

# Eigenvalues and Eigenvectors

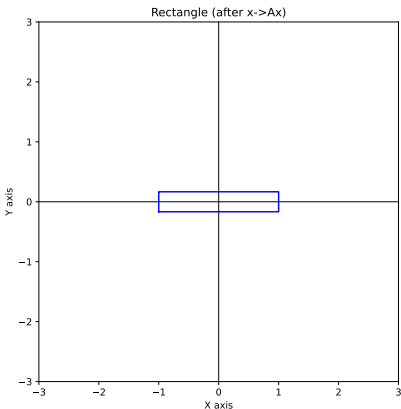
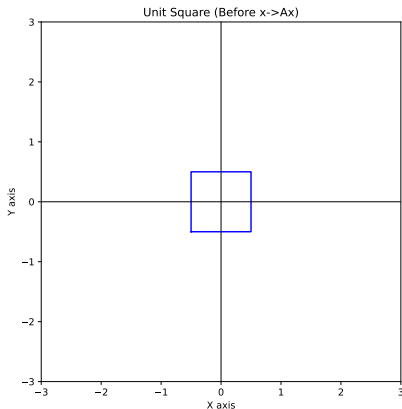
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# Eigenvalues and Eigenvectors

If  $A$  is a diagonal matrix:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

then the linear transformation  $x \mapsto Ax$  “stretches” along the  $x$ -axis and “shrinks” along the  $y$ -axis.

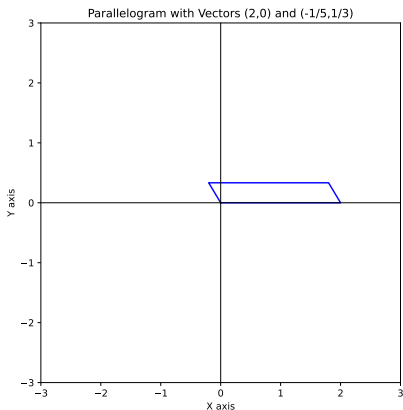
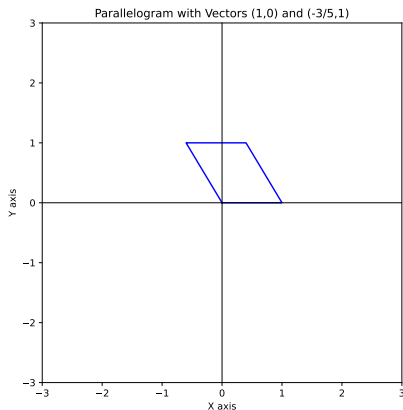


# Eigenvalues and Eigenvectors

If  $A$  is upper triangular, say

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix}$$

then  $A$  stretches along the  $x$ -axis by 2 as before. Less obviously, it shrinks along the direction given by the vector  $(-3/5, 1)$ .



# Eigenvalues and Eigenvectors

An **eigenvector** for a matrix  $A$  is a vector  $v$  which gets shrunk or lengthened by  $A$  by some factor  $\lambda$ .

The factor  $\lambda$  is called the **eigenvalue**.

More formally, a vector  $v$  is called an eigenvector for  $A$  (with eigenvalue  $\lambda$ ) if  $v$  is not zero and

$$Av = \lambda v.$$

## Eigenvalues and Eigenvectors

In the example above, the vectors  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -3/5 \\ 1 \end{bmatrix}$  are eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix}$$

with eigenvalues 2 and 1/3 respectively.

$$\begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} -3/5 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 1/3 \end{bmatrix} = (1/3) \begin{bmatrix} -3/5 \\ 1 \end{bmatrix}$$

# Triangular Matrices

If  $A$  is (upper) triangular then the diagonal entries for  $A$  are all eigenvalues. If the diagonal entries are distinct, then there are  $n$  linearly independent eigenvectors.

# Eigenspaces

Suppose that  $\lambda$  is a constant. The vectors  $v$  such that

$$Av = \lambda v$$

form a subspace called the *eigenspace* for  $\lambda$ .

This subspace is the nullspace of the matrix

$$A - \lambda I_n$$

where  $I_n$  is the  $n \times n$  identity matrix.

## Independence of Eigenvectors

If  $v_1, \dots, v_n$  are eigenvectors for a matrix  $A$  with eigenvalues  $\lambda_1, \dots, \lambda_n$ , and all the  $\lambda_i$  are different, then the  $v_i$  are linearly independent. (Note that the  $v_i$  are nonzero.)

To see this, suppose that

$$c_1 v_1 + c_2 v_2 + \dots c_n v_n = 0.$$

Then

$$A(c_1 v_1 + c_2 v_2 + \dots c_n v_n) = c_1 \lambda_1 v_1 + \dots c_n \lambda_n v_n = 0$$

Multiply the first relation by  $\lambda_1$  and subtract. You get

$$c_2(\lambda_2 - \lambda_1)v_2 + \dots + c_n(\lambda_n - \lambda_1)v_n = 0.$$

Since the differences of the  $\lambda_i$  with  $\lambda_1$  are not zero, we see that  $v_2, \dots, v_n$  are dependent.

