Basis and Linear Independence

Jeremy Teitelbaum

Basis

A set of vectors in $\mathbf{R}^n$ (or in any vector space V) is called a **basis** if

- it spans V
- it is linearly independent.

Examples: if A is an invertible $n \times n$ matrix, its columns are linearly independent and span $\mathbf{R}^n$ and therefore are a basis for $\mathbf{R}^n$.

The vectors $1, x, x^2, \ldots, x^n$ span the polynomials of degree at most n and are linearly indepenent.

The "standard vectors" e_i for $i = 1, \ldots, n$ are a basis for $\mathbf{R}^n$.

Subspace basis

The vectors $(1, 3, 2)$ and $(-1, -1, 0)$ are linearly indepedent and span a subspace H of $\mathbf{R}^3$.

Therefore they are a basis for H.

Every spanning set contains a basis

If a set S of vectors $v_1, \ldots, v_n$ spans a subspace H, then a subset of S is a basis.

Proof: If the vectors are linearly indepenent, they are already a basis.

If they are dependent, then one is a linear combination of the others. Remove that one from S. The result still spans.

Continue removing dependent vectors until the remaining vectors are independent, and you've found your basis.

A basis is a minimal spanning set

If H is a subspace of V, suppose you have a bunch of vectors in H.

Too many vectors makes them dependent. To few means they can't span. If they are a basis, there are enough to span, but not to become dependent.

Basis for $\mathrm{Nul}(A)$.

The null space of A is spanned by the vectors with weights given by the free variables in the row reduced from of A.

Those vectors are independent and therefore form a basis.

Basis for $\mathrm{Col}(A)$.

Given vectors $v_1, \ldots, v_k$, make an $m \times k$ matrix with the v_i as columns.

To find a linear relation among the columns of A, we need to solve $Ax = 0$.

But $Ax = 0$ if and only if $EAx = 0$ where E is an elementary matrix.

Put another way, row reduction doesn't change the x such that $Ax = 0$.

So we can assume A is in row reduced echelon form.

More on basis for $\mathrm{Col}(A)$.

Once A is in row reduced form, we see that:

- the columns corresponding to free variables are linear combinations of the pivot columns
- the pivot columns are linearly independent.

Basis for $\mathrm{Col}(A)$.

The columns of A corresponding to the pivot columns in the row reduced version of A are a basis for the column space.

So: a basis for the null space is made up of k vectors where k is the number of free variables, and a basis for the column space is made up of r vectors where r is the number of pivot columns.

Notice that $k + r = n$ where n is the total number of columns of A.