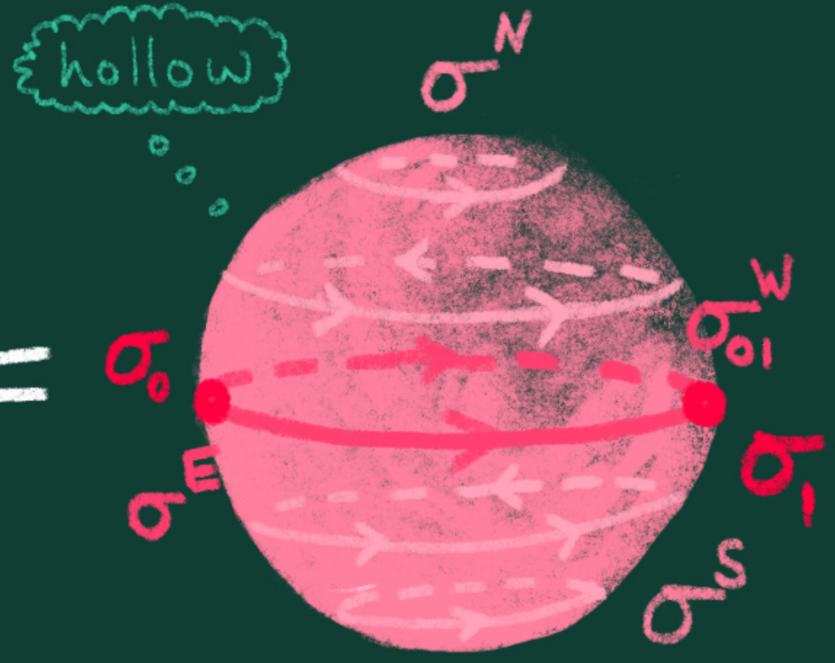
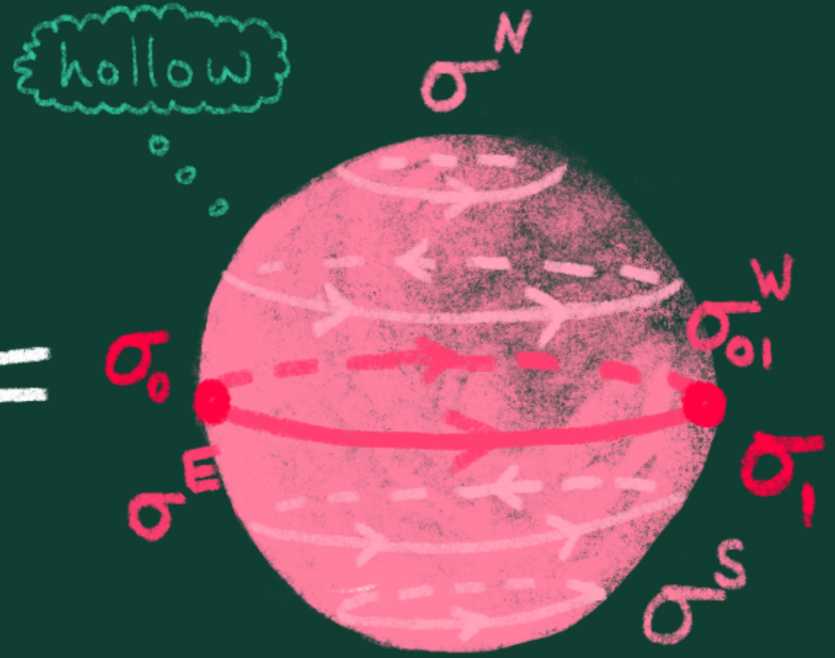
Even More Maps