By repeating this you can show that smaller and smaller collections of the  $v_i$  are dependent until you ultimately get  $v_n = 0$ .



# Characteristic Equation

Finding eigenvalues and eigenvectors of a matrix is a hard problem. We can make the following observation.

Suppose  $\lambda$  is an eigenvalue of  $A$  where  $A$  is an  $n \times n$  matrix. Then there is a vector  $v \neq 0$  so that  $Av = \lambda v$ . This means that the matrix  $A - \lambda I_n$  is *not* invertible because  $v$  is in its null space.

As a result,  $\det(A - \lambda I_n) = 0$ .

Conversely, if  $\det(A - \lambda I_n) = 0$ , then there is a vector  $v$  in the null space and that  $v$  is an eigenvector.

It turns out that  $\det(A - \lambda I_n)$  is a polynomial in  $\lambda$ , so the eigenvalues of  $A$  are the roots of this polynomial.

## Example

Let

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}.$$

The determinant of  $A - \lambda$  is

$$\det\left(\begin{bmatrix} 3 - \lambda & 5 \\ 2 & 4 - \lambda \end{bmatrix}\right) = (3 - \lambda)(4 - \lambda) - 10$$

The polynomial on the right is

$$(3 - \lambda)(4 - \lambda) - 10 = \lambda^2 - 7\lambda + 12 - 10 = \lambda^2 - 7\lambda + 2.$$

Its roots are  $\frac{7 \pm \sqrt{41}}{2}$ . These are the eigenvalues of  $A$ ; they are approximately 6.70156 and 0.29843.

## Example continued

To find the eigenvectors, we have to compute the null space of  $A - \lambda I$ . This is no fun algebraically but with some work you find that the eigenvectors are:

$$\begin{bmatrix} -\frac{\sqrt{41}}{4} - \frac{1}{4} \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{41}}{4} \\ 1 \end{bmatrix}$$

## Bigger matrices

The characteristic polynomial of an  $n \times n$  matrix is of degree  $n$ .

**Theorem:** A polynomial of degree  $n$  has  $n$  complex roots (counted correctly).

However, in practice one must use numerical methods to find roots of a polynomial of degree 3 or higher.

If one know an eigenvalue  $\lambda$  *exactly* (which usually doesn't happen) then you can find the eigenvectors by computing the null space of  $A - \lambda$ .

## Example

Let

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$$

Since  $A$  is (lower) triangular, the eigenvalues are 5, 0 and  $-2$ . The vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is the eigenvector with eigenvalue 5.

## Example continued

To find the eigenvector with eigenvalue 0, we must solve

$$Ax = 0.$$

By one of our techniques this is

$$\begin{bmatrix} -8 \\ 5 \\ 0 \end{bmatrix}$$

## Example continued

The final eigenvector is in the null space of  $A + 2I$ :

$$A = \begin{bmatrix} 7 & -8 & 1 \\ 0 & 2 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

This is

$$\begin{bmatrix} -58/7 \\ -7 \\ 2 \end{bmatrix}$$

# Similarity

Two (square) matrices  $A$  and  $B$  are *similar* if there is an invertible matrix  $P$  so that  $A = PBP^{-1}$ .

Similar matrices have the same eigenvalues because they have the same characteristic polynomial.

$$\begin{aligned}\det(PBP^{-1} - \lambda I) &= \det(P(B - \lambda I)P^{-1}) \\ &= \det(P) \det(B - \lambda I) \det(P^{-1}) \\ &= \det(B - \lambda I)\end{aligned}$$



# Diagonalization

Diagonal matrices are the simplest to work with.

**Definition:** A square matrix is diagonalizable if there is an invertible matrix  $P$  so that  $A = PDP^{-1}$  where  $P$  diagonal. In other words,  $A$  is diagonalizable if it is similar to a diagonal matrix.

**Theorem:** An  $n \times n$  matrix  $A$  is diagonalizable if and only if it has  $n$  linearly independent eigenvectors.

In fact, if  $A = PDP^{-1}$ , then the columns of  $P$  are  $n$  linearly independent eigenvectors for  $A$ , and the diagonal entries of  $D$  are the corresponding eigenvalues. This is because in this situation,

$$AP = PD.$$

## Example (from the text)

Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

1. The eigenvalues of  $A$ . The text tells us that the characteristic polynomial of  $A$  is  $-(\lambda - 1)(\lambda + 2)^2$  so the eigenvalues are 1 and  $-2$ .

## Example continued

We need three linearly independent eigenvectors. So we need the null spaces of  $A - I$  and  $A + 2I$ . The book gives us:

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

with eigenvalue 1. For the eigenvalue  $-2$ :

$$A + 2I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}$$

$$\text{Reduced form is } \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## Example continued

So null space of  $A + 2I$  is two dimensional and spanned by

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Therefore

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

## Example continued

We can compute  $AP$ :

$$AP = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} = PD$$

where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

So  $A = PDP^{-1}$  and  $A$  is diagonalizable.

