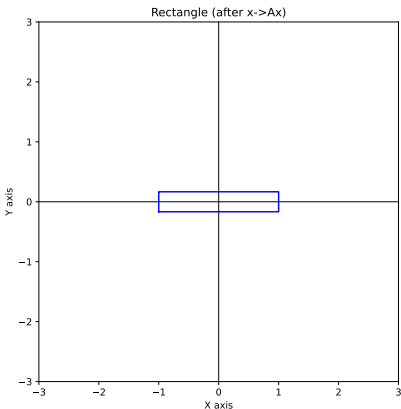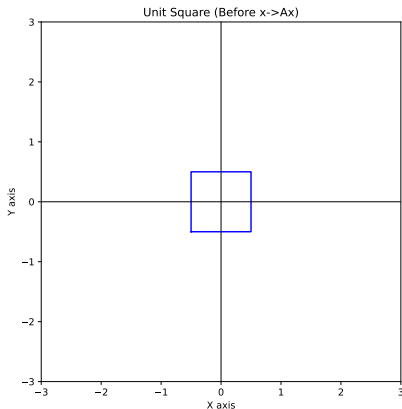# Eigenvalues and Eigenvectors

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Eigenvalues and Eigenvectors

If A is a diagonal matrix:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

then the linear transformation $x \mapsto Ax$ "stretches" along the x-axis and "shrinks" along the y-axis.
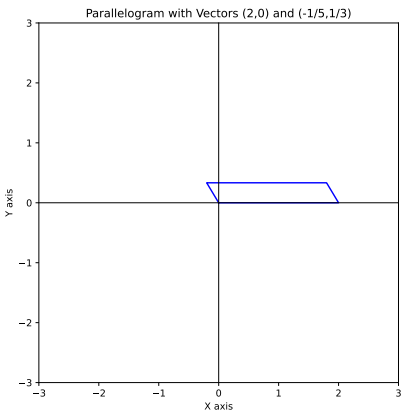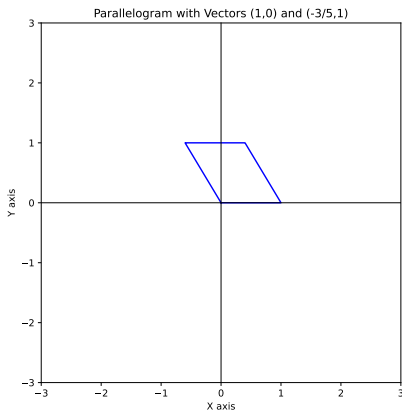
Eigenvalues and Eigenvectors

If A is upper triangular, say

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix}$$

then A stretches along the x-axis by 2 as before. Less obviously, it shrinks along the direction given by the vector $(-3/5, 1)$.

Eigenvalues and Eigenvectors

An **eigenvector** for a matrix A is a vector v which gets shrunk or lengthened by A by some factor λ.

The factor λ is called the **eigenvalue**.

More formally, a vector v is called an eigenvector for A (with eigenvalue λ) if v is not zero and

$$Av = \lambda v.$$

Eigenvalues and Eigenvectors

In the example above, the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3/5 \\ 1 \end{bmatrix}$ are eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix}$$

with eigenvalues 2 and $1/3$ respectively.

$$\begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} -3/5 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 1/3 \end{bmatrix} = (1/3) \begin{bmatrix} -3/5 \\ 1 \end{bmatrix}$$

Triangular Matrices

If A is (upper) triangular then the diagonal entries for A are all eigenvalues. There are n linearly independent eigenvectors.

Eigenspaces

Suppose that λ is a constant. The vectors v such that

$$Av = \lambda v$$

form a subspace called the *eigenspace* for λ.

This subspace is the nullspace of the matrix

$$A - \lambda I_n$$

where I_n is the $n \times n$ identity matrix.

Independence of Eigenvectors

If $v_1, \ldots, v_n$ are eigenvectors for a matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$, and all the λ_i are different, then the v_i are linearly independent. (Note that the v_i are nonzero.)

To see this, suppose that

$$c_1 v_1 + c_2 v_2 + \cdots c_n v_n = 0.$$

Then

$$A(c_1 v_1 + c_2 v_2 + \cdots c_n v_n) = c_1 \lambda_1 v_1 + \cdots c_n \lambda_n v_n = 0$$

Multiply the first relation by λ_1 and subtract. You get

$$c_2(\lambda_2 - \lambda_1)v_2 + \cdots + c_n(\lambda_n - \lambda_1)v_n = 0.$$

Since the differences of the λ_i with λ_1 are not zero, we see that $v_2, \ldots, v_n$ are dependent.

By repeating this you can show that smaller and smaller collections of the v_i are dependent until you ultimately get $v_n = 0$.

Characteristic Equation

Finding eigenvalues and eigenvectors of a matrix is a hard problem. We can make the following observation.

Suppose λ is an eigenvalue of A where A is an $n \times n$ matrix. Then there is a vector $v \neq 0$ so that $Av = \lambda v$. This means that the matrix $A - \lambda I_n$ is *not* invertible because v is in its null space.

As a result, $\det(A - \lambda I_n = 0$.

Conversely, if $\det(A - \lambda I_n) = 0$, then there is a vector v in the null space and that v is an eigenvector.

