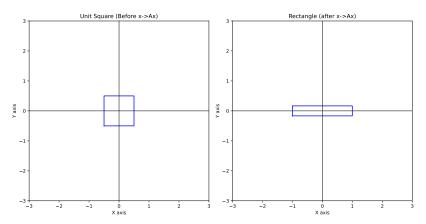
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If A is a diagonal matrix:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

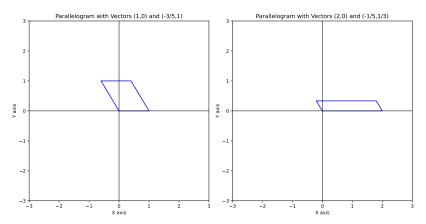
then the linear transformation $x\mapsto Ax$ "stretches" along the x-axis and "shrinks" along the y-axis.



If A is upper triangular, say

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix}$$

then A stretches along the x-axis by 2 as before. Less obviously, it shrinks along the direction given by the vector (-3/5,1).



An **eigenvector** for a matrix A is a vector v which gets shrunk or lengthened by A by some factor λ .

The factor λ is called the **eigenvalue**.

More formally, a vector v is called an eigenvector for A (with eigenvalue $\lambda)$ if v is not zero and

$$Av = \lambda v$$
.

In the example above, the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3/5 \\ 1 \end{bmatrix}$ are eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix}$$

with eigenvalues 2 and 1/3 respectively.

$$\begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} -3/5 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 1/3 \end{bmatrix} = (1/3) \begin{bmatrix} -3/5 \\ 1 \end{bmatrix}$$

Triangular Matrices

If A is (upper) triangular then the diagonal entries for A are all eigenvalues. If the diagonal entries are distinct, thenhere are n linearly independent eigenvectors.

Eigenspaces

Suppose that λ is a constant. The vectors v such that

$$Av = \lambda v$$

form a subspace called the *eigenspace* for λ .

This subspace is the nullspace of the matrix

$$A - \lambda I_n$$

where I_n is the $n \times n$ identity matrix.

Independence of Eigenvectors

If v_1,\ldots,v_n are eigenvectors for a matrix A with eigenvalues $\lambda_1,\ldots,\lambda_n$, and all the λ_i are different, then the v_i are linearly independent. (Note that the v_i are nonzero.)

To see this, suppose that

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0.$$

Then

$$A(c_1v_1+c_2v_2+\cdots c_nv_n)=c_1\lambda_1v_1+\cdots c_n\lambda_nv_n=0$$

Multiply the first relation by λ_1 and subtract. You get

$$c_2(\lambda_2 - \lambda_1)v_2 + \dots + c_n(\lambda_n - \lambda_1)v_n = 0.$$

Since the differences of the λ_i with λ_1 are not zero, we see that v_2,\dots,v_n are dependent.

By repeating this you can show that smaller and smaller collections of the v_i are dependent until you ultimately get $v_n=0$.

Characteristic Equation

Finding eigenvalues and eigenvectors of a matrix is a hard problem. We can make the following observation.

Suppose λ is an eigenvalue of A where A is an $n\times n$ matrix. Then there is a vector $v\neq 0$ so that $Av=\lambda v$. This means that the matrix $A-\lambda I_n$ is *not* invertible because v is in its null space.

As a result, $\det(A - \lambda I_n = 0$.

Conversely, if $\det(A-\lambda I_n)=0$, then there is a vector v in the null space and that v is an eigenvector.

It turns out that $\det(A-\lambda I_n)$ is a polynomial in λ , so the eigenvalues of A are the roots of this polynomial.

Example

Let

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}.$$

The determinant of $A - \lambda$ is

$$\det (\begin{bmatrix} 3-\lambda & 5 \\ 2 & 4-\lambda \end{bmatrix}) = (3-\lambda)(4-\lambda) - 10$$

The polynomial on the right is

$$(3 - \lambda)(4 - \lambda) - 10 = \lambda^2 - 7\lambda + 12 - 10 = \lambda^2 - 7\lambda + 2.$$

Its roots are $\frac{7\pm\sqrt{41}}{2}.$ These are the eigenvalues of A; they are approximately 6.70156 and 0.29843.

To find the eigenvectors, we have to compute the null space of $A-\lambda I$. This is no fun algebraically but with some work you find that the eigenvectors are:

$$\begin{bmatrix} -\frac{\sqrt{41}}{4} - \frac{1}{4} \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{41}}{4} \\ 1 \end{bmatrix}$$

Bigger matrices

The characteristic polynomial of an $n \times n$ matrix is of degree n.

Theorem: A polynomial of degree n has n complex roots (counted correctly).

However, in practice one must use numerical methods to find roots of a polynomial of degree 3 or higher.

If one know an eigenvalue λ exactly (which usually doesn't happen) then you can find the eigenvectors by computing the null space of $A-\lambda$.

Example

Let

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$$

Since A is (lower) triangular, the eigenvalues are 5.0 and -2. The vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is the eigenvector with eigenvalue 5.

To find the eigenvector with eigenvalue 0, we must solve

$$Ax = 0$$
.

By one of our techniques this is

$$\begin{bmatrix} -8\\5\\0 \end{bmatrix}$$

The final eigenvector is in the null space of A + 2I:

$$A = \begin{bmatrix} 7 & -8 & 1 \\ 0 & 2 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

This is

$$\begin{bmatrix} -58/7 \\ -7 \\ 2 \end{bmatrix}$$

Similarity

Two (square) matrices A and B are similar if there is an invertible matrix P so that $A = PBP^{-1}$.

