# Direct Proof

# Direct Proof

# From page 122 of the text

**Proposition** If $a, b, c \in \mathbb{N}$, then $\operatorname{lcm}(ca, cb) = c \cdot \operatorname{lcm}(a, b)$.

*Proof.* Assume $a, b, c \in \mathbb{N}$. Let $m = \operatorname{lcm}(ca, cb)$ and $n = c \cdot \operatorname{lcm}(a, b)$. We will show $m = n$. By definition, $\operatorname{lcm}(a, b)$ is a positive multiple of both $a$ and $b$, so $\operatorname{lcm}(a, b) = ax = by$ for some $x, y \in \mathbb{N}$. From this we see that $n = c \cdot \operatorname{lcm}(a, b) = cax = cby$ is a positive multiple of both $ca$ and $cb$. But $m = \operatorname{lcm}(ca, cb)$ is the *smallest* positive multiple of both $ca$ and $cb$. Thus $m \leq n$.

On the other hand, as $m = \operatorname{lcm}(ca, cb)$ is a multiple of both $ca$ and $cb$, we have $m = cax = cby$ for some $x, y \in \mathbb{Z}$. Then $\frac{1}{c}m = ax = by$ is a multiple of both $a$ and $b$. Therefore $\operatorname{lcm}(a, b) \leq \frac{1}{c}m$, so $c \cdot \operatorname{lcm}(a, b) \leq m$, that is, $n \leq m$.

We've shown $m \leq n$ and $n \leq m$, so $m = n$. The proof is complete. ■

Figure 1: lcm proposition

# The hidden part - check the definitions

**Definition:** Let $a$ and $b$ be positive integers. Then the least common multiple $\mathrm{lcm}(a, b)$ is the smallest positive integer $m$ such that $a|m$ and $b|m$.

# The hidden part II

Second, make sure the claim is clear. Look at some examples.

# The hidden part continued III - interpret the definition

Three ways of saying the same thing:

- $x$ is the smallest positive integer such that $a|m$ and $b|m$
- If $x$ is a positive integer so that $a|x$ and $b|x$, then $x \geq \mathrm{lcm}(a, b)$.
- If $x$ is a positive integer so that $a|x$ and $b|x$, then $\mathrm{lcm}(a, b) \leq x$.

# The hidden part IV

Read the proof to understand it's structure, without worrying about the details.

# Take the proof of the proposition apart

- Assume $a, b, c \in \mathbb{N}$.
- Let $m = \mathrm{lcm}(ca, cb)$ and $n = c\,\mathrm{lcm}(a, b)$. We will show that $m = n$.
- By definition, $\mathrm{lcm}(a, b)$ is a positive multiple of both $a$ and $b$, so $\mathrm{lcm}(a, b) = ax = by$ for some $x$ and $y$ in $\mathbb{N}$.
- From this we see that $n = c\,\mathrm{lcm}(a, b) = cax = cby$ is a positive multiple of both $ca$ and $cb$. Thus $m \leq n$.

# Taking the proof apart

- On the other hand, as $m = \mathrm{lcm}(ca, cb)$ is a multiple of both $ca$ and $cb$, we have $m = cax = cby$ for some $x, y \in \mathbb{Z}$.
- Then $\frac{1}{c}m = ax = by$ is a multiple of both $a$ and $b$.
- Therefore $\mathrm{lcm}(a, b) \leq \frac{1}{c}m$ so $c\,\mathrm{lcm}(a, b) \leq m$, that is $n \leq m$.
- Since $m \leq n$ and $n \leq m$, we have $m = n$. The proof is complete.