

# Eigenvectors and Linear Transformations

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# Linear Transformations and Matrices

Remember that a linear transformation  $T : \mathbf{R}^m \rightarrow \mathbf{R}^n$  is a function that satisfies the two conditions:

- ▶  $T(ax) = aT(x)$  for all  $x \in \mathbf{R}^m$  and  $a \in \mathbf{R}$ .
- ▶  $T(x + y) = T(x) + T(y)$  for all  $x, y \in \mathbf{R}^m$ .

We saw earlier that a linear transformation can be represented by an  $n \times m$  matrix  $A$  where

$$T(x_1, \dots, x_m) = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

# Linear transformations and bases

We can take a slightly more general point of view on matrices and linear transformations.

In the earlier version we used “standard coordinates” where  $x_1, \dots, x_n$  are relative to the “standard basis.”

Now suppose  $B = \{b_1, \dots, b_n\}$  are a basis for  $\mathbf{R}^n$ . Then if

$$x = r_1 b_1 + \dots + r_n b_n$$

we have the coordinate vector

$$[x]_B = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}.$$

## Linear transformations in other bases

By linearity

$$T(x) = T(r_1b_1 + \cdots + r_nb_n) = r_1T(b_1) + \cdots + r_nT(b_n)$$

In other words, if we make a matrix  $M$  whose *columns* are the vectors  $T(b_i)$ , then

$$T(x) = M[x]_B = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

The matrix  $M$  is called *the matrix of the linear transformation  $T$  in the basis  $B$*  and is written

$$M = [T]_B$$

## Linear transformations and change of basis

If we write  $S$  for the standard basis, the “change of basis matrix”  $P_{S \leftarrow B}$  (which the book calls just  $P_B$ ) has the property that

$$P_{S \leftarrow B}[x]_B = [x]_S$$

If  $T(x) = Ax$ , then in our notation above  $A = [T]_S$  and  $x = [x]_S$ . We can write this equation as

$$[T(x)]_S = [T]_S[x]_S$$

## Linear transformations and change of basis cont'd

So

$$[T(x)]_S = A[x]_S = AP_{S \leftarrow B}[x]_B$$

But if we want the output of  $T$  to *also* be in the  $B$ -basis, we need one more step:

$$[T(x)]_B = P_{B \leftarrow S}[T(x)]_S = P_{B \leftarrow S}AP_{S \leftarrow B}[x]_B$$

## Linear transformations and change of basis continued

If we simplify the notation and write  $P = P_{S \leftarrow B}$  then we see that

$$[T(x)]_B = [T]_B[x]_B = P^{-1}AP[x]_B$$

where  $A = [T]_S$

In other words, the matrix of  $T$  in the  $B$ -basis is *similar* to the matrix in the standard basis.

More generally, the collection of matrices that are similar to  $A = [T]_S$  are the collection of matrix representations of  $T$  in the possible bases of  $\mathbf{R}^n$ .

# Diagonal transformations

If the matrix  $A$  is diagonalizable, then one can find a basis  $B$  so that  $[T]_B$  is a diagonal matrix.



## Example

Suppose that  $B = \{b_1, b_2\}$  is a basis for a vector space  $V$  and that  $T : V \rightarrow V$  is the linear transformation defined by

$$T(b_1) = 7b_1 + 4b_2, T(b_2) = 6b_1 - 5b_2.$$

The matrix

$$[T]_B = \begin{bmatrix} 7 & 6 \\ 4 & -5 \end{bmatrix}$$

## Example continued

Is there a basis in which  $T$  is given by a diagonal matrix?

The characteristic polynomial of  $[T]_B$  is

$$\det \begin{bmatrix} 7 - \lambda & 6 \\ 4 & -5 - \lambda \end{bmatrix} = \lambda^2 - 2\lambda - 59$$

The roots are  $1 \pm 2\sqrt{15}$ . Since these are distinct, the matrix is diagonalizable.

## Example continued

The eigenvectors are

$$v_{\pm} = \begin{bmatrix} \frac{3 \pm \sqrt{15}}{2} \\ 1 \end{bmatrix}$$

and in the basis  $E$  given by these eigenvectors the matrix of  $T$  is diagonal.