Similar matrices have the same eigenvalues because they have the same characteristic polynomial.

$$\begin{array}{rcl} \det(PBP^{-1}-\lambda I) &=& \det(P(B-\lambda I)P^{-1}) \\ &=& \det(P)\det(B-\lambda I)\det(P^{-1}) \\ &=& \det(B-\lambda I) \end{array}$$

Diagonalization

Diagonal matrices are the simplest to work with.

Definition: A square matrix is diagonalizable if there is an invertible matrix P so that $A=PDP^{-1}$ where P diagonal. In other words, A is diagonalizable if it is similar to a diagonal matrix.

Theorem: An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

In fact, if $A=PDP^{-1}$, then the columns of P are n linearly independent eigenvectors for A, and the diagonal entries of D are the corresponding eigenvalues. This is because in this situation,

$$AP = PD$$
.

Example (from the text)

Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

1. The eigenvalues of A. The text tells us that the characteristic polynomial of A is $-(\lambda-1)(\lambda+2)^2$ so the eigenvalues are 1 and -2.

We need three linearly independent eigenvectors. So we need the null spaces of A-I and A+2I. The book gives us:

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

with eigenvalue 1. For the eigenvalue -2:

$$A+2I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}$$

Reduced form is
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So null space of A+2I is two dimensional and spanned by

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Therefore

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

We can compute AP:

$$AP = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} = PD$$

where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

So $A = PDP^{-1}$ and A is diagonalizable.

Matrix Powers

One application of diagonalization is that it makes it feasible to understand A^n when A is a square matrix.

If A is diagonalizable, then there is a diagonal matrix ${\cal D}$ and a matrix ${\cal P}$ so that

$$A = PDP^{-1}.$$

Then

$$A^m = (PDP^{-1})(PDP^{-1})\cdots (PDP^{-1}) = PD^mP^{-1}$$

and

$$D^n = \begin{bmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n^m \end{bmatrix}$$

Not all matrices are diagonalizable

Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The only eigenvalue is 1. The null space of A-I is only one dimensional, spanned by

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So there's no basis of eigenvectors, so \boldsymbol{A} can't be diagonalized.

n distinct eigenvalues implies diagonalizable

If A has n different eigenvalues, then it has n linearly independent eigenvectors; thus there is a basis of eigenvectors.

Therefore in this case A is diagonalizable.

But as we saw above, you can have repeated eigenvalues and still be diagonalizable.

The diagonalization Theorem

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

- a. For $1 \le k \le p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k .
- b. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (i) the characteristic polynomial factors completely into linear factors and (ii) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k .
- c. If A is diagonalizable and \mathcal{B}_k is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for \mathbb{R}^n .

Figure 1: Diagonalization Theorem

Fibonacci Numbers

The Fibonacci numbers are defined recursively by $F_0=0,\,F_1=1$, and $F_n=F_{n-1}+F_{n-2}$ for n>1.

Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F_0 & F_1 \\ F_1 & F_2 \end{bmatrix}.$$

Then

$$A^2 = AA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} F_1 & F_2 \\ F_2 & F_3 \end{bmatrix}$$

Fibonacci continued

Continuing we see that

$$A^{n} = AA^{n-1} = A \begin{bmatrix} F_{n-1} & F_{n} \\ F_{n} & F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n} & F_{n+1} \\ F_{n-1} + F_{n} & F_{n} + F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n} & F_{n+1} \\ F_{n+1} & F_{n+2} \end{bmatrix}$$
(1)

So:

$$A^n = \begin{bmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{bmatrix}$$

Computation of Fibonacci numbers

Let's try to diagonalize the matrix $A=\begin{bmatrix}0&1\\1&1\end{bmatrix}$ and use this to compute $A^n.$

The characteristic polynomial of A:

$$F(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ 1 & 1 - \lambda \end{bmatrix}$$

so $F(\lambda) = \lambda^2 - \lambda - 1$. This polynomial has two roots:

$$\lambda_{\pm} = \frac{1 \pm \sqrt{5}}{2}.$$

Therefore A is diagonalizable. To compute A^n we need to find P so that

$$A = P \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} P^{-1}$$

Computation of Fibonacci Numbers continued

The columns of the matrix P are the eigenvectors of A. To find these we must solve

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda_\pm x & \lambda_\pm y \end{bmatrix}$$

which translates into the equations

$$\begin{bmatrix} y \\ x+y \end{bmatrix} = \begin{bmatrix} \lambda_{\pm} x \\ \lambda_{\pm} y \end{bmatrix}$$

Eigenvectors are determined only up to scaling so we can set x=1. Then $y=\lambda_+.$ So

$$P = \begin{bmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{bmatrix}$$

Fibonacci numbers

Our final result is that

$$\begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix} = A^n = \frac{1}{\lambda_- - \lambda_+} \begin{bmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{bmatrix} \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix} \begin{bmatrix} \lambda_- & -1 \\ -\lambda_+ & 1 \end{bmatrix}$$

A little algebra gives:

$$F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$$

Consequences

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Let ϕ be the "Golden ratio" $\frac{1+\sqrt{5}}{2}$.

- 1. F_n is approximately $\phi^n/\sqrt{5}$.
- 2. F_n/F_{n-1} converges to ϕ .

0	0.447	0
1	0.724	1
2	1.171	1
3	1.894	2
4	3.065	3
5	4.960	5
6	8.025	8
7	12.985	13
8	21.010	21
9	33.994	34
10	55.004	55

88.998

89