Solutions to selected homework from chapter 12

12.5.10

Problem: Consider $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \frac{(-1)^n(2n-1)+1}{4}.$$

This function is bijective by a previous exercise. Find its inverse.

Solution: Because of the term $(-1)^n$, the values of this function depend heavily on whether or not n is even. If n *is* even then we can write $n = 2k$ and we have

$$f(n) = \frac{2n-1+1}{4} = \frac{n}{2} = k.$$

In other words, if n is even, then $f(n)$ is $n/2$. Since n is an even natural number, $n/2$ is a positive integer greater than or equal to 1.

To construct the inverse of this part of the function, we can start with a positive integer $k \geq 1$ and define $f^{-1}(k) = 2k$.

If n is odd, then we can write $n = 2k+1$ and we have

$$f(n) = \frac{(1-2n)+1}{4} = \frac{2-2n}{4} = \frac{2-4k-2}{4} = -k$$

so if n is odd then $f(n)$ is $(1-n)/2$. Since n is a natural number, $(1-n)/2$ will be a non-positive integer. So to reverse this part of the function, given a non-positive integer k, we can let $n = 1-2k$. This will be a positive odd natural number.

So putting the two parts together, we have

$$f^{-1}(k) = \begin{cases} 2k & k > 0 \\ 1-2k & k \leq 0 \end{cases}$$

Problem 12.6.6

Problem: Given a function $f : A \to B$ and a subset $Y \subset B$, is $f(f^{-1}(Y)) = Y$ always true? Prove or give a counterexample.

Solution: Notice that $f^{-1}(Y)$ is the subset of A consisting of elements $a \in A$ such that $f(a) \in Y$. So $f(f^{-1}(Y)) \subset Y$. The question is whether $f(f^{-1}(Y))$ might be *smaller* than all of Y; and indeed it can. Here is a simple example. Let $A = \{0\}$ and $B = \{0,1\}$. Suppose that $f = \{(0,0)\} \subset A \times B$ and let $Y = B$. Then $f^{-1}(B) = A$ since $f^{-1}(0) = 0$. But $f(A) = \{0\} \subset B$ which is smaller than all of B.

Problem 12.6.8

Problem: Given a function $f : A \to B$ and subsets $W, X \subset A$, then $f(W \cap X) = f(W) \cap f(X)$ is *false* in general. Give a counterexample.

Solution: Suppose $A = \{0, 1\}$ and $B = \{0\}$. Suppose that $f = \{(0, 0), (1, 0)\} \subset A \times B$. Let $W = \{0\}$ and $X = \{1\}$. Then $W \cap X = \emptyset$ so $f(W \cap X) = \emptyset$. On the other hand, $f(W) = B$ and $f(X) = B$ so $f(W) \cap f(X) = B \neq \emptyset$.

Notice that you can find a counterexample in both of these cases using very small sets.

Problem 12.6.12

Problem: Consider $f : A \to B$. Prove that f is injective if and only if $X = f^{-1}(f(X))$ for all $X \subset A$. Prove that f is surjective if and only if $f(f^{-1}(Y)) = Y$ for all $Y \subset B$.

Solution: Let's first notice that $X \subset f^{-1}(f(X))$ for any X and any f. To see this, suppose $x \in X$. Then $f(x) \in f(X)$. Since $f^{-1}(f(X))$ is the set of all elements a of A such that $f(a) \in X$, we have $x \in f^{-1}(f(X))$. Therefore $X \subset f^{-1}(f(X))$. Suppose that there exists $u \in f^{-1}(f(X))$ such that $u \notin X$. Then $f(u) \in f(X)$ so $f(u) = f(x)$ for some $x \in X$, and $u \neq x$ since $u \notin X$. Therefore f is not injective. Thus we've proven that if $X \neq f^{-1}(f(X))$ then f is not injective.

Now suppose f is not injective, so there are two elements a and a' in A with $f(a) = f(a')$. Let $X = \{a\}$. Then $f(a) \in f(X)$, so $a' \in f^{-1}(f(X))$, and therefore $f^{-1}(f(X)) \neq X$ for this particular X. So if f is not injective then there is an X with $X \neq f^{-1}(f(X))$.

The surjectivity argument is similar, although everything is switched around. First notice that, for any f, $f(f^{-1}(Y)) \subset Y$. This is because if $a \in f^{-1}(Y)$, then $f(a) \in Y$ by definition. So suppose there is an element y of Y that is not in $f(f^{-1}(Y))$. If there were an x with $f(x) = y$, then x would be in $f^{-1}(Y)$, and so $y = f(x)$ would be in $f(f^{-1}(Y))$. So there is no such x, and therefore f is not surjective.

On the other hand, suppose f is not surjective. Then there is a $b \in B$ for which there is no $a \in A$ with $f(a) = b$. Let $Y = \{b\}$. Then $f^{-1}(Y) = \emptyset$ and $f(f^{-1}(Y)) = f(\emptyset) = \emptyset$. Thus if f is not surjective, there is a subset Y for which $f(f^{-1}(Y)) \neq Y$.