

## 1.7 Linear Independence

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# Linear Independence

A set  $v_1, \dots, v_k$  of vectors in  $\mathbf{R}^n$  is *linearly independent* if the only solution to the equation

$$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0 \quad (1)$$

is the trivial solution where all  $x_i = 0$ .

A set of vectors that is not linearly independent is called *linearly dependent*.

A nontrivial solution to equation Equation 1 is called a *linear dependence relation* or just a *linear relation* among the vectors.

## Example

Suppose:  $v_1 = \begin{bmatrix} -13 \\ -11 \\ -1 \end{bmatrix}$ ,  $v_2 = \begin{bmatrix} 7 \\ -14 \\ 8 \end{bmatrix}$ ,  $v_3 = \begin{bmatrix} 1 \\ 3 \\ -8 \end{bmatrix}$

Consider the matrix of the homogeneous system:

$$\begin{bmatrix} -13 & 7 & 1 \\ -11 & -14 & 3 \\ -1 & 8 & -8 \end{bmatrix}$$

Apply row reduction to obtain:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

No free variables  $\Rightarrow$  no nontrivial solution

Vectors are linearly independent

## Example

$$\text{Suppose: } v_1 = \begin{bmatrix} -13 \\ -11 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} -19 \\ -36 \\ 6 \end{bmatrix}, v_3 = \begin{bmatrix} 44 \\ 97 \\ -19 \end{bmatrix}$$

Consider the matrix of the homogeneous system:

$$\begin{bmatrix} -13 & -19 & 44 \\ -11 & -36 & 97 \\ -1 & 6 & -19 \end{bmatrix}$$

Apply row reduction to obtain:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

One free variable so vectors are dependent

## Example continued

$$\text{Recall that } v_1 = \begin{bmatrix} -13 \\ -11 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} -19 \\ -36 \\ 6 \end{bmatrix}, v_3 = \begin{bmatrix} 44 \\ 97 \\ -19 \end{bmatrix}$$

The non trivial solution is

$$\begin{aligned} x_1 &= -x_3 \\ x_2 &= 3x_3 \end{aligned}$$

so (cancelling out  $x_3$ ) the linear relation is

$$-v_1 + 3v_2 + v_3 = 0$$

## Special cases

- ▶ A single vector is a linearly independent set if and only if it is zero.
- ▶ Two vectors  $v_1$  and  $v_2$  are linearly independent unless  $v_2$  is a multiple of  $v_1$ .
- ▶ If  $m < n$ , then any set of  $m$  vectors in  $\mathbf{R}^n$  is linearly dependent. This is because a reduced  $m \times n$  matrix, which has more columns than rows, must have at least one free variable.

## Another characterization

A set  $v_1, \dots, v_k$  of vectors in  $\mathbf{R}^m$  is dependent if and only if one of the vectors  $v_i$  is a linear combination of the others.

► If  $v_1 = \sum_{i=2}^k c_i v_i$  then

$$-v_1 + c_2 v_2 + \dots + c_k v_k = 0$$

is the linear relation.

► if  $c_1 v_1 + \dots + c_k v_k = 0$  is the linear relation, at least one of the  $c_i$  isn't zero, so for that  $i$  write

$$-c_i v_i = c_1 v_1 + \dots + \widehat{c_i v_i} + \dots + c_k v_k$$

where  $\widehat{c_i v_i}$  means to omit this term. Since  $c_i \neq 0$  you can divide this equation by  $-c_i$  and write  $v_i$  in terms of the other vectors.

## Example

For what value(s) of  $h$  are the following vectors linearly dependent?

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 5 \\ h \end{bmatrix}$$

They are dependent if  $v_3$  is a linear combination of  $v_1$  and  $v_2$ . In other words if

$$v_3 = x_1 v_1 + x_2 v_2$$

has a solution.

$$\begin{array}{rcl} -1 & = & x_1 + 3x_2 \\ 5 & = & -x_1 - 5x_2 \end{array}$$

and then  $h = 4x_1 + 7x_2$ .

The solution is  $x_1 = 5, x_2 = -2$  so  $h = 6$  is the only value where the three are dependent.