$S^2 =$



$$\partial(\sigma^S - \sigma^N) = 0$$

$$\mathbb{R}[2] \xrightarrow{\sigma^N - \sigma^S} C_*(S^2)$$

$S^2 =$ 

$$\partial(\sigma^S - \sigma^N) = 0$$

$$\mathbb{R}[2] \xrightarrow{\sigma^N - \sigma^S} C_*(S^2)$$

"fundametal class of S^2"

$$\mathbb{R}[K] \longrightarrow V$$
$$V \longrightarrow \mathbb{R}[K]$$

$$
\begin{array}{c}
\mathbb{R}[K] \longrightarrow V \\
V \longrightarrow \mathbb{R}[K]
\end{array}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\mathbb{R}[K] \arrow[rr, "\langle v|\varphi\rangle = \varphi(v)"] \arrow[dr, bend right, "v"'] & & \mathbb{R}[K] \\
& V \arrow[ur, bend right, "\varphi"'] &
\end{tikzcd}
$$

$$\begin{matrix} \mathbb{R}[K] \longrightarrow V \\ V \longrightarrow \mathbb{R}[K] \end{matrix} \quad \rightsquigarrow \quad \begin{tikzcd} \mathbb{R}[K] \arrow[rr, "\langle v | \varphi \rangle = \varphi(v)"] \arrow[dr, bend right, "v"'] & & \mathbb{R}[K] \\ & V \arrow[ur, bend right, "\varphi"'] & \end{tikzcd}$$

$$\mathrm{End}(\mathbb{R}[K]) \cong \mathbb{R} \ni \varphi(v)$$

$$\text{multipy by } c \longleftarrow\!\shortmid\; c$$

$$\begin{array}{l} \mathbb{R}[K] \longrightarrow V \\ V \longrightarrow \mathbb{R}[K] \end{array} \rightsquigarrow \begin{tikzcd} \mathbb{R}[K] \arrow[rr, "\langle v|\varphi\rangle = \varphi(v)"] \arrow[dr, bend right, "v"'] & & \mathbb{R}[K] \\ & V \arrow[ur, bend right, "\varphi"'] & \end{tikzcd}$$

$$\mathrm{End}(\mathbb{R}[K]) \cong \mathbb{R} \ni \varphi(v)$$

multipy by $c \longleftarrow\!\shortmid\; c$

<u>e.g.</u>

$$\begin{tikzcd} \mathbb{R}[-1] \arrow[r, "\frac{1}{z}\cdot dz"] \arrow[rr, bend right, "\text{\# of times } \gamma \text{ winds around } 0 \in \mathbb{C}"'] & \Omega^\bullet(\mathbb{C}^\times) \arrow[r, "\frac{1}{2\pi i}\int_\gamma(-)"] & \mathbb{R}[-1] \end{tikzcd}$$

$$\mathbb{D}[k] = \left(\dots 0 \to \overset{k}{\mathbb{R}} \xrightarrow{1} \overset{k-1}{\mathbb{R}} \to 0 \dots \right)$$

$$\mathbb{D}[k] = \left(\dots 0 \to \overset{k}{\mathbb{R}} \xrightarrow{1} \overset{k-1}{\mathbb{R}} \to 0 \dots \right)$$

$$\left\{ \mathbb{D}[k] \longrightarrow V \right\} \cong \left\{ \begin{array}{ccc} \mathbb{R} & \xrightarrow{w} & V_k \\ {\scriptstyle 1}\big\downarrow & \partial w = v & \big\downarrow \partial \\ \mathbb{R} & \xrightarrow{v} & V_{k-1} \end{array} \right\} \cong \left\{ \begin{array}{c} w \in V_k,\ v \in V_{k-1} \text{ s.t. } \\ \partial w = v \end{array} \right\}$$

$$\mathbb{D}[k] = \left(\dots 0 \to \overset{k}{\mathbb{R}} \xrightarrow{1} \overset{k-1}{\mathbb{R}} \to 0 \dots \right)$$

