Relations

Definition: Let A and B be sets. A relation on A and B is a subset R of the cartesian product $A \times B$. If $(a, b) \in R$ we write aRb. If $A = B$, we talk about "a relation on A" as shorthand for a relation on A and A.

Definition: If A is a set and R is a relation on A, then R is *reflexive* if, for all $a \in A$, $(a, a) \in R$. In other words, aRa for all $a \in A$.

Definition: If A is a set and R is a relation on A, then R is symmetric if for all $a, b \in A$, $(a, b) \in R \implies (b, a) \in R$. In other words, for all $a \in A$, $aRb \implies bRa$.

Definition: If A is a set and R is a relation on A, then R is transitive if, for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$. In other words, if aRb and bRc then aRc.

Definition: A relation R on a set A is called an *equivalence relation* if R is reflexive, symmetric, and transitive.

Definition: Suppose that A is a set and R is an equivalence relation on A. Then, for any $a \in A$, the *equivalence class* of a under R is the set $[a] = \{b \in A : (a, b) \in R\}$.

Definition: A *partition* of a set A is a set U of non-empty subsets of A such that the intersection of any two different elements of U is empty, and the union of all elements of U is A.

Theorem: Suppose R is an equivalence relation on a set A. Then the set of equivalence classes under R form a partition of A.

Functions

Definition: Let A and B be sets. A function $f : A \to B$ is a relation on $f \subset A \times B$ with the property that, for all $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in f$. If $(a, b) \in f$, we write $b = f(a)$. The set A is called the *domain* of f. The set B is called the *codomain* of f.

Definition: Let $f : A \to B$ be a function. The *range* of f is the subset

$$\text{range}(f) = \{b \in B : \exists a \in A, f(a) = b\}$$

Definition: Let $f : A \to B$ and $g : C \to D$ be two functions. Then f and g are equal if they are equal as sets $f \subset A \times B$ and $g \subset C \times D$.

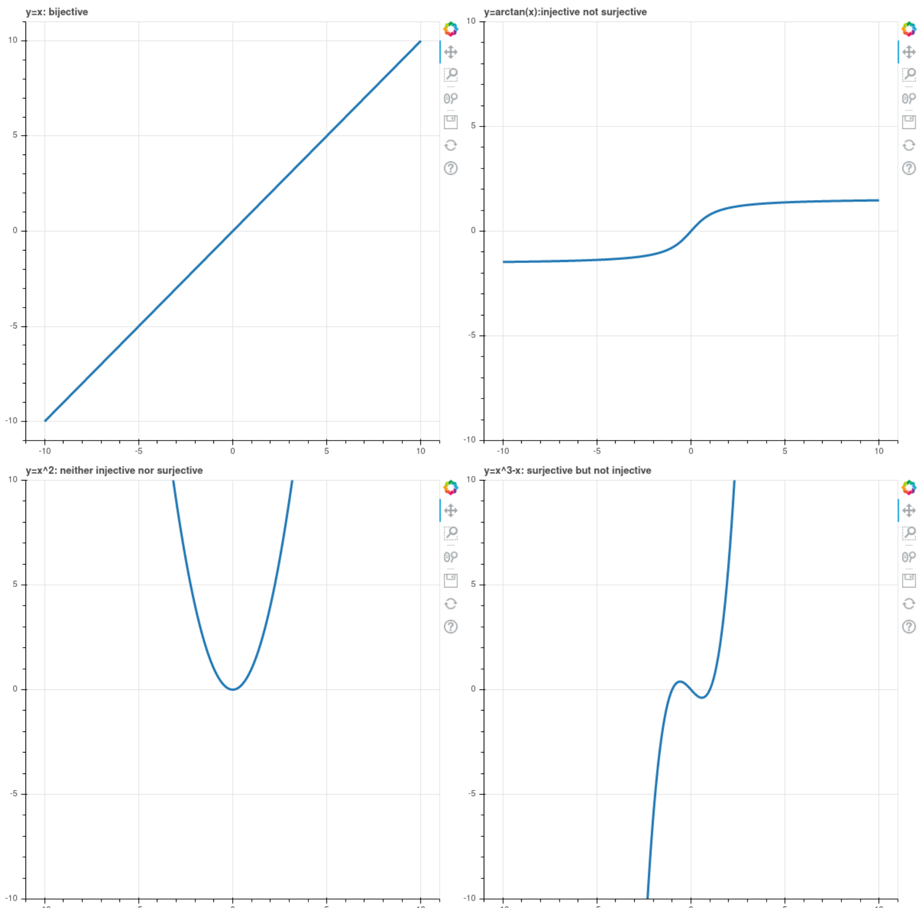
Proposition: If two functions are equal, they have the same domain and range (but not necessarily the same codomain).

Definition: Let $f : A \to B$ be a function. f is *injective* if, for all $a, a' \in A$, if $a \neq a'$ then $f(a) \neq f(a')$. Equivalently (by the contrapositive), f is injective if for all $a, a' \in A$, if $f(a) = f(a')$, then $a = a'$.

Definition: Let $f : A \to B$ be a function. f is *surjective* if, for all $b \in B$, there exists $a \in A$ such that $f(a) = b$. Equivalently, a function is surjective if its codomain equals its range.

Definition: Let $f : A \to B$ be a function. If f is both surjective and injective, then it is called *bijective.*

Examples


y=x: bijective
y=arctan(x):injective not surjective
y=x^2: neither injective nor surjective
y=x^3-x: surjective but not injective