Fibonacci numbers

Fibonacci Numbers

Figure 1: Fibonacci

The Fibonacci numbers F_n are defined by a *recursive* formula. The first two numbers are given by $F_1 = 1$ and $F_2 = 1$ and, for all $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Fibonacci Numbers and the Golden Ratio

See Donald Duck in Mathmagic Land (7 minute mark - 14 minute mark).

Fibonacci Numbers and the Golden Ratio

The golden ratio

$$\phi = \frac{1+\sqrt{5}}{2}$$

is the larger root of the quadratic polynomial $x^2 - x - 1 = 0$.

Proposition: The ratio of successive Fibonacci numbers F_{n+1}/F_n converges to the Golden ratio.

Some Data

1	1	1.000000000
1	2	2.000000000
2	3	1.500000000
3	5	1.666666667
5	8	1.600000000
8	13	1.625000000
13	21	1.615384615
21	34	1.619047619
34	55	1.617647059
55	89	1.618181818
89	144	1.617977528
144	233	1.618055556
233	377	1.618025751
377	610	1.618037135
610	987	1.618032787

Fibonacci Numbers cont'd

Proposition: $F_{n+1}^2 - F_n F_{n+1} - F_n^2 = (-1)^n$.

$$3^2 - (2)(3) - 2^2 = -1$$
$$5^2 - (3)(5) - 3^2 = 1$$
$$8^2 - (5)(8) - 5^2 = -1$$

Corollary: $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$.

Proof: Divide through by F_n^2:

$$(\frac{F_{n+1}}{F_n})^2 - (\frac{F_{n+1}}{F_n}) - 1 = \frac{(-1)^n}{F_n^2}$$

The right hand side goes to zero, so (F_{n+1}/F_n) converges to a root of the polynomial which is greater than one.

Proof of proposition

First check that $F_2^2 - F_1F_2 - F_1^2 = -1$, which is $1^2 - 1 - 1 = -1$ as we want.

- Now suppose that the formula holds for F_n, so $F_n^2 - F_nF_{n-1} - F_{n-1}^2 = (-1)^{n-1}$.
- Consider $F_{n+1}^2 - F_{n+1}F_n - F_n^2$.
- Substitute $F_{n+1} = F_n + F_{n-1}$ to get

$$(F_n + F_{n-1})^2 - (F_n + F_{n-1})F_n - F_n^2 = \\ F_n^2 + 2F_nF_{n-1} + F_{n-1}^2 - F_n^2 - F_{n-1}F_n - F_n^2$$

Then the right hand side of this equation is

$$-F_n^2 + F_nF_{n-1} + F_{n-1}^2 = -(F_n^2 - F_nF_{n-1} - F_{n-1}^2) = (-1)^n$$

where we used the inductive hypothesis to in the second-to-last step.