

2.1 Matrix Operations

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Elementary Matrices

An elementary matrix is obtained by doing a single row operation on the identity matrix.

There are three types.

Elementary Matrices - Permutations

Suppose

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Switch the first and third rows (for example) and you get

$$I = \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Elementary Matrices - Permutations

If E is the elementary matrix obtained from I_n by swapping rows i and j , and A is an $n \times m$ matrix, then EA is obtained from A by swapping rows i and j .

Such an E is an invertible matrix.

Elementary Matrices - scaling

If E is the elementary matrix obtained from I_n by multiplying row i by a , and A is any $n \times m$ matrix, then EA is obtained from A by scaling row i by a .

Such an E is an invertible matrix.

Elementary Matrices - adding

If E is the elementary matrix obtained from I_n by replacing row i by the sum of row i and row j , and A is any $n \times m$ matrix, then EA is the matrix obtained from A by adding rows i and j .

Such an E is an invertible matrix.

Theorem on elementary matrices

If A is an $n \times m$ matrix, there is a sequence of elementary matrices E_1, \dots, E_k so that

$$E_k \cdots E_2 E_1 A$$

is in row reduced echelon form.

If A is a square $n \times n$ matrix, then its row reduced form has only diagonal entries.

Invertible matrices

If the rref of a square matrix A has pivots in every column then A is invertible.

$$E_k \cdots E_2 E_1 A = I_n$$

so

$$A^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

Invertible matrices

If the rref does *not* have a pivot in every column, then it is not invertible. Because in that case there is a vector v which is not zero such that

$$E_k \cdots E_2 E_1 A v = 0$$

so $Av = 0$. But if A were invertible then $A^{-1}Av = 0$ implies $v = 0$, which is not true.

Computing the inverse

You can compute the inverse matrix of an $n \times n$ square matrix A by finding the RREF of the $n \times 2n$ matrix

$$\begin{bmatrix} A & I_n \end{bmatrix}$$

Theorem on Invertible Matrices

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A , the statements are either all true or all false.

- A is an invertible matrix.
- A is row equivalent to the $n \times n$ identity matrix.
- A has n pivot positions.
- The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- The columns of A form a linearly independent set.
- The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
- The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each \mathbf{b} in \mathbb{R}^n .
- The columns of A span \mathbb{R}^n .
- The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps \mathbb{R}^n onto \mathbb{R}^n .
- There is an $n \times n$ matrix C such that $CA = I$.
- There is an $n \times n$ matrix D such that $AD = I$.
- A^T is an invertible matrix.

Figure 1: Inverses