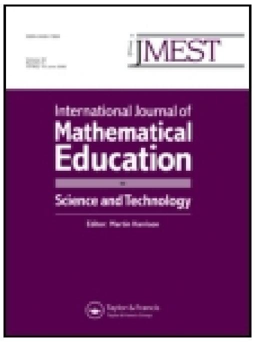


International Journal of Mathematical Educational in Science and Technology

ISSN: 0020-739X (Print) 1464-5211 (Online) Journal homepage: https://www.tandfonline.com/loi/tmes20

On the assimilation of the concept 'set' in the elementary school mathematics texts

Aron Pinker

INT. J. MATH. EDUC. SCI. TECHNOL., 1981, VOL. 12, NO. 1, 93–100

On the assimilation of the concept 'set' in the elementary school mathematics texts

by ARON PINKER
Frostburg State College, Frostburg, Maryland, U.S.A.

(*Received 12 September 1979*)

Excerpts from U.S. elementary textbooks in which the concept 'set' is incorrectly used serve as items in a questionnaire presented to students in elementary education programs. It was found that though these students had repeated instruction in sets at different academic levels, they were unable to detect incorrect uses of the concept 'set' in the excerpts. Implications of the findings are discussed.

Any survey of the conceptual framework of mathematics will result in identifying the concept of set as being of paramount importance. Writing on the foundations and fundamental concepts of mathematics, Eves and Newsom [1] say,

> "The most important and most basic term to be found in modern mathematics and logic is that of *set* or class .... The modern mathematical theory of sets is one of the most remarkable creations of the human mind. Because of the unusual boldness of some of the singular methods of proof to which it has given rise, the theory of sets is indescribably fascinating. But above this, the theory has assumed tremendous importance for almost the whole of mathematics. It has enormously enriched, clarified, extended and generalized many domains of mathematics, and its influence on the study of the foundations of mathematics has been profound."

It is not surprising, therefore, that the concept of sets was destined to become part and parcel of the school mathematics curriculum. True, many of the attributes of set theory are too advanced and too abstruse to be understood by school children. Yet, it was believed that some elementary and naïve segments of set theory do have the potential to clarify and elucidate some sections of the school mathematics curriculum, as well as provide a terminology for accurate communication.

In the mid 50s, when a concentrated effort began to improve mathematics instruction at the pre-college level, sets became a natural topic for inclusion in the 'New Math,' so much so, that the general public identified the New Math with sets, to the mathematics educator's chagrin.

Now, two decades later, can we say that the concept of sets has been well digested and assimilated? That sets have found their proper role and use? That sets are presented to the learner correctly, clearly, and effectively? Even a cursory survey of elementary texts will reveal that the situation is far from the ideal. The purpose of this paper is to identify some misconceptions about sets which prevail in currently used textbooks, and to describe the results of an experiment which aimed at discovering the sensitivity of elementary teachers to such misconceptions.

Most of the misuses of the concept of set in elementary textbooks indicate a misunderstanding of the following basic characteristics of a set:

(*a*) A set must be well defined.
(*b*) The elements of a set are distinct.
(*c*) The order in which the elements of a set are listed does not make any difference.
(*d*) When listing the elements of a set, a comma separates between two distinct elements.
(*e*) The number of subsets of a set is larger than the number of its elements.

Already George Cantor [2], the originator of set theory, explained that "By a 'set' we understand any assembly (*Zusammenfassung*) into a whole M of definite and well distinguished objects m of our perception (*Anschauung*) or thought." Though this explanation has its deficiencies, it does stress the understanding that clearly demarcated objects, subject to clear-cut criteria of identity and difference can serve as elements in a set.

A given set is well defined when for any element we can unambiguously state whether the element belongs or does not belong to the set. The following is a typical problem in which a child has to make some comparisons on sets which are not well defined.

> Write fractions to compare the number of black objects with the total number of objects.
> (1) {●●○○○}, (2) {▲▲▲▲▲△}; (3) {●●●●○○○}.

