Indexed sets

Indexed sets

Summation notation

“Recall” that we can write a long sum of a bunch of numbers using summation notation.

$$\overbrace{a_1 + a_2 + \cdots + a_n}^{n \text{ number}} = \sum_{i=1}^{n} a_i$$

$a_1 + a_2 + a_3 + \cdots + a_n$

$$\underbrace{4 + 7 + 8 + 13 + 4}_{n=5}$$

We can even write infinite sums:

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^i} + \cdots = \sum_{i=1}^{\infty} \frac{1}{2^i}$$

i takes on every value in $\mathbb{N}$

Indexed sets

Suppose we have a bunch of sets $\underbrace{A_1, A_2, \ldots, A_n}_{\text{all sets}}$. Then we can write:

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

$$\bigcup_{i=1}^{n} A_i$$

and

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

Indexed sets

If $A_1, A_2, \ldots, A_n$ are all sets, then

$$\bigcup_{i=1}^{n} A_i = \{x : x \text{ belongs to at least one set } A_i\}$$

$$Y = \bigcup_{i=1}^{3} A_i = A_1 \cup A_2 \cup A_3$$

$$= \{x : x \in A_1 \text{ OR } x \in A_2 \text{ OR } x \in A_3\}$$

- $A_1 = \{1, 4, 10, 12\}$
- $A_2 = \{5, 12, 15\}$
- $A_3 = \{1, 4, 15, 35\}$

What is $\bigcup_{i=1}^{3} A_i$?

$$Y = \{1, 4, 10, 12, 5, 15, 35\}$$

Indexed sets

$$\bigcap_{i=1}^{n} A_i = \{x : x \text{ belongs to every set } A_i\}$$

- $A_1 = \{\cancel{1}, \cancel{4}, \cancel{10}, \cancel{12}\}$
- $A_2 = \{\cancel{5}, \cancel{12}, \cancel{15}\}$
- $A_3 = \{\cancel{1}, \cancel{4}, \cancel{15}, \cancel{35}\}$

What is $\bigcap_{i=1}^{3} A_i$?

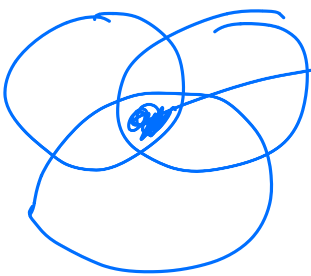
$$Y = \bigcap_{i=1}^{3} A_i = A_1 \cap A_2 \cap A_3$$

$1 \notin A_2$ $5 \notin A_1$

$4 \notin A_2$ $15 \notin A_1$

$10 \notin A_2$ $35 \notin A_2$

$12 \notin A_3$



Nothing here

$$Y = \emptyset$$

Indexed sets

One can also take the union and intersection of infinitely many sets.

$\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$.

$A_1 = \{-1, 0, 1\} \subseteq \mathbb{Z}$

$A_2 = \{-2, 0, 2\} \subseteq \mathbb{Z}$

$A_3 = \{-3, 0, 3\} \subseteq \mathbb{Z}$

$\vdots$

Example. For each $i \in \mathbb{N}$, let

$$A_i = \{-i, 0, i\}$$

What is $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$.?

$\bigcup_{i=1}^{\infty} A_i = \{x : x$ belongs to at least one of $A_i\}$.

Any integer belongs to some A_i.

Take an integer n. Then $n \in A_n = \{-n, 0, n\}$.

So every integer belongs to $\bigcup_{i=1}^{\infty} A_i$

so $\mathbb{Z} = \bigcup_{i=1}^{\infty} A_i$

$\bigcap_{i=1}^{\infty} A_i = \{x : x$ belongs to every $A_i\}$. Only zero has this property, $0 \in A_i$ for all i, $\bigcap_{i=1} A_i = \{0\}$.

Index sets

Instead of numbering the sets, one can label them with elements of any set I called an index set.

$\bigcup_{i \in I} A_i$ is the set of elements that belong to *at least one* of the sets A_i.

$\bigcap_{i \in I} A_i$ is the set of elements that belong to *every one* of the sets A_i.

