2.1 Matrix Operations

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Basic terminology

An $n \times m$ matrix A can be written as an $n \times m$ array of numbers.

It can also be written as

$$A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_m]$$

where each $\mathbf{a}_i$ is one of the m columns of A, and is a vector in $\mathbf{R}^n$.

Special matrices

The $m \times n$ zero matrix has all entries equal to zero.

The main diagonal of an $m \times n$ matrix are the entries $a_{11}, a_{22}, \ldots$ (which ends at either a_{nn} or a_{mm} depending on which is smaller).

A square matrix is diagonal if all entries off the main diagonal are zero.

Addition

Assuming matrices A and B **are the same shape** you can add them element by element:

$$C = A + B$$

and obtain another matrix of the same shape.

Scalar multiplication

You can also multiply A by a constant r (meaning multiply every element by r) to get another matrix $B = rA$ of the same shape.

Properties

Let A, B, and C be matrices of the same size, and let r and s be scalars.

a. $A + B = B + A$
b. $(A + B) + C = A + (B + C)$
c. $A + 0 = A$
d. $r(A + B) = rA + rB$
e. $(r + s)A = rA + sA$
f. $r(sA) = (rs)A$

Figure 1: properties

Matrix Multiplication

If A is and $m \times n$ matrix and B is a $k \times p$ matrix, then the product AB is defined ONLY WHEN $n = k$.

In other words you can multiply $m \times n$ times $n \times p$.

The result is an $m \times p$ matrix.

Matrix multiplication

If A is an $m \times n$ matrix and B is an $n \times p$ matrix:

$$B = \begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix}$$

then

$$AB = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}$$

This makes sense because:

- each column b_1 has n rows
- Ab_1 makes sense and has m rows
- so AB is an $m \times p$ matrix.

Example

$$A = \begin{bmatrix} 8 & 6 & -8 & -4 \\ 0 & 4 & -4 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} 8 & 7 & 1 \\ 7 & 0 & 7 \\ 2 & 1 & -7 \\ -7 & 8 & -3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 118 & 16 & 118 \\ -36 & 60 & 32 \end{bmatrix}$$

Dot product rule

If A is $m \times n$ and B is $n \times p$, then the i, j entry of AB is the dot product of row i of A with column j of B.

$$\mathrm{Row}_i(A) = [a_{i1}, a_{i2}, \dots, a_{in}]$$

$$\mathrm{Col}_j(B) = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

$$A_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Summation Notation

Here A is $m \times n$ and B is $n \times p$. $(AB)_{ij}$ means the i, j entry of the matrix product AB.

$$(AB)_{ij} = \sum_{t=1}^{n} a_{it} b_{tj}$$

Properties

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left distributive law)

c. $(B + C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$
for any scalar r

e. $I_m A = A = AI_n$ (identity for matrix multiplication)

Figure 2: MatrixMultProps

IMPORTANT: In general, $AB \neq BA$. Matrix multiplication is NOT COMMUTATIVE.

Transpose

The *transpose* of an $m \times n$ matrix A is the $n \times m$ matrix obtained from A by interchanging rows and columns.

$$A = \begin{bmatrix} -3 & -2 & -1 & 0 & 1 \\ 0 & -3 & 1 & -1 & -2 \\ -2 & -2 & -2 & -2 & -2 \end{bmatrix}$$

Transpose $A^T = \begin{bmatrix} -3 & 0 & -2 \\ -2 & -3 & -2 \\ -1 & 1 & -2 \\ 0 & -1 & -2 \\ 1 & -2 & -2 \end{bmatrix}$

Transpose and products

The transpose of a product is the product of the tranposes, in the reversed order.

$$(AB)^T = B^T A^T$$

This makes sense: If A is $m \times n$ and B is $n \times p$, then (AB) is $m \times p$ so $(AB)^T$ is $p \times m$.

On the other hand B^T is $p \times n$ and A^T is $n \times m$ so $B^T A^T$ is also $p \times m$.

Matrix Powers

If A is a matrix, then $A^n = A \cdot A \cdot A \cdots A$ makes sense (at least for integers $n \geq 0$).

Inverse Matrix

If A is an $n \times n$ matrix, and I_n is the $n \times n$ identity matrix having ones on the diagonal and zero elsewhere, then the inverse A^{-1} of A (if it exists) is the matrix such that $A^{-1}A = I_n$.

Suppose $A^{-1}A = I_n$. What about a matrix B such that $AB = I_n$?.

- Let u be $(A^{-1})^{-1}$, meaning that $uA^{-1} = I_n$. Consider $(uA^{-1})(AA^{-1})$.
- On the one hand this is AA^{-1} since $uA^{-1} = I_n$. On the other hand this is $uA^{-1} = I_n$ since the middle $A^{-1}A$ yield I_n. So $AA^{-1} = I_n$ and t he inverse works on both sides.

Not all square matrices have inverses

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. If $AB = I_2$ then

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so

$$\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is clearly impossible.

Two by two inverse

If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Just multiply them out and check that it works. Here $ad - bc \neq 0$.
If $ad - bc = 0$ then there is no inverse.

Determinant

The quantity $ad - bc$ for a 2×2 matrix is called the "determinant" of that matrix.

We will study determinants of bigger matrices later.