# 2.2-2.3 Matrix Operations

Jeremy Teitelbaum

# Elementary Matrices

An elementary matrix is obtained by doing a single row operation on the identity matrix.

There are three types.

# Elementary Matrices - Permutations

Suppose

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Switch the first and third rows (for example) and you get

$$I = \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

# Elementary Matrices - Permutations

If $E$ is the elementary matrix obtained from $I_n$ by swapping rows $i$ and $j$, and $A$ is an $n \times m$ matrix, then $EA$ is obtained from $A$ by swapping rows $i$ and $j$.

Such an $E$ is an invertible matrix.

# Elementary Matrices - scaling

If $E$ is the elementary matrix obtained from $I_n$ by multiplying row $i$ by $a$, and $A$ is any $n \times m$ matrix, then $EA$ is obtained from $A$ by scaling row $i$ by $a$.

Such an $E$ is an invertible matrix.

# Elementary Matrices - adding

If $E$ is the elementary matrix obtained from $I_n$ by replacing row $i$ by the sum of row $i$ and row $j$, and $A$ is any $n \times m$ matrix, then $EA$ is the matrix obtained from $A$ by adding rows $i$ and $j$.

Such an $E$ is an invertible matrix.

# Theorem on elementary matrices

If $A$ is an $n \times m$ matrix, there is a sequence of elementary matrices $E_1, \ldots, E_k$ so that

$$E_k \cdots E_2 E_1 A$$

is in row reduced echelon from.

If $A$ is a square $n \times n$ matrix, then its row reduced form has only diagonal entries.

# Invertible matrices

If the rref of a square matrix $A$ has pivots in every column then $A$ is invertible.

$$E_k \cdots E_2 E_1 A = I_n$$

so

$$A^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

# Invertible matrices

If the rref does *not* have a pivot in every column, then it is not invertible. Because in that case there is a vector $v$ which is not zero such that

$$E_k \cdots E_2 E_1 A v = 0$$

so $Av = 0$. But if $A$ were invertible then $A^{-1}Av = 0$ implies $v = 0$, which is not true.

# Computing the inverse

You can compute the inverse matrix of an $n \times n$ square matrix $A$ by finding the RREF of the $n \times 2n$ matrix

$$\begin{bmatrix} A & I_n \end{bmatrix}$$

# Theorem on Invertible Matrices

**The Invertible Matrix Theorem**

Let $A$ be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given $A$, the statements are either all true or all false.

a. $A$ is an invertible matrix.

b. $A$ is row equivalent to the $n \times n$ identity matrix.

c. $A$ has $n$ pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of $A$ form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of $A$ span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix $C$ such that $CA = I$.

k. There is an $n \times n$ matrix $D$ such that $AD = I$.

l. $A^T$ is an invertible matrix.

Figure 1: Inverses

# Invertible Linear Maps

Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation.

Then $T$ is *invertible* if there is an "inverse function" $S : \mathbf{R}^n \to \mathbf{R}^n$ such that $S(T(x)) = T(S(x)) = x$ for all $x \in \mathbf{R}^n$.

# Matrices and maps

Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation and let $A$ be its standard matrix. Then $T$ is invertible if and only if $A$ is an invertible matrix.

If $T$ is invertible, then $T$ is *onto*. Let $x \in \mathbf{R}^n$. Then $T(S(x)) = x$ so $S(x)$ is the element that maps to $T$. If $y = S(x)$, then $Ay = x$ so the range of $A$ is all of $\mathbf{R}^n$ and so $A$ is invertible.

Conversely if $A$ is invertible then $S(x) = A^{-1}x$ has the desired properties.

# Inverses are unique

Note: if $T$ has an inverse, it is one-to-one. Because if $T(x) = T(y)$,then $S(T(x)) = S(T(y))$ and therefore $x = y$.

Note: inverse functions are unique. If $S(T(x)) = T(S(x)) = x$ and $U(T(x)) = T(U(x)) = x$ then $T(S(x)) = T(U(x))$. Since $T$ is one-to-one, $S(x) = U(x)$.