# Math 2710

Oct 21-25

## Sequences

**Definition:** An (infinite) sequence with rational coefficients is a function $a : \mathbb{P} \to \mathbb{Q}$. Normally we view it as the sequence $a(1), a(2), \ldots$.

Some examples:

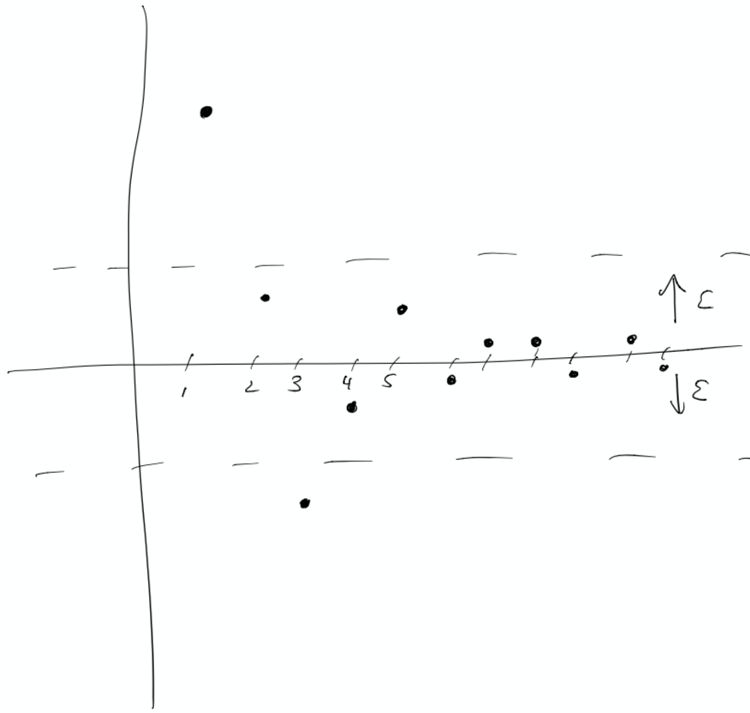
- $a(n) = 0$ for all $n$ (the zero sequence).
- $a(n) = 1/n$ for $n = 1, 2, \ldots$
- $a(n) = n$ for $n = 1, 2, \ldots$
- $a(n) = (-1)^n$ for $n = 1, 2, \ldots$

We would like to have a way to speak about what happens to sequences as $n$ gets larger and larger (so for example the sequence grows, it bounces around, or it approaches a particular number.)

## Limit of a sequence

**Definition:** Let $a(n)$ be a sequence. Then we say that the limit of $a(n)$ is $L$ if, for every $\epsilon > 0$, there is an integer $N$, so that $|a(n) - L| < \epsilon$ for all $n \geq N$. This is written:

$$\lim_{n \to \infty} a(n) = L.$$





and we say that the sequence *converges to L*.

## Limits are about estimation

### Examples

- Let $a(n)$ be the sequence defined by $a(1) = 1, a(2) = 1/2, and a(n) = 0$ for $n > 2$. Prove that the limit $\lim_{n\to\infty} a(n) = 0$.
- Let $a(n)$ be the sequence $a(n) = 1/n$. Prove that the limit of $a(n)$ as $n \to \infty$ is zero.
- Let $a(n) = (-1)^n$. Prove that the limit isn't 1. Then prove there is no limit.
- Let $a(n) = n$. Is there a limit?
- Let $a(n) = (n + 1)/n$. Prove that the limit is 1.
- Let $a(n) = 4 + (-1/2)^n$.

## Non-convergence

A sequence $a(n)$ *does not converge* to a limit $L$ means that - there exists $\epsilon > 0$ such that - for all $N$ - there exists $n \geq N$ such that - $|a(n) - L| > \epsilon$

The sequence $a(n) = (-1)^{n-1}$ does not converge to any limit because no matter what $L$ you pick and what $N$ you choose the distance $|(-1)^{n-1} - L|$ bounces back and forth between $|1 - L|$ and $|1 + L|$ so if you choose $\epsilon$ smaller than the maximum of these two you satisfy the 'non-convergence' requirement.

## Limit rules make arguments standard

**Proposition:** If $a(n)$ converges to $L$ and $b(n)$ converges to $M$ then $a(n) + b(n)$ converges to $L + M$.

**Proof:** The estimation side calculation is that we can choose $N$ large enough that $|a(n)-L| < \epsilon$ and $|b(n)-M| < \epsilon$ for $n \geq N$. Then $|a(n)+b(n)-L-M| < 2\epsilon$. So given $\epsilon$ we should choose $N$ large enought that $|a(n) - L| < \epsilon/2$ and similarly $|b(n) - M| < \epsilon/2$.

**Proposition:** Suppose that $a(n)$ is a sequence converging to $L$. Prove that there is an $N$ so that $|a(n)| < 2L$ for $n \geq N$.

**Proof:** Choose $\epsilon = L/2$. Then there is an integer $N$ such that $|a(n) - L| < L/2$ for all $n \geq N$. This means that $a(n)$ is between $L/2$ and $3L/2$ so in particular it is less than $2L$.

**Proposition:** Suppose that $a(n)$ converges to $L$. Prove that $a(n)^2$ converges to $L^2$.

**Proof:** $|a(n)^2 - L^2| = |a(n)^2 - a(n)L + a(n)L - L^2| \leq |a(n)||a(n) - L| + |L||a(n) - L|$. We can choose $N$ so that $|a(n)| < 2L$ and $N'$ so that $|a(n) - L| < \epsilon/4L$. Then for $n$ bigger than both of these we have

$$|a(n)||a(n) - L| + |L||a(n) - L| \leq (2L)\epsilon/4L + L\epsilon/4L = \epsilon/2 + \epsilon/4 = 3\epsilon/4 < \epsilon.$$

Thus for $n \geq \max(N, N')$ we have $|a(n)^2 - L^2| < \epsilon$.

## Sequences

A *sequence* is an infinite sum, but it is really a shorthand for a series. The sequence

$$a_0 + a_1 + a_2 + \dots$$

is a short hand for the sequence of partial sums $(a_0, a_1 + a_0, a_2 + a_1 + a_0, \dots)$.

A series converges to a limit $L$ means that the sequence of partial sums converges.

Key example is the geometric series $\sum_{n=0}^{\infty} ar^n$.