

Diagonalization of Symmetric Matrices

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Symmetric Matrices

A matrix A is *symmetric* if $A^T = A$. For example

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & -5 & 7 \\ 1 & 7 & 2 \end{bmatrix}$$

is symmetric.

In our discussion of least squares fitting, we ended up with the matrix $X^T X$ where X was our data matrix. This matrix is symmetric because $(X^T X)^T = X^T X$.

If U is a matrix is an $n \times m$ matrix with columns u_i for $i = 1, \dots, m$, then $U^T U$ is an $m \times m$ matrix whose i, j entry is the dot product of u_i and u_j . It is symmetric because the dot product is commutative.

Eigenvalues/vectors of symmetric matrices

Eigenvalues and vectors of symmetric matrices have special properties. Suppose A is an $n \times n$ symmetric matrix.

Theorem: If v and w are eigenvectors of A with eigenvalues λ and μ where $\lambda \neq \mu$ then $v \cdot w = 0$.

To see this, consider $Av = \lambda v$ and $Aw = \mu w$. Then $w^T A^T = \mu w^T$. But $A^T = A$ so $w^T A = \mu w^T$.

Remember that $w^T v = w \cdot v$.

This gives us

$$w^T Av = \mu(w^T v) = \mu(w \cdot v) = \lambda(w^T v) = \lambda(w \cdot v)$$

Since $\lambda \neq \mu$ this means $w \cdot v = 0$.

Orthogonal Diagonalizability

Remember that a square matrix A is diagonalizable if there is a matrix P so that $A = PDP^{-1}$ where D is a diagonal matrix (whose entries are the eigenvalues of A).

Definition: An $n \times n$ matrix A is *orthogonally diagonalizable* if is an orthonormal set u_1, \dots, u_n of eigenvectors of A .

In other words, A has n mutually orthogonal eigenvectors of length 1.

Orthogonal Diagonalizability example

Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Note that A is symmetric. The characteristic polynomial of A is

$$T^2 - 2T - 3 = (T - 3)(T + 1)$$

so the eigenvalues are 3 and -1 .

The eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Orthogonal diagonalizability example (cont'd)

Notice that v_1 and v_2 are orthogonal. To make them orthonormal, divide by their norms to get

$$u_1 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, u_2 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

We therefore have the equation:

$$A \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$A = PDP^{-1}$$

where

$$P = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

Orthogonal diagonalizability example (cont'd)

The matrix P is an *orthogonal* matrix:

- ▶ its columns are an orthonormal set.
- ▶ $P^T = P^{-1}$

The Spectral Theorem

Symmetric matrices are *special*.

The Spectral Theorem for Symmetric Matrices

An $n \times n$ symmetric matrix A has the following properties:

- A has n real eigenvalues, counting multiplicities.
- The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.
- The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.
- A is orthogonally diagonalizable.

Figure 1: Spectral Theorem

Spectral Decomposition

Let u_1, \dots, u_n be the orthonormal eigenvectors of A , where A is symmetric. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues (with multiplicity)

Then

$$A = \lambda_1 u_1 u_1^T + \dots + \lambda_n u_n u_n^T$$

This writes A as a (weighted) sum of projection operators since $u_i u_i^T v$ gives the projection of v into the u direction.

Spectral Decomposition example

In our 2×2 example we had

$$u_1 = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, u_2 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

So

$$u_1 u_1^T = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

and

$$u_2 u_2^T = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

And

$$A = 3 \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} - \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

