Math 2710

November 17-21

Finite and countable sets

Definition: For $n \in \mathbb{N}$ (where $\mathbb{N}$ are the natural numbers), we let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

- Two sets A and B *have the same cardinality* if there is a bijective map $f : A \to B$.
- A set A is finite if there exists an n so that A has the same cardinality as $[n]$. Otherwise, A is infinite.
- A set A is countable if it is finite or if it has the same cardinality as $\mathbb{N}$. In the latter case it is called "countably infinite."
- An infinite set that is not countably infinite is called *uncountable.*

Some properties of cardinality

Proposition: The relation "having the same cardinality" is an equivalence relation.

Proof: First, A has the same cardinality as A because the identity map (which sends $a \in A$ to itself) is a bijection. Second, if A and B have the same cardinality, that means there is a bijection $f : A \to B$. But then the inverse function $f^{-1} : B \to A$ is a bijection in the other direction, so the relation is symmetric. Finally, suppose A has the same cardinality as B, and B has the same cardinality as C. Then there is a bijection $f : A \to B$ and $g : B \to C$. The composition $g \circ f : A \to C$ is again a bijection, so A and C have the same cardinality.

More properties

Proposition: An infinite subset T of $\mathbb{N}$ is countably infinite.

Proof: We construct a map $\mathbb{N} \to T$. Let $f(1)$ be the smallest element of T. For $n > 1$, let $T(n) = \{x \in T : x > f(n-1)\}$. If $T(n)$ is empty, then T is finite, which isn't true; so by the well-ordering principle $T(n)$ has a smallest element.

Let $f(n)$ be the smallest element of $T(n)$ for $n \geq 2$.
f is injective because $f(1) < f(2) < \cdots$. We will show that f is surjective. Suppose not. Then there is a smallest element x of T for which there is no n with $f(n) = x$. Choose the largest element of T which is smaller than x and then there is an m with $f(m) = x$. But then x is the smallest element of $T(m+1)$ so $f(m+1) = x$, contradicting our choice of x. Thus f is surjective and therefore bijective.

more properties

Proposition: An infinite subset T of a countably infinite set S is countably infinite.

Proof: There is a bijection $f : \mathbb{N} \to S$ since S is countable. Let U be the subset of $\mathbf{N}$ consisting of $\{x \in \mathbb{N} : f(x) \in U\}$. This is an infinite subset of $\mathbb{N}$ which is therefore countably infinite and in bijection with U, so U is countably infinite.

The product of countable sets

Proposition: If A and B are countable, then so is $A \times B$.

Proof: this is the "digaonals" argument.

Propositon: If $A_1, \ldots, A_n$ are countable then so is $A_1 \times A_2 \times \cdots \times A_n$.

Proof: By induction. We know $A_1 \times A_2$ is countable. Assume $A_1 \times \cdots \times A_n$ is countable. Then $A_1 \times \cdots \times A_{n+1}$ is countable because it is $(A_1 \times \cdots \times A_n) \times A_{n+1}$.

the union of countable sets

Proposition: If A and B are countable, so is $A \cup B$.

Proof: The union of A and B is a subject of the disjoint union of A and B. Construct bijections f and g of $\mathbb{N}$ to each of A and B. Define a new function a with $a(2n) = f(n)$ and $a(2n-1) = g(n)$ for $n = 1, \ldots$. Then a is a bijection of $\mathbb{N}$ with the disjoint union of A and B, which is therefore countable, so its subset $A \cup B$ is also countable.

more properties

Proposition: The integers are countable.

Proof: $\mathbb{Z}$ is the union of $0, 1, \ldots$ and $-1, -2, \ldots$ each of which is countable.

Proposition: The rational numbers are countable..

Proof: The set of ordered pairs of integers (a, b) is countable, and the rational numbers is in bijection witha subset of this set.

Uncountable sets

- The set of infinite sequences $[a_1, a_2, a_3, \ldots]$ with $a_i \in \mathbb{N}$ is uncountable. (this is the diagonal argument)
- The real numbers are uncountable. (diagonal argument)
- The open open infinite interval $(0, \infty)$ has the same cardinality as $\mathbb{R}$. Use the $\log(x)$ and $\exp(x)$ as bijections.
- If a subset U of A is uncountable, so is A. Proof: Suppose not. Then A is countable, so U is a subset of a countable set and therefore countable.

power sets

Theorem: (Cantor) The power set of a set A has a "larger" cardinality than A.

Proof: The map $a \to \{a\}$ puts A inside $\mathcal{P}(A)$ so $\mathcal{P}(A)$ is bigger than or equal to A in cardinality. Suppose there is a bijection $f : A \to \mathcal{P}(A)$. Let $U = \{a \in A : a \notin f(a)\}$. This is a subset of A so an element of $\mathcal{P}(A)$. Suppose $f(x) = U$. Now if $x \in U$, then $x \in f(x)$, so $x \notin U$. But if $x \notin U$, then $x \notin f(x)$ so $x \in U$. Neither is possible, so U cannot be in the range of f, so f is not bijective.