Eigenvalues and Eigenvectors

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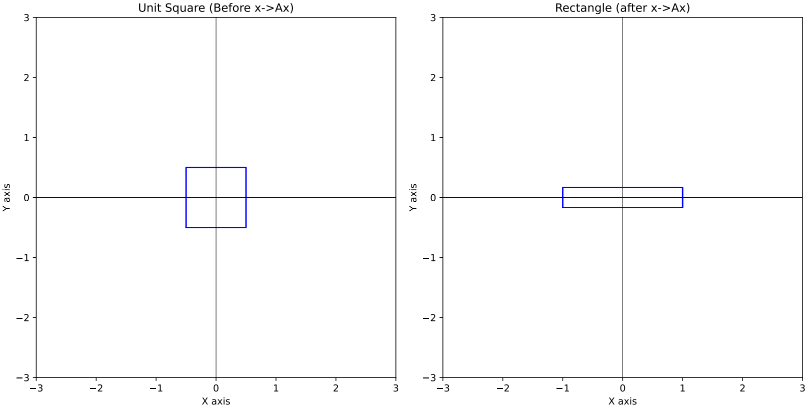
Eigenvalues and Eigenvectors

If A is a diagonal matrix:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

then the linear transformation $x \mapsto Ax$ "stretches" along the x-axis and "shrinks" along the y-axis.

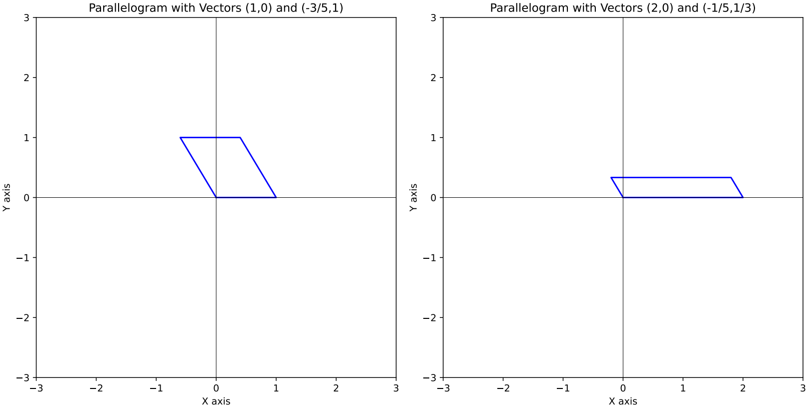


Eigenvalues and Eigenvectors

If A is upper triangular, say

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix}$$

then A stretches along the x-axis by 2 as before. Less obviously, it shrinks along the direction given by the vector $(-3/5, 1)$.



Eigenvalues and Eigenvectors

An **eigenvector** for a matrix A is a vector v which gets shrunk or lengthened by A by some factor λ.

The factor λ is called the **eigenvalue**.

More formally, a vector v is called an eigenvector for A (with eigenvalue λ) if v is not zero and

$$Av = \lambda v.$$

Eigenvalues and Eigenvectors

In the example above, the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3/5 \\ 1 \end{bmatrix}$ are eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix}$$

with eigenvalues 2 and $1/3$ respectively.

$$\begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} -3/5 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 1/3 \end{bmatrix} = (1/3) \begin{bmatrix} -3/5 \\ 1 \end{bmatrix}$$

Triangular Matrices

If A is (upper) triangular then the diagonal entries for A are all eigenvalues. There are n linearly independent eigenvectors.

Eigenspaces

Suppose that λ is a constant. The vectors v such that

$$Av = \lambda v$$

form a subspace called the *eigenspace* for λ.

This subspace is the nullspace of the matrix

$$A - \lambda I_n$$

where I_n is the $n \times n$ identity matrix.

Independence of Eigenvectors

If $v_1, \ldots, v_n$ are eigenvectors for a matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$, and all the λ_i are different, then the v_i are linearly independent. (Note that the v_i are nonzero.)

To see this, suppose that

$$c_1 v_1 + c_2 v_2 + \cdots c_n v_n = 0.$$

Then

$$A(c_1 v_1 + c_2 v_2 + \cdots c_n v_n) = c_1 \lambda_1 v_1 + \cdots c_n \lambda_n v_n = 0$$

Multiply the first relation by λ_1 and subtract. You get

$$c_2(\lambda_2 - \lambda_1)v_2 + \cdots + c_n(\lambda_n - \lambda_1)v_n = 0.$$

Since the differences of the λ_i with λ_1 are not zero, we see that $v_2, \ldots, v_n$ are dependent.

By repeating this you can show that smaller and smaller collections of the v_i are dependent until you ultimately get $v_n = 0$.

Characteristic Equation

Finding eigenvalues and eigenvectors of a matrix is a hard problem. We can make the following observation.

Suppose λ is an eigenvalue of A where A is an $n \times n$ matrix. Then there is a vector $v \neq 0$ so that $Av = \lambda v$. This means that the matrix $A - \lambda I_n$ is *not* invertible because v is in its null space.

As a result, $\det(A - \lambda I_n = 0$.

Conversely, if $\det(A - \lambda I_n) = 0$, then there is a vector v in the null space and that v is an eigenvector.

It turns out that $\det(A - \lambda I_n)$ is a polynomial in λ, so the eigenvalues of A are the roots of this polynomial.

Example

Let

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}.$$

The determinant of $A - \lambda$ is

$$\det(\begin{bmatrix} 3 - \lambda & 5 \\ 2 & 4 - \lambda \end{bmatrix}) = (3 - \lambda)(4 - \lambda) - 10$$

The polynomial on the right is

$$(3 - \lambda)(4 - \lambda) - 10 = \lambda^2 - 7\lambda + 12 - 10 = \lambda^2 - 7\lambda + 2.$$

Its roots are $\frac{7 \pm \sqrt{41}}{2}$. These are the eigenvalues of A; they are approximately 6.70156 and 0.29843.

Example continued

To find the eigenvectors, we have to compute the null space of $A - \lambda I$. This is no fun algebraically but with some work you find that the eigenvectors are:

$$\begin{bmatrix} -\frac{\sqrt{41}}{4} - \frac{1}{4} \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} -\frac{1}{4} + \frac{\sqrt{41}}{4} \\ 1 \end{bmatrix}$$

Similarity

Two (square) matrices A and B are *similar* if there is an invertible matrix P so that $A = PBP^{-1}$.

Similar matrices have the same eigenvalues because they have the same characteristic polynomial.

$$\begin{aligned} \det(PBP^{-1} - \lambda I) &= \det(P(B - \lambda I)P^{-1}) \\ &= \det(P)\det(B - \lambda I)\det(P^{-1}) \\ &= \det(B - \lambda I) \end{aligned}$$