Inner Products and Orthogonality

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The inner (dot) product.

If u and v are vectors in $\mathbf{R}^n$, then the *dot product* or *inner product* of u and v is

$$u \cdot v = u^T v = u_1 v_1 + \cdots + u_n v_n.$$

For example if

$$u = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, v = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

then

$$u \cdot v = (2)(1) + (3)(-1) + (-1)(0) = 2 - 3 = -1 \ldots$$

Key properties of the dot product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

d. $\mathbf{u} \cdot \mathbf{u} \geq 0, \quad$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

Figure 1: Theorem 1 (p. 375)

Length and distance

The *length* or *norm* of a vector (written $\|v\|$) is

$$\|v\| = \sqrt{v_1^2 + \cdots + v_n^2}$$

It is the "euclidean length" of the vector by the Pythagorean theorem.

Scaling a vector scales its length:

$$\|cv\| = |c|\|v\|$$

The distance between u and v is $\|u - v\|$ (this is the "distance formula").

Unit vectors

If v is a vector, then

$$u = \frac{v}{\|v\|}$$

is a vector of length one that "points in the same direction as v".

Such a vector is called a *unit vector*.

Orthogonality

Two vectors are "orthogonal" (or "perpendicular") if they meet at a right angle.

One way to describe this is to say that u and v are perpendicular if *the distance from u to v is the same as the distance from u to $-v$.*:

$$\|u - v\|^2 = \|u + v\|^2$$

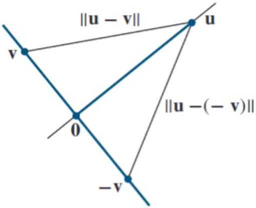

FIGURE 5

Figure 2: Perpendicular Vectors

Dot product zero means orthogonal

In other words

$$\|u\|^2 + \|v\|^2 - 2(u \cdot v) = \|u\|^2 + \|v\|^2 + 2(u \cdot v)$$

or

$$u \cdot v = 0$$

Key idea: u and v are orthognal if and only if $u \cdot v = 0$.

Orthogonal Complements

Let W be a subspace of $\mathbf{R}^n$.

The "orthogonal complement" to W, written $W^\perp$, is

$$W^\perp = \{v | v \cdot w = 0 \text{ for all w} \in \text{W}\}$$

For example, if W is the plane in $\mathbf{R}^3$ spanned by $w_1 = (2, 3, 1)$ and $w_2 = (-1, 1, 0)$, then $z \in W^\perp$ means

$$z \cdot (aw_1 + bw_2) = 0$$

for any a, b.

It's enough that $z \cdot w_1 = 0$ and $z \cdot w_2 = 0$.

Orthogonal complements continued

This gives two equations:

$$\begin{aligned} 2z_1 + 3z_2 + z_3 &= 0 \\ -z_1 + z_2 &= 0 \end{aligned}$$

which has a one dimensional solution space spanned by

$$(1, 1, -5)$$

Orthogonal complements - properties

Suppose W is a subspace of $\mathbf{R}^n$.

1. $x \in W^\perp$ if and only if $x \cdot u = 0$ for all u in a spanning set of W. (so you only need to check finitely many vectors to cover all of the elements of W).
2. $W^\perp$ is a subspace of $\mathbf{R}^n$.

Orthogonal complements and matrices

Let A be an $n \times m$ matrix. Then

$$\begin{aligned} \mathrm{Null}(A)^\perp &= \mathrm{Row}(A) \\ \mathrm{Null}(A) &= \mathrm{Row}(A)^\perp \end{aligned}$$

and

$$\begin{aligned} \mathrm{Col}(A)^\perp &= \mathrm{Nul}(A^T) \\ \mathrm{Col}(A) &= \mathrm{Null}(A^T)^\perp \end{aligned}$$

Angles

The "Law of Cosines" tells us that

$$u \cdot v = \|u\| \|v\| \cos \theta$$

where θ is the angle between u and v.

Orthogonal sets

A set $u_1, \ldots, u_k$ of vectors in $\mathbf{R}^n$ is an *orthogonal set* if any pair of (different) vectors from the set are orthogonal.

It is an *orthonormal* set if in addition the vectors have length one.

Key point: An orthogonal set is linearly independent. Therefore if S is orthogonal then it is a basis for its span.

Orthogonal basis

A basis for a subspace W is orthogonal if it is an orthogonal set.

Suppose y is any vector in W and $u_1, \dots, u_k$ are an orthogonal basis. Then

$$y = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_k}{u_k \cdot u_k} u_k$$

To see this, write

$$y = c_1 u_1 + \cdots + c_k u_k$$

and compute $y \cdot u_j$ on both sides to solve for c_j.

Orthogonal projection

Let u be a vector in $\mathbf{R}^n$. We can decompose a vector y into a part that is "parallel" to u and a part that is *perpendicular* to u.

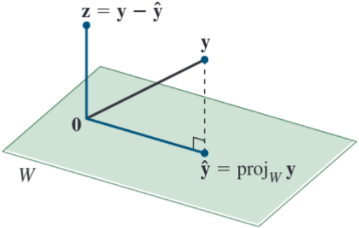

FIGURE 2

Finding α to make $\mathbf{y} - \hat{\mathbf{y}}$ orthogonal to $\mathbf{u}$.

Figure 3: Orthogonal Projection

Orthogonal projection continued

In particular:

$$\hat{y} = \frac{y \cdot u}{u \cdot u} u$$

is parallel to u, and $z = y - \hat{y}$ is perpendicular to u.

If $u_1, \ldots, u_k$ are an orthogonal basis for a subspace W, then the projection of y into W is

$$\mathrm{proj}_W(y) = \sum \frac{y \cdot u_i}{u_i \cdot u_i} u_i$$

and $y - \mathrm{proj}_W(y)$ is perpendicular to W.

Orthonormal sets

The formulae above for projections are simplified for orthonormal sets because in that case $u_i \cdot u_i = 1$.

Let U be an $m \times n$ matrix. The columns of U are orthonormal if and only if $U^T U = I$ where I is the $n \times n$ identity matrix.

If U is $m \times n$ and has orthonormal columns and x and y are vectors in $\mathbf{R}^n$ then:

1. $\|Ux\| = \|x\|$
2. $(Ux) \cdot (Uy) = x \cdot y$
3. $(Ux) \cdot (Uy) = 0$ if and only if $x \cdot y = 0$.