

## 1.8-1.9 Matrices and Linear Transformations

Jeremy Teitelbaum

## Linear Transformations and Matrices

If  $A$  is an  $n \times m$  matrix, and  $x$  is any vector in  $\mathbf{R}^m$ , then  $Ax$  is a vector in  $\mathbf{R}^n$ .

So we can define a function  $T : \mathbf{R}^m \rightarrow \mathbf{R}^n$  by

$$T(x) = Ax.$$

For example if

$$A = \begin{bmatrix} 0 & -4 \\ 4 & 1 \\ 3 & 3 \end{bmatrix} \text{ and } v = \begin{bmatrix} x \\ y \end{bmatrix}$$

then

$$T(v) = Av = \begin{bmatrix} -4y \\ 4x + y \\ 3x + 3y \end{bmatrix}$$

# Function terminology

In general if  $f : X \rightarrow Y$  is a function then  $f$  is a “rule” that associates exactly one element  $y \in Y$  to each element  $x \in X$ . The  $y$  corresponding to  $x$  is called  $f(x)$ . Furthermore:

- ▶  $X$  is called the domain of  $f$
- ▶  $Y$  is called the codomain of  $f$
- ▶ the set of  $y \in Y$  so that there is an  $x \in X$  with  $f(x) = y$  is called the *range* of  $f$ .
- ▶ if  $f(x) = y$ , then  $y$  is called the *image* of  $x$  under  $f$ .

If  $A$  is an  $n \times m$  matrix, then the domain of  $f(x) = Ax$  is  $\mathbf{R}^m$  and the codomain is  $\mathbf{R}^n$ .

## Examples of matrix transformations

If

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

is called a *projection*, in this case onto the  $xy$ -plane.

## Rotations in 2d

If

$$A(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

then

$$A(\theta) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{bmatrix}$$

rotates the vector  $(x, y)$  through an angle  $\theta$  counterclockwise.

To see this, write  $x = r \cos \phi$  and  $y = r \sin \phi$ . Then:

$$r \cos \phi \cos \theta - r \sin \phi \sin \theta = r \cos(\phi + \theta)$$

$$r \cos \phi \sin \theta + r \sin \phi \cos \theta = r \sin(\phi + \theta)$$

# Linear Transformations

Let  $T : \mathbf{R}^m \rightarrow \mathbf{R}^n$  be a function. Then  $T$  is called a *linear transformation* if

- ▶  $T(ax) = aT(x)$  for every scalar  $a$ , and
- ▶  $T(x + y) = T(x) + T(y)$  for any two vectors  $x, y \in \mathbf{R}^m$ .

Any matrix transformation  $T(x) = Ax$ , where  $A$  is  $n \times m$ , is linear.

If  $T$  is linear, then  $T(0) = 0$  (because  $T(0x) = 0T(x) = 0$ .)

# Linear Transformations

Let

$$T(x) = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix} x.$$

Find a vector  $x$  so that  $T(x) = b$  and determine if this  $x$  is unique.

Hint: The rref form for the augmented matrix  $[A \quad b]$  is

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

## Another problem

Let

$$T(x) = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix} x.$$

Find a vector  $x$  so that  $T(x) = b$  and determine if this  $x$  is unique.

Hint: The rref form for the augmented matrix  $[A \quad b]$  is

$$\begin{bmatrix} 1 & 0 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



# Linear Transformations

If  $T : \mathbf{R}^m \rightarrow \mathbf{R}^n$  is linear, and  $v_1, \dots, v_k$  are vectors in  $\mathbf{R}^m$ , then if you know

$$T(v_1), \dots, T(v_k)$$

you know

$$T(a_1 v_1 + \dots + a_k v_k)$$

for any constants  $a_i$ . In other words, you can compute  $T$  for any vector in the span of  $v_1, \dots, v_k$ .

# Linear Transformations

In particular if

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}$$

and

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} c \\ d \end{bmatrix}$$

then

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = xT\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + yT\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} ax + cy \\ bx + dy \end{bmatrix}$$

and  $T(x) = Ax$  where

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

# Matrices and Linear Transformations

We have seen that, given a matrix  $A$ , then  $T(x) = Ax$  is a linear transformation.

Now suppose  $T : \mathbf{R}^m \rightarrow \mathbf{R}^n$  is a linear transformation.

Let  $e_i \in \mathbf{R}^m$  be the vector

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

where the 1 is in row  $i$  of  $e_i$ .

# Matrices and Linear Transformations

Let

$$A(T) = [T(e_1) \quad T(e_2) \cdots T(e_m)]$$

whose columns are the  $T(e_i)$ . This is an  $n \times m$  matrix because each  $T(e_i) \in \mathbf{R}^n$ .

Notice that  $Ae_i = T(e_i)$  for  $i = 1, \dots, m$ . As a result, by linearity,  $Av = T(v)$  for any vector  $v \in \mathbf{R}^m$ .

Therefore *every linear transformation comes from multiplication by a matrix.*

## The identity map

The map  $T : \mathbf{R}^m \rightarrow \mathbf{R}^m$  given by  $Tx = x$  is called the identity map.

Since  $T(e_i) = e_i$  for  $i = 1, \dots, m$  the matrix of  $T$  is the  $m \times m$  matrix with 1's on the diagonal and zeros elsewhere.

