

# Proofs Without Words

Exercises in Visual Thinking

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# Introduction

see (sē) *v.*, *saw*, *seen*, *seeing*. —*v.t.*

- ...  
5. to perceive (things) mentally; discern;  
understand: *to see the point of an argument.*

...  
—THE RANDOM HOUSE DICTIONARY  
OF THE ENGLISH LANGUAGE (2<sup>nd</sup> ED.)  
UNABRIDGED.

"Proofs without words" (PWWs) have become regular features in the journals published by the Mathematical Association of America — notably *Mathematics Magazine* and *The College Mathematics Journal*. PWWs began to appear in *Mathematics Magazine* about 1975, and, in an editors' note in the January 1976 issue of the *Magazine*, J. Arthur Seebach and Lynn Arthur Steen encouraged further contributions of PWWs to the *Magazine*. Although originally solicited for "use as end-of-article fillers," the editors went on to ask "What could be better for this purpose than a pleasing illustration that made an important mathematical point?"

A few years earlier Martin Gardner, in his popular "Mathematical Games" column in the October 1973 issue of the *Scientific American*, discussed PWWs as "look-see" diagrams. Gardner points out that "in many cases a dull proof can be supplemented by a geometric analogue so simple and beautiful that the truth of a theorem is almost seen at a glance." This dramatically illustrates the dictionary quote above: in English "to see" is often "to understand."

In the same vein, the editorial policy of *The College Mathematics Journal* throughout most of the 1980s stated that, in addition to expository articles, "The Journal also invites other types of contributions, most notably: *proofs without words*, mathematical poetry, quotes, ..." (their italics). But PWWs are not recent innovations — they have a long history. Indeed, in this volume you will find modern renditions of proofs without words from ancient China, classical Greece, and India of the twelfth century.

Of course, “proofs without words” are not really proofs. As Theodore Eisenberg and Tommy Dreyfus note in their paper “On the Reluctance to Visualize in Mathematics” [in *Visualization in Teaching and Learning Mathematics*, MAA Notes Number 19], some consider such visual arguments to be of little value, and “that there is one and only one way to communicate mathematics, and ‘proofs without words’ are not acceptable.” But to counter this viewpoint, Eisenberg and Dreyfus go on to give us some quotes on the subject:

[Paul] Halmos, speaking of Solomon Lefschetz (editor of the *Annals*), stated: “He saw mathematics not as logic but as pictures.” Speaking of what it takes to be a mathematician, he stated: “To be a scholar of mathematics you must be born with ... the ability to visualize” and most teachers try to develop this ability in their students. [George] Pólya’s “Draw a figure ...” is classic pedagogic advice, and Einstein and Poincaré’s views that we should use our visual intuitions are well known.

So, if “proofs without words” are not proofs, what are they? As you will see from this collection, this question does not have a simple, concise answer. But generally, PWWs are pictures or diagrams that help the observer see *why* a particular statement may be true, and also to see *how* one might begin to go about proving it true. In some an equation or two may appear in order to guide the observer in this process. But the emphasis is clearly on providing visual clues to the observer to stimulate mathematical thought.

I should note that this collection is not intended to be complete. It does not include all PWWs which have appeared in print, but is rather a sample representative of the genre. In addition, as readers of the Association’s journals are well aware, new PWWs appear in print rather frequently, and I anticipate that this will continue. Perhaps some day a second volume of PWWs will appear!

I hope that the readers of this collection will find enjoyment in discovering or rediscovering some elegant visual demonstrations of certain mathematical ideas; that teachers will want to share many of them with their students; and that all will find stimulation and encouragement to try to create new “proofs without words.”

*Acknowledgment.* I would like to express my appreciation and gratitude to the many people who have played a part in the publication of this collection: to Gerald Alexanderson and Martha Siegel, who, as editors of *Mathematics Magazine*, gave me encouragement over the years as I learned to read and write PWWs; to Doris Schattschneider, Eugene Klotz, and Richard Guy for sharing with me their collections of PWWs; and finally, to all those individuals who have contributed “proofs without words” to the mathematical literature (see the *Index of Names* on pp. 151-152), without whom this collection simply would not exist.

*Note.* All the drawings in this collection were redone to create a uniform appearance. In a few instances titles were changed, and shading or symbols were added (or deleted) for clarity. Any errors resulting from that process are entirely my responsibility.

Roger B. Nelsen  
Lewis and Clark College  
Portland, Oregon



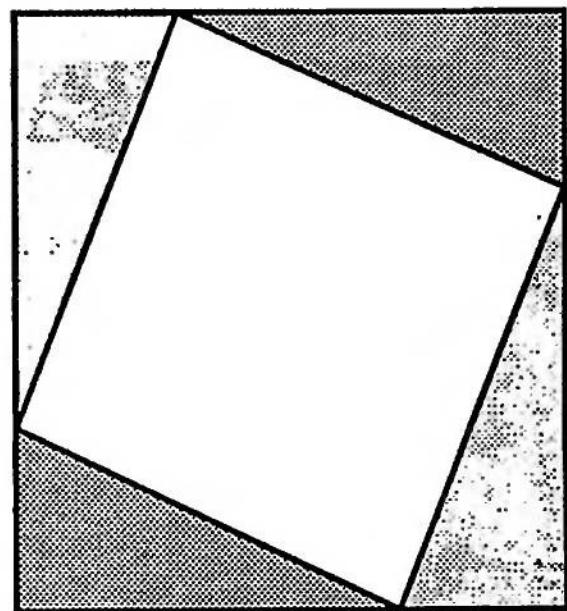
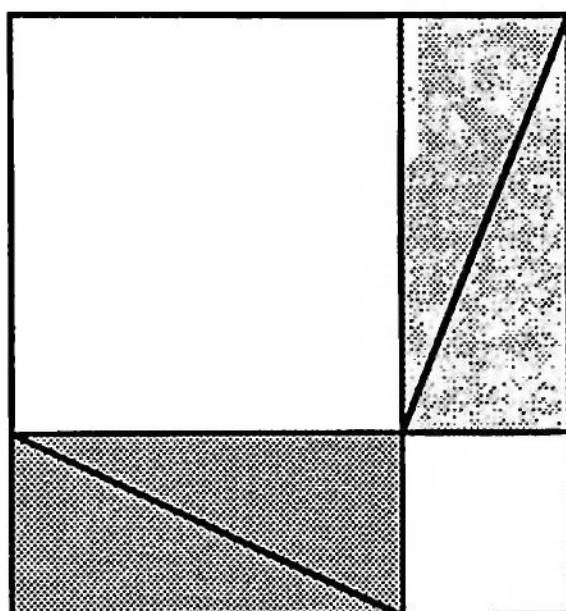
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# Geometry & Algebra

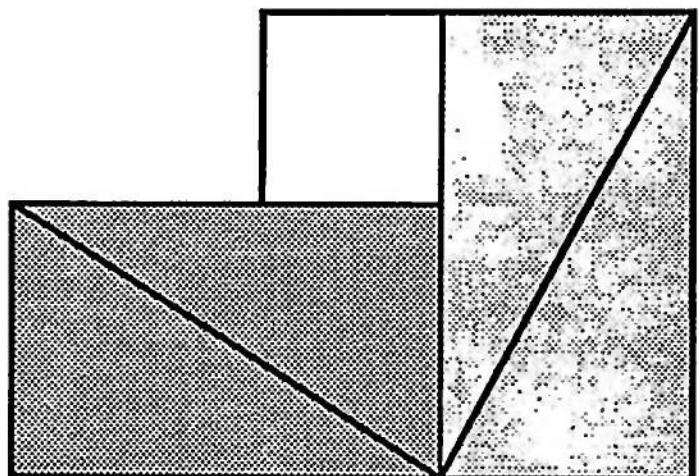
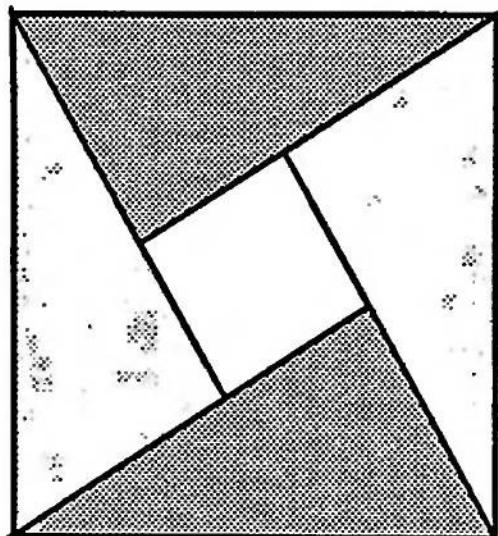
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# The Pythagorean Theorem I



—adapted from the *Chou pei suan ching*  
(author unknown, circa B.C. 200?)

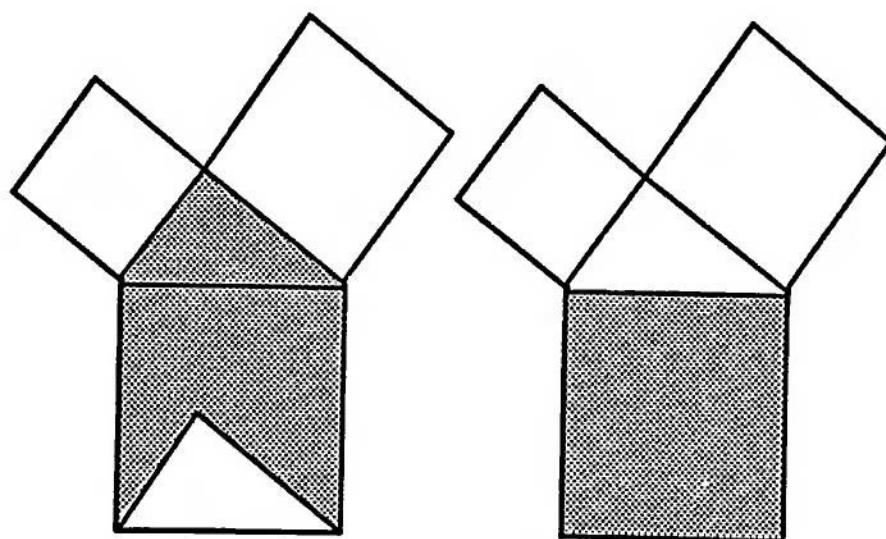
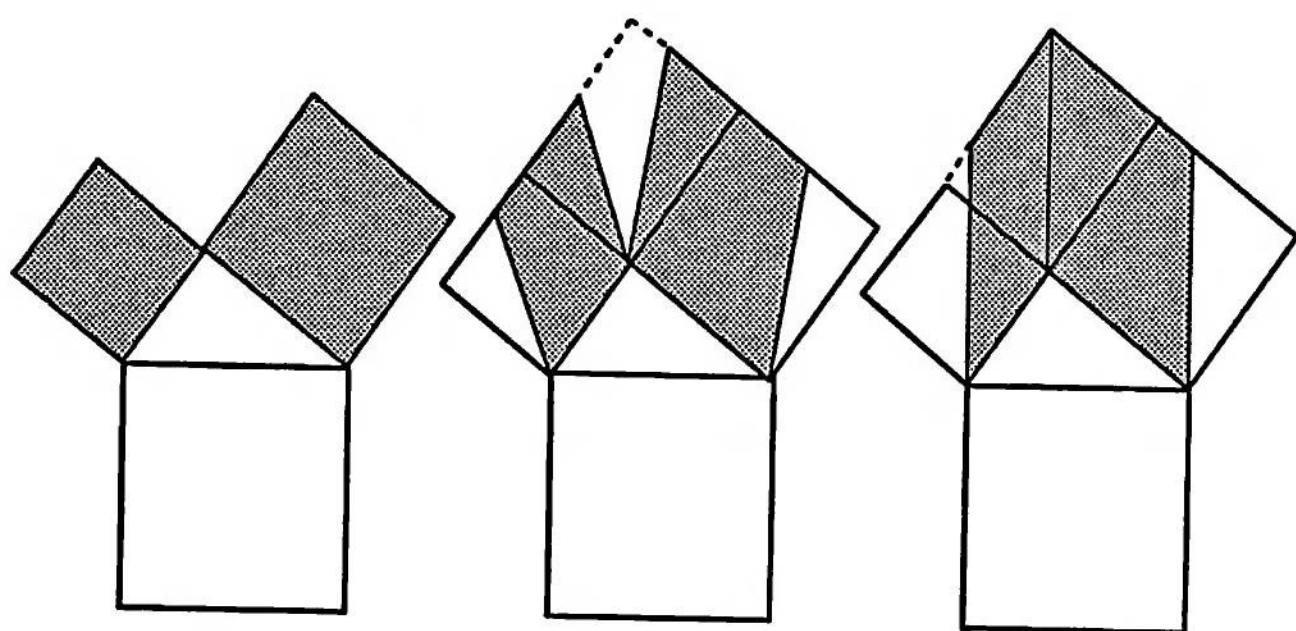
## The Pythagorean Theorem II



*Behold!*

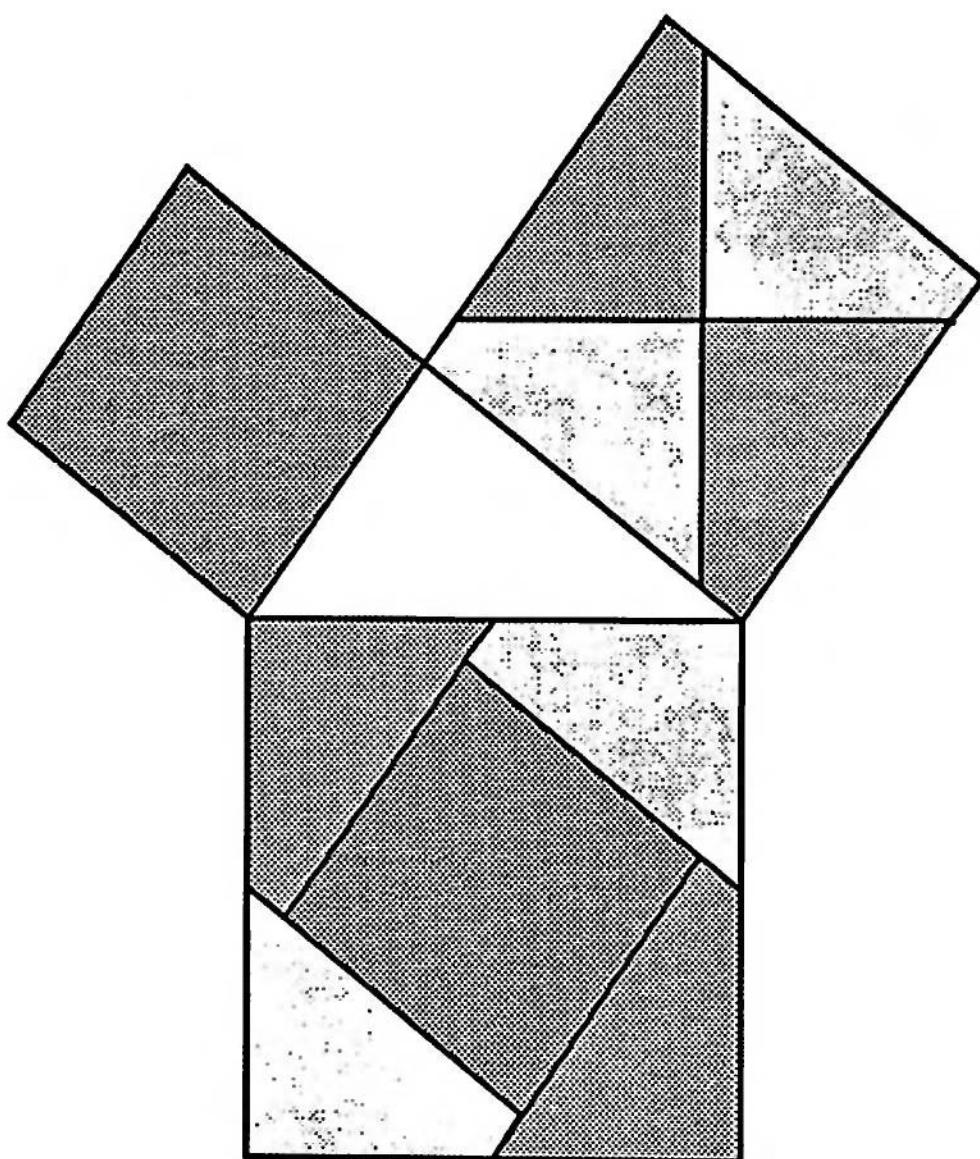
—Bhāskara (12<sup>th</sup> century)

## The Pythagorean Theorem III



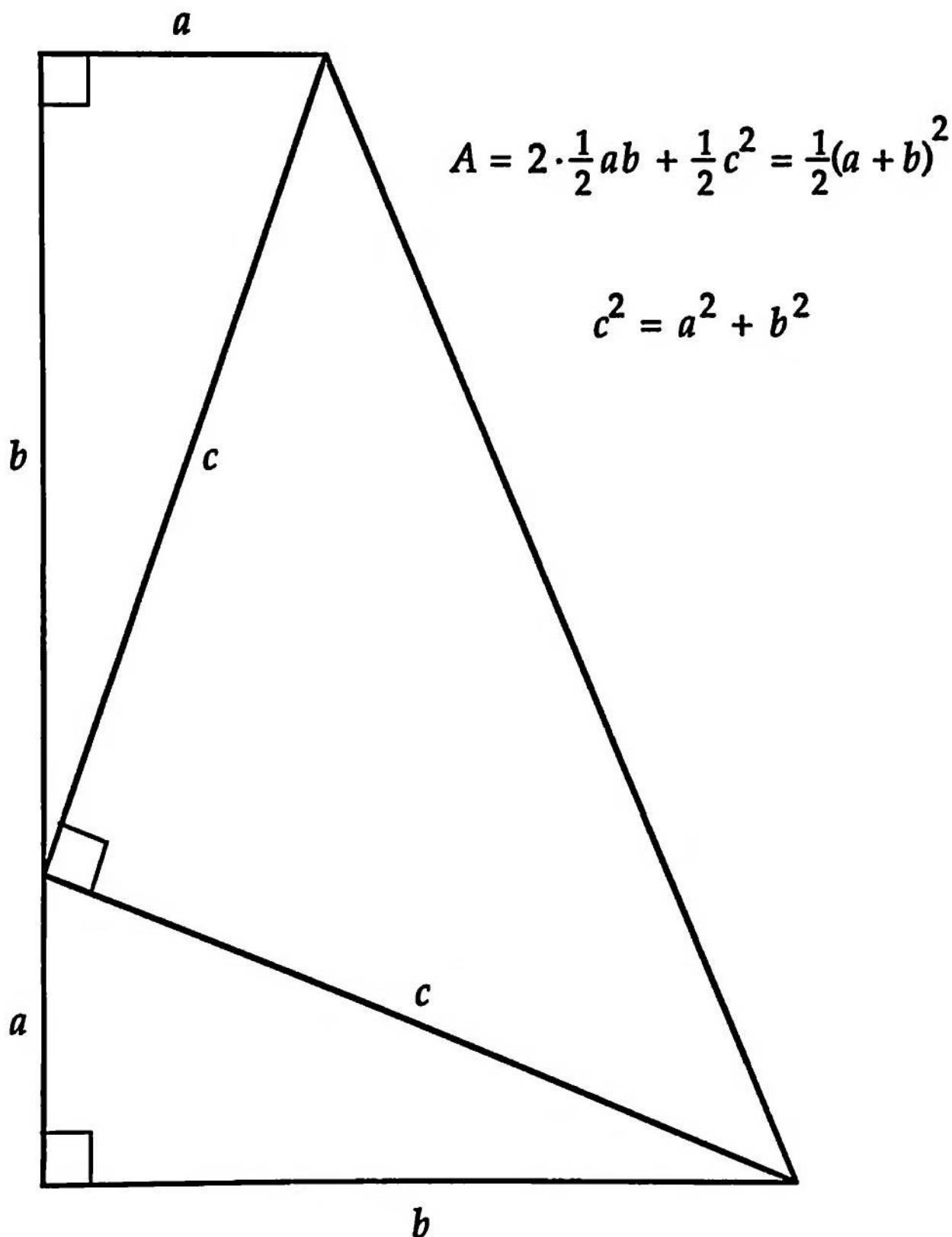
—based on Euclid's proof

## The Pythagorean Theorem IV



—H. E. Dudeney (1917)

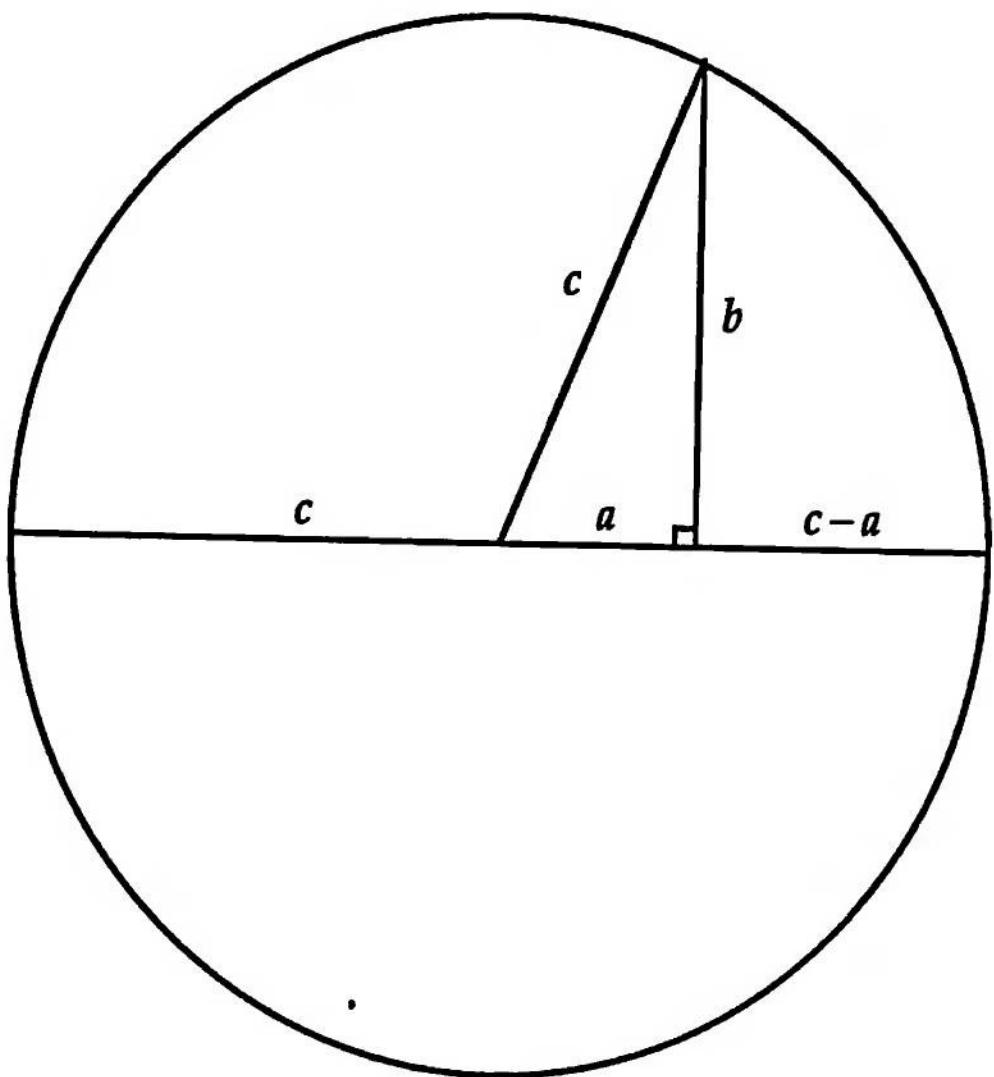
## The Pythagorean Theorem V



—James A. Garfield (1876)  
20<sup>th</sup> President of the United States

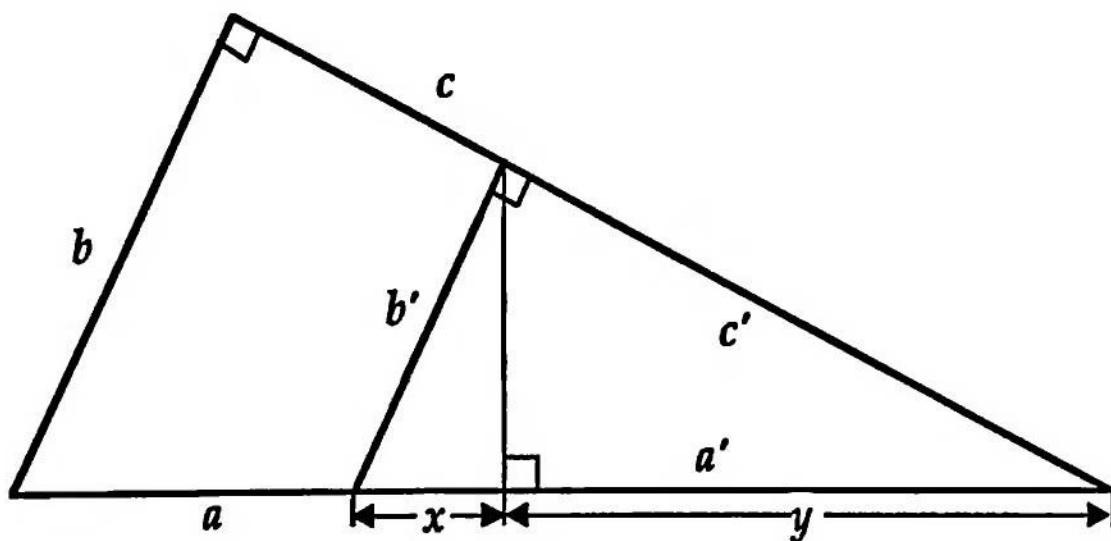
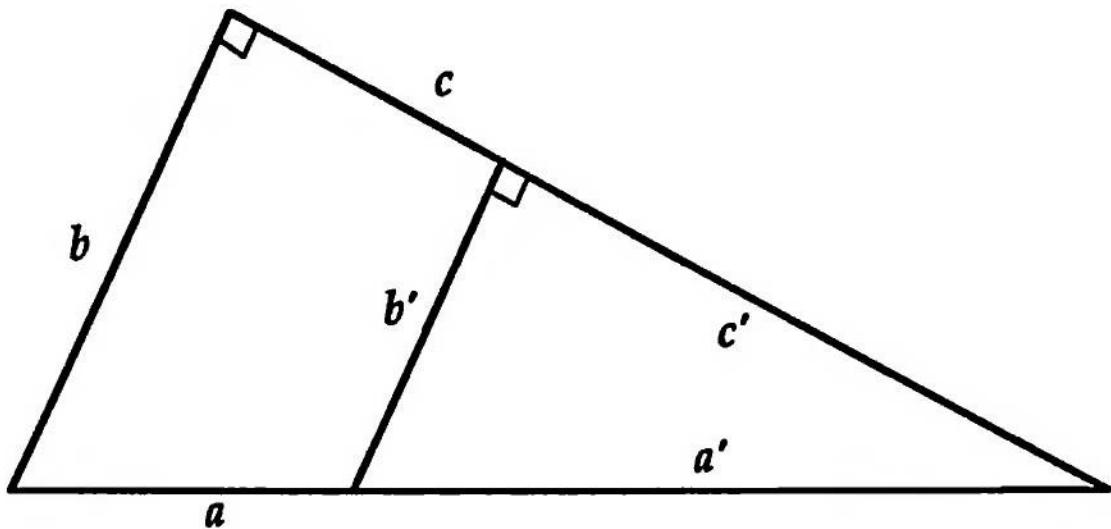
# The Pythagorean Theorem VI

$$\frac{c+a}{b} = \frac{b}{c-a}$$
$$a^2 + b^2 = c^2$$



—Michael Hardy

# A Pythagorean Theorem: $a \cdot a' = b \cdot b' + c \cdot c'$

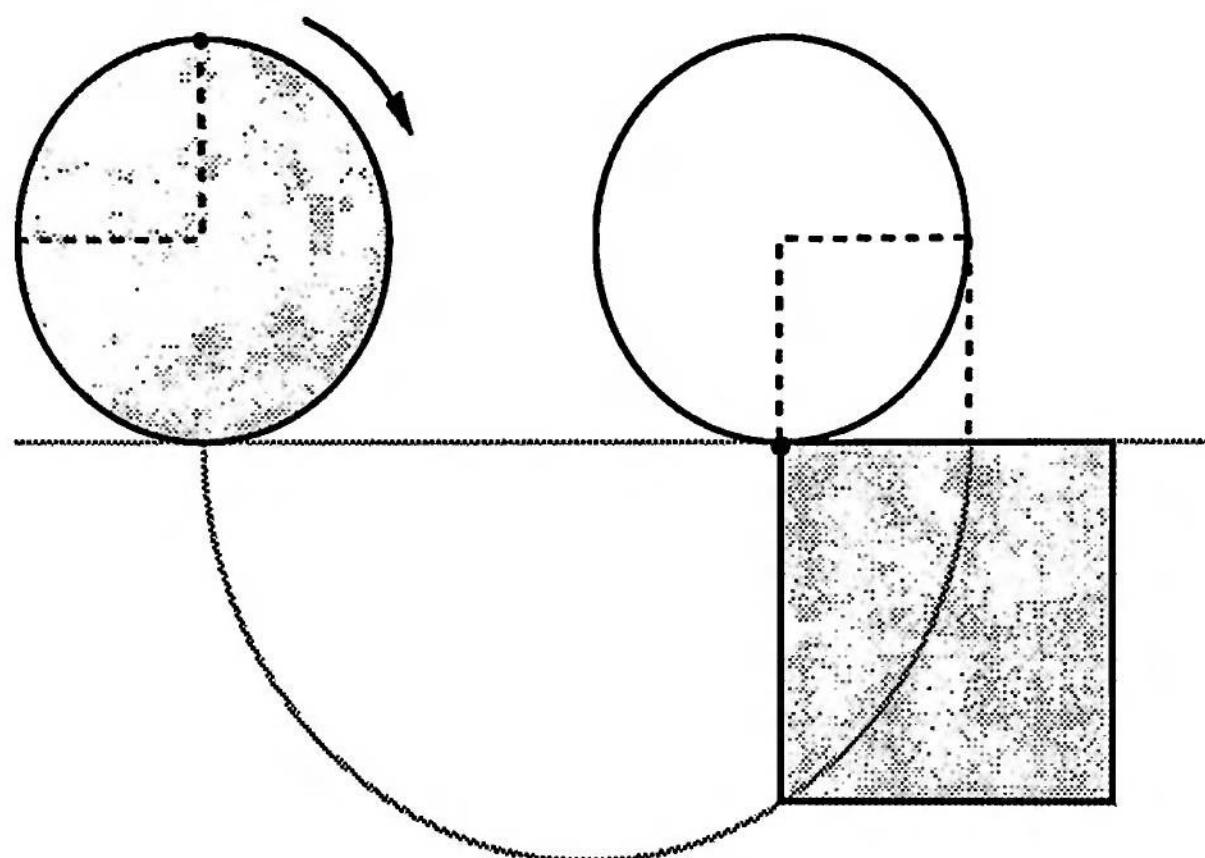


$$\frac{x}{b} = \frac{b'}{a} \Rightarrow a \cdot x = b \cdot b';$$

$$\frac{y}{c} = \frac{c'}{a} \Rightarrow a \cdot y = c \cdot c';$$

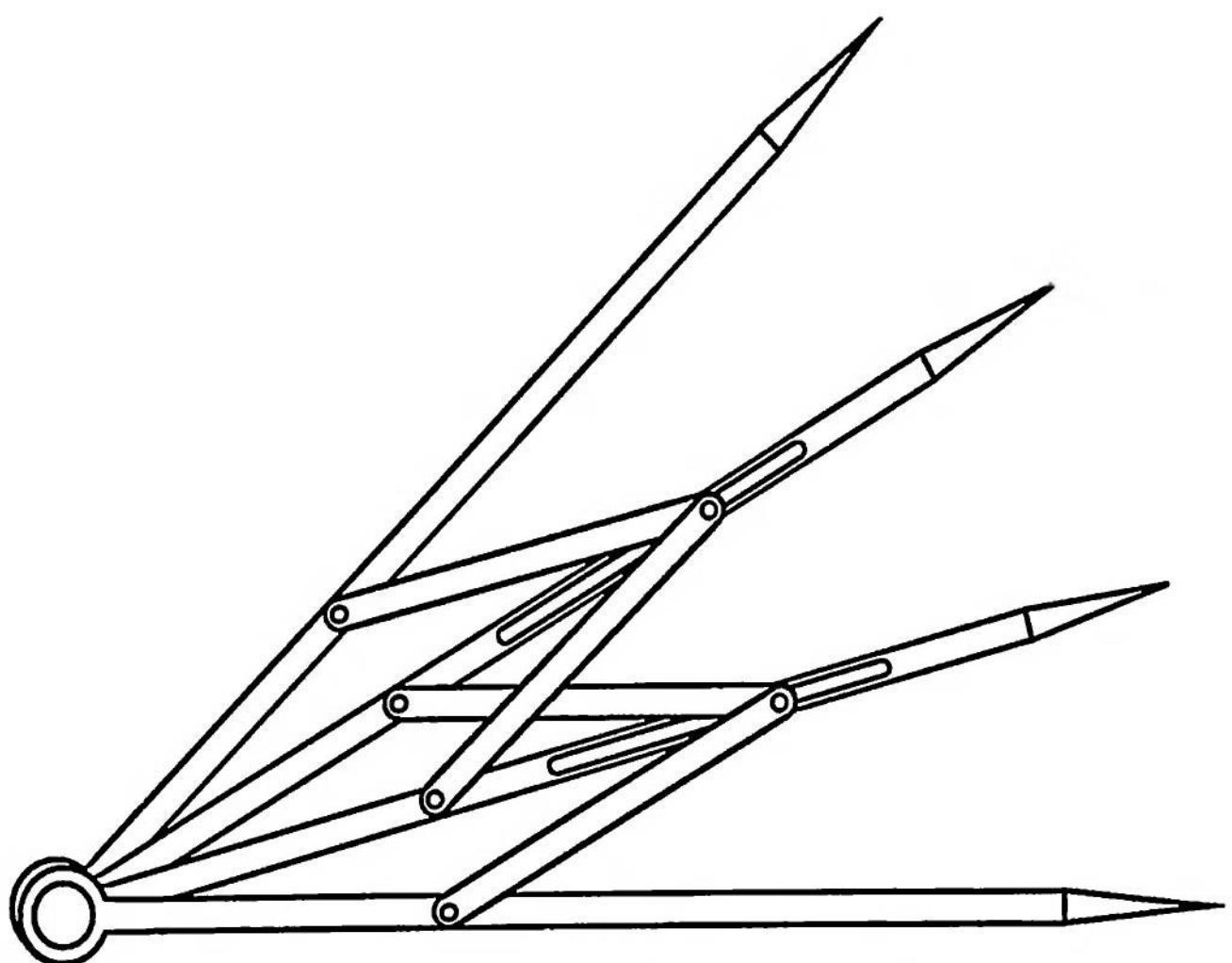
$$\therefore a \cdot a' = a \cdot (x + y) = b \cdot b' + c \cdot c'.$$

# The Rolling Circle Squares Itself



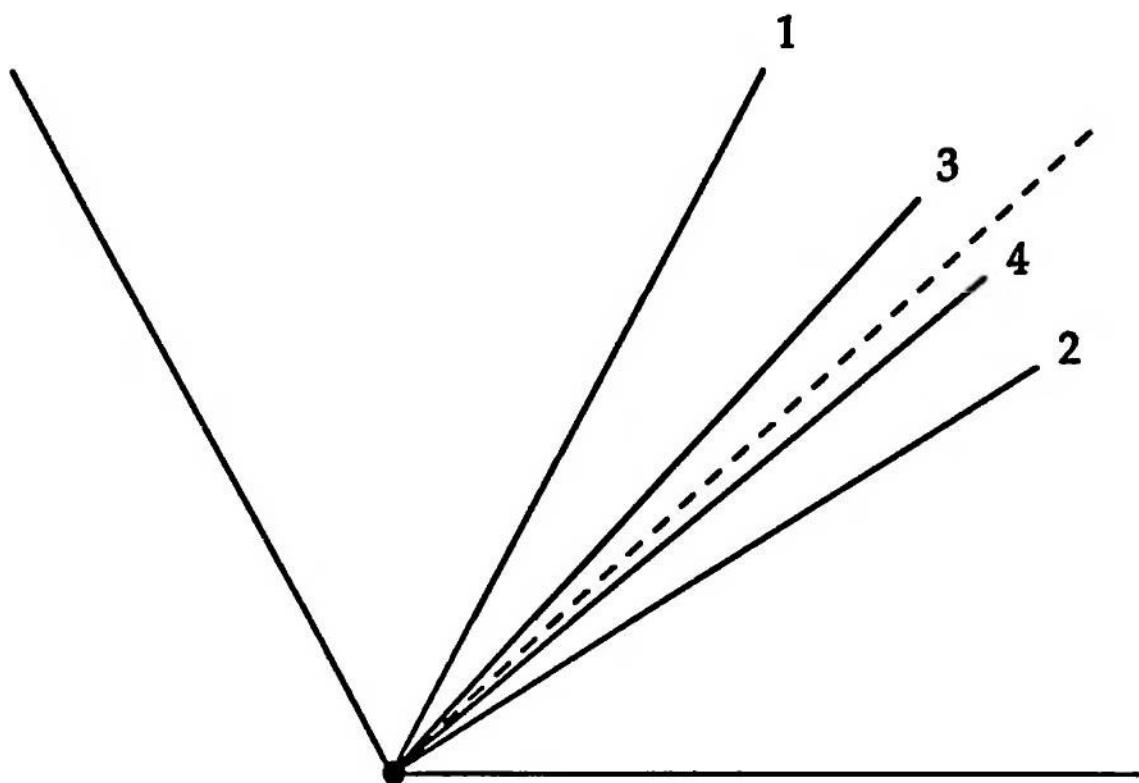
—Thomas Elsner

## On Trisecting an Angle



—Rufus Isaacs

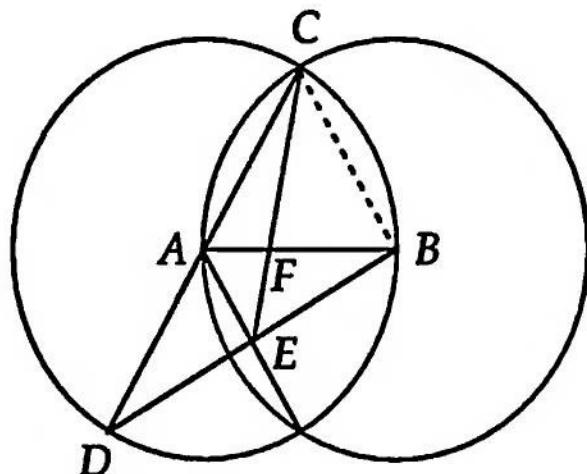
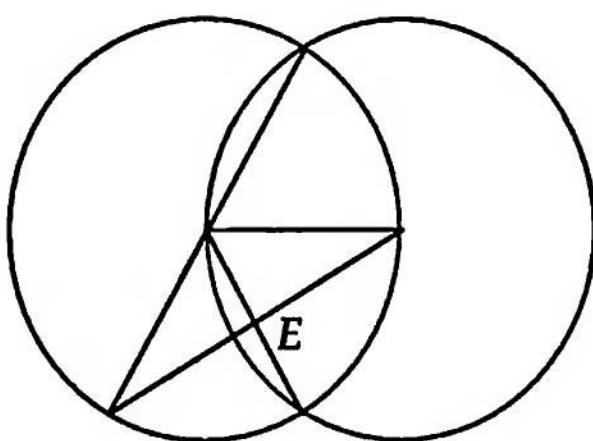
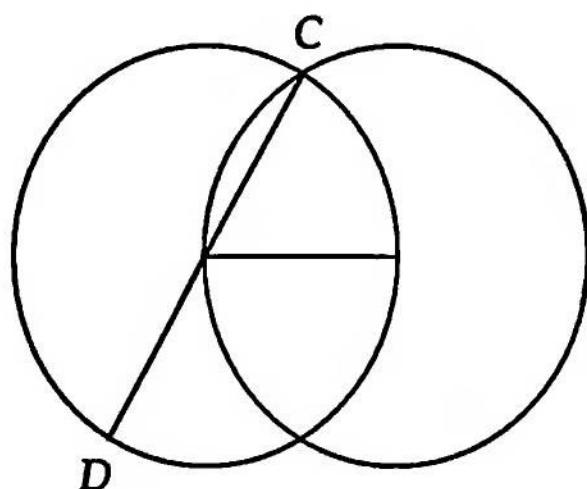
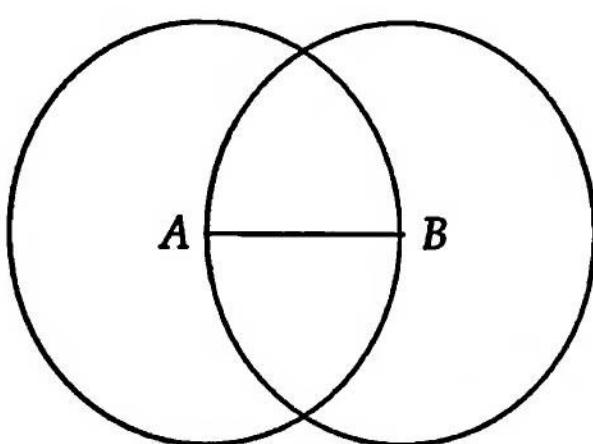
## Trisection of an Angle in an Infinite Number of Steps



$$\frac{1}{3} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots$$

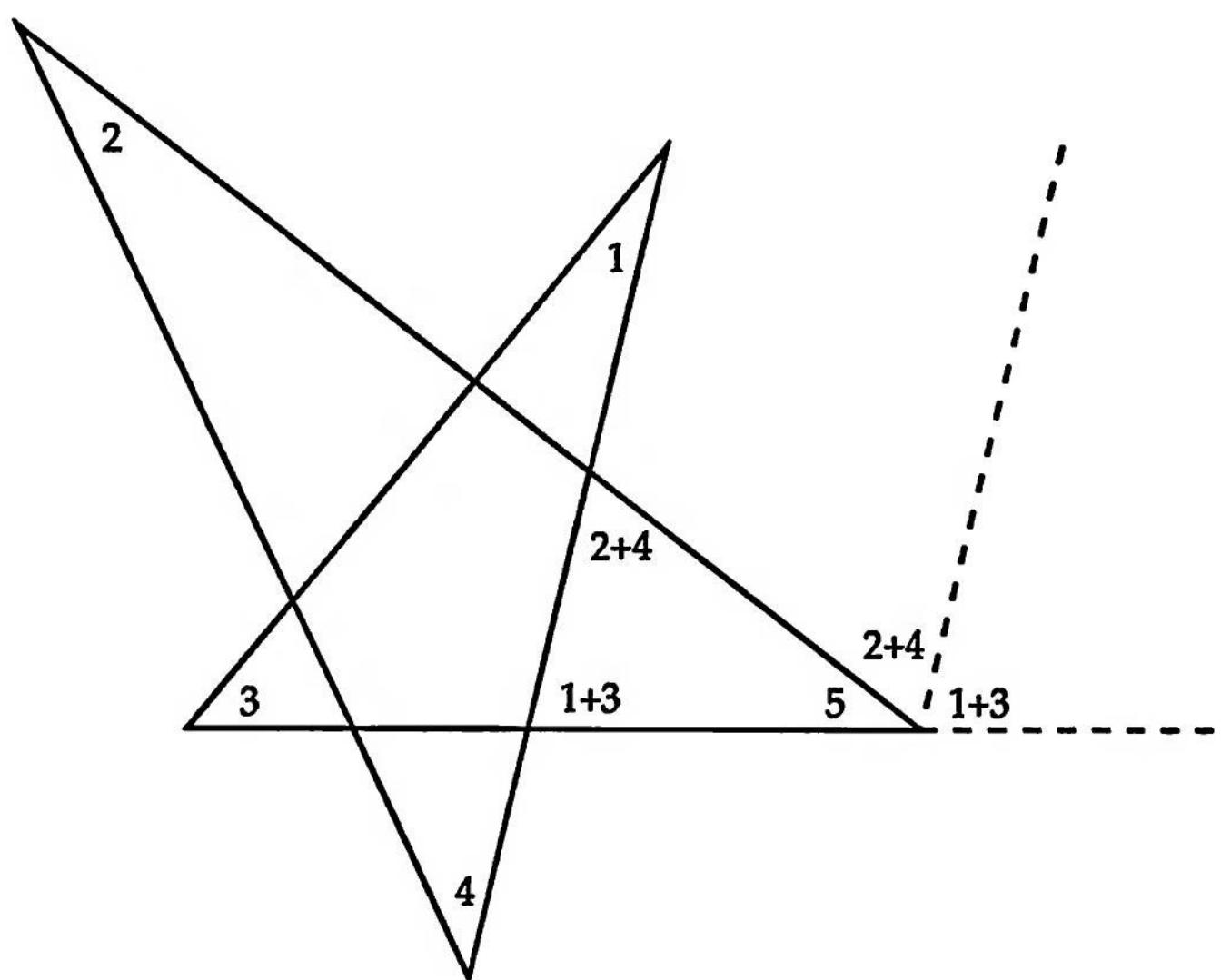
—Eric Kincanon

## Trisection of a Line Segment



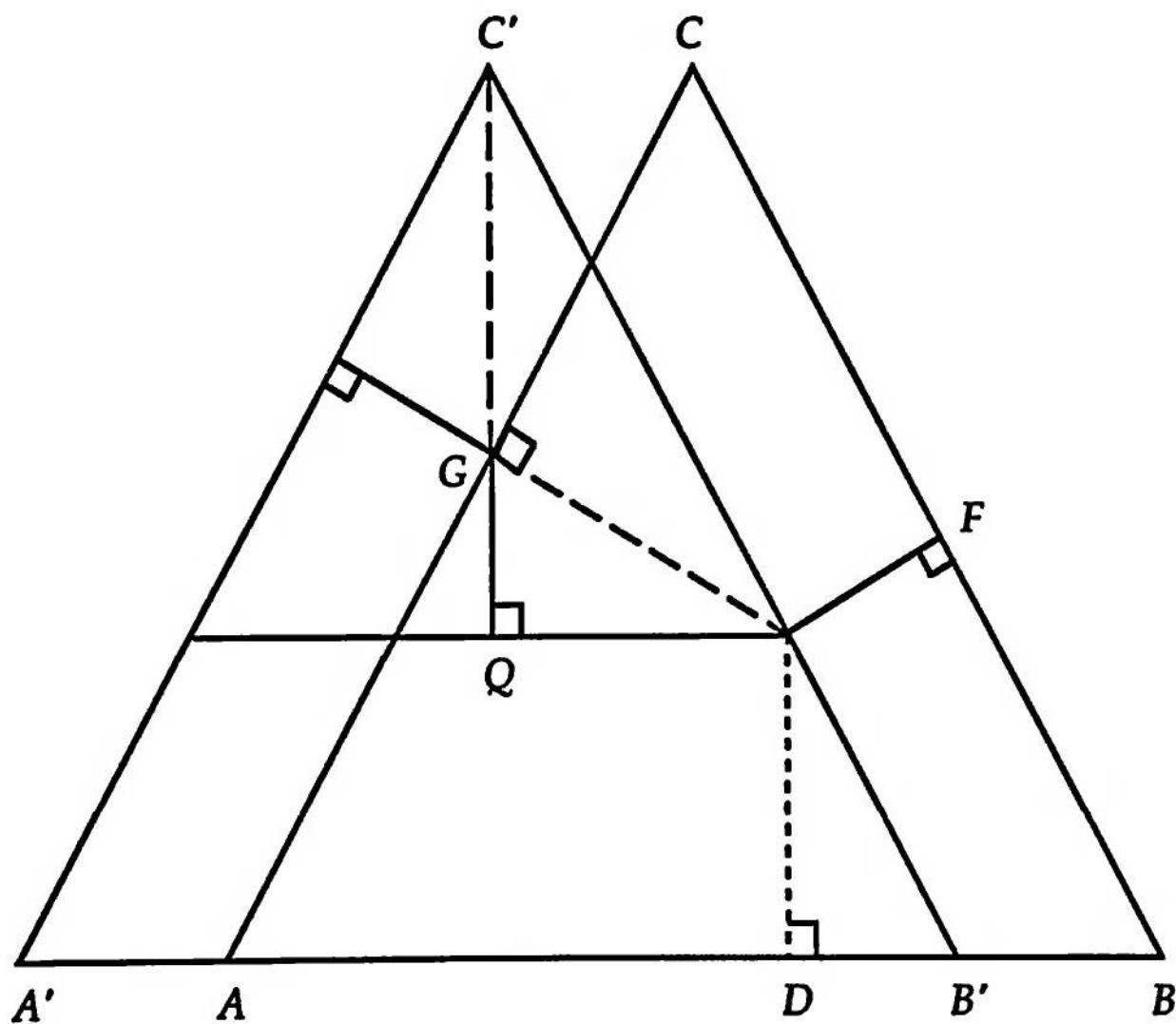
$$\overline{AF} = \frac{1}{3} \cdot \overline{AB}$$

## The Vertex Angles of a Star Sum to $180^\circ$



## Viviani's Theorem

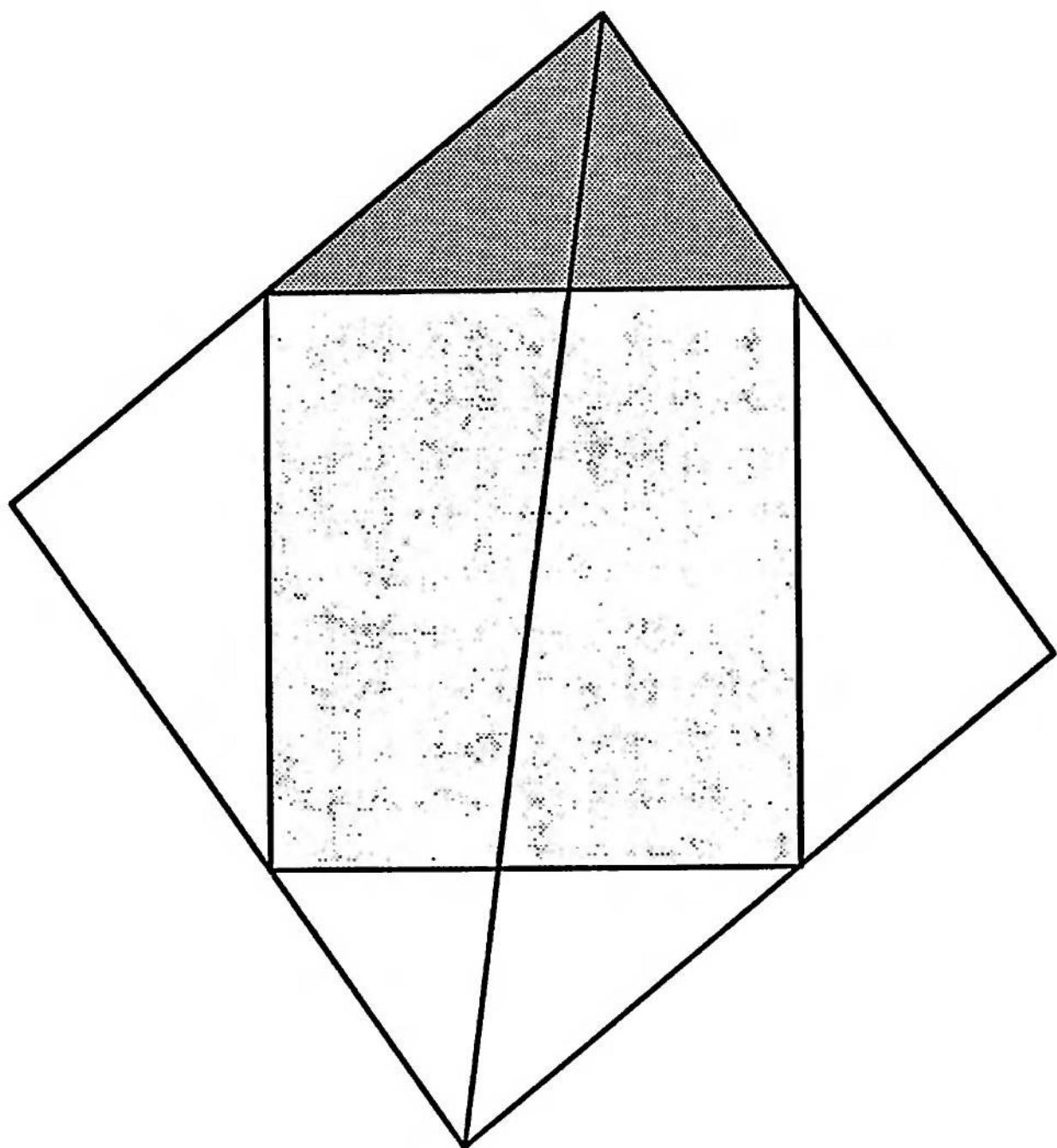
The perpendiculars to the sides from a point on the boundary or within an equilateral triangle add up to the height of the triangle.



—Samuel Wolf

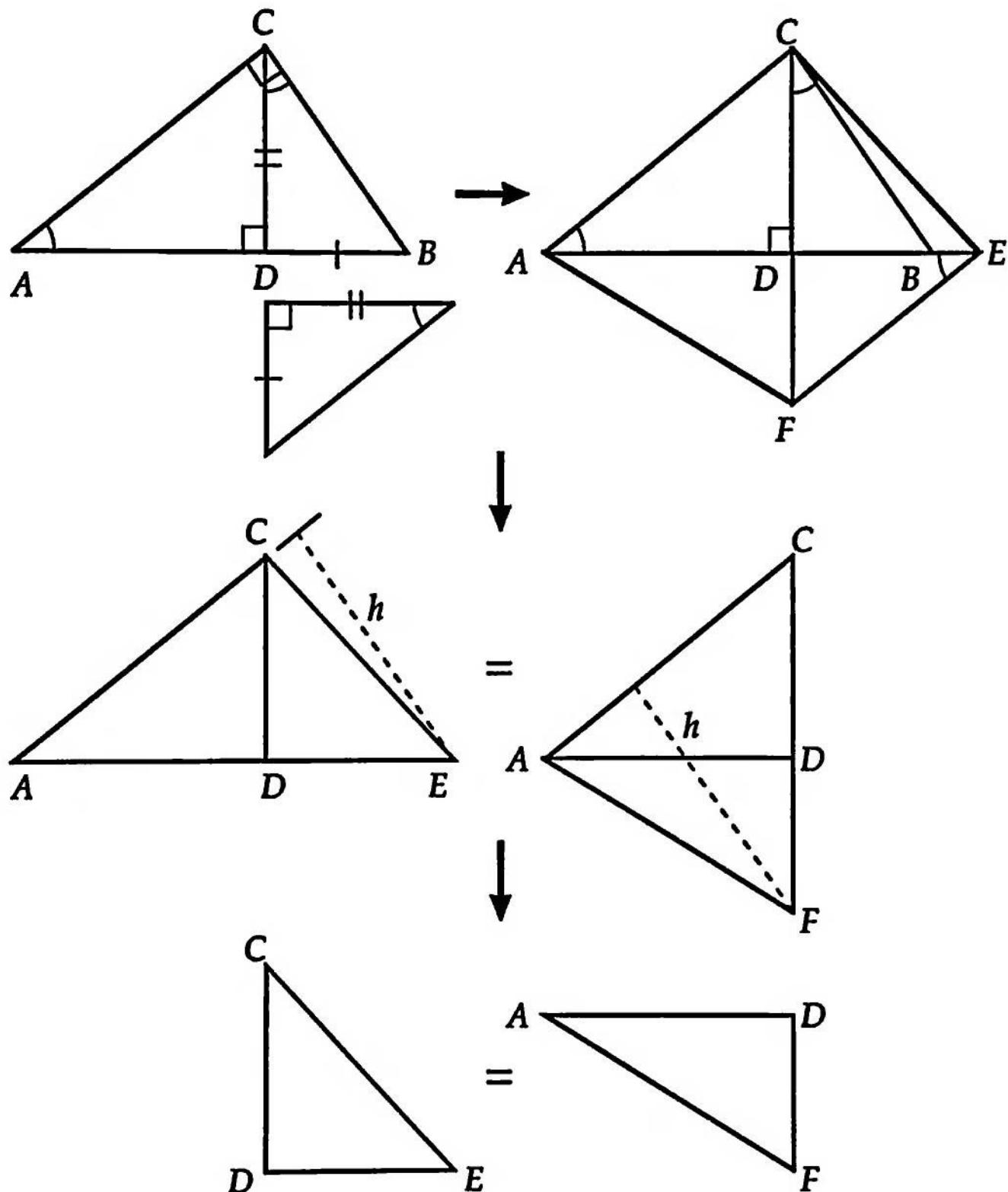
## A Theorem About Right Triangles

The internal bisector of the right angle of a right triangle bisects the square on the hypotenuse.



