If and only if

If and only if statements

A theorem that asserts that two statements P and Q are equivalent requires you to prove *both* $P \implies Q$ and $Q \implies P$.

Chapter 7 exercise 3.

Proposition: If a is an integer, then a is even if and only if $a^3 + a^2 + a$ is even.

There are two claims:

a even implies $a^3 + a^2 + a$ is even.

$a^3 + a^2 + a$ is even implies a is even.

Each requires proof.

Proof: First we show that, if a is even, then $a^3 + a^2 + a$ is even.

Now we show that, if $a^3 + a^2 + a$ is even, then a is even.

Chapter 7, exercise 9.

Proposition: Suppose that a is an integer. Then $14|a$ if and only if $7|a$ and $2|a$.

Proof: First we suppose that $14|a$ and show that both 7 and 2 divide a.

Now we show that, if both 7 and 2 divide a, then 14 divides a.

Equivalence

Theorem Suppose A is an $n \times n$ matrix. The following statements are equivalent:

(a) The matrix A is invertible.

(b) The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$.

(c) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(d) The reduced row echelon form of A is I_n.

(e) $\det(A) \neq 0$.

(f) The matrix A does not have 0 as an eigenvalue.

Figure 1: Theorem from page 149

Cycle proofs

If each step in the circle of implications:

$$P_1 \implies P_2 \implies P_3 \implies \cdots \implies P_n \implies P_1$$

is true, then all of the statements are equivalent – that is, all true or all false together.