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

# Matrices and Linear Transformations

If  $T : \mathbf{R}^m \rightarrow \mathbf{R}^n$  is linear, then  $T$  is determined by what it does to the standard basis vectors  $e_i$ .

For example, if  $m = n = 2$ , and  $T$  is the reflection map  $T(x, y) = (y, x)$ , then  $T(e_1) = e_2$  and  $T(e_2) = e_1$  and therefore  $Tx = Ax$  where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

# Reflections

**TABLE 1** Reflections

Transformation	Image of the Unit Square	Standard Matrix
Reflection through the $x_2$ -axis		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection through the $x_1$ -axis		$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Reflection through the line $x_2 = -x_1$		$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
Reflection through the origin		$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Figure 1: reflections

# Shears

**TABLE 3** Shears

Transformation	Image of the Unit Square	Standard Matrix
Horizontal shear	 <p style="text-align: center;"><math>k &lt; 0</math></p>	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
	 <p style="text-align: center;"><math>k &gt; 0</math></p>	
Vertical shear	 <p style="text-align: center;"><math>k &lt; 0</math></p>	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$
	 <p style="text-align: center;"><math>k &gt; 0</math></p>	

Figure 2: shears



# Contractions/Expansions

**TABLE 2** Contractions and Expansions

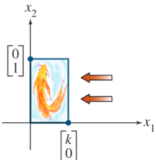
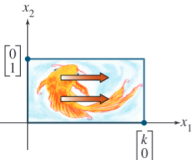
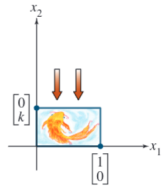
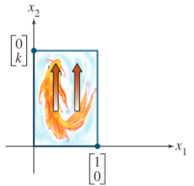
Transformation	Image of the Unit Square	Standard Matrix
Horizontal contraction and expansion	<div style="display: flex; justify-content: space-around; align-items: flex-start;"> <div style="text-align: center;">  <p><math>0 &lt; k &lt; 1</math></p> </div> <div style="text-align: center;">  <p><math>k &gt; 1</math></p> </div> </div>	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Vertical contraction and expansion	<div style="display: flex; justify-content: space-around; align-items: flex-start;"> <div style="text-align: center;">  <p><math>0 &lt; k &lt; 1</math></p> </div> <div style="text-align: center;">  <p><math>k &gt; 1</math></p> </div> </div>	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

Figure 3: contractions and expansions

# Projections

**TABLE 4** Projections

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the $x_1$ -axis		$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Projection onto the $x_2$ -axis		$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Figure 4: projections

## One-to-one and onto maps

A function  $T : A \rightarrow B$  is *one-to-one* if the only way that  $T(x) = T(y)$  is if  $x = y$ .

Eg the function  $f(x) = x^2$  is *not* one-to-one, because  $f(-1) = f(1)$  even though  $-1 \neq 1$ . But the function  $f(x) = 3x$  is one-to-one, because if  $3x = 3y$  then  $x$  and  $y$  must be equal.

A function  $T : A \rightarrow B$  is *onto* if, for any  $b \in B$ , there is an  $a \in A$  so that  $T(a) = b$ .

The function  $f(x) = x^2$  is not *onto*, because the equation  $-1 = x^2$  does not have a solution (at least in real numbers.) The function  $f(x) = 3x$  is *onto*, because the equation  $y = 3x$  always has a solution ( $x = y/3$ ).

## One-to-one linear maps

If  $T$  is linear, then  $T(x) = T(y)$  if and only if  $T(x) - T(y) = T(x - y) = 0$ . So  $T$  is one-to-one if the only solution to  $T(v) = 0$  is  $v = 0$ .

Since  $T$  comes from a matrix  $A$ , the map is one-to-one if and only if the matrix equation  $Ax = 0$  has only zero as its solution.

This happens if and only if *the columns of  $A$  are linearly independent*.

## Onto linear maps

If  $T : \mathbf{R}^m \rightarrow \mathbf{R}^n$  is linear, then  $T(x)$  is onto if only if  $T(x) = b$  has a solution for any  $b \in \mathbf{R}^n$ . This means that the matrix equation

$$Ax = b$$

has a solution for any  $b \in \mathbf{R}^n$ .

Since  $Ax$  is a linear combination of the columns of  $A$ , every equation  $Ax = b$  has a solution only if every  $b$  is a linear combination of the columns of  $A$ . In other words,  $A$  is onto if and only if the columns of  $A$  span  $\mathbf{R}^n$ .

## Theorem 12

Theorem 12 in the book summarizes these two key facts.

**Theorem:** Let  $T(x) = Ax$  be a linear map from  $\mathbf{R}^m$  to  $\mathbf{R}^n$ , where  $A$  is an  $n \times m$  matrix.

1.  $T$  is one-to-one if and only if the columns of  $A$  are linearly independent vectors in  $\mathbf{R}^n$ .
2.  $T$  is onto if and only if the columns of  $A$  span  $\mathbf{R}^n$ .

## Algebraic version

Algebraically:

1.  $T(x) = Ax$  is one-to-one if and only if the rref of  $A$  has no free variables - in other words, if every column has a pivot.
2.  $T(x) = Ax$  is onto if and only if every row of  $A$  has a pivot.

Note that if  $A$  is an  $n \times m$  matrix, then:

- ▶ if  $m > n$ , the map cannot be one-to-one.
- ▶ if  $n > m$ , the map cannot be onto.