Set Proofs Continued

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Inclusion

Basic principle: $A \subseteq B$ is equivalent to the statement $x \in A \implies x \in B$. One can prove this both directly and as $x \notin B \implies x \notin A$.

Proposition: Let $A = \{4x + 2 : x \in \mathbb{Z}\}$. Let $B = \{2x : x \in \mathbb{Z}\}$. Then $A \subseteq B$.

$A = \{4x+2 : x \in \mathbb{Z}\}$

$-2 \quad 2 = 4 \cdot 0 + 2$

$-6 \quad 6$

$-10 \quad 10$

$\vdots \quad 14$

$B = \{2x : x \in \mathbb{Z}\}$

$\ldots -4, -2, 0, 2, 4, 6, 8, \ldots$

Is $A \subseteq B$?

if $x \in A$, then $x \in B$.

if $x = 4k+2$, $k \in \mathbb{Z}$ then $x = 2y$, $y \in \mathbb{Z}$

Note $x = 2(2k+1)$

So $x = 2(y)$ where $y = 2k+1$

therefore $x \in B$

More examples

Proposition: For all $k \in \mathbb{Z}$, let $A = \{n \in \mathbb{Z} : n|k\}$ and let $B = \{n \in \mathbb{Z} : n|k^2\}$. Then $A \subseteq B$. (Note: this is problem 3 on page 171.)

Let $A = \{n \in \mathbb{Z} : n|k\}$

$B = \{n \in \mathbb{Z} : n|k^2\}$

k is fixed at the beginning

$\boxed{\text{For all } k \in \mathbb{Z}, \quad n|k \Rightarrow n|k^2}$

Since $n|k$, $k = nd$.

$k^2 = n^2 d^2 = n(nd^2)$

$= ny$ where $y = nd^2$.

so $n|k^2$.

this shows $A \subseteq B$.

More examples

Proposition: Suppose A, B, and C are sets. If $B \subseteq C$, then $A \times B \subseteq A \times C$. (This is problem 7 on page 171.)

Suppose A, B, C are any sets.
if $B \subseteq C$, then $A \times B \subseteq A \times C$.

if $[x \in B \Rightarrow x \in C]$ then $[x \in A \times B \Rightarrow x \in A \times C]$

Proof: Assume $x \in B \Rightarrow x \in C$. $(B \subseteq C)$.
and we assume $x \in A \times B$.
$x = (a, b)$ where $a \in A$ and $b \in B$.
We must show $(a, b) \in A \times C$.
in other words, ~~(a, b)~~ we must show
that $(a, b) = (a', c')$ where $a' \in A$
$c' \in C$
Since $b \in B$, $b \in C$ by hypothesis $B \subseteq C$. Therefore $c' = b$
$a' = a$
$(a, b) \in A \times C$.

One more

Proposition: Let A and B be sets. Prove that $A \subseteq B$ if and only if $A - B = \emptyset$. (This is problem 21 on page 171.)

$A \subseteq B \iff A - B = \emptyset$

Recall: $A - B = \{x \in A : x \notin B\}$.

$x \in A - B \iff x \in A$ and $x \notin B$.

$\underbrace{[x \in A \Rightarrow x \in B]}_{P} \iff$ $\underbrace{\text{there does not exist an } x \text{ such that } x \in A \text{ and } x \notin B.}_{Q}$

$P \Rightarrow Q$: not $Q \Rightarrow$ not P.

there does exist an x such that $x \in A$ and $x \notin B$.

$\Longrightarrow$ "$x \in A \Rightarrow x \in B$" is false

$Q \Rightarrow P$: not $P \Rightarrow$ not Q

there is an $x \in A$ but not in B $\Longrightarrow$ not Q.