$$\{\mathbb{D}[k] \to V\} \cong \left\{ \begin{tikzcd} \mathbb{R} \arrow[r, "w"] \arrow[d, "1"'] \arrow[dr, phantom, "\partial w = v"] & V_k \arrow[d, "\partial"] \\ \mathbb{R} \arrow[r, "v"'] & V_{k-1} \end{tikzcd} \right\} \cong \left\{ \begin{matrix} w \in V_k,\ v \in V_{k-1} \text{ s.t.} \\ \partial w = v \end{matrix} \right\}$$

$$\mathbb{R}[k] \to \mathbb{D}[k+1]$$

$$\begin{tikzcd} k+1 & 0 \arrow[r] \arrow[d] & \mathbb{R} \arrow[d] \\ k & \mathbb{R} \arrow[r] & \mathbb{R} \end{tikzcd}$$

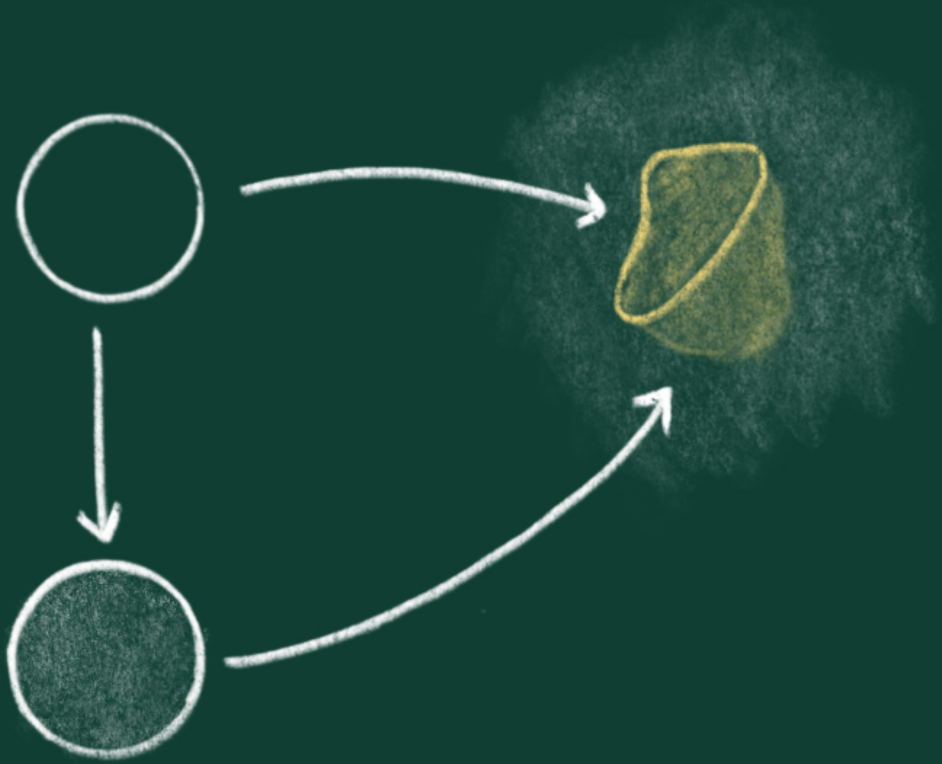
$$\mathbb{D}[k] \to \mathbb{R}[k]$$

$$\begin{tikzcd} k & \mathbb{R} \arrow[r, "1"] \arrow[d, "1"'] & \mathbb{R} \arrow[d] \\ k-1 & \mathbb{R} \arrow[r, "0"'] & 0 \end{tikzcd}$$

