

## 3.1 Determinants

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## Recall the 2x2 case

If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a  $2 \times 2$  matrix, then the “determinant” of  $A$  (written  $\det(A)$ ) is

$$\det(A) = ad - bc$$

The matrix  $A$  is invertible if and only if  $\det(A) \neq 0$ ; if it's non-zero then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

## Generalization

The determinant of an  $n \times n$  matrix is defined inductively starting from the  $2 \times 2$  case.

Let  $A$  be an  $n \times n$  matrix and let  $A_i$  be the  $(n-1) \times (n-1)$  submatrix obtained by deleting the  $i^{th}$  row and column.

Then (by definition)

$$\det(A) = \sum_{i=1}^n (-1)^{n-1} a_{1i} \det(A_i).$$

This works because you can use the same formula on  $A_i$  (which is smaller) to find its determinant. When you get to the  $2 \times 2$  case you know the answer.

## 3x3 case

$$\det(A_1) = a_{22}a_{33} - a_{23}a_{32}:$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A_2) = a_{21}a_{33} - a_{23}a_{31}:$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A_3) = a_{21}a_{32} - a_{22}a_{31}:$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11} \det(A_1) - a_{12} \det(A_2) + a_{13} \det(A_3)$$

# Cofactors

We can generalize the work above by introducing the submatrix  $A_{ij}$  obtained by deleting row  $i$  and column  $j$  from our matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

The  $i, j$  cofactor  $C_{ij}$  of  $A$  is  $(-1)^{i+j} \det(A_{ij})$ . The sign here is important!

## Cofactor Expansion

The determinant can be expanded along any row or column yielding the same result.

Fix  $i$  and compute along row  $i$ :

$$\det(A) = \sum_{j=1}^n a_{ij} C_{ij}$$

or fix  $j$  and compute along column  $j$ :

$$\det(A) = \sum_{i=1}^n a_{ij} C_{ij}$$

## Example (first row expansion)

$$M = \begin{bmatrix} -7 & 4 & 6 \\ 6 & 5 & -7 \\ -2 & -1 & -8 \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} 5 & -7 \\ -1 & -8 \end{bmatrix}, C_{11} = -47$$

$$M_{12} = \begin{bmatrix} 6 & -7 \\ -2 & -8 \end{bmatrix}, C_{12} = 62$$

$$M_{13} = \begin{bmatrix} 6 & 5 \\ -2 & -1 \end{bmatrix}, C_{13} = 4$$

$$\det(M) = 601$$

## Example (second row expansion)

$$M = \begin{bmatrix} -7 & 4 & 6 \\ 6 & 5 & -7 \\ -2 & -1 & -8 \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} 4 & 6 \\ -1 & -8 \end{bmatrix}, C_{21} = 26$$

$$M_{22} = \begin{bmatrix} -7 & 6 \\ -2 & -8 \end{bmatrix}, C_{22} = 68$$

$$M_{23} = \begin{bmatrix} -7 & 4 \\ -2 & -1 \end{bmatrix}, C_{23} = -15$$

$$\det(M) = 601$$



# Triangular Matrices

Suppose that  $A$  is an  $n \times n$  triangular matrix, meaning that all of the entries  $a_{ij}$  where  $i > j$  are zero (or all of its entries  $a_{ij}$  where  $i < j$  are zero)

Then the determinant of  $A$  is the product of its diagonal entries.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 7 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\det(A) = (1)(7)(3) = 21$$

Note: this is a nice result to see by mathematical induction.

# Row operations and determinants

The three row operations on a square matrix  $A$  have the following effects:

1. Adding a multiple of one row to another does not affect the determinant.
2. Interchanging two rows changes the sign of the determinant.
3. Multiplying a row by a constant  $k$  multiplies the determinant by  $k$ .

Note: the same is true of “column” operations.

## Computing determinants by reduction

$$A = \begin{bmatrix} -5 & 4 & 4 \\ -5 & 2 & -4 \\ 0 & 4 & -4 \end{bmatrix}$$

Replace row 2 with row 2 minus row 1.

$$\begin{bmatrix} -5 & 4 & 4 \\ 0 & -2 & -8 \\ 0 & 4 & -4 \end{bmatrix}$$

Replace row 3 by row 3 plus 2 row 2.

$$\begin{bmatrix} -5 & 4 & 4 \\ 0 & -2 & -8 \\ 0 & 0 & -20 \end{bmatrix}$$

So determinant is  $(-5)(-2)(-20) = -200$ .

# Fundamental properties

Let  $A$  and  $B$  be  $n \times n$  matrices.

1.  $\det(AB) = \det(A) \det(B)$ .
2.  $A$  is invertible if and only if  $\det(A)$  is nonzero.
3.  $\det(A^T) = \det(A)$

## Cramer's Rule

Look at the equation  $Ax = b$  where  $A$  is an  $n \times n$  matrix.

If  $Y$  is an  $n \times n$  matrix, Let  $Y_i(x)$  be the matrix obtained by replacing column  $i$  with  $x$  and let  $Y_i(b)$  be the matrix obtained from  $Y$  by replacing the  $i^{th}$  column by  $b$ .

Then  $AI_i(x) = A_i(b)$ .

A  $2 \times 2$  example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & y \end{bmatrix} = \begin{bmatrix} a & ax + by \\ c & cx + dy \end{bmatrix} = \begin{bmatrix} a & u \\ c & v \end{bmatrix}$$

so  $\det(A)y = \det(A_2(b))$  and therefore

$$y = \det(A_2(b)) / \det(A)$$

## More on Cramer's Rule

The general form of Cramer's rule is:

$$x_i = \frac{\det(A_i(b))}{\det(A)}$$

This is true because  $A_i(x) = AI_i(x)$ . The determinant of  $I_i(x)$  is  $x_i$ . So

$$\det(A)x_i = \det(A_i(b))$$

# Volumes

Let  $A$  be a square matrix of size  $n \times n$ . The linear map  $x \mapsto Ax$  expands volumes by a factor of  $|\det(A)|$ .

This is a generalization of the fact that the volume of a parallelogram is the base times the height.