$A_1, A_2, \ldots$

what about A_r for every real number r.

$$\bigcup_{i=1}^{\infty} A_i \xleftrightarrow{\text{Instead}} \bigcup_{i \in \mathbb{N}} A_i$$

$$\cdots \cup A_{-3} \cup A_{-2} \cup A_{-1} \cup A_0 \cup \cdots$$

$$\bigcup_{i \in \mathbb{Z}} A_i$$

Index sets example

Let C be the set of Counties in the state of Connecticut (there are 8 of these). For each county $c \in C$, let $T(c)$ be the set of Towns in that County.

For example, if c is Tolland County, then the elements of $T(c)$ are Andover, Bolton, Columbia, Coventry, Ellington, Hebron, Mansfield, Somers, Stafford, Tolland, Union, Vernon, and Willington.

What is $\bigcup_{c \in C} T(c)$?

$$T_c = T(c) = \{\text{towns in county } c\}.$$

$$\bigcup_{c \in C} T(c) = \{\text{elements in any of the counties}\}$$
$$= \{\text{all towns in CT}\}$$

$$\bigcap_{c \in C} T(c) = \emptyset$$

Index sets

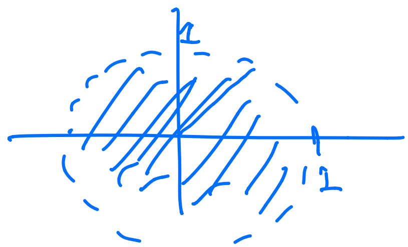
Let

$$\mathbb{R}_+ = \{r : r \in \mathbb{R}, r > 0\}. \quad = (0,\infty)$$

For every real number $r \in \mathbb{R}_+$, let

$$A_r = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}.$$

A_1 :



A_r = interior of circle of radius r in plane

A_2 :

$x^2+y^2<4$

Index sets

What is $\bigcap_{r\in\mathbb{R}_+} A_r$?

What, if any, pair (x,y) is in A_r for every $r>0$?

$x^2+y^2<r^2$ for every $r\in\mathbb{R}^+$

$r>0$

Inside circle of radius r no matter how small r gets.

$(0,0)\in\bigcap_{r\in\mathbb{R}_+} A_r$

$(0,0)\in A_r$ because

$0^2+0^2<r^2$

$0<r^2$

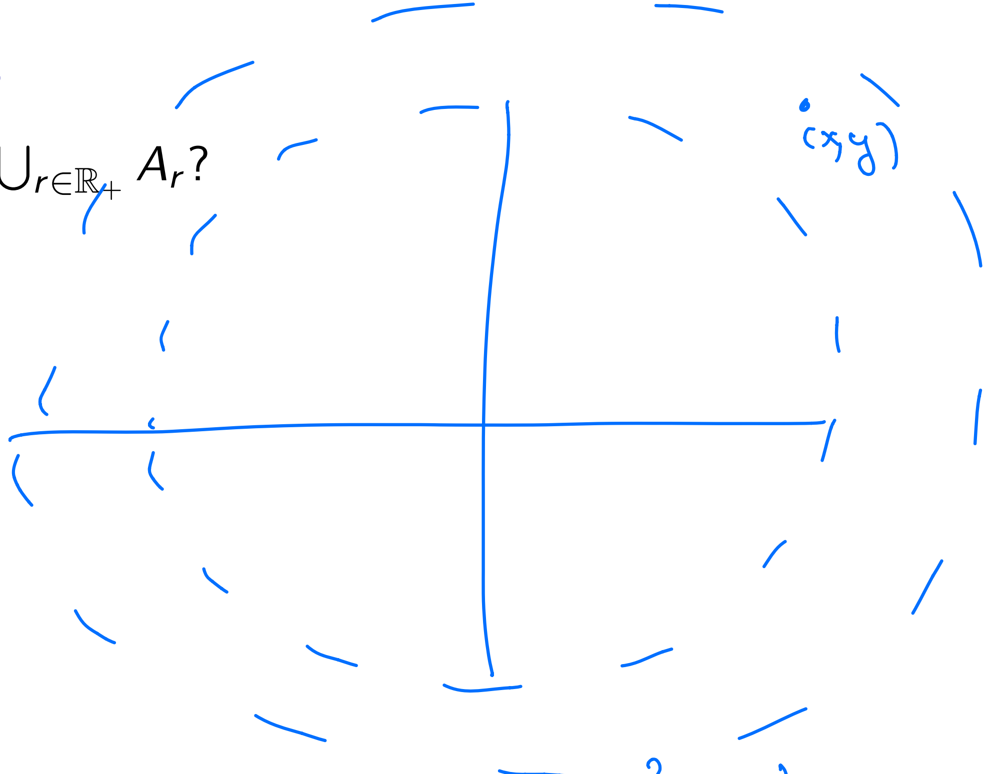
(x,y) choose $r' < x^2+y^2$

then $(x,y)\notin A_{r'}$, so not in $\bigcap_{r\in\mathbb{R}^+} A_r$.

