Practice Problems for in-class work

Hammack, Section 12.2:

- Problem 18, then prove that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality. (the relevant function is $f(n) = ((-1)^n(2n-1)+1)/4$)
- Problems 11, 12: consider the functions $\theta(a,b) = (-1)^a b$ and $\theta(a,b) = a - 2ab + b$ from from $\{0,1\} \times \mathbb{N}$ to $\mathbb{Z}$. Are they injective? surjective? bijective?

other problems

- Prove that if $g \circ f$ is injective, then f is injective. Prove that if $g \circ f$ is surjective, then g is surjective. Give counterexamples showing that $g \circ f$ can be injective but g not injective, and $g \circ f$ is surjective by f is not surjective.
- Give an example of a countable subset of the irrational numbers, or prove that no such subset exists.
- How many surjective maps are there from a 3 element set to a 2 element set? From a 4 element set to a 3 element set?

More problems for in-class discussion

- Go through the argument that $\mathbb{N} \times \mathbb{N}$ is countable (Figure 14.2)
- Go through Cantor's diagonalization argument proving that (for example) infinite sequences form an uncountable set. See pp. 271-272 and also Gilbert and Vanstone Theorem 6.67.

More problems still

- Let A be the set of sequences of natural numbers $a_1, a_2, a_3, \ldots$ that are eventually constant at zero – in other words, a sequence $a_1, a_2, \ldots$ is in A if there exists an N such that $a_i = 0$ for all $i \geq N$. Is A countable?
- Prove that the intervals $(0,1)$ and (a,b) in $\mathbb{R}$ have the same cardinality for any a, b with $b > a$.
- Give 3 examples of functions that are surjective but not injective, and 3 examples of functions that are injective but not surjective

And still more problems

Two tricky problems from Gilbert and Vanstone.

- Prove that a function $f : X \to Y$ is injective if and only if, given two maps $g : T \to X$ and $h : T \to X$,

$$f \circ g : T \to X = f \circ h : T \to X \implies g = h.$$

- If $f : A \to B$ and $g : B \to C$ are bijective maps, prove that the inverse map to $f \circ g$ is $g^{-1} \circ f^{-1}$.