# 1.8-1.9 Matrices and Linear Transformations

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# Linear Transformations and Matrices

If $A$ is an $n \times m$ matrix, and $x$ is any vector in $\mathbf{R}^m$, then $Ax$ is a vector in $\mathbf{R}^n$.

So we can define a function $T : \mathbf{R}^m \to \mathbf{R}^n$ by

$$T(x) = Ax.$$

For example if

$$A = \begin{bmatrix} 0 & -4 \\ 4 & 1 \\ 3 & 3 \end{bmatrix} \text{ and } v = \begin{bmatrix} x \\ y \end{bmatrix}$$

then

$$T(v) = Av = \begin{bmatrix} -4y \\ 4x + y \\ 3x + 3y \end{bmatrix}$$

# Function terminology

In general if $f : X \to Y$ is a function then $f$ is a "rule" that associates exactly one element $y \in Y$ to each element $x \in X$. The $y$ corresponding to $x$ is called $f(x)$. Furthermore:

- $X$ is called the domain of $f$
- $Y$ is called the codomain of $f$
- the set of $y \in Y$ so that there is an $x \in X$ with $f(x) = y$ is called the *range* of $f$.
- if $f(x) = y$, then $y$ is called the *image* of $x$ under $f$.

If $A$ is an $n \times m$ matrix, then the domain of $f(x) = Ax$ is $\mathbf{R}^m$ and the codomain is $\mathbf{R}^n$.

# Examples of matrix transformations

If

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

is called a *projection*, in this case onto the $xy$-plane.

# Rotations in 2d

If

$$A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

then

$$A(\theta) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

rotates the vector $(x, y)$ through an angle $\theta$ counterclockwise.

To see this, write $x = r\cos\phi$ and $y = r\sin\phi$. Then:

$$\begin{aligned} r\cos\phi\cos\theta - r\sin\phi\sin\theta &= r\cos(\phi+\theta) \\ r\cos\phi\sin\theta + r\sin\phi\cos\theta &= r\sin(\phi+\theta) \end{aligned}$$

# Linear Transformations

Let $T : \mathbf{R}^m \to \mathbf{R}^n$ be a function. Then $T$ is called a *linear transformation* if

- $T(ax) = aT(x)$ for every scalar $a$, and
- $T(x + y) = T(x) + T(y)$ for any two vectors $x, y \in \mathbf{R}^m$.

Any matrix transformation $T(x) = Ax$, where $A$ is $n \times m$, is linear.

If $T$ is linear, then $T(0) = 0$ (because $T(0x) = 0T(x) = 0$.)

# Linear Transformations

Let

$$T(x) = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix} x.$$

Find a vector $x$ so that $T(x) = b$ and determine if this $x$ is unique.

Hint: The rref form for the augmented matrix $[A \quad b]$ is

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

# Another problem

Let

$$T(x) = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix} x.$$

Find a vector $x$ so that $T(x) = b$ and determine if this $x$ is unique.

Hint: The rref form for the augmented matrix $[A \quad b]$ is

$$\begin{bmatrix} 1 & 0 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Linear Transformations

If $T : \mathbf{R}^m \to \mathbf{R}^n$$ is linear, and $v_1, \ldots, v_k$ are vectors in $\mathbf{R}^m$, then if you know

$$T(v_1), \ldots, T(v_k)$$

you know

$$T(a_1 v_1 + \cdots + a_k v_k)$$

for any constants $a_i$. In other words, you can compute $T$ for any vector in the span of $v_1, \ldots, v_k$.

# Linear Transformations

In particular if

$$T(\begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}a\\b\end{bmatrix}$$

and

$$T(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}c\\d\end{bmatrix}$$

then

$$T(\begin{bmatrix}x\\y\end{bmatrix}) = T(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}) = xT(\begin{bmatrix}1\\0\end{bmatrix}) + yT(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}ax+cy\\bx+dy\end{bmatrix}$$

and $T(x) = Ax$ where

$$A = \begin{bmatrix}a & c\\b & d\end{bmatrix}$$

# Matrices and Linear Transformations

We have seen that, given a matrix $A$, then $T(x) = Ax$ is a linear transformation.

Now suppose $T : \mathbf{R}^m \to \mathbf{R}^n$ is a linear transformation.

Let $e_i \in \mathbf{R}^m$ be the vector

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

where the $1$ is in row $i$ of $e_i$.

# Matrices and Linear Transformations

Let

$$A(T) = [T(e_1) \quad T(e_2) \cdots T(e_m))]$$

whose columns are the $T(e_i)$. This is an $n \times m$ matrix because each $T(e_i) \in \mathbf{R}^n$.

Notice that $Ae_i = T(e_i)$ for $i = 1, \ldots, m$. As a result, by linearlity, $Av = T(v)$ for any vector $v \in \mathbf{R}^m$.

Therefore *every linear transformation comes from multiplication by a matrix.*

# The identity map

The map $T : \mathbf{R}^m \to \mathbf{R}^m$ given by $Tx = x$ is called the identity map.

Since $T(e_i) = e_i$ for $i = 1, \dots, m$ the matrix of $T$ is the $m \times m$ matrix with $1$'s on the diagonal and zeros elsewhere.

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

# Matrices and Linear Transformations

If $T : \mathbf{R}^m \to \mathbf{R}^n$ is linear, then $T$ is determined by what it does to the standard basis vectors $e_i$.

