

# 1.1-1.2 Systems of Linear Equations

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# Linear Equations

A *linear equation* in variables  $x_1, \dots, x_n$  with constants  $a_1, \dots, a_n$  and  $b$  is an equation where the variables all appear to the first power (only).

$$a_1x_1 + a_2x_2 + \cdots a_nx_n = b$$

If  $n = 2$ , the solution set to an equation

$$a_1x_1 + a_2x_2 = b$$

is a line (hence the name linear). In higher dimensions, the solution set is a “hyperplane”.

# Systems

A system of linear equations is a collection

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\&\vdots \\a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n &= b_k\end{aligned}$$

Note the indexing:

- ▶ there are  $k$  equations in  $n$  unknowns, so there are  $k \times n$  coefficients  $a_{ij}$
- ▶ there are  $k$  constants  $b_j$ .

# Solutions

Given two equations in two unknowns there are three possibilities:

- ▶ the two equations have infinitely many common solutions.
- ▶ the two equations have one common solution.
- ▶ the two equations have no common solutions.

# Infinitely many common solutions



# One common solution



# No common solutions



## Goal: Generalize

What can we say about systems with more equations and more unknowns?

*Spoiler alert:* the same three possibilities hold:

- ▶ no solutions
- ▶ one solution
- ▶ infinitely many solutions



# Matrix Equation

We can simplify the writing by replacing this information:

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\&\vdots \\a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n &= b_k\end{aligned}$$

with a “matrix” consisting of just the coefficients.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} & b_k \end{pmatrix}$$

# Augmented and Coefficient Matrices

This is called the *augmented matrix* of the system of equations.

If you drop the final “b” column then it’s called the coefficient matrix.

# Row operations

Given a system of equations, you can:

1. Rearrange the equations into any order.
2. Replace any equation in the system by a multiple by a non-zero constant.
3. Replace any equation  $e$  by  $e + f$  where  $f$  is another equation in the system.

All of these operations are reversible and so the solutions of the transformed system are the same as the original.

These are called *elementary row operations*.

The algorithm for solving a system involves using these row operations to reduce the system to one where the solutions are easy to see.

## Row Reduction 1

$$\begin{bmatrix} 0. & 1. & 4. & -4. \\ 1. & 3. & 3. & -2. \\ 3. & 7. & 5. & 6. \end{bmatrix}$$

Swap row 3 and row 1

$$\begin{bmatrix} 3. & 7. & 5. & 6. \\ 1. & 3. & 3. & -2. \\ 0. & 1. & 4. & -4. \end{bmatrix}$$

Row[2]  $\rightarrow$   $-3 \cdot \text{Row}[2] + \text{Row}[1]$

## Row Reduction Continued

```
[[ 3.  7.  5.  6.]  
 [ 0. -2. -4. 12.]  
 [ 0.  1.  4. -4.]]
```

Row[3]  $\rightarrow$  2\*Row[3]+Row[2]

```
[[ 3.  7.  5.  6.]  
 [ 0. -2. -4. 12.]  
 [ 0.  0.  4.  4.]]
```

Divide rows by leading coeffs

```
[[ 1.          2.33333333  1.66666667  2.          ]  
 [-0.          1.          2.          -6.          ]  
 [ 0.          0.          1.          1.          ]]
```

## Row Reduction 1 continued

- ▶ The reduced matrix tells us that  $x_3 = 1$ .
- ▶ Then from the second row we get:  $x_2 + 2x_3 = -6$ , so  $x_2 + 2 = -6$  and  $x_2 = -8$ .
- ▶ Then from the first row we get  $x_1 + (7/3)x_2 + (5/3)x_3 = 2$ , so  $x_1 - 56/3 + 5/3 = 2$  and  $x_1 = 19$ .

We should check.

$$x_2 + 4x_3 = -8 + 4 = -4$$

$$x_1 + 3x_2 + 3x_3 = 19 - 24 + 3 = -2$$

$$3x_1 + 7x_2 + 5x_3 = 57 - 56 + 5 = 6$$

## Row Reduction 2

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

$$\text{Row}[2] \rightarrow -2 * \text{Row}[2] + \text{Row}[3]$$

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

Swap Row 3 and Row 1

## Row Reduction 2 continued

$$\begin{bmatrix} 4 & -8 & 12 & 1 \\ 0 & -2 & 8 & -1 \\ 0 & 1 & -4 & 8 \end{bmatrix}$$

$$\text{Row}[3] \rightarrow 2 * \text{Row}[3] + \text{Row}[2]$$

$$\begin{bmatrix} 4 & -8 & 12 & 1 \\ 0 & -2 & 8 & -1 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

This system has no solutions, it is inconsistent - the last row would mean  $0=15$



# Echelon form

A matrix is in *echelon form* (row echelon form) if:

- ▶ The zero rows are at the bottom of the matrix
- ▶ Each leading non-zero entry in a row is to the right of any leading entry above it.
- ▶ The entries below a leading entry are zero.

$$\begin{pmatrix} \square & * & \dots & * & * & \dots & * \\ 0 & \square & * \dots & * & * & \dots & * \\ 0 & 0 & 0 & \square & * & \dots & * \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 \end{pmatrix}$$

Here  $\square$  is non-zero, and  $*$  is anything.