—Roland H. Eddy

# Area and the Projection Theorem of a Right Triangle

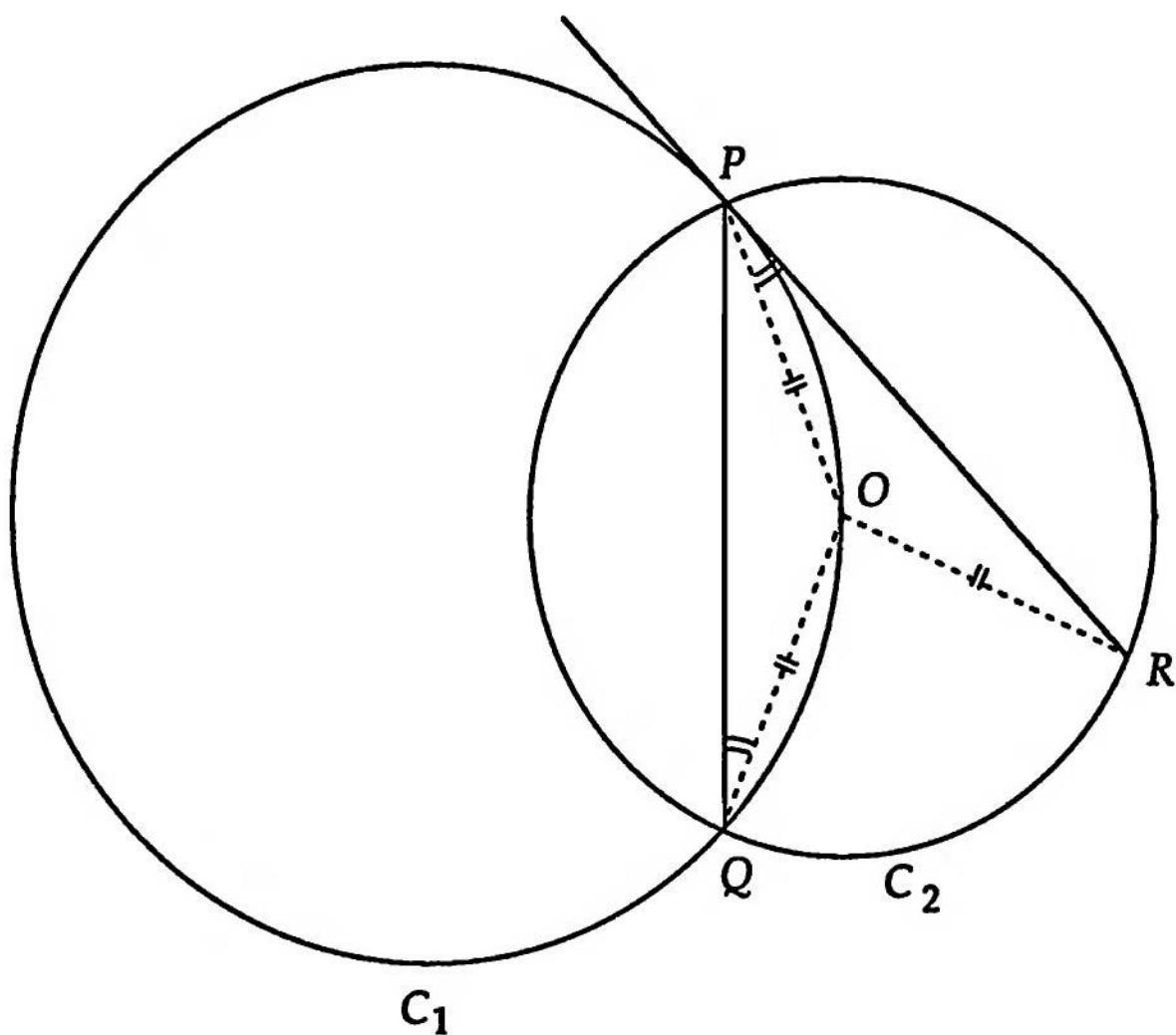


$$CD^2 = AD \cdot DB$$

—Sidney H. Kung

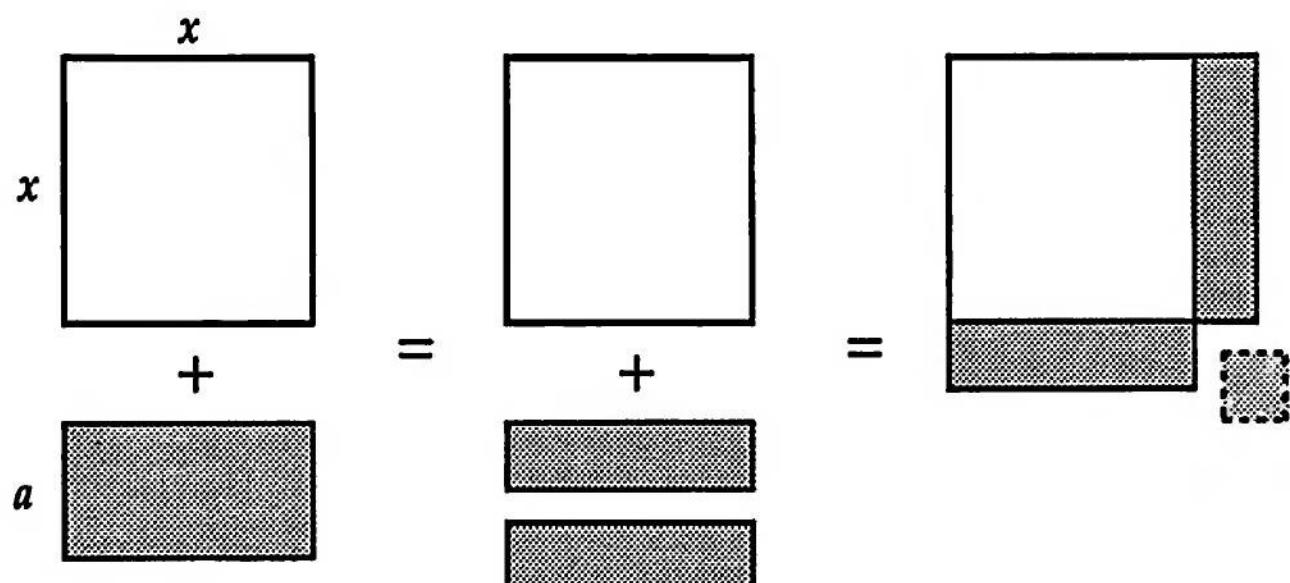
## Chords and Tangents of Equal Length

If circle  $C_1$  passes through the center  $O$  of circle  $C_2$ , the length of the common chord  $\overline{PQ}$  is equal to the tangent segment  $\overline{PR}$ .



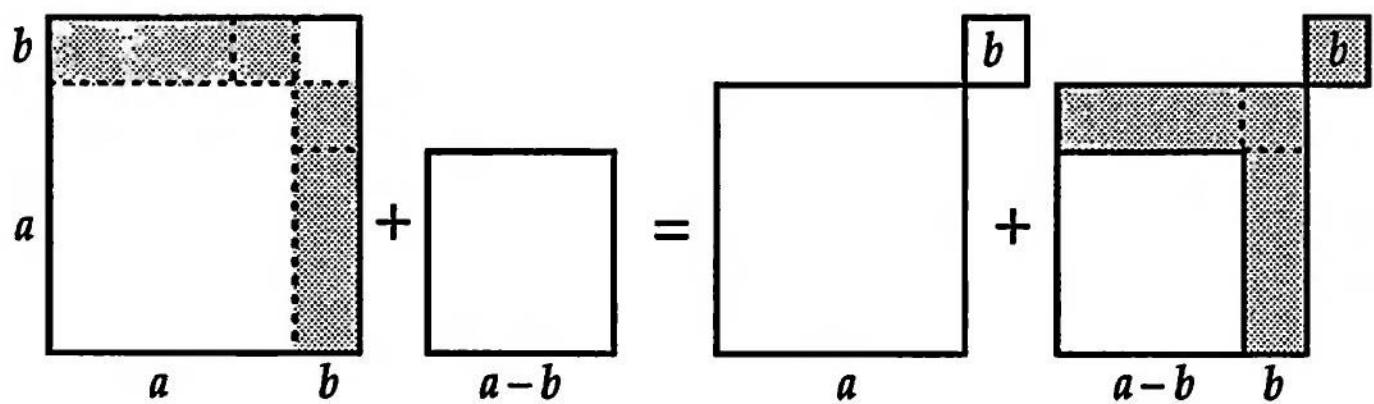
## Completing the Square

$$x^2 + ax = (x + a/2)^2 - (a/2)^2$$



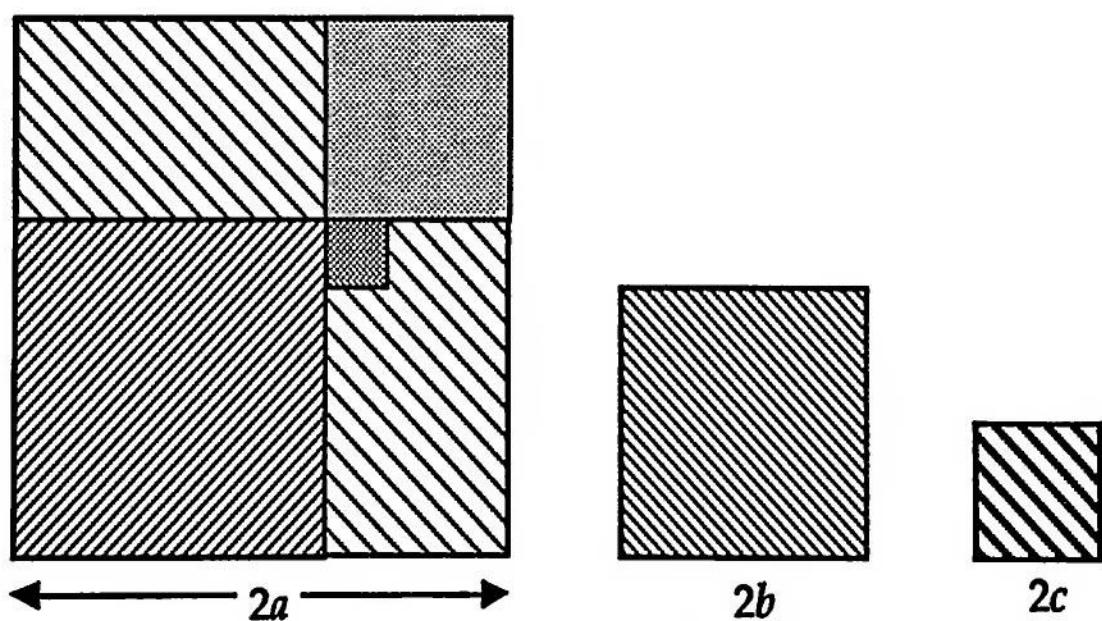
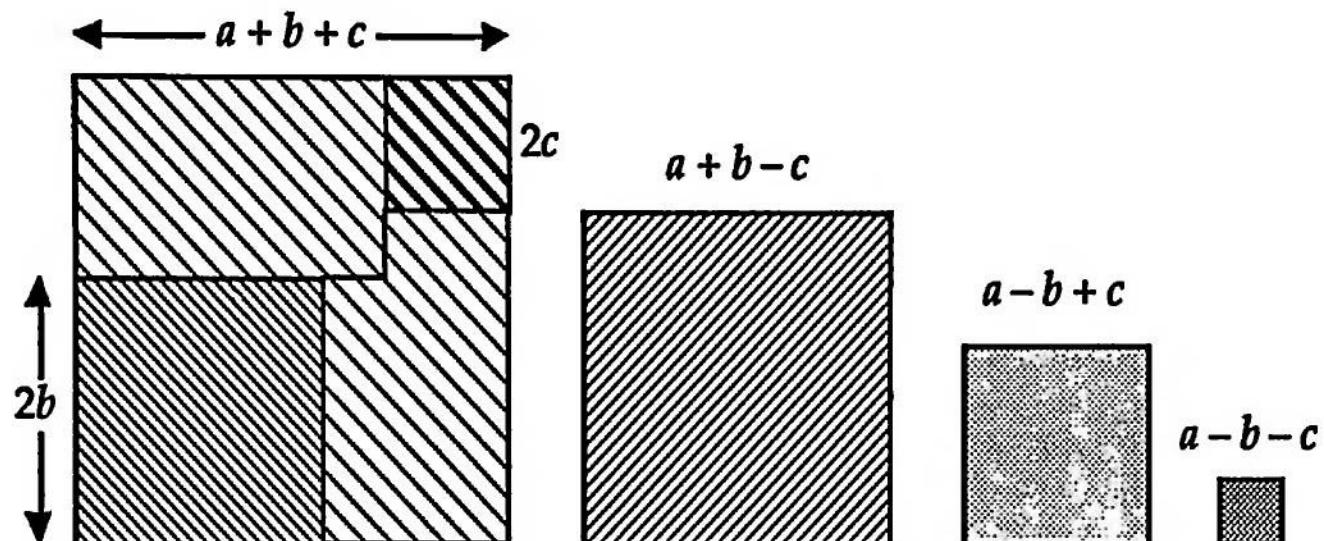
# Algebraic Areas I

$$(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$$



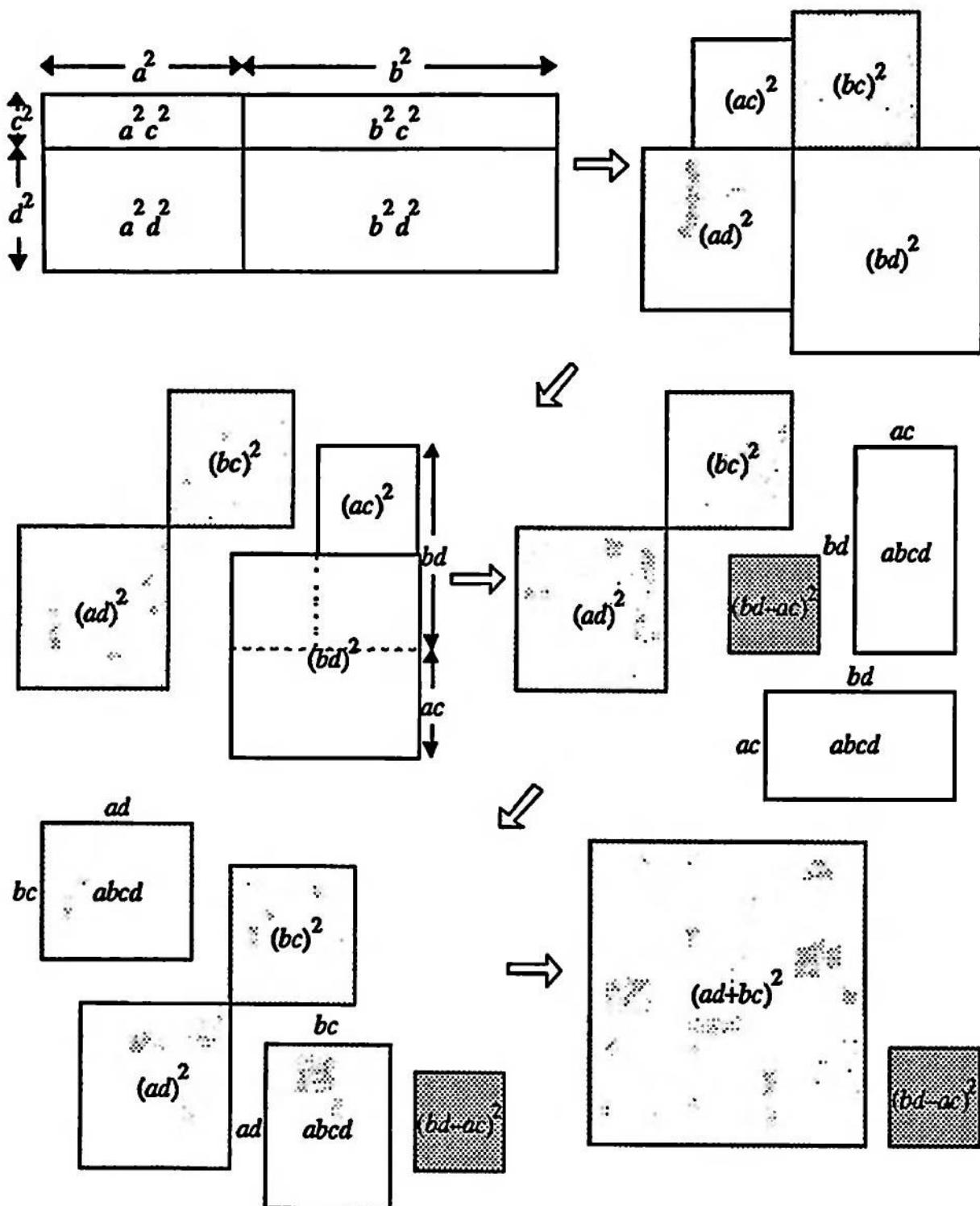
## Algebraic Areas II

$$(a + b + c)^2 + (a + b - c)^2 + (a - b + c)^2 + (a - b - c)^2 \\ = (2a)^2 + (2b)^2 + (2c)^2$$



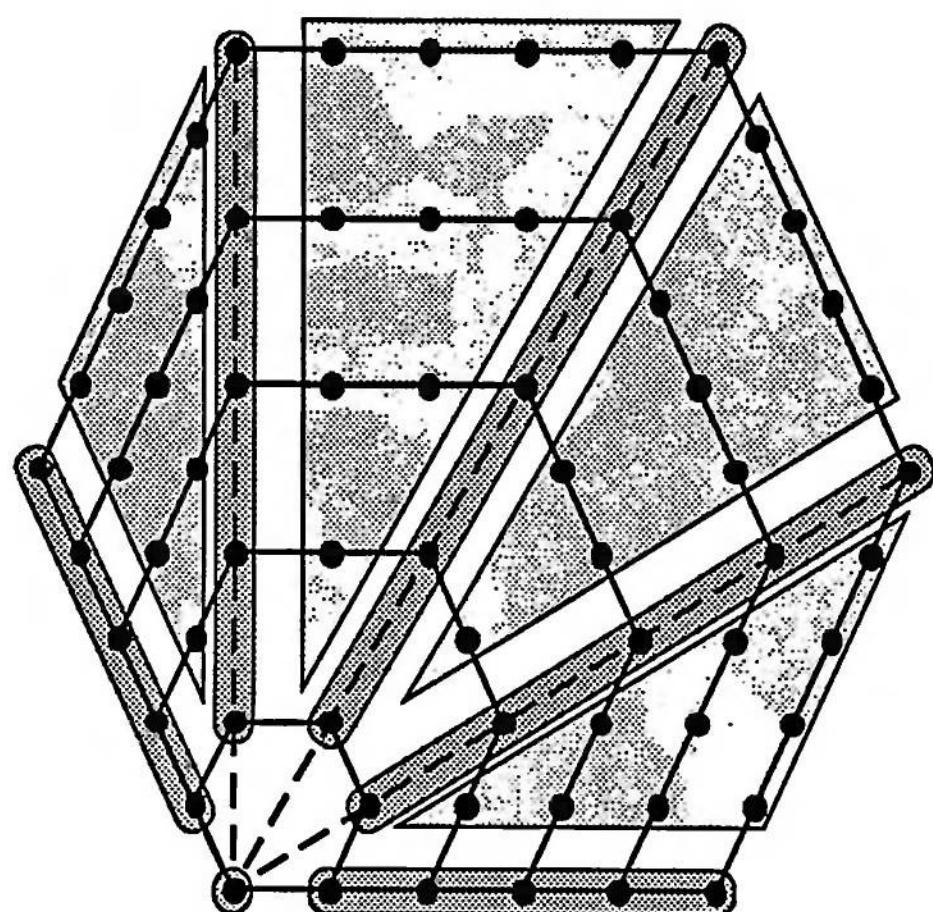
# Diophantus of Alexandria's "Sum of Squares" Identity

$$(a^2 + b^2)(c^2 + d^2) = (ad + bc)^2 + (bd - ac)^2$$



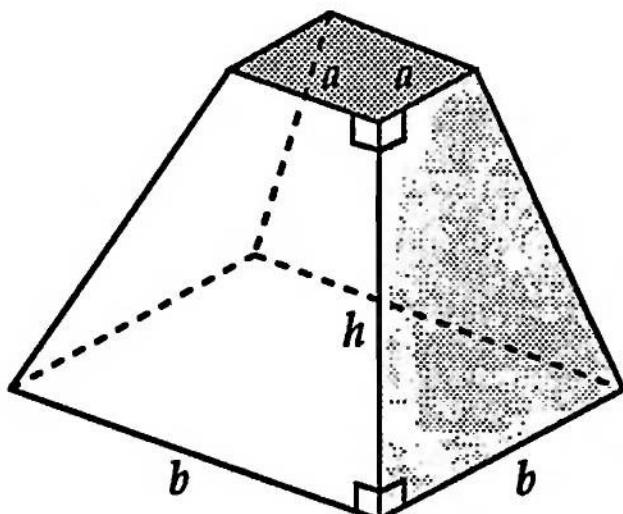
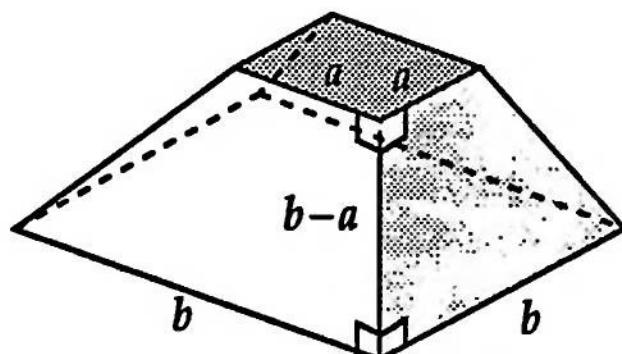
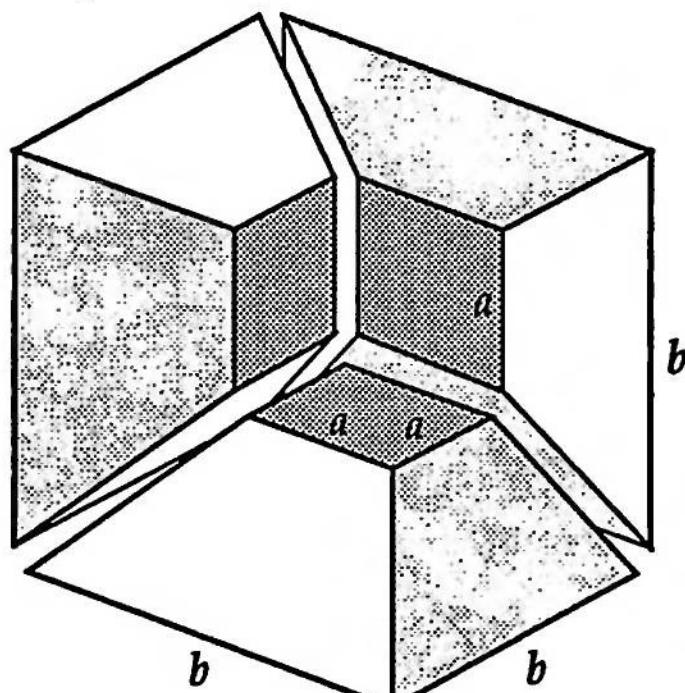
The  $k$ th  $n$ -gonal Number is

$$1 + (k - 1)(n - 1) + \frac{1}{2}(k - 2)(k - 1)(n - 2)$$



## The Volume of a Frustum of a Square Pyramid

[Problem 14, *The Moscow Papyrus*, circa 1850 b.c.]

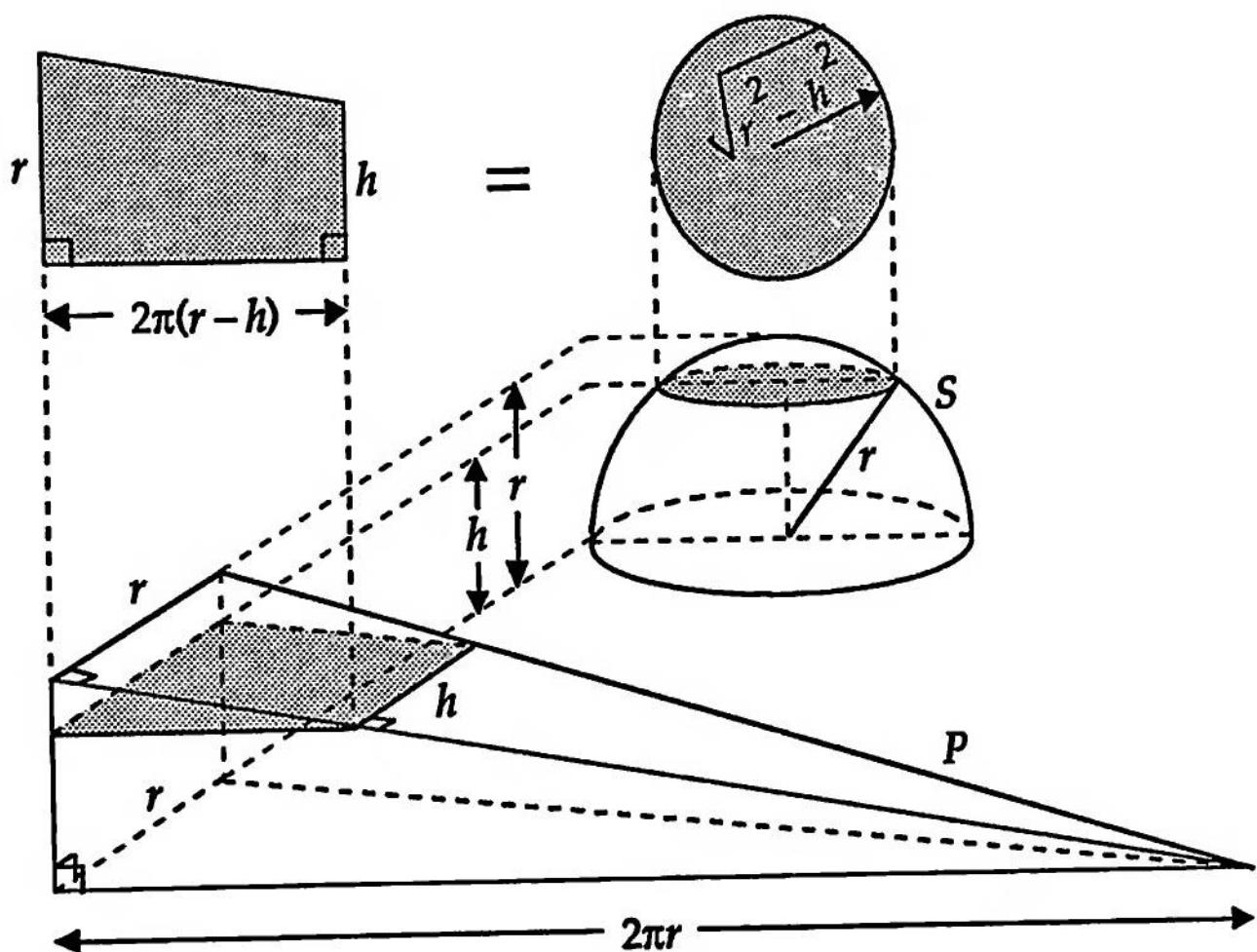
 $P_1$  $P_2$  $3P_2$ 

$$V(P_1) = \frac{h}{b-a} V(P_2) = \frac{h}{b-a} \cdot \frac{1}{3} (b^3 - a^3) = \frac{h}{3} (a^2 + ab + b^2)$$

### REFERENCES

1. C. B. Boyer, *A History of Mathematics*, John Wiley & Sons, New York, 1968, pp. 20-22.
2. R. J. Gillings, *Mathematics in the Time of the Pharaohs*, The MIT Press, Cambridge, 1972, pp. 187-193.

## The Volume of a Hemisphere via Cavalieri's Principle\*



$$V_S = V_P = \frac{1}{3} r^2 \cdot 2\pi r = \frac{2}{3} \pi r^3$$

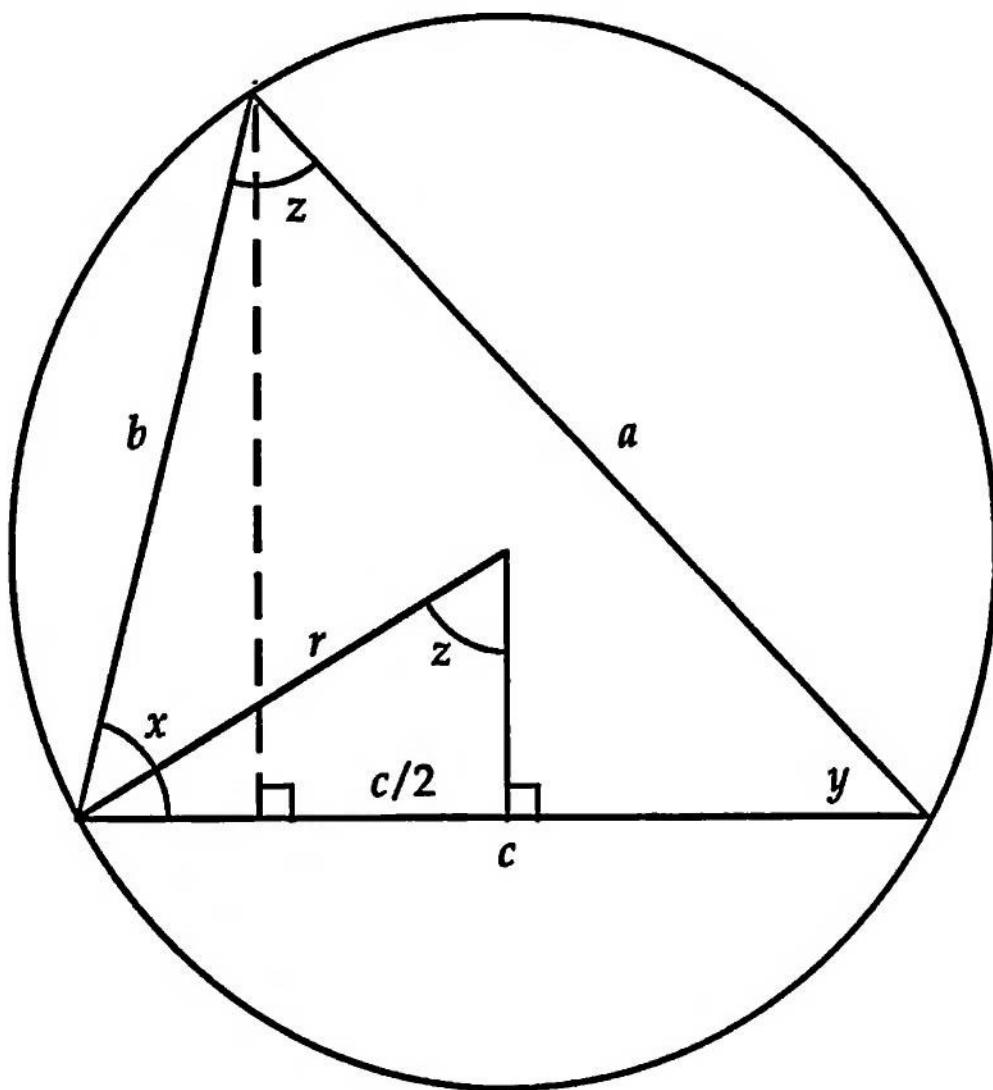
\*Tzu Geng, son of the most celebrated mathematician Tzu Chung Chih in ancient China, was believed to be the first to develop the principle in the 5th century A. D.

# Trigonometry, Calculus & Analytic Geometry

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## Sine of the Sum

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \text{ for } x+y < \pi$$

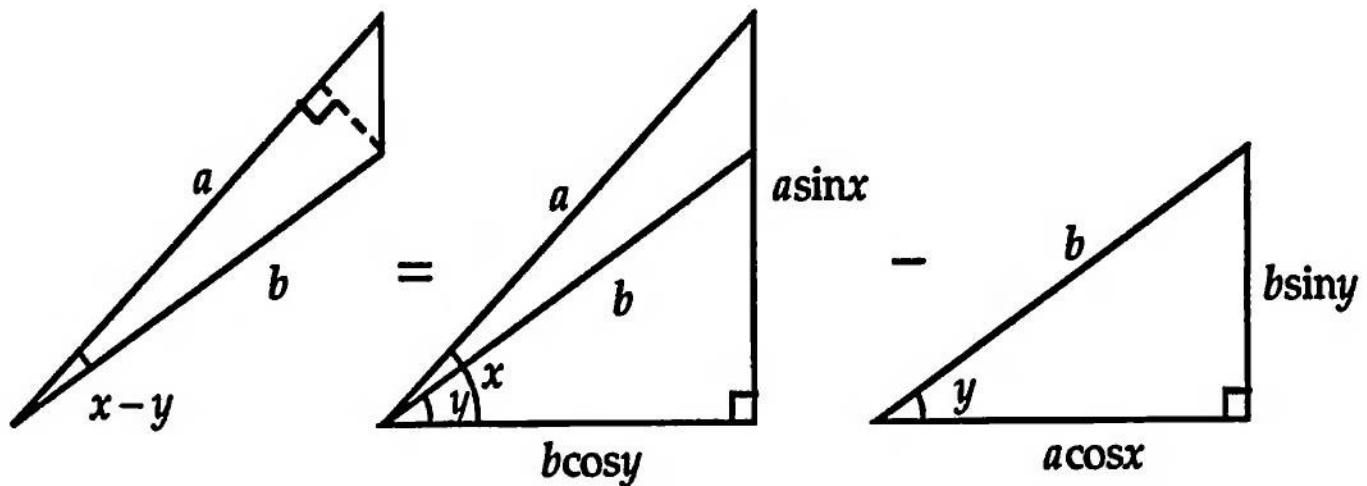


$$c = a \cos y + b \cos x$$

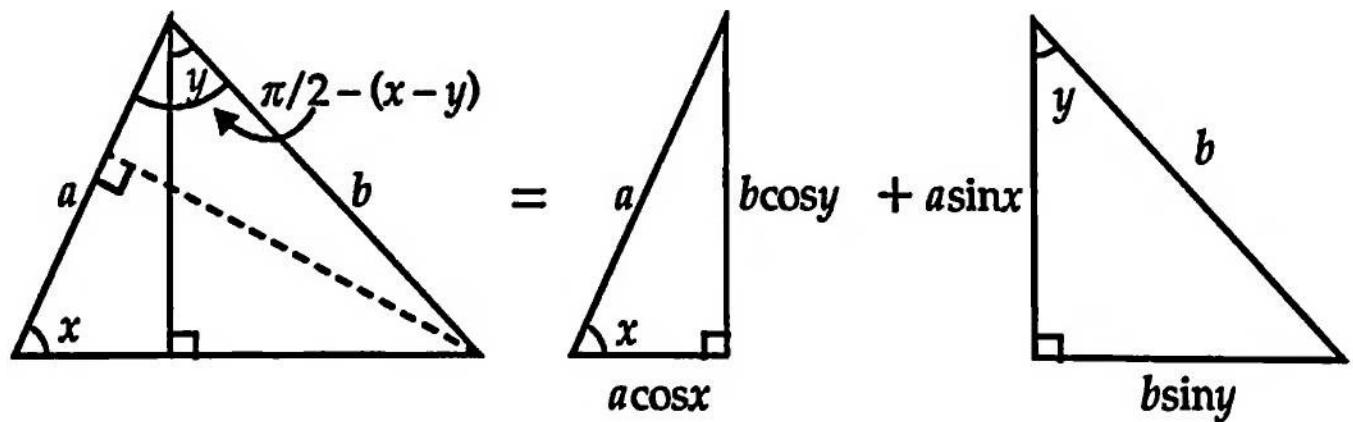
$$r = 1/2 \Rightarrow \sin z = (c/2)/(1/2) = c, \sin x = a, \sin y = b;$$

$$\sin(x+y) = \sin(\pi - (x+y)) = \sin z = \sin x \cos y + \sin y \cos x$$

## Area and Difference Formulas



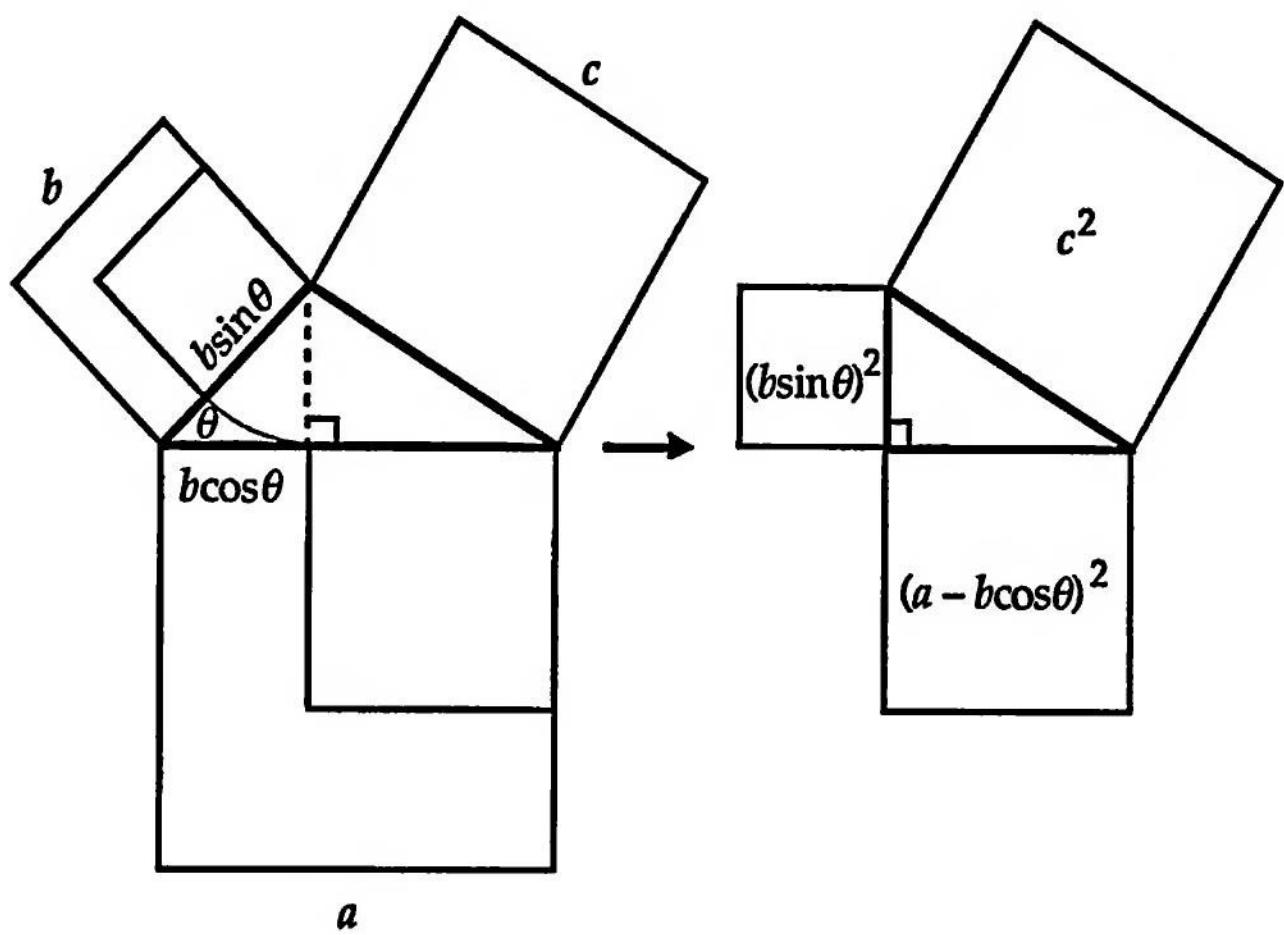
$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$



$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

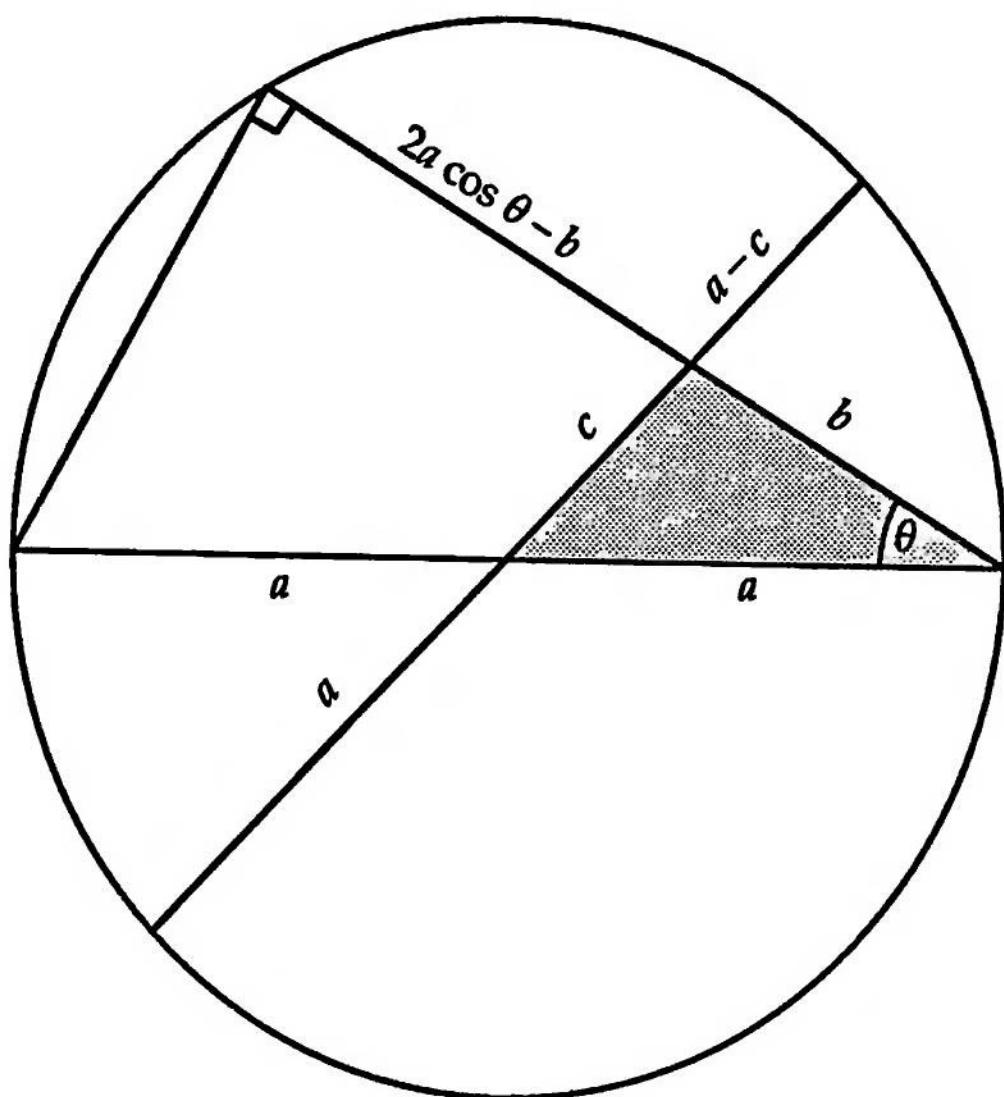
—Sidney H. Kung

## The Law of Cosines I



$$\begin{aligned}c^2 &= (b \sin \theta)^2 + (a - b \cos \theta)^2 \\&= a^2 + b^2 - 2ab \cos \theta\end{aligned}$$

## The Law of Cosines II

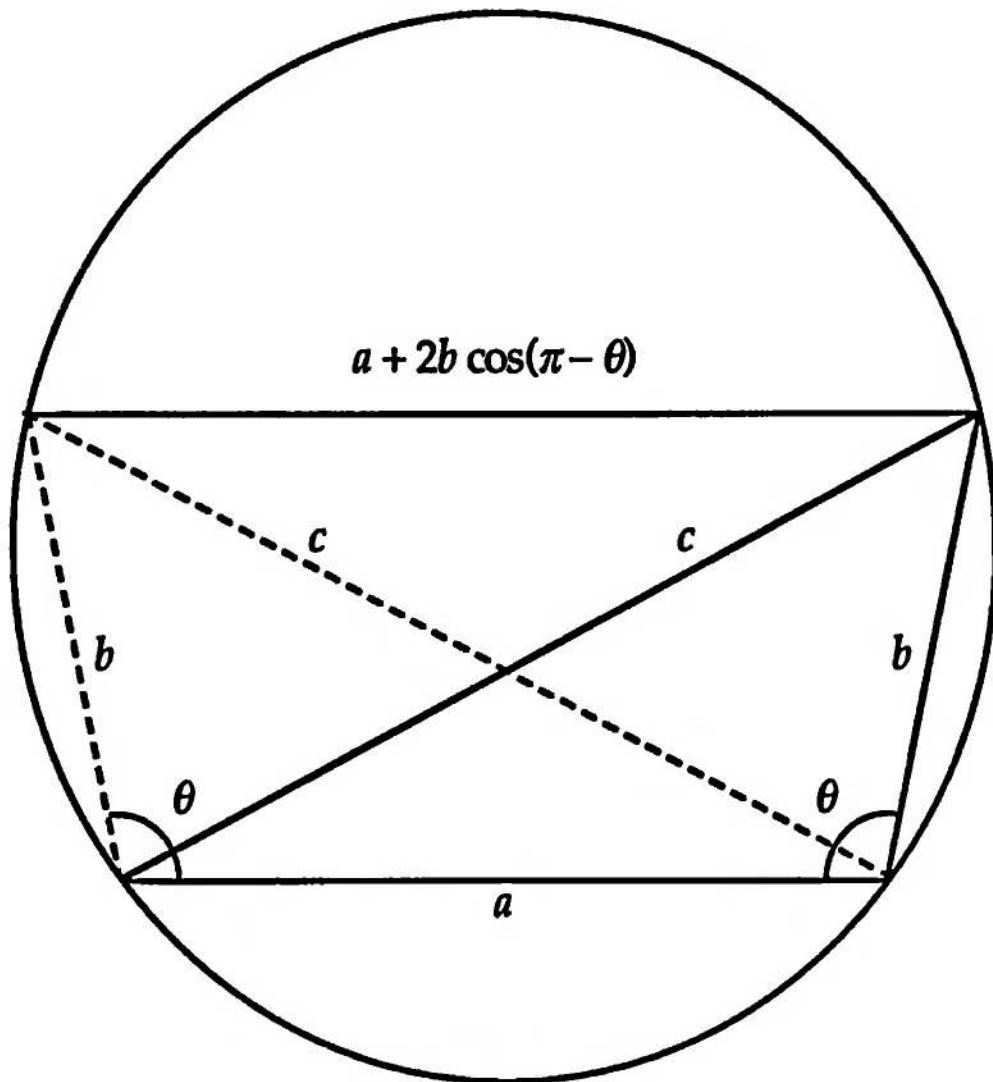


$$(2a \cos \theta - b)b = (a - c)(a + c)$$

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

—Sidney H. Kung

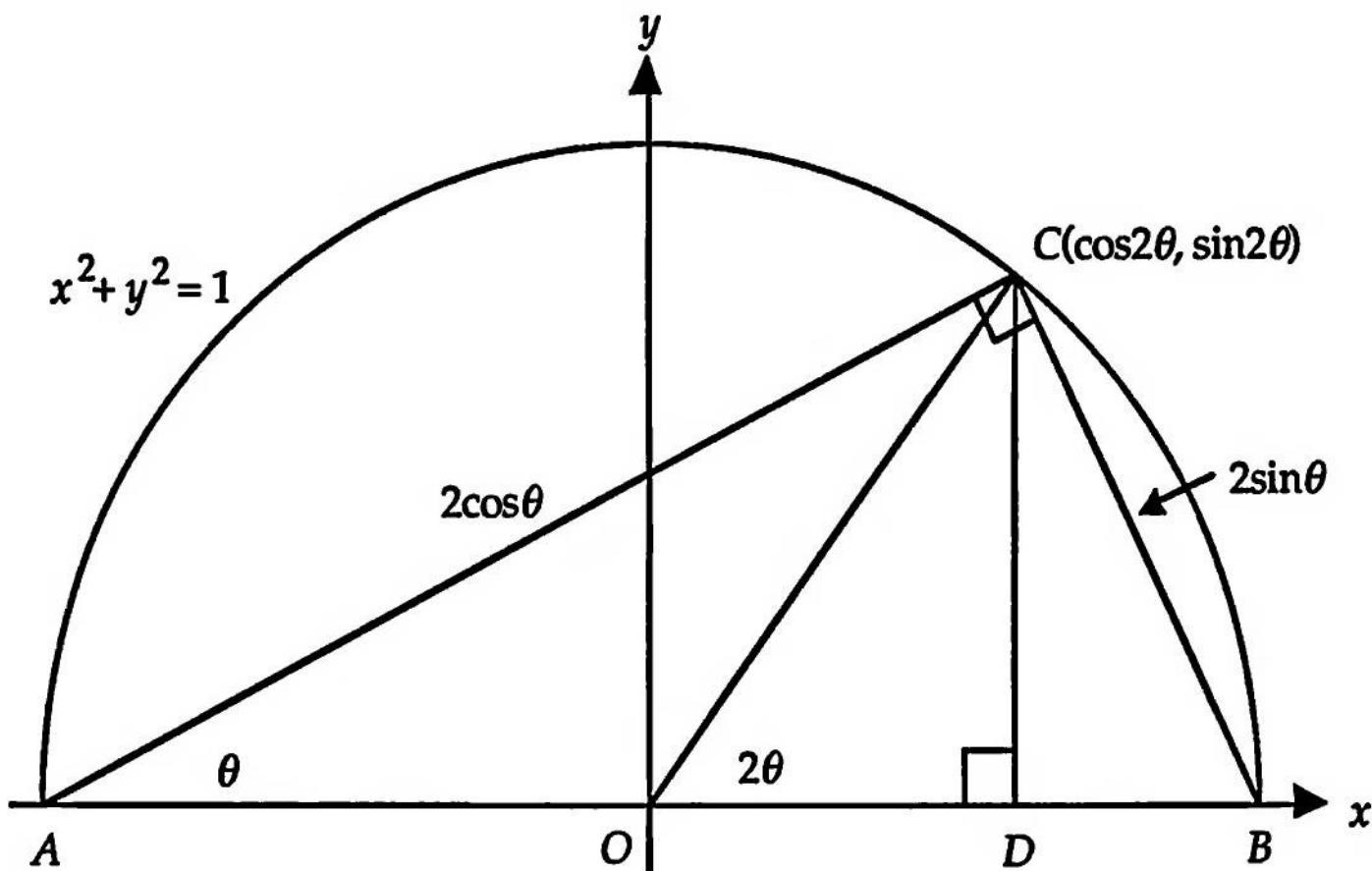
## The Law Of Cosines III (via Ptolemy's Theorem)



$$c \cdot c = b \cdot b + (a + 2b \cos(\pi - \theta)) \cdot a$$

$$c^2 = a^2 + b^2 - 2ab \cdot \cos\theta$$

# The Double-Angle Formulas



$$\Delta ACD \sim \Delta ABC$$

$$\overline{CD} / \overline{AC} = \overline{BC} / \overline{AB}$$

$$\sin 2\theta / 2\cos\theta = 2\sin\theta / 2$$

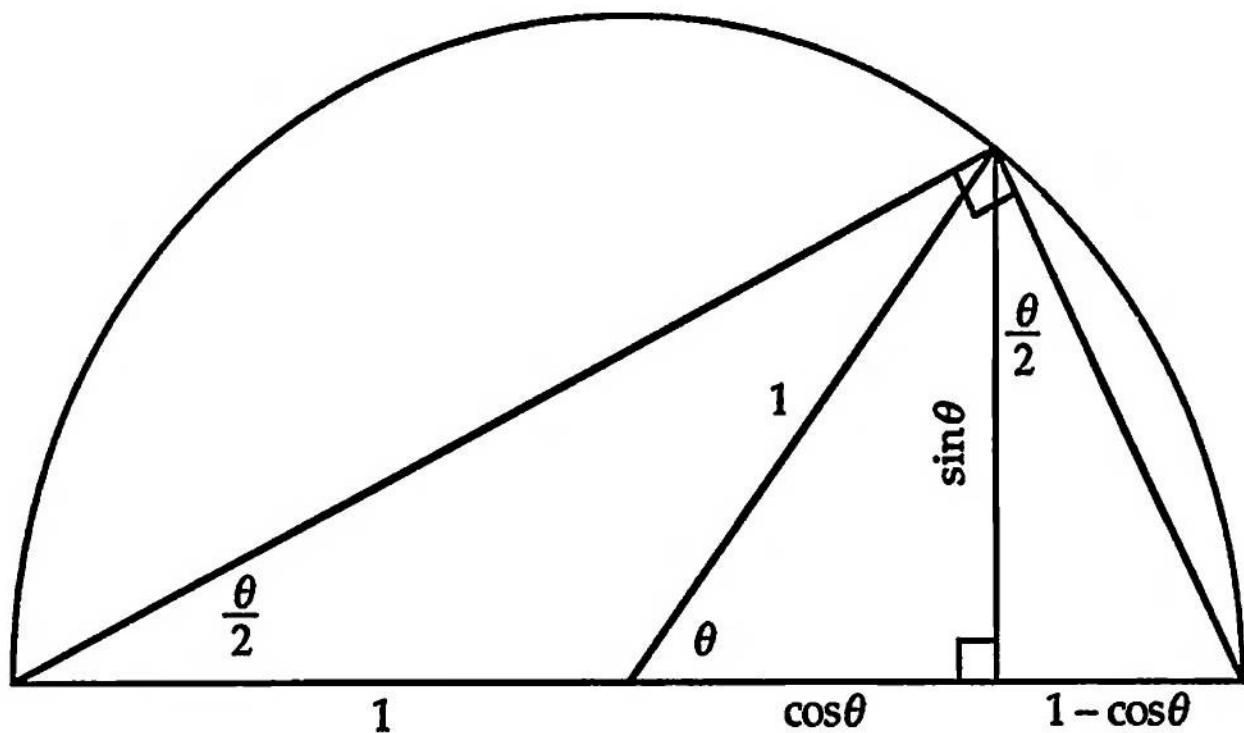
$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\overline{AD} / \overline{AC} = \overline{AC} / \overline{AB}$$

$$(1 + \cos 2\theta) / 2\cos\theta = 2\cos\theta / 2$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