$$\bigcap_{r\in\mathbb{R}_+} A_r = \{(0,0)\}$$

Index Sets

What is $\bigcup_{r \in \mathbb{R}_+} A_r$?



Every point $(x,y) \in A_r$ once $r^2 > x^2 + y^2$

So every point in $\mathbb{R}^2$ in $\bigcup_{r \in \mathbb{R}_+} A_r$

so $\mathbb{R}^2 = \bigcup_{r \in \mathbb{R}_+} A_r$

Example

What is $\bigcap_{i\in\mathbb{N}}[0, i+1]$?

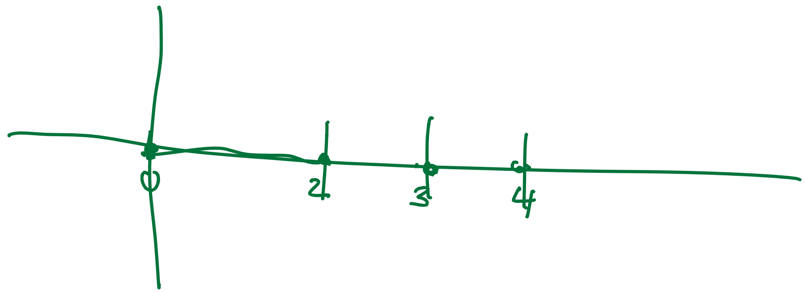
$[0,2] \subseteq \mathbb{R}$

$A_1 = [0,2]$

$A_2 = [0,3]$

$A_3 = [0,4]$

$\vdots$



$A_1 \underset{=}{\subseteq} \bigcap_{i\in\mathbb{N}} [0,i+1]$

$A_1 \subseteq A_2 \subseteq A_3 \cdots$

$x \in A_1$ is in $\bigcap_{i\in\mathbb{N}} [0,i+1]$

if $x \in \bigcap_{i\in\mathbb{N}} [0,i+1]$ then $x \in [0,2]$

$A_1 = \bigcap_{i\in\mathbb{N}} [0,i+1]$

Example

$J \subseteq I$

Suppose that I and J are sets, that $J \neq \emptyset$, and that ~~$I \subseteq J$~~. Is

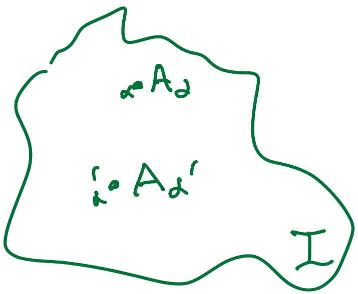
$$\bigcap_{a \in I} A_a \subseteq \bigcap_{a \in J} A_a?$$

$\{x : x \in A_\alpha \text{ for every } \alpha \in J\}$

fewer conditions on the right

Explain.

$\{x : x \in A_a \text{ for every } a \in I\}$.



$J \ni \alpha_1 \quad A_{\alpha_1}$

$\alpha_2 \quad A_{\alpha_2}$

$J \ni \alpha_3 \quad \vdots$

α_b

If $x \in A_\alpha$ for every $\alpha \in I$, then

Since $J \subseteq I$, $x \in A_\alpha$ for every $\alpha \in J$

Therefore $x \in \bigcap_{\alpha \in J} A_\alpha$

$I = \{1, 2, 3\}$ $\quad J = \{1\}$

$$\bigcap_{i \in I} A_i = A_1 \cap A_2 \cap A_3$$

$$\bigcap_{i \in J} A_i = A_1$$

Is $A_1 \cap A_2 \cap A_3 \subseteq A_1$

yes!

Ch 1.8 Examples.