For example, if $m = n = 2$, and $T$ is the reflection map $T(x, y) = (y, x)$, then $T(e_1) = e_2$ and $T(e_2) = e_1$ and therefore $Tx = Ax$ where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

# Reflections

**TABLE 1 Reflections**

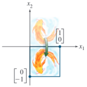
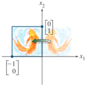
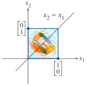
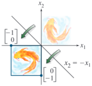
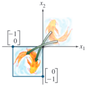
| Transformation | Image of the Unit Square | Standard Matrix |
|---|---|---|
| Reflection through the $x_1$-axis |  | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ |
| Reflection through the $x_2$-axis |  | $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| Reflection through the line $x_2 = x_1$ |  | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ |
| Reflection through the line $x_2 = -x_1$ |  | $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ |
| Reflection through the origin |  | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ |

Figure 1: reflections

# Shears

**TABLE 3 Shears**

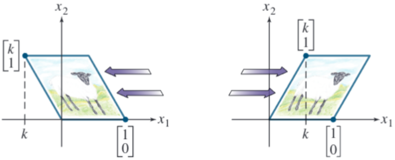
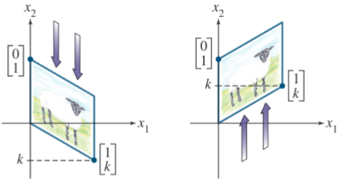
| Transformation | Image of the Unit Square | Standard Matrix |
|---|---|---|
| Horizontal shear |   $k < 0$ $k > 0$ | $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ |
| Vertical shear |   $k < 0$ $k > 0$ | $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ |

Figure 2: shears

# Contractions/Expansions

**TABLE 2 Contractions and Expansions**

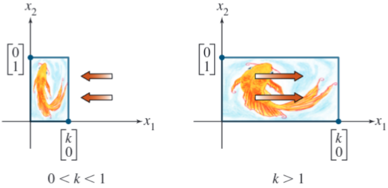
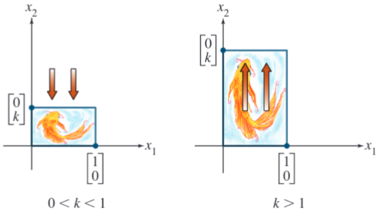
| Transformation | Image of the Unit Square | Standard Matrix |
|---|---|---|
| Horizontal contraction and expansion |   | $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ |
| Vertical contraction and expansion |   | $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ |

Figure 3: contractions and expansions

# Projections

**TABLE 4 Projections**

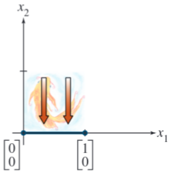
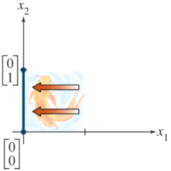
| **Transformation** | **Image of the Unit Square** | **Standard Matrix** |
|---|---|---|
| Projection onto the $x_1$-axis |  | $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ |
| Projection onto the $x_2$-axis |  | $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ |

Figure 4: projections

# One-to-one and onto maps

A function $T : A \to B$ is *one-to-one* if the only way that $T(x) = T(y)$ is if $x = y$.

Eg the function $f(x) = x^2$ is *not* one-to-one, because $f(-1) = f(1)$ even though $-1 \neq 1$. But the function $f(x) = 3x$ is one-to-one, because if $3x = 3y$ then $x$ and $y$ must be equal.

A function $T : A \to B$ is *onto* if, for any $b \in B$, there is an $a \in A$ so that $T(a) = b$.

The function $f(x) = x^2$ is not *onto*, because the equation $-1 = x^2$ does not have a solution (at least in real numbers.) The function $f(x) = 3x$ is *onto*, because the equation $y = 3x$ always has a solution ($x = y/3$).

# One-to-one linear maps

If $T$ is linear, then $T(x) = T(y)$ if and only if $T(x) - T(y) = T(x - y) = 0$. So $T$ is one-to-one if the only solution to $T(v) = 0$ is $v = 0$.

Since $T$ comes from a matrix $A$, the map is one-to-one if and only if the matrix equation $Ax = 0$ has only zero as its solution.

This happens if and only if *the columns of $A$ are linearly independent.*

# Onto linear maps

If $T : \mathbf{R}^m \to \mathbf{R}^n$ is linear, then $T(x)$ is onto if only if $T(x) = b$ has a solution for any $b \in \mathbf{R}^n$. This means that the matrix equation

$$Ax = b$$

has a solution for any $b \in \mathbf{R}^n$.

Since $Ax$ is a linear combination of the columns of $A$, every equation $Ax = b$ has a solution only if every $b$ is a linear combination of the columns of $A$. In other words, $A$ is onto if and only if the columns of $A$ span $\mathbf{R}^n$.

# Theorem 12

Theorem 12 in the book summarizes these two key facts.

**Theorem:** Let $T(x) = Ax$ be a linear map from $\mathbf{R}^m$ to $\mathbf{R}^n$, where $A$ is an $n \times m$ matrix.

1. $T$ is one-to-one if and only if the columns of $A$ are linearly independent vectors in $\mathbf{R}^n$.
2. $T$ is onto if and only if the columns of $A$ span $\mathbf{R}^n$.

# Algebraic version

Algebraically:

1. $T(x) = Ax$ is one-to-one if and only if the rref of $A$ has no free variables - in other words, if every column has a pivot.
2. $T(x) = Ax$ is onto if and only if every row of $A$ has a pivot.

Note that if $A$ is an $n \times m$ matrix, then:

- if $m > n$, the map cannot be one-to-one.
- if $n > m$, the map cannot be onto.