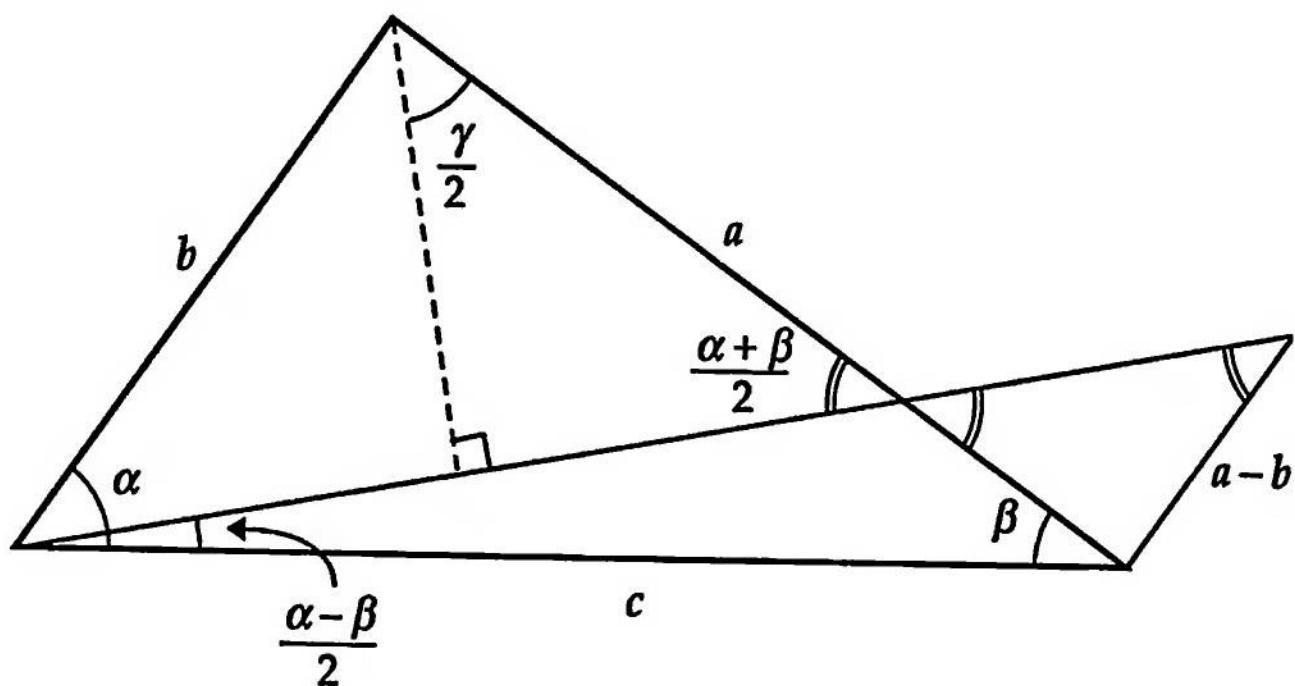
## The Half-Angle Tangent Formulas



$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta}$$

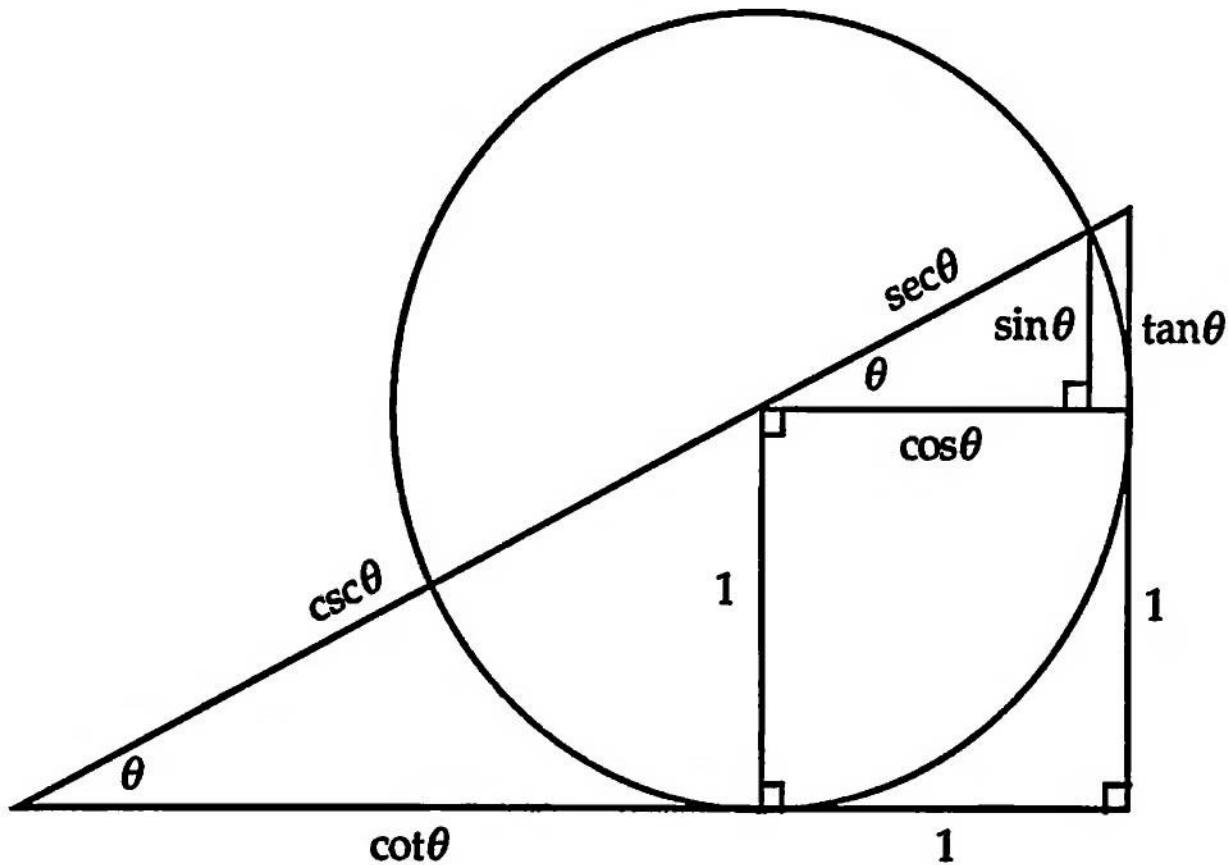
## Mollweide's Equation

$$(a - b) \cos \frac{\gamma}{2} = c \sin \left( \frac{\alpha - \beta}{2} \right)$$



—H. Arthur DeKleine

$$(\tan\theta + 1)^2 + (\cot\theta + 1)^2 = (\sec\theta + \csc\theta)^2$$



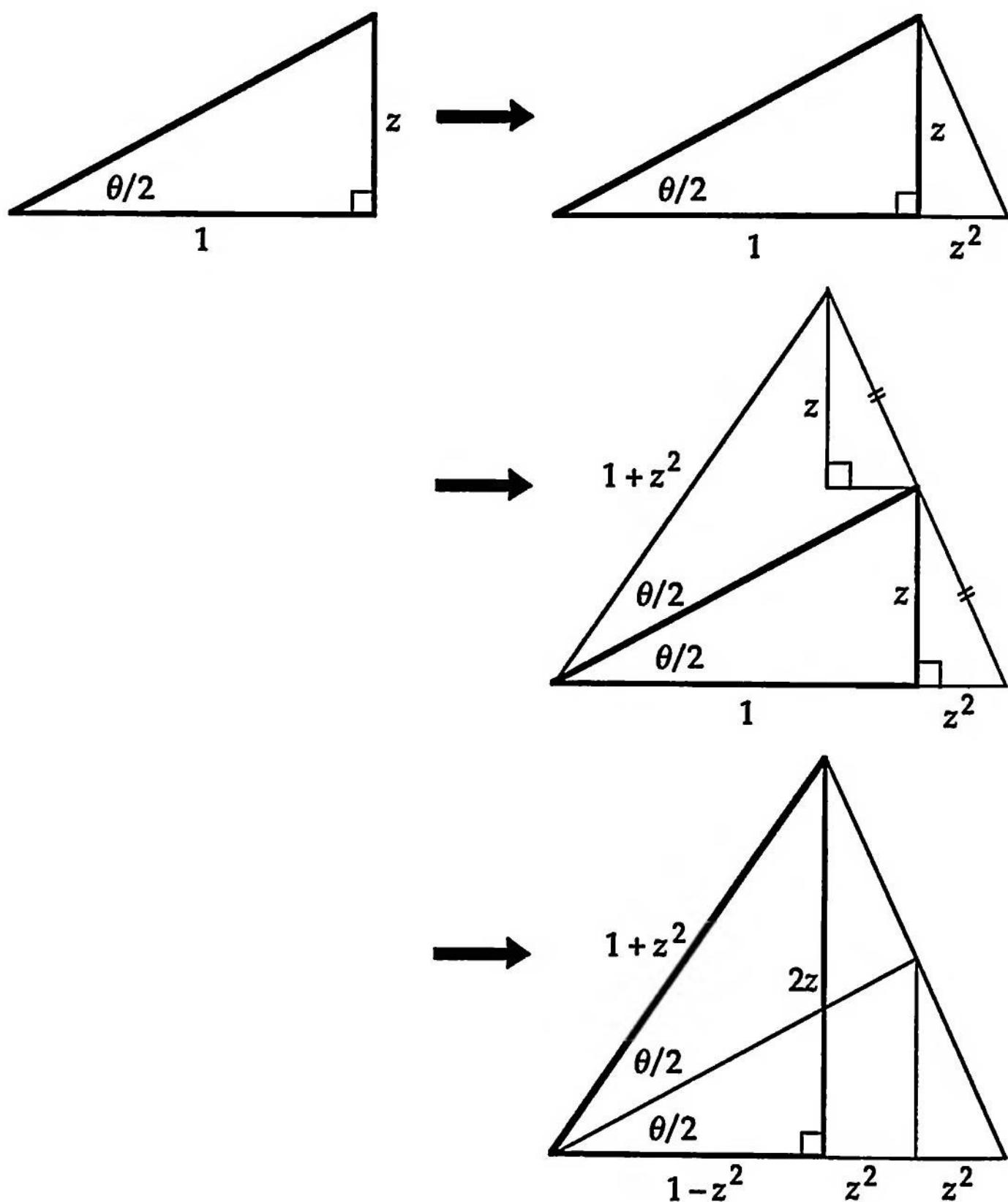
$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

$$(\tan\theta + 1)^2 + (\cot\theta + 1)^2 = (\sec\theta + \csc\theta)^2$$

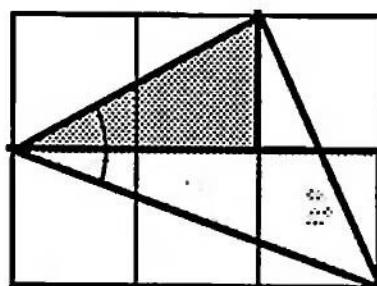
$$\left( \text{also } \tan\theta = \frac{\tan\theta + 1}{\cot\theta + 1} \right)$$

## The Substitution to Make a Rational Function of the Sine and Cosine

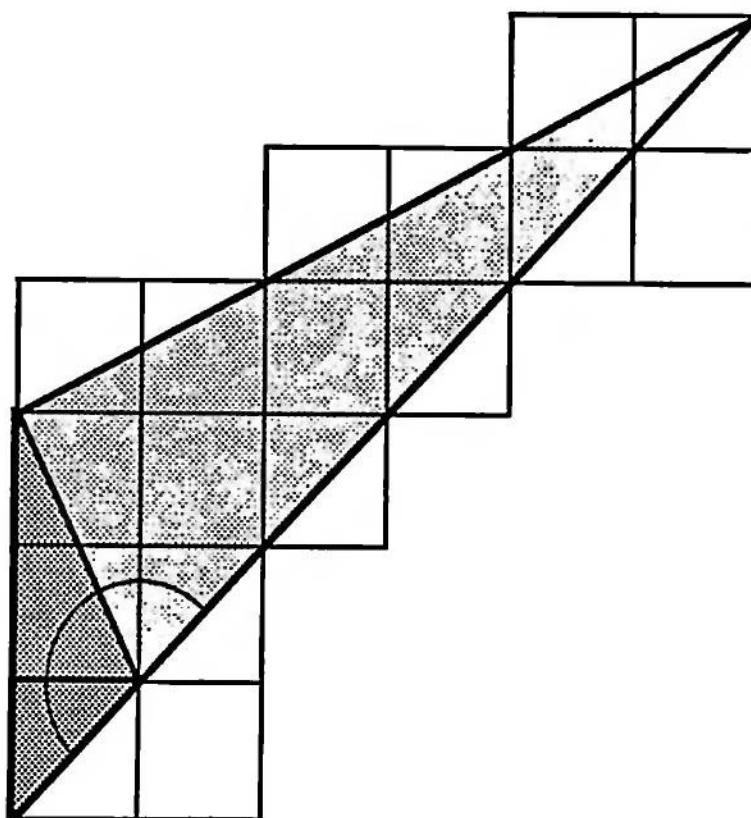


$$z = \tan \frac{\theta}{2} \Rightarrow \sin \theta = \frac{2z}{1+z^2} \text{ and } \cos \theta = \frac{1-z^2}{1+z^2}$$

## Sums of Arctangents



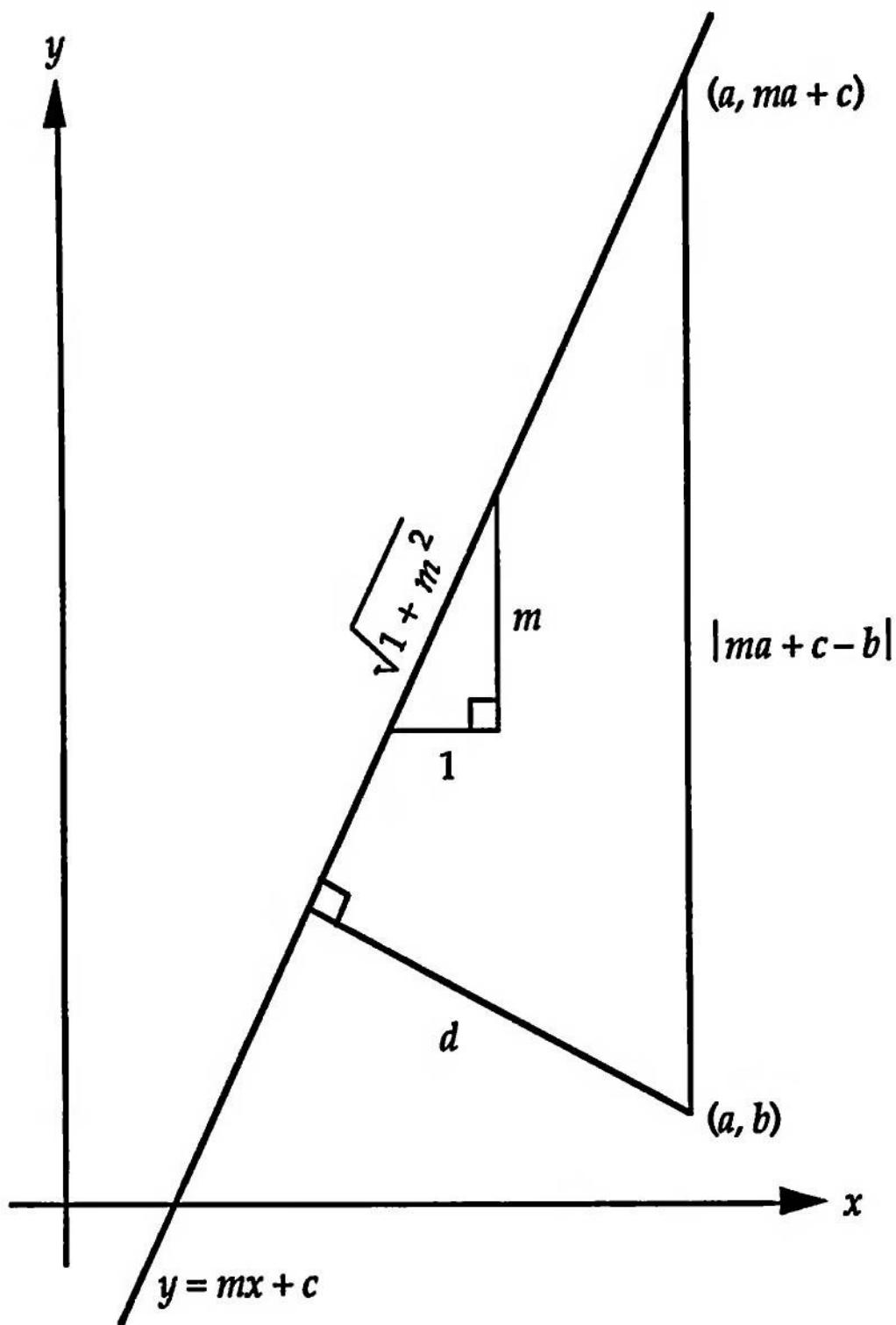
$$\arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}$$



$$\arctan 1 + \arctan 2 + \arctan 3 = \pi$$

—Edward M. Harris

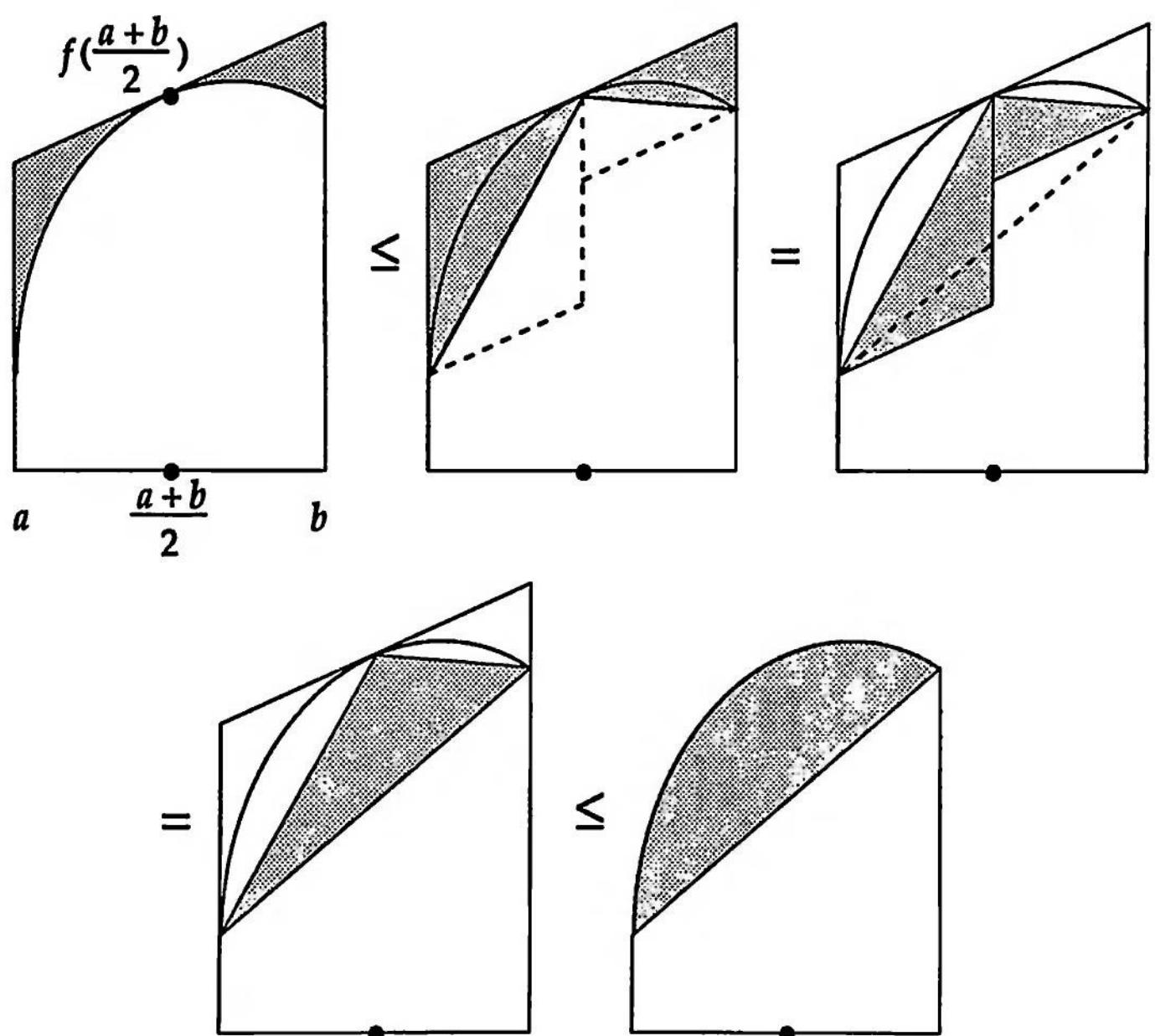
# The Distance Between a Point and a Line



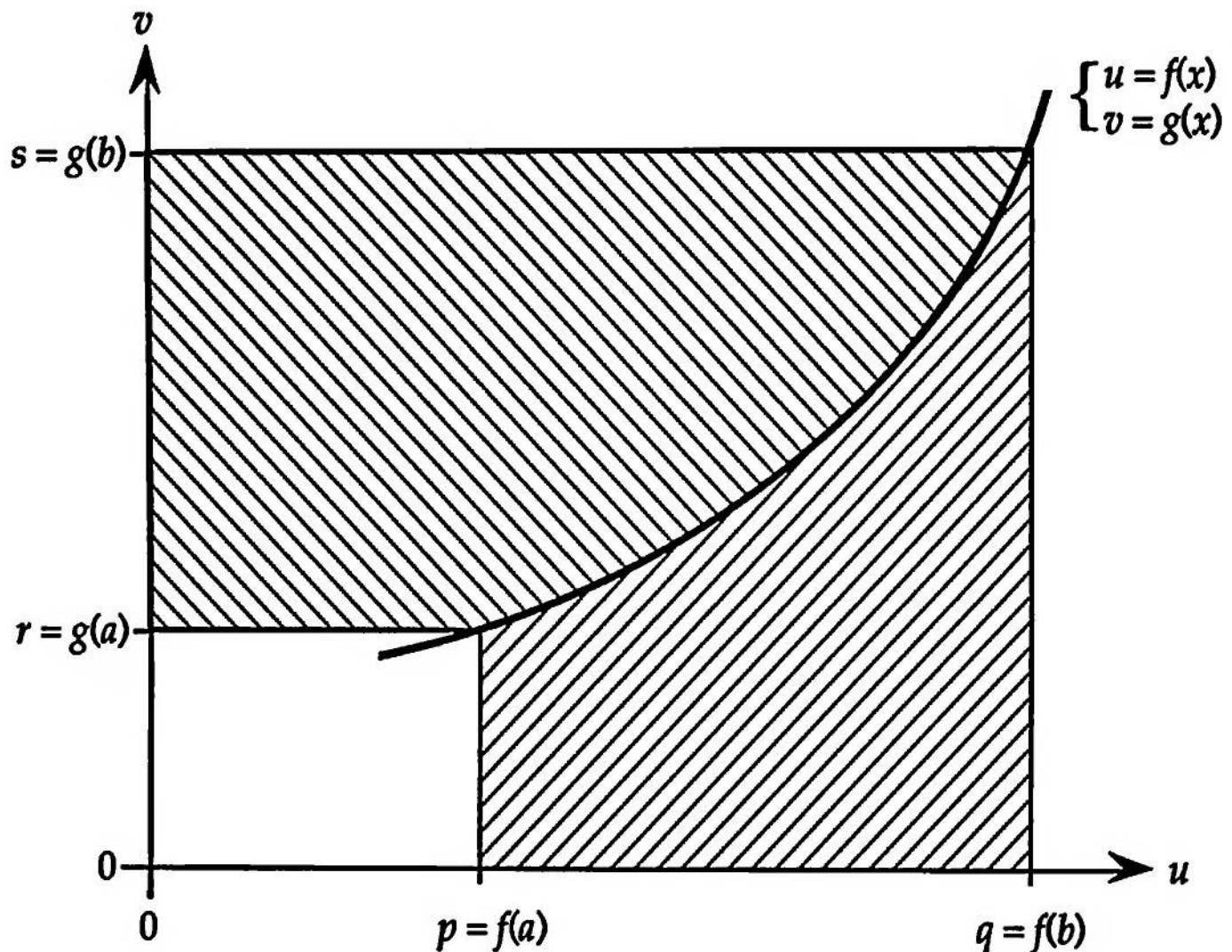
$$\frac{d}{1} = \frac{|ma + c - b|}{\sqrt{1 + m^2}}$$

—R. L. Eisenman

## The Midpoint Rule is Better than the Trapezoidal Rule for Concave Functions



# Integration by Parts



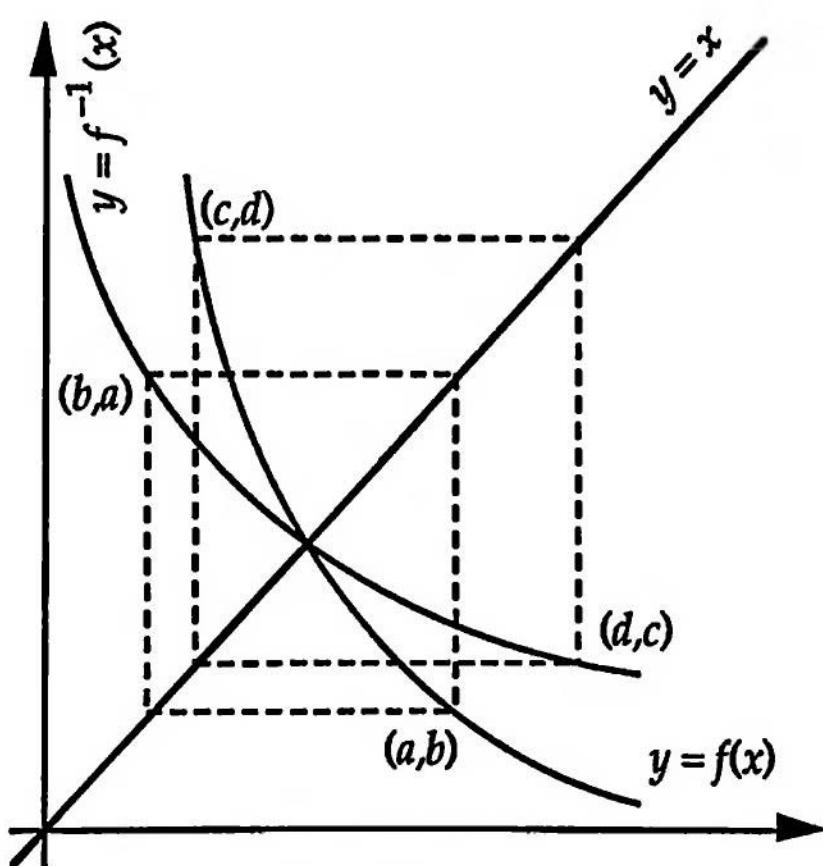
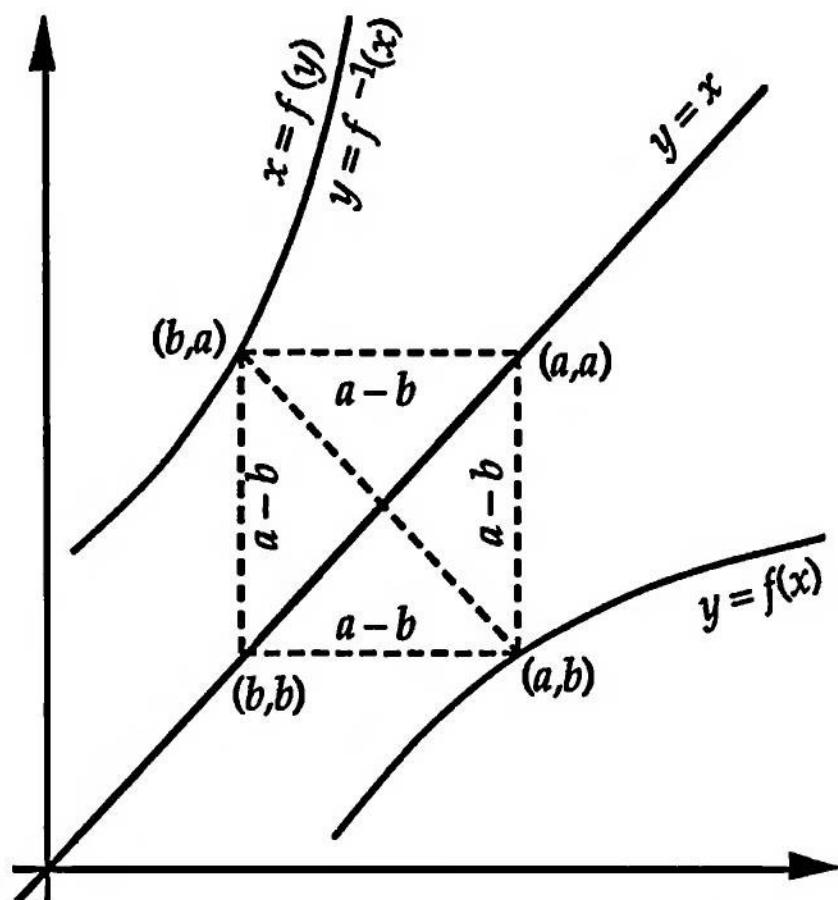
$$\text{Area } \boxed{\text{white}} + \text{Area } \boxed{\text{hatched}} = qs - pr$$

$$\int\limits_r^s u \, dv + \int\limits_p^q v \, du = uv \Big|_{(p,r)}^{(q,s)}$$

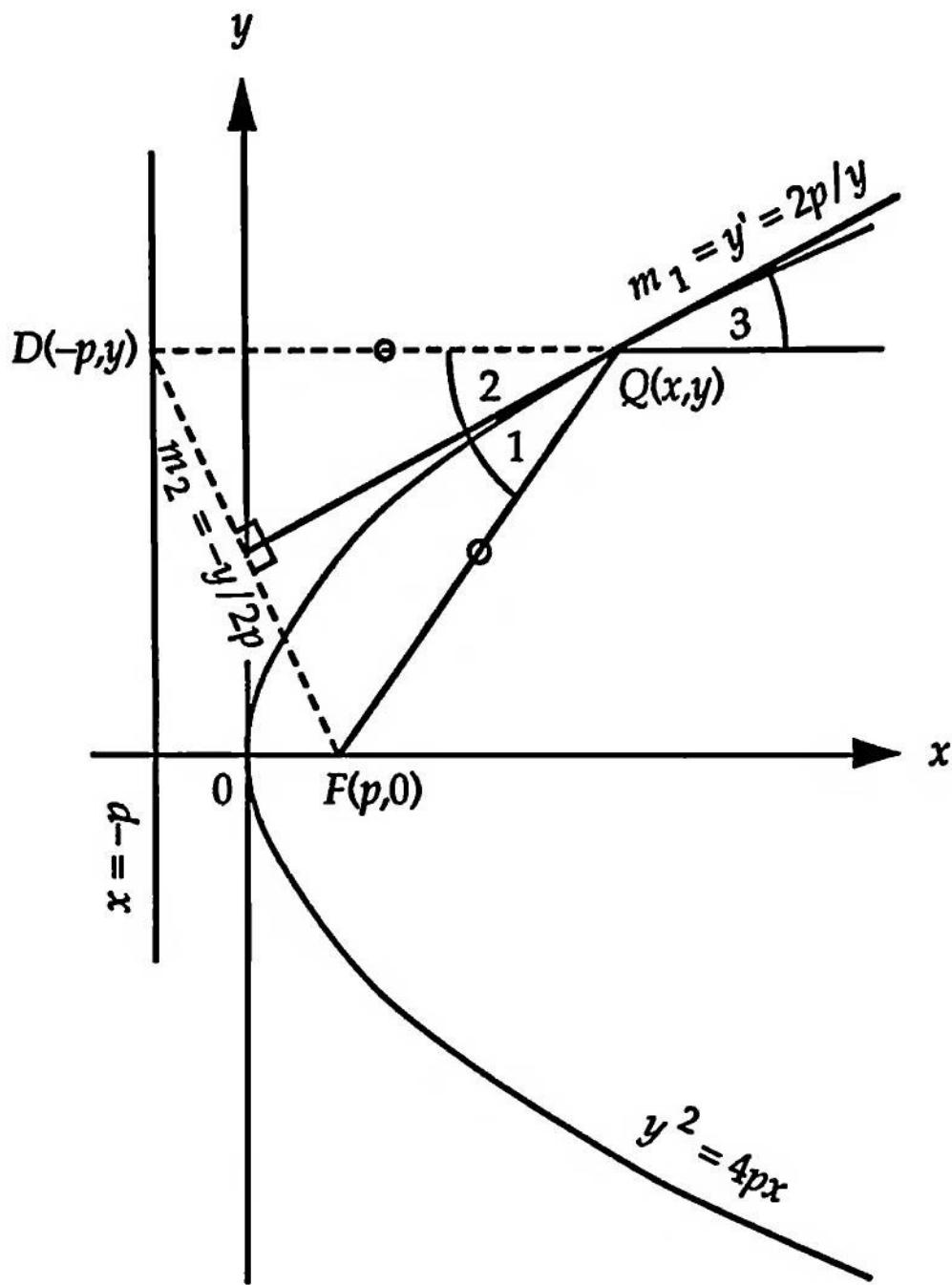
$$\int\limits_a^b f(x)g'(x)dx = f(x)g(x) \Big|_a^b - \int\limits_a^b g(x)f'(x)dx$$

—Richard Courant

The Graphs of  $f$  and  $f^{-1}$  are Reflections about the Line  $y = x$



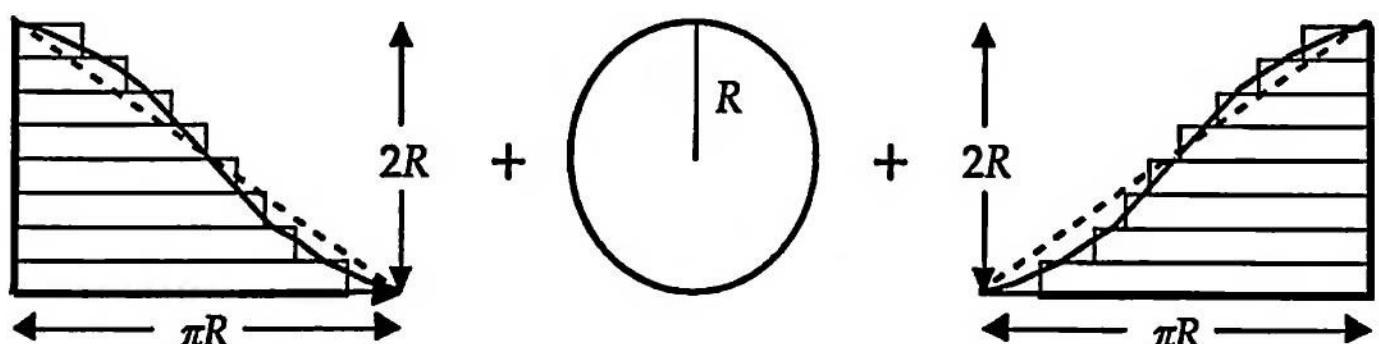
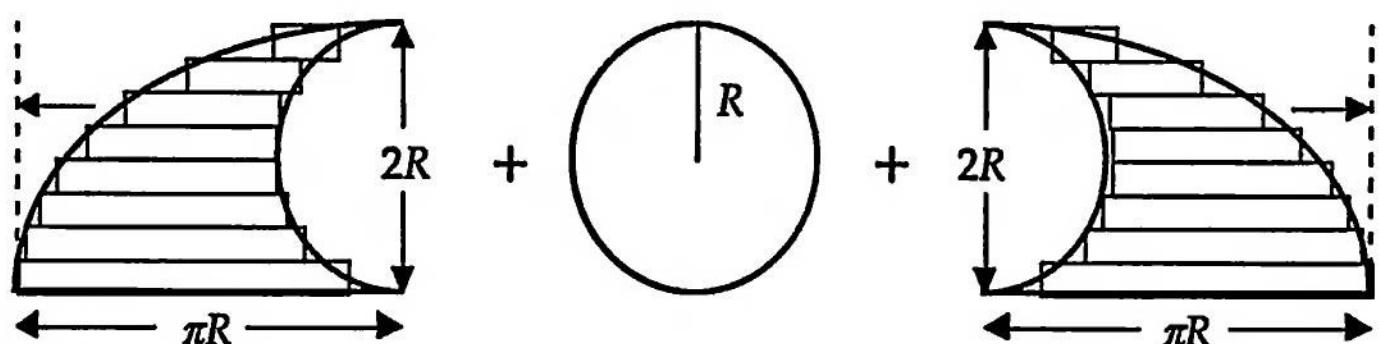
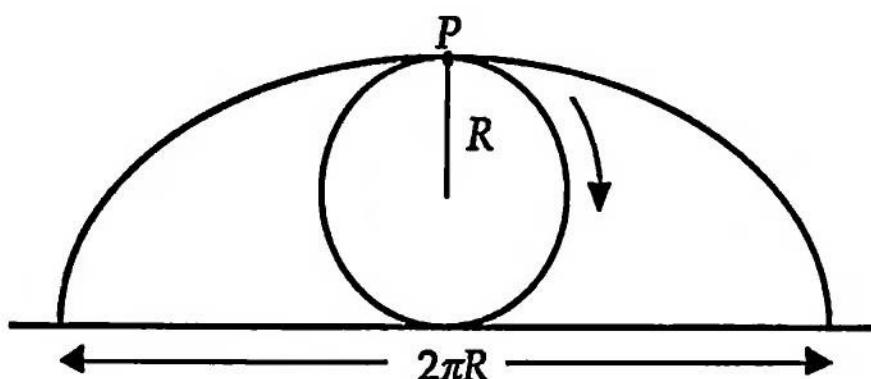
# The Reflection Property of the Parabola



$$QF = QD \quad \& \quad m_1 \cdot m_2 = -1 \quad \Rightarrow \quad \angle 1 = \angle 2 = \angle 3$$

—Ayoub B. Ayoub

## Area under an Arch of the Cycloid



$$\frac{1}{2} \pi R \cdot 2R + \pi R^2 + \frac{1}{2} \pi R \cdot 2R \\ \Rightarrow A = 3 \pi R^2$$

—Richard M. Beekman

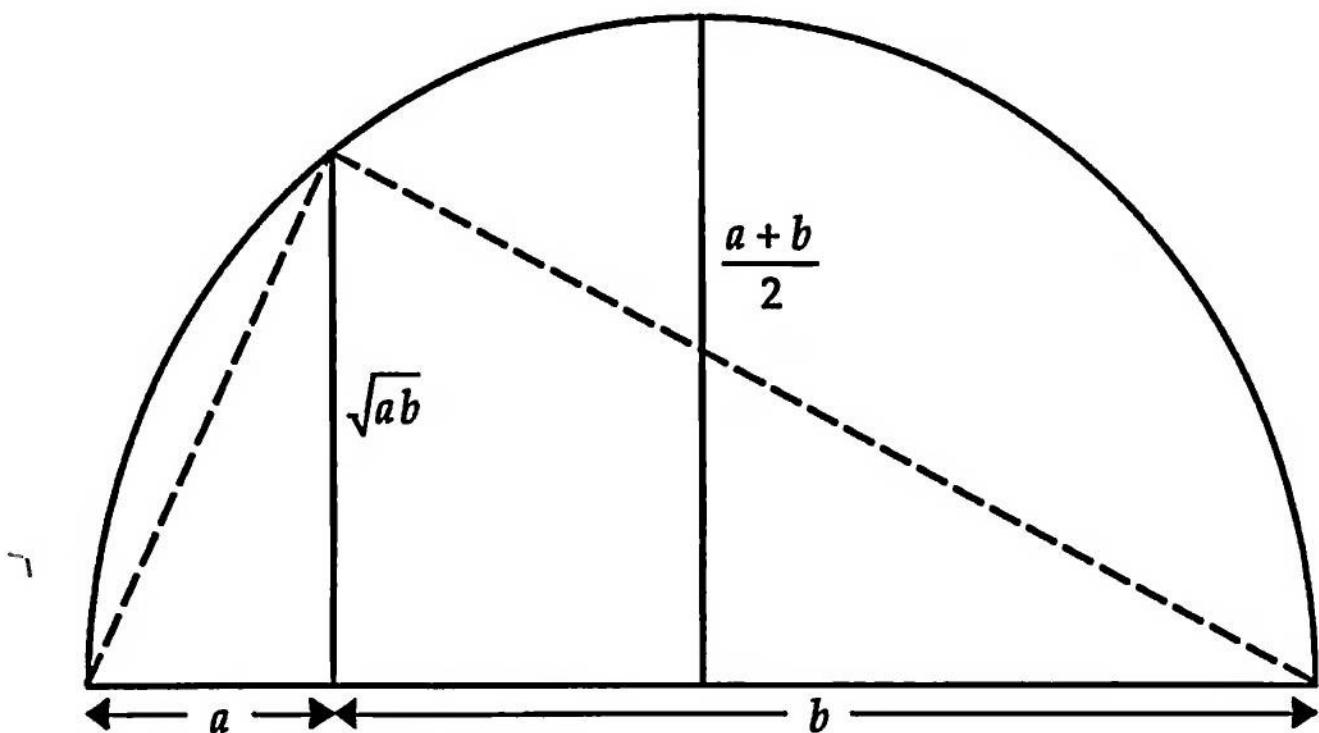


# Inequalities

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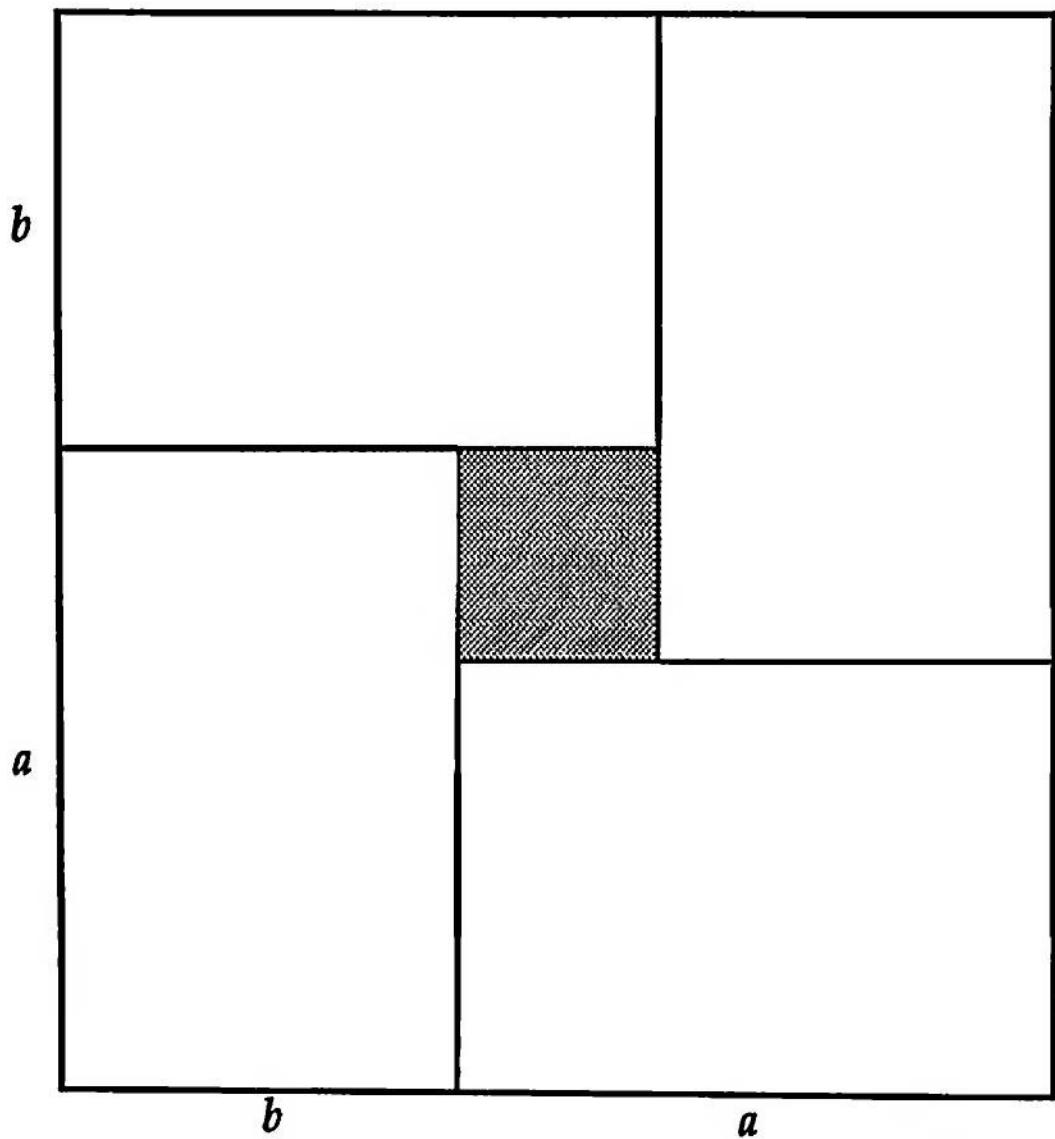


## The Arithmetic Mean—Geometric Mean Inequality I



$$\sqrt{ab} \leq \frac{a+b}{2}$$

## The Arithmetic Mean—Geometric Mean Inequality II



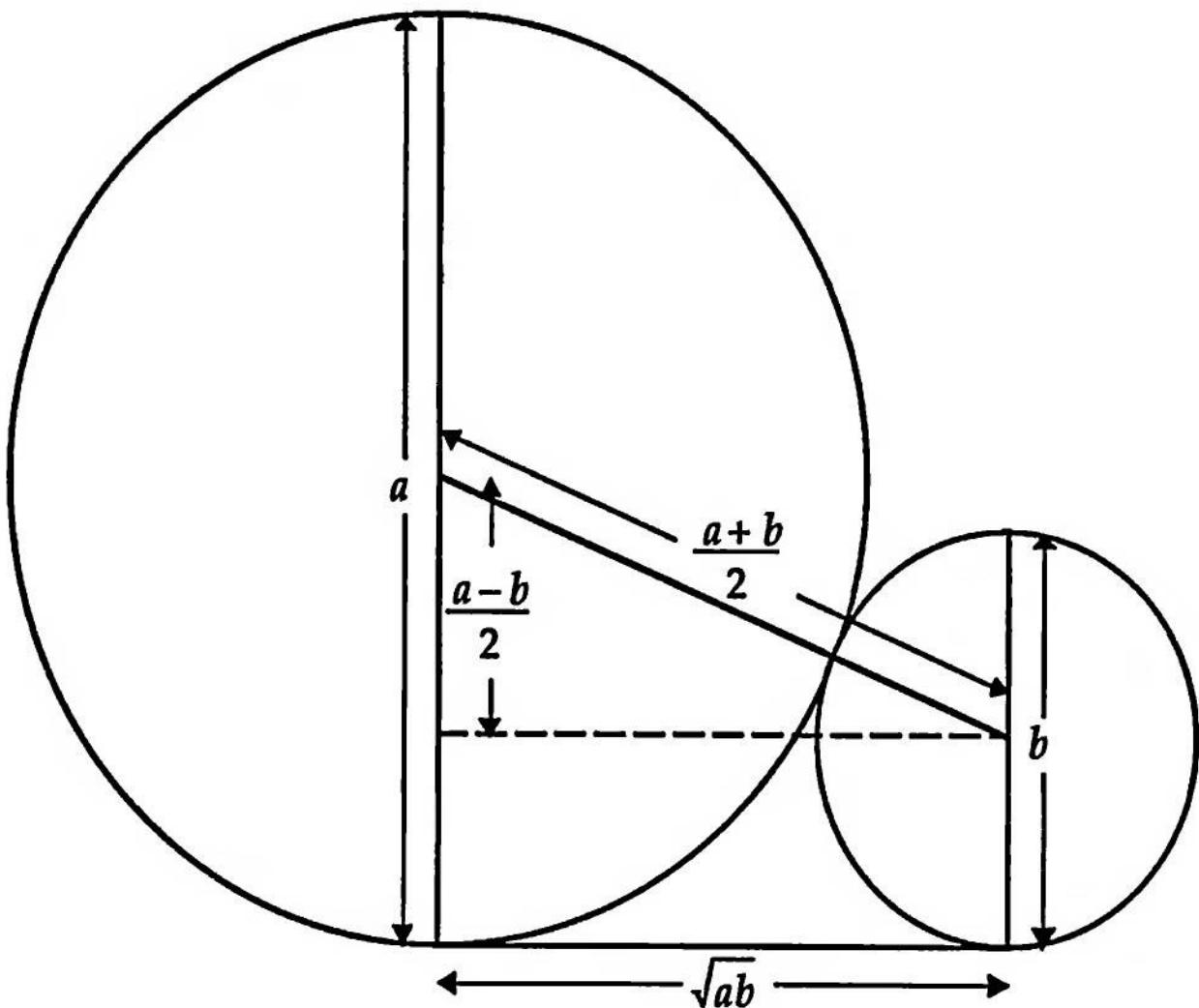
$$(a + b)^2 - (a - b)^2 = 4ab$$

$$\frac{a + b}{2} \geq \sqrt{ab}$$

—Doris Schattschneider

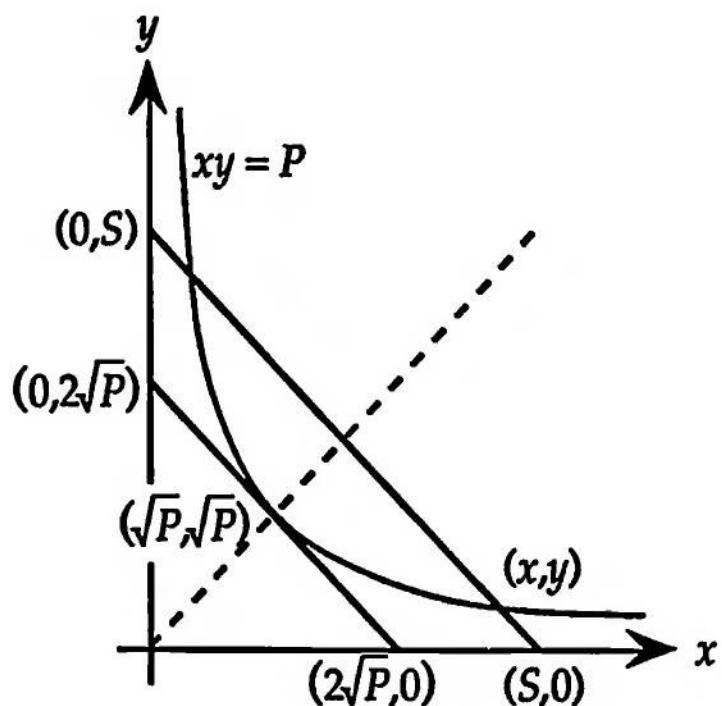
## The Arithmetic Mean—Geometric Mean Inequality III

$$\frac{a+b}{2} \geq \sqrt{ab}, \text{ with equality if and only if } a = b$$

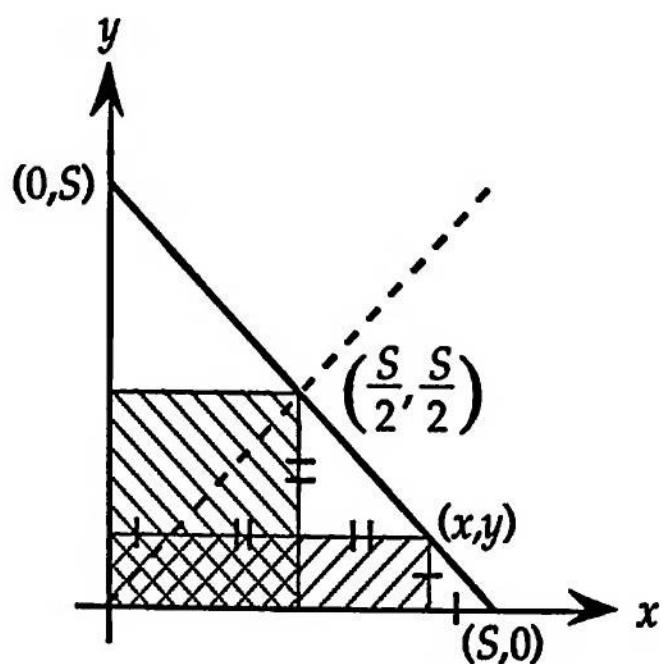


## Two Extremum Problems

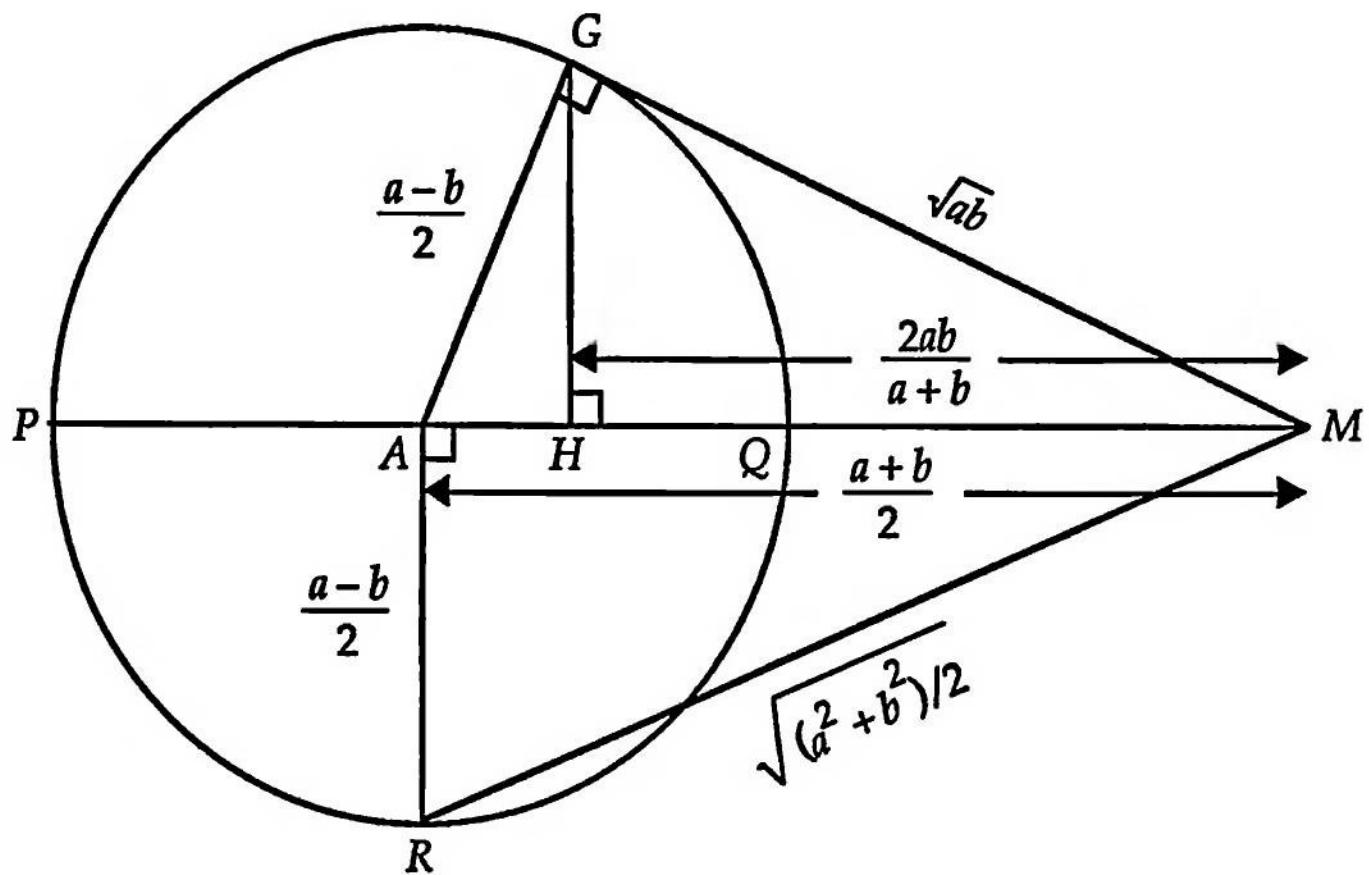
For a given product, the sum of two positive numbers is minimal when the numbers are equal.



For a given sum, the product of two positive numbers is maximal when the numbers are equal.



