

# Gram Schmidt

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In our discussion so far we have been handed orthogonal bases for various subspaces.

How do we find such a thing?

**Problem:** Given a set of vectors  $v_1, \dots, v_k$  in  $\mathbf{R}^n$ , find an orthogonal basis (or an orthonormal basis) for the span  $W$  of those vectors.

Strategy: Work systematically:

- ▶ Start with  $v_1$ ; it becomes  $u_1$ .
- ▶ Subtract the component of  $v_2$  in the  $v_1$  direction from  $v_2$ ; call this  $u_2$ .
- ▶ Subtract the projection of  $v_3$  into the span of  $u_1$  and  $u_2$  from  $v_3$ , and call that  $u_3$ .
- ▶ Continue in this way, subtracting the projection of  $v_n$  from the span of  $u_1, \dots, u_{n-1}$ , to obtain  $u_n$ .

If you normalize these vectors  $u_i$  you get an orthonormal basis.

## Gram Schmidt (Example)

Suppose

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

The first two vectors in the sequence of G-S vectors is

$$u_1 = v_1, u_2 = v_2 - 3/4 v_1 = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

## Example (continued)

The third vector

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2}$$

Now  $u_1 \cdot u_1 = 4$  and

$$u_2 \cdot u_2 = v_2 \cdot v_2 - 3/2 v_2 \cdot v_1 + 9/16 v_1 \cdot v_1 = 3 - 9/2 + 9/4 = 3/4$$

Also  $v_3 \cdot u_2 = v_3 \cdot v_2 - 3/4 v_3 \cdot v_1 = 1/2$ . So

$$u_3 = v_3 - \frac{2}{4} u_1 - \frac{2}{3} u_2 = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

# The QR decomposition

Suppose that  $A$  is an  $n \times m$  matrix with **linearly independent columns**. Then there is an orthogonal matrix  $Q$  (of size  $n \times m$ ) and an upper triangular matrix  $R$  of size  $m \times m$  so that

$$A = QR$$

The columns of  $Q$  form an orthonormal basis for the column space of  $A$ ;  $Q^T Q = I$ ; and the diagonal entries of  $R$  are positive.

(This is called the “QR” decomposition of  $A$ ).

It's really a restatement of the Gram-Schmidt process.

# The QR decomposition

Let  $A$  be an  $n \times m$  matrix. To compute the  $QR$  decomposition, we apply Gram-Schmidt to the columns of  $A$ .

Each step in  $GS$  corresponds to multiplying  $A$  on the right by an upper triangular matrix.

## GS and QR example

Suppose that

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -5 & 2 \\ 0 & 2 & -4 \end{bmatrix}$$

We wish to apply Gram-Schmidt to the columns of  $A$ . We leave the first column alone. Multiplying  $A$  on the right by

$$e = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

extracts the first column:

$$Ae_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}.$$

## GS/QR continued

The next step is to compute

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1$$

Since  $v_2 \cdot u_1 = 2 - 15 = -13$  and  $u_1 \cdot u_1 = 9 + 1 = 10$  this means

$$u_2 = v_2 + \frac{13}{10} u_1 = \begin{bmatrix} 33/10 \\ -11/10 \\ 2 \end{bmatrix}.$$

This second vector can be obtained by multiplying  $A$  on the right by

$$e_2 = \begin{bmatrix} 13/10 \\ 1 \\ 0 \end{bmatrix}$$

so that

$$Ae_2 = \begin{bmatrix} 33/10 \\ -11/10 \\ 2 \end{bmatrix}$$



## GS and QR continued

Combining steps 1 and 2 we have

$$A \begin{bmatrix} 1 & 13/10 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 33/10 \\ 3 & -11/10 \\ 0 & 2 \end{bmatrix}$$

The last step is to compute

$$u_3 = v_3 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1$$

This gives

$$u_3 = v_3 - \frac{-27/2}{161/10} u_2 - \frac{5}{10} u_1 = \frac{1}{322} \begin{bmatrix} 408 \\ -136 \\ -748 \end{bmatrix}.$$

## QR and GS continuedf

In terms of the matrix  $A$ , computing  $u_3$  comes from multiplying  $A$  on the right by

$$e_3 = \begin{bmatrix} -1/2 \\ 270/322 \\ 1 \end{bmatrix}$$

So we've shown that

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -5 & 2 \\ 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 & 13/10 & -1/2 \\ 0 & 1 & 270/322 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 33/10 & 408/322 \\ 3 & -11/10 & -136/322 \\ 0 & 2 & -748/322 \end{bmatrix}$$

## QR and GS continued

If we let

$$Q = \begin{bmatrix} 1 & 33/10 & 408/322 \\ 3 & -11/10 & -136/322 \\ 0 & 2 & -748/322 \end{bmatrix}$$

and

$$R = \begin{bmatrix} 1 & 13/10 & -1/2 \\ 0 & 1 & 270/322 \\ 0 & 0 & 1 \end{bmatrix}$$

then  $Q^T Q = I$  and  $AR = Q$ .

Also  $R$  is invertible (it's diagonal with ones on the diagonal) so  $A = QR^{-1}$ .

## Orthogonal decomposition

The  $QR$  decomposition usually has  $A$  a square matrix and  $Q$  an *orthogonal* matrix meaning that its columns aren't only orthogonal but orthonormal. We can do this by normalizing the columns.

We have

$$u_1 \cdot u_1 = 10, u_2 \cdot u_2 = 161/10, u_3 \cdot u_3 = 2312/322$$

$$Q' = QZ$$

where

$$Z = \begin{bmatrix} 1/\sqrt{10} & 0 & 0 \\ 0 & 1/\sqrt{161/10} & 0 \\ 0 & 0 & 1/\sqrt{2312/322} \end{bmatrix}$$

## QR decomposition concluded

Then  $Q'$  satisfies  $Q'^{-1} = Q'^T$  and

$$A = Q' Z^{-1} R^{-1} = Q' R'$$

where

$$R' = Z^{-1} R^{-1}$$

is still upper triangular.

## Geometric interpretation

A linear transformation like  $x \mapsto Ax$  splits into two parts – a shear (coming from the  $R$ ) and a rotation (coming from  $Q$ ).