Functions via set theory

A typical "function" is given by a formula of the form

$$\underline{f(x) = \sin(x)}$$

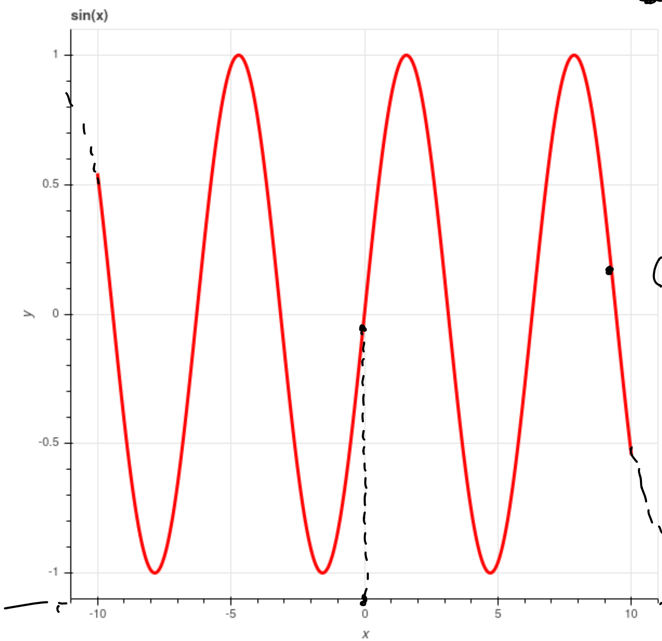
Rule: plug in $x \to \sin(x)$

$f : \mathbb{R} \longrightarrow \mathbb{R}$

domain codomain

and we visualize it with its graph:

$\mathbb{R}$



$\mathbb{R}$

$\mathbb{R} \times \mathbb{R} = \{(x,y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$

$(x, f(x)) = (x, \sin(x))$

graph of f = red points $= \{(x,y) \mid y = \sin(x)\}$

Figure 1: sin graph

Functions as (special) relations

The key insight in abstracting the idea of "function" is to understand what the graph of a function really is.

If $f : A \to B$ is a function, then the graph of f is the set of points $G(f) = \{(a, b) \in A \times B : f(a) = b\}$.

Two observations:

1. G is a relation from the set A to the set B since $G \subseteq A \times B$.
   graph of f
2. Everything we need to know about f is stored in G.

A is called the **domain** of f. B is called the **codomain** of f.

Given G. What is $f(3)$? Look for the ordered pair $(3, y) \in G$.

$y = f(3)$.

$G \subseteq A \times B$.

Functions as (special) relations continued

The key property that makes a general relation G a function is the fact that

for all $a \in A$, there exists a unique $b \in B$ so that the pair $(a, b) \in G(\cancel{f})$. (note the quantifiers here).

Notice that for a general relation, there is no such condition – *any* subset R of $A \times B$ is a relation.

A general relation vs a function

$A, B = [0, 1, 2, \ldots, 9]$

For all $a \in A$, there exists a unique $b \in B$ so that $(a,b) \in G$.

