Math 2710

Sep 2-6

Some catch-up from prior classes

Two notes

Definitions

Although I never said this explicitly, a **definition** is an ‘if and only if’ statment. When I write:

Definition: An integer x is “5-ish” if there is an integer n so that $x = 5n$

I am actually saying that "x is 5-ish IF AND ONLY IF there is an integer n so that $x = 5n$.

Negation of implication

The easiest way to express NOT (A implies B) is as (A and NOT B). Check the truth tables.

1.4 Variable statements and quantifiers

First examples

Compare the following three statements

- *Helen* is a UConn student who has watched every minute of Game of Thrones.
- *There is a UConn student* who has watched every minute of Game of Thrones.
- *Every UConn student* has watched every minute of Game of Thrones.

All make assertions about the set U of UConn students

- The first asserts that a *particular* named element of U has a certain property (...has watched every minute of GoT)
- The second asserts that *There exists* an element of U with that property.
- The third asserts that *Every* element of U has that property.

Universal quantifier (For all, for every, for each)

A statement that includes a universal quantifier makes a claim about ALL objects of a particular type.

- For all x in the real numbers, $(x^2 - 1) = (x + 1)(x - 1)$.
- Every declared democratic presidential candidate will appear in the next official television debate.
- Each midterm exam in this course counts as 25% of your final grade.

Symbolic Form

- For all X, $P(X)$
- $\forall x$, $P(X)$.

Existential quantifier (There is, there exists, for some)

- There is a real number y so that $y^2 = 11$.
- There exists a car for sale in the United States that gets 50 mpg.
- There are some dogs that you should be afraid of.

Symbolic Form

- There exists X such that $P(X)$
- $\exists x$ such that $P(x)$.

Relation between universal and existential quantifiers

To show that the statement *Every UConn student has watched every minute of Game of Thrones* is FALSE, you must produce an example of a UConn student who has NOT watched every minute. So the negation of this claim is:

Some UConn student has not watched every minute of Game of Thrones or **There is a UConn student who has not watched every minute of Game of Thrones**

To show that the statement *There is a UConn student who has watched every minute of Game of Thrones* is FALSE, you must show that: **No student has watched every minute of Game of Thrones** or **All students at UConn have NOT watched every minute of Game of Thrones.**

Symbolic Form (page 11 of the text)

- $\text{NOT}(\forall x, P(x)) \leftrightarrow \exists x, \mathrm{NOT}\ P(x)$
- $\text{NOT}(\exists x, P(x)) \leftrightarrow \forall x, \mathrm{NOT}\ P(x)$

Examples

The statement $(S \cap T) \subset U$ is the statement that

$\forall x, ((x \in S) and (x \in T)) \implies x \in U$

Write the negation of this statement in a simple form.

Second order statements

Second order statements have two quantifiers.

- For all x, there exists y, so that....
- There exists x, so that for all y, ...

For all x, there exists y.

- For every even integer x, there exists an integer y so that $x = 2y$.
- For every real positive number x, there exists a real number y so that $x = y^2$.
- For every real $\epsilon > 0$, there exists a real $\delta > 0$ so that if $|x| < \delta$ then $x^2 < \epsilon$.

There exists y, so that for all x

- There exists an integer x so that, for all integers y, $xy = 0$.

An example

Definition: Given two integers n and d, we say that

-*n is divisible by d*

or

-*n is a multiple of d*

or

-*d divides n*

if there exists an integer m so that $n = dm$.

Divisibility examples

A. Let $X = \{n \in \mathbb{Z} : 3|n\}$ and let $Y = \{n \in \mathbb{Z} : 5|n\}$. Show that $X \cap Y = \{n \in \mathbb{Z} : 15|n\}$.

B. Let $X = \{n \in \mathbb{Z} : 6|n\}$ and let $Y = \{n \in \mathbb{Z} : 4|n\}$. Show that $X \cap Y$ is not equal to $W = \{n \in \mathbb{Z} : 24|z\}$.

Section 1.5: Proofs

Main ingredients

Remember that a mathematical proof is a careful explanation of the logical reasons for the truth of a proposition. Good proofs are:

-*rigorous*, meaning that they present a completely, logically correct argument -*informative*, meaning that they convey the reasoning behind the truth of the proposition being proved -*efficient*, meaning that they are as short as possible while still being rigorous and informative.

Things to try

Faced with a proposition to be provided:

- Make sure you understand the definitions of all the terms in the statement
- Carefully review the logical structure of the proposition so you know what you need to establish.
- If it's not clear how to proceed, consider some special cases or examples. Review carefully what you know already. We will see more approaches later.
- Finding a proof of a proposition can be hard. It can take many people working for centuries. For example, the Clay Millenium problems are a series of propositions to be proved (or disproved); successfully solving one of these problems brings a $1M dollar prize as well as world-wide fame.

Examples 1: Direct Implication

Proposition: Let S and T be sets. Prove that if $S \cap T = S$ then $S \subset T$.

Analysis: This is a direct implication $P \implies Q$.

- P is the statement $S \cap T = S$.
- Q is the statement $S \subset T$.

$S \cap T = S$ means that $x \in S \text{ and } x \in T$ if and only if $x \in S$.

$S \subset T$ means that $(x \in S) \implies (x \in T)$.

Look at the truth tables and compare with paragraph on page 14.

Examples 2: If and only if

Proposition: $S \cap T = S \cup T$ if and only if $S = T$.

- P is the statement $x \in (S \cap T) \Leftrightarrow x \in (S \cup T)$.
- Q is the statement $x \in S \Leftrightarrow x \in T$.

Look at truth tables and compare with paragraph on page 14.

Examples 3: Contrapositive

Proposition: If x is a real number such that $x^3 + 7x^2 < 9$ then $x < 1.1$.

- P is the statement $x^3 + 7x^2 < 9$
- Q is the statement $x < 1.1$.

$P \implies Q$ is equivalent to $\sim Q \implies \sim P$.

Must show: $x \geq 1.1$ implies $x^3 + 7x^2 \geq 9$.

Example 4: Contradiction.

Proposition: There is no largest integer.

Suppose that this statement P is false. Then there is a largest integer; call it n. Since n is the largest integer, $n + 1$ must be less than or equal to n. Therefore $n + 1 \leq n$ or $1 \leq 0$. This is false.

Let Q be the statement "There is no largest integer." Let P be the statement $1 \leq 0$.

Then we have shown that $\quad Q \implies P$. Since P is false, this implication can only be true if $\quad Q$ is false, so Q is true.

This is called *PROOF BY CONTRADICTION.*

Example 5: Compound implications

Proposition: If x is a real number, then $(x - a)(x - b) = 0$ if and only if either $x = a$ or $x = b$.

Here P is $(x - a)(x - b) = 0$. The conclusion is of the form $Q \text{ or } R$ where Q is $x = a$ and R is $x = b$.

One approach: $P \implies (Q \text{ or } R)$ is equivalent to $P \text{ and } R \implies Q$. So $(x - a)(x - b) = 0$ and $x \neq a$ means we can divide by $x - a$ to get $x = b$.

In the other direction, try each possibility.

Discussion Problems

Selected problems from 55-61 on page 21 and 65-70 on page 22 of the book.