One observes that the curled brackets indicate that (1), (2), and (3) are sets. If we choose a black object in set (1) we cannot decide whether it belongs or does not belong to the set in (3). Indeed, if any black or white object is set (1) belongs to the set in (3) and vice versa, then (1) and (2) are identical exercises. We have then to understand that there is some distinction between the objects of the two sets undiscernable by the naked eye, or perhaps even some more refined tool of observation. Clearly, here the sets are not well defined.

The preceding example also illustrates the confusion regarding the distinction between the elements within a set. In (1) it seems as though the two black circles are the same, and any two white circles are the same. Thus the set in (1) contains a single black circle and a single white circle. Similarly for (2) and (3). True, we may say that the very fact of including objects in a set implies their distinctness. Such an argument would presuppose that children who are asked to solve such problems are capable of such a sophisticated *a priori* type of reasoning.

The example under discussion is also confusing with regard to the number of elements in each set. Because commas do not separate between the elements we have actually one element in each of the sets (1), (2) and (3). Yet, this is not apparently the intent of the problem.

The following brainteaser is typical of problems in which sets are confused with sequences.

Brainteaser

Anne partitioned the whole numbers into 6 sets as follows:

$A=\{0, 6, 12, 18, \ldots\}$ $\quad D=\{3, 9, 15, 21, \ldots\}$
$B=\{1, 7, 13, 19, \ldots\}$ $\quad E=\{4, 10, 16, 22, \ldots\}$
$C=\{2, 8, 14, 20, \ldots\}$ $\quad F=\{5, 11, 17, 23, \ldots\}$

(1) Anne was correct when she said that 24 and 30 were the next two numbers in set A. Can you tell why she was right?
(2) In each of the other sets, what are the next two numbers?

Since the order in which the elements of a set are listed is immaterial $\{0, 6, 12, 18, \ldots\} = \{18, 0, 6, 12, 30, \ldots\} = \{x | x = 6n \text{ and } n \in W\}$, where W is the set of whole numbers. Was Anne correct when she said that 24 and 30 were *the next* two numbers in set A? No! She should have said: "Any whole numbers which are multiples of 6 and have not yet been listed could be added to the text's list."

In sequences, the order in which the elements are listed indicates which element is first, second, etc. Even in this case Anne's answer would not have been correct because she did not state the rule according to which she made the partition (namely, equivalence classes mod 6).

Explanatory statements such as:

> Symbols like *1st, 2nd, 3rd* and *first, second, third* refer to the order of the members of a set and are called *ordinals*. The *ordinal number* associated with the sixth member of a set would be 6.

only formalize the confusion of sets with sequences. The problem of finding the number of subsets each having k elements of a set which has n elements is a simple exercise in combinations. This number is

$$\binom{n}{k} = n!/(n-k)!k!.$$

For instance, to find how many sets of 7 may be formed from a set of 31 we would compute

$$\binom{31}{7} = \frac{31!}{24!\,7!} = 2629575$$

and not as in the following excerpt:

> To find how many sets of 7 may be formed from a set of 31, you may think like this:
>
> $4 \times 7 = 28 \quad 5 \times 7 = 35$
>
> 31 is between 28 and 35.
> There are more than 4 sets of 7 in 31,
> but not enough for 5 sets of 7.
>
> $31 = (4 \times 7) + 3$
> A set of 31 forms 4 sets of 7 with a remainder of 3.

That subsets of a set need not be disjoint is largely neglected in elementary texts though the concept of disjoint sets is usually mentioned. It is easy to see that if we insist on taking disjoint sets the procedure suggested in the excerpt becomes a valid one. Similarly, neglect to mention that A and B *must* be disjoint sets, faults the following instruction to the teacher.

Illustrate joining set A to set B. Then review with pupils that joining set A to set B can be undone by removing the set A from $A \cup B$.

Numerous variants on these five 'set themes' can be found in almost every elementary mathematics textbook. The question then arises whether the elementary school teacher is sensitive enough to discover these flaws. To this aim the following experiment was conducted.

Sixty-two students in the elementary education programme of a four-year liberal arts college, who were enrolled in mathematics courses specifically designed for them, were asked to respond to the following questionnaire.