The Harmonic Mean—Geometric Mean—  
 Arithmetic Mean—Root Mean Square  
 Inequality I

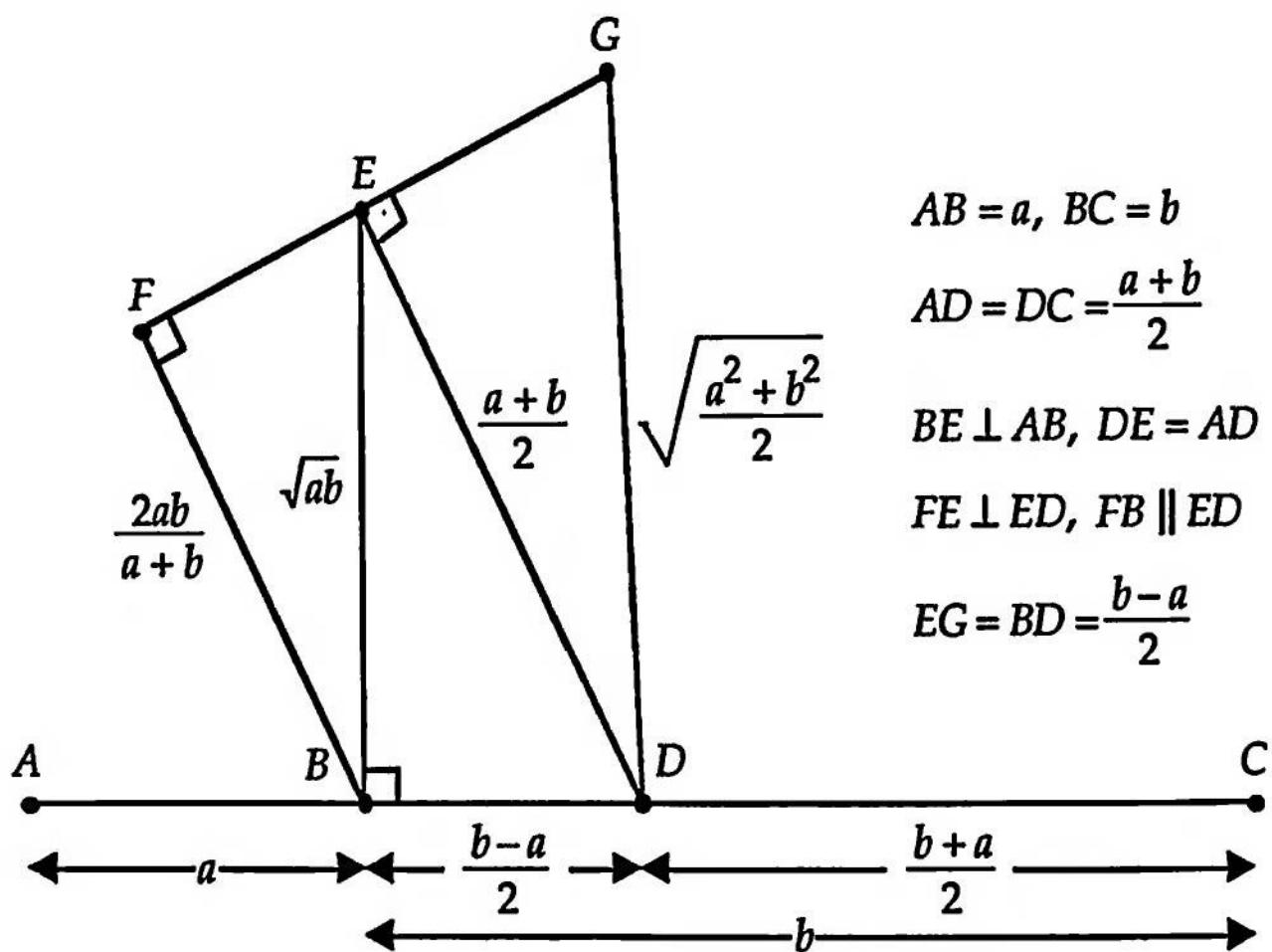


$$PM = a, QM = b, a > b > 0$$

$$HM < GM < AM < RM$$

$$\frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{(a^2 + b^2)/2}$$

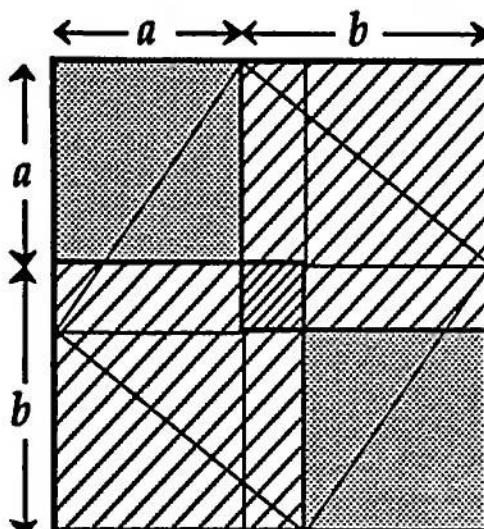
The Harmonic Mean—Geometric Mean—  
 Arithmetic Mean—Root Mean Square  
 Inequality II



—Sidney H. Kung

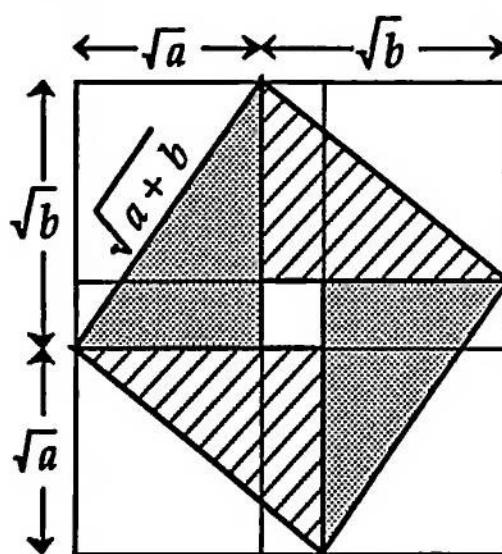
# The Harmonic Mean—Geometric Mean—Arithmetic Mean—Root Mean Square Inequality III

$$a, b > 0 \Rightarrow \sqrt{(a^2 + b^2)/2} \geq \frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2ab}{a+b}$$



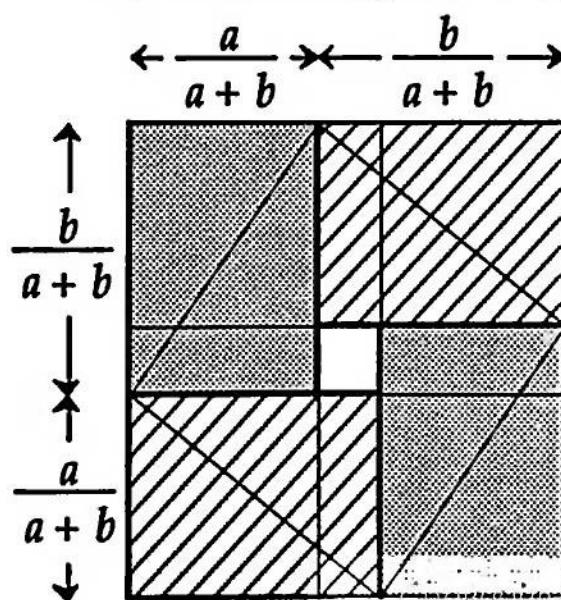
$$2a^2 + 2b^2 \geq (a+b)^2$$

$$\sqrt{\frac{a^2 + b^2}{2}} \geq \frac{a+b}{2}$$



$$(\sqrt{a+b})^2 \geq 4 \cdot \frac{1}{2} \sqrt{a} \sqrt{b}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$



$$1 \geq 4 \frac{a}{a+b} \cdot \frac{b}{a+b}$$

$$\sqrt{ab} \geq \frac{2ab}{a+b}$$

## Five Means — and Their Means

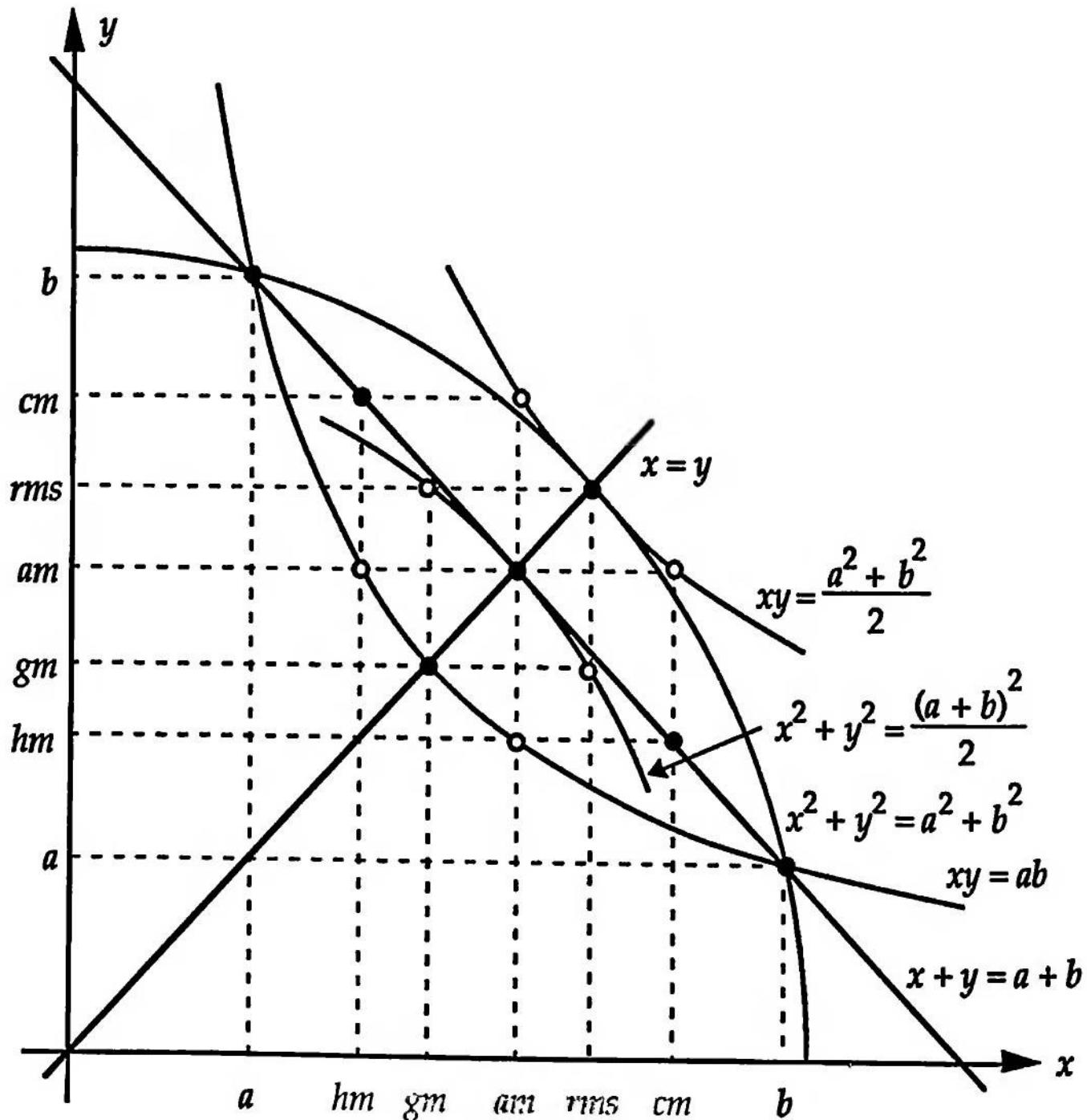
Arithmetic:  $am = AM(a,b) = \frac{a+b}{2}$

Contraharmonic:  $cm = CM(a,b) = \frac{a^2 + b^2}{a+b}$

Geometric:  $gm = GM(a,b) = \sqrt{ab}$

Harmonic:  $hm = HM(a,b) = \frac{2ab}{a+b}$

Root Mean Square:  $rms = RMS(a,b) = \sqrt{\frac{a^2 + b^2}{2}}$



I.  $0 < a < b \Rightarrow$

$$a < \frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{\frac{a^2+b^2}{2}} < \frac{a^2+b^2}{a+b} < b$$

II.  $hm + cm = a + b \Rightarrow AM(hm, cm) = am.$

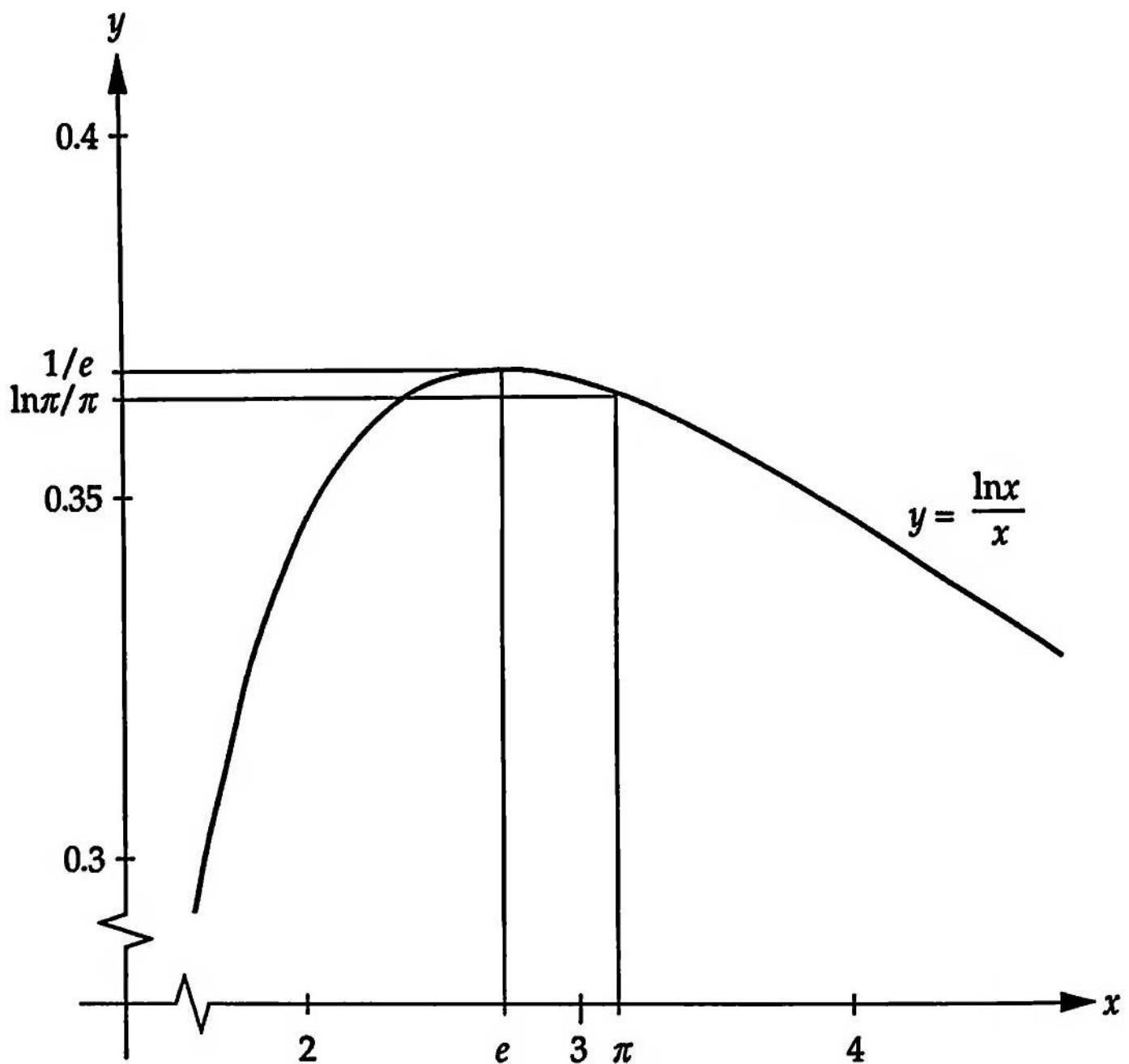
III.  $hm \cdot am = a \cdot b \Rightarrow GM(hm, am) = gm.$

IV.  $am \cdot cm = \frac{a^2 + b^2}{2} \Rightarrow GM(am, cm) = rms.$

V.  $gm^2 + rms^2 = \frac{(a+b)^2}{2} \Rightarrow RMS(gm, rms) = am.$

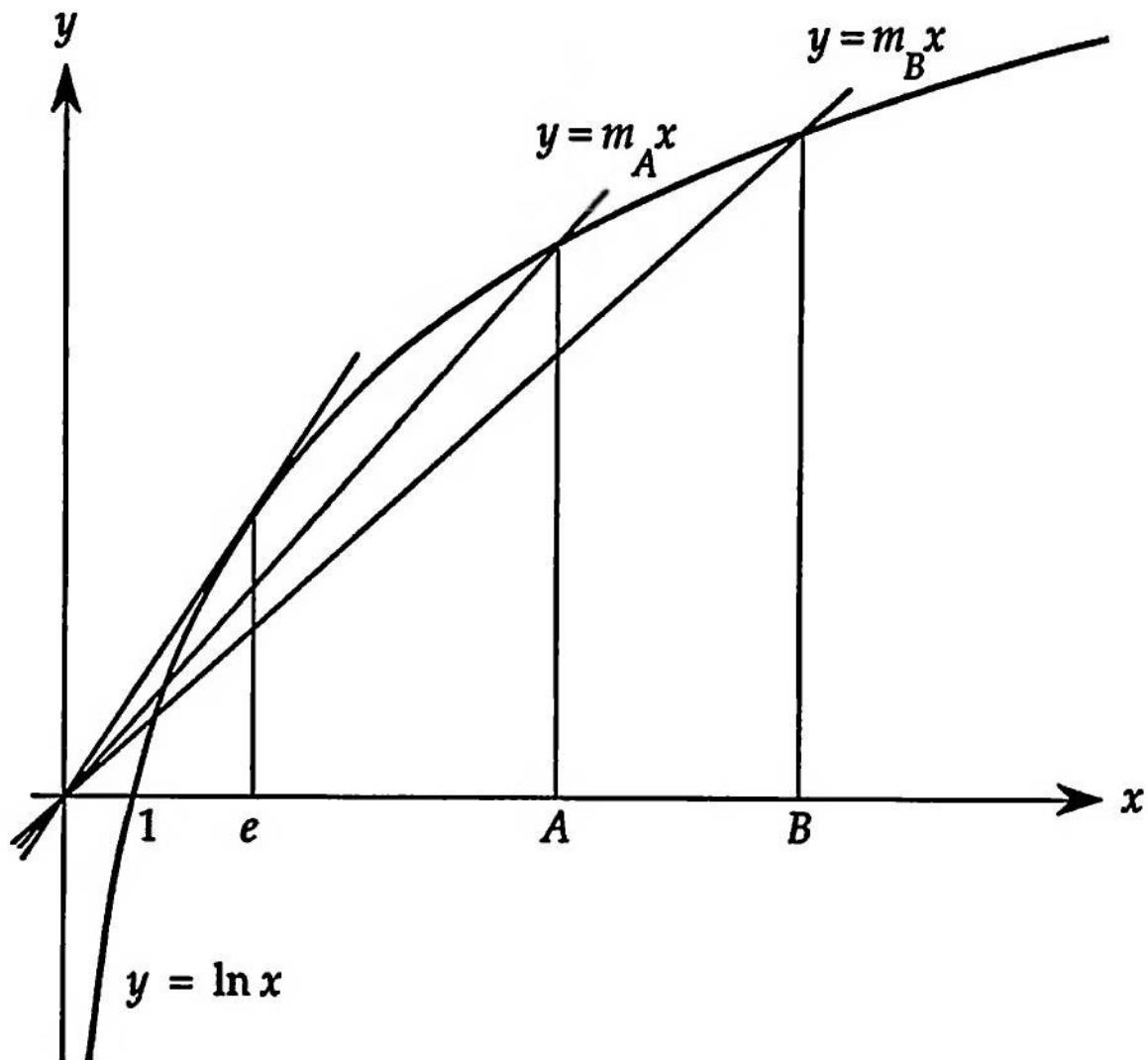
$$a < \frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{\frac{a^2+b^2}{2}} < \frac{a^2+b^2}{a+b} < b$$

$$e^\pi > \pi^e$$



—Fouad Nakhli

$A^B > B^A$  for  $e \leq A < B$



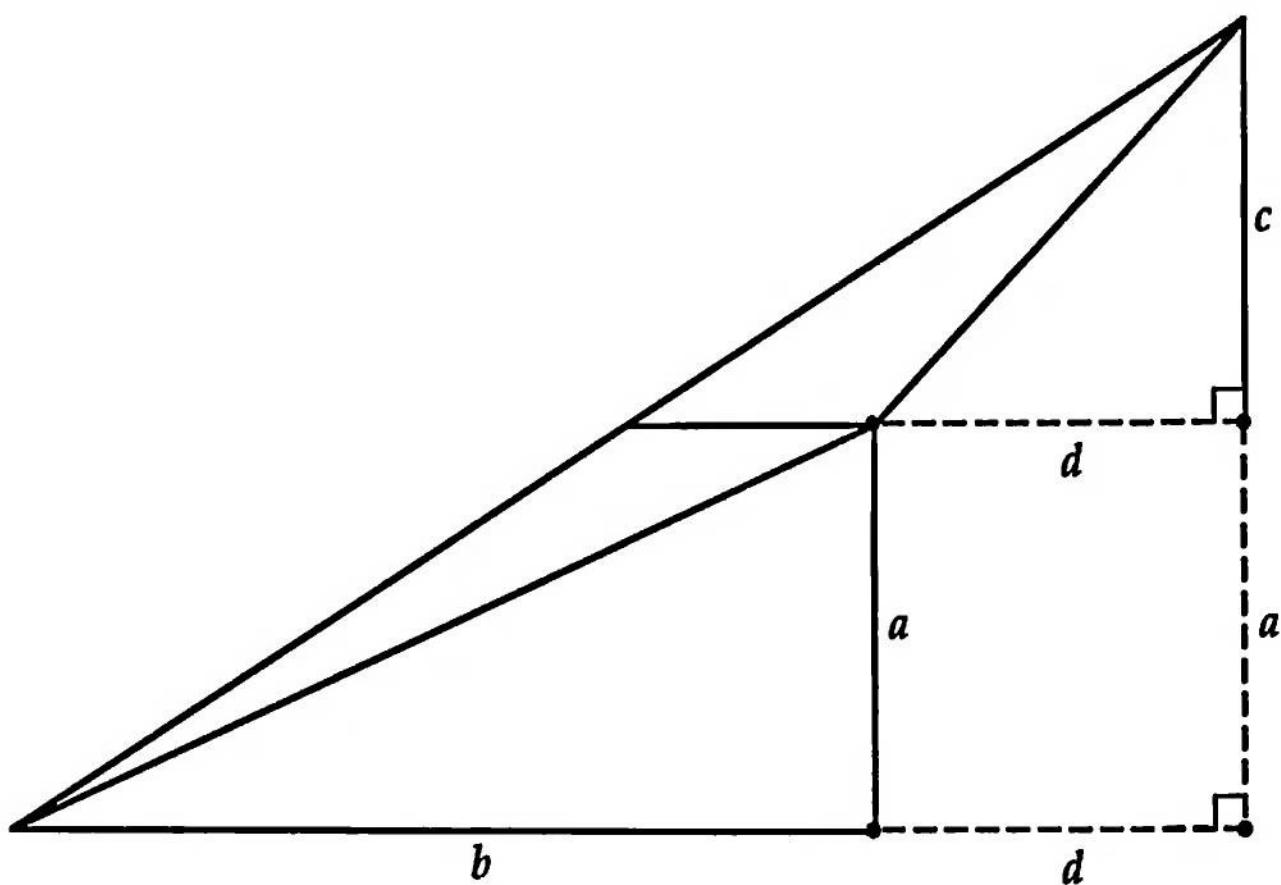
$$e \leq A < B \Rightarrow m_A > m_B$$

$$\Rightarrow \frac{\ln A}{A} > \frac{\ln B}{B}$$

$$\Rightarrow A^B > B^A$$

## The Mediant Property

$$\frac{a}{b} < \frac{c}{d} \Rightarrow \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

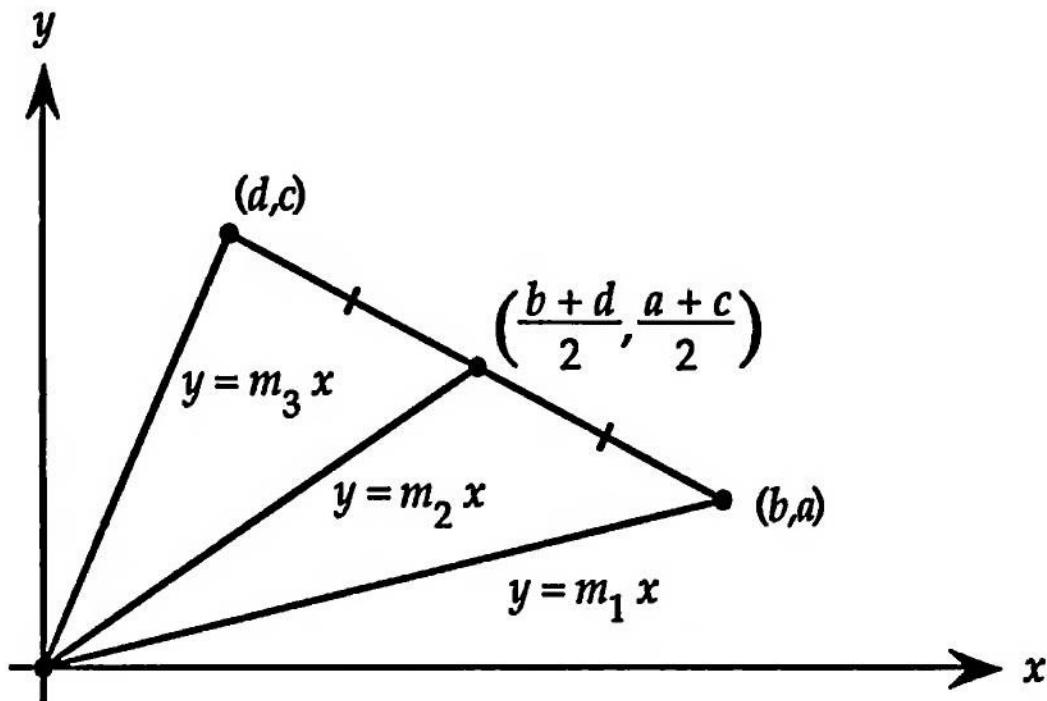


## Regle des Nombres Moyens (Two Proofs)

[Nicolas Chuquet, *Le Triparty en la Science des Nombres*, 1484]

$$a, b, c, d > 0; \frac{a}{b} < \frac{c}{d} \Rightarrow \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

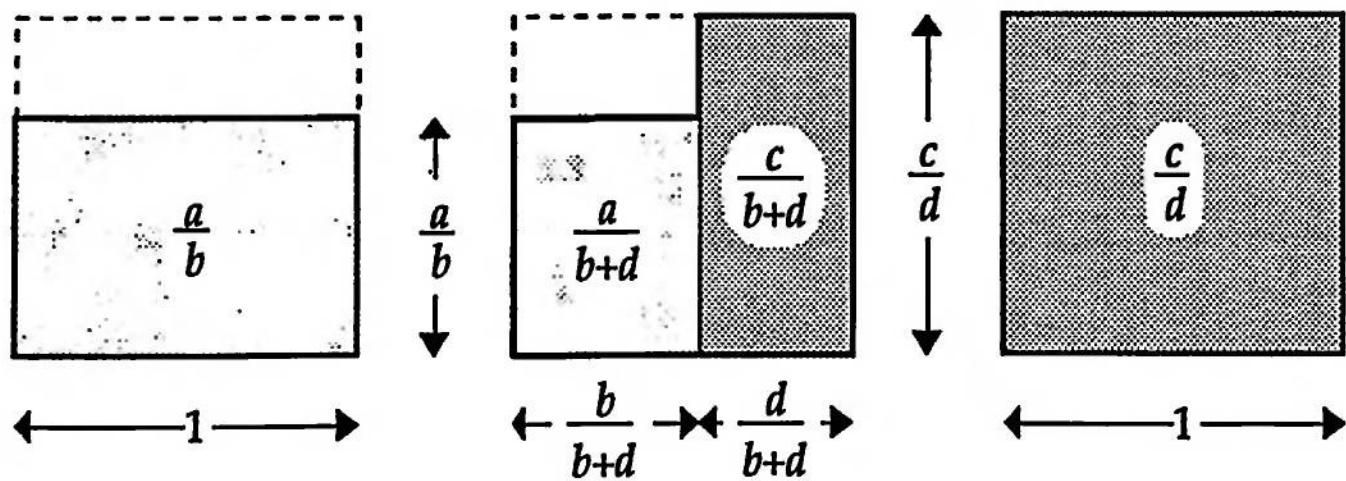
I.



$$m_1 < m_3 \Rightarrow m_1 < m_2 < m_3$$

—Li Changming

II.

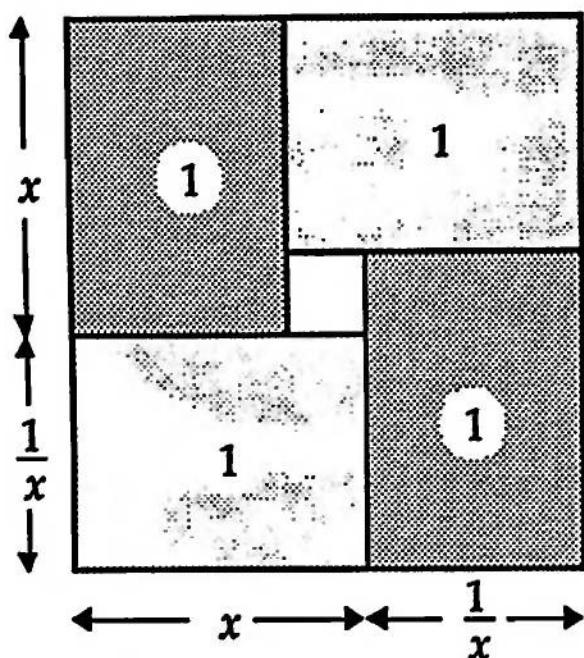


$$\frac{a}{b} < \frac{a}{b+d} + \frac{c}{b+d} < \frac{c}{d}$$

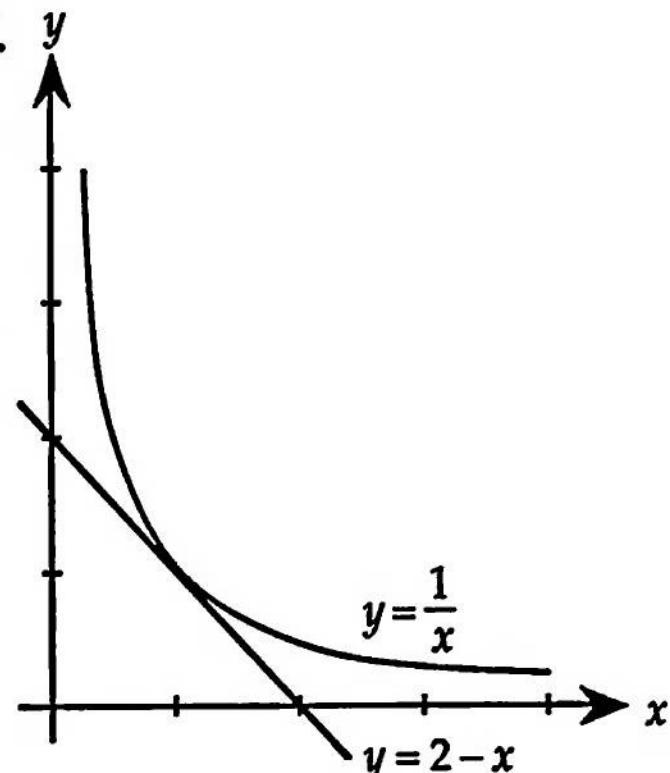
—RBN

# The Sum of a Positive Number and its Reciprocal is at least Two (Four Proofs)

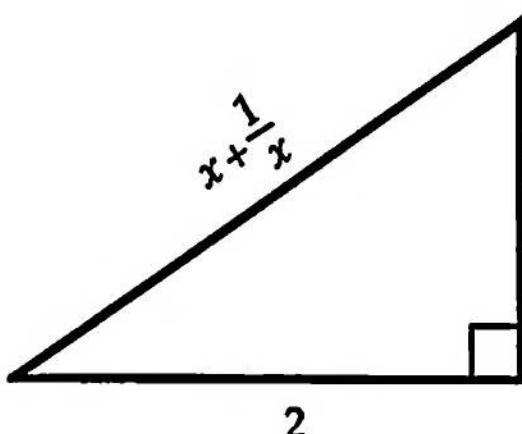
I.



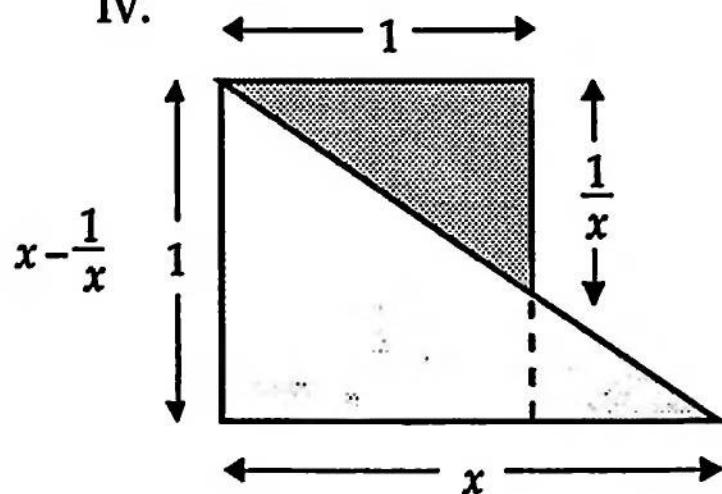
II.



III.



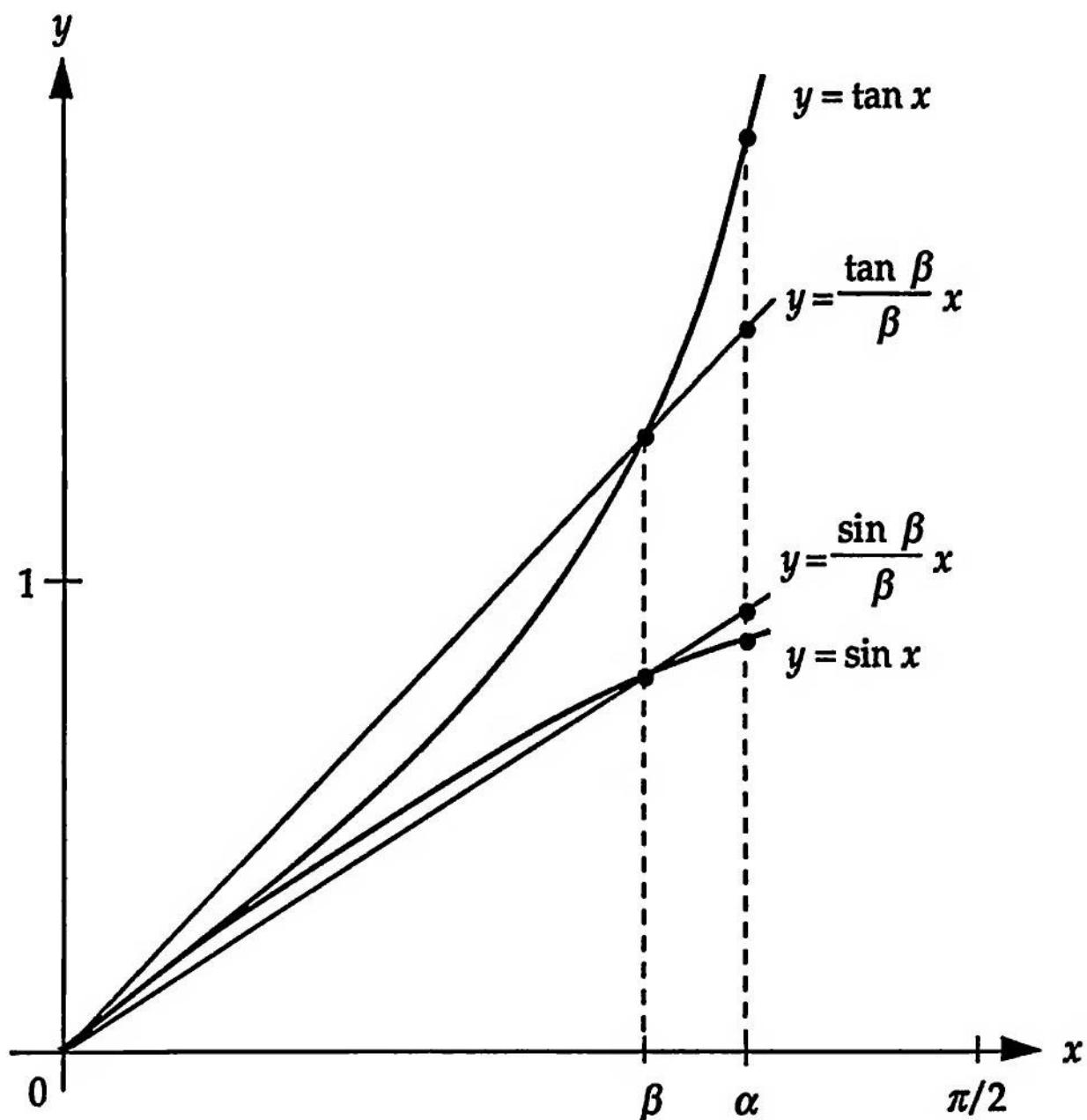
IV.



$$x \geq 1 \Rightarrow x + \frac{1}{x} \geq 2$$

## Aristarchus' Inequalities

$$0 < \beta < \alpha < \frac{\pi}{2} \Rightarrow \frac{\sin \alpha}{\sin \beta} < \frac{\alpha}{\beta} < \frac{\tan \alpha}{\tan \beta}$$

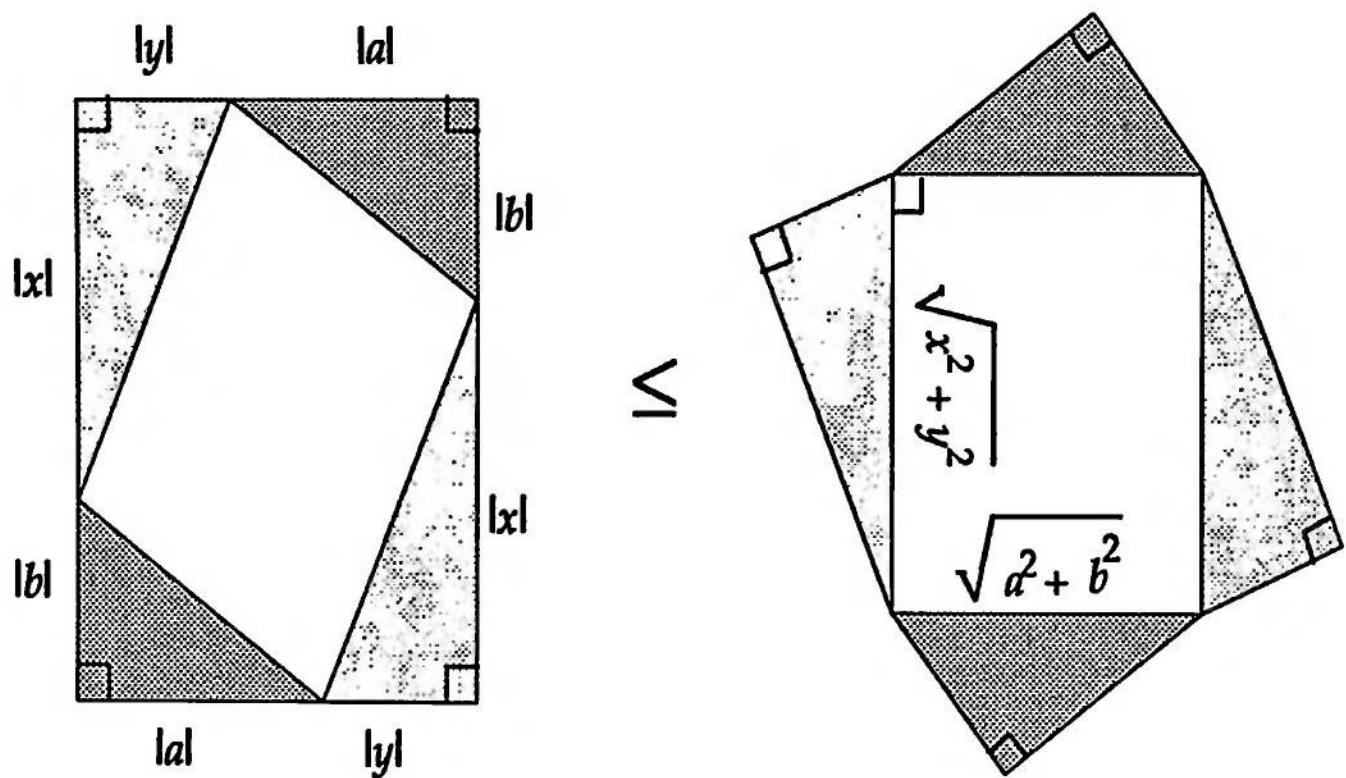


$$\sin \alpha < \frac{\sin \beta}{\beta} \alpha; \quad \frac{\tan \beta}{\beta} \alpha < \tan \alpha$$

$$\therefore \frac{\sin \alpha}{\sin \beta} < \frac{\alpha}{\beta} < \frac{\tan \alpha}{\tan \beta}$$

# The Cauchy-Schwarz Inequality

$$|\langle a, b \rangle \cdot \langle x, y \rangle| \leq \|\langle a, b \rangle\| \|\langle x, y \rangle\|$$



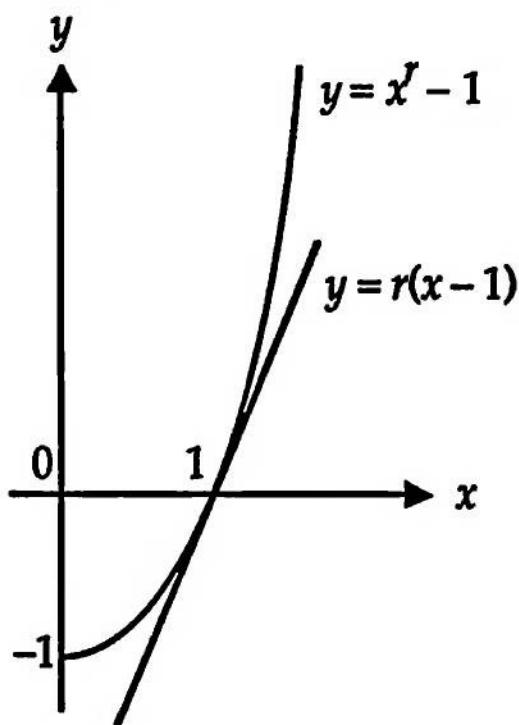
$$(|a| + |b|)(|b| + |x|) \leq 2\left(\frac{1}{2}|a||b| + \frac{1}{2}|x||y|\right) + \sqrt{a^2 + b^2} \sqrt{x^2 + y^2}$$

$$\therefore |ax + by| \leq |a||x| + |b||y| \leq \sqrt{a^2 + b^2} \sqrt{x^2 + y^2}$$

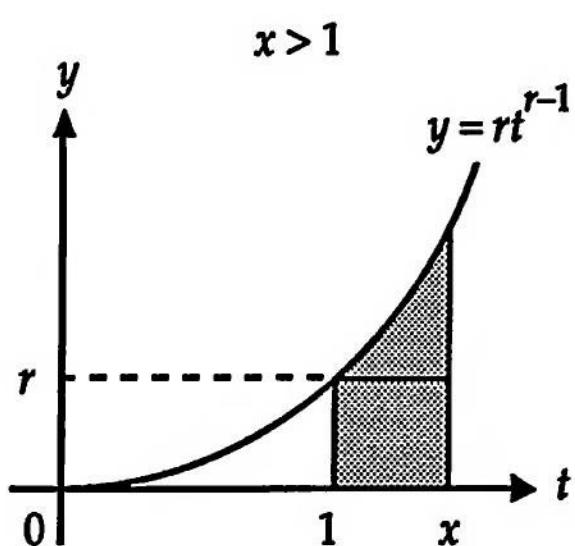
## Bernoulli's Inequality (two proofs)

$$x > 0, x \neq 1, r > 1 \Rightarrow x^r - 1 > r(x - 1)$$

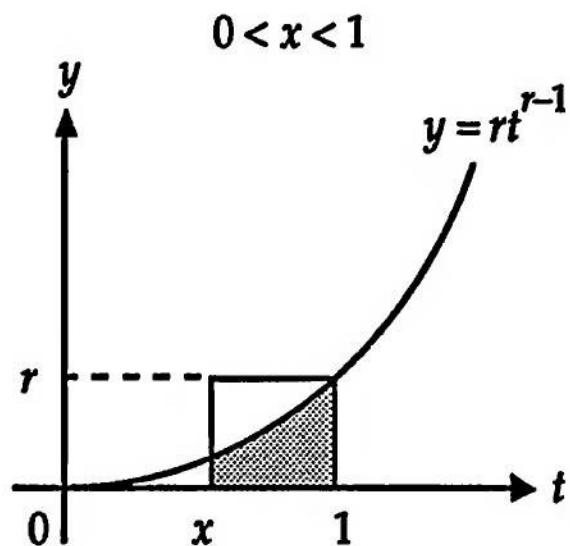
### I. (first semester calculus)



### II. (second semester calculus)



$$x^r - 1 = \int_1^x rt^{r-1} dt > r(x - 1)$$

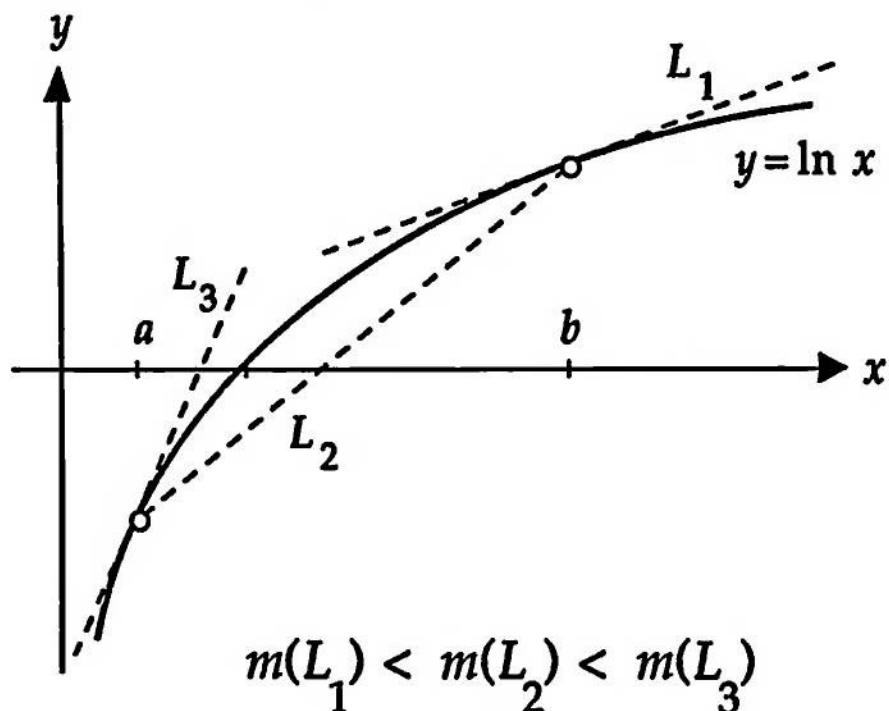


$$1 - x^r = \int_x^1 rt^{r-1} dt < r(1 - x)$$

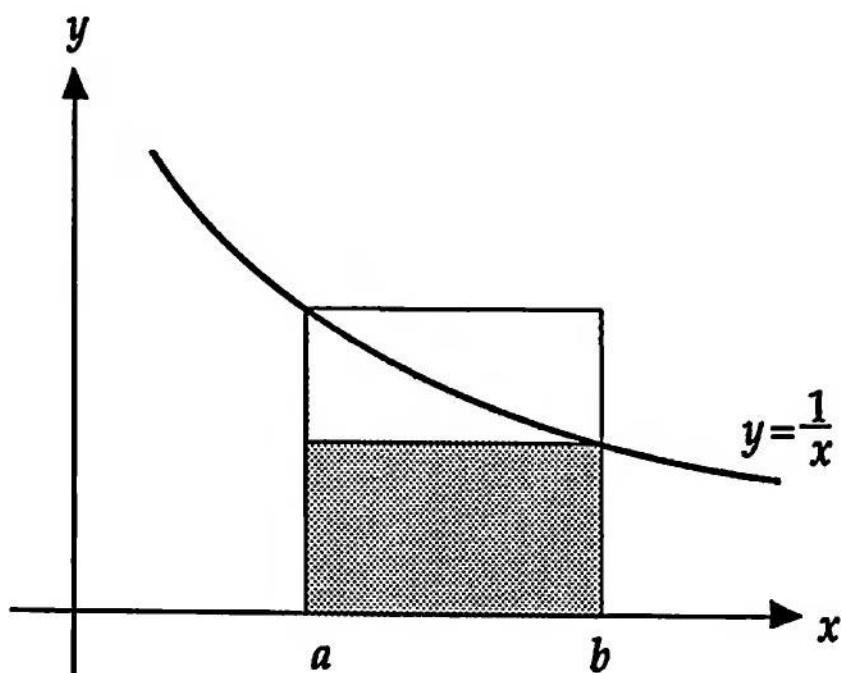
## Napier's Inequality (two proofs)

$$b > a > 0 \Rightarrow \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$$

### I. (first semester calculus)



### II. (second semester calculus)



$$\frac{1}{b}(b-a) < \int_a^b \frac{1}{x} dx < \frac{1}{a}(b-a)$$

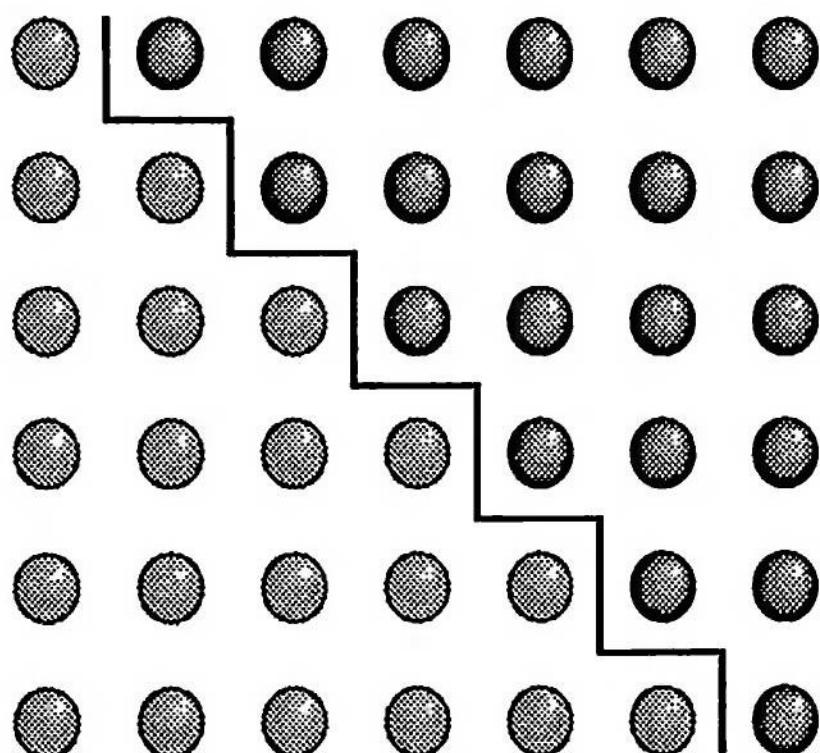
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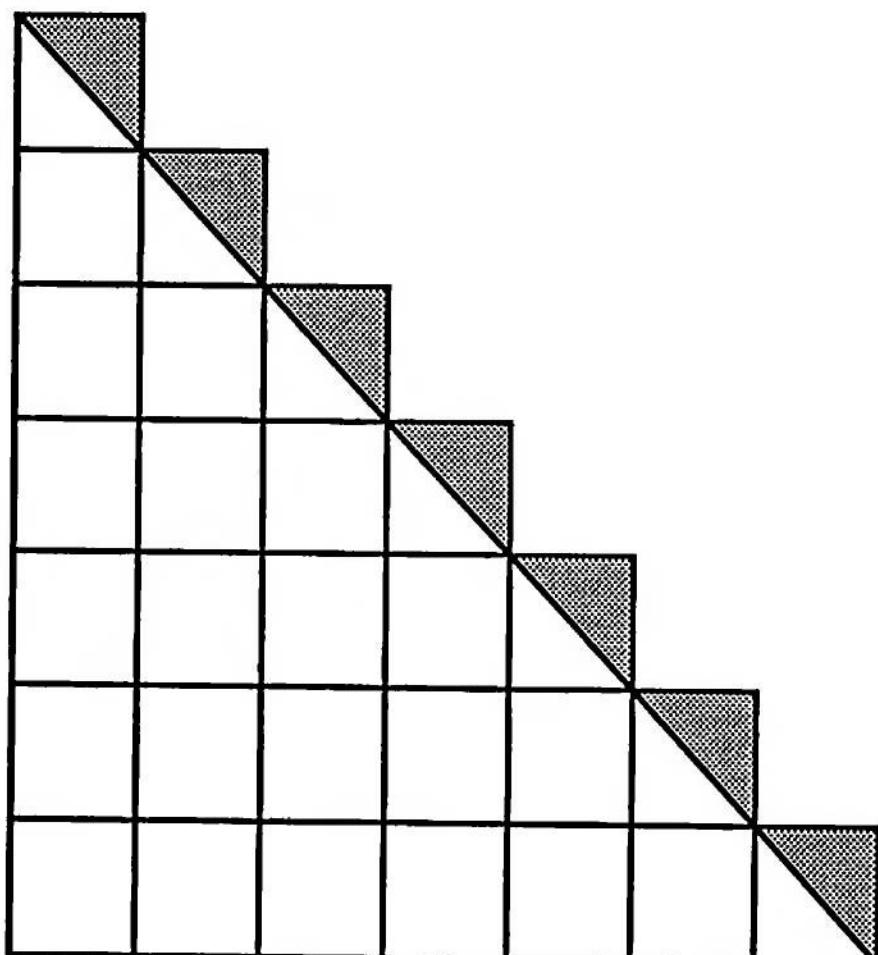
## Sums of Integers I



$$1 + 2 + \dots + n = \frac{1}{2}n(n + 1)$$

—“The ancient Greeks”  
(as cited by Martin Gardner)

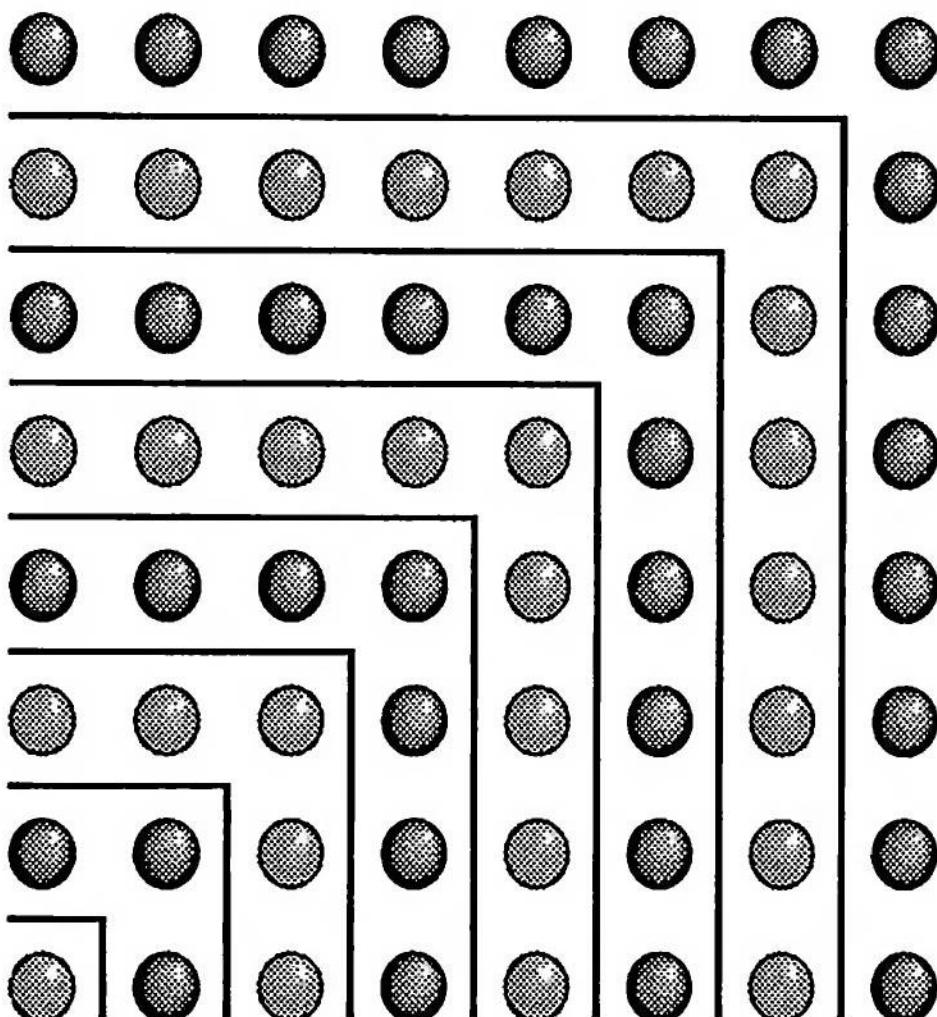
## Sums of Integers II



$$1 + 2 + \cdots + n = \frac{n^2}{2} + \frac{n}{2}$$

—Ian Richards

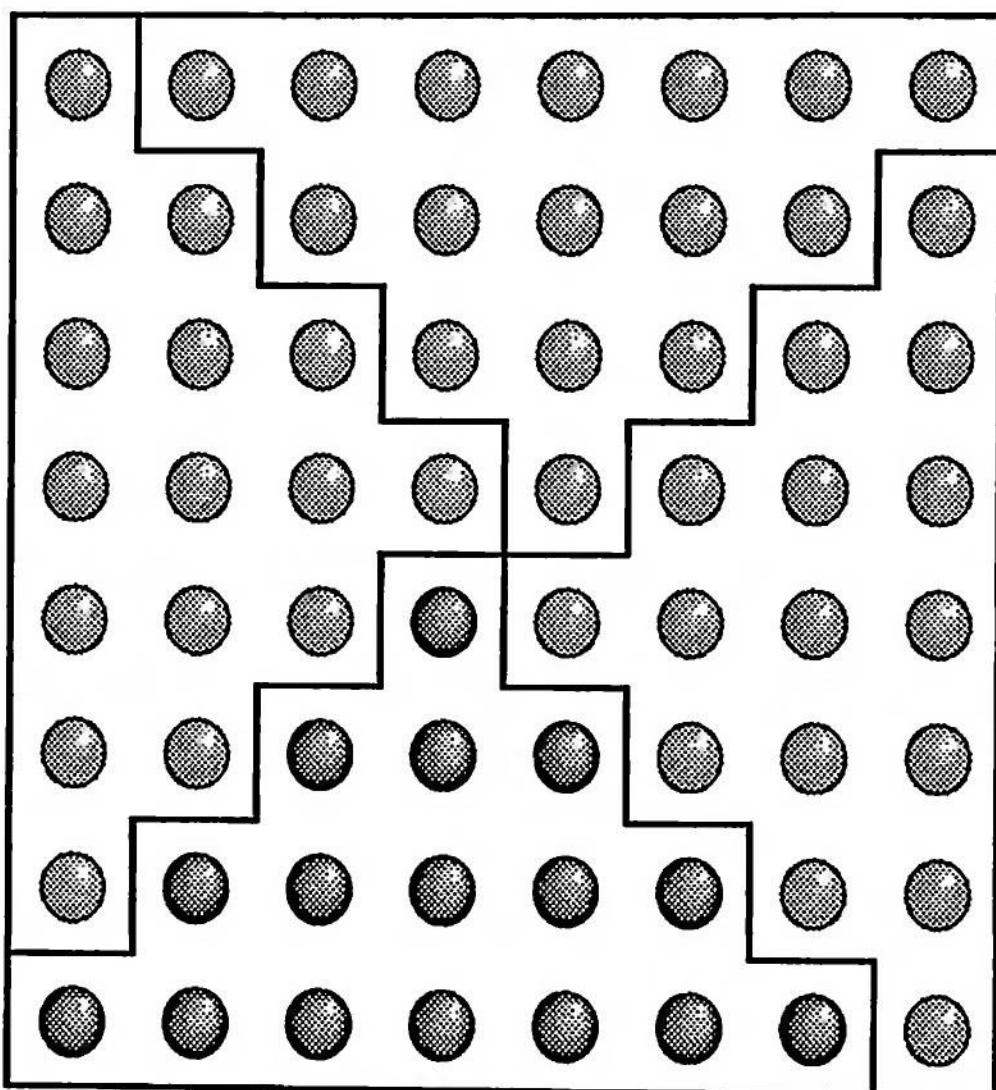
## Sums of Odd Integers I



$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

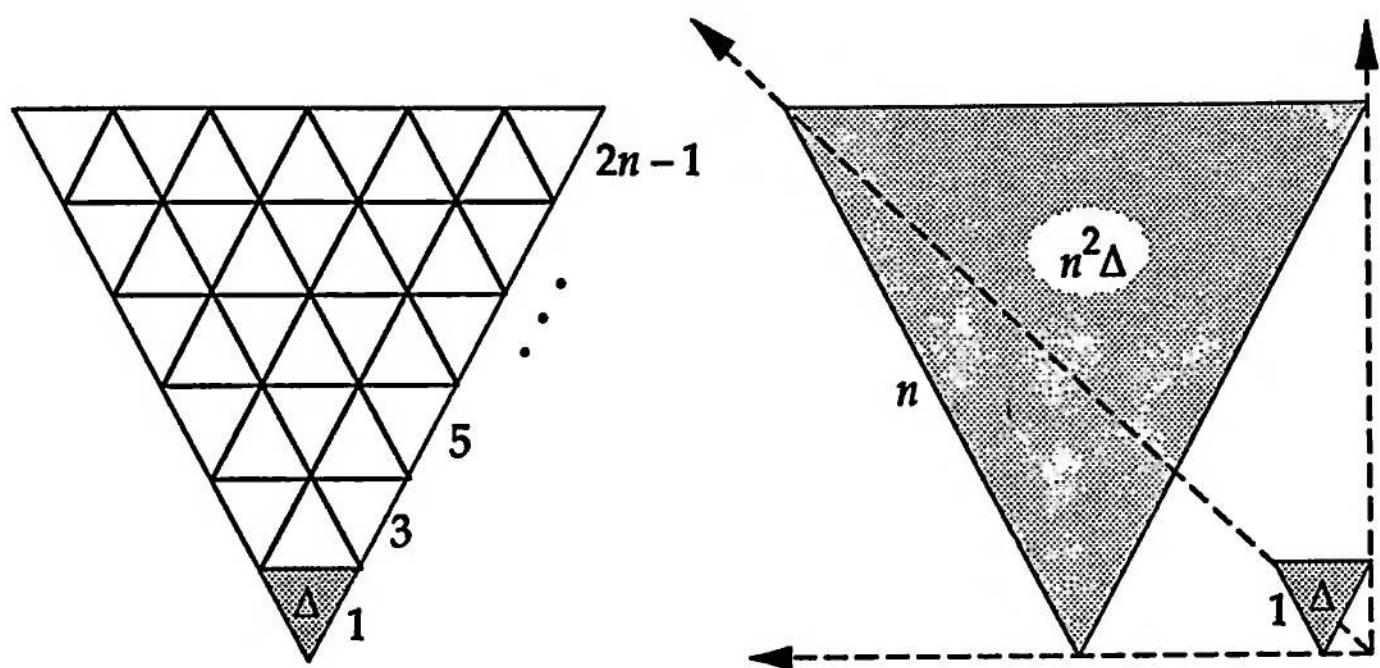
—Nicomachus of Gerasa (circa A.D. 100)

## Sums of Odd Integers II



$$1 + 3 + \cdots + (2n - 1) = \frac{1}{4}(2n)^2 = n^2$$

## Sums of Odd Integers III

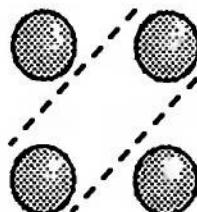


$$\Delta + 3\cdot\Delta + \dots + (2n-1)\cdot\Delta = A = n^2\cdot\Delta$$

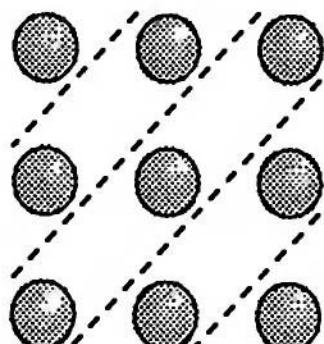
$$\sum_{i=1}^n (2i-1) = n^2$$

## Squares and Sums of Integers

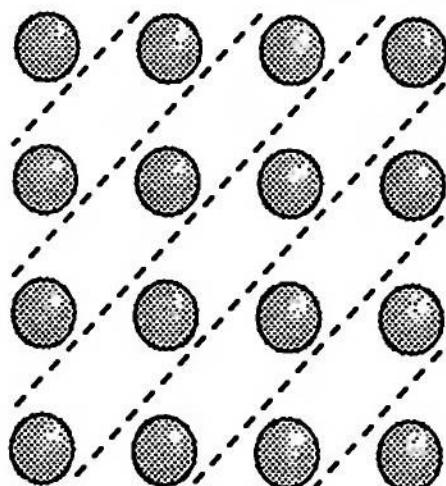
I.



$$1 + 2 + 1 = 2^2$$



$$1 + 2 + 3 + 2 + 1 = 3^2$$

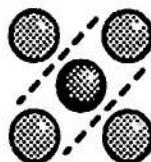


$$1 + 2 + 3 + 4 + 3 + 2 + 1 = 4^2$$

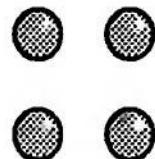
$$1 + 2 + \dots + (n-1) + n + (n-1) + \dots + 2 + 1 = n^2$$

—“The ancient Greeks”  
(as cited by Martin Gardner)

II.



