Show that $x^2+y^2-3=0$ has no rational solutions.

Can't find fractions x, y with $x^2+y^2=3$.

Assume you can.

$x = a/c \qquad y = b/c$ a, b, c have no common factor.

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 3$$ a, b, c are integers

$$a^2 + b^2 = 3c^2$$

If n any integer
$n = 3k$ $n \equiv 0 \ (3)$
$n = 3k+1$ $n \equiv 1 \ (3)$
$n = 3k+2$ $n \equiv 2 \ (3)$.

Case 1: $3|a$. Then $a = 3s$ for some $s \in \mathbb{Z}$.

$$3^2 s^2 + b^2 = 3c^2$$

$$b^2 = 3(c^2 - 9s^2)$$

b^2 is divisible by 3, so b is divisible by 3.

$$b = 3t$$

$$(3s)^2 + (3t)^2 = 3c^2$$

$$3s^2 + 3t^2 = c^2$$

so c is divisible by 3

But a, b, c have no common factor.

so $3 \nmid a$. also $3 \nmid b$.

$a = 3k+1$ or $a = 3k+2$ for some $k, s \in \mathbb{Z}$.
$b = 3s+1$ or $b = 3s+2$

a^2 if $a = 3k+1$, then $a^2 = \underline{9k^2+6k}+1$ of of form "$3k+1$".
$a = 3k+2$ $a^2 = 9k^2+12k+4$
$= 9k^2+12k+3+1$ $3k+1$.

If neither a nor b is divisible by 3 then both a^2 and b^2 have remainder 1 when divided by 3.

(If a, b are not congruent to 0 mod 3, then)
$a^2 \equiv b^2 \equiv 1 \mod 3$

$a^2 = 3u + 1$
$b^2 = 3v + 1$
$a^2 + b^2 = 3(u+v) + 2$

$a^2 + b^2 \equiv 2 \mod 3$

$a^2 + b^2 = 3c^2 \equiv 0 \mod 3$

this is a contradiction.

Therefore $a^2 + b^2 = 3c^2$ has no integer solutions and
$x^2 + y^2 = 3$ has no rational ones.

Which integers n can be written
$n = x^2 + y^2$ with $x, y \in \mathbb{Z}$??