Existence Proofs

Review of universal quantifiers

A theorem asserting the truth of a conditional statement is typically a "for all" statement.

Theorem: If a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable, it is continuous.

Here there is an implicit *universal quantifier.*

Theorem: For all functions $f : \mathbb{R} \to \mathbb{R}$, f differentiable implies f continuous.

Another example

Theorem: An $n \times n$ matrix A with real entries is invertible if and only if $det(A) \neq 0$.

This is asserting that:

For all $n \times n$ matrices A with real entries, A is invertible if and only if $det(A) \neq 0$.

Existence claims

Some theorems assert the existence of an object with particular properties.

Proof of an existence theorem requires you to present an example.

Definition: A Pythagorean Triple is an element (a, b, c) of $\mathbb{Z}^3$ such that

$$c^2 = a^2 + b^2.$$

Theorem: A Pythagorean triple exists.

Proof: Let $a = 3$, $b = 4$, and $c = 5$. Then $c^2 = 25 = a^2 + b^2$.

Existence claims can be hard to establish

Theorem: There exist integers A, B, and C so that

$$A/(B+C) + B/(A+C) + C/(A+B) = 4.$$

Proof: Let

$A = 1544768021087461664419513150199198374856643256695654317000$

$B = 3687513179412999982719781156522547482549297996897197099628$

$C = 437361267792869725786125260237139015281653755816161361862$

Then these values satisfy the given equation. (Check this if you can!)

- verification requires work
- no clue given as to how to find this; and, in fact, it's hard.