

## 1.3 Vector Equations

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# Vectors

A vector (in  $\mathbf{R}^n$ ) is an  $n$ -tuple of real numbers. For example

$$v = (-1, 3, 2, 5, 0, 7)$$

is a vector in  $\mathbf{R}^6$ .

Vectors can be written as matrices with one column (column vectors).

$$v = \begin{pmatrix} -1 \\ 3 \\ 2 \\ 5 \\ 0 \\ 7 \end{pmatrix}$$

## Vector arithmetic

Vectors can be added:

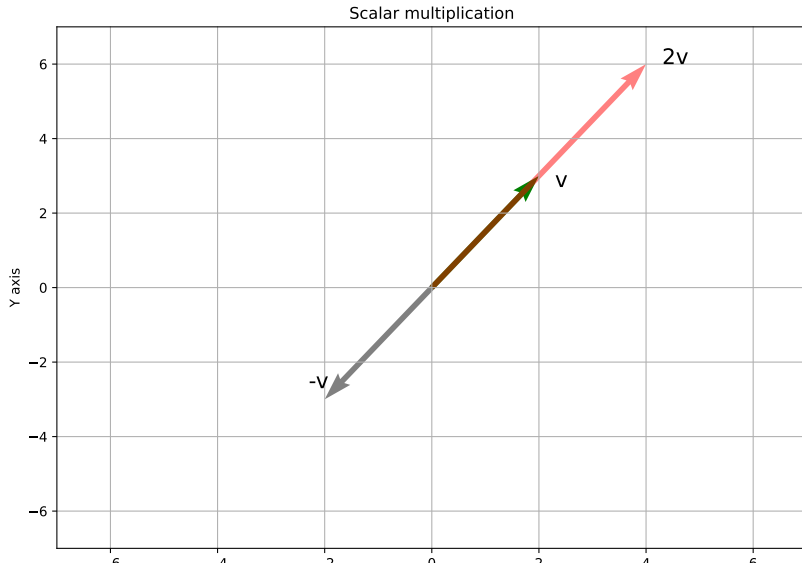
$$\begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \\ 11 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 16 \end{pmatrix}$$

Vectors can be multiplied by a number (a *scalar*):

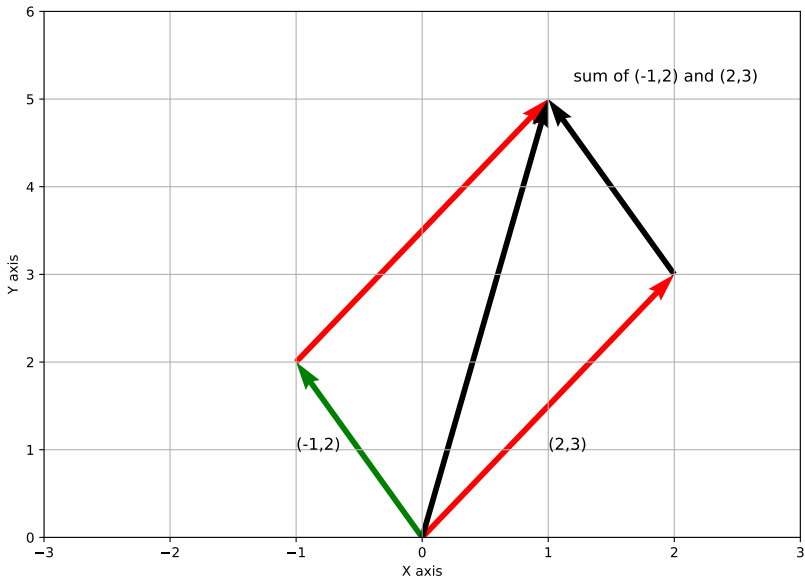
$$3 \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \\ 15 \end{pmatrix}$$

## Geometry: Scalar Multiplication

If  $\mathbf{v}$  is a vector, and  $a$  is a scalar (a real number) then  $a\mathbf{v}$  “points in the same direction” but it’s length is “scaled by  $a$ .”



## Geometry: Addition



This illustrates the “parallelogram law.” A similar picture holds in

# Linear combinations

If  $v_1, v_2, \dots, v_k$  are  $k$  vectors in  $\mathbf{R}^n$ , and  $a_1, \dots, a_k$  are constants (scalars), then the vector

$$y = a_1 v_1 + a_2 v_2 + \dots + a_k v_k$$

is called a *linear combination* of  $v_1, \dots, v_k$ .

## Linear Combination Example

If

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

then the linear combinations of  $v_1$  and  $v_2$  are

$$y = a_1 v_1 + a_2 v_2 = \begin{pmatrix} a_1 \\ a_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a_2 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_1 + a_2 \\ a_2 \end{pmatrix}$$

# Span

The *span* of a set of vectors in  $\mathbf{R}^n$  is the collection of all possible linear combinations of the set.

In the previous example, the span of  $v_1$  and  $v_2$  is all vectors  $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$  such that you can find  $a_1$  and  $a_2$  so that:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a_2 \\ a_2 \end{pmatrix}$$



## Span (continued)

This means that  $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$  is in the span of  $v_1$  and  $v_2$  if, **given**  $x, y, z$ , you can **find**  $a_1, a_2$  so that

$$\begin{array}{rcl} a_1 & = & x \\ a_1 + a_2 & = & y \\ a_2 & = & z \end{array}$$

## Span (continued)

In augmented matrix form this is

$$\left(\begin{array}{ccc} 1 & 0 & x \\ 1 & 1 & y \\ 0 & 1 & z \end{array}\right)$$

## Span (continued)

Using row reduction this yields

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y - x \\ 0 & 0 & z - y + x \end{pmatrix}$$

This system has a solution (is consistent) exactly when  $z - y + x = 0$  or  $y = z + x$ .

## Span (continued)

So

$$v = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

is in the span, but

$$w = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

is not.

## Span (continued)

More generally if  $v_1, \dots, v_k$  are vectors in  $\mathbf{R}^n$ , then a vector  $w$  in  $\mathbf{R}^n$  belongs to the span of the  $v_i$  if and only if the linear system with augmented matrix

$$M = [v_1 \quad v_2 \quad \cdots \quad v_k \quad w]$$

has a solution (is consistent).

Here the matrix  $M$  has the indicated vectors as its columns. It is an  $n \times (k + 1)$  matrix.

## Examples

Is

$$\begin{pmatrix} -5 \\ 11 \\ -7 \end{pmatrix}$$

in the span of the vectors

$$\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 8 \end{pmatrix}$$

## Matrix form

Associated matrix

$$\begin{bmatrix} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{bmatrix}$$

Echelon Form:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & \frac{4}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So the last column is *not* in the span of the first three.