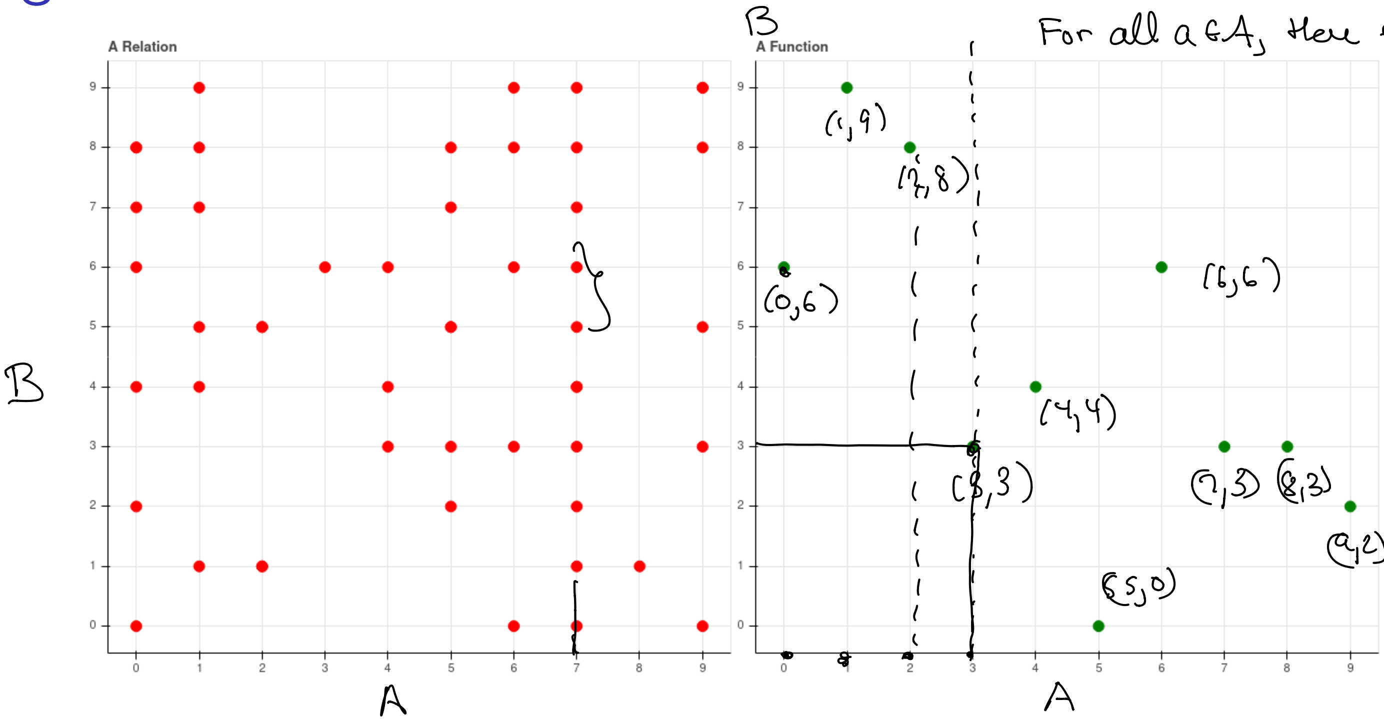


Figure 2: A relation and a function on (0..9)x(0..9)

Drawn in this way, a relation $R \subset A \times B$ is a function if it passes the *vertical line test* - every vertical line hits exactly one point in B.

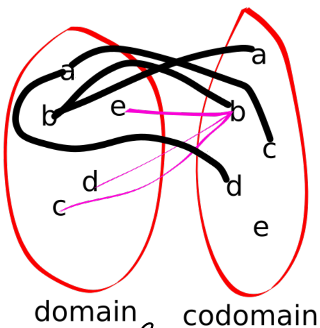
relations vs functions continued

We can also explore the special properties of functions among relations using the other way of representing functions.

Not a function:
$\exists\, a \in$ domain such that for all $b \in$ codomain. either $(a,b) \notin G$ or there exist 2 b, b' with $b \neq b'$ and $(a,b) \in G$ and (a,b')

For every $a \in$ domain, $\exists$ a unique $b \in$ codomain such that $(a,b) \in F$

A relation not a function

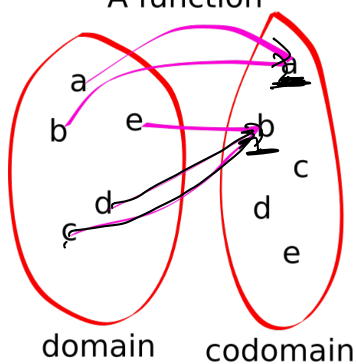


(a,d)
(a,e) $\in G$

G

NOT A FUNCTION

A function



(a,a)
(b,a)
(e,b)
(d,b)
(c,b)