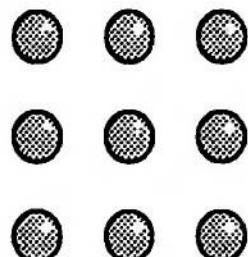
$$= \bullet +$$



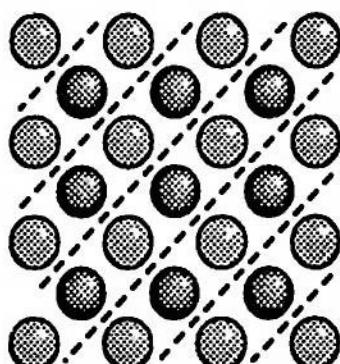
$$1 + 3 + 1 = 1^2 + 2^2$$



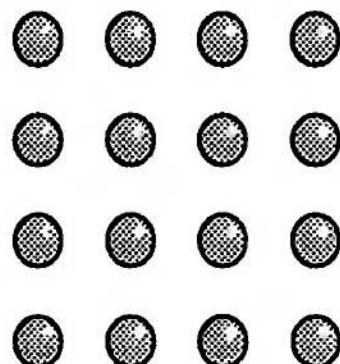
$$= \bullet \bullet +$$



$$1 + 3 + 5 + 3 + 1 = 2^2 + 3^2$$



$$= \bullet \bullet \bullet +$$



$$1 + 3 + 5 + 7 + 5 + 3 + 1 = 3^2 + 4^2$$

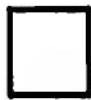
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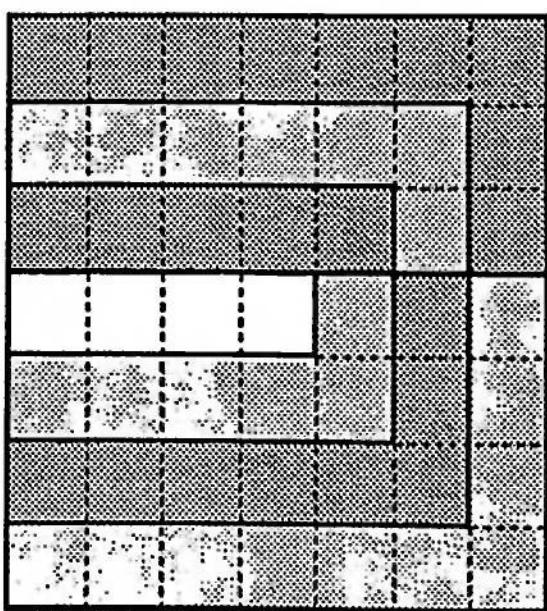
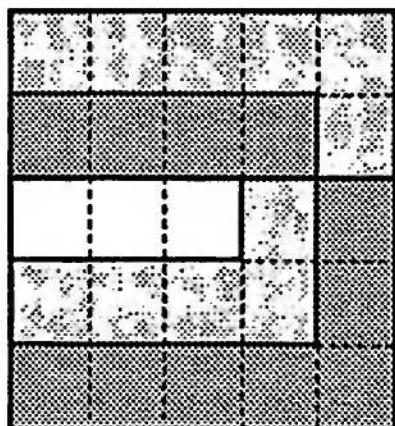
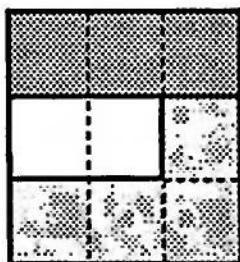
•

$$1 + 3 + \dots + (2n-1) + (2n+1) + (2n-1) + \dots + 3 + 1 = n^2 + (n+1)^2$$

## Arithmetic Progressions with Sum Equal to the Square of the Number of Terms



$$\sum_{k=n}^{3n-2} k = (2n-1)^2; \quad n=1, 2, 3, \dots.$$



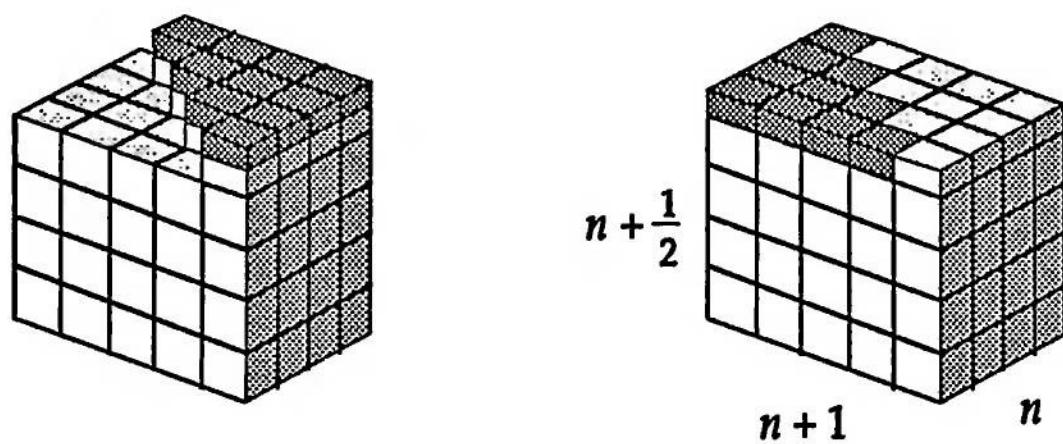
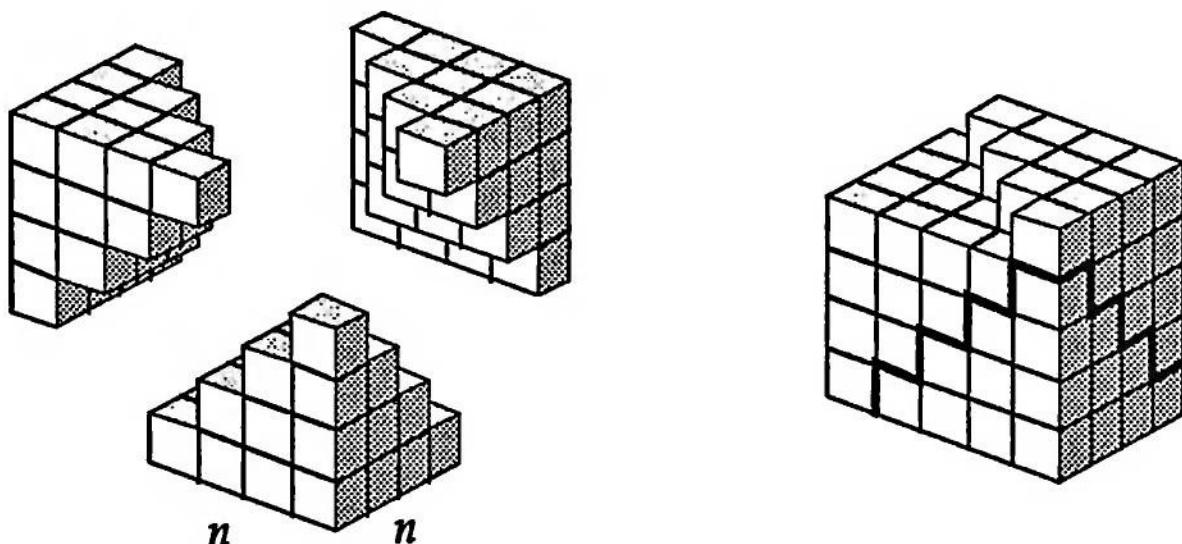
$$n = 4$$

$$4 + 5 + 6 + 7 + 8 + 9 + 10 = 7^2$$

—James O. Chilaka

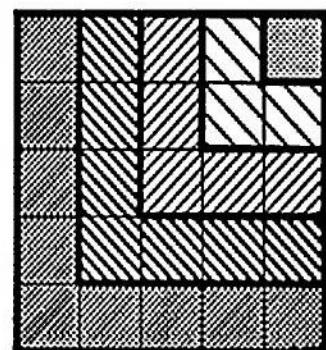
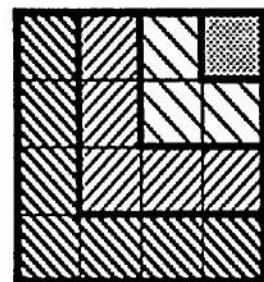
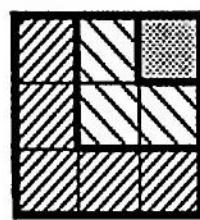
## Sums of Squares I

$$1^2 + 2^2 + \dots + n^2 = \frac{1}{3}n(n+1)(n+\frac{1}{2})$$

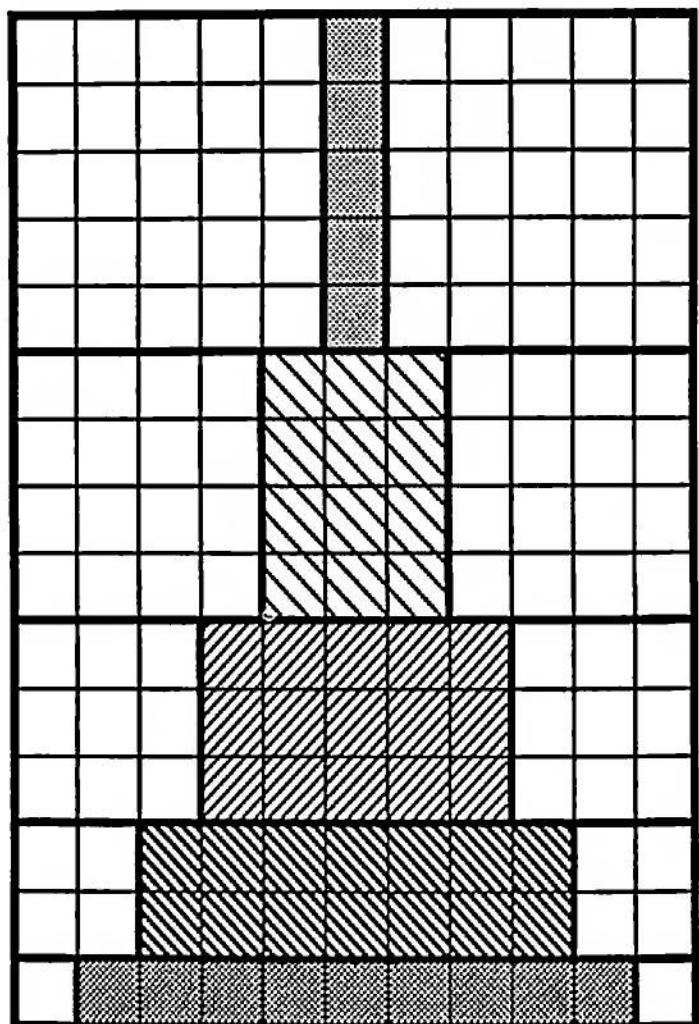


## Sums of Squares II

$$3(1^2 + 2^2 + \dots + n^2) = (2n + 1)(1 + 2 + \dots + n)$$



1 + 2 + ⋯ + n



$2n + 1$

—Martin Gardner and Dan Kalman  
(independently)

## Sums of Squares III

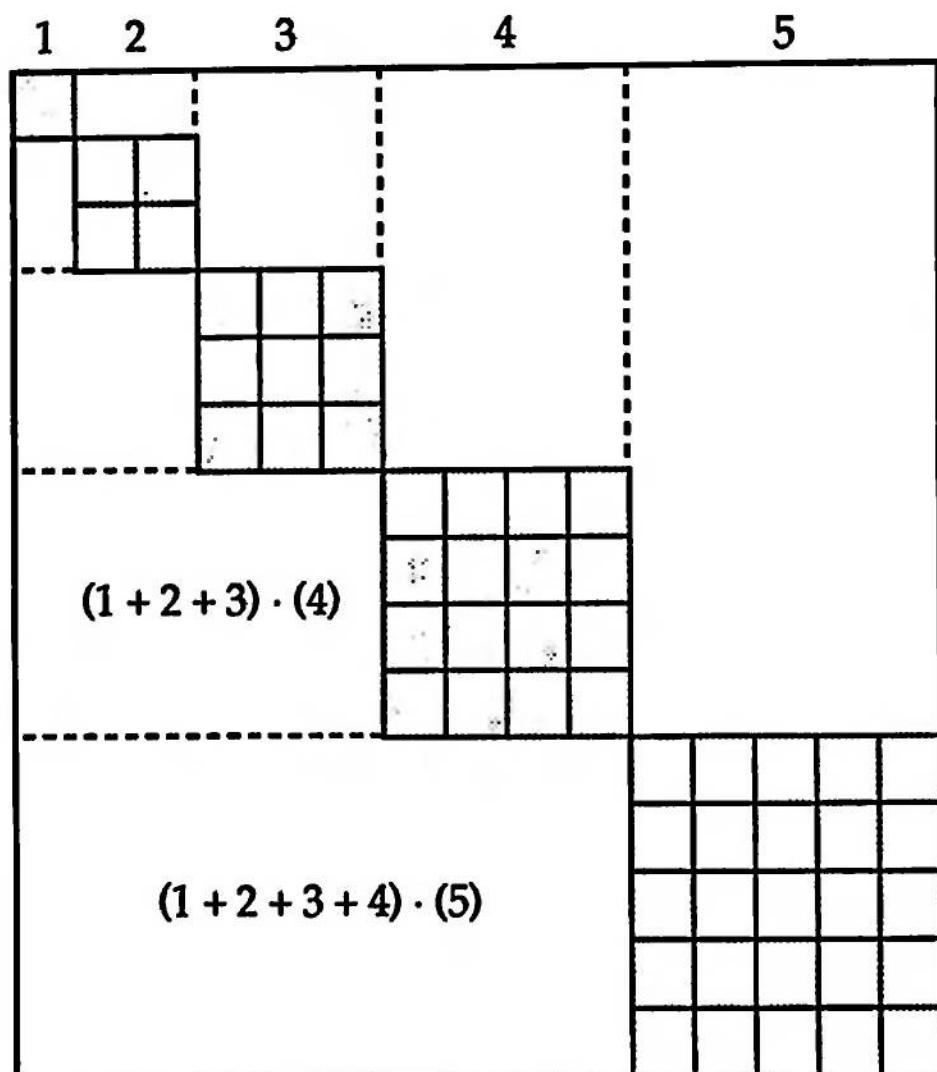
$$3(1^2 + 2^2 + \dots + n^2) = \frac{1}{2}n(n+1)(2n+1)$$

$$\begin{array}{ccccccccc}
 n & n & \cdots & n & n & n & n-1 & \cdots & 2 & 1 & 1 & 2 & \cdots & n-1 & n \\
 n-1 & n-1 & \cdots & n-1 & n & n-1 & \cdots & 2 & & 2 & 3 & \cdots & n \\
 \cdot & & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & & + & \cdot & \cdot & \cdot & & + & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot \\
 2 & 2 & & & n & n-1 & & & & n-1 & n & & \\
 1 & & & & n & & & & & n & & &
 \end{array}$$

$$\begin{array}{cccccc}
 & & & & & \\
 & 2n+1 & 2n+1 & \cdots & 2n+1 & 2n+1 \\
 & 2n+1 & 2n+1 & \cdots & 2n+1 & \\
 & \cdot & \cdot & & \cdot & \\
 = & \cdot & \cdot & \cdot & \cdot & \\
 & \cdot & \cdot & \cdot & & \\
 & 2n+1 & 2n+1 & & & \\
 & 2n+1 & & & &
 \end{array}$$

## Sums of Squares IV

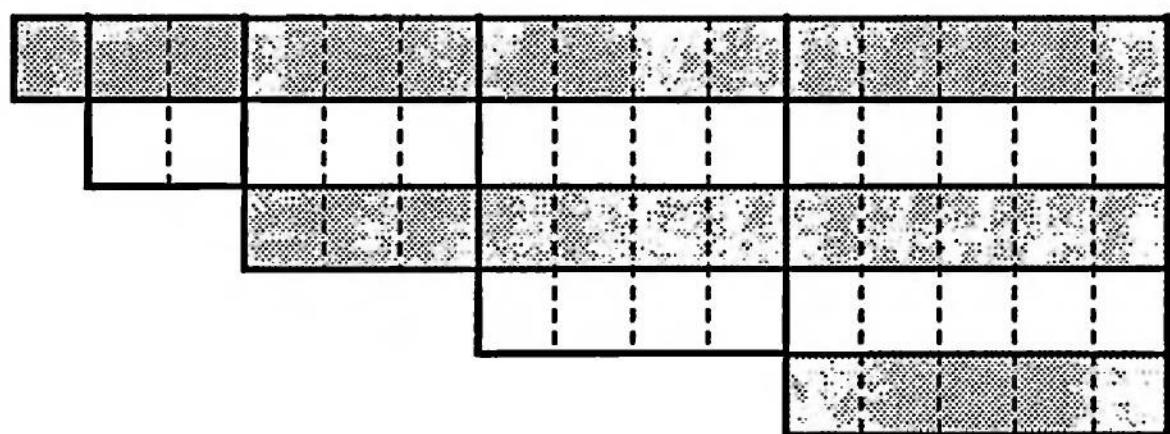
$$\sum_{k=1}^n k^2 = \left( \sum_{k=1}^n k \right)^2 - 2 \sum_{k=1}^{n-1} \left[ \left( \sum_{i=1}^k i \right) (k+1) \right]$$



—James O. Chilaka

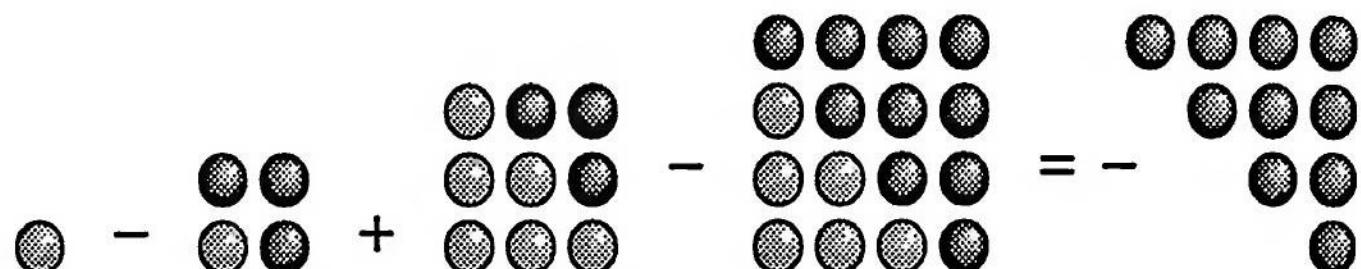
## Sums of Squares V

$$\sum_{i=1}^n \sum_{j=i}^n j = \sum_{i=1}^n i^2$$



## Alternating Sums of Squares

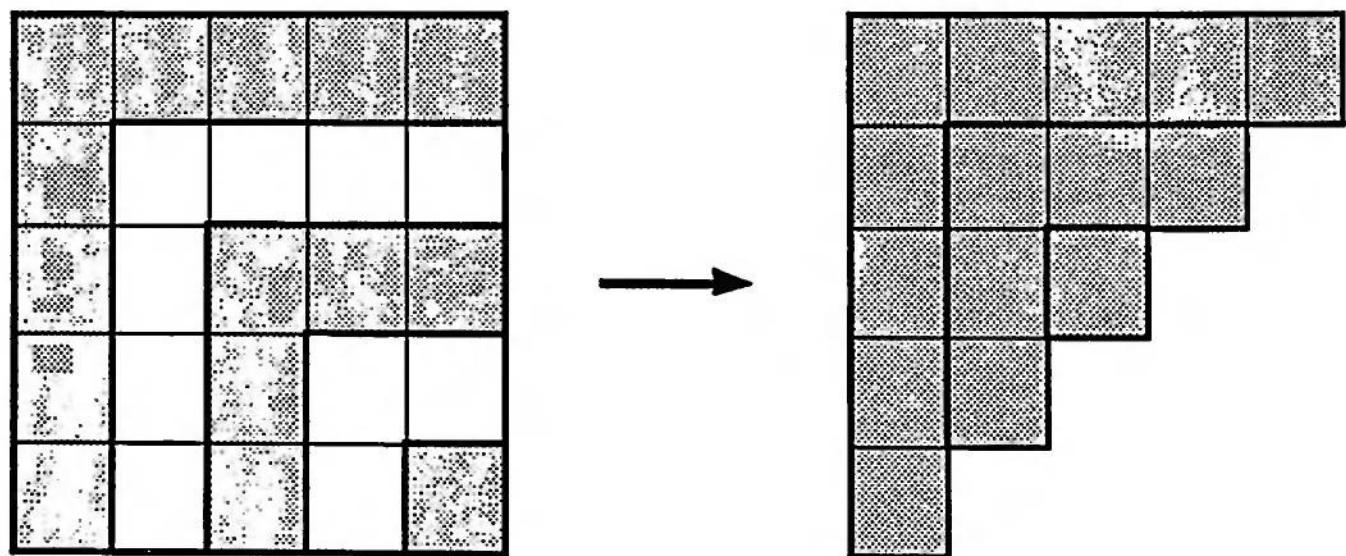
I.



$$\sum_{k=1}^n (-1)^{k+1} k^2 = (-1)^{n+1} T_n = (-1)^{n+1} \frac{n(n+1)}{2}$$

—Dave Logothetti

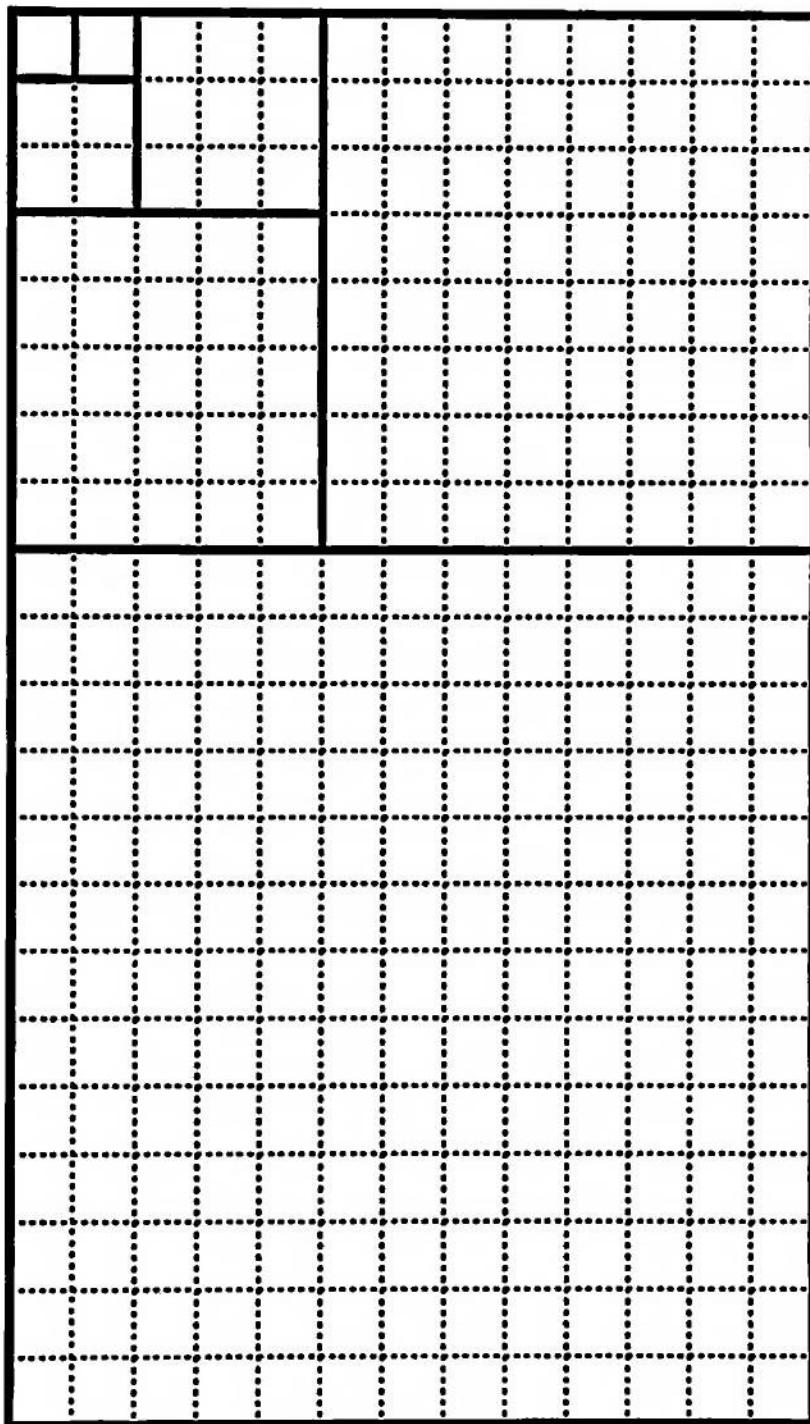
II.



$$n^2 - (n-1)^2 + \dots + (-1)^{n-1} (1)^2 = \sum_{k=0}^n (-1)^k (n-k)^2 = \frac{n(n+1)}{2}$$

—Steven L. Snover

## Sums of Squares of Fibonacci Numbers

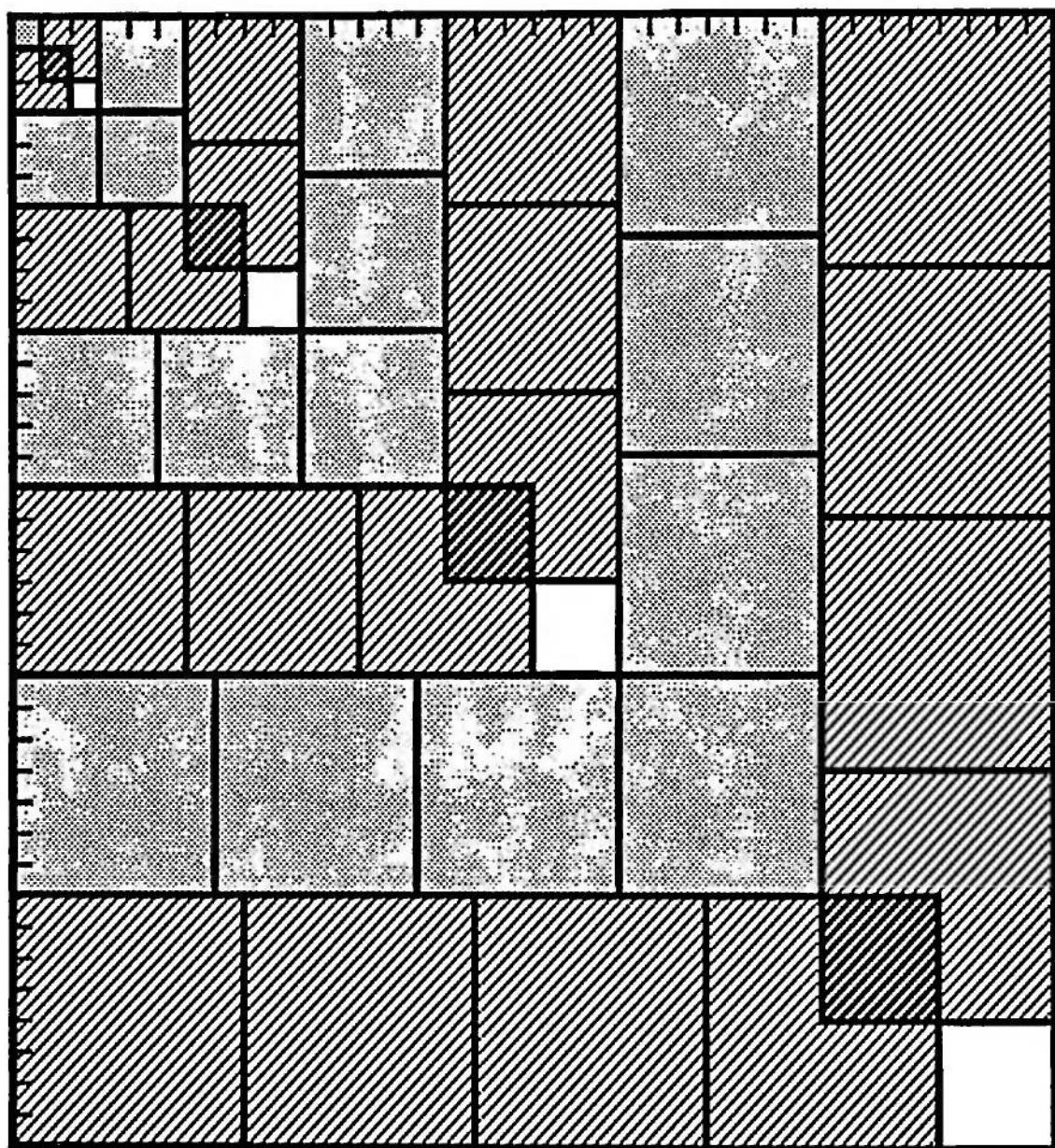


$$F_1 = F_2 = 1; F_{n+2} = F_{n+1} + F_n \Rightarrow F_1^2 + F_2^2 + \dots + F_n^2 = F_n F_{n+1}$$

—Alfred Brousseau

## Sums of Cubes I

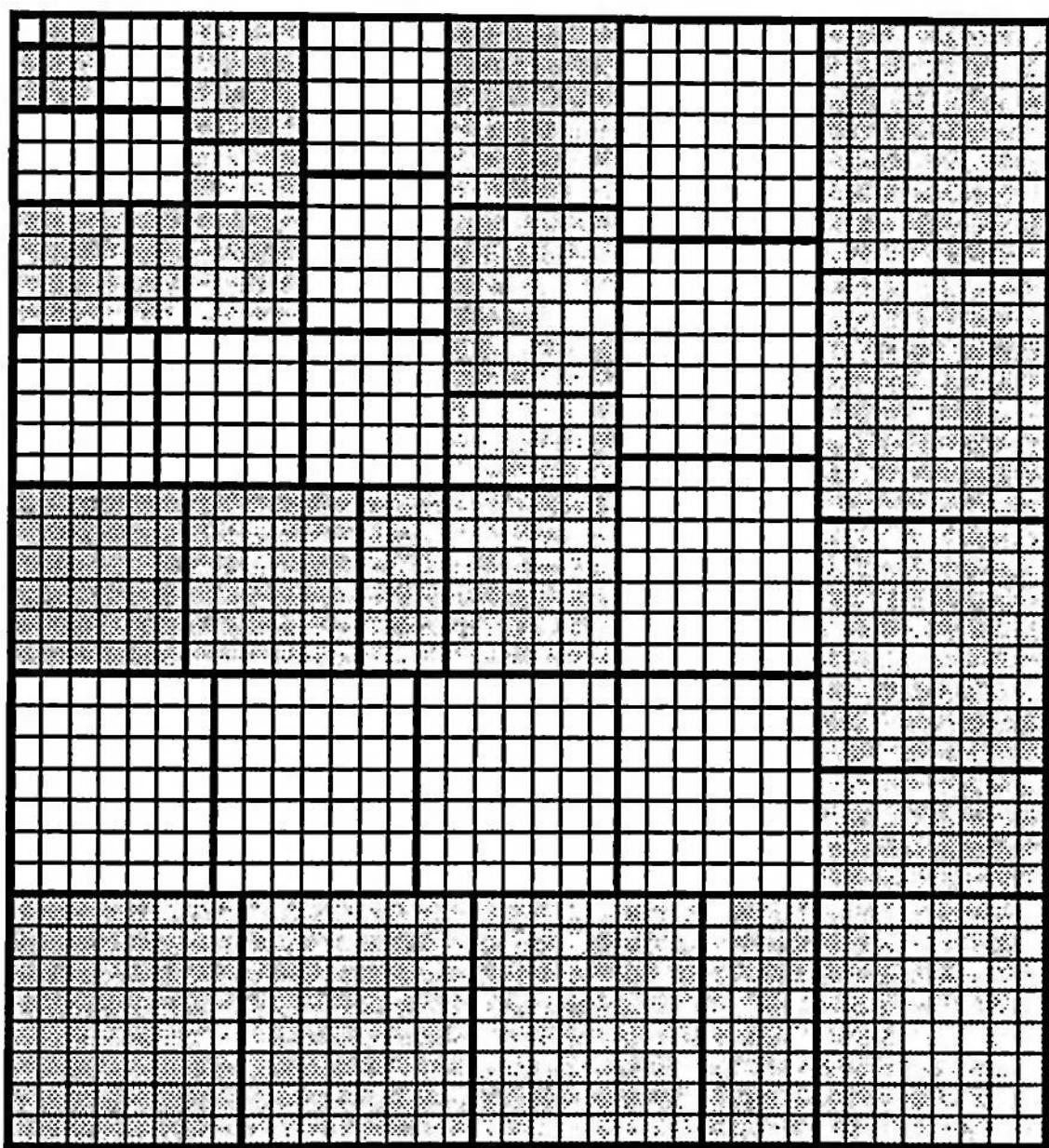
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



—Solomon W. Golomb

## Sums of Cubes II

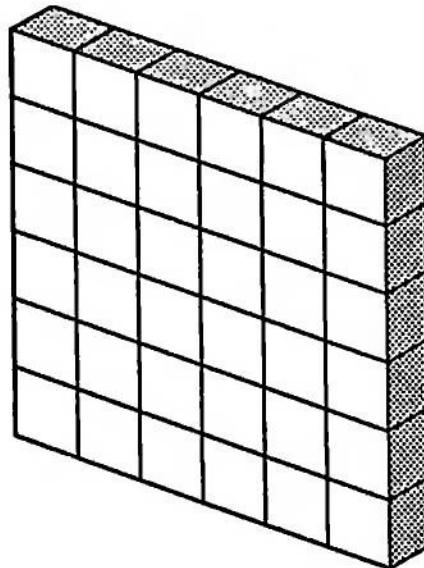
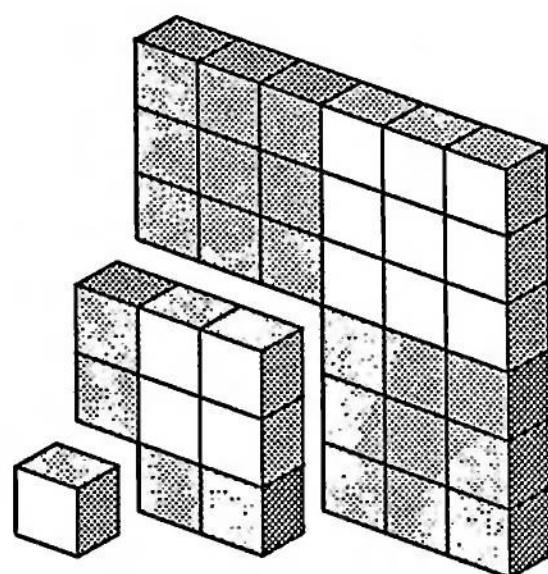
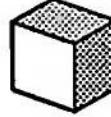
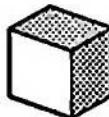
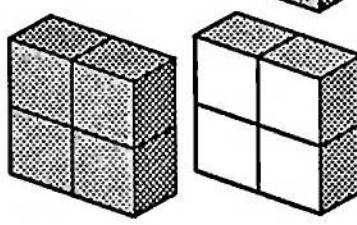
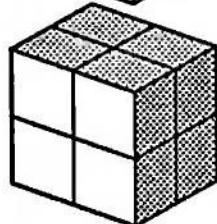
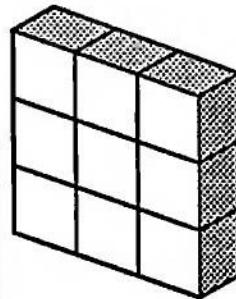
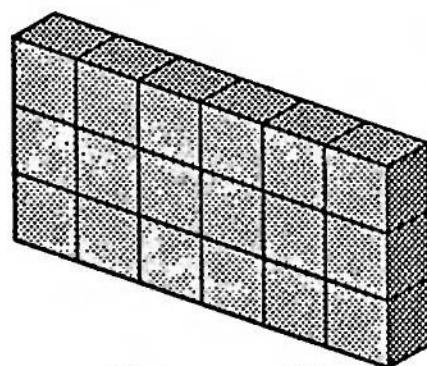
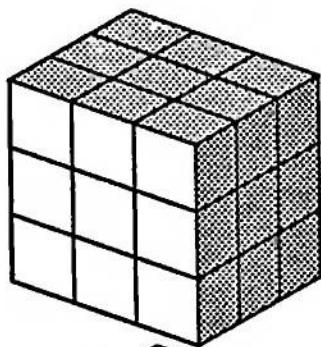
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



—J. Barry Love

## Sums of Cubes III

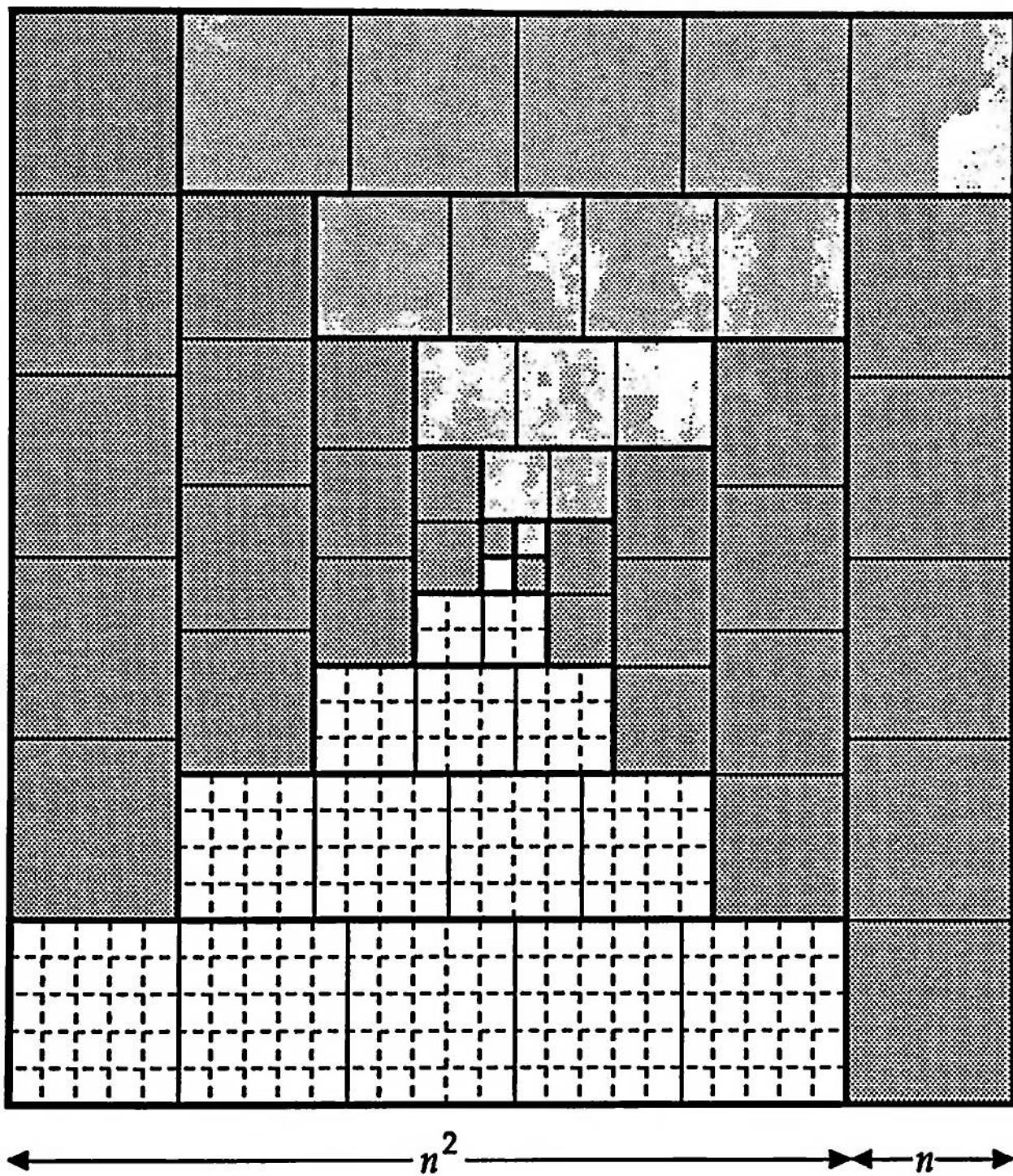
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



—Alan L. Fry

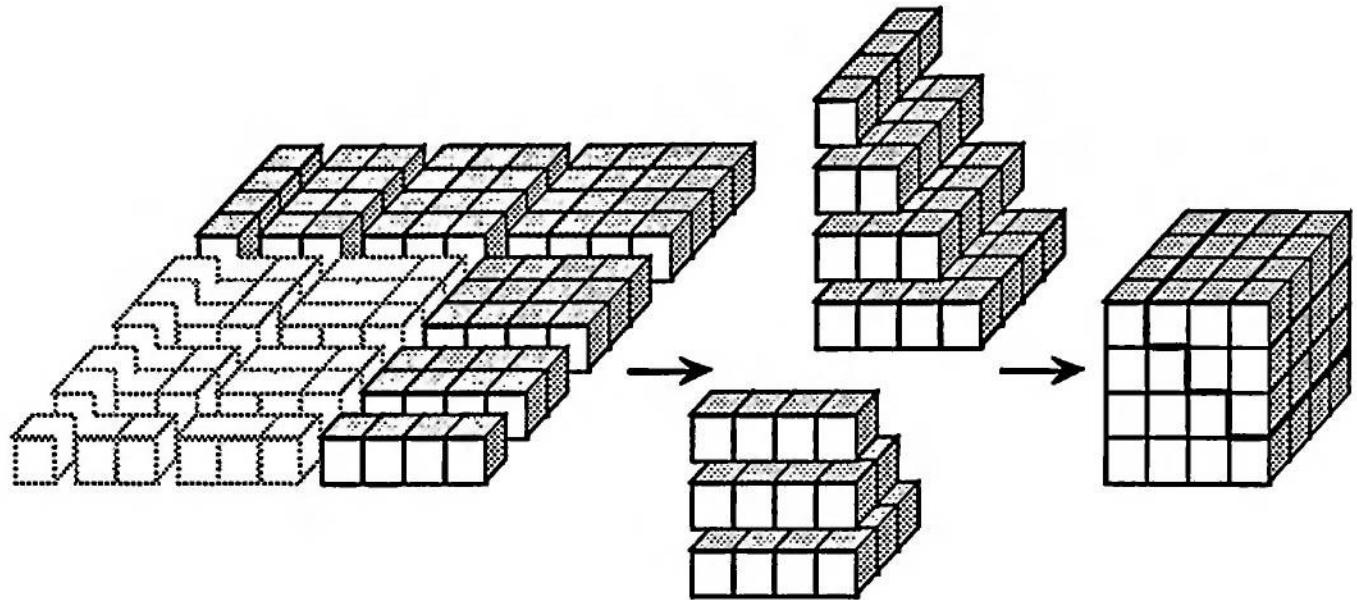
## Sums of Cubes IV

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}[n(n+1)]^2$$

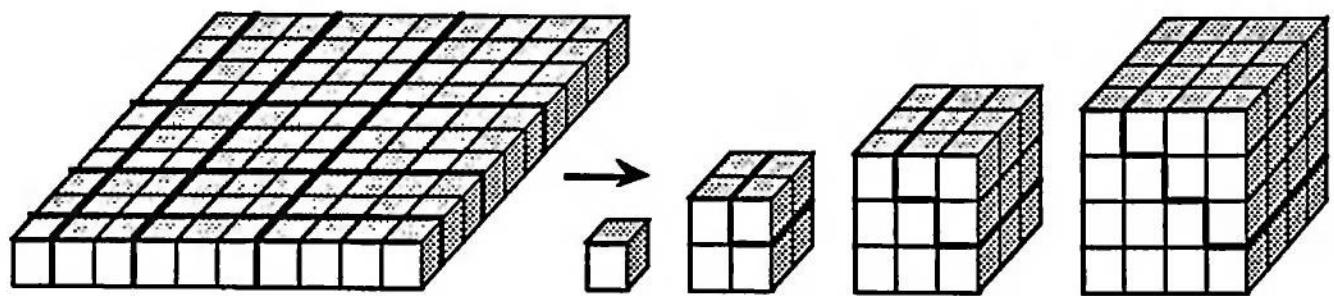


—Antonella Cupillari and Warren Lushbaugh  
(independently)

## Sum of Cubes V



$$t_n = 1 + 2 + \cdots + n \Rightarrow t_n^2 - t_{n-1}^2 = n^3$$



$$t_n^2 = (1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$$

## Sum of Cubes VI

	1	2	3	.	.	.	n
+	2	4	6	.	.	.	2n
+	3	6	9	.	.	.	3n
+	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
+	n	2n	3n	.	.	.	n <sup>2</sup>

	1	2	3	.	.	.	n
+	2	4	6	.	.	.	2n
+	3	6	9	.	.	.	3n
+	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
+	n	2n	3n	.	.	.	n <sup>2</sup>

$$= \sum_{i=1}^n i + 2 \sum_{i=1}^n i + \dots + n \sum_{i=1}^n i$$

$$= 1(1)^2 + 2(2)^2 + \dots + n(n)^2$$

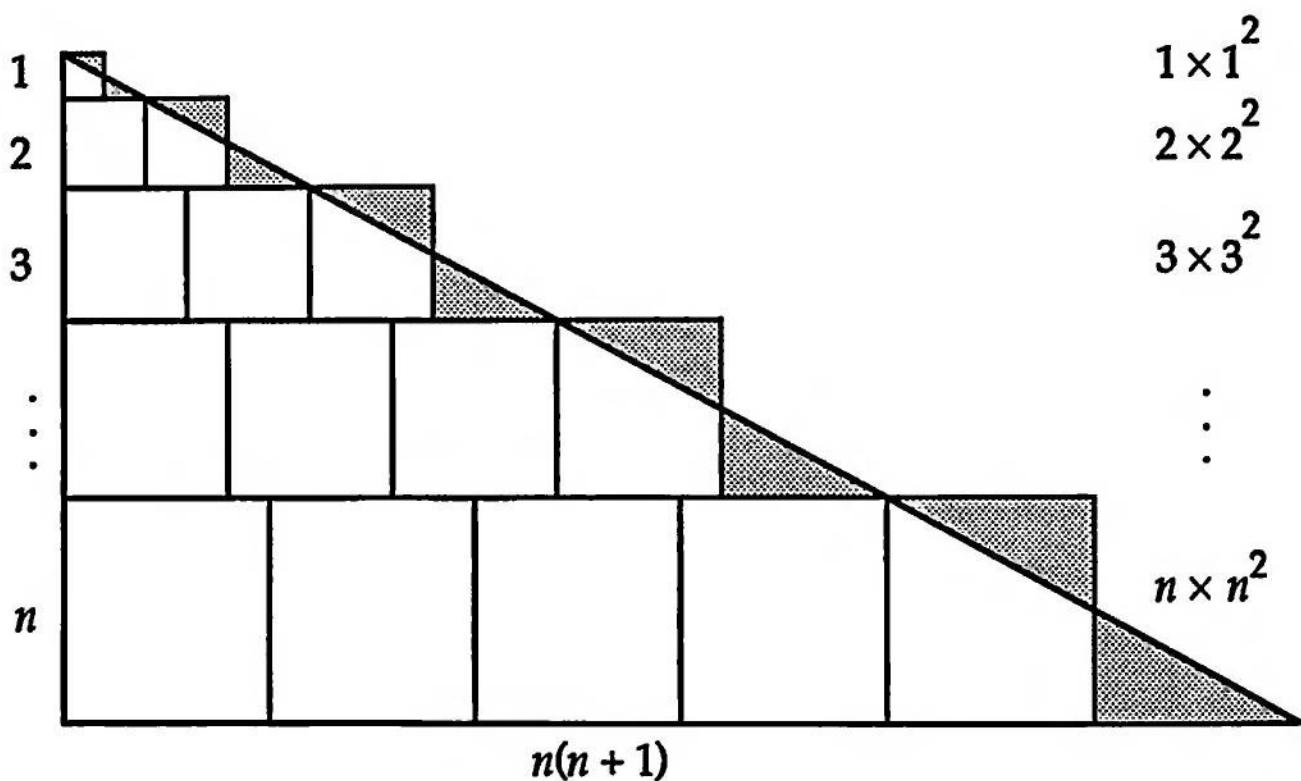
$$= \left( \sum_{i=1}^n i \right)^2$$

$$= \sum_{i=1}^n i^3$$

## Sums of Integers and Sums of Cubes

$$1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$$

$$1^3 + 2^3 + \cdots + n^3 = \left( \frac{1}{2}n(n + 1) \right)^2$$



—Georg Schrage

# Sums of Odd Cubes are Triangular Numbers

$$1^3 = \square$$

$$3^3 = 3(3^2) = \boxed{9} + \boxed{9} + \boxed{9} = \begin{array}{c} \text{A 3x3 grid of squares. The first two columns are shaded black, and the third column is white. A double-headed arrow below the grid indicates its width is } 1+2(3). \\ \hline \end{array}$$

$$5^3 = 5(5^2) = \boxed{\text{grid}} + \boxed{\text{grid}} + \boxed{\text{grid}}$$

$$\leftarrow 2 + 3(5) \rightarrow$$

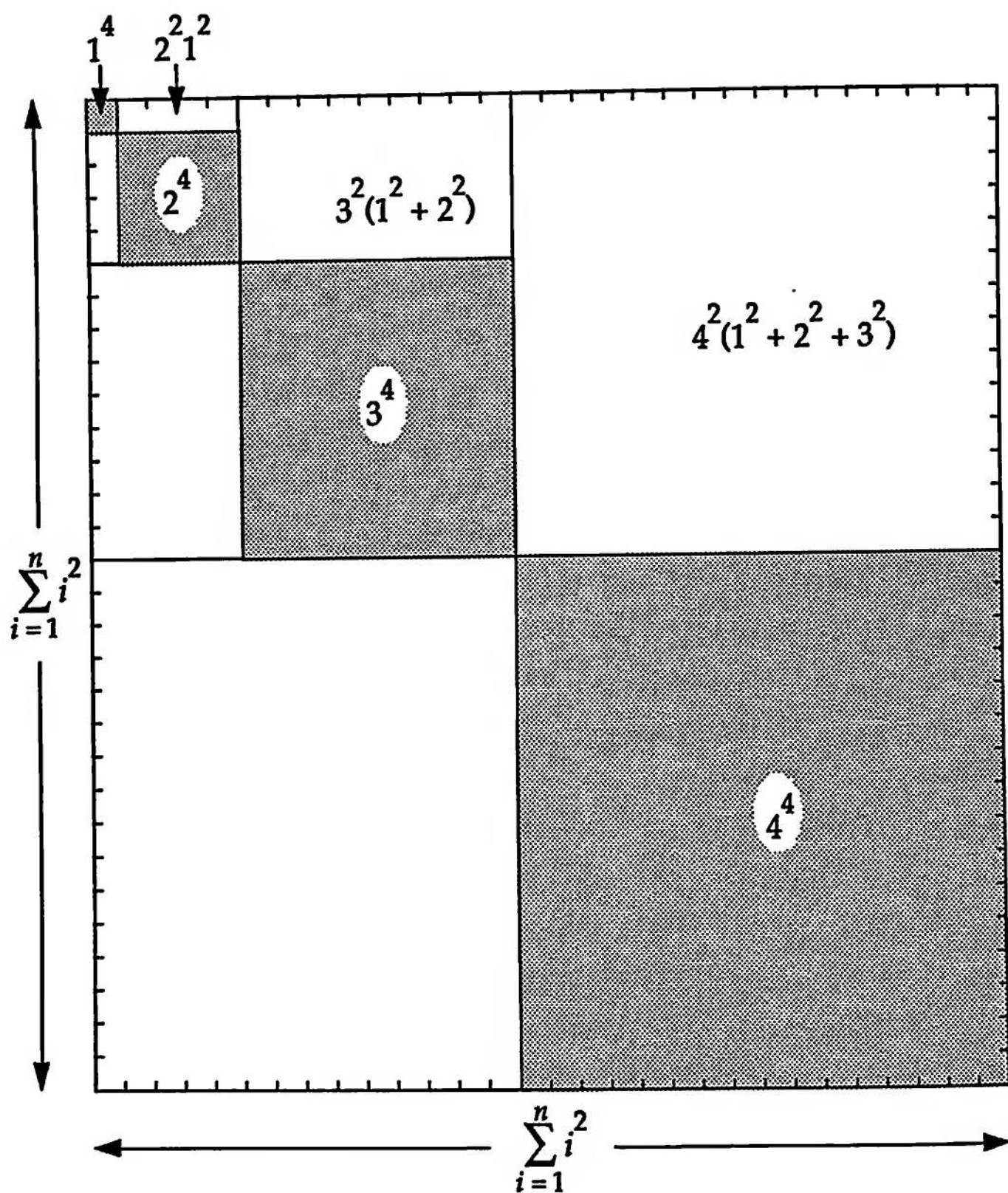
$$(2n - 1)^3 = (2n - 1)(2n - 1)^2 = \dots$$

$$1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = 1 + 2 + 3 + \dots + (2n^2-1) = n^2(2n^2-1)$$

