Set equality

The assertion that two sets A and B are equal is equivalent to saying that

$$x \in A \Leftrightarrow x \in B.$$

In other words, x is in A if and only if x is in B. Now $(x \in A) \Leftrightarrow (x \in B)$ is the same as

$$[(x \in A) \implies (x \in B)] \text{ AND } [(x \in B) \implies (x \in A)]$$

and this is just $A \subseteq B$ and $B \subseteq A$.

So we prove two sets are equal by proving BOTH $A \subseteq B$ and $B \subseteq A$.

Euclid's algorithm

Here's what we proved in the discussion in Chapter 7.

Proposition: Let $d = \gcd(a, b)$ and let m be any integer. Then there exist k and l such that $m = ak + bl$ if and only if $d|m$.

Set version:

Proposition: Let a and b be natural numbers, and let $d = \gcd(a, b)$. Define sets $A = \{dn : n \in \mathbb{Z}\}$ and $B = \{ax + by : x, y \in \mathbb{Z}\}$. Then $A = B$.

Here:

- $A \subseteq B$ means that every multiple of d can be written in the form $ax + by$.
- $B \subseteq A$ means that every number of the form $ax + by$ is a multiple of d.

More examples

Proposition: Let a and b be prime numbers. Let $A = \{da : d \in \mathbb{Z}\}$ and $B = \{db : d \in \mathbb{Z}\}$. Then $A \cap B = \{dab : d \in \mathbb{Z}\}$.

More examples

Proposition: If A, B, and C are sets then
$A \times (B \cap C) = (A \times B) \cap (A \times C)$. (this is problem 17 on page 171)

More examples

Proposition: Prove that $\{12a + 4b : a, b \in \mathbb{Z}\} = \{4c : c \in \mathbb{Z}\}$.

More examples

Proposition: Let $A = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$. Let B be the set of real numbers z such that there exists $x \in \mathbb{R}$ such that $(x, z) \in A$. Then $B = \{z \in \mathbb{R} : z \geq 0\}$.