### Math 2710

Aug 26-28

## Course Info

# Key links

- Syllabus
- Tests
- ► Homework
- Piazza

## Grading

- ► Two midterms (25 points) tentatively Sep 30 and Nov 5.
  - Notify me by Sep 20 if you need an alternate date for the first exam because of Rosh Hashanah.
- Final Exam (40 points)
- ► Homework + participation (including Piazza) (10 points)

#### Homework

- daily assignments
- periodically collected and graded with short lead time
- assorted short quizzes or other assignments from time to time

# 1.1 What is this course about?

### Mathematics as a discipline

This course is about

- how mathematics is done
- how mathematics is communicated.

The actual mathematics we will learn in this course is less important than the approach

## A very simple example

**Assertion:** The sum of two even numbers is an even number.

Goal: find a mathematical proof of this fact.

#### Mathematical Proof

A mathematical proof of this assertion is an argument that starts from known facts and definitions and establishes the truth of the assertion using the tools of logic.

A proof in *formal logic* starts from explicit hypotheses or axioms and applies the rules of deductive logic to reach a conclusion. Proofs of even simple facts in formal logic are extremely long and mostly not readable by humans.

In principle, a mathematical proof contains enough information to produce a formal logical proof.

#### Good Mathematical Proofs

#### A good mathematical proof is

- rigorous, meaning it gives a complete logical argument,
- informative, meaning that it provides enough information to explain why the assertion is true
- efficient, meaning that it is as short as possible while still being rigorous and informative.

#### Example, continued

#### To construct a proof of this assertion, we need:

- ▶ to know exactly what the terms mean (what is an even integer?)
- to establish in our own minds that the assertion IS true, and figure out why
- communicate our understanding of why the assertion is true rigorously and efficiently.

#### Discussion

- Define even number.
- ► Explain why the assertion about even numbers is true, as rigorously and efficiently as you can.

# Key Vocabulary

- theorem
- lemma
- proposition
- corollary
- example
- algorithm
- definition
- proof

- statement
- proposition
- converse
- contrapositive
- conditional statement

1.2 Logic

#### Statements

A statement is a sentence that is either **True** or **False** 

Compound statements are built up using logical operators **AND**, **OR**, **NOT**, and others.

The truth of a compound statement depends on the individual statements and the properties of the operators.

# AND, OR, NOT

Р	Q	AND
Т	Т	Т
Т	F	F
F	Т	F
F	F	F

P	Q	OR
T	Т	Т
Τ	F	Τ
F	Т	Τ
F	F	F

P	NOT
Т	F
F	T
	,

## Implications/Conditionals

A great deal of reasoning is about *conditional statments*.

- ▶ If I get a vaccine for flu today, then I will not get the flu this year.
- ▶ If there is a recession in the next six months, President Trump will not be re-elected.
- If the nucleotide at a certain genomic position is switched from A to T, the affected individual will have a certain genetic disease.

The truth of an implication depends on the Truth/Falsehood of BOTH components. But if the first clause is FALSE, the statement is TRUE. So the interesting cases are when the first clause is TRUE.

**For discussion:** Compare how other disciplines think about implications such as those above with how mathematicians do. Which of the statements above might be susceptible to proof in "real life"?

## Truth Tables for conditionals

T T T T F F T T F F T	Р	Q	$\Longrightarrow$
F T T	Т	Т	Т
	Т	F	F
F F T	F	Т	Т
	F	F	Т

#### Alternate formulations

- P implies Q
- ▶ If P, then Q
- ▶ Q if P
- ► P only if Q
- P suffices for Q, or is sufficient for Q
- Q is necessary for P.

### Equivalence

The claim that two statements are *equivalent* is the claim that they are either both True or both False.

- P is equivalent to Q
- P if and only if Q
- P is necessary and sufficient for Q.

Р	Q	$\iff$
T	Т	Т
Т	F	F
F	Т	F
F_	F	Т

### **Example Computation**

 $(X \text{ and } (Y \text{ or } Z)) \iff ((X \text{ and } Y) \text{ or } (X \text{ and } Z))$ 

X	Y	Z	Y or Z	X and (Y or Z)	X and Y	X and Z	((X and Y
Т	Т	Т	Т	Т	Т	Т	
Т	Т	F	Т	Т	T	F	
Т	F	Τ	Т	Т	F	T	
Т	F	F	F	F	F	F	
F	Т	Τ	Т	F	F	F	

(X and (Y or Z)) have the same truth values as ((X and Y) or (X and Z)) and so the statements are equivalent.

#### Discussion

- Give some examples (in English) of statements where P implies
  Q (meaning P implies Q is TRUE), but Q does not imply P.
- Give some examples of statements that are equivalent.
- Are their statements P and Q so that neither P implies Q nor Q implies P are True?
- ▶ P and Q are equivalent means P ⇔ Q or "P if and only if Q".
- ▶ Show:  $P \implies Q$  is equivalent to (Q OR NOT P).
- Exclusive OR is the operator that is TRUE when one of two statements is True, but not both. Express it in terms of AND, OR, and NOT.
- If my basement is wet, then it is either very rainy or a pipe has broken. Express this using the various operators and test its truth under different conditions.

### 1.3 Sets

#### Sets

We rely on a "naive" notion of set, meaning a collection of objects. For example:

- the set of integers
- the set of words in the English language
- the set of people in the world
- the empty set

For a discussion of why this is naive, see Russell's paradox.

#### Subsets

We can construct sets by selecting elements of another set, yielding a *subset* of the original set.

# **Explicit specification**

 $\textit{A} = \{1, 3, 5, 8, 9\}\text{, a subset of the integers.}$ 

## Selection by a property

Suppose P is the set of people. Then

$$\{p\in P: p \text{ is a legal resident of Chicago}\}$$

is shorthand for the set consisting of people p for which the statement "p is a legal resident of Chicago" is True.

### Set operations

If A and B are both subsets of some huge (and usually unmentioned) set U, then:

- ▶  $A \subset B$  means that every element of A is also an element of B or  $x \in A \implies x \in B$ .
- ▶ A = B means that A and B have the same elements, or  $x \in A \iff x \in B$ .

## More set operations

- ▶  $A \cup B$ , the union of A and B, is  $\{x \in U : x \in A \text{ OR } x \in B\}$ .
- ▶  $A \cap B$ , the intersection of A and B, is  $\{x \in U : x \in A \text{ AND } x \in B\}$ .
- ▶  $A \times B$ , the product of A and B, is the set of all ordered pairs (x, y) where  $x \in A$  and  $y \in B$ .

#### Discussion

Suppose A and B are two sets contained in some big set U. Prove the following by a truth table:

▶  $((A \cap B) = A)$  implies  $A \subset B$ .

Hint: Start with the statements  $X = (x \in A)$  and  $Y = (x \in B)$ . Then  $A \subset B$  is  $X \Longrightarrow Y$ . Express the left hand side similarly and work out the truth table.