

## 2.1 Matrix Operations

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## Basic terminology

An  $n \times m$  matrix  $A$  can be written as an  $n \times m$  array of numbers.

It can also be written as

$$A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_m]$$

where each  $\mathbf{a}_i$  is one of the  $m$  columns of  $A$ , and is a vector in  $\mathbf{R}^n$ .

# Special matrices

The  $m \times n$  zero matrix has all entries equal to zero.

The main diagonal of an  $m \times n$  matrix are the entries  $a_{11}, a_{22}, \dots$  (which ends at either  $a_{nn}$  or  $a_{mm}$  depending on which is smaller).

A square matrix is diagonal if all entries off the main diagonal are zero.

# Addition

Assuming matrices  $A$  and  $B$  **are the same shape** you can add them element by element:

$$C = A + B$$

and obtain another matrix of the same shape.

## Scalar multiplication

You can also multiply  $A$  by a constant  $r$  (meaning multiply every element by  $r$ ) to get another matrix  $B = rA$  of the same shape.

# Properties

Let  $A$ ,  $B$ , and  $C$  be matrices of the same size, and let  $r$  and  $s$  be scalars.

a.  $A + B = B + A$

b.  $(A + B) + C = A + (B + C)$

c.  $A + 0 = A$

d.  $r(A + B) = rA + rB$

e.  $(r + s)A = rA + sA$

f.  $r(sA) = (rs)A$

Figure 1: properties

# Matrix Multiplication

If  $A$  is an  $m \times n$  matrix and  $B$  is a  $k \times p$  matrix, then the product  $AB$  is defined ONLY WHEN  $n = k$ .

In other words you can multiply  $m \times n$  times  $n \times p$ .

The result is an  $m \times p$  matrix.

# Matrix multiplication

If  $A$  is an  $m \times n$  matrix and  $B$  is an  $n \times p$  matrix:

$$B = [b_1 \quad b_2 \quad \cdots \quad b_p]$$

then

$$AB = [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_p]$$

This makes sense because:

- ▶ each column  $b_1$  has  $n$  rows
- ▶  $Ab_1$  makes sense and has  $m$  rows
- ▶ so  $AB$  is an  $m \times p$  matrix.



## Example

$$A = \begin{bmatrix} 8 & 6 & -8 & -4 \\ 0 & 4 & -4 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} 8 & 7 & 1 \\ 7 & 0 & 7 \\ 2 & 1 & -7 \\ -7 & 8 & -3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 118 & 16 & 118 \\ -36 & 60 & 32 \end{bmatrix}$$

## Dot product rule

If  $A$  is  $m \times n$  and  $B$  is  $n \times p$ , then the  $i, j$  entry of  $AB$  is the dot product of row  $i$  of  $A$  with column  $j$  of  $B$ .

$$\text{Row}_i(A) = [a_{i1}, a_{i2}, \dots, a_{in}]$$

$$\text{Col}_j(B) = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

$$A_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

# Summation Notation

Here  $A$  is  $m \times n$  and  $B$  is  $n \times p$ .  $(AB)_{ij}$  means the  $i, j$  entry of the matrix product  $AB$ .

$$(AB)_{ij} = \sum_{t=1}^n a_{it} b_{tj}$$

# Properties

Let  $A$  be an  $m \times n$  matrix, and let  $B$  and  $C$  have sizes for which the indicated sums and products are defined.

- a.  $A(BC) = (AB)C$  (associative law of multiplication)
- b.  $A(B + C) = AB + AC$  (left distributive law)
- c.  $(B + C)A = BA + CA$  (right distributive law)
- d.  $r(AB) = (rA)B = A(rB)$   
for any scalar  $r$
- e.  $I_m A = A = A I_n$  (identity for matrix multiplication)

Figure 2: MatrixMultProps

**IMPORTANT:** In general,  $AB \neq BA$ . Matrix multiplication is NOT COMMUTATIVE.

# Transpose

The *transpose* of an  $m \times n$  matrix  $A$  is the  $n \times m$  matrix obtained from  $A$  by interchanging rows and columns.

$$A = \begin{bmatrix} -3 & -2 & -1 & 0 & 1 \\ 0 & -3 & 1 & -1 & -2 \\ -2 & -2 & -2 & -2 & -2 \end{bmatrix}$$

$$\text{Transpose } A^T = \begin{bmatrix} -3 & 0 & -2 \\ -2 & -3 & -2 \\ -1 & 1 & -2 \\ 0 & -1 & -2 \\ 1 & -2 & -2 \end{bmatrix}$$

## Transpose and products

The transpose of a product is the product of the transposes, in the reversed order.

$$(AB)^T = B^T A^T$$

This makes sense: If  $A$  is  $m \times n$  and  $B$  is  $n \times p$ , then  $(AB)$  is  $m \times p$  so  $(AB)^T$  is  $p \times m$ .

On the other hand  $B^T$  is  $p \times n$  and  $A^T$  is  $n \times m$  so  $B^T A^T$  is also  $p \times m$ .

# Matrix Powers

If  $A$  is a matrix, then  $A^n = A \cdot A \cdot A \cdots A$  makes sense (at least for integers  $n \geq 0$ ).

# Inverse Matrix

If  $A$  is an  $n \times n$  matrix, and  $I_n$  is the  $n \times n$  identity matrix having ones on the diagonal and zero elsewhere, then the inverse  $A^{-1}$  of  $A$  (if it exists) is the matrix such that  $A^{-1}A = I_n$ .

Suppose  $A^{-1}A = I_n$ . What about a matrix  $B$  such that  $AB = I_n$ ?

- ▶ Let  $u$  be  $(A^{-1})^{-1}$ , meaning that  $uA^{-1} = I_n$ . Consider  $(uA^{-1})(AA^{-1})$ .
- ▶ On the one hand this is  $AA^{-1}$  since  $uA^{-1} = I_n$ . On the other hand this is  $uA^{-1} = I_n$  since the middle  $A^{-1}A$  yield  $I_n$ . So  $AA^{-1} = I_n$  and the inverse works on both sides.



## Not all square matrices have inverses

Let  $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ . If  $AB = I_2$  then

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so

$$\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is clearly impossible.

## Two by two inverse

If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Just multiply them out and check that it works. Here  $ad - bc \neq 0$ .  
If  $ad - bc = 0$  then there is no inverse.

# Determinant

The quantity  $ad - bc$  for a  $2 \times 2$  matrix is called the “determinant” of that matrix.

We will study determinants of bigger matrices later.