Basis and Linear Independence

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Basis

A set of vectors in $\mathbf{R}^n$ (or in any vector space V) is called a **basis** if

- it spans V
- it is linearly independent.

Examples: if A is an invertible $n \times n$ matrix, its columns are linearly independent and span $\mathbf{R}^n$ and therefore are a basis for $\mathbf{R}^n$.

The vectors $1, x, x^2, \dots, x^n$ span the polynomials of degree at most n and are linearly indepenent.

The "standard vectors" e_i for $i = 1, \dots, n$ are a basis for $\mathbf{R}^n$.

Subspace basis

The vectors $(1, 3, 2)$ and $(-1, -1, 0)$ are linearly indepedent and span a subspace H of $\mathbf{R}^3$.

Therefore they are a basis for H.

Every spanning set contains a basis

If a set S of vectors $v_1, \ldots, v_n$ spans a subspace H, then a subset of S is a basis.

Proof: If the vectors are linearly indepenent, they are already a basis.

If they are dependent, then one is a linear combination of the others. Remove that one from S. The result still spans.

Continue removing dependent vectors until the remaining vectors are independent, and you've found your basis.

A basis is a minimal spanning set

If H is a subspace of V, suppose you have a bunch of vectors in H.

Too many vectors makes them dependent. To few means they can't span. If they are a basis, there are enough to span, but not to become dependent.

Basis for $\mathrm{Nul}(A)$.

The null space of A is spanned by the vectors with weights given by the free variables in the row reduced from of A.

Those vectors are independent and therefore form a basis.

Basis for $\mathrm{Col}(A)$.

Given vectors $v_1, \ldots, v_k$, make an $m \times k$ matrix with the v_i as columns.

To find a linear relation among the columns of A, we need to solve $Ax = 0$.

But $Ax = 0$ if and only if $EAx = 0$ where E is an elementary matrix.

Put another way, row reduction doesn't change the x such that $Ax = 0$.

So we can assume A is in row reduced echelon form.

More on basis for $\mathrm{Col}(A)$.

Once A is in row reduced form, we see that:

- the columns corresponding to free variables are linear combinations of the pivot columns
- the pivot columns are linearly independent.

Basis for $\mathrm{Col}(A)$.

The **columns of** A corresponding to the pivot columns in the row reduced version of A are a basis for the column space. (note that these are *not* the columns of the reduced matrix).

So: a basis for the null space is made up of k vectors where k is the number of free variables, and a basis for the column space is made up of r vectors where r is the number of pivot columns.

Notice that $k + r = n$ where n is the total number of columns of A.

Example

Suppose that

$$A = \begin{bmatrix} 2 & 4 & 5 & 1 \\ 1 & -3 & -2 & 0 \\ 0 & -2 & -3 & 1 \end{bmatrix}$$

The row reduced form of A is

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Since the first three columns are pivot columns, the first three columns of A span the column space of A, and the last column satisfies $c_4 = c_1 + c_2 - c_3$.

Example continued

The nullspace of A is the solution to the homogeneous system, and it is given by the equations

$$\begin{aligned} x_1 &= -x_4 \\ x_2 &= -x_4 \\ x_3 &= x_4 \end{aligned}$$

so the null space is spanned by

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

Null Space and Col Space

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\, A \quad \mathbf{0} \,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\, A \quad \mathbf{v} \,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

Figure 1: Null Space vs Col Space

Row space

The *row space* of a matrix is the span of its rows.

Row operations do not change the row space, so one can find a basis for the row space of A by putting A in reduced form.

The rows with a pivot (that is, the nonzero rows) form a basis for the row space.

This is because they are clearly linearly independent (and they span by definition).

Linear Transformations

A linear transformation (or linear map) $T : V \to W$, where V and W are vector spaces, is a function that satisfies $T(u+v) = T(u) + T(v)$ and $T(cv) = cT(v)$ for all $u, v \in V$ and $c \in \mathbf{R}$.

The *kernel* of a linear transformation is the set of vectors that map to zero:

$$\mathrm{kernel}(T) = \{x \in V : T(x) = 0\}$$

The *range* or *image* of a linear transformation is the set of vectors $w \in W$ such that there is a $v \in V$ with $T(v) = w$.

Coordinate systems

Unique representation: Suppose that $B = \{b_1, \ldots, b_n\}$ are a basis for a vector space V. Then any vector v can be written in exactly one way as a linear combination of the b_i:

$$v = c_1 b_1 + \ldots + c_n b_n$$

The coefficients $c_1, \ldots, c_n$ are called the *coordinates* of v relative to the basis B.

The vector

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is called the *coordinate vector* for v relative to B.

Coordinates (example)

Suppose that

$$e_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

These form a basis of $\mathbf{R}^2$. If

$$v = c_1 e_1 + c_2 e_2$$

then

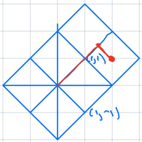
$$v = \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix}$$

Coordinates continued

Suppose

$$v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

What are the coordinates of v in the e_1, e_2 basis?



• - (2,1) in "usual coords"

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$a + b = 2$

$a - b = 1$

$a = 3/2$

$b = 1/2$

Coordinates

In general, each choice of basis for a vector space gives a different system of coordinates on that vector space.

Consider the polynomials with degree at most 2. This vector space has basis $1, x, x^2$.

Consider the polynomials $a(x) = \frac{x(x-1)}{2}$, $b(x) = 1 - x^2$, and $c(x) = \frac{x(x+1)}{2}$.

They form another basis for the degree 2 polynomials.

Coordinates continued

If

$$f = c_0 + c_1 x + c_2 x^2$$

then the coordinates of f in terms of a, b, c are $f(-1), f(0), f(1)$:

$$f(x) = f(-1)a(x) + f(0)b(x) + f(1)c(x).$$

Coordinates

If $b_1, \ldots, b_n$ is a basis, let B be the matrix whose columns are the vectors b_i. Then if we write

$$w = B \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

we have $w = c_1 b_1 + \cdots + c_n b_n$ and so the c_i are the coordinates of w relative to B. To *find* the c_i for a given w, we need the inverse of B:

$$B^{-1} w = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Coordinates

In our 2-d example, we have

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

so

$$B^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

In particular

$$B^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

as above.