Math 2710

Oct 21-25

Sequences

Definition: An (infinite) sequence with rational coefficients is a function $a : \mathbb{P} \to \mathbb{Q}$. Normally we view it as the sequence $a(1), a(2), \ldots$.

Some examples:

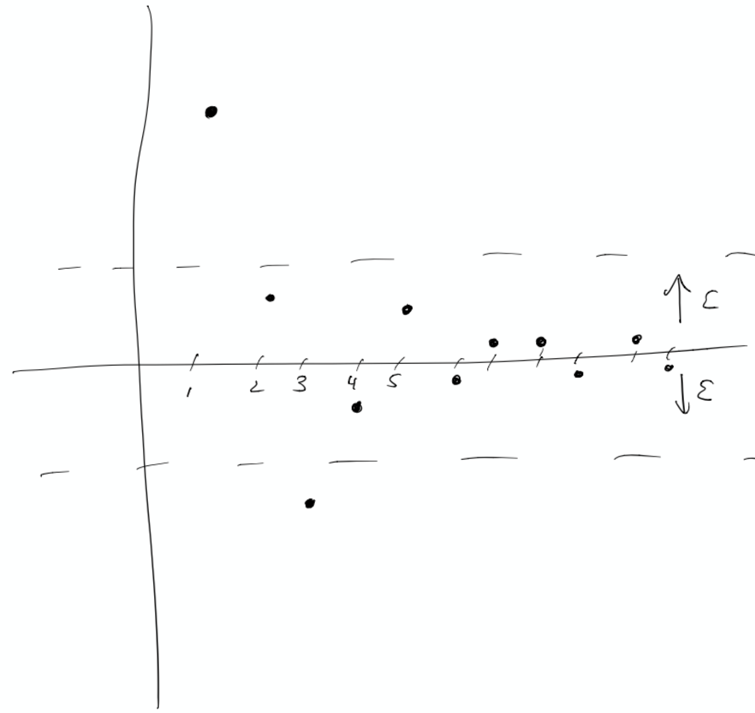
- $a(n) = 0$ for all n (the zero sequence).
- $a(n) = 1/n$ for $n = 1, 2, \ldots$
- $a(n) = n$ for $n = 1, 2, \ldots$
- $a(n) = (-1)^n$ for $n = 1, 2, \ldots$

We would like to have a way to speak about what happens to sequences as n gets larger and larger (so for example the sequence grows, it bounces around, or it approaches a particular number.)

Limit of a sequence

Definition: Let $a(n)$ be a sequence. Then we say that the limit of $a(n)$ is L if, for every $\epsilon > 0$, there is an integer N, so that $|a(n) - L| < \epsilon$ for all $n \geq N$. This is written:

$$\lim_{n\to\infty} a(n) = L.$$



and we say that the sequence *converges to* L.

Limits are about estimation

Examples

- Let $a(n)$ be the sequence defined by $a(1) = 1, a(2) = 1/2, and a(n) = 0$ for $n > 2$. Prove that the limit $\lim_{n\to\infty} a(n) = 0$.
- Let $a(n)$ be the sequence $a(n) = 1/n$. Prove that the limit of $a(n)$ as $n \to \infty$ is zero.
- Let $a(n) = (-1)^n$. Prove that the limit isn't 1. Then prove there is no limit.
- Let $a(n) = n$. Is there a limit?
- Let $a(n) = (n+1)/n$. Prove that the limit is 1.
- Let $a(n) = 4 + (-1/2)^n$.

Non-convergence

A sequence $a(n)$ *does not converge* to a limit L means that - there exists $\epsilon > 0$ such that - for all N - there exists $n \geq N$ such that - $|a(n) - L| > \epsilon$

The sequence $a(n) = (-1)^{n-1}$ does not converge to any limit because no matter what L you pick and what N you choose the distance $|(-1)^{n-1} - L|$ bounces back and forth between $|1 - L|$ and $|1 + L|$ so if you choose ϵ smaller than the maximum of these two you satisfy the 'non-convergence' requirement.

Limit rules make arguments standard

Proposition: If $a(n)$ converges to L and $b(n)$ converges to M then $a(n) + b(n)$ converges to $L + M$.

Proof: The estimation side calculation is that we can choose N large enough that $|a(n)-L| < \epsilon$ and $|b(n)-M| < \epsilon$ for $n \geq N$. Then $|a(n)+b(n)-L-M| < 2\epsilon$. So given ϵ we should choose N large enought that $|a(n) - L| < \epsilon/2$ and similarly $|b(n) - M| < \epsilon/2$.

Proposition: Suppose that $a(n)$ is a sequence converging to L. Prove that there is an N so that $|a(n)| < 2L$ for $n \geq N$.

Proof: Choose $\epsilon = L/2$. Then there is an integer N such that $|a(n) - L| < L/2$ for all $n \geq N$. This means that $a(n)$ is between $L/2$ and $3L/2$ so in particular it is less than $2L$.

Proposition: Suppose that $a(n)$ converges to L. Prove that $a(n)^2$ converges to L^2.

Proof: $|a(n)^2-L^2| = |a(n)^2-a(n)L+a(n)L-L^2| \leq |a(n)||a(n)-L|+|L||a(n)-L|$. We can choose N so that $|a(n)| < 2L$ and N' so that $|a(n) - L| < \epsilon/4L$. Then for n bigger than both of these we have

$$|a(n)||a(n) - L| + |L||a(n) - L| \leq (2L)\epsilon/4L + L\epsilon/4L = \epsilon/2 + \epsilon/4 = 3\epsilon/4 < \epsilon.$$

Thus for $n \geq \max(N, N')$ we have $|a(n)^2 - L^2| < \epsilon$.