1.4 Matrix Equation Ax=b

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Matrix Equations

A system of n linear equations in k unknowns can be written in matrix form

$$Ax = b$$

Here A is the $n \times k$ matrix of coefficients

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nk} \end{pmatrix}$$

Matrix multiplication (Matrix x Vector)

The vectors x and b are

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

The matrix product Ax is *by definition* the linear combination of the columns of A with weights given by x.

Notice that if A is $n \times m$, then b must be in $\mathbf{R}^m$ and the product is in $\mathbf{R}^n$.

Matrix Multiplication

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ 13 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 13 \\ 13 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Matrix Multiplication

The "dot product" $a \cdot b$ of two vectors a and b in $\mathbf{R}^m$, where

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

is the sum

$$a \cdot b = a_1 b_1 + a_2 b_2 + \cdots a_m b_m$$

Matrix Multiplication

The entries of the product Ab (where A is $n \times m$ and b is in $\mathbf{R}^m$) are the successive dot products of the rows of A with b.

Each row of A has m entries, and b has m entries; there are n dot products, so the product is in $\mathbf{R}^n$.

Solving a matrix equation

Given the matrix equation $Mx = b$, to solve it, use row reduction on the augmented matrix $[Mb]$.

For M and b as above the augmented matrix is

$$\begin{bmatrix} 1 & 3 & 2 & 13 \\ 2 & 4 & 1 & 13 \\ -1 & -1 & 0 & -3 \end{bmatrix}$$

Applying row reduction yields

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This gives $x_1 = 1, x_2 = 2, x_3 = 3$ as the only solution to this equation.

Another example

Consider the following M and b.

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

Augmented form

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix}$$

Example continued

Reduced form

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution

x_3 is a free variable.

$$\begin{aligned} x_2 &= 2 \\ x_1 &= -1 - \tfrac{4}{3}x_3 \end{aligned}$$

Vector form

$$x = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{pmatrix}$$

Inconsistency

A system is inconsistent if the only non zero entry in a row occurs by itself in the last column. Consider $Mx = b$.

Matrix M

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Vector b

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Inconsistency example

Reduced form - last row shows inconsistent.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Summary

Let A be an $n \times m$ matrix. Then the following statements are either all true or all false:

1. $Ax = b$ has a solution for any $b \in \mathbf{R}^m$
2. Any b in $\mathbf{R}^m$ is a linear combination of the columns of A.
3. The columns of A span $\mathbf{R}^m$.
4. The rref of A has a pivot in every row.

Homogeneous Systems

A *homogeneous system* is a matrix equation $Ax = 0$, so the target vector b is zero.

In general:

- The solutions are parameterized by s vectors, where s is the number of free variables in the reduced matrix A.
- The values of the variables corresponding to pivots are determined by the free variables.
- If there are no free variables, the only solution to the inhomogeneous system is zero.

Also notice that if v and w satisfy $Av = Aw = 0$ then also $A(v + w) = 0$ and $A(cv) = 0$.

Some examples

$$\text{Matrix } A = \begin{bmatrix} 11 & -9 & -3 \\ -14 & 3 & -9 \\ -11 & -7 & -16 \\ 8 & -20 & -15 \\ -8 & 9 & -1 \end{bmatrix}$$

$$\text{Reduced form} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

There are no free variables, so the zero vector is the only solution.

Example

$$\text{Matrix } A = \begin{bmatrix} 11.0 & -9.0 & -3.0 & -14.0 & 3.0 \\ -9.0 & -11.0 & -7.0 & -16.0 & 8.0 \\ -20.0 & -15.0 & -8.0 & 9.0 & -1.0 \end{bmatrix}$$

$$\text{Reduced form} = \begin{bmatrix} 1.0 & 0 & 0 & -1.7 & 0.55 \\ 0 & 1.0 & 0 & -4.3 & 2.0 \\ 0 & 0 & 1.0 & 11.0 & -5.0 \end{bmatrix}$$

Here there are two free variables (x_4 and x_5). The solutions are

$$\begin{aligned} x_1 &= -1.7x_4 + .55x_5 \\ x_2 &= -4.3x_4 + 2.0x_5 \\ x_3 &= 11x_4 - 5x_4 \end{aligned}$$

Parametric form for solutions

In vector form this is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} -1.7 \\ -4.3 \\ 11 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} .55 \\ 2.0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

Some special cases

Notice that:

- if a matrix has more columns than rows (A is $n \times m$ and $m > n$) – or more variables than equations –
- then the homogeneous system $Ax = b$ always has infinitely many solutions, and in fact there are always at least $m - n$ free variables.

Nonhomogeneous systems

If $b \neq 0$ then $Ax = b$ is called a nonhomogeneous or inhomogeneous system.

Key Observation:

1. If v is a solution to the homogeneous system $Ax = 0$, and w is a solution to the inhomogeneous system $Ax = b$, then $v + w$ is also a solution to $Ax = b$ because $A(v + w) = Av + Aw = 0 + b = b$.

2. If v and w are two solutions to $Ax = b$, then $v - w$ is a solution to $Ax = 0$ because $A(v - w) = Av - Aw = b - b = 0$.

Therefore the solutions (if any) to the inhomogeneous system are of the form $v + w$ where v is any *one* solution to $Ax = b$ and w is any solution to $Ax = 0$.

Examples

Matrix $A = \begin{bmatrix} -20.0 & -6.0 & -18.0 \\ -11.0 & -12.0 & -12.0 \end{bmatrix}$

b$= \begin{bmatrix} -11.0 \\ -5.0 \end{bmatrix}$

Reduced form of augmented$= \begin{bmatrix} 1.0 & 0 & 0.83 & 0.59 \\ 0 & 1.0 & 0.24 & -0.12 \end{bmatrix}$

Solution:

$$\begin{aligned} x_1 &= -.83x_3 - .59 \\ x_2 &= -.24x_3 + .12 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -.59 \\ .12 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -.83 \\ -.24 \\ 1 \end{bmatrix}$$

An example

Propane combusion: C_3H_8 and O_2 combine to yield CO_2 and H_2O.

Balancing:

$$(x_1)C_3H_8 + (x_2)O_2 = x_3(CO_2) + x_4(H_2O)$$

This means

$$\begin{aligned} 3x_1 &= x_3 \\ 8x_1 &= 2x_4 \\ 2x_2 &= x_4 \end{aligned}$$

Matrix form $Ax = 0$ where

$$A = \begin{bmatrix} 3 & 0 & -1 & 0 \\ 8 & 0 & 0 & -2 \\ 0 & 2 & 0 & -1 \end{bmatrix}$$

Solution

A$=\begin{bmatrix}3 & 0 & -1 & 0\\ 8 & 0 & 0 & -2\\ 0 & 2 & 0 & -1\end{bmatrix}$

reduced form$=\begin{bmatrix}1 & 0 & 0 & -\frac{1}{4}\\ 0 & 1 & 0 & -\frac{1}{2}\\ 0 & 0 & 1 & -\frac{3}{4}\end{bmatrix}$

The solution has one free variable x_4 and we have

$$\begin{aligned} x_1 &= 1/4x_4 \\ x_2 &= 1/2x_4 \\ x_3 &= 3/4x_4 \end{aligned}$$

Parametric form

You get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 1/4 \\ 1/2 \\ 3/4 \\ 1 \end{bmatrix}$$

You can rescale this to get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$