Gram Schmidt

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In our discussion so far we have been handed orthogonal bases for various subspaces.

How do we find such a thing?

Problem: Given a set of vectors $v_1, \ldots, v_k$ in $\mathbf{R}^n$, find an orthogonal basis (or an orthonormal basis) for the span W of those vectors.

Strategy: Work systematically:

- Start with v_1; it becomes u_1.
- Subtract the component of v_2 in the v_1 direction from v_2; call this u_2.
- Subtract the projection of v_3 into the span of u_1 and u_2 from v_3, and call that u_3.
- Continue in this way, subtracting the projection of v_n from the span of $u_1, \ldots, u_{n-1}$, to obtain u_n.

If you normalize these vectors u_i you get an orthonormal basis.

Gram Schmidt (Example)

Suppose

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

The first two vectors in the sequence of G-S vectors is

$$u_1 = v_1, u_2 = v_2 - 3/4 v_1 = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

Example (continued)

The third vector

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2}$$

Now $u_1 \cdot u_1 = 4$ and

$u_2 \cdot u_2 = v_2 \cdot v_2 - 3/2 v_2 \cdot v_1 + 9/16 v_1 \cdot v_1 = 3 - 9/2 + 9/4 = 3/4$

Also $v_3 \cdot u_2 = v_3 \cdot v_2 - 3/4 v_3 \cdot v_1 = 1/2$. So

$$u_3 = v_3 - \frac{2}{4} u_1 - \frac{2}{3} u_2 = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$