Quantifiers

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Remember that an open sentence is a sentence that includes variables; when you specify the variables, the open sentence becomes a statement that can be true or false.

Quantifiers

Most equations that we want to “solve” are really open sentences. For example,

$$\begin{aligned} 3x &= 7 \\ x^2 + 5x + 6 &= 0 \end{aligned}$$

are open sentences whose truth depends on the choice of x.

Whether or not these equations even *have* solutions depends on what kind of values x is allowed to have.

Quantifiers

For example:

- neither of these equations have solutions if x is required to be a natural number.
- if x is allowed to be an integer, then the second equation has two solutions, but the first one still has none.
- if x is allowed to be a rational number, then both equations have solutions.

Quantifiers

Quantifiers are an element of the logical language that put a scope on the possible values of a variable in an open sentence, and in the process convert the open sentence into a statement.

The are two quantifiers: - "there exists" makes the statement about *some* x in a particular set, - "for all" makes the statement about *all* x in a particular set.

Existential quantifier (there exists)

"There exists $x \in \mathbb{Q}$ such that $3x = 7$"

This statement is true if and only if the subset

$$X = \{x : x \in \mathbb{Q}, 3x = 7\}$$

has at least one element – there is *some* x so that $3x = 7$ among the $x \in \mathbb{Q}$.

- "There exists $x \in \mathbb{Q}$ such that $3x = 7$" is True
- "There exists $x \in \mathbb{Z}$ such that $3x = 7$" is False

More generally, if X is any set, and $P(x)$ is an open sentence, then the statement "There exists $x \in X$ so that $P(x)$" (in symbols "$\exists x, P(x)$") is true exactly when the set

$$Y = \{x : x \in X, P(x)\}$$

has at least one element.

Univeral quantifier (for all)

The statement "For all $x \in \mathbb{N}$, $x^2 > 0$" is true if and only if

$$X = \{x : x \in \mathbb{N}, x^2 > 0\} = \mathbb{N}.$$

It claims something is true for *all* $x \in \mathbb{N}$. This is in fact a true statement.

On the other hand, the statement "For all $x \in \mathbb{Z}$, xˆ2>0" is false since $0^2 = 0$ and $0 \in \mathbb{Z}$.

More generally, the statement "For all $x \in X$, $P(x)$" (in symbols "$\forall x, P(x)$") is true exactly when

$$X = \{x \in X : P(x)\}.$$

This is a statement about *all* $x \in X$.

A few more examples

- There exists $x \in \mathbb{R}$ such that $x^2 = 15$.
- For all $y \in \mathbb{R}$, $|\sin(y)| \leq 1$.
- There exists a subset X of $\mathbb{N}$ which has 5 elements.

Negating quantified statements

The statement "There exists $x \in X$ such that $P(x)$" is false exactly when "For all $x \in X$, not $P(x)$" is true.

For example, "There exists $x \in \mathbb{R}$ such that $x^2 < 0$" is false because "For all $x \in \mathbb{R}$, $x^2 \geq 0$" is true.

The statement "For all x, $P(x)$" is false exactly when "There exists x such that not $P(x)$" is true.

For example, the statement "For all $x \in \mathbb{N}$, $x^2 > 0$" is true because "There exists $x \in \mathbb{N}$ with $x^2 \leq 0$." is false.

Existence and "OR", For all and "AND"

There exists $x \in X$ such that $P(x)$ is a kind of "OR" statement.

For all $x \in X$ such that $P(x)$ is a kind of "AND" statement.