_{1,1}v_{3,1} = (v_{3,1} - v_{1,1})(v_{3,1} - v_{1,2})$$

and so D_1 is full rank unless $v_{3,1} = v_{1,1}$ or $v_{3,1} = v_{1,2}$. But, from the definition of $v_{1,1}$, $v_{2,1}$, and $v_{3,1}$ we know this is not the case. Thus, D_1 is full rank. We now show that D_2 is full rank.

$$D_2 \sim \begin{bmatrix} I & V_3^{-1}\iota \\ v_{1,1}v_{2,2}\mathbf{e}'_1 & 1 \end{bmatrix} \sim \begin{bmatrix} I & V_3^{-1}\iota \\ 0 & 1 - \frac{v_{1,1}v_{2,2}}{v_{1,1}v_{1,2}} \end{bmatrix} = \begin{bmatrix} I & V_3^{-1}\iota \\ 0 & 1 - \frac{v_{2,2}}{v_{1,2}} \end{bmatrix} \quad (82)$$

Thus, D_2 is full rank if $v_{1,2} \neq v_{2,2}$. But, this is true from the definition of $v_{1,2}$ and $v_{2,2}$ and so D_2 is full rank. So, we see that D is full rank and so, by Lemma 19, S is distinguishing.

10.5 Derivatives and CLT

Let f satisfy (8) and let $f(\mathbf{q}|Q; \boldsymbol{\theta})$ satisfy (17). Let \mathbf{c} be a random vector with pmf given by f and let S, S_1, S_2, \dots, S_I be signal functions. Let $\mathcal{S} = S(\mathbf{c})$ and let $\mathcal{S}_i = S_i(\mathbf{c})$ for each $i \in \{1, \dots, I\}$. Let $v(\mathbf{q}; \boldsymbol{\theta})$ be defined by

$$v(\mathbf{q}, \mathbf{p}; \boldsymbol{\theta}) = \frac{\partial U(\mathbf{q}, \mathbf{p}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \quad (83)$$

and let $V(\mathbf{q}, \mathbf{p}; \boldsymbol{\theta})$ be defined by

$$V(\mathbf{q}, \mathbf{p}; \boldsymbol{\theta}) = \frac{\partial^2 U(\mathbf{q}, \mathbf{p}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \quad (84)$$

We have the following.

Lemma 21 *The following equations are true.*

$$\begin{aligned} \frac{\partial \ln f(\mathbf{c}|\mathcal{S}, \boldsymbol{\rho}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} &= v(\mathbf{c}, \boldsymbol{\rho}; \boldsymbol{\theta}) - \mathbf{E} \left[v(\mathbf{c}, \boldsymbol{\rho}; \boldsymbol{\theta}) \middle| \mathcal{S}, \boldsymbol{\rho} \right] \\ \frac{\partial^2 \ln f(\mathbf{c}|\mathcal{S}, \boldsymbol{\rho}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} &= V(\mathbf{c}, \boldsymbol{\rho}; \boldsymbol{\theta}) - \mathbf{E} \left[V(\mathbf{c}, \boldsymbol{\rho}; \boldsymbol{\theta}) \middle| \mathcal{S}, \boldsymbol{\rho} \right] - \mathbf{Var} \left[v(\mathbf{c}, \boldsymbol{\rho}; \boldsymbol{\theta}) \middle| \mathcal{S} \right] \\ \mathbf{Var} \left[\frac{1}{I} \sum_{i=1}^I \frac{\partial \ln f(\mathbf{c}|\mathcal{S}, \boldsymbol{\rho}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right] &= \mathbf{E} \left[\mathbf{Var} \left[v(\mathbf{c}, \boldsymbol{\rho}; \boldsymbol{\theta}) \middle| \mathcal{S}, \boldsymbol{\rho} \right] \right] \\ &\quad - \left(\frac{I-1}{I} \right) \mathbf{Var} \left[\mathbf{E} \left[\mathbf{E} \left[v(\mathbf{c}, \boldsymbol{\rho}; \boldsymbol{\theta}) \middle| \mathcal{S}, \boldsymbol{\rho} \right] \middle| \mathbf{c}, \boldsymbol{\rho} \right] \right] \end{aligned}$$

For $Q \subseteq \mathbb{N}_0^L$ let $V(Q; \boldsymbol{\theta})$ be defined by

$$V(Q; \boldsymbol{\theta}) = \sum_{\mathbf{q} \in Q} f(\mathbf{q}|Q; \boldsymbol{\theta}) \frac{\partial U(\mathbf{q}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$$

Define $W_1(\mathbf{q}|Q; \boldsymbol{\theta})$ by

$$W_1(\tilde{\mathbf{q}}|Q; \boldsymbol{\theta}) = \sum_{\mathbf{q} \in Q} f(\mathbf{q}|Q; \boldsymbol{\theta}) \left[\frac{\partial^2 U(\tilde{\mathbf{q}}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} - \frac{\partial^2 U(\mathbf{q}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \right]$$

and define $W_2(Q; \boldsymbol{\theta})$ by

$$W_2(Q; \boldsymbol{\theta}) = \sum_{\mathbf{q} \in Q} f(\mathbf{q}|Q; \boldsymbol{\theta}) \left(\frac{\partial U(\mathbf{q}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} - V(Q; \boldsymbol{\theta}) \right) \left(\frac{\partial U(\mathbf{q}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} - V(Q; \boldsymbol{\theta}) \right)'$$

It can be verified that

$$\frac{\partial \ln f(\tilde{\mathbf{q}}|Q; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \frac{\partial U(\tilde{\mathbf{q}}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} - V(Q; \boldsymbol{\theta})$$

and that

$$\frac{\partial^2 \ln f(\tilde{\mathbf{q}}|Q; \boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} = W_1(\tilde{\mathbf{q}}|Q; \boldsymbol{\theta}) - W_2(Q; \boldsymbol{\theta})$$