

## 1.4-1.5 Matrix Equation $Ax=b$

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# Matrix Equations

A system of  $n$  linear equations in  $k$  unknowns can be written in matrix form

$$Ax = b$$

Here  $A$  is the  $n \times k$  matrix of coefficients

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nk} \end{pmatrix}$$

## Matrix multiplication (Matrix x Vector)

The vectors  $x$  and  $b$  are

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

The matrix product  $Ax$  is *by definition* the linear combination of the columns of  $A$  with weights given by  $x$ .

Notice that if  $A$  is  $n \times m$ , then  $b$  must be in  $\mathbf{R}^m$  and the product is in  $\mathbf{R}^n$ .

## Matrix Multiplication

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ 13 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 13 \\ 13 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

# Matrix Multiplication

The “dot product”  $a \cdot b$  of two vectors  $a$  and  $b$  in  $\mathbf{R}^m$ , where

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

is the sum

$$a \cdot b = a_1 b_1 + a_2 b_2 + \cdots a_m b_m$$

# Matrix Multiplication

The entries of the product  $Ab$  (where  $A$  is  $n \times m$  and  $b$  is in  $\mathbf{R}^m$ ) are the successive dot products of the rows of  $A$  with  $b$ .

Each row of  $A$  has  $m$  entries, and  $b$  has  $m$  entries; there are  $n$  dot products, so the product is in  $\mathbf{R}^n$ .

## Solving a matrix equation

Given the matrix equation  $Mx = b$ , to solve it, use row reduction on the augmented matrix  $[Mb]$ .

For  $M$  and  $b$  as above the augmented matrix is

$$\begin{bmatrix} 1 & 3 & 2 & 13 \\ 2 & 4 & 1 & 13 \\ -1 & -1 & 0 & -3 \end{bmatrix}$$

Applying row reduction yields

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This gives  $x_1 = 1, x_2 = 2, x_3 = 3$  as the only solution to this equation.



## Another example

Consider the following  $M$  and  $b$ .

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

Augmented form

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix}$$

## Example continued

Reduced form

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution

$x_3$  is a free variable.

$$x_2 = 2$$

$$x_1 = -1 - \frac{4}{3}x_3$$

## Vector form

$$x = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{pmatrix}$$

# Inconsistency

A system is inconsistent if the only non zero entry in a row occurs by itself in the last column. Consider  $Mx = b$ .

Matrix M

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Vector b

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

## Inconsistency example

Reduced form - last row shows inconsistent.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Summary

Let  $A$  be an  $n \times m$  matrix. Then the following statements are either all true or all false:

1.  $Ax = b$  has a solution for any  $b \in \mathbf{R}^m$
2. Any  $b$  in  $\mathbf{R}^m$  is a linear combination of the columns of  $A$ .
3. The columns of  $A$  span  $\mathbf{R}^m$ .
4. The rref of  $A$  has a pivot in every row.

# Homogeneous Systems

A *homogeneous system* is a matrix equation  $Ax = 0$ , so the target vector  $b$  is zero.

In general:

- ▶ The solutions are parameterized by  $s$  vectors, where  $s$  is the number of free variables in the reduced matrix  $A$ .
- ▶ The values of the variables corresponding to pivots are determined by the free variables.
- ▶ If there are no free variables, the only solution to the inhomogeneous system is zero.

Also notice that if  $v$  and  $w$  satisfy  $Av = Aw = 0$  then also  $A(v + w) = 0$  and  $A(cv) = 0$ .

## Some examples

$$\text{Matrix } A = \begin{bmatrix} 11 & -9 & -3 \\ -14 & 3 & -9 \\ -11 & -7 & -16 \\ 8 & -20 & -15 \\ -8 & 9 & -1 \end{bmatrix}$$

$$\text{Reduced form} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

There are no free variables, so the zero vector is the only solution.



