

# Eigenvalues and Eigenvectors

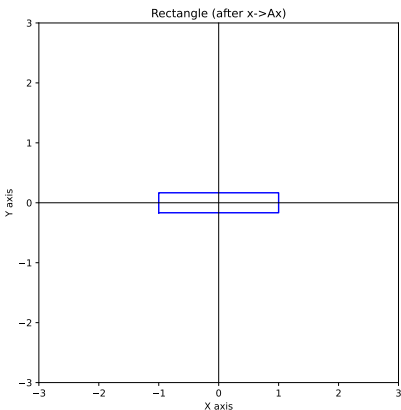
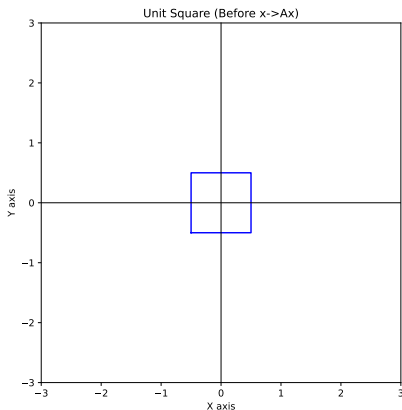
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# Eigenvalues and Eigenvectors

If  $A$  is a diagonal matrix:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

then the linear transformation  $x \mapsto Ax$  “stretches” along the  $x$ -axis and “shrinks” along the  $y$ -axis.

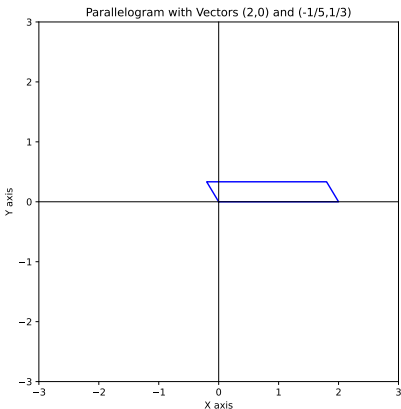
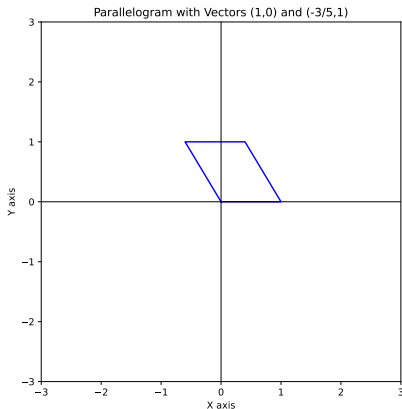


# Eigenvalues and Eigenvectors

If  $A$  is upper triangular, say

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix}$$

then  $A$  stretches along the  $x$ -axis by 2 as before. Less obviously, it shrinks along the direction given by the vector  $(-3/5, 1)$ .



# Eigenvalues and Eigenvectors

An **eigenvector** for a matrix  $A$  is a vector  $v$  which gets shrunk or lengthened by  $A$  by some factor  $\lambda$ .

The factor  $\lambda$  is called the **eigenvalue**.

More formally, a vector  $v$  is called an eigenvector for  $A$  (with eigenvalue  $\lambda$ ) if  $v$  is not zero and

$$Av = \lambda v.$$

## Eigenvalues and Eigenvectors

In the example above, the vectors  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -3/5 \\ 1 \end{bmatrix}$  are eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix}$$

with eigenvalues 2 and 1/3 respectively.

$$\begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} -3/5 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 1/3 \end{bmatrix} = (1/3) \begin{bmatrix} -3/5 \\ 1 \end{bmatrix}$$

# Triangular Matrices

If  $A$  is (upper) triangular then the diagonal entries for  $A$  are all eigenvalues. There are  $n$  linearly independent eigenvectors.

# Eigenspaces

Suppose that  $\lambda$  is a constant. The vectors  $v$  such that

$$Av = \lambda v$$

form a subspace called the *eigenspace* for  $\lambda$ .

This subspace is the nullspace of the matrix

$$A - \lambda I_n$$

where  $I_n$  is the  $n \times n$  identity matrix.

## Independence of Eigenvectors

If  $v_1, \dots, v_n$  are eigenvectors for a matrix  $A$  with eigenvalues  $\lambda_1, \dots, \lambda_n$ , and all the  $\lambda_i$  are different, then the  $v_i$  are linearly independent. (Note that the  $v_i$  are nonzero.)

To see this, suppose that

$$c_1 v_1 + c_2 v_2 + \dots c_n v_n = 0.$$

Then

$$A(c_1 v_1 + c_2 v_2 + \dots c_n v_n) = c_1 \lambda_1 v_1 + \dots c_n \lambda_n v_n = 0$$

Multiply the first relation by  $\lambda_1$  and subtract. You get

$$c_2(\lambda_2 - \lambda_1)v_2 + \dots + c_n(\lambda_n - \lambda_1)v_n = 0.$$

Since the differences of the  $\lambda_i$  with  $\lambda_1$  are not zero, we see that  $v_2, \dots, v_n$  are dependent.

By repeating this you can show that smaller and smaller collections of the  $v_i$  are dependent until you ultimately get  $v_n = 0$ .



## Characteristic Equation

Finding eigenvalues and eigenvectors of a matrix is a hard problem. We can make the following observation.

Suppose  $\lambda$  is an eigenvalue of  $A$  where  $A$  is an  $n \times n$  matrix. Then there is a vector  $v \neq 0$  so that  $Av = \lambda v$ . This means that the matrix  $A - \lambda I_n$  is *not* invertible because  $v$  is in its null space.

As a result,  $\det(A - \lambda I_n) = 0$ .

Conversely, if  $\det(A - \lambda I_n) = 0$ , then there is a vector  $v$  in the null space and that  $v$  is an eigenvector.

It turns out that  $\det(A - \lambda I_n)$  is a polynomial in  $\lambda$ , so the eigenvalues of  $A$  are the roots of this polynomial.

## Example

Let

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}.$$

The determinant of  $A - \lambda$  is

$$\det\left(\begin{bmatrix} 3 - \lambda & 5 \\ 2 & 4 - \lambda \end{bmatrix}\right) = (3 - \lambda)(4 - \lambda) - 10$$

The polynomial on the right is

$$(3 - \lambda)(4 - \lambda) - 10 = \lambda^2 - 7\lambda + 12 - 10 = \lambda^2 - 7\lambda + 2.$$

Its roots are  $\frac{7 \pm \sqrt{41}}{2}$ . These are the eigenvalues of  $A$ ; they are approximately 6.70156 and 0.29843.

## Example continued

To find the eigenvectors, we have to compute the null space of  $A - \lambda I$ . This is no fun algebraically but with some work you find that the eigenvectors are:

$$\begin{bmatrix} -\frac{\sqrt{41}}{4} - \frac{1}{4} \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{41}}{4} \\ 1 \end{bmatrix}$$

# Similarity

Two (square) matrices  $A$  and  $B$  are *similar* if there is an invertible matrix  $P$  so that  $A = PBP^{-1}$ .

Similar matrices have the same eigenvalues because they have the same characteristic polynomial.

$$\begin{aligned}\det(PBP^{-1} - \lambda I) &= \det(P(B - \lambda I)P^{-1}) \\ &= \det(P) \det(B - \lambda I) \det(P^{-1}) \\ &= \det(B - \lambda I)\end{aligned}$$