# Vector Spaces and Subspaces

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# Vector Spaces

Part of the power of linear algebra comes from the observation that many problems can be recast in terms of vectors from $\mathbf{R}^n$.

This process of abstraction is based on the idea of a *vector space.*

**Definition:** A (real) vector space is a set $V$ (whose elements are called *vectors*) with two operations:

- addition, which works on pairs of vectors, converting two vectors into a third: $(v, w) \mapsto v + w$
- scalar multiplication, which works on a real number $a$ and a vector $v$, yielding a vector $av$.

# Vector Space Axioms

The operations must satisfy the following properties:

- Addition is commutative $u + v = v + u$ and associative $(u + v) + w = u + (v + w)$.
- Scalar multiplication is distributve so $a(u + v) = au + av$ and $(a + b)u = au + bu$.
- Scalar multiplication satisfies $a(bv) = (ab)v$ and $1v = v$.
- There is a zero vector $0 \in V$ satisfying $0 + v = v$ for all $v$, and every vector $v$ has an inverse $-v$ so that $v + (-v) = 0$.

Clearly the "usual" vectors $\mathbf{R}^n$ satisfy all these conditions.

# Other examples of vector spaces

1. The polynomials of degree at most $n$.
2. The solutions to the differential equation $x'' + x = 0$.
3. The possible prices for a stock on the first of each month from January 2019 through December 2023. (Here each stock gives a vector of 60 prices).

# Subspaces

A subspace of a vector space is a subset that is also a vector space. If $W$ is a subset of $V$ that contains $0$ and has the closure properties:

- If $w, w' \in W$ then $w + w' \in W$
- If $w \in W$ then $aw \in W$

then $W$ is a subspace.

- The vectors in $\mathbf{R}^n$ whose last entry is zero is a subspace.
- It's silly but the set consisting of just $0$ is a subspace of any vector space.
- The polynomials of degree at most $3$ are a subspace of the polynomials of degree at most $10$.

# Subspaces and spans

If $v_1, \ldots, v_k$ are vectors in $\mathbf{R}^n$, then the span of the set of $v_i$ is a subspace.

This is called the subspace spanned by the $v_i$.

- The span of $(1, 0, 0)$ and $(0, 1, 0)$ in $\mathbf{R}^3$ is the subspace of vectors whose last entry is zero.
- The span of $(1, 1, 0)$ and $(1, -1, 0)$ is the same.
- The span of $(2, 3, 1)$ and $(-1, -1, 0)$ is a plane in $\mathbf{R}^3$ that is a vector space in its own right.

# Subspaces related to matrices

Let $A$ be an $m \times n$ matrix. So $x \mapsto Ax$ is a linear map from $\mathbf{R}^n \to \mathbf{R}^m$.

The set of vectors $v$ such that $Av = 0$ is called *the null space of* $A$ written $\mathrm{Nul}(A)$. The null space is a subspace of $\mathbf{R}^n$.

This follows because:

- $A(0) = 0$
- $A(u + v) = Au + Av = 0$ if so $u + v \in \mathrm{Nul}(A)$ if $u$ and $v$ are.
- $A(av) = aAv = 0$ so $av \in \mathrm{Nul}(A)$ if $v$ is.

Put another way, the solution to a system of *homogeneous* equations is a subspace.

# Finding the null space

To find the Null space of $A$, use row reduction to put $A$ in row reduced echelon form. Then write the basic variables in terms of the free variables, and give the general solution as a linear combination of vectors where the weights are the free variables.

Let

$$A = \begin{bmatrix} -2 & -2 & 0 & 1 & 2 \\ 1 & -2 & 2 & -1 & -2 \\ 2 & -2 & -3 & 2 & -3 \end{bmatrix}$$

Apply row reduction yielding:

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{4}{17} & -\frac{22}{17} \\ 0 & 1 & 0 & -\frac{9}{34} & \frac{5}{17} \\ 0 & 0 & 1 & -\frac{11}{17} & -\frac{1}{17} \end{bmatrix}$$

# Null Space computation

This gives

$$\begin{aligned} x_1 &= \tfrac{4}{17}x_4 + \tfrac{22}{17}x_5 \\ x_2 &= \tfrac{9}{34}x_4 - \tfrac{5}{17}x_5 \\ x_3 &= \tfrac{11}{17}x_4 + \tfrac{1}{17}x_5 \end{aligned}$$

Parametrically

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} \frac{4}{17} \\ \frac{9}{34} \\ \frac{11}{17} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \frac{22}{17} \\ -\frac{5}{17} \\ \frac{1}{17} \\ 0 \\ 1 \end{bmatrix}$$

# Conclusion

Notice that the two vectors *span the null space* and that they are *linearly independent* (look at the last two cooordinates).

Two observations:

- this algorithm will *always* produce a linearly independent spanning set for the null space
- The number of vectors in this spanning set corresponds to the number of free variables in $Ax = 0$.

# Column Space

The column space of an $m \times n$ matrix $A$ is the span of the column vectors; that is, the set of all linear combinations of the columns.

$$\mathrm{Col}(A) = \{Ax : x \in \mathbf{R}^n\}$$

The column space is a subspace because:

- 0 is a linear combination of the columns (all zero coefficients)
- if $y = Ax_1$ and $z = Ax_2$ then $y + z = A(x_1 + x_2)$
- if $y = Ax_1$ then $ay = A(ax_1)$.

The column space of $A$ is all of $\mathbf{R}^m$ means that the map $T(x) = Ax$ is onto and that $Ax = b$ has a solution for any $b$.

# Elements of $\mathrm{Col}(A)$.

The columns of $A$ are "obvious" members of $\mathrm{Col}(A)$.

Given another vector $b \in \mathrm{R}^m$, to tell if $b$ is in $\mathrm{Col}(A)$ requires finding an $x$ so that $Ax = b$.

# The Row Space

The row space is the span of the rows of a matrix.