Analysis of Elementary Mathematics Textbook Content

Questionnaire

Instructions: The following are excerpts from mathematics textbooks which are currently used in the elementary schools. Analyse these excerpts according to the following criteria:

(1) Incorrect use of the English language.
(2) Mathematically incorrect.
(3) Very clear explanation.
(4) Nice graphical or pictorial display.
(5) Well posed problem.
(6) Confusing graphical or pictorial display.

On the answer sheet, in the 'Analysis' column, place *one* or *more* of the numbers 1, 2, 3, 4, 5, and 6, identifying which of the six criteria it does satisfy. Write in column "Reasons" the reasons which guided your judgment.

A. To find how many sets of 7 may be formed from a set of 31, you may think like this:

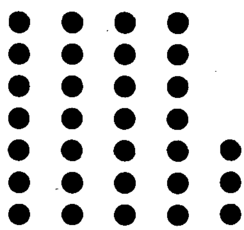
$4 \times 7 = 28 \quad 5 \times 7 = 35$

31 is between 28 and 35.

There are more than 4 sets of 7 in 31, but not enough for 5 sets of 7.

$31 = (4 \times 7) + 3$

A set of 31 forms 4 sets of 7 with a remainder of 3.

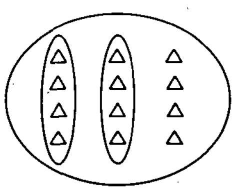


B. Write fractions to compare the number of red objects with the total number of objects.

(1) {●●○○○}; (2) {▲▲▲▲▲△}; (3) {●●●●○○○}.

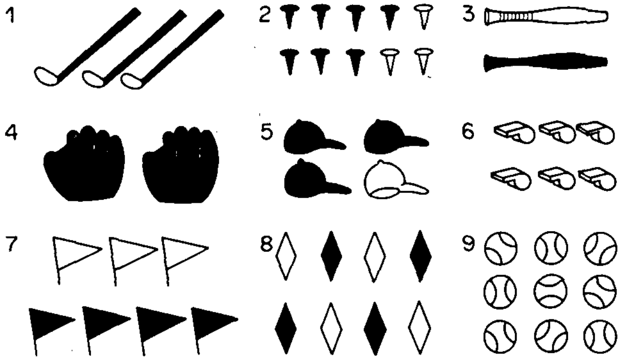
C. ● Separating sets

Study the sets and the equations.



$$\tfrac{1}{3} \times 12 = 4$$
$$\tfrac{2}{3} \times 12 = (2 \times \tfrac{1}{3}) \times 12$$
$$= 2 \times (\tfrac{1}{3} \times 12)$$
$$= 2 \times 4$$

D. Write a fractional numeral for the shaded part of each set.



E. *Prebook activities*. Illustrate joining set A to set B. Then review with pupils that joining set A to set B can be undone by removing the set A from $A \cup B$. Then join 4 sets of 3 members each to form a set of 12. Ask: How can we undo this joining so that we will again have 4 sets of 3 members each? (*Separate the set of 12 into 4 equivalent subsets of 3 members each.*)

F. Ratio and per cent.
The ratio of △'s to □'s for the two sets below can be expressed as follows:

{△△} {□□□□□□}
2 to 6 2 : 6 $\frac{2}{6}$

G. Match the members of the first set to the members of the second set. Express this matching as a ratio in three different ways.

{■■■■}, {☆☆☆}

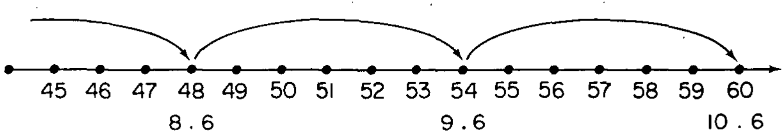
H. If a set has 7 members, — will be the ordinal of the middle member.

I. A fraction is a numeral for a fractional number. A fractional number compares a subset with a set. There are many fractions for each fractional number.

J. A function is a set of ordered pairs together with a rule. If we know the first member of a pair, the rule tells us how to find the second member.