extension problem

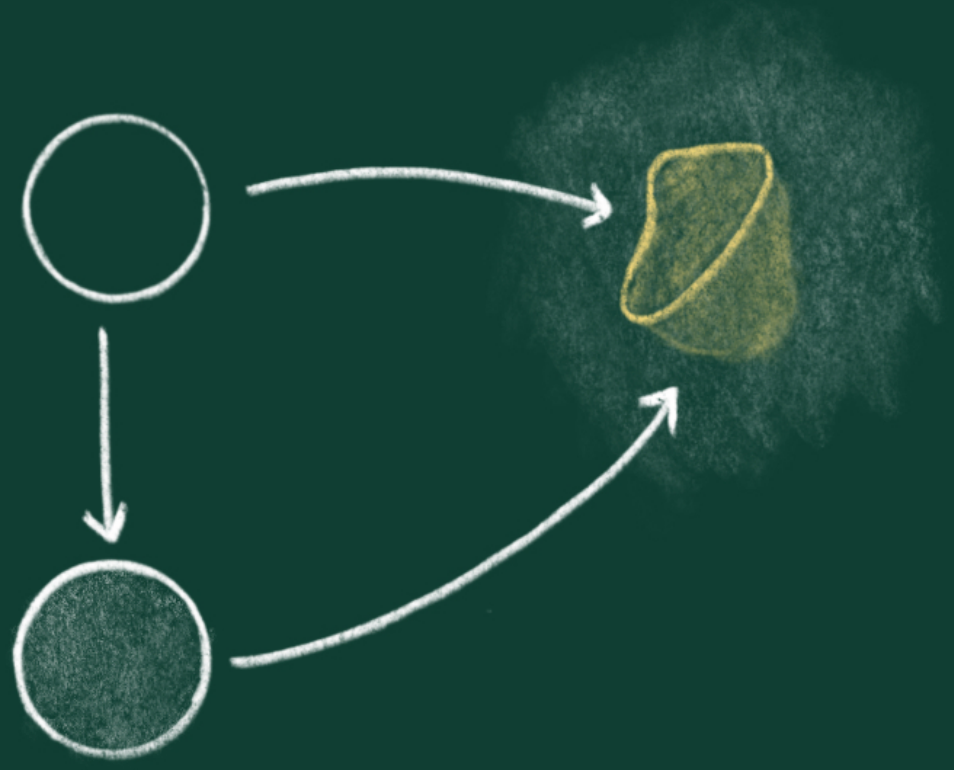
$$\begin{tikzcd} \mathbb{R}[k] \arrow[r, "r"] \arrow[d] \arrow[dr, phantom, "\partial w = v"] & V \\ \mathbb{D}[k+1] \arrow[ur, "w"'] & \end{tikzcd}$$

we call a cycle v
a **boundary** when $\exists$ sol'n
to $\partial \tilde{v} = v$



extension problem

$$\begin{tikzcd} \mathbb{R}[k] \arrow[r, "v"] \arrow[d] \arrow[dr, phantom, "\partial w = v"] & V \\ \mathbb{D}[k+1] \arrow[ur, "w"'] & \end{tikzcd}$$

