

# Inner Products and Orthogonality

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## The inner (dot) product.

If  $u$  and  $v$  are vectors in  $\mathbf{R}^n$ , then the *dot product* or *inner product* of  $u$  and  $v$  is

$$u \cdot v = u^T v = u_1 v_1 + \cdots + u_n v_n.$$

For example if

$$u = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, v = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

then

$$u \cdot v = (2)(1) + (3)(-1) + (-1)(0) = 2 - 3 = -1 \dots$$

# Key properties of the dot product

Let  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  be vectors in  $\mathbb{R}^n$ , and let  $c$  be a scalar. Then

- a.  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- b.  $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- c.  $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$
- d.  $\mathbf{u} \cdot \mathbf{u} \geq 0$ , and  $\mathbf{u} \cdot \mathbf{u} = 0$  if and only if  $\mathbf{u} = \mathbf{0}$

Figure 1: Theorem 1 (p. 375)

## Length and distance

The *length* or *norm* of a vector (written  $\|v\|$ ) is

$$\|v\| = \sqrt{v_1^2 + \cdots + v_n^2}$$

It is the “euclidean length” of the vector by the Pythagorean theorem.

Scaling a vector scales its length:

$$\|cv\| = |c|\|v\|$$

The distance between  $u$  and  $v$  is  $\|u - v\|$  (this is the “distance formula”).

# Unit vectors

If  $v$  is a vector, then

$$u = \frac{v}{\|v\|}$$

is a vector of length one that “points in the same direction as  $v$ ”.

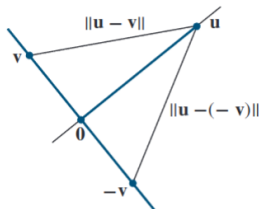
Such a vector is called a *unit vector*.

## Orthogonality

Two vectors are “orthogonal” (or “perpendicular”) if they meet at a right angle.

One way to describe this is to say that  $u$  and  $v$  are perpendicular if *the distance from  $u$  to  $v$  is the same as the distance from  $u$  to  $-v$* .

$$\|u - v\|^2 = \|u + v\|^2$$



**FIGURE 5**

Figure 2: Perpendicular Vectors

# Dot product zero means orthogonal

In other words

$$\|u\|^2 + \|v\|^2 - 2(u \cdot v) = \|u\|^2 + \|v\|^2 + 2(u \cdot v)$$

or

$$u \cdot v = 0$$

**Key idea:**  $u$  and  $v$  are orthogonal if and only if  $u \cdot v = 0$ .

# Orthogonal Complements

Let  $W$  be a subspace of  $\mathbf{R}^n$ .

The “orthogonal complement” to  $W$ , written  $W^\perp$ , is

$$W^\perp = \{v \mid v \cdot w = 0 \text{ for all } w \in W\}$$

For example, if  $W$  is the plane in  $\mathbf{R}^3$  spanned by  $w_1 = (2, 3, 1)$  and  $w_2 = (-1, 1, 0)$ , then  $z \in W^\perp$  means

$$z \cdot (aw_1 + bw_2) = 0$$

for any  $a, b$ .

It's enough that  $z \cdot w_1 = 0$  and  $z \cdot w_2 = 0$ .



## Orthogonal complements continued

This gives two equations:

$$2z_1 + 3z_2 + z_3 = 0$$

$$-z_1 + z_2 = 0$$

which has a one dimensional solution space spanned by

$$(1, 1, -5)$$

# Orthogonal complements - properties

Suppose  $W$  is a subspace of  $\mathbf{R}^n$ .

1.  $x \in W^\perp$  if and only if  $x \cdot u = 0$  for all  $u$  in a spanning set of  $W$ . (so you only need to check finitely many vectors to cover all of the elements of  $W$ ).
2.  $W^\perp$  is a subspace of  $\mathbf{R}^n$ .

# Orthogonal complements and matrices

Let  $A$  be an  $n \times m$  matrix. Then

$$\text{Null}(A)^\perp = \text{Row}(A)$$

$$\text{Null}(A) = \text{Row}(A)^\perp$$

and

$$\text{Col}(A)^\perp = \text{Nul}(A^T)$$

$$\text{Col}(A) = \text{Null}(A^T)^\perp$$

# Angles

The “Law of Cosines” tells us that

$$u \cdot v = \|u\| \|v\| \cos \theta$$

where  $\theta$  is the angle between  $u$  and  $v$ .

# Orthogonal sets

A set  $u_1, \dots, u_k$  of vectors in  $\mathbf{R}^n$  is an *orthogonal set* if any pair of (different) vectors from the set are orthogonal.

It is an *orthonormal* set if in addition the vectors have length one.

*Key point:* An orthogonal set is linearly independent. Therefore if  $S$  is orthogonal then it is a basis for its span.

## Orthogonal basis

A basis for a subspace  $W$  is orthogonal if it is an orthogonal set.

Suppose  $y$  is any vector in  $W$  and  $u_1, \dots, u_k$  are an orthogonal basis. Then

$$y = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \dots + \frac{y \cdot u_k}{u_k \cdot u_k} u_k$$

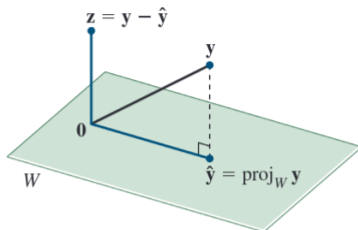
To see this, write

$$y = c_1 u_1 + \dots + c_k u_k$$

and compute  $y \cdot u_j$  on both sides to solve for  $c_j$ .

# Orthogonal projection

Let  $u$  be a vector in  $\mathbf{R}^n$ . We can decompose a vector  $y$  into a part that is “parallel” to  $u$  and a part that is *perpendicular* to  $u$ .



**FIGURE 2**

Finding  $\alpha$  to make  $y - \hat{y}$   
orthogonal to  $u$ .

Figure 3: Orthogonal Projection

## Orthogonal projection continued

In particular:

$$\hat{y} = \frac{y \cdot u}{u \cdot u} u$$

is parallel to  $u$ , and  $z = y - \hat{y}$  is perpendicular to  $u$ .

If  $u_1, \dots, u_k$  are an orthogonal basis for a subspace  $W$ , then the projection of  $y$  into  $W$  is

$$\text{proj}_W(y) = \sum \frac{y \cdot u_i}{u_i \cdot u_i} u_i$$

and  $y - \text{proj}_W(y)$  is perpendicular to  $W$ .



# Orthonormal sets

The formulae above for projections are simplified for orthonormal sets because in that case  $u_i \cdot u_i = 1$ .

Let  $U$  be an  $m \times n$  matrix. The columns of  $U$  are orthonormal if and only if  $U^T U = I$  where  $I$  is the  $n \times n$  identity matrix.

If  $U$  is  $m \times n$  and has orthonormal columns and  $x$  and  $y$  are vectors in  $\mathbf{R}^n$  then:

1.  $\|Ux\| = \|x\|$
2.  $(Ux) \cdot (Uy) = x \cdot y$
3.  $(Ux) \cdot (Uy) = 0$  if and only if  $x \cdot y = 0$ .