Why 2710?

Basics of 2710

- Math 2710 is intended as preparation for the core, proof-based courses in the math major
- Aimed at students who have completed the basic calculus and linear algebra classes
- Usually students at end of second or beginning of third year of studies.

Who *must* take 2710

- Students seeking the BA or BS degree in Mathematics
- Students seeking the BS degree in Applied Mathematics
- Students seeking the BS degree in Math-Physics with the Math Emphasis.

Which math programs DO NOT require 2710?

- Applied Math B.A. (2710 can be used toward major, but is not required)
- Math-Physics B.S. (Physics Emphasis) (2710 can be used toward major, but is not required)
- Math-Stat B.A. (2710 can be used toward major, but is not required)
- Math-Stat B.S. (2710 can be used toward major, but is not required)
- Actuarial Science B.A.
- Actuarial Science B.S.
- Actuarial Science-Finance B.A.
- Actuarial Science-Finance B.S.

What is 2710 about?

- The required upper division courses in the math major are **very different** than the Calculus sequence in their approach to the subject.

- Their focus is on **theory**
- They proceed through a sequence of **theorems** and **proofs.**
- The homework problems and test problems are primarily of the form: **Show that such and such is true** or **Prove such and such a statement.**
- Core subjects are *Analysis*, *Algebra*, and *Topology* as well as *Number Theory* and related topics such as *Combinatorics* and *Differential Geometry*.

- Math 2710 is an introduction to this language of abstract mathematics: logic, sets, quantifiers, induction, theorems, and proofs. It is designed to give you the tools to pursue further study in abstract mathematics.

Example: Analysis

Analysis is Calculus done right. It starts with a careful definition of the real numbers and builds up the theory of derivatives and integrals, *proving everything as you go.*

3. Use Lemma 2.3.2 to prove that if $f'(x_0) > 0$, there is a $\delta > 0$ such that

$$f(x) < f(x_0) \text{ if } x_0 - \delta < x < x_0 \text{ and } f(x) > f(x_0) \text{ if } x_0 < x < x_0 + \delta.$$

4. Suppose that p is continuous on $(a, c]$ and differentiable on (a, c), while q is continuous on $[c, b)$ and differentiable on (c, b). Let

$$f(x) = \begin{cases} p(x), & a < x \le c, \\ q(x), & c < x < b. \end{cases}$$

(a) Show that

$$f'(x) = \begin{cases} p'(x), & a < x < c, \\ q'(x), & c < x < b. \end{cases}$$

(b) Under what additional conditions on p and q does $f'(c)$ exist? Prove that your stated conditions are necessary and sufficient.

5. Find all derivatives of $f(x) = x^{n-1}|x|$, where n is a positive integer.

6. Suppose that $f'(0)$ exists and $f(x + y) = f(x)f(y)$ for all x and y. Prove that f' exists for all x.

7. Suppose that $c'(0) = a$ and $s'(0) = b$ where $a^2 + b^2 \ne 0$, and

$$c(x + y) = c(x)c(y) - s(x)s(y)$$
$$s(x + y) = s(x)c(y) + c(x)s(y)$$

for all x and y.

(a) Show that c and s are differentiable on $(-\infty, \infty)$, and find c' and s' in terms of c, s, a, and b.

(b) (For those who have studied differential equations.) Find c and s explicitly.

from Intro to Real Analysis by Trench.

Example: Algebra

Algebra is the study of abstract algebraic systems such as groups and rings. Here one starts with basic algebraic properties and uncovers deeper structure. Linear Algebra is one part of algebra, and the upper division linear algebra course redoes the results from Linear Algebra from an abstract point of view.

19. Show that
$$0 + a \equiv a + 0 \equiv a \pmod{n}$$
for all $a \in \mathbb{Z}_n$.

20. Prove that there is a multiplicative identity for the integers modulo n:
$$a \cdot 1 \equiv a \pmod{n}.$$

21. For each $a \in \mathbb{Z}_n$ find an element $b \in \mathbb{Z}_n$ such that
$$a + b \equiv b + a \equiv 0 \pmod{n}.$$

22. Show that addition and multiplication mod n are well defined operations. That is, show that the operations do not depend on the choice of the representative from the equivalence classes mod n.

