Proof by contradiction

Proposition: The square root of 2 is not a rational number.

Lemma: If a^2 is even, then a is even.

Proof: We will prove the contrapositive, which says that if a is odd, then a^2 is odd. Suppose a is odd. Then $a = 2k + 1$ for some k. Therefore $a^2 = (2k + 1)^2 = 4k^2 + 4k + 1$ is odd.

Proof of Proposition: Suppose that $\sqrt{2}$ is a rational number. Then we can find positive integers a and b with $b \neq 0$ so that

$$2 = (\frac{a}{b})^2.$$

Proof of Proposition: Suppose that $\sqrt{2}$ is a rational number. Then we can find positive integers a and b with $b \neq 0$ so that

$$a^2 - 2b^2 = 0.$$

Logical Structure of proof by contradiction

- A contradiction is a statement of the form (C and $\sim C$) which is *always false.*
- The strategy of proof by contradiction is that if $A \implies B$ is true, and B is false, then A is false.

Proposition: $P \implies Q$.

- Assume P is true.
- Assume (P and $\sim Q$) is true and

$$P \text{ and } \sim Q \implies C \text{ and } \sim C$$

  for some statement C.
- If (P and $\sim Q$) implies (C and $\sim C$) is a true implication yielding a false conclusion, then the hypothesis must be false.
- Therefore (P and $\sim Q$) is false.
- If (P and $\sim Q$) is false, and P is true then $\sim Q$ is false
- Q is true.