Inverse functions

Inverse functions

Let A and B be sets and let $f \subset A \times B$ be a function ($f : A \to B$ in the alternative notation). Since f is a relation, one can consider the inverse relation f^{-1}.

Sometimes the inverse relation f^{-1} *is* a function, and sometimes it *is not* a function.

Examples

Let R be the relation $\{(x, x^2) : x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}$.

- R is a function because for every $x \in \mathbb{R}$ there is a unique $y = x^2$ in $\mathbb{R}$ so that $(x, y) \in R$.

- R^{-1} is *not* a function because both $(1, -1)$ and $(1, 1)$ are in R^{-1}.

Example

Let R be the relation $\{(x, \frac{1}{1+x^2}) : x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}$.

- R is a function.

- R^{-1} is *not* a function because $0 < \frac{1}{1+x^2} \leq 1$ for all x, and therefore there is no pair $(x, y) \in R^{-1}$ with $x = 2$.

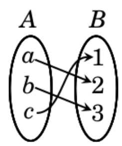
Examples

Let R be the relation $\{(x, x^3) : x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}$.

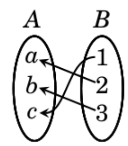
- R is a function because for every $x \in \mathbb{R}$ there is a unique $y = x^3$ in $\mathbb{R}$ so that $(x, y) \in R$.

- R^{-1} is also a function because for every $x \in \mathbb{R}$ there is a unique $y = x^{1/3}$ for every $x \in \mathbb{R}$ so that $(x, y) \in R^{-1}$.

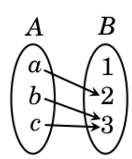
Examples (p. 239)



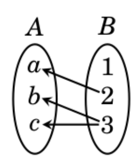
$f = \{(a,2),(b,3),(c,1)\}$



$f^{-1} = \{(2,a),(3,b),(1,c)\}$



$g = \{(a,2),(b,3),(c,3)\}$



$g^{-1} = \{(2,a),(3,b),(3,c)\}$

The Inverse Function Theorem

Theorem: Let $F \subset A \times B$ be a function. The inverse relation $F^{-1} \subset B \times A$ is also a function if and only if F is bijective.

Inverse functions (definition)

Definition: If $f : A \to B$ is bijective, then its **inverse** is the function

$$f^{-1} : B \to A.$$

We have

$$f^{-1} \circ f : A \to A = i_A.$$

and

$$f \circ f^{-1} : B \to B = i_B$$