## Solutions from echelon form

$$\begin{array}{rcccc} x_1 & & +5x_3 & +x_4 & = 11 \\ & 2x_2 & & -x_4 & = 5 \\ & & x_3 & +x_4 & = 1 \end{array}$$

This yields:

$$x_3 = 1 - x_4$$

$$x_2 = 5/2 + x_4/2$$

$$x_1 = 11 - 5(1 - x_4) + x_4 = 6 + 6x_4$$

There are infinitely many solutions;  $x_4$  can be anything and the others follow.

## Reduced echelon form

A matrix is in reduced echelon form if it is in echelon form and:

- ▶ the leading entries are 1
- ▶ each leading entry is the only nonzero entry in its column.

**Theorem:** Given a  $k \times n$  matrix, there is a sequence of row operations that will change it into a matrix in reduced row echelon form. A matrix has only *one* reduced row echelon form.

## Reduced echelon form continued

Remember our echelon matrix from before

```
[[ 1.          2.33333333  1.66666667  2.          ]
 [-0.          1.          2.          -6.          ]
 [ 0.          0.          1.          1.          ]]
```

We can reduce this

$X[1] \rightarrow X[1] - 7/3X[2]$

## Reduced Echelon Form continued

$$\begin{bmatrix} 1. & 0. & -3. & 16. \\ -0. & 1. & 2. & -6. \\ 0. & 0. & 1. & 1. \end{bmatrix}$$
$$X[1] \rightarrow X[1] + 3X[3]$$
$$\begin{bmatrix} 1. & 0. & 0. & 19. \\ -0. & 1. & 2. & -6. \\ 0. & 0. & 1. & 1. \end{bmatrix}$$
$$X[2] \rightarrow X[2] - 2X[3]$$
$$\begin{bmatrix} 1. & 0. & 0. & 19. \\ -0. & 1. & 0. & -8. \\ 0. & 0. & 1. & 1. \end{bmatrix}$$

Notice that this “solves” the system explicitly (look at the last column)

## Row reduction algorithm (forward pass)

### Forward Pass:

1. Find the leftmost column with a nonzero entry. Swap rows to make the top entry in that column nonzero. (This nonzero entry in the top leftmost position is called a pivot).
2. Use row operations to zero out all of the entries below the pivot.
3. Look at the submatrix below the pivot. Carry out steps 1 and 2 on this submatrix. Continue moving down and to the right, applying steps 1 and 2 to smaller and smaller submatrices until you reach the last row.

## Row reduction algorithm (backward pass)

4. Now start at the last row which a nonzero entry. Scale that row so its left most nonzero entry is 1.
5. Use row operations to make all the entries in the column above this 1 equal to zero.
6. Now move up and to the left, scaling the leading entry to 1 and eliminating non-zero entries above, until you reach the upper left corner.

## Extracting solutions

Let  $M$  be the augmented matrix of a linear system. Put  $M$  in reduced row echelon form. Then:

0. If there is a row with a non-zero final entry but zeros before that, the system is inconsistent. In other words, if the last column is a pivot column, the system is inconsistent. Otherwise:
  1. Columns with a nonzero pivot correspond to *basic variables*.
  2. Columns without a pivot correspond to *free variables*.

The free variables can take any value, and the basic variables can be computed for any choice of the free variables.



# Classification

- ▶ The system has no solutions if the last column of the augmented matrix is a pivot column.
- ▶ The system has infinitely many solutions if it is consistent and has at least one free variable.
- ▶ The system has a unique solution if every column (except the last one) is a pivot column, and therefore it has no free variables.

## Example

Matrix is

1	2	3	4
---	---	---	---

4	5	6	7
---	---	---	---

6	7	8	9
---	---	---	---

Reduced matrix is

1	0	-1	-2
---	---	----	----

0	1	2	3
---	---	---	---

0	0	0	0
---	---	---	---

Pivot columns are (1, 2)

Column 3 is a free variable

# Solutions

$$x_2 = 3 - 2x_3$$

$$x_1 = -2 + x_3$$

## Example

Matrix is

1 3 5 7

3 5 7 9

5 7 9 1

Reduced matrix is

1 0 -1 0

0 1 2 0

0 0 0 1

Pivot columns are (1, 2)

Column 3 is a free variable

This is an inconsistent system