Strong induction

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Axiom of Induction: For all $n \in \mathbb{N}$, Let $P(n)$ be a statement. If $P(1)$ is true and, for all $n \in \mathbb{N}$, $P(n) \implies P(n+1)$, then $P(n)$ is true for all $n \in \mathbb{N}$.

Strong induction changes the hypothesis slightly.

Strong Induction: For all $n \in \mathbb{N}$, let $P(n)$ be a statement. If $P(1)$ is true and, for all n, the statement

$$P(1) \wedge P(2) \wedge \cdots \wedge P(n) \implies P(n+1)$$

is true, then $P(n)$ is true for all $n \in \mathbb{N}$.

This means that if you prove $P(1)$ true, and then, by assuming *all* of the preceeding statements $P(1), P(2), \ldots, P(n)$ true you can prove $P(n+1)$ true, then all $P(n)$ are true.

Strong induction cont'd

Why does strong induction hold? It holds because it can be changed into regular induction.

Suppose $P(n)$ is a sequence of statements that satisfy the conditions of strong induction, so $P(1)$ is true and $P(n+1)$ is a consequence of *all* of the preceeding statements $P(1), \dots, P(n)$.

Let $S(1) = P(1)$, and let $S(n) = P(1) \wedge P(2) \cdots \wedge P(n)$. We apply regular induction to the set of statements $S(n)$.

- So $S(n)$ is a sequence of statements, and $S(1)$ is true.
- Also, we know that $S(n) \implies P(n+1)$ by our hypothesis.
- But $S(n) \wedge P(n+1) = S(n+1)$, and since $S(n)$ is true and $P(n+1)$ is true, so is $S(n+1)$.
- Therefore we've shown that $S(1)$ is true and $S(n) \implies S(n+1)$ for all $n \in \mathbb{N}$.
- By *regular* induction, $S(n)$ is true for all n.
- But the only way $S(n)$ is true is if all $P(j)$ for $1 \leq j \leq n$ are true.
- So all $P(n)$ are also true.