Proofs homework

I'd like you to carefully write up proofs for the following problems: 6, 10, 24, 26, 28. For each of these problems, your solution should include:

- A statement of any definitions that you will rely on in the proof.
- A couple of examples of the statement that is being proved.
- A proof in paragraph form using complete sentences, proper grammar, and a minimum of symbols.

Problems

Problem 6. If $a|b$ and $a|c$ then $a|(b+c)$.

Proof: Since $a|b$, there is an integer x such that $b = ax$. Since $a|c$, there is an integer y so that $c = ay$. Then $(b+c) = (ax+ay) = a(x+y)$. Therefore there is an integer $z = (x+y)$ such that $(b+c) = az$ and so $z|(b+c)$.

Problem 10. Suppose that a and b are integers. if $a|b$, then $a|(3b^3 - b^2 + 5b)$.

Proof: Since $a|b$, there is an integer x such that $b = ax$. Then

$$(3b^3 - b^2 + 5b) = (3a^3x^3 - a^2x^2 + 5ax) = a(3a^2x^3 - ax^2 + 5x).$$

Therefore there is an integer $z = (3a^2x^3 - ax^2 + 5x)$ such that

$$(3b^3 - b^2 + 5b) = az$$

and therefore $a|(3b^3 - b^2 + 5b)$.

Problem 24. If $m \in \mathbb{N}$ and $n \geq 2$, then the numbers $n!+2, n!+3, \ldots, n!+n$ are all composite.

Proof: We will show that $i|n!+i$ for $i = 2, \ldots, n$. Remember that $n!$ is the product of the numbers from 1 up to n. Therefore, since i is an integer less than n and greater than 1, it is a divisor of $n!$, and so $n! = ix$ for some integer x. Then $n!+i = ix+i = i(x+1)$ and so i is also a divisor of $n!+i$. At the same time $n!+i$ is greater than i. Therefore $n!+i$ has a proper divisor i, so $n!+i$ is composite.

Problem 26. Every odd integer is a difference of two squares.

Proof: Let n be an odd integer. Then there is an integer k so that $n = 2k+1$. Notice that

$$(k+1)^2 - k^2 = k^2 + 2k + 1 - k^2 = 2k + 1 = n.$$

Therefore $n = (k+1)^2 - k^2$.

Problem 28. Suppose that a, b, c are integers, that a and b are not both zero, and $c \neq 0$. Prove that $c \cdot \gcd(a,b) \leq \gcd(ca, cb)$.

Proof: Let d be the greatest common divisor of a and b. Then d is a divisor of both a and b. Then cd is a divisor of both ca and cb. Therefore cd is a common divisor of ca and cb. Since $\gcd(ca, cb)$ is the *greatest* common divisor of ca and cb, we must have $cd \leq \gcd(ca, cb)$.