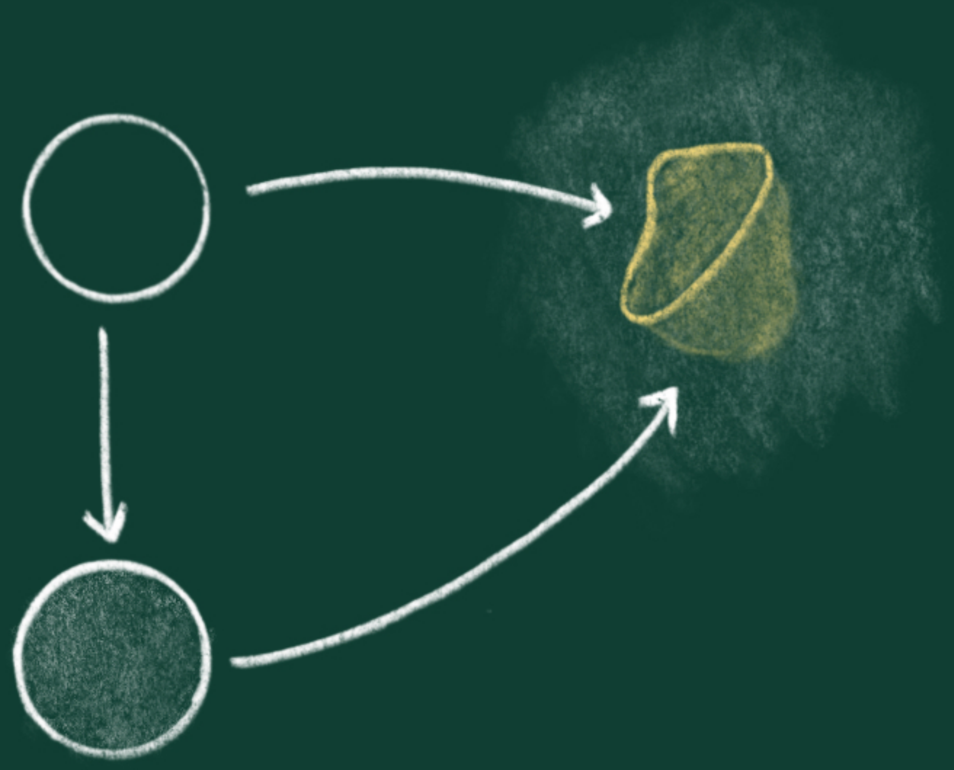


we call a cycle v a <u>boundary</u> when $\exists$ sol'n to $\partial \tilde{v} = v$

$$\underbrace{\mathbb{R}[k] \to \overbrace{\mathbb{D}[k+1] \to V}^{\text{every "boundary"}}}_{\text{is a cycle:}}$$

$$\begin{tikzcd}
\mathbb{R}[k] \arrow[r, "v"] \arrow[d] \arrow[dr, phantom, "\partial w = v"] & V \\
\mathbb{D}[k+1] \arrow[ur, "w"'] &
\end{tikzcd}$$

extension problem



we call a cycle v a <u>boundary</u> when ∃ sol'n to $\partial \tilde{v} = v$

$$\underbrace{\mathbb{R}[k] \to \overbrace{\mathbb{D}[k+1] \to V}^{\text{every ``boundary''}}}_{\text{is a cycle:}}$$

<u>Ex:</u> $\mathbb{R}[2] \to C_*(S^2)$ is not a boundary

all maps $\mathbb{D}[K+1] \to \mathbb{R}[K]$ are 0

all maps $\mathbb{D}[k+1] \to \mathbb{R}[k]$ are 0

$$
\begin{tikzcd}
\mathbb{R}[k] \arrow[rr, "v"] \arrow[dr, bend right, "0"'] \arrow[rrr, bend right=70, "0"'] & & Y \arrow[r, "\varphi"] & \mathbb{R}[k] \\
& \mathbb{D}[k+1] \arrow[ur, bend right, "w"'] \arrow[u, phantom, "\partial v = w"] & &
\end{tikzcd}
$$

cocycle (boundary) $= 0$

$$\left\{ V \xrightarrow{\breve{\varphi}} \mathbb{D}[K] \right\} \simeq \left\{ \begin{matrix} V_k \xrightarrow{\breve{\varphi}_k} \mathbb{R} \\ V_{k-1} \xrightarrow{\hat{\varphi}_{k-1}} R \end{matrix} \;\middle|\; \breve{\varphi}_k(v) = \breve{\varphi}_{k-1}(\partial v) \right\}$$

$$\left\{ V \xrightarrow{\hat{\check{\varphi}}} \mathbb{D}[K] \right\} \simeq \left\{ \begin{matrix} V_k \xrightarrow{\hat{\check{\varphi}}_k} \mathbb{R} \\ V_{k-1} \xrightarrow{\hat{\check{\varphi}}_{k-1}} R \end{matrix} \;\middle|\; \hat{\check{\varphi}}_k(v) = \hat{\check{\varphi}}_{k-1}(\partial v) \right\}$$

