1.3 Vector Equations

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Vectors

A vector (in $\mathbf{R}^n$) is an n-tuple of real numbers. For example

$$v = (-1, 3, 2, 5, 0, 7)$$

is a vector in $\mathbf{R}^6$.

Vectors can be written as matrices with one column (column vectors).

$$v = \begin{pmatrix} -1 \\ 3 \\ 2 \\ 5 \\ 0 \\ 7 \end{pmatrix}$$

Vector arithmetic

Vectors can be added:

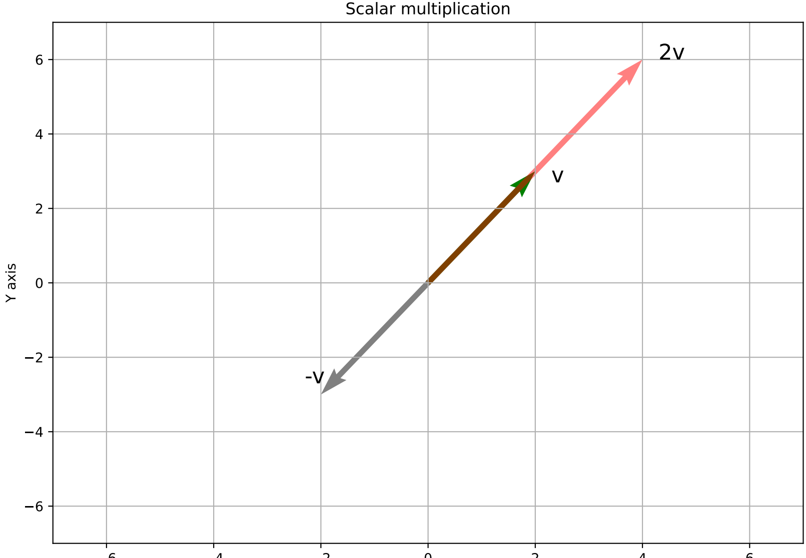
$$\begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \\ 11 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 16 \end{pmatrix}$$

Vectors can be multiplied by a number (a *scalar*):

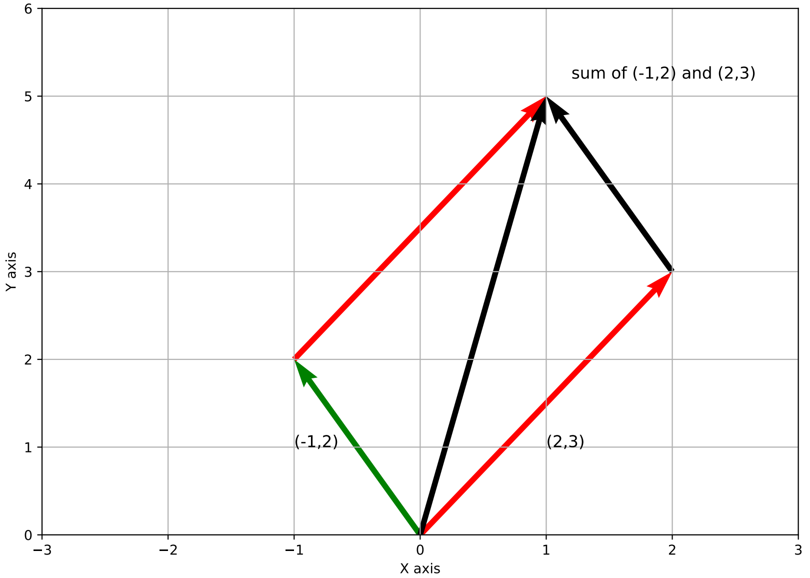
$$\begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \\ 15 \end{pmatrix}$$

Geometry: Scalar Multiplication

If $\mathbf{v}$ is a vector, and a is a scalar (a real number) then av “points in the same direction” but it’s length is “scaled by a.”



Geometry: Addition



This illustrates the “parallelogram law.” A similar picture holds in

Linear combinations

If $v_1, v_2, \dots, v_k$ are k vectors in $\mathbf{R}^n$, and $a_1, \dots, a_k$ are constants (scalars), then the vector

$$y = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$$

is called a *linear combination* of $v_1, \dots, v_k$.

Linear Combination Example

If

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

then the linear combinations of v_1 and v_2 are

$$y = a_1 v_1 + a_2 v_2 = \begin{pmatrix} a_1 \\ a_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a_2 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_1 + a_2 \\ a_2 \end{pmatrix}$$

Span

The *span* of a set of vectors in $\mathbf{R}^n$ is the collection of all possible linear combinations of the set.

In the previous example, the span of v_1 and v_2 is all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that you can find a_1 and a_2 so that:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a_2 \\ a_2 \end{pmatrix}$$

Span (continued)

This means that $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in the span of v_1 and v_2 if, **given** x, y, z, you can **find** a_1, a_2 so that

$$\begin{array}{rcl} a_1 & = & x \\ a_1 + a_2 & = & y \\ a_2 & = & z \end{array}$$

Span (continued)

In augmented matrix form this is

$$\begin{pmatrix} 1 & 0 & x \\ 1 & 1 & y \\ 0 & 1 & z \end{pmatrix}$$

Span (continued)

Using row reduction this yields

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y-x \\ 0 & 0 & z-y+x \end{pmatrix}$$

This system has a solution (is consistent) exactly when $z - y + x = 0$ or $y = z + x$.

Span (continued)

So

$$v = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

is in the span, but

$$w = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

is not.

Span (continued)

More generally if $v_1, \ldots, v_k$ are vectors in $\mathbf{R}^n$, then a vector w in $\mathbf{R}^n$ belongs to the span of the v_i if and only if the linear system with augmented matrix

$$M = \begin{bmatrix} v_1 & v_2 & \cdots & v_k & w \end{bmatrix}$$

has a solution (is consistent).

Here the matrix M has the indicated vectors as its columns. It is an $n \times (k+1)$ matrix.

Examples

Is

$$\begin{pmatrix} -5 \\ 11 \\ -7 \end{pmatrix}$$

in the span of the vectors

$$\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 8 \end{pmatrix}$$

Matrix form

Associated matrix

$$\begin{bmatrix} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{bmatrix}$$

Echelon Form:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & \frac{4}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So the last column is *not* in the span of the first three.