## Matrix Powers

One application of diagonalization is that it makes it feasible to understand  $A^n$  when  $A$  is a square matrix.

If  $A$  is diagonalizable, then there is a diagonal matrix  $D$  and a matrix  $P$  so that

$$A = PDP^{-1}.$$

Then

$$A^m = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1}) = PD^m P^{-1}$$

and

$$D^n = \begin{bmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^n \end{bmatrix}$$

## Not all matrices are diagonalizable

Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The only eigenvalue is 1. The null space of  $A - I$  is only one dimensional, spanned by

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So there's no basis of eigenvectors, so  $A$  can't be diagonalized.

$n$  distinct eigenvalues implies diagonalizable

If  $A$  has  $n$  different eigenvalues, then it has  $n$  linearly independent eigenvectors; thus there is a basis of eigenvectors.

Therefore in this case  $A$  is diagonalizable.

But as we saw above, you can have repeated eigenvalues and still be diagonalizable.



# The diagonalization Theorem

Let  $A$  be an  $n \times n$  matrix whose distinct eigenvalues are  $\lambda_1, \dots, \lambda_p$ .

- a. For  $1 \leq k \leq p$ , the dimension of the eigenspace for  $\lambda_k$  is less than or equal to the multiplicity of the eigenvalue  $\lambda_k$ .
- b. The matrix  $A$  is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals  $n$ , and this happens if and only if (i) the characteristic polynomial factors completely into linear factors and (ii) the dimension of the eigenspace for each  $\lambda_k$  equals the multiplicity of  $\lambda_k$ .
- c. If  $A$  is diagonalizable and  $\mathcal{B}_k$  is a basis for the eigenspace corresponding to  $\lambda_k$  for each  $k$ , then the total collection of vectors in the sets  $\mathcal{B}_1, \dots, \mathcal{B}_p$  forms an eigenvector basis for  $\mathbb{R}^n$ .

Figure 1: Diagonalization Theorem

# Fibonacci Numbers

The Fibonacci numbers are defined recursively by  $F_0 = 0$ ,  $F_1 = 1$ , and  $F_n = F_{n-1} + F_{n-2}$  for  $n > 1$ .

Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F_0 & F_1 \\ F_1 & F_2 \end{bmatrix}.$$

Then

$$A^2 = AA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} F_1 & F_2 \\ F_2 & F_3 \end{bmatrix}$$

## Fibonacci continued

Continuing we see that

$$\begin{aligned} A^n &= AA^{n-1} = A \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix} = \\ &\begin{bmatrix} F_n & F_{n+1} \\ F_{n-1} + F_n & F_n + F_{n+1} \end{bmatrix} = \\ &\begin{bmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{bmatrix} \quad (1) \end{aligned}$$

So:

$$A^n = \begin{bmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{bmatrix}$$

## Computation of Fibonacci numbers

Let's try to diagonalize the matrix  $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$  and use this to compute  $A^n$ .

The characteristic polynomial of  $A$ :

$$F(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & 1 - \lambda \end{bmatrix}$$

so  $F(\lambda) = \lambda^2 - \lambda - 1$ . This polynomial has two roots:

$$\lambda_{\pm} = \frac{1 \pm \sqrt{5}}{2}.$$

Therefore  $A$  is diagonalizable. To compute  $A^n$  we need to find  $P$  so that

$$A = P \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} P^{-1}$$

## Computation of Fibonacci Numbers continued

The columns of the matrix  $P$  are the eigenvectors of  $A$ . To find these we must solve

$$A \begin{bmatrix} x \\ y \end{bmatrix} = [\lambda_{\pm} x \quad \lambda_{\pm} y]$$

which translates into the equations

$$\begin{bmatrix} y \\ x + y \end{bmatrix} = \begin{bmatrix} \lambda_{\pm} x \\ \lambda_{\pm} y \end{bmatrix}$$

Eigenvectors are determined only up to scaling so we can set  $x = 1$ . Then  $y = \lambda_{\pm}$ . So

$$P = \begin{bmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{bmatrix}$$

# Fibonacci numbers

Our final result is that

$$\begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix} = A^n = \frac{1}{\lambda_- - \lambda_+} \begin{bmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{bmatrix} \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix} \begin{bmatrix} \lambda_- & -1 \\ -\lambda_+ & 1 \end{bmatrix}$$

A little algebra gives:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

# Consequences

Let  $\phi$  be the “Golden ratio”  $\frac{1+\sqrt{5}}{2}$ .

1.  $F_n$  is approximately  $\phi^n / \sqrt{5}$ .
2.  $F_n / F_{n-1}$  converges to  $\phi$ .

0	0.447	0
1	0.724	1
2	1.171	1
3	1.894	2
4	3.065	3
5	4.960	5
6	8.025	8
7	12.985	13
8	21.010	21
9	33.994	34
10	55.004	55
11	88.998	89