Proof by contradiction

Proposition: My dog is not an elephant.

Proof: Suppose my dog is an elephant.

Elephants have trunks, so my dog has a trunk

My dog does not have a trunk.

This is a contradiction.

So my dog is NOT an elephant.

Proposition: The square root of 2 is not a rational number.

$$(\sqrt{2})^2 = 2.$$

$\sqrt{2}$ a rational number means

$$\sqrt{2} = \frac{a}{b}$$ where a, b are integers and $b \neq 0$.

$$2 = \frac{a^2}{b^2}$$

Therefore $a^2 - 2b^2 = 0$.

If $\sqrt{2}$ is rational then $a^2 - 2b^2 = 0$ has a solution where $a, b \in \mathbb{Z}$ and $b \neq 0$.

Lemma: If a^2 is even, then a is even. $(a \in \mathbb{Z})$

$(\sim Q \Rightarrow \sim P)$

Proof: We will prove the contrapositive, which says that if a is odd, then a^2 is odd. Suppose a is odd. Then $a = 2k + 1$ for some k. $k \in \mathbb{Z}$.
Therefore $a^2 = (2k+1)^2 = 4k^2 + 4k + 1$ is odd.

Suppose a is odd

a odd $\Rightarrow$ a^2 odd

a even $\Leftarrow$ a^2 even

Proof of Proposition: Suppose that $\sqrt{2}$ is a rational number. Then we can find positive integers a and b with $b \neq 0$ so that

$$2 = \left(\frac{a}{b}\right)^2. = a^2/b^2$$

We can choose a, b NOT BOTH EVEN. We can put a/b in lowest terms. Either a or b is odd.

$$a^2 = 2b^2$$

so a^2 is even

Lemma: ~~tells~~ tells a is even.

so b is odd.

On the other hand, $a = 2m$ for some $m \in \mathbb{Z}$.

$$(2m)^2 = 2b^2$$

$$4m^2 = 2b^2$$

$$2m^2 = b^2$$

So b^2 is even.

By lemma, b is even.

(b is odd and b is even)

That is a contradiction.

Proof of Proposition: Suppose that $\sqrt{2}$ is a rational number. Then we can find positive integers a and b with $b \neq 0$ so that

$$a^2 - 2b^2 = 0.$$

a, b are both positive (not zero).

Among all solutions, choose (a_0, b_0) where a_0 is as small as possible.

$a_0^2 = 2b_0^2$ so a_0^2 is even.

Lemma: a_0 is even

$a_0 = 2m$ for some $m \in \mathbb{Z}$.

$m < a_0$

$4m^2 = 2b_0^2$

$2m^2 = b_0^2$

b_0^2 is even, so b_0 is even, $b_0 = 2n$

$2m^2 = 4n^2$

$m^2 = 2n^2$

m, n solves $a^2 = 2b^2$

$m < a_0$.

Therefore we have a contradiction so $\sqrt{2}$ is not rational

Logical Structure of proof by contradiction

- A contradiction is a statement of the form (C and $\sim C$) which is *always false.*

- The strategy of proof by contradiction is that if $A \implies B$ is true, and B is false, then A is false.

Proposition: $P \implies Q$.

- Assume P is true.
- Assume (P and $\sim Q$) is true and

$$P \text{ and } \sim Q \implies C \text{ and } \sim C$$

  for some statement C.
- If (P and $\sim Q$) implies (C and $\sim C$) is a true implication yielding a false conclusion, then the hypothesis must be false.
- Therefore (P and $\sim Q$) is false.
- If (P and $\sim Q$) is false, and P is true then $\sim Q$ is false
- Q is true.