Composition of functions

Suppose that $f : A \to B$ and $g : B \to C$ are functions. The *composition* of f and g is a new function $g \circ f : A \to C$ defined by $(g \circ f)(x) = g(f(x))$.

Composition cont'd

In terms of ordered pairs, if $f \subseteq A \times B$ and $g \subseteq B \times C$ are functions, then $g \circ f$ is the set of ordered pairs $(a, c) \in A \times C$ such that there exists $b \in B$ with $(a, b) \in f$ and $(b, c) \in g$.

Variations

- Suppose $f : A \to B$ and $g : C \to D$ are functions and $B \subseteq C$. Then we can still define $g \circ f$ by the same formula $(g \circ f)(x) = g(f(x))$.
- Suppose $f : A \to B$ and $g : C \to D$ are functions and the range of f is a subset of C. Then we can still define $(g \circ f)$ by the same formula.

A warning

Warning: $g \circ f$ means *first* f*, then* g, NOT *first* g*, then* f, which is what our normal left-to-right instincts (at least in English) might suggest.

Examples

Problem 12.4.1: Suppose $A = \{5, 6, 8\}$, $B = \{0, 1\}$, and $C = \{1, 2, 3\}$. Let $f = \{(5, 1), (6, 0), (8, 1)\} \subseteq A \times B$ and let $g = \{(0, 1), (1, 1)\} \subseteq B \times C$. Find $g \circ f$.

Examples continued

Problem 12.4.3: Let $A = \{1, 2, 3\}$ and let $f \subseteq A \times A$ be the function $f = \{(1, 3), (2, 1), (3, 2)\}$. Find $g \circ f$ and $f \circ g$.

Examples continued

Problem 12.4.9: Let $f : \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(m, n) = m + n$ and $g : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be the function $g(m) = (m, m)$. Find the formulae for $g \circ f$ and $f \circ g$.

Proposition: Suppose that $f : A \to B$, $g : B \to C$ and $h : C \to D$ are functions. Then $(h \circ g) \circ f) = h \circ (g \circ f)$. In other words, composition of functions is associative.

Theorem: Suppose $f : A \to B$ and $g : B \to C$ are functions.

- If f and g are injective, then $g \circ f$ is injective.
- If f and g are surjective, then $g \circ f$ is surjective.