Equivalence Relations

Equivalence Relation: Definition

Definition: Let A be a set. A relation R on A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Examples

Let $A = \{-2, -1, 1, 2, 3, 4\}$.

- The relation $=$ is an equivalence relation.
- The relation "has the same parity as" is an equivalence relation.
- The relation "has the same sign as" is an equivalence relation.
- The relation "has the same sign and parity" is an equivalence relation.

Let X be the set of books in Babbidge Library with one author. Here are some equivalence relations:

- Has the same author.
- Has the same number of pages.
- Are located on the same floor of the library.

Equivalence Classes

Definition: Let A be a set and R a relation on A. For any $a \in A$, the *equivalence class* of a under R, written $[a]$ or $[a]_R$, is the set

$$[a] = \{b \in A : bRa\}.$$

If $A = \{-2, -1, 1, 2, 3, 4\}$ and R is the relation "has the same parity as" then:

- $[-2]$ is the set $\{-2, 2, 4\}$
- $[2]$ is the same set $\{-2, 2, 4\}$
- $[1]$ is the set $\{-1, 1, 3\}$
- $[3]$ is the set $\{-1, 1, 3\}$

Equivalence Classes - Examples

If X is the set of books in Babbidge Library with one author, and R is the relation “has the same author” then

- [Ray Bradbury] is the set of books in Babbidge Library with only one author, and that author is Ray Bradbury.

If R is the relation “has the same number of pages”, then

- [War and Peace] is the set of books in Babbidge Library (with one author) that have the same number of pages as War and Peace.

Question: why do I insist on books with one author?

Example 11.12 - polynomials

Let P be the set of polynomials with real coefficients. Define a relation R on P by saying that fRg if f and g have the same degree. Then R is an equivalence relation.

The equivalence class $[x]$ of the polynomial x consists of all polynomials of degree one.

More generally there is one equivalence class for each degree, and the equivalence class consists of all polynomials of that degree.

Example 11.13 - Congruence

We have seen that $\equiv \pmod{n}$ is an equivalence relation on $\mathbb{Z}$.

What are the equivalence classes $[x]$ for $x \in \mathbb{Z}$?

Rational numbers

Let M be the set of pairs (m, n) where m and n are integers and $n \neq 0$. Define a relation $(m, n)R(m', n')$ if $mn' - m'n = 0$. What are the equivalence classes?