—Monte J. Zerger

## Sums of Fourth Powers

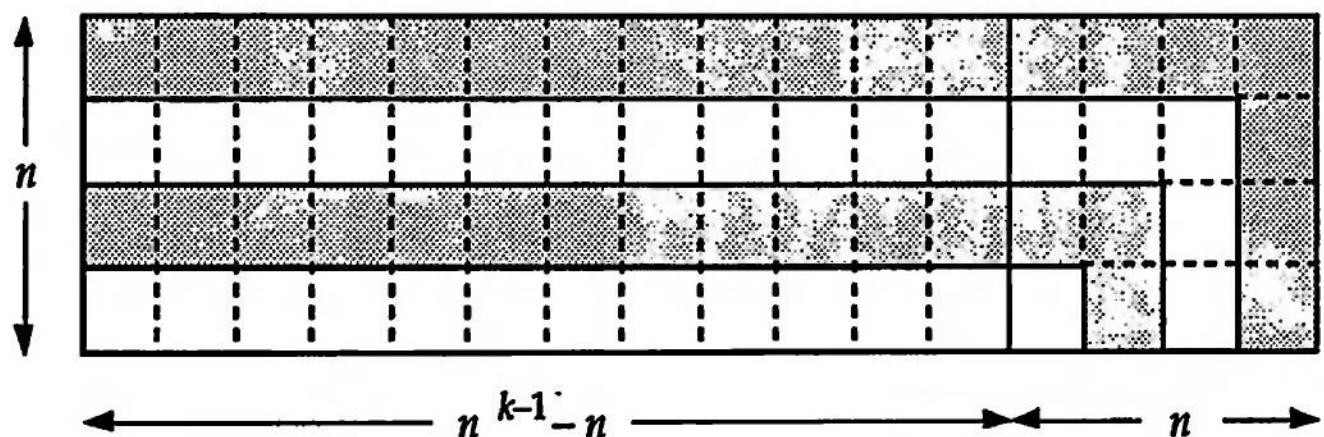
$$\sum_{i=1}^n i^4 = \left( \sum_{i=1}^n i^2 \right)^2 - 2 \left[ \sum_{k=2}^n \left( k^2 \sum_{i=1}^{k-1} i^2 \right) \right]$$



—Elizabeth M. Markham

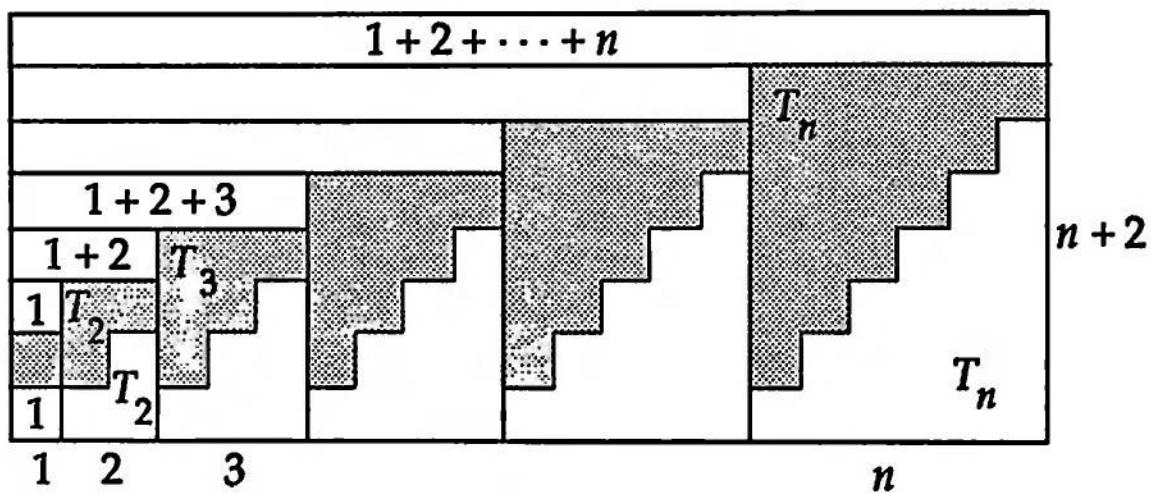
## $k^{\text{th}}$ Powers as Sums of Consecutive Odd Numbers

$$n^k = (n^{k-1} - n + 1) + (n^{k-1} - n + 3) + \dots + (n^{k-1} - n + 2n - 1); \\ k = 2, 3, \dots.$$



## Sums of Triangular Numbers I

$$T_n = 1 + 2 + \cdots + n \Rightarrow T_1 + T_2 + \cdots + T_n = \frac{n(n+1)(n+2)}{6}$$

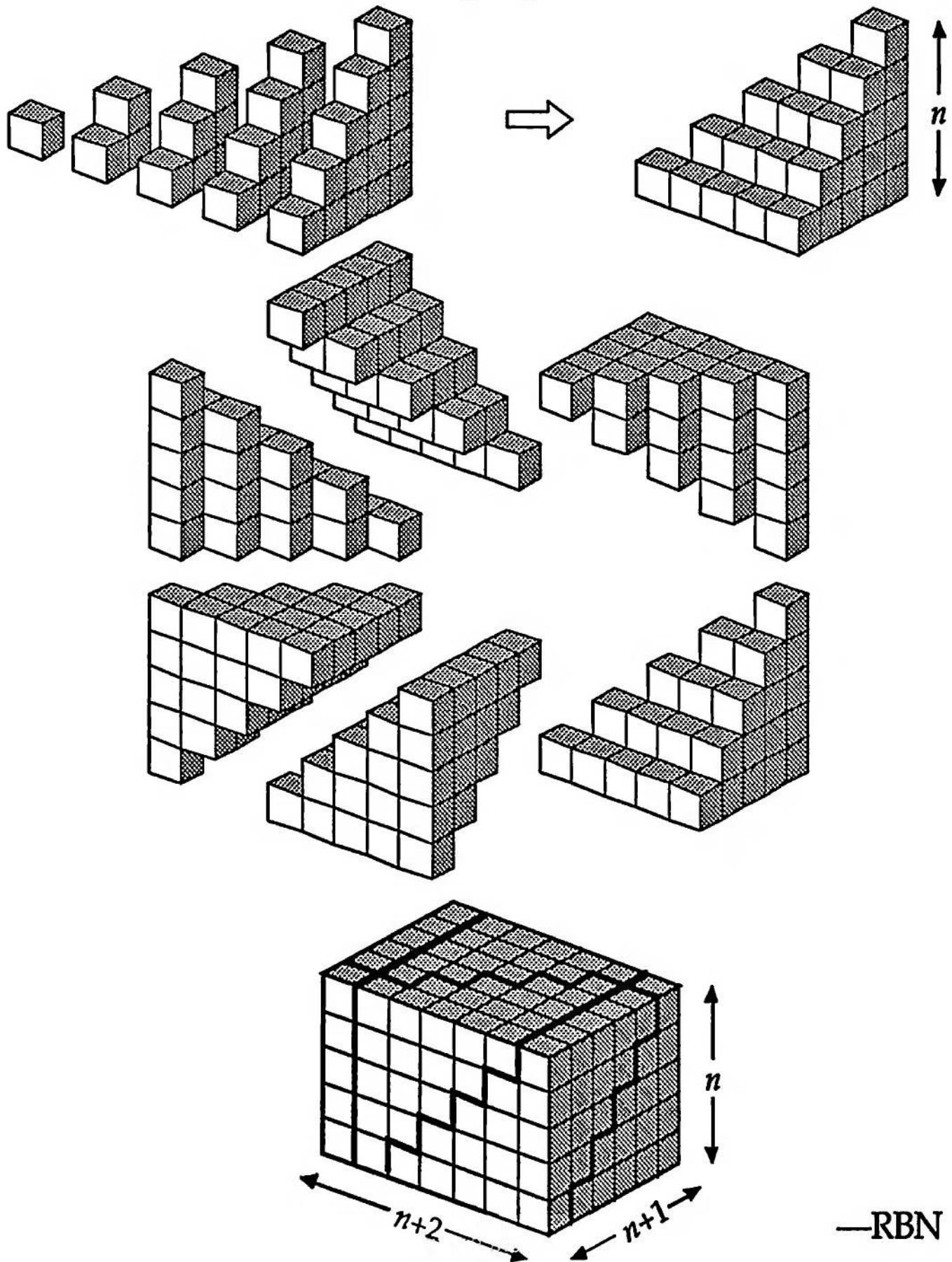


$$3(T_1 + T_2 + \cdots + T_n) = (n+2) \cdot T_n$$

$$T_1 + T_2 + \cdots + T_n = \frac{(n+2)}{3} \cdot \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

## Sums of Triangular Numbers II

$$T_k = 1 + 2 + \cdots + k \Rightarrow \sum_{k=1}^n T_k = \frac{1}{6} n(n+1)(n+2)$$



## Sums of Triangular Numbers III

$$T_k = 1 + 2 + \cdots + k \Rightarrow 3 \sum_{k=1}^n T_k = \frac{1}{2} n(n+1)(n+2)$$

$$\begin{array}{ccccccc}
 1 & & 1 & & & & n \\
 1 2 & & 2 1 & & & & n-1 n-1 \\
 1 2 3 & & 3 2 1 & & & & n-2 n-2 n-2 \\
 \cdot \cdot \cdot & + & \cdot \cdot \cdot & + & \cdot \cdot \cdot & & \cdot \cdot \cdot \\
 \cdot \cdot \cdot & & \cdot \cdot \cdot & & & & \cdot \cdot \cdot \\
 1 2 \cdots n-1 & & n-1 n-2 & \cdots & 1 & 2 2 & \cdots 2 \\
 1 2 \cdots n-1 n & & n n-1 & \cdots & 2 1 & 1 1 & \cdots 1 1
 \end{array}$$

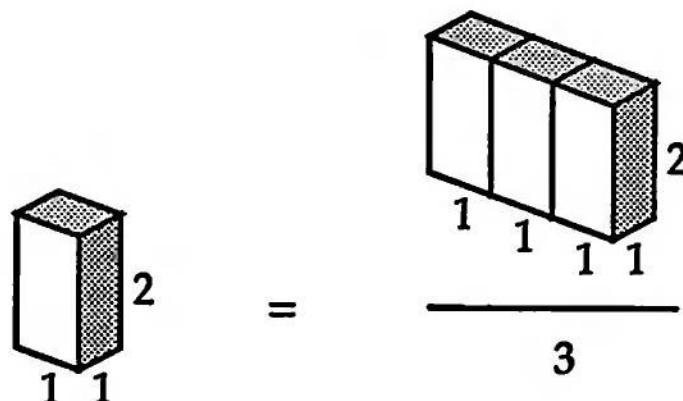
$$\begin{array}{c}
 n+2 \\
 n+2 n+2 \\
 n+2 n+2 n+2 \\
 = \cdot \cdot \cdot \\
 \cdot \cdot \cdot \\
 n+2 n+2 \cdots n+2 \\
 n+2 n+2 \cdots n+2 n+2
 \end{array}$$

$$3(T_1 + T_2 + \cdots + T_n) = T_n \cdot (n+2)$$

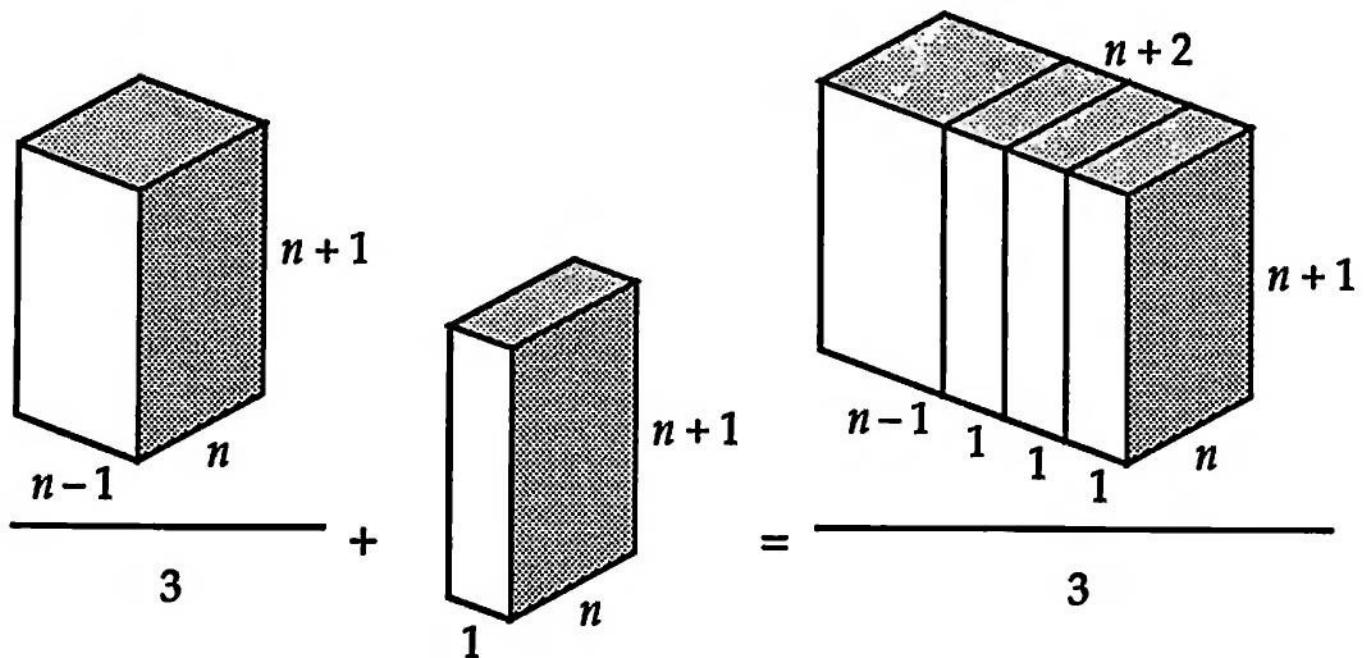
# Sums of Oblong Numbers I

$$(1 \times 2) + (2 \times 3) + (3 \times 4) + \cdots + (n-1)n = \frac{(n-1)n(n+1)}{3}$$

(i)

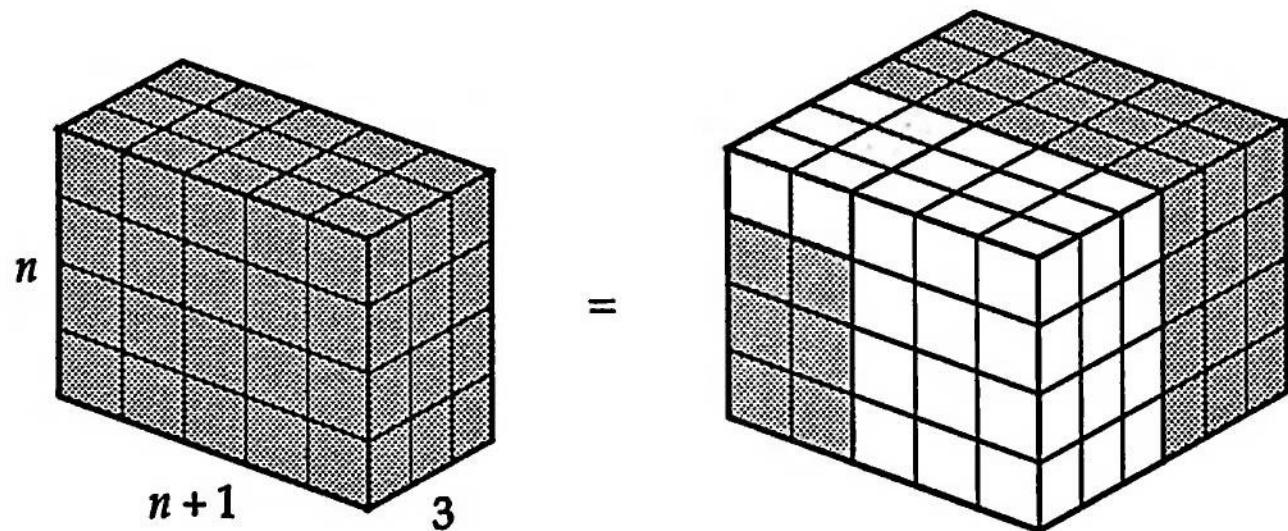
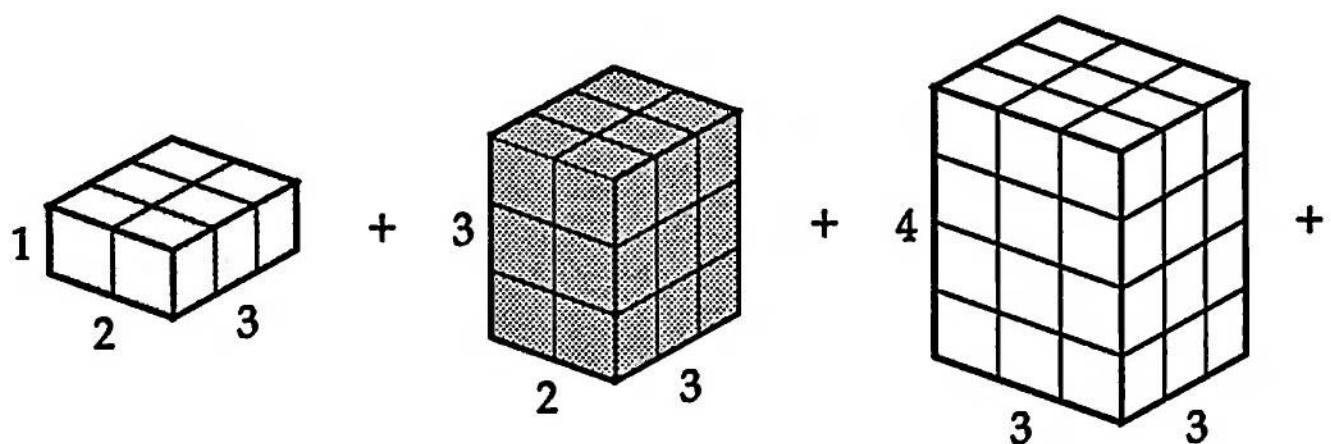


(ii)



## Sums of Oblong Numbers II

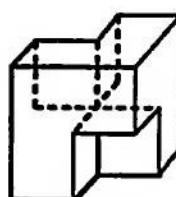
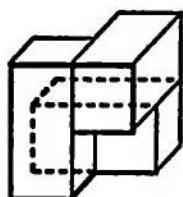
$$3(1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1)) = n(n+1)(n+2)$$



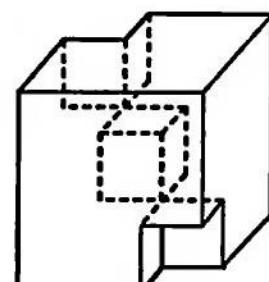
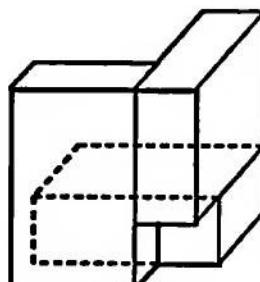
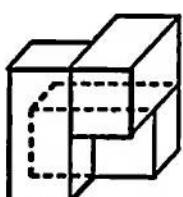
—Sidney H. Kung

## Sums of Oblong Numbers III

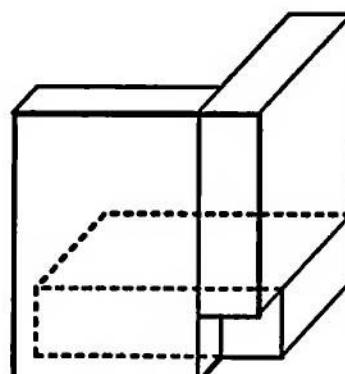
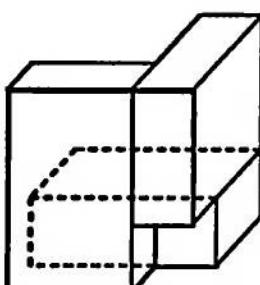
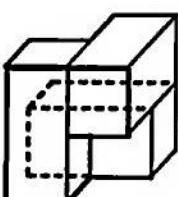
$$(1 \times 2) + (2 \times 3) + \cdots + (n - 1) \times n = \frac{1}{3}[n^3 - n]$$



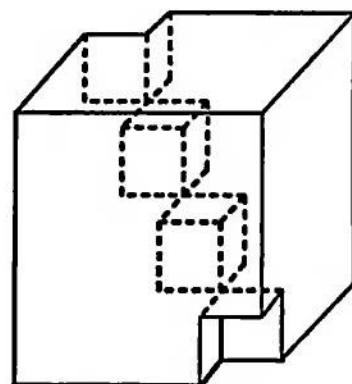
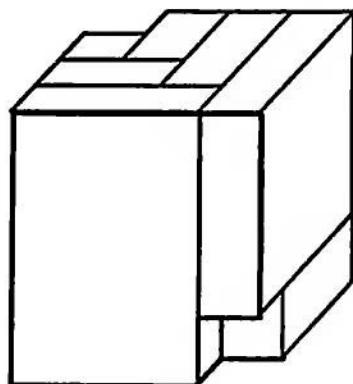
$$3(1 \times 2) = 2^3 - 2$$



$$3(1 \times 2) + 3(2 \times 3) = 3^3 - 3$$



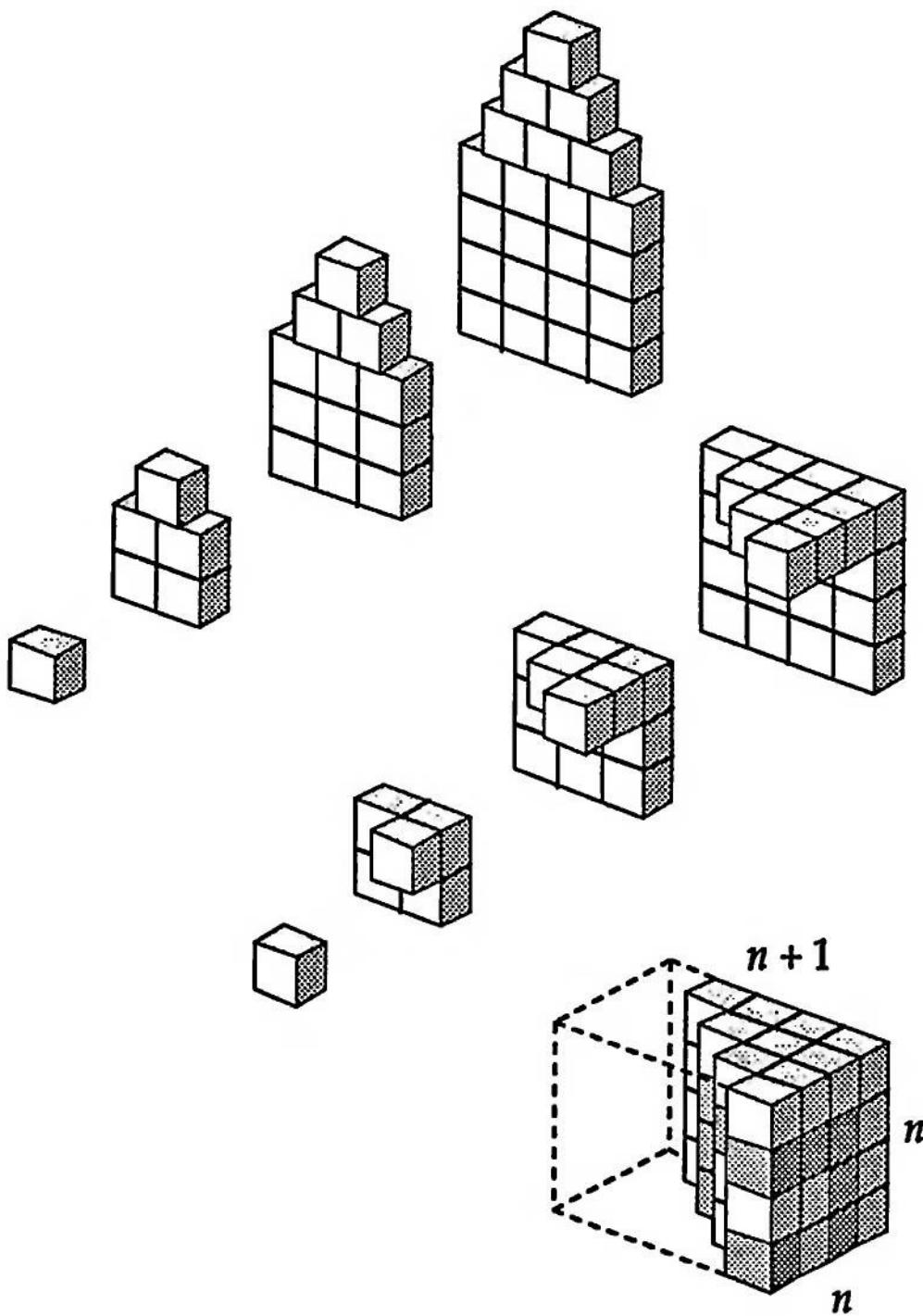
$$3(1 \times 2) + 3(2 \times 3) + 3(3 \times 4) =$$



$$4^3 - 4$$

## Sums of Pentagonal Numbers

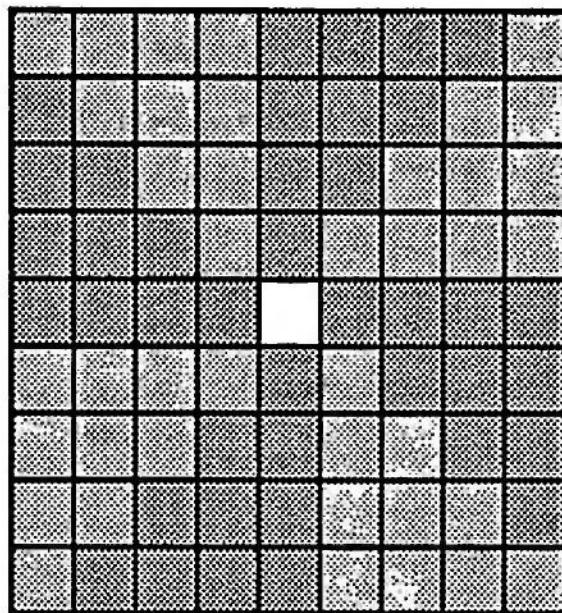
$$\frac{1 \cdot 2}{2} + \frac{2 \cdot 5}{2} + \frac{3 \cdot 8}{2} + \cdots + \frac{n(3n - 1)}{2} = \frac{n^2(n + 1)}{2}$$



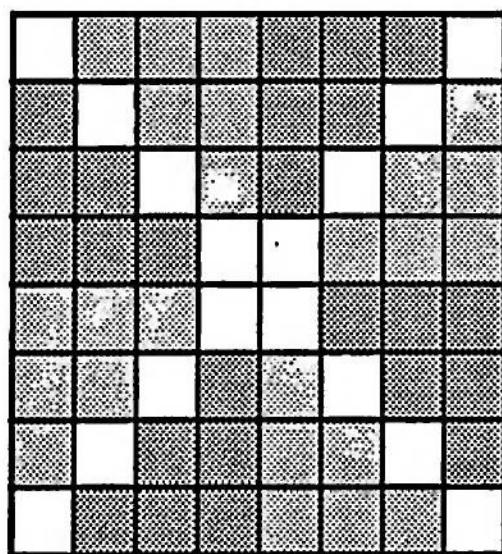
—William A. Miller

## On Squares of Positive Integers

$$T_n = 1 + 2 + \cdots + n \Rightarrow$$

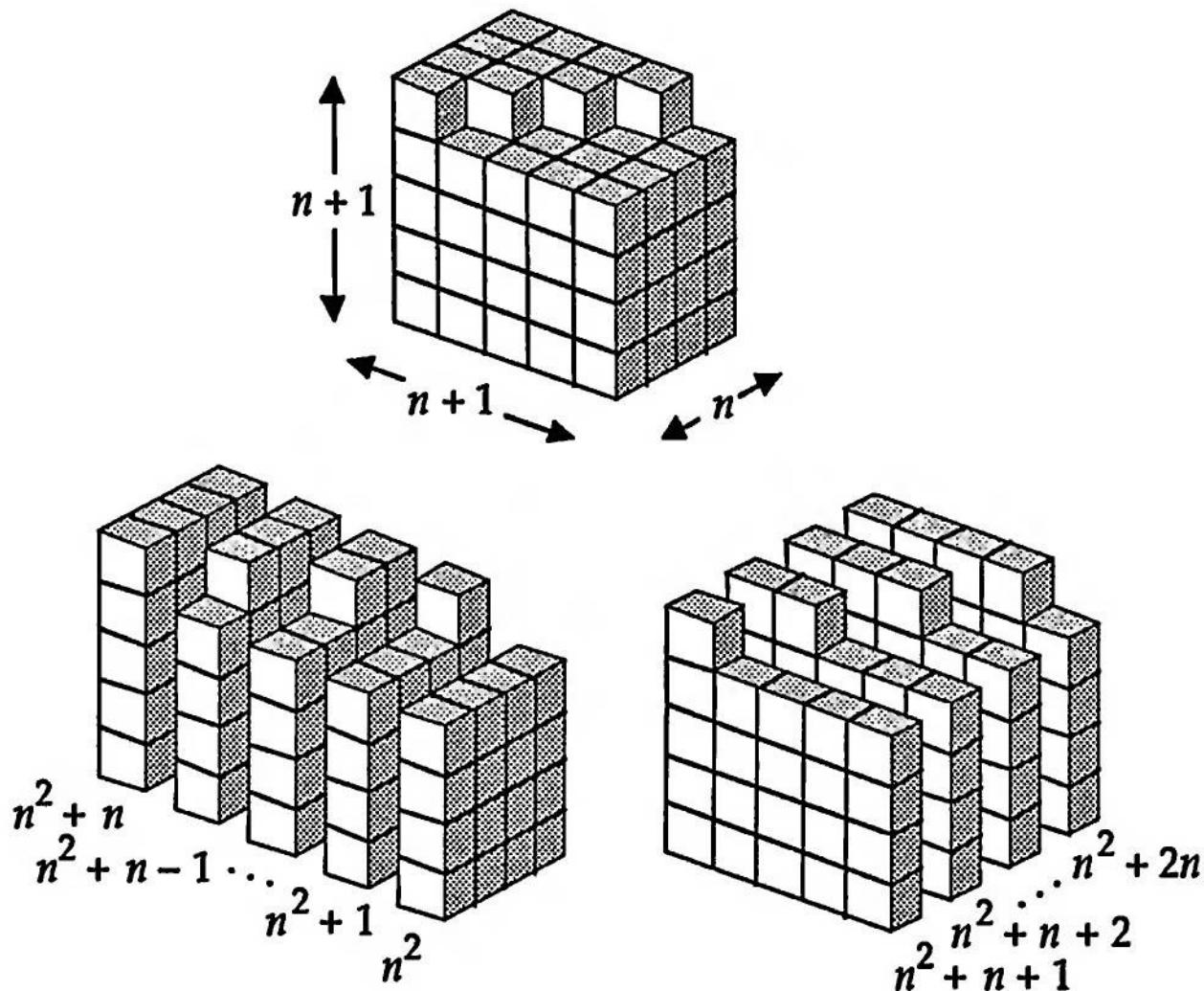


$$(2n+1)^2 = 8T_n + 1$$



$$(2n)^2 = 8T_{n-1} + 4n$$

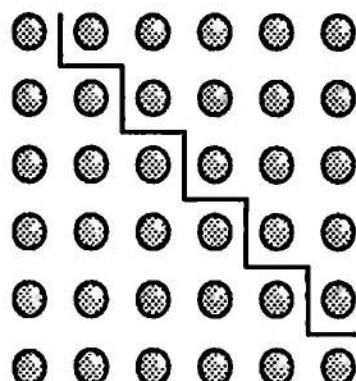
# Consecutive Sums of Consecutive Integers



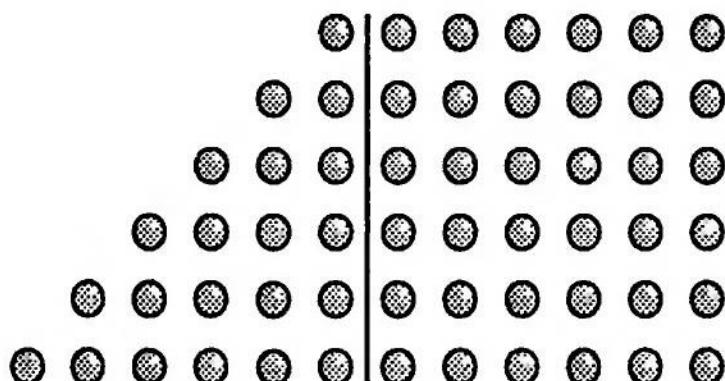
$$\begin{aligned}
 1 + 2 &= 3 \\
 4 + 5 + 6 &= 7 + 8 \\
 9 + 10 + 11 + 12 &= 13 + 14 + 15 \\
 16 + 17 + 18 + 19 + 20 &= 21 + 22 + 23 + 24
 \end{aligned}$$

$$n^2 + (n^2 + 1) + \dots + (n^2 + n) = (n^2 + n + 1) + \dots + (n^2 + 2n)$$

## Count the Dots



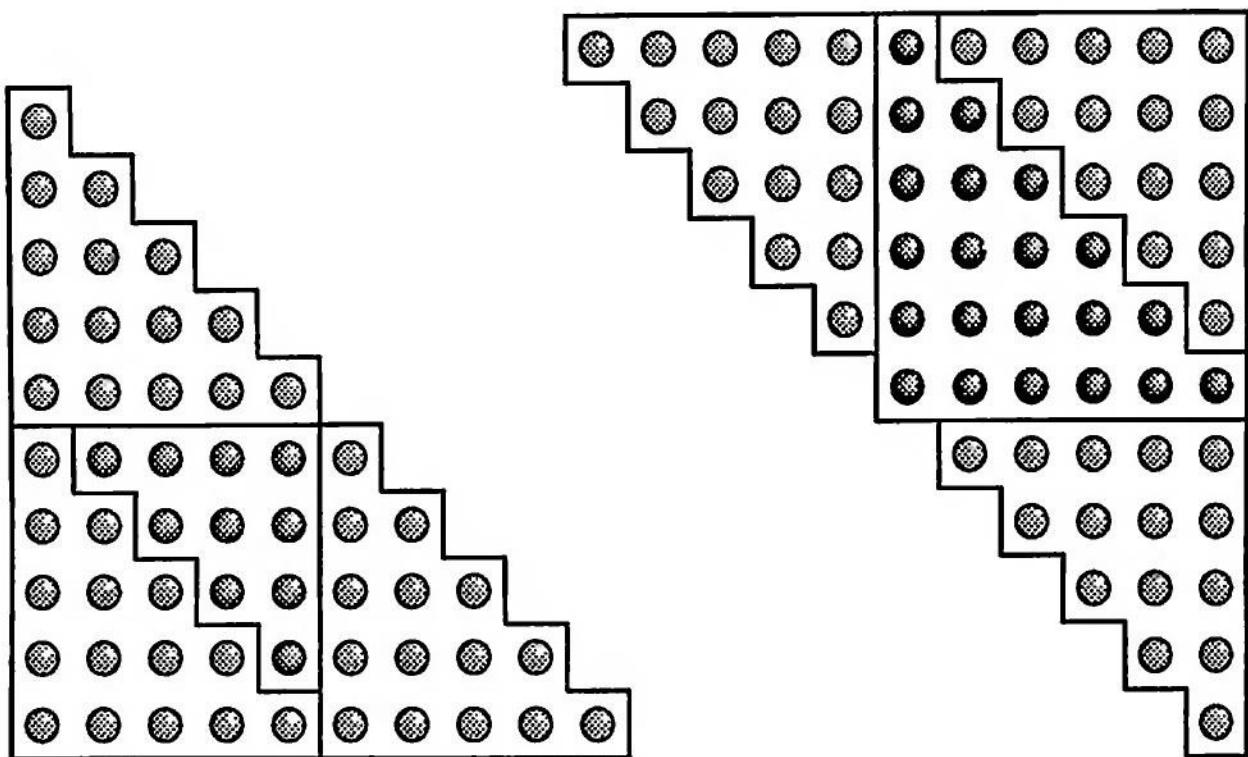
$$\sum_{k=1}^n k + \sum_{k=1}^{n-1} k = n^2$$



$$\sum_{k=1}^n k + n^2 = \sum_{k=n+1}^{2n} k$$

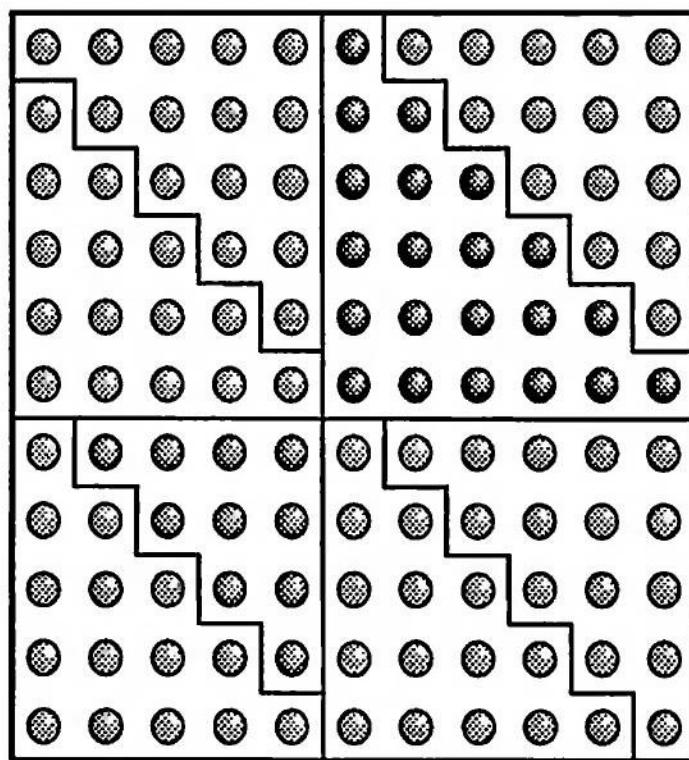
## Identities for Triangular Numbers

$$T_n = 1 + 2 + \cdots + n \Rightarrow$$



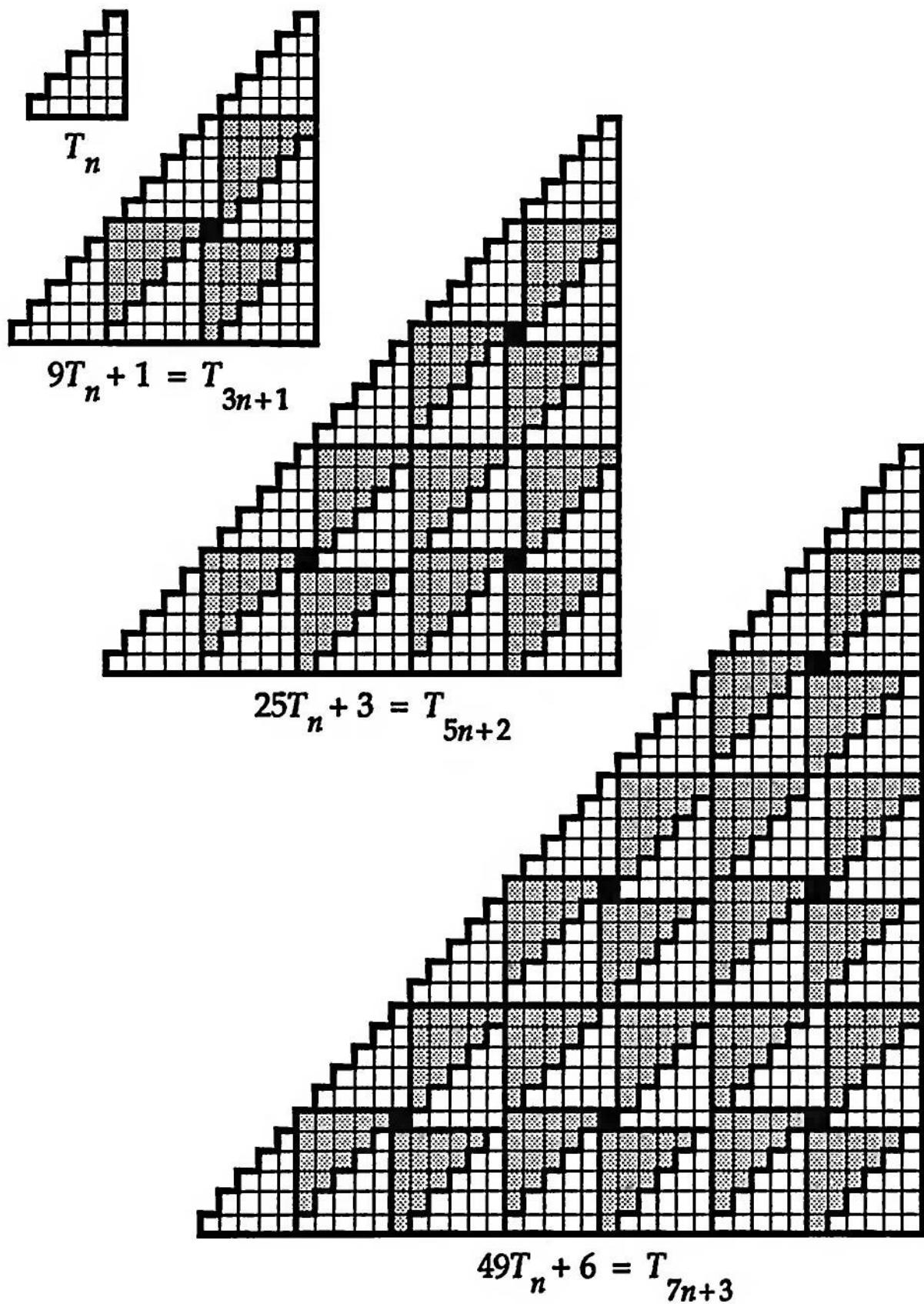
$$3T_n + T_{n-1} = T_{2n}$$

$$3T_n + T_{n+1} = T_{2n+1}$$



$$T_{n-1} + 6T_n + T_{n+1} = (2n+1)^2$$

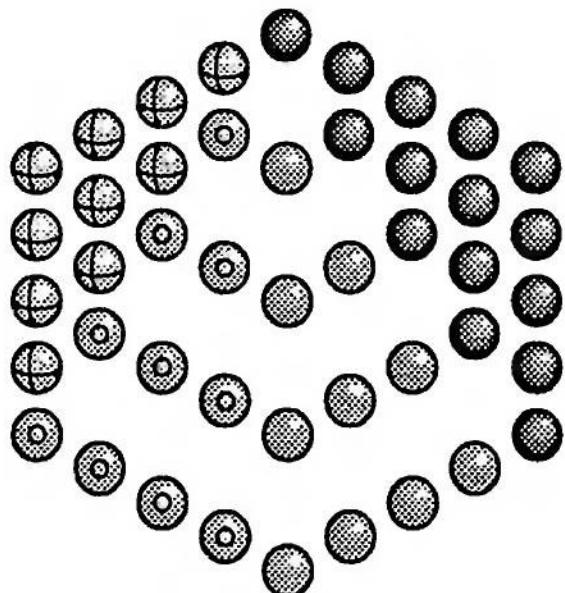
## A Triangular Identity



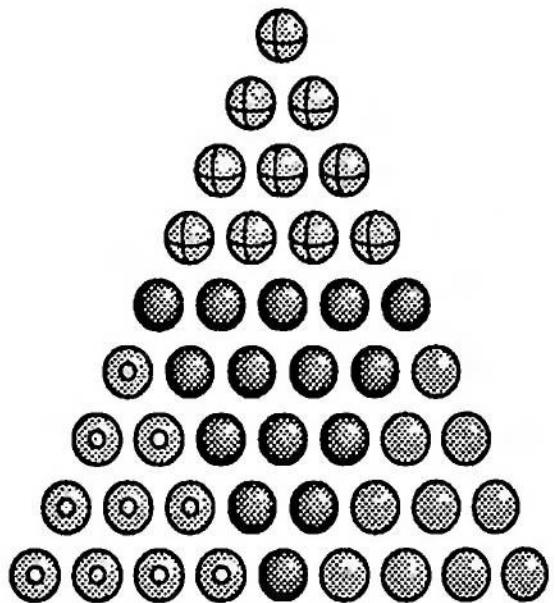
$$T_n = 1 + 2 + \dots + n \Rightarrow (2k+1)^2 T_n + T_k = T_{(2k+1)n+k}$$

## Every Hexagonal Number is a Triangular Number

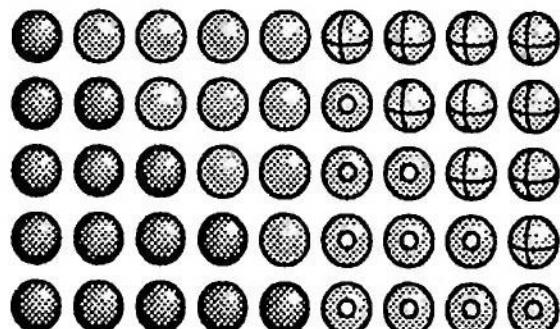
$$\left. \begin{array}{l} H_n = 1+5+\dots+(4n-3) \\ T_n = 1+2+\dots+n \end{array} \right\} \Rightarrow H_n = 3T_{n-1} + T_n = T_{2n-1} = n(2n-1)$$



$H_5$

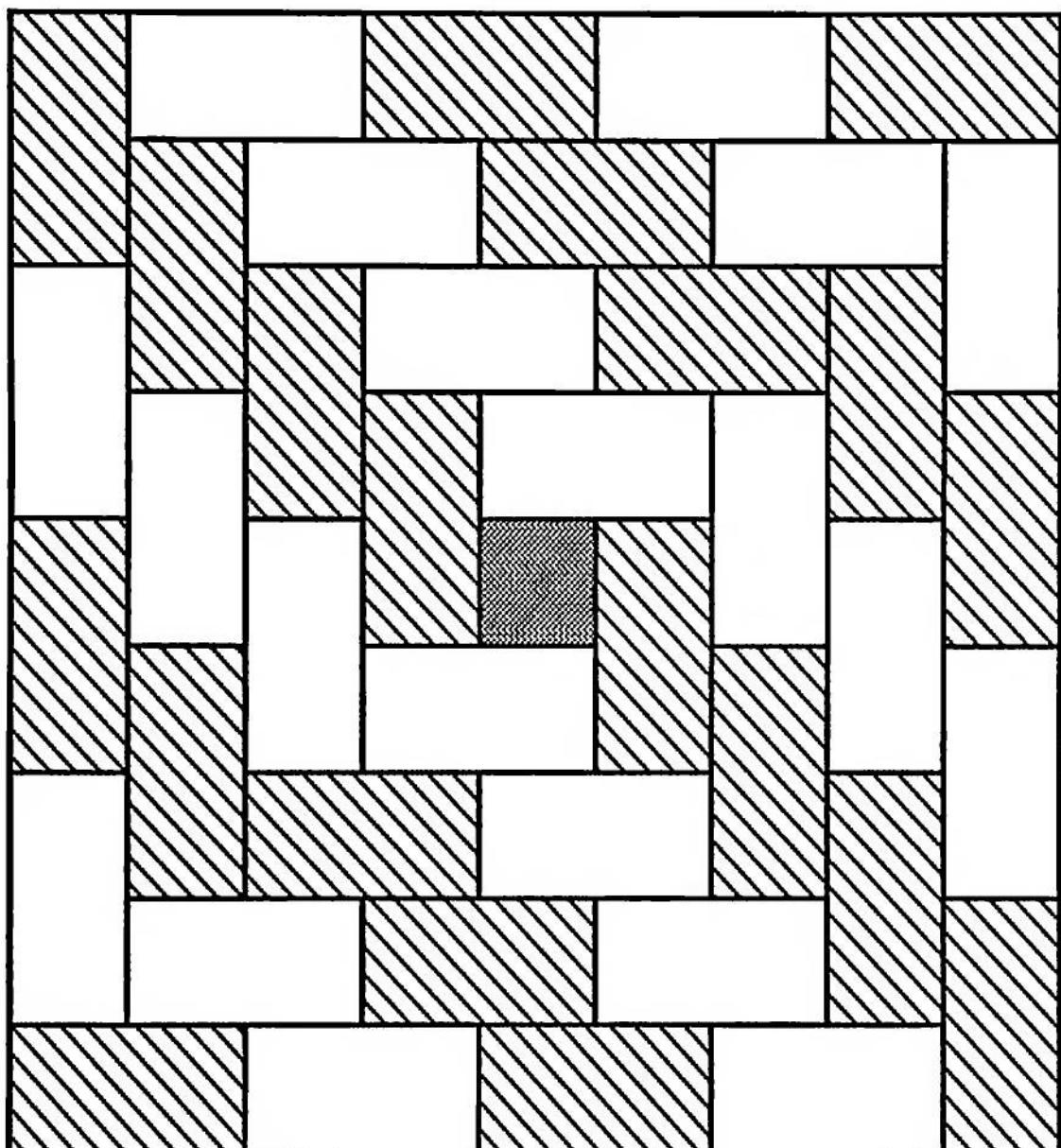


$T_9$



5.9

## One Domino = Two Squares: Concentric Squares

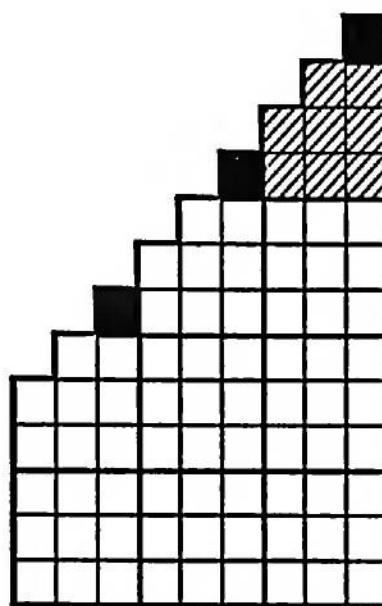
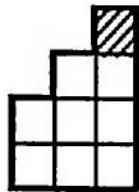
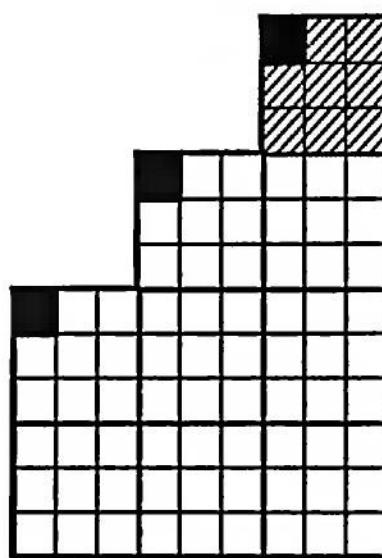
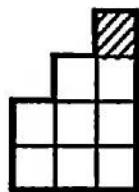
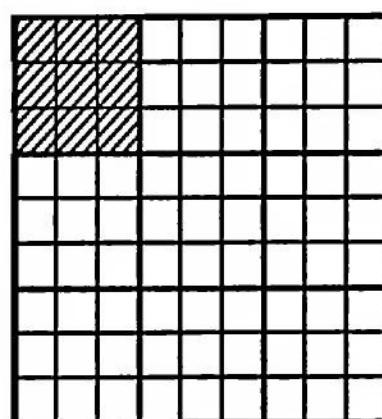
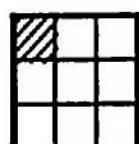


$$1 + 4 \cdot 2 + 8 \cdot 2 + 12 \cdot 2 + 16 \cdot 2 = 9^2$$

$$1 + 2 \sum_{k=1}^n 4k = (2n+1)^2$$

—Shirley A. Wakin

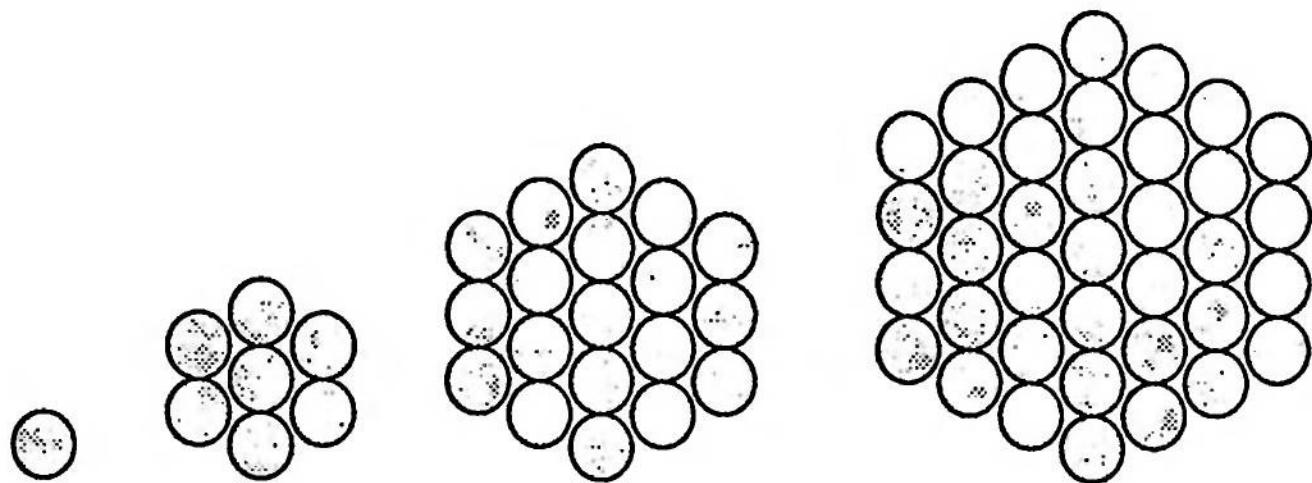
## Sums of Consecutive Powers of Nine are Sums of Consecutive Integers