K. Anne partitioned the whole numbers into 6 sets as follows:
$A = \{0, 6, 12, 18, \ldots\}$ $D = \{3, 9, 15, 21, \ldots\}$
$B = \{1, 7, 13, 19, \ldots\}$ $E = \{4, 10, 16, 22, \ldots\}$
$C = \{2, 8, 14, 20, \ldots\}$ $F = \{5, 11, 17, 23, \ldots\}$
(1) Anne was correct when she said that 24 and 30 were the next two numbers in set A. Can you tell why she was right?
(2) In each of the other sets, what are the next two numbers?

L.



To find how many sets of 6 may be formed from a set of 55, we may count by sixes on the number line as above. 55 is between 9×6 and 10×6. There are 9 sets of 6 in 55 and 1 remaining. 9 is the quotient and 1 is the remainder.

M. Write a fractional numeral for the shaded part of each set.

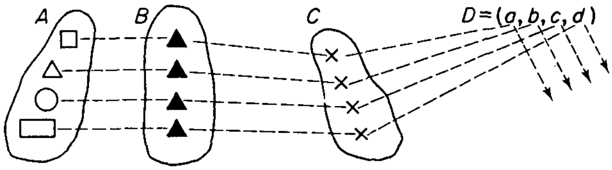
(*a*) △△△ (*b*) ○○●○ (*c*) ■■■
△△△ ●○○● ■■

N. When we join subsets we add the fractional numbers. When we separate subsets we subtract the fractional numbers.

O. Supply the missing members in the following sets, where each number is 100 greater than the one before it.

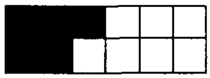
(a) $\{8, 108, 208, \ldots 908, 1008\}$
(b) $\{17, 117, 217, \ldots 1017, 1117\}$
(c) $\{21, 121, 221, \ldots 1321, 1421\}$

P. The number of a set



Sets A, B, C, and D are called *equivalent sets.* Describe other sets that are equivalent to each set above. What do all of these sets have in common?

Q. Name the fractional numbers shown by the red subsets and region.

(1) {● ● ○ ○ ○ ○} (2) {▼ ▲ ▼ ▲ ▼ △ ▽ △} (3) 

R. Symbols like *1st, 2nd, 3rd* and *first, second, third* refer to the order of the members of a set and are called *ordinals.* The *ordinal number* associated with the sixth member of a set would be 6.

All these students have been exposed to the concept 'set' in their high school and college studies, and perhaps even in the elementary school. The questionnaire was presented to them some time after they had studied sets in the college mathematics course. Since this (two-semester) mathematics course for elementary education majors is the only one they are required to take for graduation, it was assumed that they would adequately represent the mathematics level of an average elementary school teacher. Being asked to respond to this questionnaire so soon after their encounter with sets would perhaps make them more sensitive than 'oldtimers' to any misrepresentations of the concept.

Students were given 30 minutes to complete the questionnaire and were asked to make some explanatory remark for each item. The questionnaire consisted of 18 excerpts from a number of elementary school mathematics textbooks. These were reproduced (except for colour) as they appear in the textbook. The background story for the questionnaire was that as future teachers they would be required to express opinions on the texts they are using. The questionnaire represents a hypothetical sample which they have to judge according to the given criteria.

The experiment aimed at determining the proportion of the responses 2. Since any questionnaire is amenable to misinterpretations, it was felt that the remarks would allow a better determination of the respondent's intentions. For each item in the questionnaire more then one response was allowed. This perhaps dissipated the suspicion that the students are tested on some mathematical topic, and well fits the background story.

The results obtained are given in table 1.

Only in two cases did about 25 per cent of the students think the excerpts were mathematically incorrect, in items I and R. The reasons given for 'I' being incorrect were: "Very confusing," "Definition didn't tell one thing—doesn't sound right to me," "In defining a word you cannot put the word in the definition." Indeed, this definition *is* confusing because the criterion by which a subset is compared with a set is not specified, and perhaps it does not provide for fractions larger than 1. In this case, "confusing" was felt by some as being equivalent to "mathematically incorrect." The reason given for 'R' being incorrect were almost uniformly: "The ordinal number should be 6th not 6." Obviously, none of the respondents sensed that in a set order is immaterial and therefore the explanation is incorrect.

For all the other items, the proportion of responses '2,' was below 16 per cent.

