Direct Proof

Direct Proof

From page 122 of the text

Proposition If $a, b, c \in \mathbb{N}$, then $\operatorname{lcm}(ca, cb) = c \cdot \operatorname{lcm}(a, b)$.

Proof. Assume $a, b, c \in \mathbb{N}$. Let $m = \operatorname{lcm}(ca, cb)$ and $n = c \cdot \operatorname{lcm}(a, b)$. We will show $m = n$. By definition, $\operatorname{lcm}(a, b)$ is a positive multiple of both a and b, so $\operatorname{lcm}(a, b) = ax = by$ for some $x, y \in \mathbb{N}$. From this we see that $n = c \cdot \operatorname{lcm}(a, b) = cax = cby$ is a positive multiple of both ca and cb. But $m = \operatorname{lcm}(ca, cb)$ is the *smallest* positive multiple of both ca and cb. Thus $m \leq n$.

On the other hand, as $m = \operatorname{lcm}(ca, cb)$ is a multiple of both ca and cb, we have $m = cax = cby$ for some $x, y \in \mathbb{Z}$. Then $\frac{1}{c}m = ax = by$ is a multiple of both a and b. Therefore $\operatorname{lcm}(a, b) \leq \frac{1}{c}m$, so $c \cdot \operatorname{lcm}(a, b) \leq m$, that is, $n \leq m$.

We've shown $m \leq n$ and $n \leq m$, so $m = n$. The proof is complete. ■

Figure 1: lcm proposition

The hidden part

First, remember the definition.

Definition: Let a and b be positive integers. Then the least common multiple $\mathrm{lcm}(a, b)$ is the smallest positive integer m such that $a|m$ and $b|m$.

The hidden part continued

"x is the smallest positive integer such that $a|m$ and $b|m$" is equivalent to "If x is a positive integer so that $a|x$ and $b|x$, then $x \geq \mathrm{lcm}(a, b)$."

The hidden part II

Second, make sure the claim is clear, for example by doing examples.

The hidden part III

Read the proof to understand it's structure, without worrying about the details.

Take the proof of the proposition apart

- Assume $a, b, c \in \mathbb{N}$.
- Let $m = \operatorname{lcm}(ca, cb)$ and $n = c\operatorname{lcm}(a, b)$. We will show that $m = n$.
- By definition, $\operatorname{lcm}(a, b)$ is a positive multiple of both a and b, so $\operatorname{lcm}(a, b) = ax = by$ for some x and y in $\mathbb{N}$.
- From this we see that $n = c\operatorname{lcm}(a, b) = cax = cby$ is a positive multiple of both ca and cb. Thus $m \leq n$.

Taking the proof apart

- On the other hand, as $m = \mathrm{lcm}(ca, cb)$ is a multiple of both ca and cb, we have $m = cax = cby$ for some $x, y \in \mathbb{Z}$.
- Then $\frac{1}{c}m = ax = by$ is a multiple of both a and b.
- Therefore $\mathrm{lcm}(a, b) \leq \frac{1}{c}m$ so $c\,\mathrm{lcm}(a, b) \leq m$, that is $n \leq m$.
- Since $m \leq n$ and $n \leq m$, we have $m = n$. The proof is complete.