$$1 + 9 + \dots + 9^n = 1 + 2 + 3 + \dots + (1 + 3 + \dots + 3^n)$$

—RBN

## Sums of Hex Numbers Are Cubes

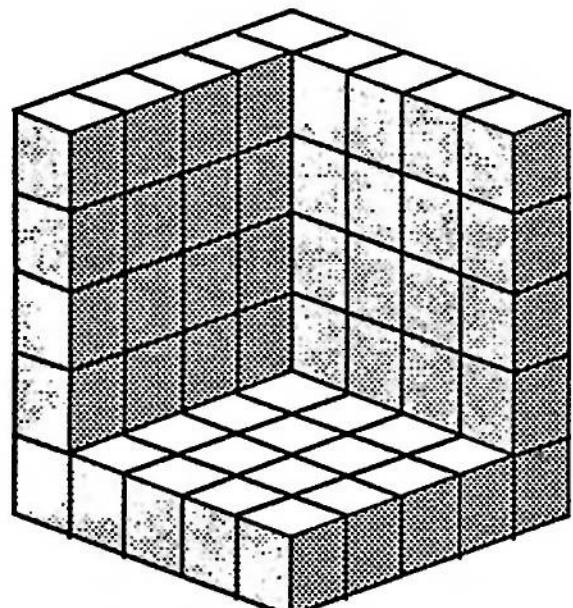
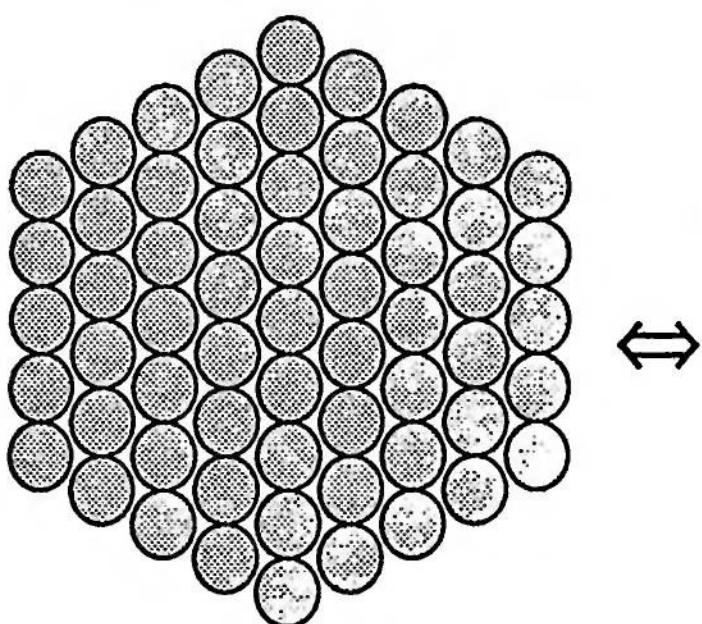


$$h_1 = 1$$

$$h_2 = 7$$

$$h_3 = 19$$

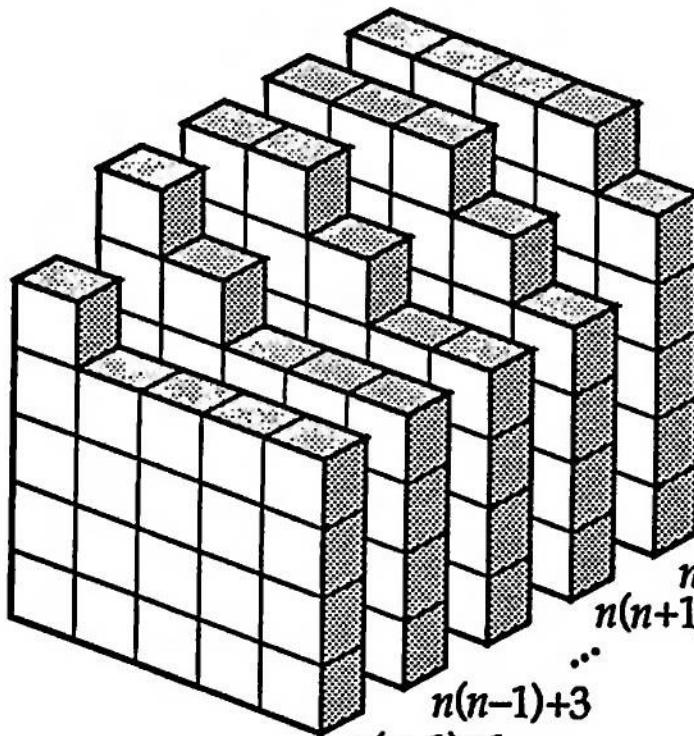
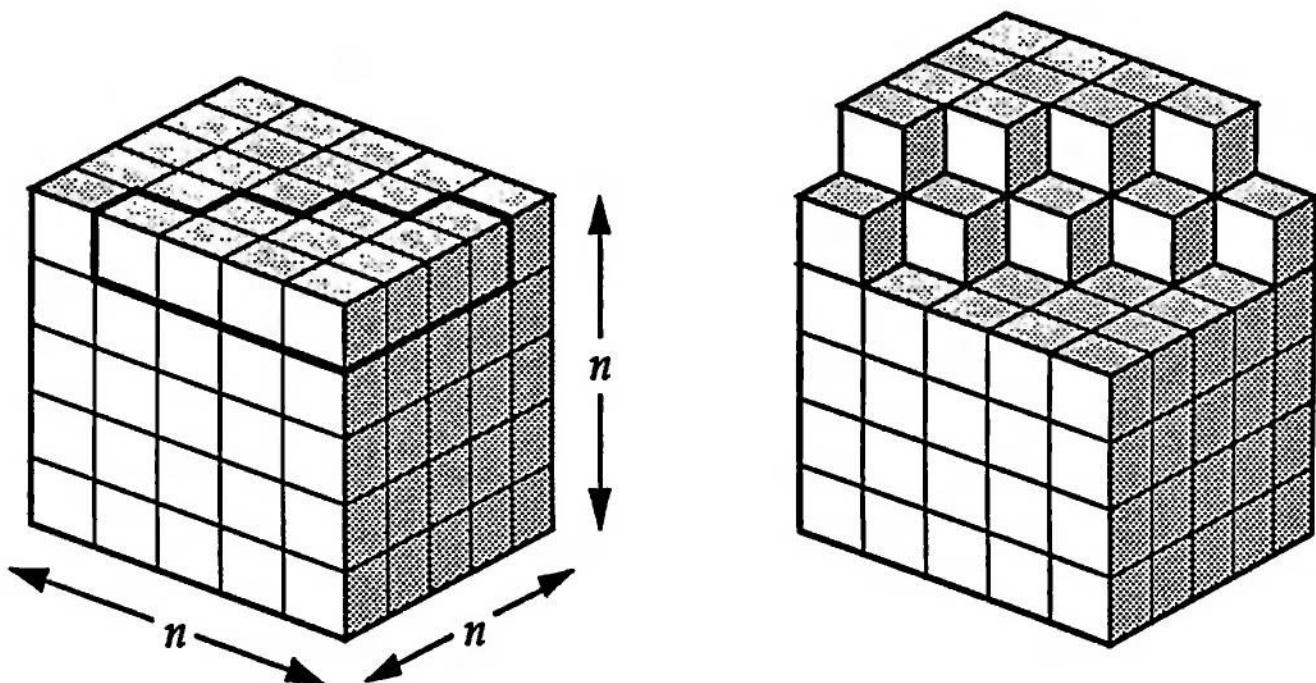
$$h_4 = 37$$



$$h_n = n^3 - (n-1)^3$$

$$\therefore h_1 + h_2 + \dots + h_n = n^3.$$

## Every Cube is the Sum of Consecutive Odd Numbers



$$1^3 = 1$$

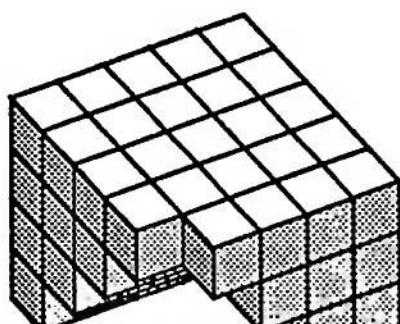
$$2^3 = 3 + 5$$

$$3^3 = 7 + 9 + 11$$

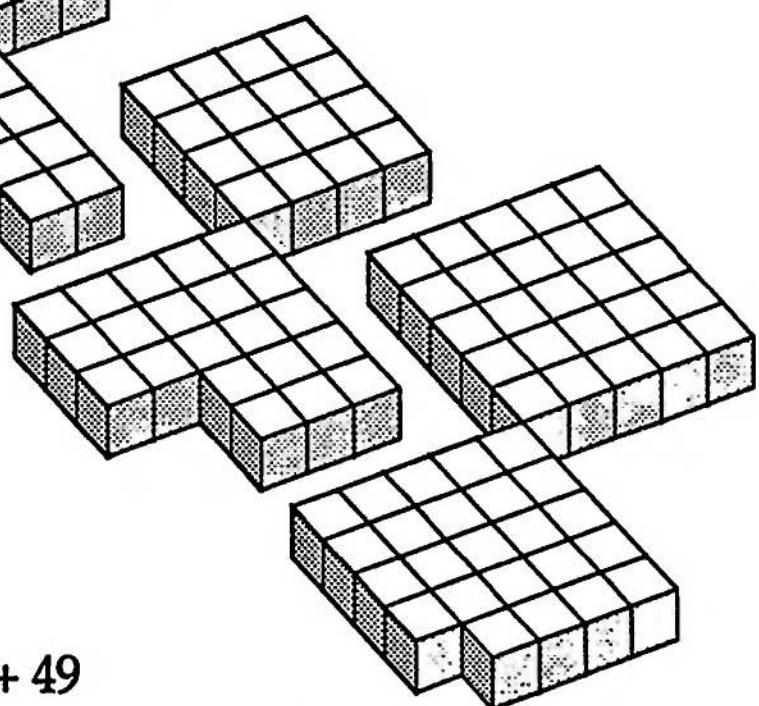
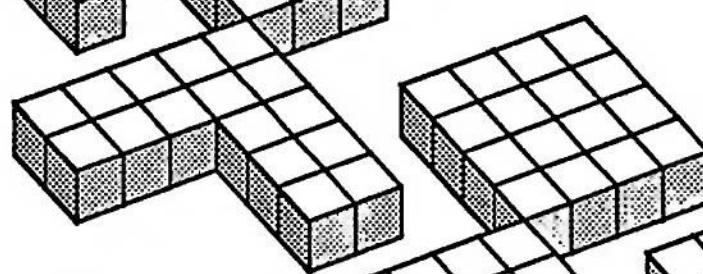
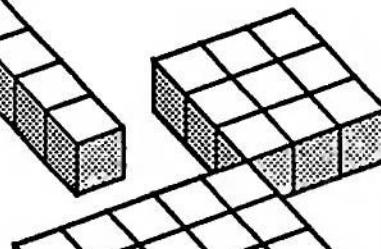
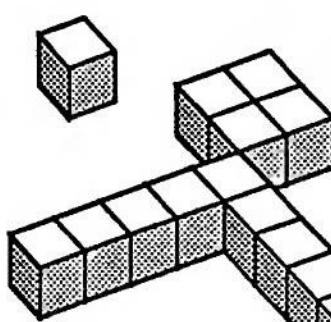
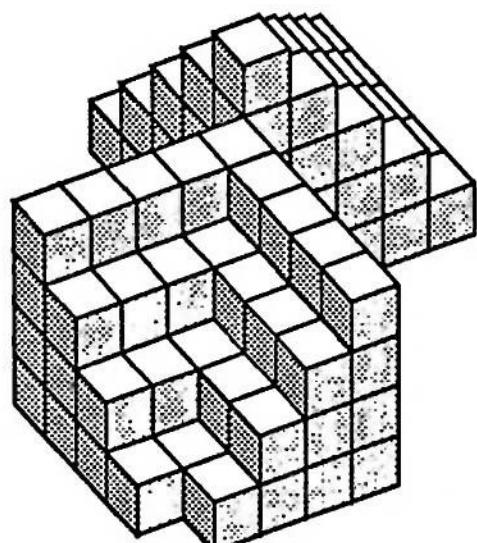
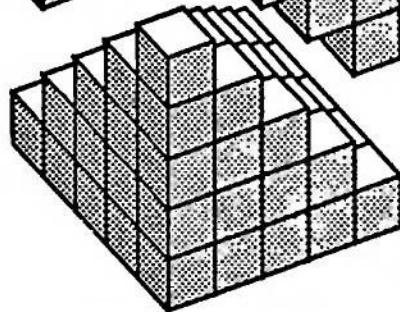
⋮

$$n^3 = [n(n-1)+1] + \dots + [n(n+1)-1]$$

## The Cube as an Arithmetic Sum



$$n^3 = \sum_{i=0}^{n-1} 2i(n+1) + 1$$



$$5^3 = 1 + 13 + 25 + 37 + 49$$

—Robert Bronson and  
Christopher Brueningsen



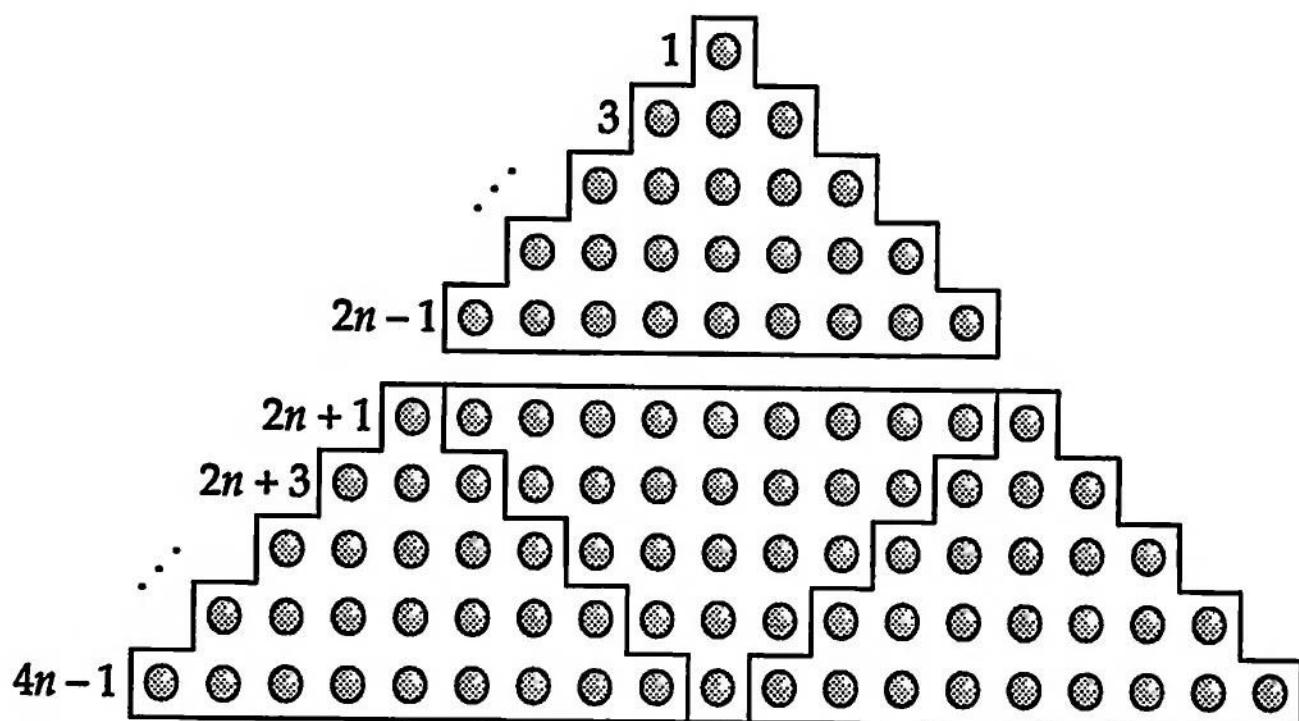
# Sequences & Series

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## On a Property of the Sequence of Odd Integers (Galileo, 1615)

$$\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \dots .$$



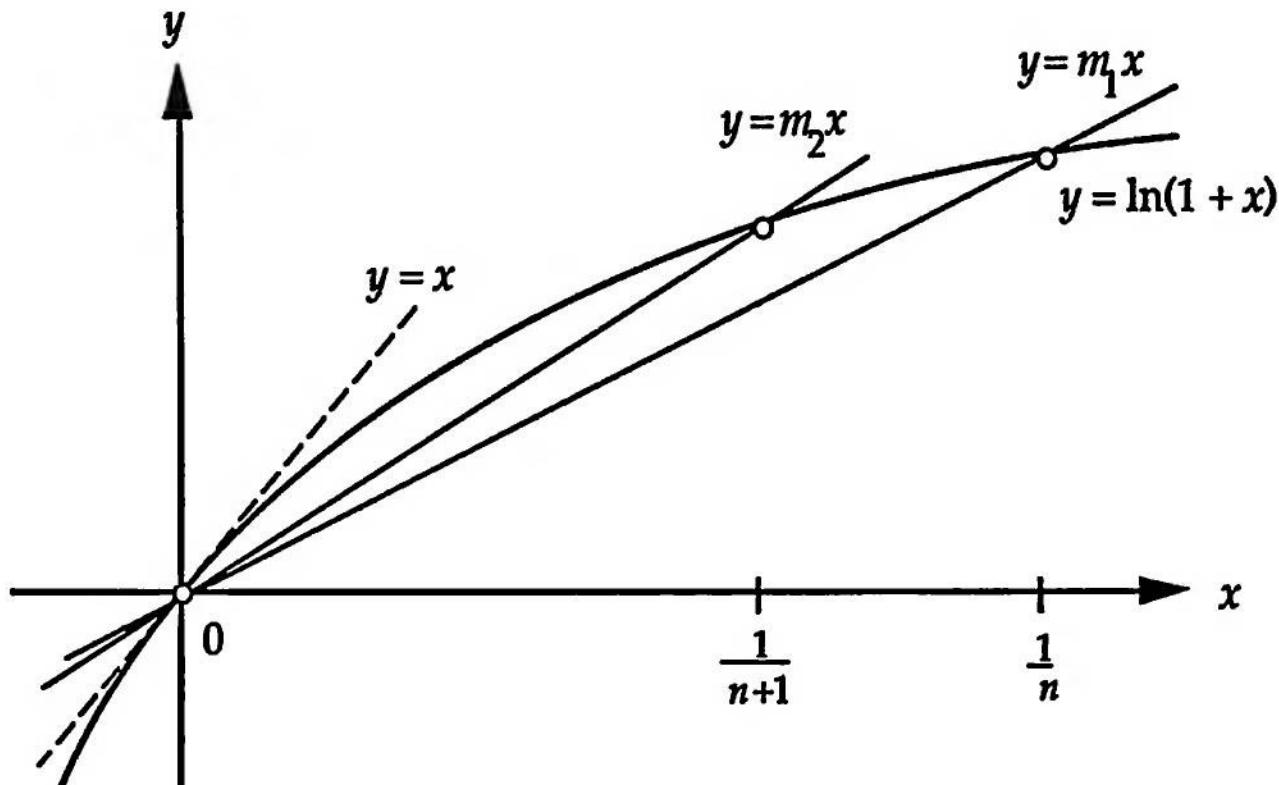
$$\frac{1 + 3 + \dots + (2n - 1)}{(2n + 1) + (2n + 3) + \dots + (4n - 1)} = \frac{1}{3}$$

### REFERENCE

S. Drake, *Galileo Studies*, The University of Michigan Press, Ann Arbor, 1970, pp. 218-219.

## A Monotone Sequence Bounded by $e$

$$\forall n \geq 1, \left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1} < e.$$

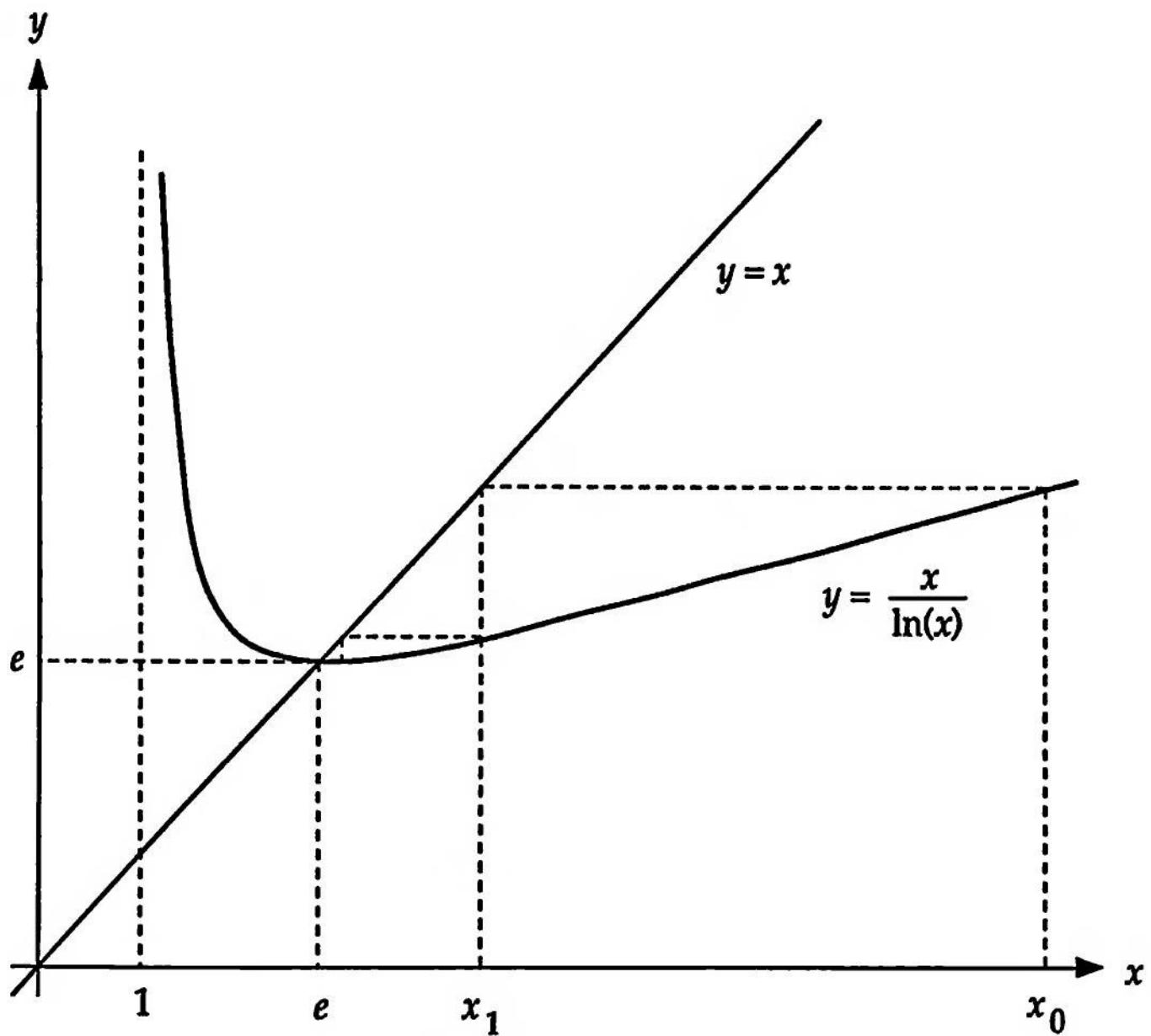


$$n \geq 1 \Rightarrow m_1 < m_2 < 1$$

$$\Rightarrow \frac{\ln(1 + \frac{1}{n})}{\frac{1}{n}} < \frac{\ln(1 + \frac{1}{n+1})}{\frac{1}{n+1}} < 1$$

$$\Rightarrow \left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1} < e$$

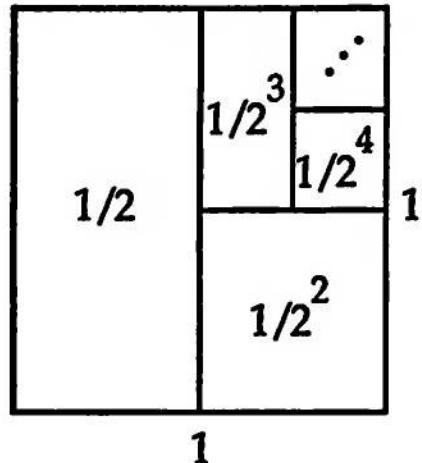
## A Recursively Defined Sequence for $e$



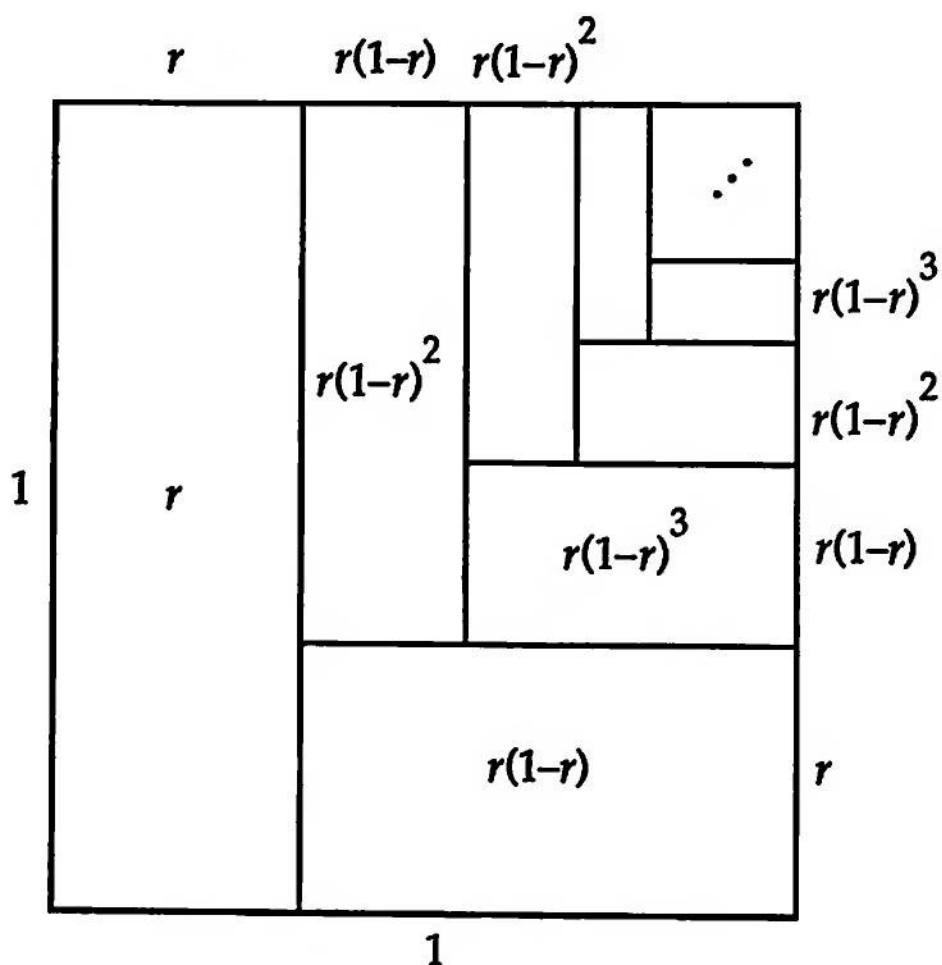
$$x_0 > 1 \text{ } \& \text{ } x_{n+1} = \frac{x_n}{\ln(x_n)} \Rightarrow \lim x_n = e$$

—Thomas P. Dence

## Geometric Sums



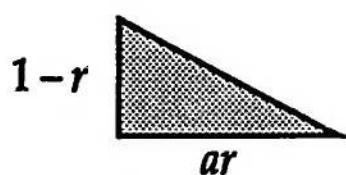
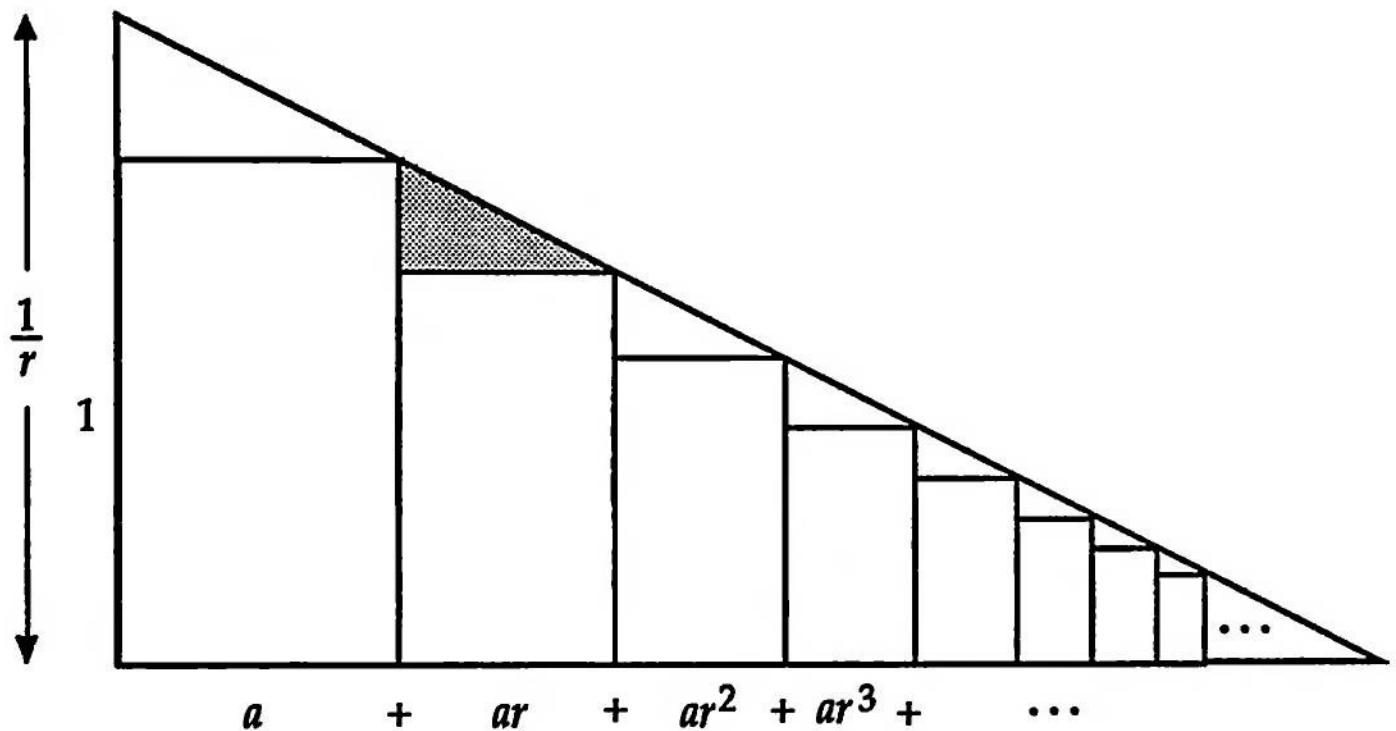
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$$



$$r + r(1-r) + r(1-r)^2 + \dots = 1$$

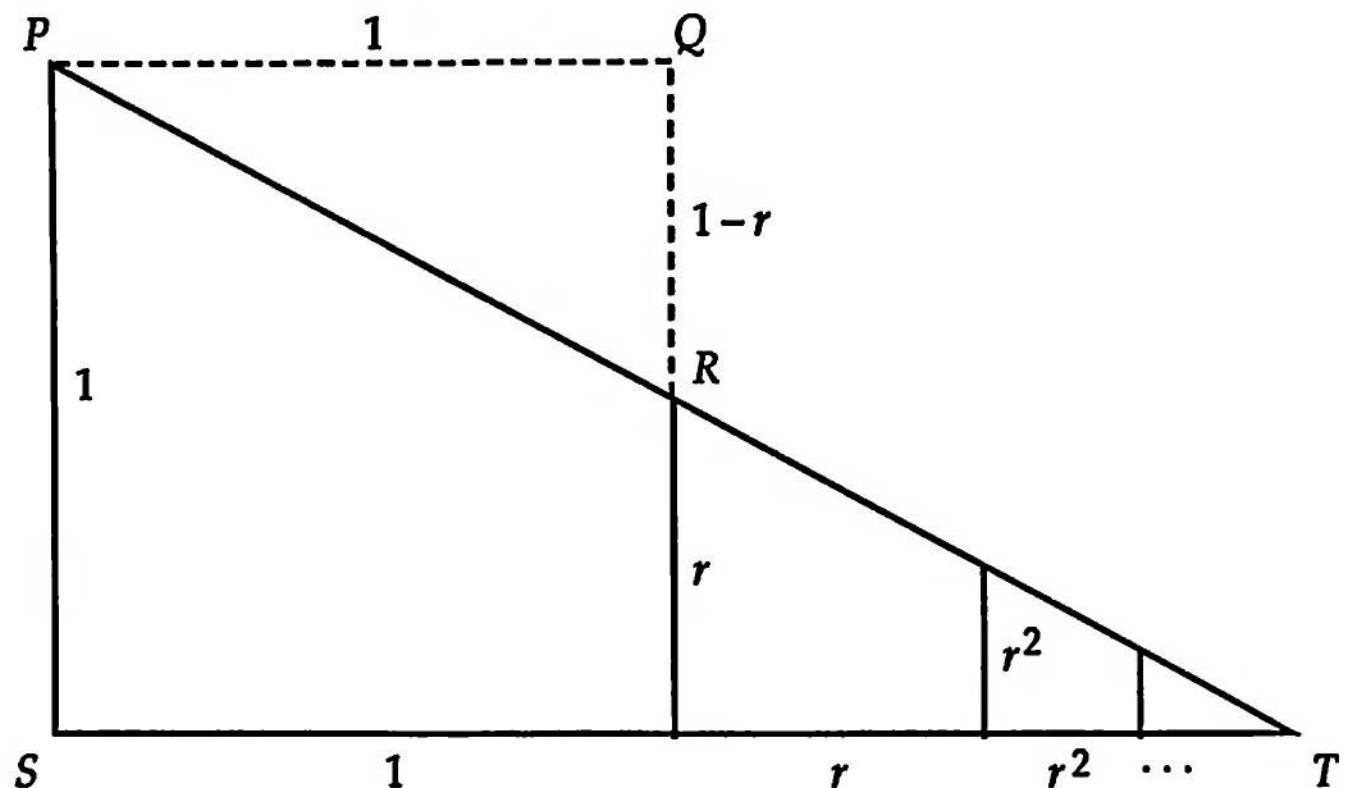
## Geometric Series I

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$



$$\frac{a + ar + ar^2 + ar^3 + \dots}{1/r} = \frac{ar}{1-r}$$

## Geometric Series II

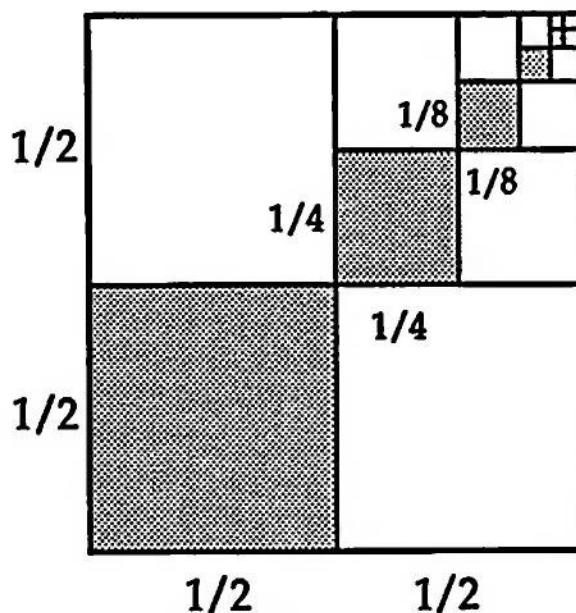


$$\Delta PQR \approx \Delta TSP$$

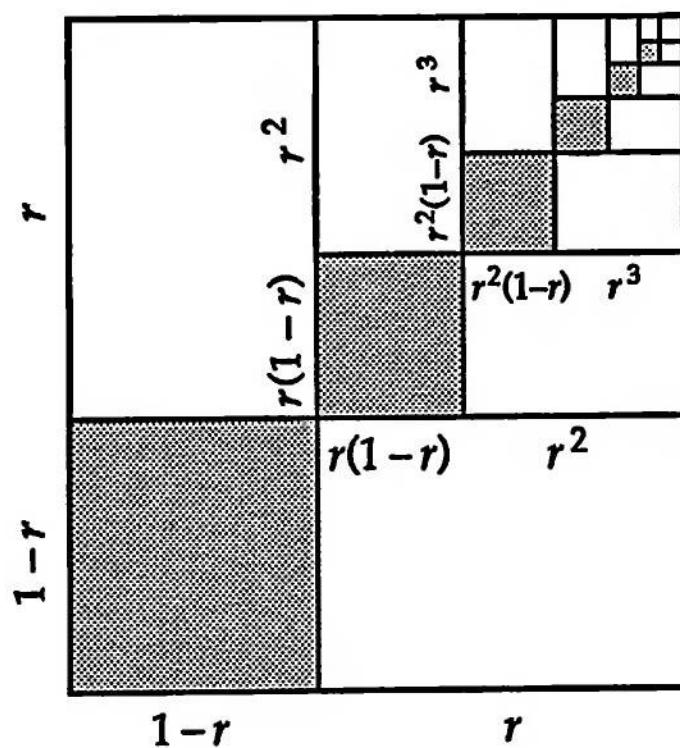
$$\therefore 1 + r + r^2 + \dots = \frac{1}{1-r}.$$

—Benjamin G. Klein and Irl C. Bivens

## Geometric Series III



$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots = \frac{1}{3}$$

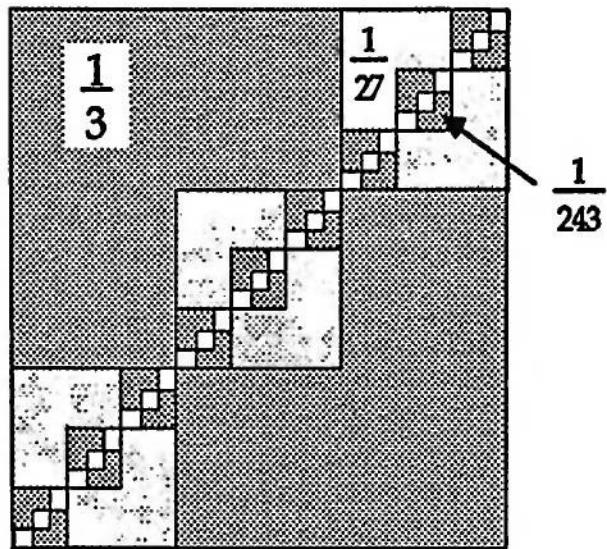


$$(1-r)^2 + r^2(1-r)^2 + r^4(1-r)^2 + \dots = \frac{(1-r)^2}{(1-r)^2 + 2r(1-r)} = \frac{1-r}{1+r}$$

$$1 + r^2 + r^4 + \dots = \frac{1}{1-r^2}$$

$$a + ar + ar^2 + \dots = \frac{a}{1-r}$$

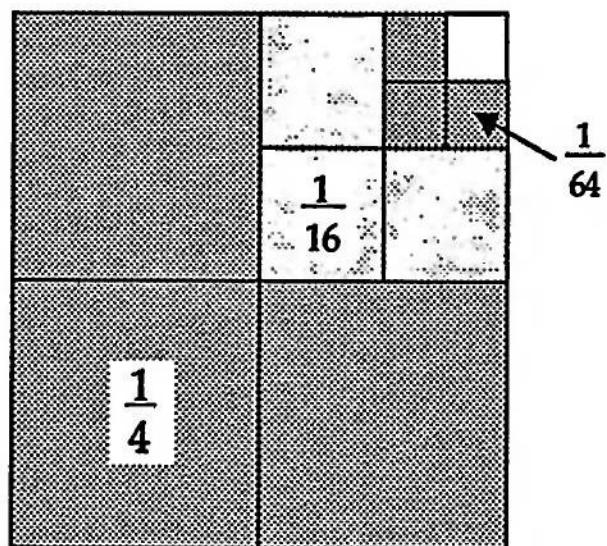
## Geometric Series IV



$$2 \left( \frac{1}{3} + 3 \cdot \frac{1}{27} + 9 \cdot \frac{1}{243} + \dots \right) = 1$$

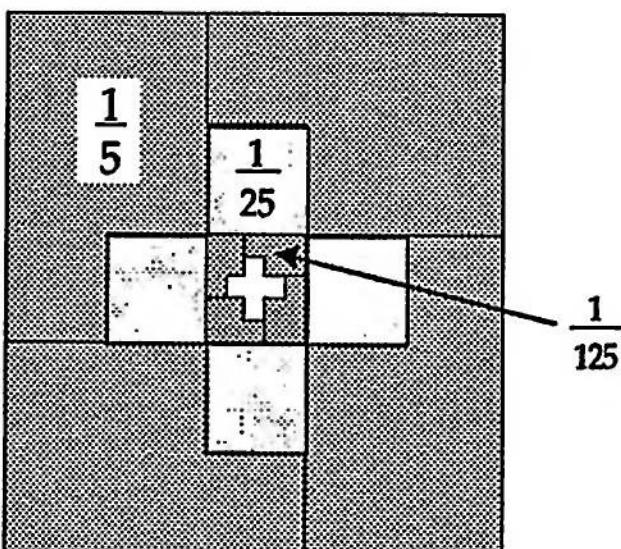
$$2 \sum_{n=1}^{\infty} \frac{1}{3^n} = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}$$



$$3 \sum_{n=1}^{\infty} \frac{1}{4^n} = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1}{3}$$



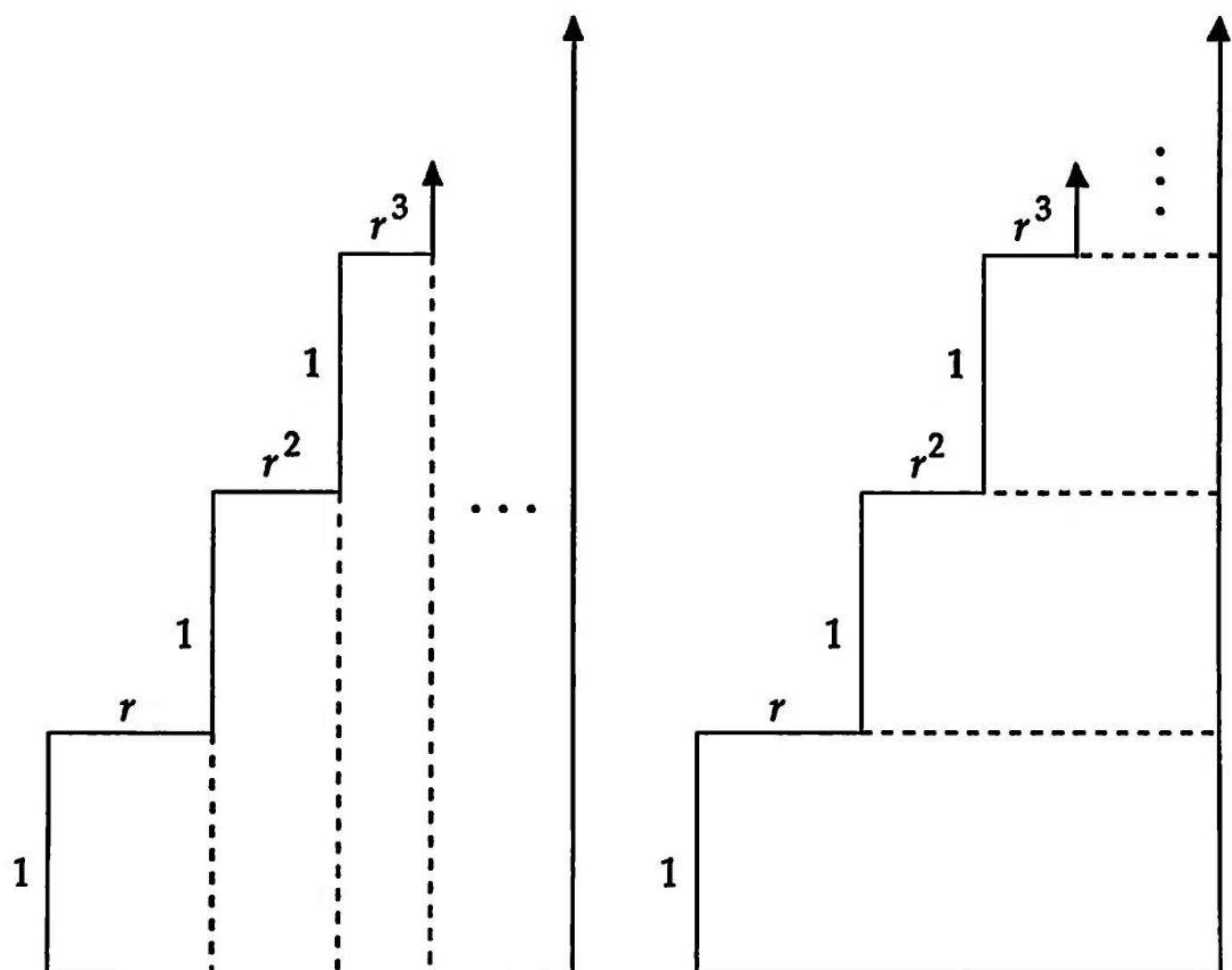
$$4 \sum_{n=1}^{\infty} \frac{1}{5^n} = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{4}$$

—Elizabeth M. Markham

## Gabriel's Staircase

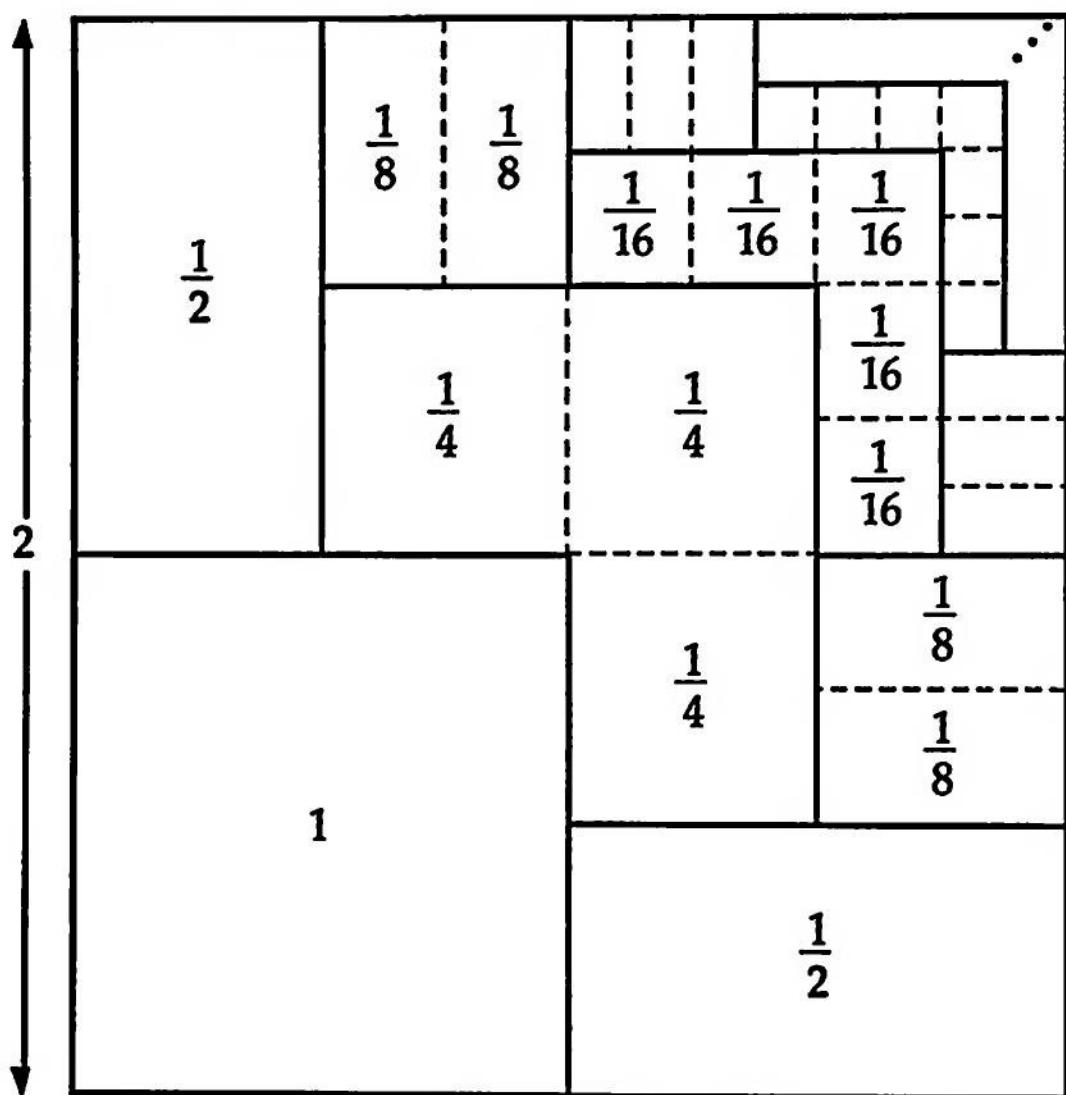
$$\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2} \text{ for } 0 < r < 1$$



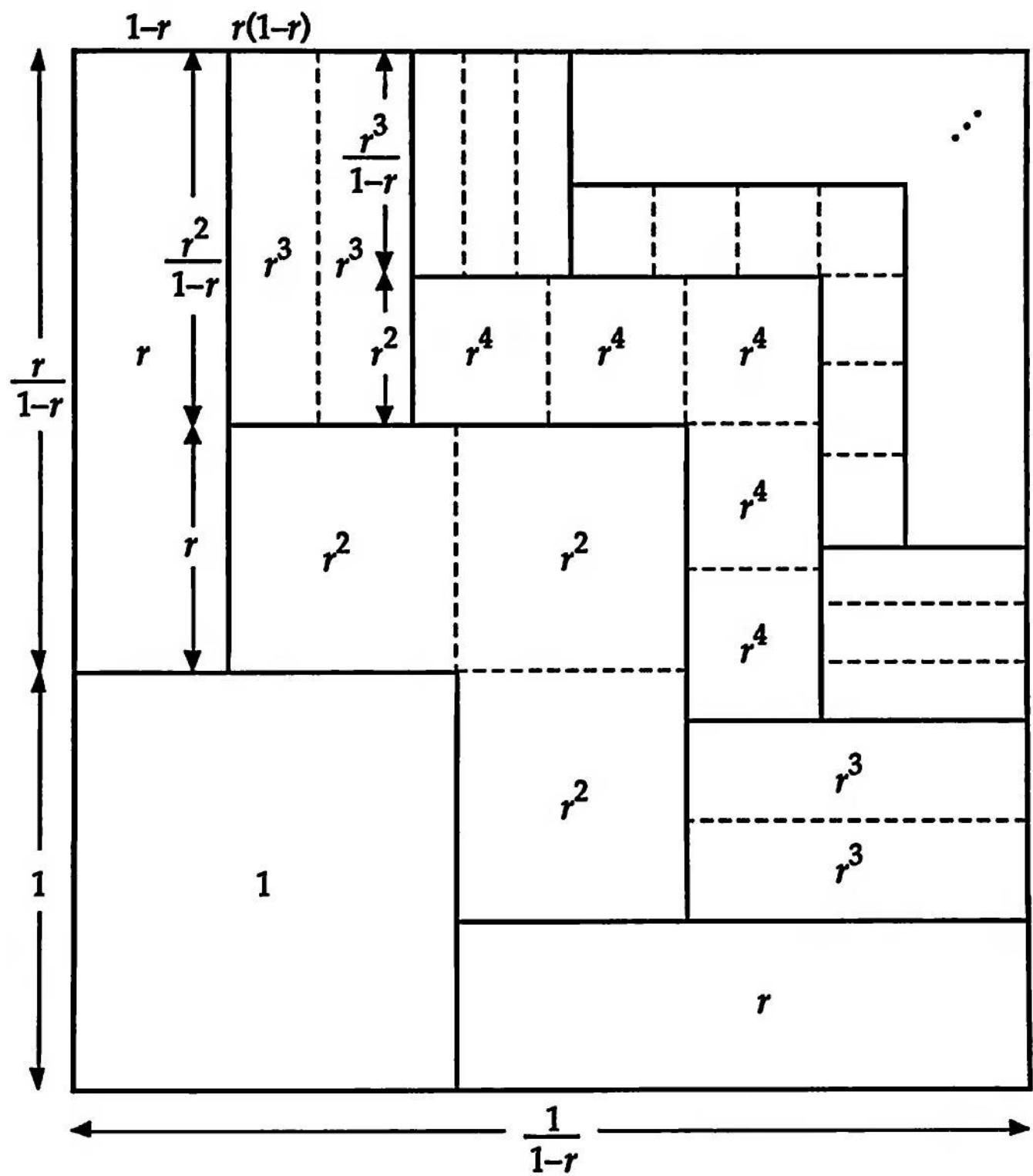
$$\sum_{k=1}^{\infty} kr^k = \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} r^i = \frac{r}{(1-r)^2}$$

## Differentiated Geometric Series

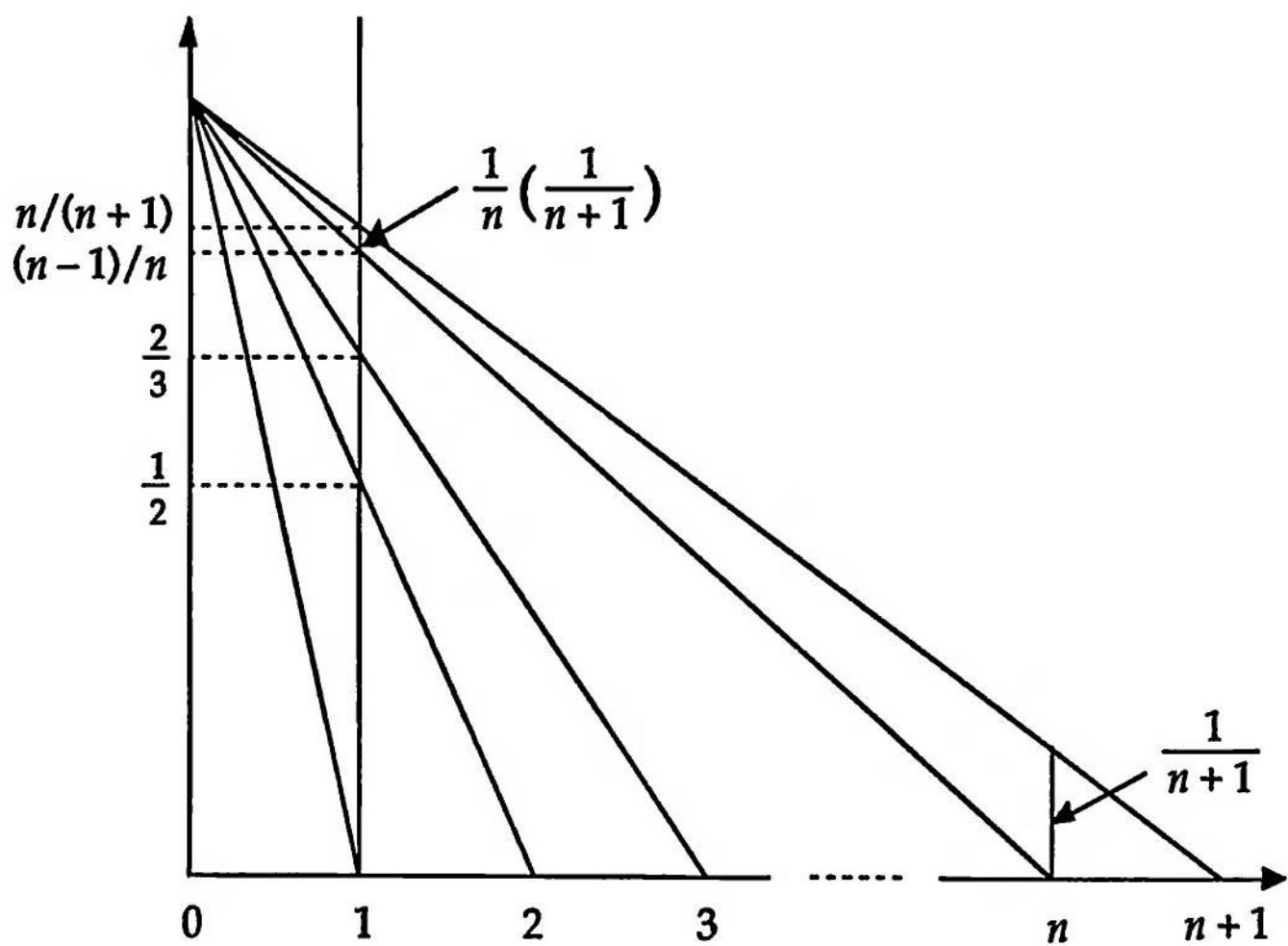
$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \dots = 4$$



$$1 + 2r + 3r^2 + 4r^3 + \dots = \left(\frac{1}{1-r}\right)^2, \quad 0 \leq r < 1$$