Table 1. Percent of students who gave the corresponding letter response.

Excerpt letter	1 %	2 %	3 %	4 %	5 %	6 %
A	23	5	50	47	26	16
	(14)	(3)	(31)	(29)	(16)	(10)
B	16	5	44	50	47	11
	(10)	(3)	(27)	(31)	(29)	(7)
C	0	8	13	13	16	68
	(0)	(5)	(8)	(8)	(10)	(42)
D	10	2	32	47	34	44
	(6)	(1)	(20)	(29)	(21)	(27)
E	44	6	16	2	23	15
	(33)	(4)	(10)	(1)	(14)	(9)
F	5	16	65	56	40	11
	(3)	(10)	(40)	(35)	(25)	(7)
G	19	11	31	40	31	23
	(12)	(7)	(19)	(25)	(19)	(14)
H	48	2	5	2	15	26
	(30)	(1)	(3)	(1)	(9)	(16)
I	61	26	15	0	2	35
	(38)	(16)	(9)	(0)	(1)	(22)
J	24	5	40	5	8	23
	(15)	(3)	(25)	(3)	(5)	(14)
K	16	8	42	24	52	34
	(10)	(5)	(26)	(15)	(32)	(21)
L	8	8	32	32	44	26
	(5)	(5)	(20)	(20)	(27)	(16)
M	3	2	15	61	40	13
	(2)	(1)	(9)	(38)	(25)	(8)
N	11	10	48	8	10	10
	(7)	(6)	(30)	(5)	(6)	(6)
O	10	3	48	29	32	34
	(6)	(2)	(30)	(18)	(20)	(21)
P	3	5	24	35	32	34
	(2)	(3)	(15)	(22)	(20)	(21)
Q	11	3	11	16	13	23
	(7)	(2)	(7)	(10)	(8)	(14)
R	5	24	32	18	16	3
	(3)	(15)	(20)	(11)	(10)	(2)

N=62, (x) is the absolute number of students who gave a particular response.

Students' insensitivity to mathematical correctness of the questionnaire items was further revealed in their positive assertions. Thus, item K was considered by 52 per cent to be a well-posed problem, though in it sets are confused with sequences. Also, 47 per cent thought that item *B* was well-posed, though it is not clear relative to what the comparison has to be made and what are the elements in the set.

Many (68 per cent) felt that item C was graphically confusing for a variety of reasons, such as: it does not distinguish between 1/3 and 2/3 of 12; complicated equation; mathematically incorrect, since $2/3 \times 12 \neq (2 \times 1/3) \times 12$; does not indicate what child is supposed to know, etc. Very few felt that the set language makes it senseless.

Students' remarks reveal a high degree of confusion as to the aims of each of the items as mathematics exercises for children, and their own mathematical immaturity. They simply do not understand nor can they guess what are the writer's objectives.

This pilot study indicates that in spite of a high measure of exposure to sets in school and college, future elementary teachers are essentially insensitive to abuses of the concept of set as found in many elementary textbooks. Authors cannot rely on the elementary teacher's understanding and ability to supplement the text and correct its errors and inadequacies.

If sets are to be kept in the elementary curriculum, authors have to realize that there is a great distinction between the conversational concept 'set' and the mathematical concept of 'set,' and have to adhere to this distinction in all the arts and explanations. Further, the instructions to the teachers have clearly to state and demonstrate what is the goal and how it is achieved. Methods instructors may need to include the discovery of abuses of sets in elementary texts as assignments in their courses.

It is quite clear that the concept of set is not an easy one, and that it was not well assimilated in spite of a protracted effort. Would it not be sensible to limit its use to cases where it is of some significance? Should it be introduced in the elementary school at all?

References

[1] EVES, H., and NEWSOM, C. V., 1958, *An Introduction to the Foundations and Fundamental Concepts of Mathematics*, p. 226.

[2] CANTOR, G., 1962, *Gesammette Abhandlungen* (Hildesheim), p. 282, translation.

Additional references

VAUGHAN, 1970, What sets are not, *Arithmetic Teacher*, January, 55.

SANDERS, W., 1966, Cardinal numbers and sets, *Arithmetic Teacher*, January, 26.