

# Dimension

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# Dimension

**Theorem:** Suppose  $B = \{b_1, \dots, b_n\}$  is a basis for a vector space  $V$ . Then any set  $u_1, \dots, u_m$  of  $n + 1$  or more vectors is linearly dependent.

**Proof:** Write each  $u_i$  in coordinates using  $b_i$ . There are  $n$  coordinates  $[u_i]_B$  for each  $i$ , and  $m > n$  such coordinate vectors, so these coordinate vectors are linearly dependent.

More explicitly, since  $B$  spans, we can write:

$$\begin{aligned} u_1 &= a_{11}b_1 + a_{12}b_2 + \cdots + a_{1n}b_n \\ &\vdots \\ u_m &= a_{m1}b_1 + a_{m2}b_2 + \cdots + a_{mn}b_n \end{aligned}$$

## Dimension continued

Our goal is to find a non trivial solution to  $\sum_{j=1}^m c_j u_j = 0$ . The coefficient of  $b_i$  in this linear combination is

$$\sum_{j=1}^m c_j a_{ji}.$$

Since the  $b_i$  are linearly independent, we must have

$$\sum_{j=1}^m c_j a_{ji} = 0$$

for each  $i$ .

This is a homogeneous linear system of  $n$  equations in  $m$  unknowns where  $m > n$ . Thus it must have a nontrivial solution.

# Dimension

**Theorem:** If  $V$  has a basis  $B$  with  $n$  vectors, then every basis of  $V$  has  $n$  vectors.

**Proof:** If  $B'$  is another basis, it must have  $n$  or fewer elements, because if it had more than  $n$  it would be linearly dependent. If it had fewer than  $n$ , then the original basis  $B$  would be dependent. So the only possibility is that  $B'$  also has  $n$  elements.

**Definition:** If  $V$  has a finite basis, it is called finite dimensional and the dimension of  $V$  is the number of elements in a finite basis. Otherwise we say  $V$  is infinite dimensional.

## Dimension of subspaces

If  $H$  is a subspace of a finite dimensional space  $V$ , then the dimension of  $H$  is at most that of  $V$ ; if they have the same dimension, then  $H = V$ .

Also:

- ▶ Any linearly independent subset of a vector space  $H$  can be extended to a basis. (Assume  $H$  finite dimensional)
- ▶ Any spanning set contains a basis.

## Construction of a basis

Suppose  $b_1, \dots, b_k$  are linearly independent in a vector space  $V$ . Either they span  $V$ , in which case they are already a basis, or they don't span  $V$ .

If they don't span  $V$  there is a  $b_{k+1}$  in  $V$  that is not in the span of the  $b_i$  for  $i \leq k$ . Then  $b_1, \dots, b_{k+1}$  is still linearly independent.

If  $V$  is finite dimensional of dimension  $n$ , this process cannot continue indefinitely because once you have  $n$  linearly independent vectors you have a basis.

## Every (finite) spanning set contains a basis

Suppose  $B$  is a (finite) set of vectors that span  $V$ . If  $B$  is linearly independent, it is already a basis. If not, then one vector in  $B$  is dependent on the others, so you can delete it and the remaining vectors still span.

Repeating this process reduces the size of the set of spanning vectors; eventually this has to reach a basis.

## Example

Suppose that

$$H = \left\{ \begin{bmatrix} s - 2t \\ s + t \\ 3t \end{bmatrix} : s, t \in \mathbf{R} \right\}$$

What is the dimension of this space? Find a basis.

We can rewrite this as

$$H = \left\{ s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} : s, t \in \mathbf{R} \right\}$$

so  $H$  is spanned by the two given vectors.

Since they are linearly independent (look at the last entry) the dimension of  $H$  is 2.



## Example

The polynomials  $1$ ,  $2t$ ,  $4t^2 - 2$  and  $8t^3 - 12t$  are called the first four Hermite polynomials. They come up in the solution of certain differential equations.

Show that they form a basis for the polynomials of degree 3, so every such polynomial has a unique expansion in terms of the Hermite polynomials.

## Example

Relative to the basis  $1, t, t^2, t^3$  the Hermite polynomials have coordinate vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -12 \\ 0 \\ 8 \end{bmatrix}$$

These are four vectors in a four dimensional space, so they are a basis if they are linearly independent. Check this. (note that the associated matrix is already upper triangular)