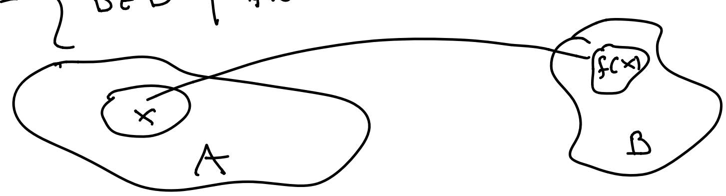
Consider $f: A \to B$. Prove that

- f is injective if and only if $X = f^{-1}(f(X))$ for all $X \subseteq A$.
- f is surjective if and only if $f(f^{-1}(Y)) = Y$ for all $Y \subseteq B$.

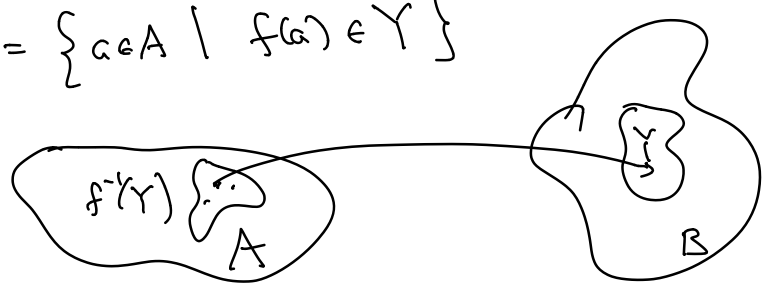
Reminder: So if $X \subseteq A$

$$f(X) = \{ b \in B \mid \text{there exists } a \in A \text{ with } f(a) = b \}.$$



If $Y \subseteq B$

$$f^{-1}(Y) = \{ a \in A \mid f(a) \in Y \}$$



① f injective $\Leftrightarrow$ $f^{-1}(f(X)) = X$ for all $X \subseteq A$.

① if f injective then $f^{-1}(f(X)) = X$ for all $X \subseteq A$.

Assume f injective.

i) $X \subseteq f^{-1}(f(X))$

Choose $a \in X$. To show $a \in f^{-1}(f(X))$ we need $f(a) \in f(X)$.

We need to know $f(a) = f(a')$ for $a' \in X$.

But $a \in X$ so $a = a'$.

ii) $f^{-1}(f(X)) \subseteq X$.

Suppose $a \in f^{-1}(f(X))$. Then $f(a) \in f(X)$

So $f(a) = f(a')$ for some $a' \in X$.

Since f injective, $a = a'$ so $a \in X$

f injective $\Rightarrow$ $f^{-1}(f(X)) = X$

② $f^{-1}(f(X)) = X$ then f injective.

$X = \{a\}$ $\quad f(X) = \{f(a)\}$.

$f^{-1}(f(X)) = \{a' \mid f(a') = f(a)\} = \{a\}$

The only a' with $f(a') = f(a)$ is $a' = a$.

So f is injective.

② f surjective $\Leftrightarrow$ $f(f^{-1}(Y)) = Y$ for all $Y \subseteq B$.

i) f surjective $\Rightarrow$ $f(f^{-1}(Y)) = Y$.

$b \in f^{-1}(Y) \Rightarrow f(b) \in Y$

so. $f(f^{-1}(Y)) \subseteq Y$.

$f(f^{-1}(Y)) \neq Y$ then there is a $b \in Y$

so that $b \notin f(f^{-1}(Y))$.

If $f(a) = b$ for some $a \in A$, then

$a \in f^{-1}(Y)$ so $f(a) = b \in f(f^{-1}(Y))$

So b is not in the image of f so f

is not surjective.

ii) $f(f^{-1}(Y)) = Y \Rightarrow f$ surjective.

Choose $b \in B$.

if $f^{-1}(\{b\}) = \emptyset$ then $f(f^{-1}(b)) = \emptyset$

But $f(f^{-1}(\{b\})) = \{b\}$

so $f^{-1}(\{b\}) \neq \emptyset$ so b is in image.

of f. Therefore f is surjective.