1.3 Matrix Equations

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Matrix Equations

A system of n linear equations in k unknowns can be written in matrix form

$$Ax = b$$

Here A is the $n \times k$ matrix of coefficients

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nk} \end{pmatrix}$$

Matrix multiplication

The vectors x and b are

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

The matrix product Ax is *by definition* the linear combination of the columns of A with weights given by x.

Matrix Multiplication

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ 13 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 13 \\ 13 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Important: In general a matrix equation $Ax = b$ has a solution x if b is in the span of the columns of A.

Solving a matrix equation

Given the matrix equation $Mx = b$, to solve it, use row reduction on the augmented matrix $[Mb]$.

For M and b as above the augmented matrix is

$$\begin{bmatrix} 1 & 3 & 2 & 13 \\ 2 & 4 & 1 & 13 \\ -1 & -1 & 0 & -3 \end{bmatrix}$$

Applying row reduction yields

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This gives $x_1 = 1, x_2 = 2, x_3 = 3$ as the only solution to this equation.

Another example

Consider the following M and b.

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

Augmented form

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix}$$

Example continued

Reduced form

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution

x_3 is a free variable.

$$\begin{aligned} x_2 &= 2 \\ x_1 &= -1 - \tfrac{4}{3}x_3 \end{aligned}$$

Vector form

$$x = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{pmatrix}$$

Inconsistency

A system is inconsistent if the only non zero entry in a row occurs by itself in the last column. Consider $Mx = b$.

Matrix M

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Vector b

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Iconsistency example

Reduced form - last row shows inconsistent.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Systems

A *homogeneous system* is a matrix equation $Ax = b$ where the target vector b is zero.

The solutions are parameterized by s vectors, where s is the number of free variables in the reduced matrix A.

The values of the variables corresponding to pivots are determined by the free variables.

If there are no free variables, the only solution to the inhomogeneous system is zero.