Proof by Contrapositive

The contrapositive.

Important: The contrapositive of an implication $P \implies Q$ is $\sim Q \implies \sim P$.

- If $\sim Q$ is false (meaning Q is true) the implication $\sim Q \implies \sim P$ is automatically true.
- So we assume $\sim Q$ is true – that is, that Q is false – and try to conclude that $\sim P$ is true – meaning that P is false.

Contrapositive.

Proposition: Suppose that $x \in \mathbb{Z}$. Suppose $x^2 - 4x + 3$ is even. Then x is odd.

Contrapositive

Proof: Suppose x is even. Then $x = 2m$ for some integer m. Therefore

$$B = x^2 - 4x + 3 = 4m^2 - 8x + 3 = 2(2m^2 - 2m + 1) + 1.$$

Since B is of the form $2k + 1$ with $k = 2m^2 - 2m + 1$, we conclude that B is odd. Therefore B is not even. We have shown that if x is not odd, then B is not even, and therefore if B is even, x is odd.

Contrapositive

Proposition: Suppose that $x \in \mathbb{Z}$, that a is even, and that b is odd. If $x^2 - ax + b$ is even, then x is odd.

An example from calculus

Theorem: Let $f : [a, b] \to \mathbb{R}$ be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f'(x) = 0$ for all $x \in [a, b]$, then f is constant.

Proof:

- We will show that if f is not constant, then there is an $x \in [a, b]$ with $f'(x) \neq 0$.
- Suppose that $f(x)$ is not constant. Then there are two (different) points u and v in $[a, b]$ such that $f(u) \neq f(v)$.

calculus cont'd

- $f : [u, v] \to \mathbb{R}$ is continuous on $[u, v]$ and differentiable on (u, v). Therefore, by the mean value theorem, there is a point $c \in (u, v)$ such that

$$f'(c) = \frac{f(v) - f(u)}{v - u}.$$

  Since $f(v) \neq f(u)$, the quantity on the right is not zero, and so $f'(c) \neq 0$.

- Therefore $f'(x)$ is not zero for all $x \in [a, b]$. This proves our result.