Introduction to mathematical induction

A sample problem

Proposition: The sum of the first n odd natural numbers is n^2.

The first n odd numbers are $1, 3, \ldots, 2n-1$. So in summation notation this is the claim that, for all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n}(2i-1) = n^2$$

.

Sample problem continued.

The proposition above is infinitely many statements.

n	sum of the first n odd natural numbers	n^2
1	$1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$	1
2	$1+3 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$	4
3	$1+3+5 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$	9
4	$1+3+5+7 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$	16
5	$1+3+5+7+9 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$	25
$\vdots$	$\vdots$	$\vdots$
n	$1+3+5+7+9+11+\cdots+(2n-1) = \ldots\ldots$	n^2
$\vdots$	$\vdots$	$\vdots$

Figure 1: From pg. 180 of the text

Sample continued

We can prove any *one* of these statements.

How do we prove *all* of them?

Mathematical induction

Mathematical induction *extends* our system of logic by adding an axiom.

Axiom of Induction: Let $P(n)$ be a collection of statements, one for each natural number. Suppose that $P(1)$ is true and, for all n, the implication $P(n) \implies P(n+1)$ is true. Then $P(n)$ is true for all n.

The book calls this a *method of proof* but it is really an axiom.

A prototype

Proposition: Suppose that S is a set such that $1 \in S$ and, for all n, if $n \in S$, then also $n+1 \in S$. Then $\mathbb{N} \subseteq S$.

Proof: Let $P(n)$ be the statement $n \in S$. The hypotheses say that $P(1)$ is true, and that $P(n) \implies P(n+1)$. Therefore $P(n)$ is true for all n, and so every natural number is in S, so $\mathbb{N} \subset S$.

Proof of the result on sum of odd numbers

Proposition: For all n, we have

$$\sum_{i=1}^{n}(2n-1) = n^2.$$

Proof: We apply mathematical induction. The statement $P(n)$ is

$$\sum_{i=1}^{n}(2n-1) = n^2.$$

So $P(1)$ is the claim that $1 = 1^2$, which is true. To prove that $P(n) \implies P(n+1)$, we assume $P(n)$ true:

$$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

Proof, continued

$$\begin{aligned}1+3+5+\cdots+(2n-1)+(2(n+1)-1) &= \\ 1+3+5+\cdots+(2n-1)+2n+1 &= \\ n^2+2n+1 &= (n+1)^2.\end{aligned}$$

Therefore, if $P(n)$ is true then $P(n+1)$ is also true. By mathematical induction $P(n)$ is true for all n.