$$\begin{tikzcd}
 & \mathbb{D}[K] \arrow[d] \\
V \arrow[ur, "\hat{\check{\varphi}}"] \arrow[r, "\varphi"'] & \mathbb{R}[K]
\end{tikzcd}$$

we call such a cocycle φ a <u>coboundary</u> when

$$\varphi(v) = \hat{\check{\varphi}}(\partial v)$$

All maps $\mathbb{R}[K] \longrightarrow \mathbb{D}[K]$ are 0

All maps $\mathbb{R}[K] \longrightarrow \mathbb{D}[K]$ are 0

$$\begin{tikzcd}
\mathbb{R}[K] \arrow[r, "v"] \arrow[drr, bend right, "0"'] \arrow[rrr, bend right=60, "0"'] & V \arrow[rr, "\varphi"] \arrow[dr] & & \mathbb{R}[K] \\
& & \mathbb{D}[K] \arrow[ur] &
\end{tikzcd}$$

coboundary (cycle) = 0

Exercise

$$\begin{tikzcd}
\Omega^\bullet(\mathbb{R}^3) \arrow[r, "\int_\gamma"] \arrow[dr, "\Theta"'] & \mathbb{R}[-1] \\
& \mathbb{D}[K] \arrow[u]
\end{tikzcd}$$

Use $\partial \mathcal{U} = \gamma$ to construct Θ

A $n \times (n+k)$ matrix of rank n $\rightsquigarrow$ $C(A) = (\dots 0 \to \overset{1}{\mathbb{R}^n} \xrightarrow{A} \overset{0}{\mathbb{R}^{n+k}} \to 0 \dots)$

$\underline{\text{eg}}$ $\quad A = \iota_{\leq n} = \begin{array}{c} \begin{bmatrix} 1 & & 0 \\ & \ddots & 1 \\ 0 & & 0 \\ & & \ddots \; 0 \end{bmatrix} \end{array} \begin{array}{l} \}\, n \\ \}\, k \end{array} =$ include 1^{st} n coord.

A $n \times (n+k)$ matrix of rank n $\rightsquigarrow$ $C(A) = \big(\ldots 0 \longrightarrow \overset{1}{\mathbb{R}^n} \xrightarrow{A} \overset{0}{\mathbb{R}^{n+k}} \longrightarrow 0 \ldots \big)$

<u>eg</u> $A = \iota_{\leq n} = \begin{array}{cc} \begin{bmatrix} 1 \; \ddots \; 1 & 0 \\ 0 & 0 \; \ddots \; 0 \end{bmatrix} & \begin{array}{l} \}\, n \\ \}\, k \end{array} \end{array} =$ include 1^{st} n coord.

$$\left\{ \mathbb{D}[1] \longrightarrow C(A) \right\} \simeq \left\{ \begin{array}{c} \text{solutions to the} \\ \text{inhomogenous equation} \\ Ax = y \end{array} \right\}$$

A $n\times(n+k)$ matrix of rank n $\rightsquigarrow$ $C(A) = (\ldots 0 \to \overset{1}{\mathbb{R}^n} \xrightarrow{A} \overset{0}{\mathbb{R}^{n+k}} \to 0 \ldots)$

$\underline{\text{eg}}$ $A = \iota_{\leq n} = \begin{bmatrix} 1 & & & 0 \\ & \ddots & 1 & \\ 0 & & 0 & \\ & & & \ddots 0 \end{bmatrix}\begin{matrix}\}n \\ \}k\end{matrix} =$ include 1^{st} n coord.

$$\{\mathbb{D}[1] \longrightarrow C(A)\} \simeq \left\{\begin{matrix}\text{solutions to the} \\ \text{inhomogenous equation} \\ Ax = y\end{matrix}\right\}$$

Gaussian Elimination $\exists$ $C(A) \xrightarrow[S]{\simeq} C(\iota_{\leq n})$