Problem 11.3.14

Problem: Suppose that R is a reflexive and symmetric relation on a finite set A. Define a relation S on A by declaring that xSy if and only if, for some $n \in \mathbb{N}$, there are elements $x_1, \ldots, x_n \in A$ satisfying $xRx_1, x_1Rx_2, \ldots, x_{n-1}Rx_n$ and x_nRy.

1. Show that S is an equivalence relation.
2. Show that $R \subseteq S$.
3. Show that S is the unique smallest equivalence relation on A containing R.

Discussion: The idea of this problem is to show that, given a reflexive and symmetric relation R which isn't necessarily transitive, you can make a relation that is consistent with the original relation but which *is* transitive. The way you do this is to add in all the ordered pairs (x, y) that *should* be related if the relation were transitive. For example, if $(a, b) \in R$ and $(b, c) \in R$, and R were transitive, then (a, c) should be in R. In making our transitive relation, then, we keep all the ordered pairs in R and also add (a, c).

The condition that we declare xSy – meaning that we include (x, y) in S if "there exist $x_1, \ldots, x_n \in A$ so that $xRx_1, x_1Rx_2, \ldots, x_{n-1}Rx_n, x_nRy$" expresses in formal terms the idea that we include (x, y) in S if x and y *should* be related if R *were* transitive.

In particular, suppose R were in fact already transitive. Then if there were a sequence of x_i as above, the transitive property would force (x, y) to already be in R. So if R is transitive, then the S constructed in this problem would already be R.

Proof: First we prove that S is an equivalence relation. We must show that S is reflexive, symmetric, and transitive.

S is reflexive. Given $x \in A$, let $x_1 = x$ and $y = x$. Then because R is reflexive we know that xRx_1 and x_1Ry. So x_1 is a sequence of length one that meets the condition for xSx to be true.

S is symmetric. Let $x, y \in A$ and suppose xSy. Then there is a sequence $x_1, \ldots, x_n$ so that

$$xRx_1, \ldots, x_nRy$$

as in the defining property for S. Since R is symmetric, we can reverse all of these ordered pairs to obtain a sequence

$$yRx_n, \ldots, x_1Rx.$$

If we renumber the sequence $x_1, \ldots, x_n$ in reverse order, with $x'_i = x_{n-i+1}$ for $i = 1, \ldots, n$, then we have a sequence

$$yRx'_1, \ldots, x'_nRx$$

and therefore ySx.

S *is transitive.* Let $x, y, z \in A$ and suppose xSy and ySz. Then we have sequences $x_1, \ldots, x_n$ and $x'_1, \ldots, x'_m$ so that

$$xRx_1, \ldots, x_nRy$$

and

$$yRx'_1, \ldots, x'_nRz.$$

If we combine the two sequences into a long sequence of length $n+m$ where $x''_i = x_i$ for $i = 1, \ldots, n$ and $x''_i = x'_{i-n}$ then we have

$$xRx''_1, \ldots, xRx''_n, xRx''_{n+1}, \ldots, x''_{n+m}Rz$$

and so xSz. Therefore S is transitive.

Next we must show that $R \subseteq S$. In other words, we must show that if x and y are in A, and xRy, then xSy. For this, make a sequence where $x_1 = x$ and $x_2 = y$. Then xRx_1 since R is reflexive, x_1Rx_2 by hypothesis, and x_2Ry by reflexivity again. Therefore

$$xRx_1, x_1Rx_2, x_2Ry$$

gives a sequence that tells us that xSy.

Finally, we need to show that S is the *unique smallest equivalence relation on A containing* R. Here, *smallest* means that any other equivalence relation that contains R also contains S. In other words, if you want to make R transitive, the very least you can do is add the relations that create S. So we must show that if T is an equivalence relation that contains R, then $S \subseteq T$.

Suppose therefore that $(x, y) \in S$. This means that there is a sequence $x_1, \ldots, x_n$ so that

$$xRx_1, \ldots, x_nRy$$

as in the defining property for S. Since $R \subseteq T$, we have

$$xTx_1, \ldots, x_nTy$$

and since T is transitive, this means that xTy. Therefore $(x, y) \in T$ and we have shown that $S \subseteq T$.

Finally, we must show that S is the *unique* smallest equivalence relation. This means that if S and S' are two equivalence relations containing R, and having the property that, if T is another equivalence relation containing R, then $S \subseteq T$ and also $S' \subseteq T$, then $S = S'$. But if S has this property, it means that $S \subseteq S'$ since S' is an equivalence relation; and if S' has this property it means that $S' \subseteq S$. Therefore indeed $S = S'$.