Direct Proofs in-class Exercises

- Identify the hypothesis and the conclusion.
- Isolate any relevant definitions.
- Given an example illustrating the truth of the statement.
- Write a proof of the statement.
- Verify that all hypotheses were used (or not).

1. Suppose $x, y \in \mathbb{Z}$. If x and y are odd, then xy is odd.

2. Suppose $x, y \in \mathbb{Z}$. If x is even, then xy is even.

3. If $x \in \mathbb{R}$ and $0 < x < 4$ then $\frac{4}{x(4-x)} \geq 1$.

4. If a is an integer and $a^2|a$, then $a \in \{-1, 0, 1\}$

Analyze the following proof along the given lines.

Proposition If $a, b, c \in \mathbb{N}$, then $\operatorname{lcm}(ca, cb) = c \cdot \operatorname{lcm}(a, b)$.

Proof. Assume $a, b, c \in \mathbb{N}$. Let $m = \operatorname{lcm}(ca, cb)$ and $n = c \cdot \operatorname{lcm}(a, b)$. We will show $m = n$. By definition, $\operatorname{lcm}(a, b)$ is a positive multiple of both a and b, so $\operatorname{lcm}(a, b) = ax = by$ for some $x, y \in \mathbb{N}$. From this we see that $n = c \cdot \operatorname{lcm}(a, b) = cax = cby$ is a positive multiple of both ca and cb. But $m = \operatorname{lcm}(ca, cb)$ is the *smallest* positive multiple of both ca and cb. Thus $m \leq n$.

On the other hand, as $m = \operatorname{lcm}(ca, cb)$ is a multiple of both ca and cb, we have $m = cax = cby$ for some $x, y \in \mathbb{Z}$. Then $\frac{1}{c}m = ax = by$ is a multiple of both a and b. Therefore $\operatorname{lcm}(a, b) \leq \frac{1}{c}m$, so $c \cdot \operatorname{lcm}(a, b) \leq m$, that is, $n \leq m$.

We've shown $m \leq n$ and $n \leq m$, so $m = n$. The proof is complete. ■

Figure 1: LCM theorem