Eigenvectors and Linear Transformations

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Linear Transformations and Matrices

Remember that a linear transformation $T : \mathbf{R}^m \to \mathbf{R}^n$ is a function that satisfies the two conditions:

- $T(ax) = aT(x)$ for all $x \in \mathbf{R}^m$ and $a \in \mathbf{R}$.
- $T(x + y) = T(x) + T(y)$ for all $x, y \in \mathbf{R}^m$.

We saw earlier that a linear transformation can be represented by an $n \times m$ matrix A where

$$T(x_1, \dots, x_m) = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Linear transformations and bases

We can take a slightly more general point of view on matrices and linear transformations.

In the earlier version we used "standard coordinates" where $x_1, \ldots, x_n$ are relative to the "standard basis."

Now suppose $B = \{b_1, \ldots, b_n\}$ are a basis for $\mathbf{R}^n$. Then if

$$x = r_1 b_1 + \cdots + r_n b_n$$

we have the coordinate vector

$$[x]_B = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}.$$

Linear transformations in other bases

By linearity

$$T(x) = T(r_1 b_1 + \cdots + r_n b_n) = r_1 T(b_1) + \cdots + r_n T(b_n)$$

Furthermore, each $T(b_i)$ has coordinates $[T(b_i)]_B$ so that

$$T(b_i) = t_{i1} b_1 + t_{i2} b_2 + \cdots + t_{in} b_n$$

If we make a matrix M whose *columns* are the vectors $[T(b_i)]_B$, then

$$[T(x)]_B = [T(\sum_{i=1}^{n} r_i b_i)]_B = \sum r_i [T(b_i)]_B = M[x]_B$$

The matrix M is called *the matrix of the linear transformation T in the basis B* and is written

$$M = [T]_B$$

Linear transformations and change of basis

If we write S for the standard basis, the "change of basis matrix" $P_{S\leftarrow B}$ (which the book calls just P_B has the property that

$$P_{S\leftarrow B}[x]_B = [x]_S$$

If $T(x) = Ax$, then in our notation above $A = [T]_S$ and $x = [x]_S$. We can write this equation as

$$[T(x)]_S = [T]_S[x]_S$$

Linear transformations and change of basis cont'd

So

$$[T(x)]_S = A[x]_S = AP_{S \leftarrow B}[x]_B$$

But if we want the output of T to *also* be in the B-basis, we need one more step:

$$[T(x)]_B = P_{B \leftarrow S}[T(x)]_S = P_{B \leftarrow S}AP_{S \leftarrow B}[x]_B$$

Linear transformations and change of basis continued

If we simplify the notation and write $P = P_{S \leftarrow B}$ then we see that

$$[T(x)]_B = [T]_B[x]_B = P^{-1}AP[x]_B$$

where $A = [T]_S$

In other words, the matrix of T in the B-basis is *similar* to the matrix in the standard basis.

More generally, the collection of matrices that are similar to $A = [T]_S$ are the collection of matrix representations of T in the possible bases of $\mathbf{R}^n$.

Diagonal transformations

If the matrix A is diagonalizable, then one can find a basis B so that $[T]_B$ is a diagonal matrix.

Example

Suppose that $B = \{b_1, b_2\}$ is a basis for a vector space V and that $T : V \to V$ is the linear transformation defined by

$$T(b_1) = 7b_1 + 4b_2, T(b_2) = 6b_1 - 5b_2.$$

The matrix

$$[T]_B = \begin{bmatrix} 7 & 6 \\ 4 & -5 \end{bmatrix}$$

Example continued

Is there a basis in which T is given by a diagonal matrix?

The characteristic polynomial of $[T]_B$ is

$$\det \begin{bmatrix} 7-\lambda & 6 \\ 4 & -5-\lambda \end{bmatrix} = \lambda^2 - 2\lambda - 59$$

The roots are $1 \pm 2\sqrt{15}$. Since these are distinct, the matrix is diagonalizable.

Example continued

The eigenvectors are

$$v_{\pm} = \begin{bmatrix} \frac{3\pm\sqrt{15}}{2} \\ 1 \end{bmatrix}$$

and in the basis E given by these eigenvectors the matrix of T is diagonal.