## Example

$$\text{Matrix } A = \begin{bmatrix} 11.0 & -9.0 & -3.0 & -14.0 & 3.0 \\ -9.0 & -11.0 & -7.0 & -16.0 & 8.0 \\ -20.0 & -15.0 & -8.0 & 9.0 & -1.0 \end{bmatrix}$$

$$\text{Reduced form} = \begin{bmatrix} 1.0 & 0 & 0 & -1.7 & 0.55 \\ 0 & 1.0 & 0 & -4.3 & 2.0 \\ 0 & 0 & 1.0 & 11.0 & -5.0 \end{bmatrix}$$

Here there are two free variables ( $x_4$  and  $x_5$ ). The solutions are

$$x_1 = -1.7x_4 + .55x_5$$

$$x_2 = -4.3x_4 + 2.0x_5$$

$$x_3 = 11x_4 - 5x_5$$

## Parametric form for solutions

In vector form this is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} -1.7 \\ -4.3 \\ 11 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} .55 \\ 2.0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

## Some special cases

Notice that:

- ▶ if a matrix has more columns than rows ( $A$  is  $n \times m$  and  $m > n$ ) – or more variables than equations –
- ▶ then the homogeneous system  $Ax = b$  always has infinitely many solutions, and in fact there are always at least  $m - n$  free variables.

## Nonhomogeneous systems

If  $b \neq 0$  then  $Ax = b$  is called a nonhomogeneous or inhomogeneous system.

### Key Observation:

1. If  $v$  is a solution to the homogeneous system  $Ax = 0$ , and  $w$  is a solution to the inhomogeneous system  $Ax = b$ , then  $v + w$  is also a solution to  $Ax = b$  because  $A(v + w) = Av + Aw = 0 + b = b$ .
2. If  $v$  and  $w$  are two solutions to  $Ax = b$ , then  $v - w$  is a solution to  $Ax = 0$  because  $A(v - w) = Av - Aw = b - b = 0$ .

Therefore the solutions (if any) to the inhomogeneous system are of the form  $v + w$  where  $v$  is any *one* solution to  $Ax = b$  and  $w$  is any solution to  $Ax = 0$ .

## Examples

$$\text{Matrix } A = \begin{bmatrix} -20.0 & -6.0 & -18.0 \\ -11.0 & -12.0 & -12.0 \end{bmatrix}$$

$$b = \begin{bmatrix} -11.0 \\ -5.0 \end{bmatrix}$$

$$\text{Reduced form of augmented} = \begin{bmatrix} 1.0 & 0 & 0.83 & 0.59 \\ 0 & 1.0 & 0.24 & -0.12 \end{bmatrix}$$

Solution:

$$x_1 = -.83x_3 - .59$$

$$x_2 = -.24x_3 + .12$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -.59 \\ .12 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -.83 \\ -.24 \\ 1 \end{bmatrix}$$

## An example

Propane combustion:  $C_3H_8$  and  $O_2$  combine to yield  $CO_2$  and  $H_2O$ .

Balancing:



This means

$$\begin{aligned} 3x_1 &= x_3 \\ 8x_1 &= 2x_4 \\ 2x_2 &= x_4 \end{aligned}$$

Matrix form  $Ax = 0$  where

$$A = \begin{bmatrix} 3 & 0 & -1 & 0 \\ 8 & 0 & 0 & -2 \\ 0 & 2 & 0 & -1 \end{bmatrix}$$

## Solution

$$A = \begin{bmatrix} 3 & 0 & -1 & 0 \\ 8 & 0 & 0 & -2 \\ 0 & 2 & 0 & -1 \end{bmatrix}$$

$$\text{reduced form} = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{4} \end{bmatrix}$$

The solution has one free variable  $x_4$  and we have

$$\begin{aligned} x_1 &= 1/4x_4 \\ x_2 &= 1/2x_4 \\ x_3 &= 3/4x_4 \end{aligned}$$

## Parametric form

You get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 1/4 \\ 1/2 \\ 3/4 \\ 1 \end{bmatrix}$$

You can rescale this to get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$