

Rank and Change of Basis

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Rank

Let A be an $n \times m$ matrix.

The row rank of A is the dimension of the space spanned by the rows of A .

The column rank of A is the dimension of the space spanned by the columns of A .

The nullity of A is the dimension of the null space of A .

Row rank = column rank

Perhaps surprisingly, the row and column ranks are the same.

To see this, put A into row reduced echelon form. The dimension of the column space is the number of linearly independent columns, which is the number of columns containing a pivot.

The rows of A containing a pivot form a basis for the row space of A .

Since the number of pivots is the same whether you look at rows or columns, the ranks are the same.

The Rank Theorem

Let A be an $n \times m$ matrix. Then:

$$\text{rank}(A) + \text{nullity}(A) = \text{number of columns}(A) = m$$

This is because:

- ▶ the rank of A is the number of pivot columns
- ▶ the dimension of the null space is the dimension of the solution space to $Ax = 0$ which is the number of free variables in the row reduced form of A .

These two numbers (pivots plus free variables) add up to the total number of columns.

More on invertible matrices

If A is square of size $n \times n$, then:

- ▶ A is invertible if and only if $\text{rank}(A) = n$.
- ▶ A is invertible if and only if $\text{nullity}(A) = 0$.

These are restatements of earlier conditions; the first says that the columns of A are linearly independent, the second says that there are no free variables in the rref for A .

A few things to think about

- ▶ If V has dimension n , and H is a subspace of V of dimension n , then $H = V$.
- ▶ Suppose that A is a 4×7 matrix. Then the rank of A is *at most 4* and the nullity of A is *at least 3*.
- ▶ Suppose that A is a 7×4 matrix. Then the rank of A is at most 4. The nullity is between 0 and 4.

Change of basis

A choice of a basis for a vector space gives a set of coordinates for that vector space.

If we have *two* bases, then we have two sets of coordinates. How are they related?

Suppose x_1, \dots, x_n and y_1, \dots, y_n are both bases of V .

We can write each x_i in terms of the y_j to get a matrix.

Change of basis

$$\begin{aligned}x_1 &= a_{11}y_1 + a_{21}y_2 + \cdots + a_{n1}y_n \\&\vdots \\x_n &= a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{nn}y_n\end{aligned}$$

If a vector $v = c_1x_1 + \cdots + c_nx_n$ then, written in terms of the y_i we have

$$\begin{aligned}v &= (c_1a_{11} + c_2a_{12} + \cdots + c_na_{1n})x_1 + \cdots \\&\quad + (c_1a_{n1} + \cdots + c_na_{nn})x_n\end{aligned}$$

Change of basis continued

The coordinates $[v]_x$ of v in the x basis are computed from the coordinates $[v]_y$ in the y -basis as:

$$[v]_x = A[v]_y$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

NOTE: The columns of A are the coordinates $[x_i]_y$ of the x -basis elements in terms of the y -basis.

Example

If e_1, e_2 are the standard basis for \mathbf{R}^2 and y_1, y_2 are the vectors $(1, 1)$ and $(-1, 1)$ then to convert from the y_1, y_2 basis to the standard basis we should make the matrix A whose columns are the y_1, y_2 in terms of the standard basis.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

So if $v = ay_1 + by_2$ then

$$A \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a - b \\ a + b \end{bmatrix}$$

and so $v = (a - b)e_1 + (a + b)e_2$.

Example continued

To go backwards, suppose we have $v = ae_1 + be_2$. The inverse of A is

$$A^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (a+b)/2 \\ (b-a)/2 \end{bmatrix}$$

To check:

$$(a+b)/2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b-a)/2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Notice also that the columns of A^{-1} are the standard basis written in the y_1, y_2 coordinates.

Change of basis theorem

Theorem 15 (page 275 of the text): Let $B = \{b_1, \dots, b_n\}$ and $C = \{c_1, \dots, c_n\}$ be two bases of a vector space V . Then there is a (unique) $n \times n$ matrix $P_{C \leftarrow B}$ such that

$$[x]_C = P_{C \leftarrow B} [x]_B.$$

This matrix is called the “change of basis” or “change of coordinates” matrix from B to C .

The columns of $P_{C \leftarrow B}$ are the C -coordinates of the B -basis vectors. That is, the i^{th} column of $P_{C \leftarrow B}$ is $[b_i]_C$; its entries are a_1, \dots, a_n where

$$b_i = a_1 c_1 + a_2 c_2 + \dots + a_n c_n.$$

Computing the change of basis matrix

To compute the change of basis matrix $P_{C \leftarrow B}$, suppose we have b_1, \dots, b_n and c_1, \dots, c_n as vectors in \mathbf{R}^n .

Let B be the matrix whose columns are the b_i , and C be the matrix whose columns are the c_i .

Note that B is the change of basis matrix from B to the standard basis, and C is the change of basis matrix from C to the standard basis. So C^{-1} is the change of basis matrix from the standard basis to C .

Form the augmented matrix $[C|B]$ and do row reduction. Since C is invertible this will yield $[I|P]$ where I is the identity matrix.

The matrix P is the desired change of basis matrix $P_{C \leftarrow B}$. In fact $P = C^{-1}B$.

Numerical Example

Suppose that $B = \{(1, 1), (1, -1)\}$ and $C = \{(2, 1), (1, -2)\}$. We want $P_{C \leftarrow B}$. The augmented matrix is

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 \end{bmatrix}$$

The row reduced version is:

$$\begin{bmatrix} 1 & 0 & \frac{3}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$$

So the change of basis matrix is

$$\begin{bmatrix} 3/5 & 1/5 \\ -1/5 & 3/5 \end{bmatrix}$$

Checking the example.

Let $v = (2, 0) = b_1 + b_2$. The coordinates of v in the b -basis are $(1, 1)$. Multiplying:

$$\begin{bmatrix} 3/5 & 1/5 \\ -1/5 & 3/5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 2/5 \end{bmatrix}$$

Now $4/5(2, 1) + 2/5(1, -2) = (2, 0)$ so the coordinates of $(2, 0)$ in the c -basis are in fact $(4/5, 2/5)$.