Direct Proofs Example

The arithmetic/geometric mean inequality

Definition: The arithmetic mean of two real numbers a and b is $(a+b)/2$.

Definition: The geometric mean of two positive real numbers a and b is $\sqrt(ab)$.

Proposition: If a and b are positive real numbers, then the geometric mean of a and b is less than or equal to their arithmetic mean.

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$$\sqrt{(}ab) \leq \frac{(a+b)}{2}$$

Problem Solving Phase

Isolating the needed lemma

Lemma: If a and b are positive, and $a \leq b$, then $\sqrt{a} \leq \sqrt{b}$ where $\sqrt{x}$ denotes the positive square root of x.

The proof

Proposition: If a and b are positive real numbers, then:

$$\sqrt(ab) \leq \frac{(a+b)}{2}$$

Proof:

We know that $(a-b)^2 \geq 0$. Therefore $a^2 + b^2 - 2ab \geq 0$ and so $a^2 + b^2 \geq 2ab$. Add $2ab$ to both sides to obtain $a^2 + 2ab + b^2 \geq 4ab$ so $(a+b)^2 \geq 4ab$. Both sides of this inequality are positive, since the left side is a square the right side is a product of positive numbers. Now apply the lemma to take the square root of both sides to obtain

$$(a+b) \geq 2\sqrt{ab}.$$

Dividing both sides by 2 yields the desired result.