23. Show that addition and multiplication mod n are associative operations.

24. Show that multiplication distributes over addition modulo n:
$$a(b + c) \equiv ab + ac \pmod{n}.$$

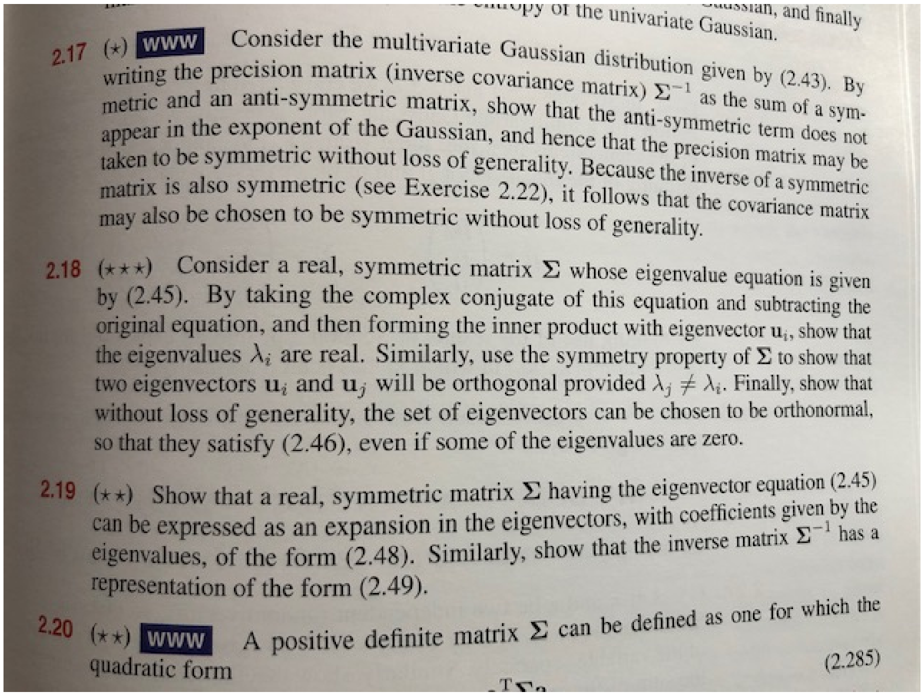
25. Let a and b be elements in a group G. Prove that $ab^n a^{-1} = (aba^{-1})^n$ for $n \in \mathbb{Z}$.

26. Let $U(n)$ be the group of units in $\mathbb{Z}_n$. If $n > 2$, prove that there is an element $k \in U(n)$ such that $k^2 = 1$ and $k \neq 1$.

from Abstract Algebra: Theory and Applications by Judson.

Example: Machine Learning

The language of theorems and proofs is fundamental to advanced work in many of the more theoretical areas of the mathematical sciences including computer science and statistics. Here are some problems from a famous book on Machine Learning.



…py of the univariate Gaussian, and finally Gaussian.

2.17 ($\star$) **www** Consider the multivariate Gaussian distribution given by (2.43). By writing the precision matrix (inverse covariance matrix) $\mathbf{\Sigma}^{-1}$ as the sum of a symmetric and an anti-symmetric matrix, show that the anti-symmetric term does not appear in the exponent of the Gaussian, and hence that the precision matrix may be taken to be symmetric without loss of generality. Because the inverse of a symmetric matrix is also symmetric (see Exercise 2.22), it follows that the covariance matrix may also be chosen to be symmetric without loss of generality.

2.18 ($\star\star\star$) Consider a real, symmetric matrix $\mathbf{\Sigma}$ whose eigenvalue equation is given by (2.45). By taking the complex conjugate of this equation and subtracting the original equation, and then forming the inner product with eigenvector $\mathbf{u}_i$, show that the eigenvalues λ_i are real. Similarly, use the symmetry property of $\mathbf{\Sigma}$ to show that two eigenvectors $\mathbf{u}_i$ and $\mathbf{u}_j$ will be orthogonal provided $\lambda_j \neq \lambda_i$. Finally, show that without loss of generality, the set of eigenvectors can be chosen to be orthonormal, so that they satisfy (2.46), even if some of the eigenvalues are zero.

2.19 ($\star\star$) Show that a real, symmetric matrix $\mathbf{\Sigma}$ having the eigenvector equation (2.45) can be expressed as an expansion in the eigenvectors, with coefficients given by the eigenvalues, of the form (2.48). Similarly, show that the inverse matrix $\mathbf{\Sigma}^{-1}$ has a representation of the form (2.49).

2.20 ($\star\star$) **www** A positive definite matrix $\mathbf{\Sigma}$ can be defined as one for which the quadratic form (2.285)

from Pattern Recognition and Machine Learning by Bishop

A (very simple) proof.

Theorem: Prove that the sum of two even integers is even.

Proof:

note: To prove this, you need to start with a very basic question – what is an even number?

Success in mathematics

There is a myth that success in mathematics is primarily driven by talent. In fact, I have worked with some of the greatest mathematicians in the world and what really distinguishes them is that they have incredible stamina. They are willing to struggle for years to understand something very complicated and to find the right way to express it.

This course is an introduction to this kind of mathematical work.

Hard-learned truths

Learning the language of theorems and proofs is very rewarding. It can open up whole worlds of new ideas as well as help you write more clearly and discipline your thinking. But it is *hard.*

- Many people – even those who think of themselves as *good at math* – find this course *very difficult.*
- Success in the course requires *close reading* of the text and *reflection* on the ideas in it.
- The assignments require both *problem solving* AND *writing.*
- This course is only the *first step* on a long road to learning to speak the language of mathematics.

Ask yourself these questions

- Are you clear in your own mind about why you are taking this course?
- Have you acknowledged that this course may be very challenging for you and you have accepted that challenge?
- Do you accept the fact that this course by itself is only a gateway to further study, and it may require further study for you to fully appreciate what you learn in this course?
- Are you committed to working on problems that are quite abstract and often not immediately related to applied questions?
- In addition to watching the course videos and participating in the discussions, are you prepared to read the textbook closely, ask questions about it, and reflect on it?
- Are you open to respectfully giving and willingly receiving constructive criticism from me and from your classmates?

If you do not feel that you can give a definitive yes to each of these questions, and you want to discuss this with me further, please get in touch by email and we can talk.