It turns out that $\det(A - \lambda I_n)$ is a polynomial in λ, so the eigenvalues of A are the roots of this polynomial.

Example

Let
$$A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}.$$

The determinant of $A - \lambda$ is

$$\det(\begin{bmatrix} 3 - \lambda & 5 \\ 2 & 4 - \lambda \end{bmatrix}) = (3 - \lambda)(4 - \lambda) - 10$$

The polynomial on the right is

$$(3 - \lambda)(4 - \lambda) - 10 = \lambda^2 - 7\lambda + 12 - 10 = \lambda^2 - 7\lambda + 2.$$

Its roots are $\frac{7 \pm \sqrt{41}}{2}$. These are the eigenvalues of A; they are approximately 6.70156 and 0.29843.

Example continued

To find the eigenvectors, we have to compute the null space of $A - \lambda I$. This is no fun algebraically but with some work you find that the eigenvectors are:

$$\begin{bmatrix} -\frac{\sqrt{41}}{4} - \frac{1}{4} \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{41}}{4} \\ 1 \end{bmatrix}$$

Bigger matrices

The characteristic polynomial of an $n \times n$ matrix is of degree n.

Theorem: A polynomial of degree n has n complex roots (counted correctly).

However, in practice one must use numerical methods to find roots of a polynomial of degree 3 or higher.

If one know an eigenvalue λ *exactly* (which usually doesn't happen) then you can find the eigenvectors by computing the null space of $A - \lambda$.

Example

Let

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$$

Since A is (lower) triangular, the eigenvalues are $5, 0$ and -2. The vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is the eigenvector with eigenvalue 5.

Example continued

To find the eigenvector with eigenvalue 0, we must solve

$$Ax = 0.$$

By one of our techniques this is

$$\begin{bmatrix} -8 \\ 5 \\ 0 \end{bmatrix}$$

Example continued

The final eigenvector is in the null space of $A + 2I$:

$$A = \begin{bmatrix} 7 & -8 & 1 \\ 0 & 2 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

This is

$$\begin{bmatrix} -58/7 \\ -7 \\ 2 \end{bmatrix}$$

Similarity

Two (square) matrices A and B are *similar* if there is an invertible matrix P so that $A = PBP^{-1}$.

Similar matrices have the same eigenvalues because they have the same characteristic polynomial.

$$
\begin{aligned}
\det(PBP^{-1} - \lambda I) &= \det(P(B - \lambda I)P^{-1}) \\
&= \det(P)\det(B - \lambda I)\det(P^{-1}) \\
&= \det(B - \lambda I)
\end{aligned}
$$

Diagonalization

Diagonal matrices are the simplest to work with.

Definition: A square matrix is diagonalizable if there is an invertible matrix P so that $A = PDP^{-1}$ where P diagonal. In other words, A is diagonalizable if it is similar to a diagonal matrix.

Theorem: An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

In fact, if $A = PDP^{-1}$, then the columns of P are n linearly independent eigenvectors for A, and the diagonal entries of D are the corresponding eigenvalues. This is because in this situation,

$$AP = PD.$$

Example (from the text)

Let
$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

1. The eigenvalues of A. The text tells us that the characteristic polynomial of A is $-(\lambda - 1)(\lambda + 2)^2$ so the eigenvalues are 1 and -2.

Example continued

We need three linearly independent eigenvectors. So we need the
null spaces of $A - I$ and $A + 2I$. The book gives us:

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

with eigenvalue 1. For the eigenvalue -2:

$A+2I= \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}$

Reduced form is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

So null space of $A + 2I$ is two dimensional and spanned by

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Therefore

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Example continued

We can compute AP:

$$AP = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} = PD$$

where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

So $A = PDP^{-1}$ and A is diagonalizable.

Not all matrices are diagonalizable

Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The only eigenvalue is 1. The null space of $A - I$ is only one dimensional, spanned by

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So there's no basis of eigenvectors, so A can't be diagonalized.

n distinct eigenvalues implies diagonalizable

If A has n different eigenvalues, then it has n linearly independent eigenvectors; thus there is a basis of eigenvectors.

Therefore in this case A is diagonalizable.

But as we saw above, you can have repeated eigenvalues and still be diagonalizable.

The diagonalization Theorem

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

a. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

b. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (i) the characteristic polynomial factors completely into linear factors and (ii) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

c. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

Figure 1: Diagonalization Theorem

A numerical example

Let

$$M = \begin{bmatrix} -6 & 4 & 0 & 9 \\ -3 & 0 & 1 & 6 \\ -1 & -2 & 1 & 0 \\ -4 & 4 & 0 & 7 \end{bmatrix}$$

```python
import numpy as np
M = Matrix([[-6, 4, 0, 9], [-3, 0, 1, 6],...
M = np.array(M).astype(np.float64)
L, P = np.linalg.eig(M)
```

The results are

Eigenvalues (list L) are [5. 1. -2. -2.]

P matrix (rounded) is
$$\begin{bmatrix} -0.63 & 0.26 & -0.84 & -0.57 \\ -0.32 & -0.13 & 0.23 & -0.58 \\ 0.32 & -0.92 & -0.13 & -0.58 \\ -0.63 & 0.26 & -0.48 & 0 \end{bmatrix}$$