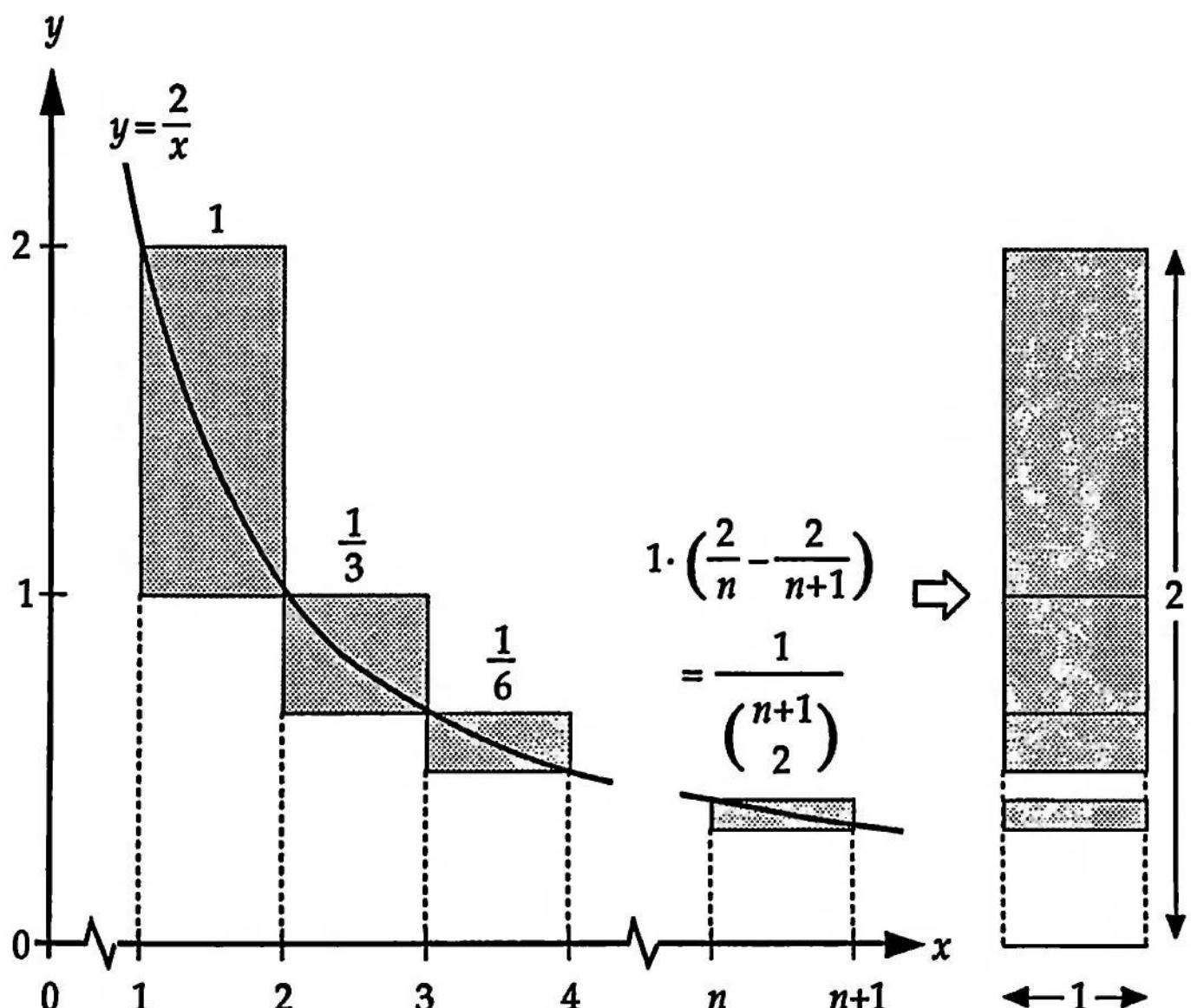
$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$



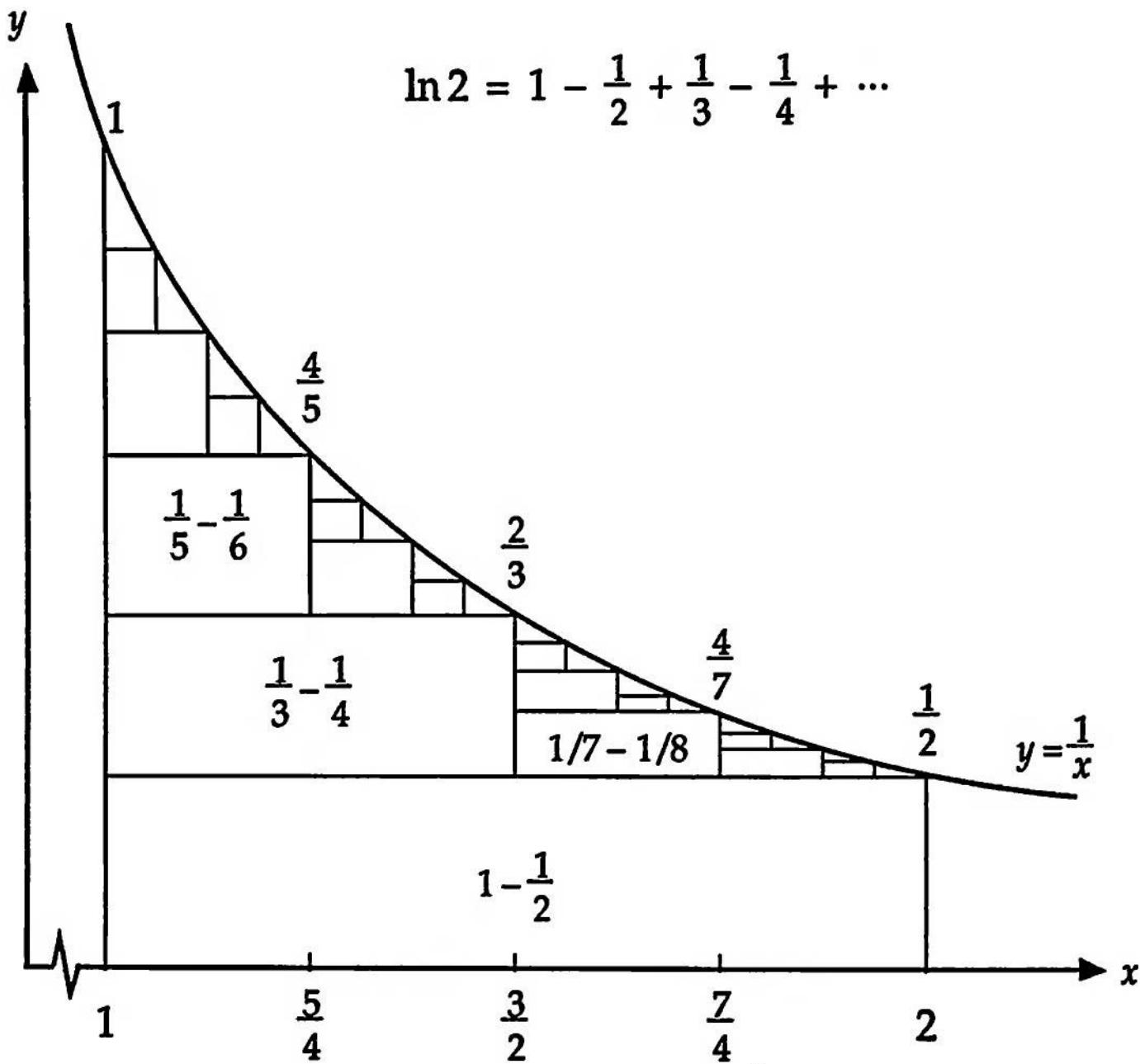
—Roman W. Wong

## The Series of Reciprocals of Triangular Numbers

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \cdots + \frac{1}{\binom{n+1}{2}} + \cdots = 2$$



## The Alternating Harmonic Series



$$\frac{1}{2} \left( \frac{2}{3} - \frac{2}{4} \right) = \frac{1}{3} - \frac{1}{4};$$

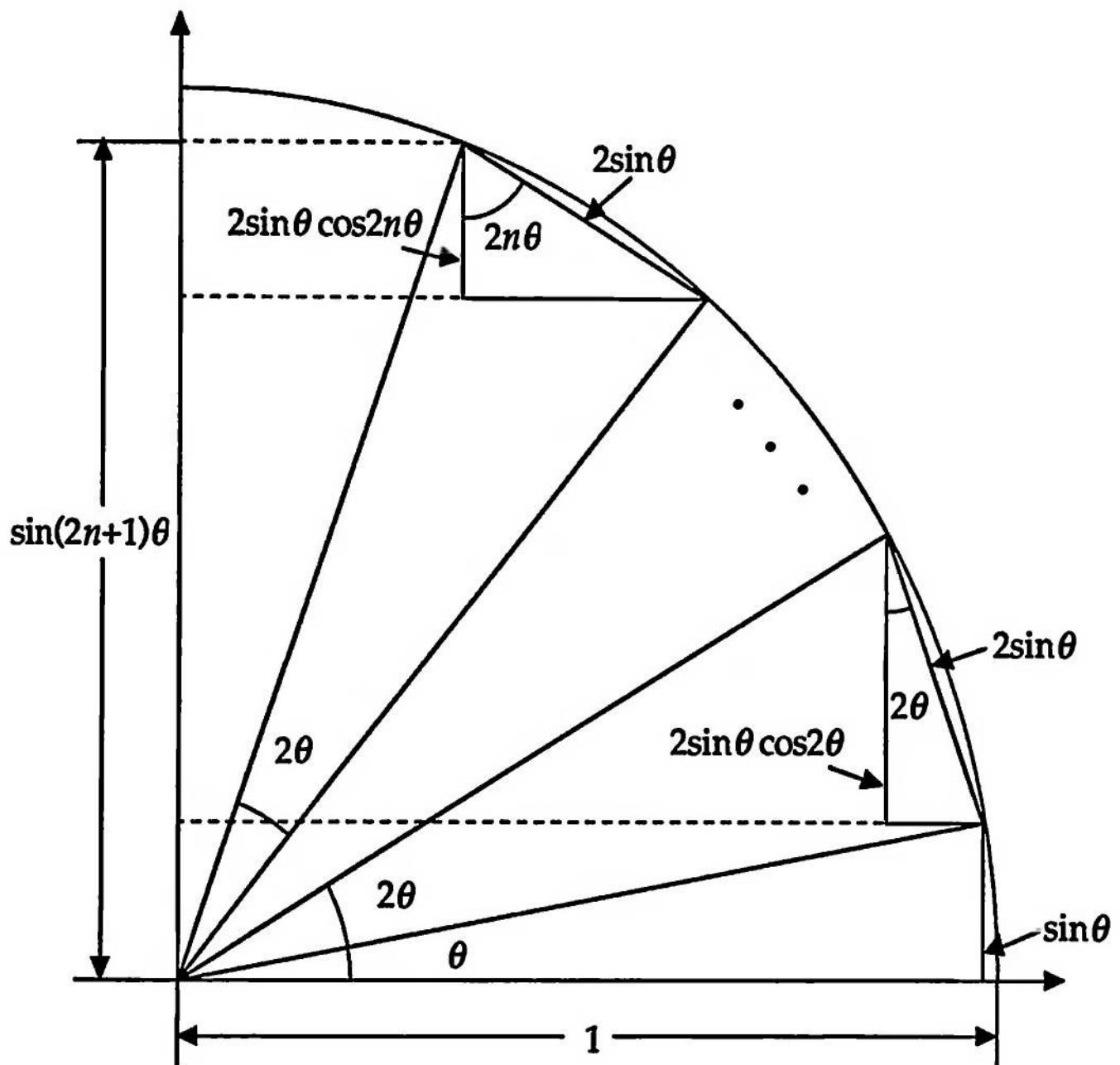
$$\frac{1}{4} \left( \frac{4}{5} - \frac{4}{6} \right) = \frac{1}{5} - \frac{1}{6}, \quad \frac{1}{4} \left( \frac{4}{7} - \frac{4}{8} \right) = \frac{1}{7} - \frac{1}{8};$$

$$\frac{1}{2^n} \left( \frac{2^n}{2^n + 2k-1} - \frac{2^n}{2^n + 2k} \right) = \frac{1}{2^n + 2k-1} - \frac{1}{2^n + 2k}, \quad k = 1, 2, \dots, 2^{n-1}; \\ n = 1, 2, \dots.$$

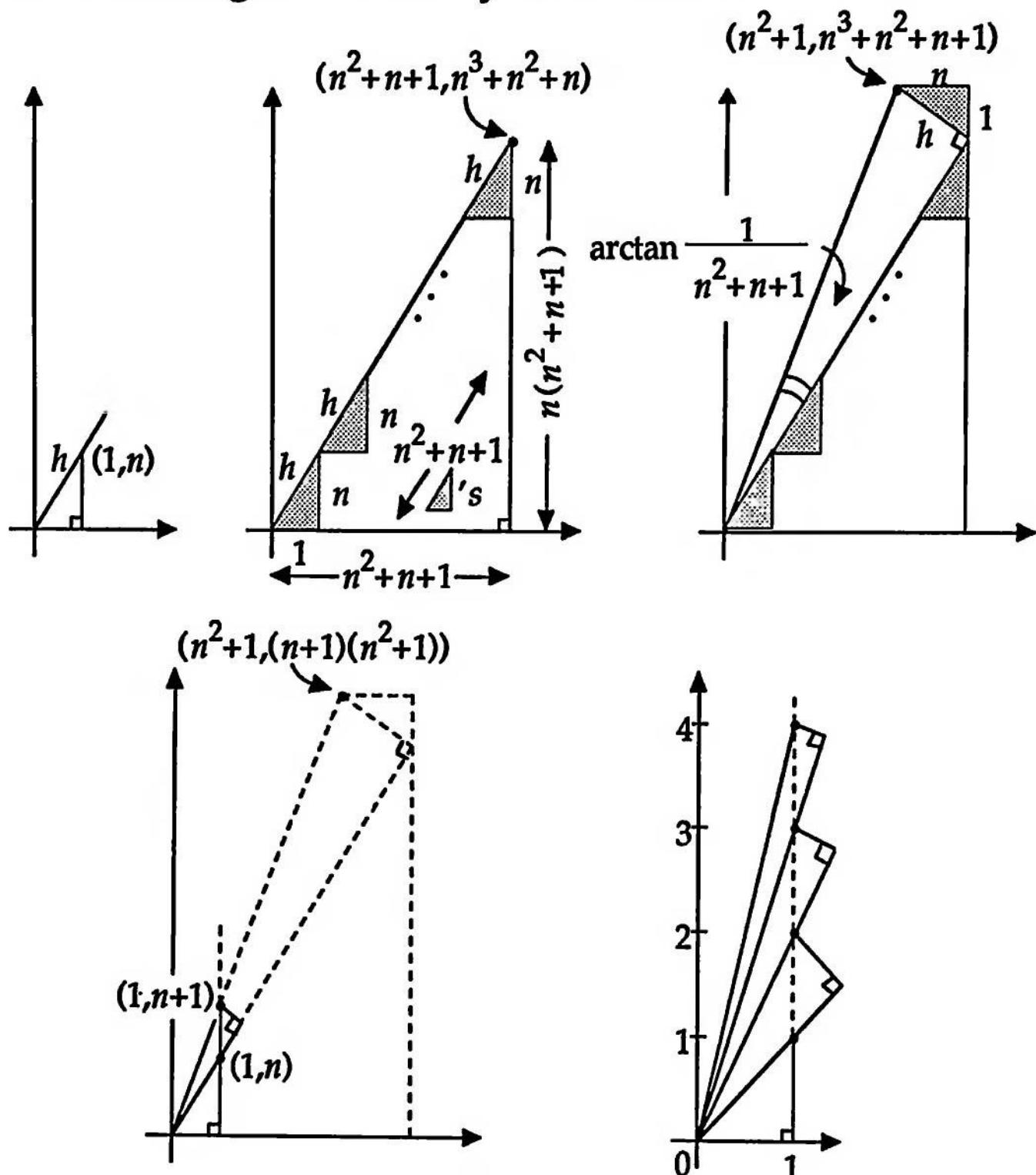
$$\ln 2 = \int_1^2 \frac{dx}{x} = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \dots = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

—Mark Finkelstein

$$\sin(2n+1)\theta = \sin\theta + 2\sin\theta \sum_{k=1}^n \cos 2k\theta$$



# An Arctangent Identity and Series



$$\arctan n + \arctan \frac{1}{n^2 + n + 1} = \arctan(n + 1)$$

$$\arctan \frac{1}{n^2 + n + 1} = \arctan(n + 1) - \arctan n$$

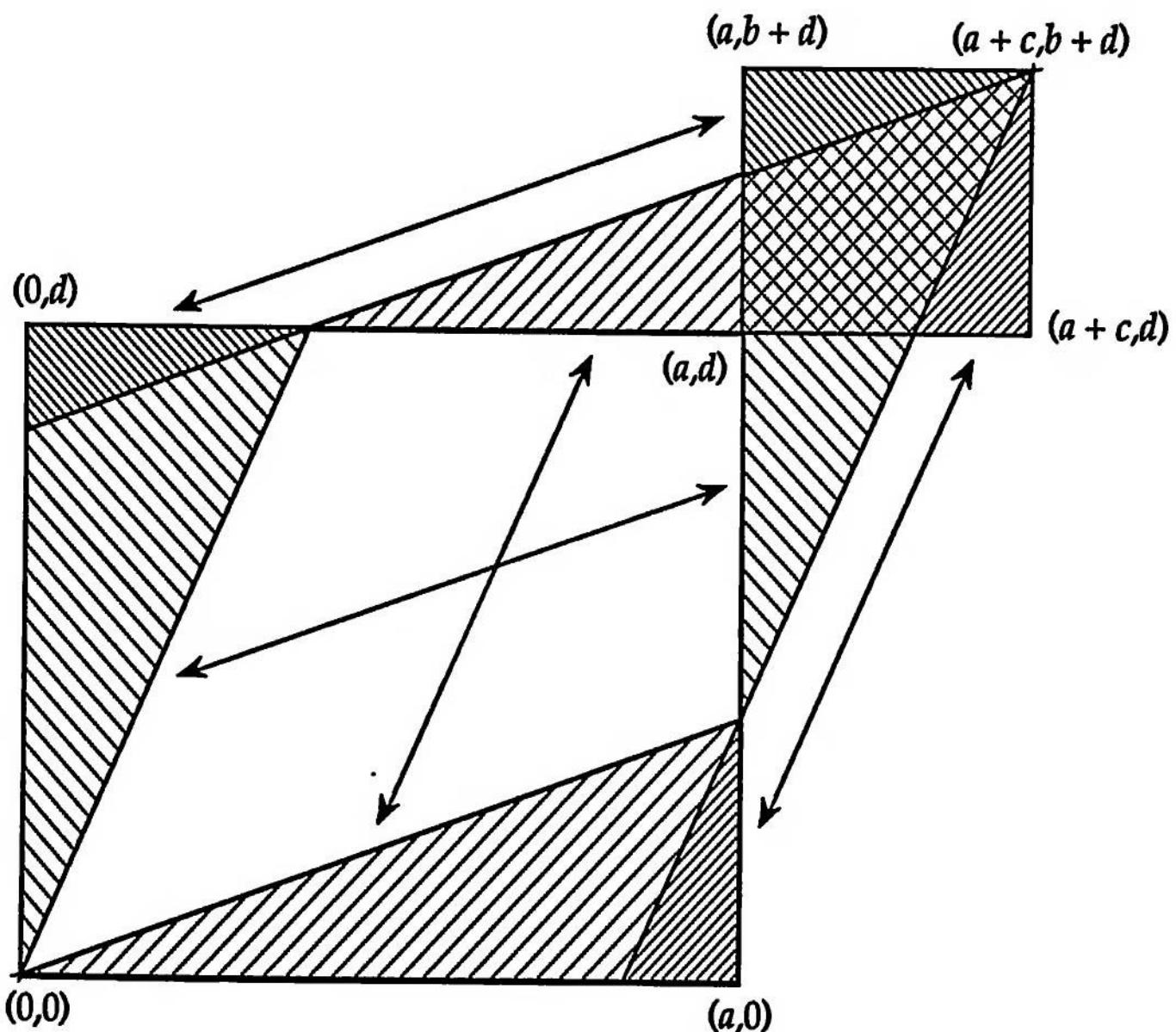
$$\sum_{n=0}^{\infty} \arctan \frac{1}{n^2 + n + 1} = \lim_{N \rightarrow \infty} \arctan(N + 1) = \frac{\pi}{2}$$

# Miscellaneous

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## A $2 \times 2$ Determinant is the Area of a Parallelogram

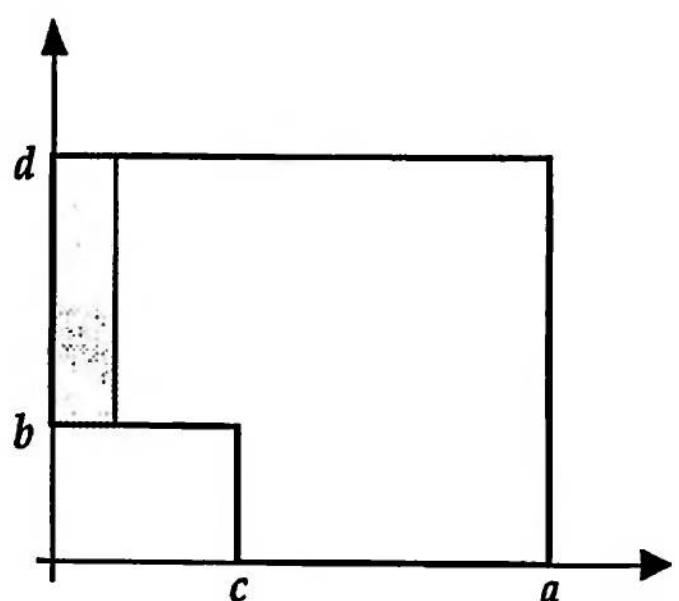
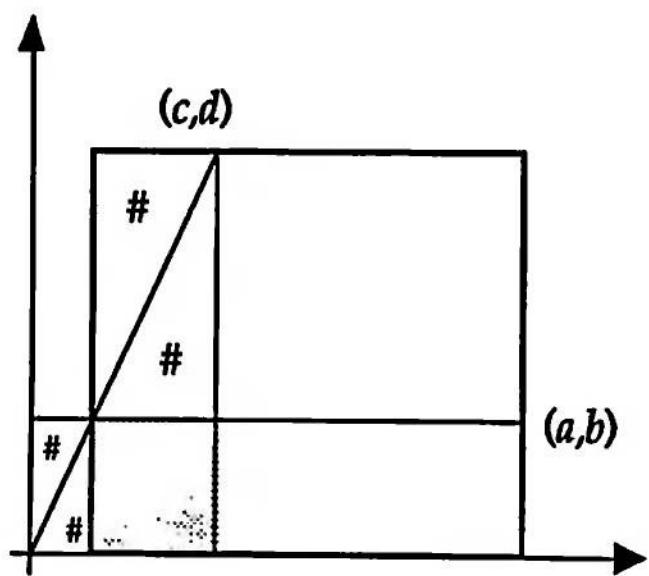
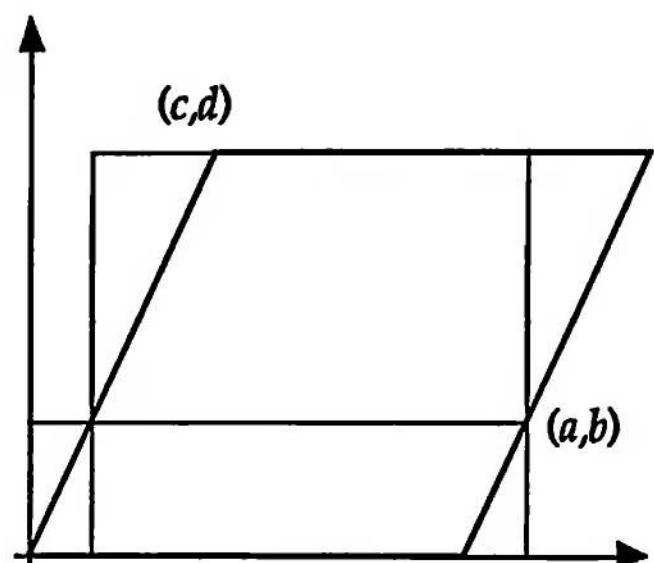
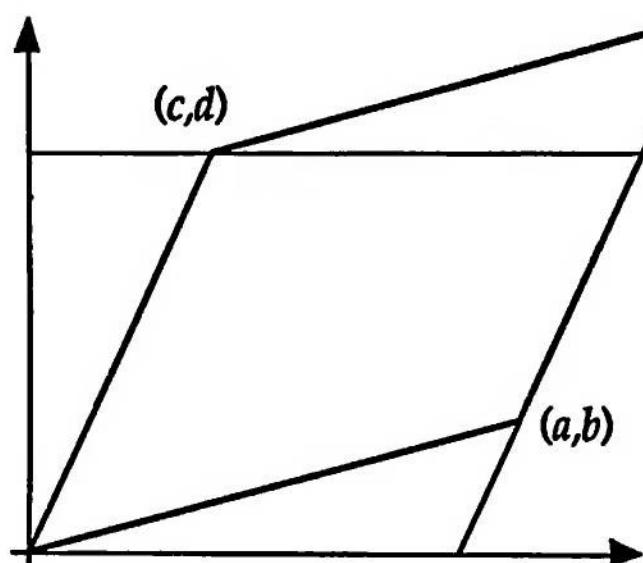


$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \left\| \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right\| - \left\| \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right\| = \left\| \begin{array}{|c|c|} \hline \diagup & \diagdown \\ \hline \end{array} \right\|$$

—Solomon W. Golomb

## Area of the Parallelogram Determined by

$$\text{Vectors } (a,b) \text{ and } (c,d) = \pm \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm (ad - bc)$$



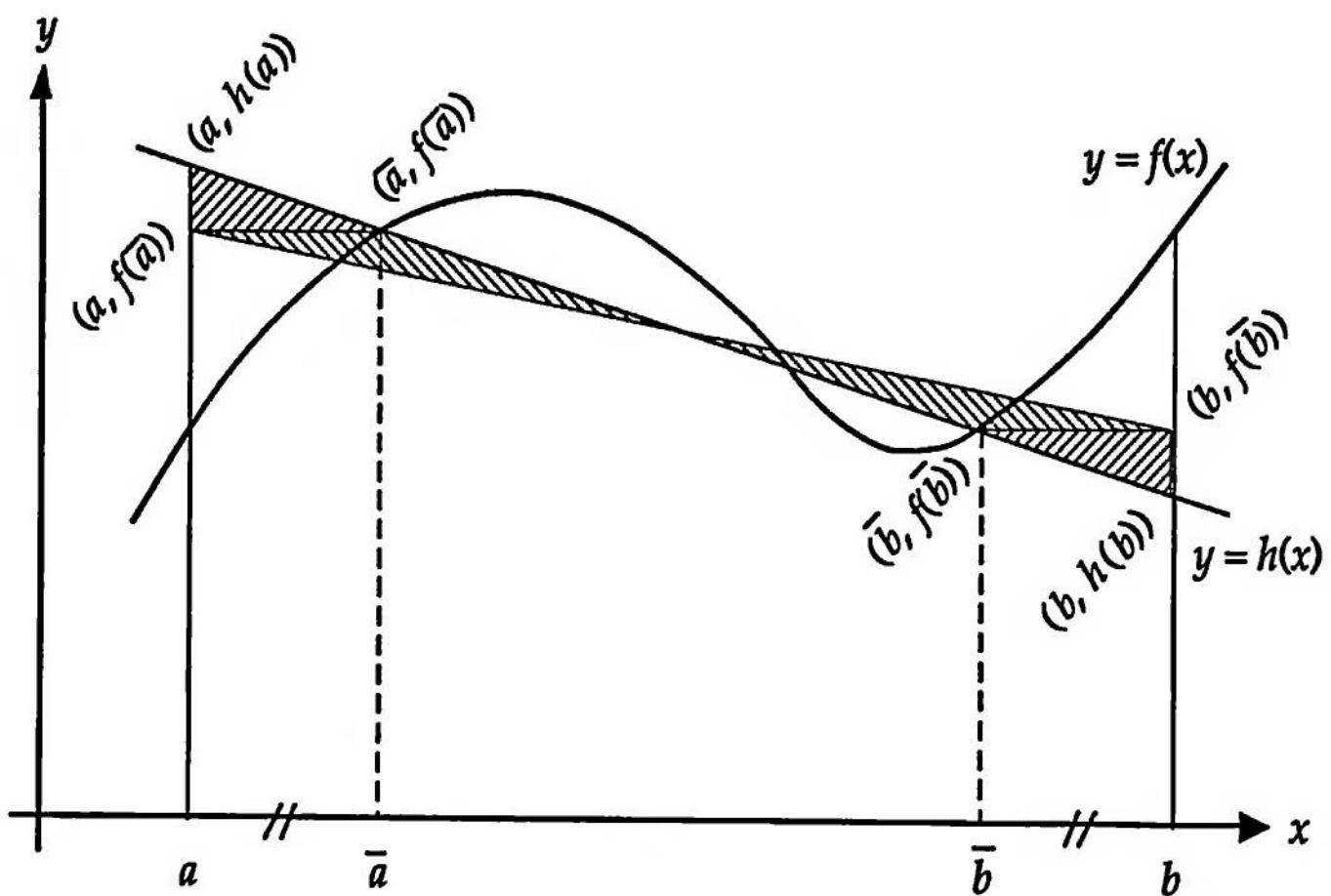
## The Characteristic Polynomials of $AB$ and $BA$ are Equal

$$-\lambda^n |AB - \lambda I| = \left| \begin{pmatrix} A & AB - \lambda I \\ \lambda I & 0 \end{pmatrix} \right| = \left| \begin{pmatrix} A & I \\ \lambda I & B \end{pmatrix} \begin{pmatrix} I & B \\ 0 & -\lambda I \end{pmatrix} \right| = \left| \begin{pmatrix} A & I \\ \lambda I & B \end{pmatrix} \right| (-\lambda)^n$$

$$-\lambda^n |BA - \lambda I| = \left| \begin{pmatrix} 0 & \lambda I \\ BA - \lambda I & \lambda B \end{pmatrix} \right| = \left| \begin{pmatrix} A & I \\ \lambda I & B \end{pmatrix} \begin{pmatrix} -I & 0 \\ A & \lambda I \end{pmatrix} \right| = \left| \begin{pmatrix} A & I \\ \lambda I & B \end{pmatrix} \right| (-\lambda)^n$$

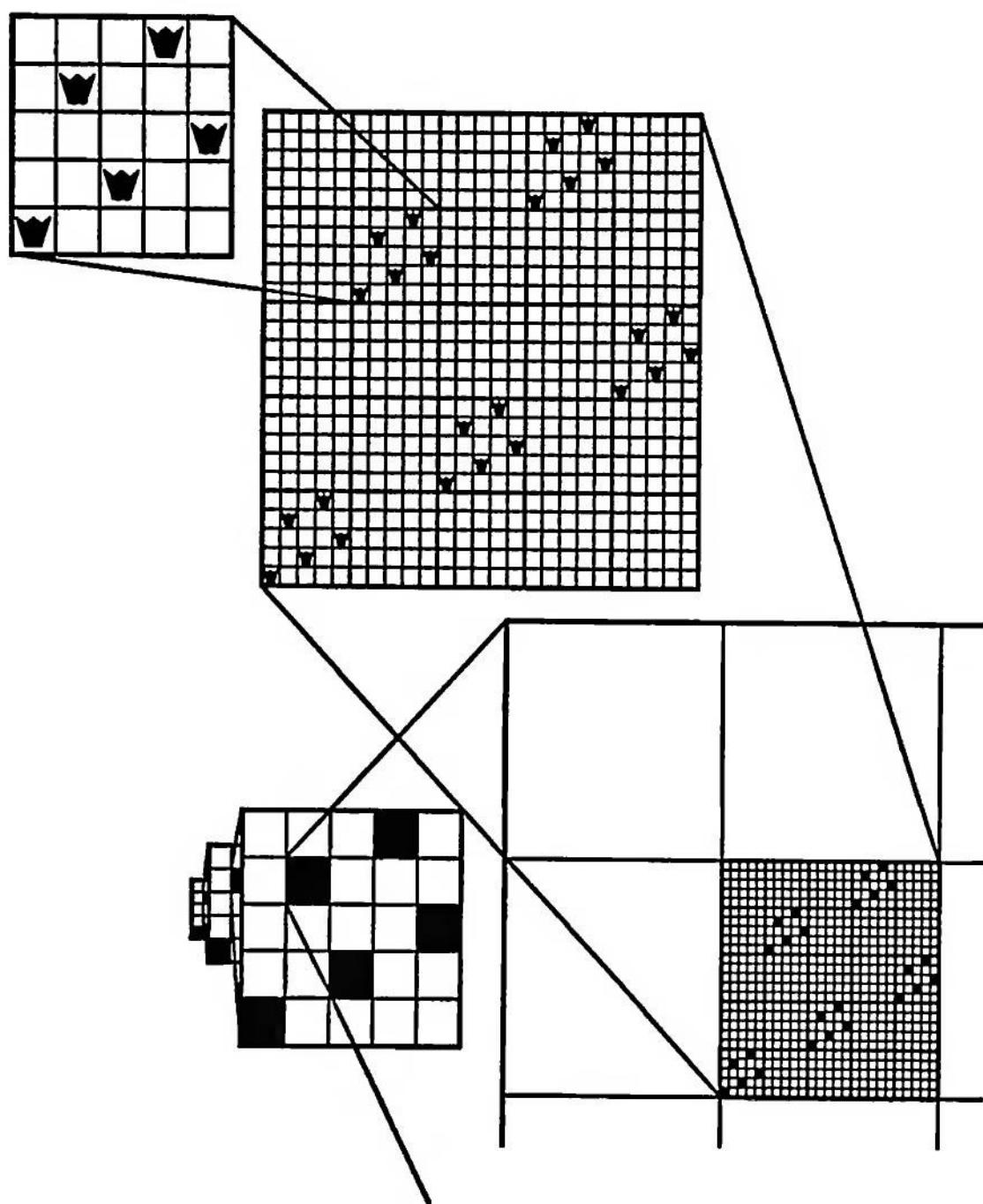
## The Gaussian Quadrature as the Area of Either Trapezoid

$$\frac{1}{2}(b - a)(f(\bar{a}) + f(\bar{b})) = \frac{1}{2}(b - a)(h(a) + h(b))$$



—Mike Akerman

## Inductive Construction of an Infinite Chessboard with Maximal Placement of Nonattacking Queens



### REFERENCES

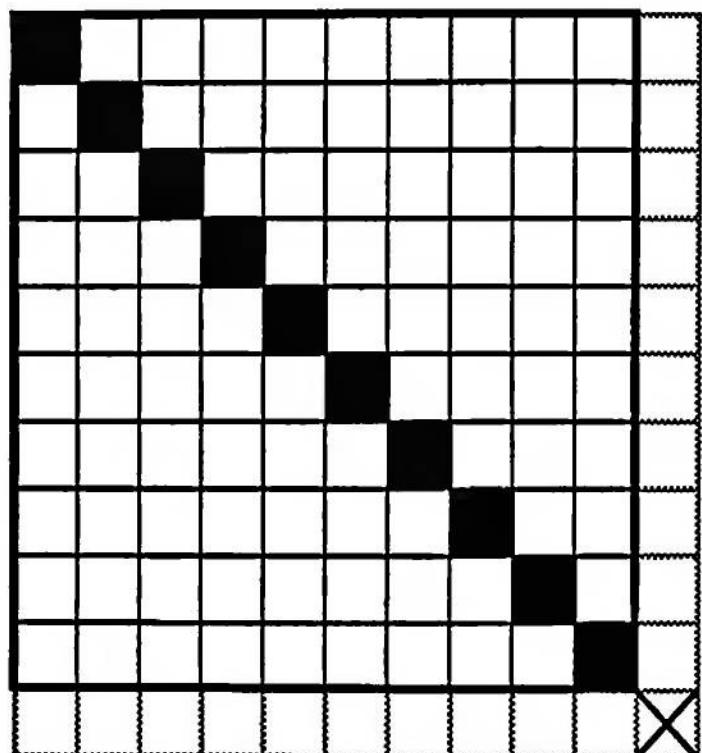
1. Dean S. Clark and Oved Shisha, Invulnerable Queens on an Infinite Chessboard, *Annals of the New York Academy of Sciences, The Third International Conference on Combinatorial Mathematics*, 1989, 133-139.
2. M. Kraitchik, *La Mathématique des Jeux ou Récréations Mathématiques*, Imprimerie Stevens Frères, Bruxelles, 1930, 349-353.

—Dean S. Clark and Oved Shisha

## Combinatorial Identities

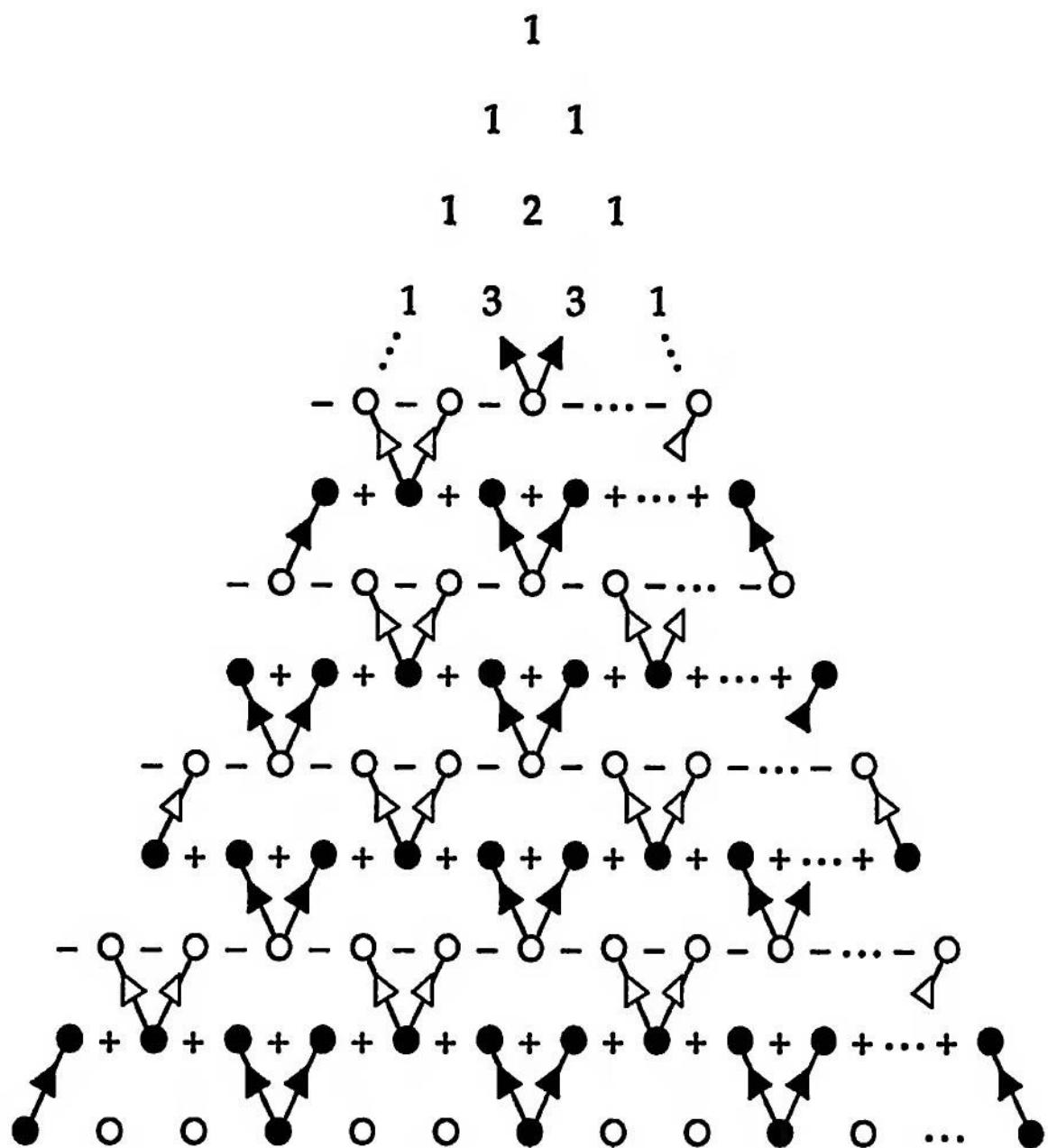
$$\binom{n}{2} = \frac{1}{2}(n^2 - n) = \sum_{i=1}^{n-1} i$$

$$\binom{n+1}{2} = \binom{n}{2} + n$$



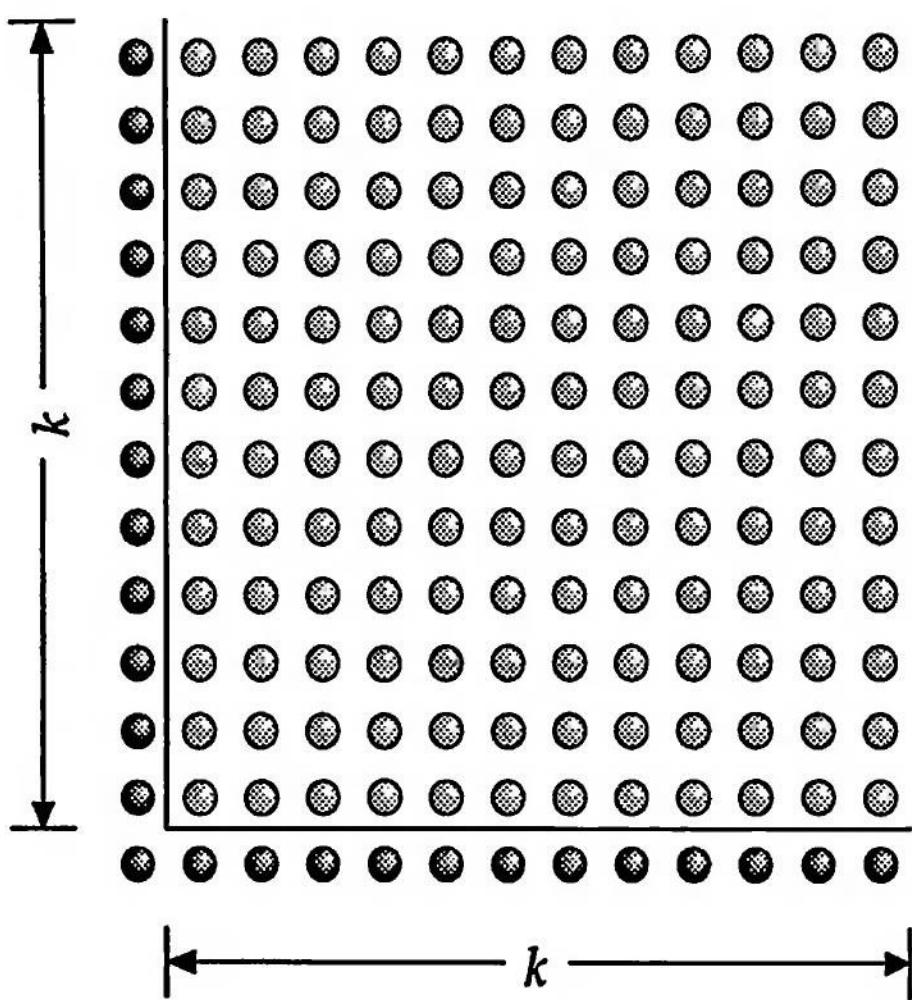
—James O. Chilaka

$3 \sum_{j=0}^n \binom{3n}{3j} = 8^n + 2(-1)^n$ , by Inclusion-Exclusion in Pascal's Triangle



$$\sum_{j=0}^n \binom{3n}{3j} = \sum_{j=1}^{3n-1} (-1)^{j-1} 2^{3n-j} = -2^{3n} \sum_{j=1}^{3n-1} \left(-\frac{1}{2}\right)^j = \frac{8^n + 2(-1)^n}{3}.$$

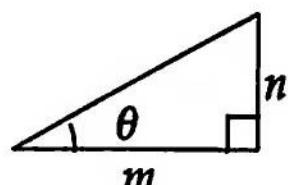
# The Existence of Infinitely Many Primitive Pythagorean Triples



$$n^2 = 2k+1 \Rightarrow k^2 + n^2 = (k+1)^2 \quad \& \quad (k, k+1) = 1$$

—Charles Vanden Eynden

## Pythagorean Triples via Double Angle Formulas

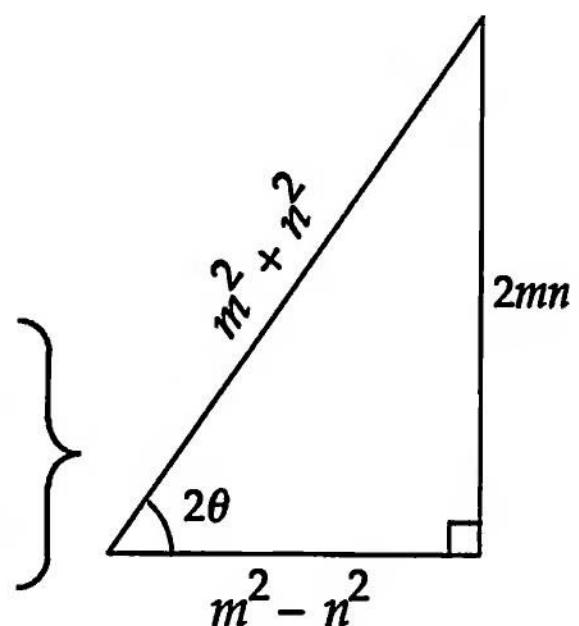


$$\begin{aligned}m &> n > 0 \\ m, n &\in \mathbb{I}\end{aligned}$$

$$\left\{ \begin{array}{l} \sin \theta = \frac{n}{\sqrt{m^2 + n^2}} \\ \cos \theta = \frac{m}{\sqrt{m^2 + n^2}} \end{array} \right.$$

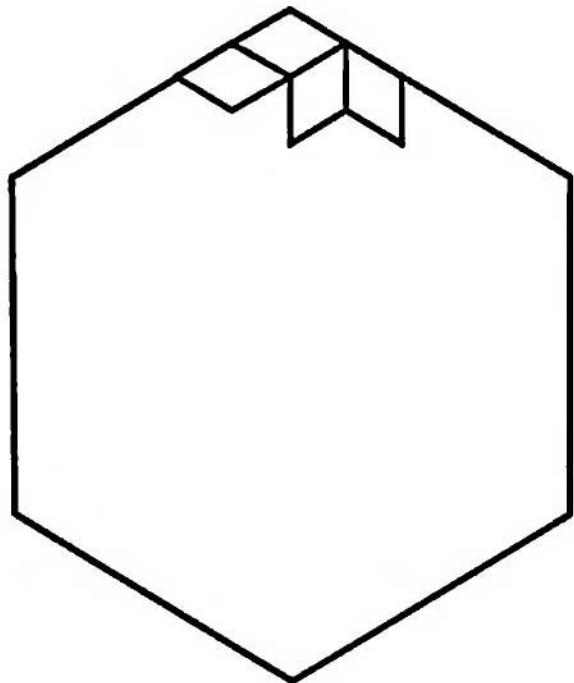
$$\sin 2\theta = \frac{2mn}{m^2 + n^2}$$

$$\cos 2\theta = \frac{m^2 - n^2}{m^2 + n^2}$$



## The Problem of the Calissons

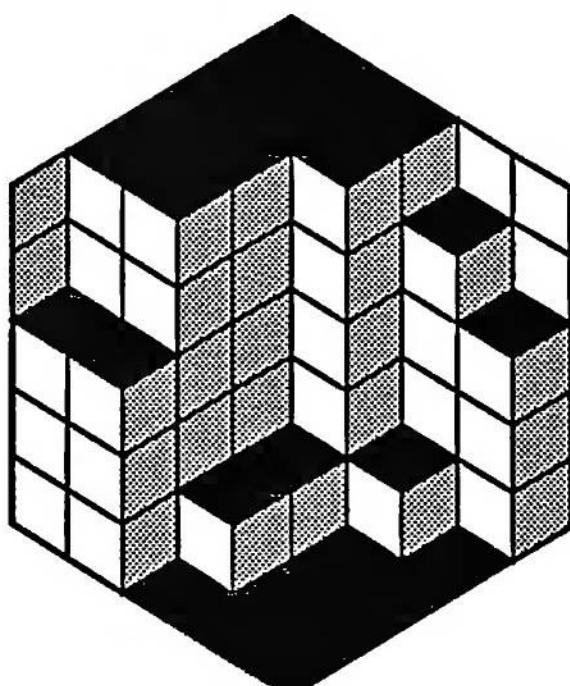
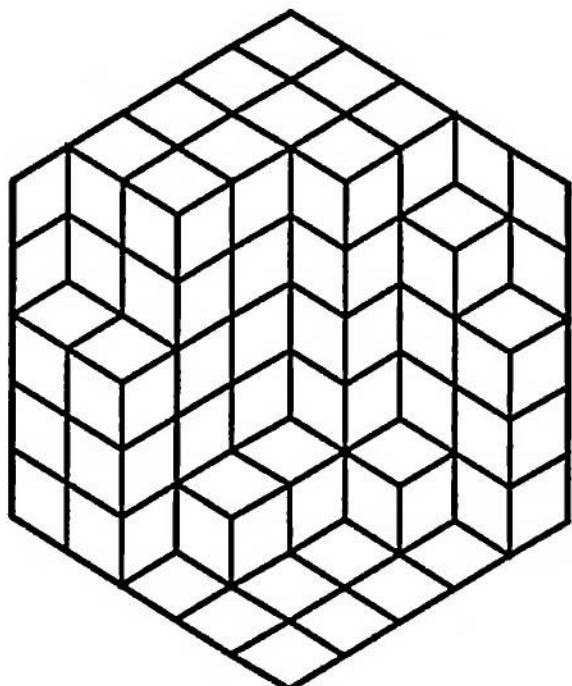
A *calisson* is a French sweet that looks like two equilateral triangles meeting along an edge. Calissons could come in a box shaped like a regular hexagon, and their packing would suggest an interesting combinatorial problem. Suppose a box with side of length  $n$  is filled with sweets of sides of length 1. The short diagonal of each calisson in the box is parallel to a pair of sides of the box.



We refer to these three possibilities by saying that a calisson admits three distinct orientations.

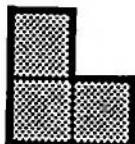
**THEOREM:** *In any packing, the number of calissons with a given orientation is one-third of the total number of calissons in the box.*

**PROOF:**

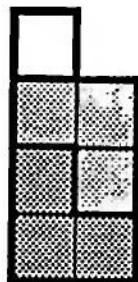


—Guy David and Carlos Tomei

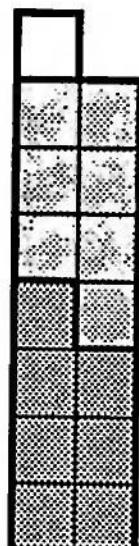
## Recursion



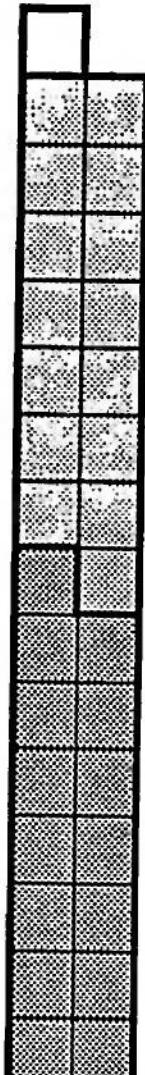
$$A_2 = 3$$



$$A_3 = 2A_2 + 1$$



$$A_4 = 2A_3 + 1$$

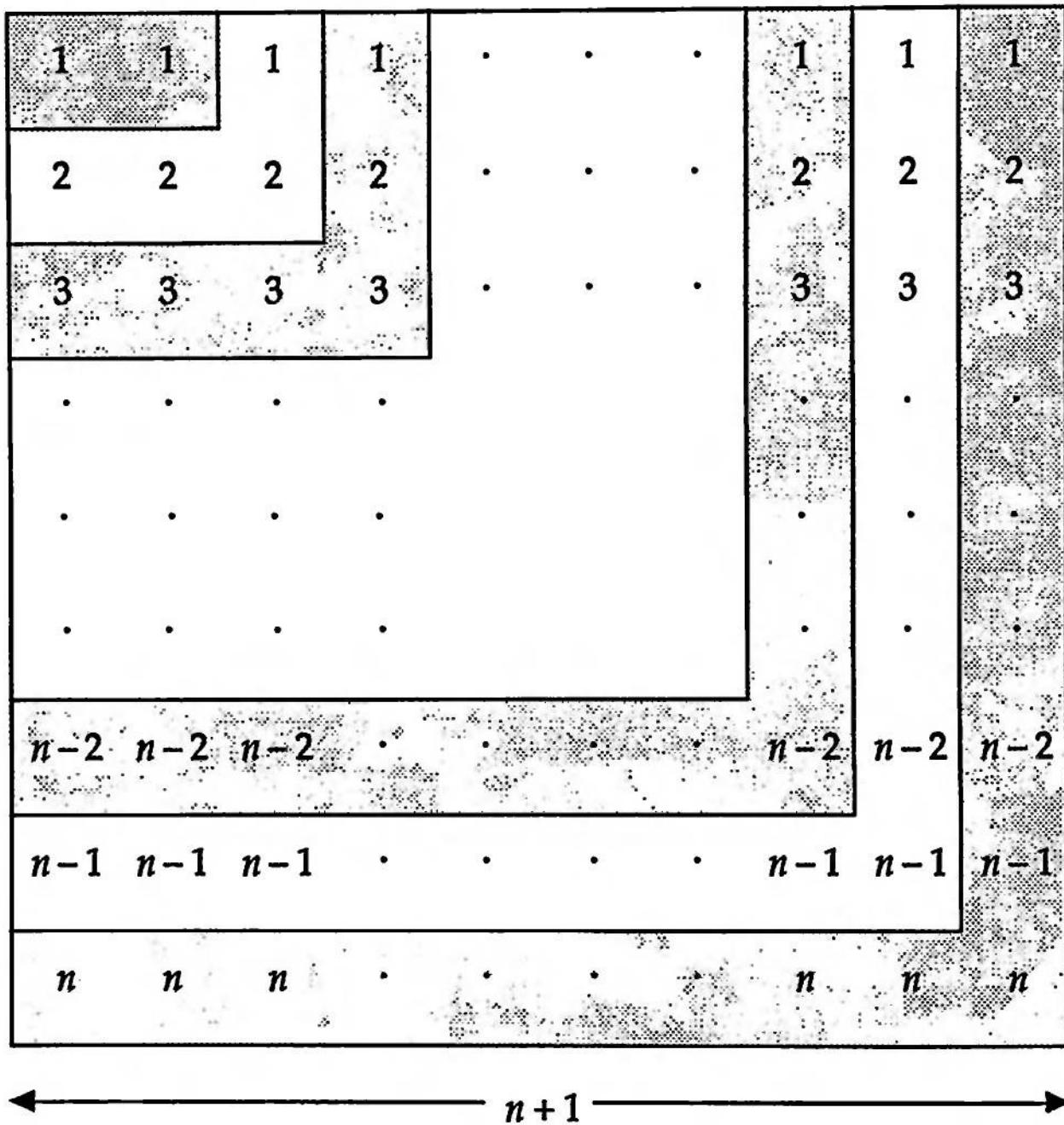


$$A_5 = 2A_4 + 1$$

$$A_2 = 3 \text{ & } A_n = 2A_{n-1} + 1 \Leftrightarrow A_n = 2(2^{n-1}) - 1 = 2^n - 1$$

—Shirley Wakin

$$\prod_{k=1}^n k^k \cdot k! = (n!)^{n+1}$$



—Edward T. H. Wang

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# PROOFS WITHOUT WORDS

## EXERCISES IN VISUAL THINKING



Roger Nelsen received his Ph.D. in Mathematics from Duke University. Since 1969 he has taught at Lewis and Clark College, in Portland Oregon, where he is a professor of mathematics. He is currently an Associate Editor of the "Problems and Solutions" section of the *College Mathematics Journal*.

*Journal*. His expository and research papers in mathematics have appeared in the *American Mathematical Monthly*, *Mathematics Magazine*, *The College Mathematics Journal*, *Nature*, *Journal of Applied Probability*, *Communications in Statistics*, *Probability Theory and Related Fields*, *Statistics and Probability Letters*, *Sankhyā*, and the *Journal of Nonparametric Statistics*. His main research interests are in probability and mathematical statistics.

Just what are "proofs without words?" First of all, most mathematicians would agree that they certainly are not "proofs" in the formal sense. Indeed, the question does not have a simple answer. But, as you will see in this book, proofs without words are generally pictures or diagrams that help the reader see why a particular mathematical statement is true, and also to see how one could begin to go about proving it true. While in some proofs without words an equation or two may appear to help guide that process, the emphasis is clearly on providing visual clues to stimulate mathematical thought. Proofs without words bear witness to the observation that often in the English language to see means to understand, as in "to see the point of an argument."

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The proofs in this collection are arranged by topic into six chapters: Geometry and Algebra; Trigonometry, Calculus and Analytic Geometry; Inequalities; Integer Sums; Sequences and Series, and Miscellaneous. Teachers will find that many of the proofs without words in this collection are well suited for classroom discussion and for helping students to think visually in mathematics.



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