

Existence Proofs

Review of universal quantifiers

A theorem asserting the truth of a conditional statement is typically a “for all” statement.

Theorem: If a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable, it is continuous.

Here there is an implicit *universal quantifier*.

Theorem: For all functions $f : \mathbb{R} \rightarrow \mathbb{R}$, f differentiable implies f continuous.

Another example

Theorem: An $n \times n$ matrix A with real entries is invertible if and only if $\det(A) \neq 0$.

This is asserting that:

For all $n \times n$ matrices A with real entries, A is invertible if and only if $\det(A) \neq 0$.

Existence claims

Some theorems assert the existence of an object with particular properties.

Proof of an existence theorem generally requires you to present an example.

Definition: A Pythagorean Triple is an element (a, b, c) of \mathbb{N}^3 such that

$$c^2 = a^2 + b^2.$$

Theorem: A Pythagorean triple exists.

Proof: Let $a = 3$, $b = 4$, and $c = 5$. Then $c^2 = 25 = a^2 + b^2$.

Existence claims can be hard to establish

Theorem: There exist natural numbers A , B , and C so that

$$A/(B + C) + B/(A + C) + C/(A + B) = 4.$$

Proof: Let

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A = 154476802108746166441951315019919837485664325669565431700026634898253202035277999  
B = 36875131794129999827197811565225474825492979968971970996283137471637224634055579  
C = 4373612677928697257861252602371390152816537558161613618621437993378423467772036
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Figure 1: Big Numbers

Then these values satisfy the given equation. (Check this if you can!)

- ▶ verification requires work
- ▶ no clue given as to how to find this; and, in fact, it's hard.

Millenium Problems: Navier-Stokes

One of the million dollar millenium problems is an existence claim about solutions to the Navier Stokes equation for fluid flow.