

## 2.2-2.3 Matrix Operations

Jeremy Teitelbaum

# Elementary Matrices

An elementary matrix is obtained by doing a single row operation on the identity matrix.

There are three types.

# Elementary Matrices - Permutations

Suppose

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Switch the first and third rows (for example) and you get

$$I = \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

## Elementary Matrices - Permutations

If  $E$  is the elementary matrix obtained from  $I_n$  by swapping rows  $i$  and  $j$ , and  $A$  is an  $n \times m$  matrix, then  $EA$  is obtained from  $A$  by swapping rows  $i$  and  $j$ .

Such an  $E$  is an invertible matrix.

## Elementary Matrices - scaling

If  $E$  is the elementary matrix obtained from  $I_n$  by multiplying row  $i$  by  $a$ , and  $A$  is any  $n \times m$  matrix, then  $EA$  is obtained from  $A$  by scaling row  $i$  by  $a$ .

Such an  $E$  is an invertible matrix.

## Elementary Matrices - adding

If  $E$  is the elementary matrix obtained from  $I_n$  by replacing row  $i$  by the sum of row  $i$  and row  $j$ , and  $A$  is any  $n \times m$  matrix, then  $EA$  is the matrix obtained from  $A$  by adding rows  $i$  and  $j$ .

Such an  $E$  is an invertible matrix.

## Theorem on elementary matrices

If  $A$  is an  $n \times m$  matrix, there is a sequence of elementary matrices  $E_1, \dots, E_k$  so that

$$E_k \cdots E_2 E_1 A$$

is in row reduced echelon form.

If  $A$  is a square  $n \times n$  matrix, then its row reduced form has only diagonal entries.

# Invertible matrices

If the rref of a square matrix  $A$  has pivots in every column then  $A$  is invertible.

$$E_k \cdots E_2 E_1 A = I_n$$

so

$$A^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$



# Invertible matrices

If the rref does *not* have a pivot in every column, then it is not invertible. Because in that case there is a vector  $v$  which is not zero such that

$$E_k \cdots E_2 E_1 A v = 0$$

so  $Av = 0$ . But if  $A$  were invertible then  $A^{-1}Av = 0$  implies  $v = 0$ , which is not true.

## Computing the inverse

You can compute the inverse matrix of an  $n \times n$  square matrix  $A$  by finding the RREF of the  $n \times 2n$  matrix

$$\begin{bmatrix} A & I_n \end{bmatrix}$$

# Theorem on Invertible Matrices

## The Invertible Matrix Theorem

Let  $A$  be a square  $n \times n$  matrix. Then the following statements are equivalent. That is, for a given  $A$ , the statements are either all true or all false.

- $A$  is an invertible matrix.
- $A$  is row equivalent to the  $n \times n$  identity matrix.
- $A$  has  $n$  pivot positions.
- The equation  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.
- The columns of  $A$  form a linearly independent set.
- The linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is one-to-one.
- The equation  $A\mathbf{x} = \mathbf{b}$  has at least one solution for each  $\mathbf{b}$  in  $\mathbb{R}^n$ .
- The columns of  $A$  span  $\mathbb{R}^n$ .
- The linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$  maps  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ .
- There is an  $n \times n$  matrix  $C$  such that  $CA = I$ .
- There is an  $n \times n$  matrix  $D$  such that  $AD = I$ .
- $A^T$  is an invertible matrix.

Figure 1: Inverses

# Invertible Linear Maps

Let  $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a linear transformation.

Then  $T$  is *invertible* if there is an “inverse function”  $S : \mathbf{R}^n \rightarrow \mathbf{R}^n$  such that  $S(T(x)) = T(S(x)) = x$  for all  $x \in \mathbf{R}^n$ .

## Matrices and maps

Let  $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a linear transformation and let  $A$  be its standard matrix. Then  $T$  is invertible if and only if  $A$  is an invertible matrix.

If  $T$  is invertible, then  $T$  is *onto*. Let  $x \in \mathbf{R}^n$ . Then  $T(S(x)) = x$  so  $S(x)$  is the element that maps to  $T$ . If  $y = S(x)$ , then  $Ay = x$  so the range of  $A$  is all of  $\mathbf{R}^n$  and so  $A$  is invertible.

Conversely if  $A$  is invertible then  $S(x) = A^{-1}x$  has the desired properties.

## Inverses are unique

Note: if  $T$  has an inverse, it is one-to-one. Because if  $T(x) = T(y)$ , then  $S(T(x)) = S(T(y))$  and therefore  $x = y$ .

Note: inverse functions are unique. If  $S(T(x)) = T(S(x)) = x$  and  $U(T(x)) = T(U(x)) = x$  then  $T(S(x)) = T(U(x))$ . Since  $T$  is one-to-one,  $S(x) = U(x)$ .

Note: One can show that the inverse of a linear transformation is linear. This is problem 48 on page 149 of the text.

## Some T/F problems from the homework

Suppose that  $A$  is an  $n \times n$  matrix. True or False:

1. If the columns of  $A$  span  $\mathbf{R}^n$ , then they are linearly independent.
2. The equation  $Ax = b$  always has at least one solution for each  $b \in \mathbf{R}^n$ .
3. If the linear transformation  $T(x) = Ax$  is one-to-one from  $\mathbf{R}^n$  to  $\mathbf{R}^n$ , then  $A$  has  $n$  pivots.
4. If  $A$  has two identical columns, it is not invertible.
5. If  $A^T$  is invertible, so is  $A$ .