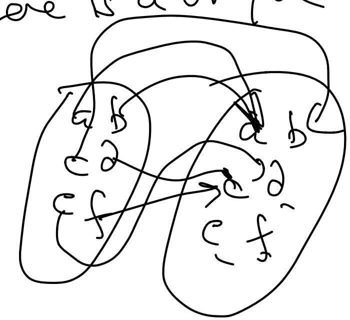
F

The range of a function

Definition: The range of a function F is the set of $b \in B$ such that there exists $a \in A$ with $(a, b) \in F$.

In "old fashioned" terms, the range of F is the set of b for which there exists a with $F(a) = b$.

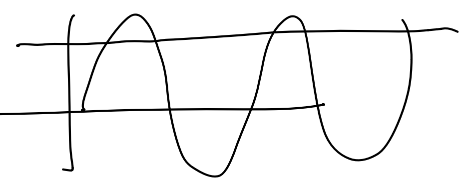
Definition: A function f is a subset of the Cartesian Product of $A \times B$ where, for all $a \in A$, there is a unique $b \in B$ such that $(a, b) \in f$.



A is called the domain of f.

B is called the codomain of f

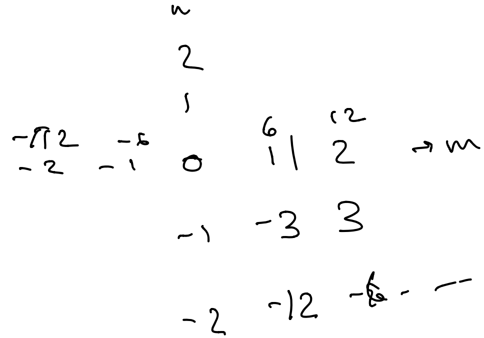
Range of $\sin(x) = [-1, 1]$



Example of the range of a function

(Example 12.3 from the book). We define $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by the formula $\phi(m,n) = 6m - 9n$. As a set, this is the function $\{(m,n), 6m - 9n\}$ as a subset of $\mathbb{Z}^2 \times \mathbb{Z}$.

What is its range?



$\varphi : (m,n) = 6m - 9n.$

$\varphi \subseteq (\mathbb{Z}\times\mathbb{Z})\times\mathbb{Z} = \{((m,n), 6m-9n\}$

$6m - 9n = \underline{3}(2m - \underline{3}n)$

Range of φ includes only multiples of 3

$\{x \mid 3|x\} \overset{?}{\subseteq} \mathrm{range}(\varphi).$

$\{6m-9n\} = \{\text{multiples of } \gcd(6,9)\}$

$=$ multiples of 3.

By Euclid's algorithm,

$\mathrm{range}(\varphi) \subseteq \{x \mid 3|x\}$

range of φ = multiples of 3.

Equality of functions

Since functions are defined to be sets, two functions are equal if they are the same set.

Proposition: If two functions F and G are equal, they have the same domain.

Proof: The set of a such that $(a, x) \in F$ is the domain of F. Since $F = G$, we know that $(a, x) \in G$, so a is in the domain of G. This proves that the domain of F is a subset of the domain of G. But the same argument shows the opposite inclusion.

Proposition: If two functions are equal, then F and G have the same range.

Proof: Let x be in the range of F. Then there exists an a in the domain of $F so that $(a, x) \in F$. Since $F = G$, we have $(a, x) \in G$, so x in the range of G. This proves that the range of F is contained in the range of G. The opposite argument is the same.

We've proved that if $F = G$ then the domain and range of F and G are the same. The converse is false; there are lots of different functions with the same domain and range.

What is true is this:

Proposition: If F and G are functions with the same domain, then $F = G$ if and only if $F(x) = G(x)$ for all x in that domain.