Counting

The multiplication principle

From the text, p. 69: Suppose in making a list of length n there are a_1 possible choices for the first entry, a_2 possible choices for the second entry, a_3 possible choices for the third entry, and so on. Then the total number of lists that can be made is this way is the product $a_1 \cdot a_2 \cdots a_n$.

We will treat this as a proposition, put it in more formal language, and prove it by induction.

Notice that a "list" where the first element is chosen from X_1, the second from X_2, and so on is precisely an element of $X_1 \times X_2 \times \cdots X_n$, so we are really trying to compute the number of elements in a cartesian product of finite sets.

Proposition: Let $X_1, X_2, \ldots, X_n$ be finite sets and suppose that $|X_i| = a_i$ for $i = 1, \ldots, n$. Then the number of elements

$$|X_1 \times X_2 \times \cdots \times X_n| = a_1 \cdot a_2 \cdots a_n$$

Proof: (by induction). First we consider the case where $n = 2$. We have two sets X_1 and X_2 with a_1 and a_2 elements respectively. Explicitly we have

$$X_1 \times X_2 = \{(x, y) : x \in X_1, y \in X_2\}.$$

For each element $b \in X_2$, we have a_1 elements $(x, b) \in X_1 \times X_2$. Counting up these a_1 elements for each of the a_2 elements b, we see that $X_1 \times X_2$ has $a_1 a_2$ elements.

Now suppose we know that $X_1 \times \cdots \times X_n$ has $a_1 \cdots a_n$ elements. We must show that this implies that $X_1 \times \cdots \times X_n \times X_{n+1}$ has $(a_1 \cdots a_n)a_{n+1}$ elements. Let

$$Y = X_1 \times \cdots \times X_n.$$

Then $Y \times X_{n+1}$ consists of pairs (y, x) where y is an element of the cartesian product of the X_i and x is an element of X_{n+1}. Strictly speaking $Y \times X_{n+1}$ and $X_1 \times \cdots \times X_n \times X_{n+1}$ are not equal sets, but they do have the same number of elements.

By the inductive hypothesis $|Y \times X_{n+1}| = a_1 \cdots a_n a_{n+1}$ which is equal to $|X_1 \times \cdots X_{n+1}|$, as we hoped to prove.