Problem 11.4.4

Problem: Suppose that P is a partition of a set A. Define a relation R on A by declaring that xRy if and only if $x, y \in X$ for some set $X \in P$. Prove that

- R is an equivalence relation on A.
- P is the set of equivalence classes of R.

Discussion: Remember that a partition P of A is a set of subsets $X \subseteq A$ such that each element of A belongs to exactly one of the subsets and the union of all of the subsets is A.

In other words, the partition divides A up into a family of disjoint subsets that together cover all of A.

Proof: An ordered pair (x, y) belongs to R if and only if x and y belong to the same set X of the partition P. We need to show that this definition yields a relation that is reflexive, symmetric, and transitive.

R is reflexive. By definition, (x, x) belongs to R if and only if x belongs to the same set X as x. This is clearly true since there is only one element x involved.

R is symmetric. Suppose that $(x, y) \in R$. This means that x and y belong to the same set $X \in P$. This condition doesn't care about the order of x and y, so R is symmetric.

R is transitive. Suppose that $(x, y) \in R$ and $(y, z) \in R$. This means there are two sets $X_1, X_2 \in P$ so that $x, y \in X_1$ and $y, z \in X_2$. But in this case, $y \in X_1 \cap X_2$, and since the elements of a partition are either disjoint or equal, we must have $X_1 = X_2 = X$. Therefore $x, z \in X$ so $(x, z) \in R$.

Next, we have to prove that the equivalence classes of the relation R are exactly the sets of the partition P. This is a question about *equality of sets* – if we let Q be the set of equivalence classes of R, we are trying to show that $Q = P$. As usual with such proofs, we need to prove that $Q \subseteq P$ and $P \subseteq Q$. In other words, we must show both of the following:

- every equivalence class of R is an element of the set P.
- every element of the set P is an equivalence class of R.

For the first, let $a \in A$ and let

$$[a] = \{x \in A : xRa\}.$$

Let X be the element of P containing a. (We know there is only one such X since P is a partition). If $z \in X$, then zRa, so $z \in [a]$. Therefore $X \subseteq [a]$. If $z \in [a]$, then zRa, so z belongs to X, so $[a] \subseteq X$. Therefore $[a] = X$. We have shown that every equivalence class is one of the sets of the partition P.

Now let $X \in P$ be one of the elements of the partition. Choose $a \in X$ and let $[a]$ be the equivalence class of a under R. If $z \in X$, then zRa so $z \in [a]$; therefore $X \subseteq [a]$. If $z \in [a]$, then zRa which means z and a belong to the same set of the partition; since $a \in X$, we have $z \in X$. Therefore $[a] \subseteq X$. Thus $[a] = X$ and so every $X \in P$ is one of the equivalence classes.

Thus the two partitions (the one given by P, and the one given by equivalence classes of R) are actually the same.