

# Rank and Change of Basis

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# Rank

Let  $A$  be an  $n \times m$  matrix.

The row rank of  $A$  is the dimension of the space spanned by the rows of  $A$ .

The column rank of  $A$  is the dimension of the space spanned by the columns of  $A$ .

The nullity of  $A$  is the dimension of the null space of  $A$ .

## Row rank = column rank

Perhaps surprisingly, the row and column ranks are the same.

To see this, put  $A$  into row reduced echelon form. The dimension of the column space is the number of linearly independent columns, which is the number of columns containing a pivot.

The rows of  $A$  containing a pivot form a basis for the row space of  $A$ .

Since the number of pivots is the same whether you look at rows or columns, the ranks are the same.

# The Rank Theorem

Let  $A$  be an  $n \times m$  matrix. Then:

$$\text{rank}(A) + \text{nullity}(A) = \text{number of columns}(A) = m$$

This is because:

- ▶ the rank of  $A$  is the number of pivot columns
- ▶ the dimension of the null space is the dimension of the solution space to  $Ax = 0$  which is the number of free variables in the row reduced form of  $A$ .

These two numbers (pivots plus free variables) add up to the total number of columns.

## More on invertible matrices

If  $A$  is square of size  $n \times n$ , then:

- ▶  $A$  is invertible if and only if  $\text{rank}(A) = n$ .
- ▶  $A$  is invertible if and only if  $\text{nullity}(A) = 0$ .

These are restatements of earlier conditions; the first says that the columns of  $A$  are linearly independent, the second says that there are no free variables in the rref for  $A$ .

## A few things to think about

- ▶ If  $V$  has dimension  $n$ , and  $H$  is a subspace of  $V$  of dimension  $n$ , then  $H = V$ .
- ▶ Suppose that  $A$  is a  $4 \times 7$  matrix. Then the rank of  $A$  is *at most 4* and the nullity of  $A$  is *at least 3*.
- ▶ Suppose that  $A$  is a  $7 \times 4$  matrix. Then the rank of  $A$  is at most 4. The nullity is between 0 and 4.

## Change of basis

A choice of a basis for a vector space gives a set of coordinates for that vector space.

If we have *two* bases, then we have two sets of coordinates. How are they related?

Suppose  $x_1, \dots, x_n$  and  $y_1, \dots, y_n$  are both bases of  $V$ .

We can write each  $x_i$  in terms of the  $y_j$  to get a matrix.

## Change of basis

$$\begin{aligned}x_1 &= a_{11}y_1 + a_{21}y_2 + \cdots + a_{n1}y_n \\&\vdots \\x_n &= a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{nn}y_n\end{aligned}$$

If a vector  $v = c_1x_1 + \cdots + c_nx_n$  then, written in terms of the  $y_i$  we have

$$\begin{aligned}v &= (c_1a_{11} + c_2a_{12} + \cdots + c_na_{1n})x_1 + \cdots \\&\quad + (c_1a_{n1} + \cdots + c_na_{nn})x_n\end{aligned}$$



## Change of basis continued

The coordinates  $[v]_x$  of  $v$  in the  $x$  basis are computed from the coordinates  $[v]_y$  in the  $y$ -basis as:

$$[v]_x = A[v]_y$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

**NOTE:** The columns of  $A$  are the coordinates  $[x_i]_y$  of the  $x$ -basis elements in terms of the  $y$ -basis.

## Example

If  $e_1, e_2$  are the standard basis for  $\mathbf{R}^2$  and  $y_1, y_2$  are the vectors  $(1, 1)$  and  $(-1, 1)$  then to convert from the  $y_1, y_2$  basis to the standard basis we should make the matrix  $A$  whose columns are the  $y_1, y_2$  in terms of the standard basis.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

So if  $v = ay_1 + by_2$  then

$$A \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a - b \\ a + b \end{bmatrix}$$

and so  $v = (a - b)e_1 + (a + b)e_2$ .

## Example continued

To go backwards, suppose we have  $v = ae_1 + be_2$ . The inverse of  $A$  is

$$A^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (a+b)/2 \\ (b-a)/2 \end{bmatrix}$$

To check:

$$(a+b)/2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b-a)/2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Notice also that the columns of  $A^{-1}$  are the standard basis written in the  $y_1, y_2$  coordinates.

## Change of basis theorem

**Theorem 15** (page 275 of the text): Let  $B = \{b_1, \dots, b_n\}$  and  $C = \{c_1, \dots, c_n\}$  be two bases of a vector space  $V$ . Then there is a (unique)  $n \times n$  matrix  $P_{C \leftarrow B}$  such that

$$[x]_C = P_{C \leftarrow B} [x]_B.$$

This matrix is called the “change of basis” or “change of coordinates” matrix from  $B$  to  $C$ .

The columns of  $P_{C \leftarrow B}$  are the  $C$ -coordinates of the  $B$ -basis vectors. That is, the  $i^{\text{th}}$  column of  $P_{C \leftarrow B}$  is  $[b_i]_C$ ; its entries are  $a_1, \dots, a_n$  where

$$b_i = a_1 c_1 + a_2 c_2 + \dots + a_n c_n.$$

## Computing the change of basis matrix

To compute the change of basis matrix  $P_{C \leftarrow B}$ , suppose we have  $b_1, \dots, b_n$  and  $c_1, \dots, c_n$  as vectors in  $\mathbf{R}^n$ .

Let  $B$  be the matrix whose columns are the  $b_i$ , and  $C$  be the matrix whose columns are the  $c_i$ .

Note that  $B$  is the change of basis matrix from  $B$  to the standard basis, and  $C$  is the change of basis matrix from  $C$  to the standard basis. So  $C^{-1}$  is the change of basis matrix from the standard basis to  $C$ .

Form the augmented matrix  $[C|B]$  and do row reduction. Since  $C$  is invertible this will yield  $[I|P]$  where  $I$  is the identity matrix.

The matrix  $P$  is the desired change of basis matrix  $P_{C \leftarrow B}$ . In fact  $P = C^{-1}B$ .

## Numerical Example

Suppose that  $B = \{(1, 1), (1, -1)\}$  and  $C = \{(2, 1), (1, -2)\}$ . We want  $P_{C \leftarrow B}$ . The augmented matrix is

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 \end{bmatrix}$$

The row reduced version is:

$$\begin{bmatrix} 1 & 0 & \frac{3}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$$

So the change of basis matrix is

$$\begin{bmatrix} 3/5 & 1/5 \\ -1/5 & 3/5 \end{bmatrix}$$

## Checking the example.

Let  $v = (2, 0) = b_1 + b_2$ . The coordinates of  $v$  in the  $b$ -basis are  $(1, 1)$ . Multiplying:

$$\begin{bmatrix} 3/5 & 1/5 \\ -1/5 & 3/5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 2/5 \end{bmatrix}$$

Now  $4/5(2, 1) + 2/5(1, -2) = (2, 0)$  so the coordinates of  $(2, 0)$  in the  $c$ -basis is in fact  $(4/5, 2/5)$ .