The inverse function theorem

Definition: Let A be a set. The identity function $\mathbf{id}_A \subset A \times A$ is the function

$$A = \{(a,a) : a \in A\}.$$

Definition: Let $F \subset A \times B$ be a relation. The inverse relation $F^{-1} \subset B \times A$ consists of all pairs (b,a) where $(a,b) \in F$.

Definition: Let $F \subset A \times B$ be a function. Then a function $G : B \to A$ is called an *inverse function* for F if $F \circ G : B \to B$ is the identity function on B, and $G \circ F : A \to A$ is the identity function on A.

Theorem: Let $F \subset A \times B$ be a function. Then F has an inverse function if and only if F is bijective. If an inverse function exists, it is unique and is given by the inverse relation F^{-1}.

Problems - inverse functions

- Why is $f(x) = x^3$ from $\mathbb{R} \to \mathbb{R}$ injective? Why is it surjective?
- (From the homework) Let $f : \mathbb{N} \to \mathbb{Z}$ be the function

$$f(n) = \frac{(-1)^n(2n-1)+1}{4}.$$

  This function is bijective; what is its inverse?

Harder:

- Suppose f and g are injective. Is $f \circ g$ necessarily injective?
- Suppose $f : A \to B$ and $g : B \to C$ are functions and $g \circ f : A \to C$ is injective. What can you say about f and g?
- Suppose $f : A \to B$ and $g : B \to C$ are surjective. What can you say about $g \circ f$?
- Suppose $f : A \to B$ and $g : B \to C$ are functions and $g \circ f$ is surjective. What can you say about f and g?

Problems - image and preimage

- (from the homework) Given $f : A \to B$ and subsets $W, X \subset A$, show that $f(W \cap X)$ need not equal $f(W) \cap f(X)$.
- Given $f : A \to B$ and subsets $W, X \subset A$, show that $f(W \cup X) = f(W) \cup f(X)$.
- (from the homework) Given $f : A \to B$, show that:
  - f is injective iff $f^{-1}(f(X)) = X$ for all subsets $X \subset A$.
  - f is surjective iff $f(f^{-1}(Y)) = Y$ for all subsets $Y \subset B$.