13.7 Sequences

Our final two sections treat sequences and series, topics usually covered in a second semester of calculus.

Recall that a **sequence** is an infinitely long list of real numbers

$$a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ \ldots$$

The number a_1 is called the *first term*, a_2 is the *second term*, a_3 is the *third term*, and so on. For example, the sequence

$$2,\ \tfrac{3}{4},\ \tfrac{4}{9},\ \tfrac{5}{16},\ \tfrac{6}{25},\ \tfrac{7}{36},\ \ldots$$

has nth term $a_n = \frac{n+1}{n^2}$. The nth term is sometimes called the **general term**.

We can define a sequence by giving a formula for its general term. The sequence with general term $a_n = \frac{(-1)^{n+1}(n+1)}{n}$ is

$$2,\ -\tfrac{3}{2},\ \tfrac{4}{3},\ -\tfrac{5}{4},\ \tfrac{6}{5},\ -\tfrac{7}{5},\ \cdots.$$

We denote a sequence with nth term a_n as $\{a_n\}$. For example, the three sequences displayed above are denoted compactly as $\{a_n\}$ and $\{\frac{n+1}{n^2}\}$ and $\left\{\frac{(-1)^{n+1}(n+1)}{n}\right\}$, respectively. In this manner, the sequence $\{n^2+1\}$ is

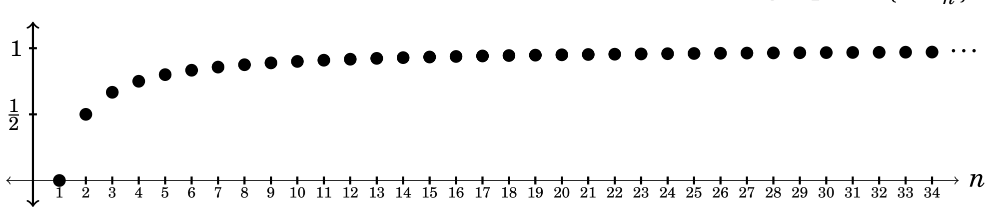
$$2,\ 5,\ 10,\ 17,\ 26,\ 37,\ \cdots.$$

Sometimes we define a sequence by writing down its first several terms, with the agreement that the general term is implied by the number pattern. For instance, the sequence

$$1,\ 4,\ 9,\ 16,\ 25,\ \cdots$$

is understood to be $\{n^2\}$ because n^2 is the most obvious formula that matches the first six terms. But be alert to the fact that a finite number of terms can *never* completely and unambiguously specify an infinite sequence. For all we know, the nth term of $1,4,9,16,25,\ldots$ might not be $a_n = n^2$, but actually $a_n = n^2+(n-1)(n-2)(n-3)(n-4)(n-5)$. This agrees with the first five listed terms, but the sixth term is $a_6 = 156$, not the expected $a_6 = 36$.

A sequence $\{a_n\}$ can be regarded as a function $f:\mathbb{N}\to\mathbb{R}$, where $f(n) = a_n$. For example, the sequence $\{1-\frac{1}{n}\}$ is the function $f(n) = 1-\frac{1}{n}$. In this sense we can graph a sequence; but the graph looks like a string of beads rather than a curve, because the domain is $\mathbb{N}$, not $\mathbb{R}$. Here is the graph of $\{1-\frac{1}{n}\}$.



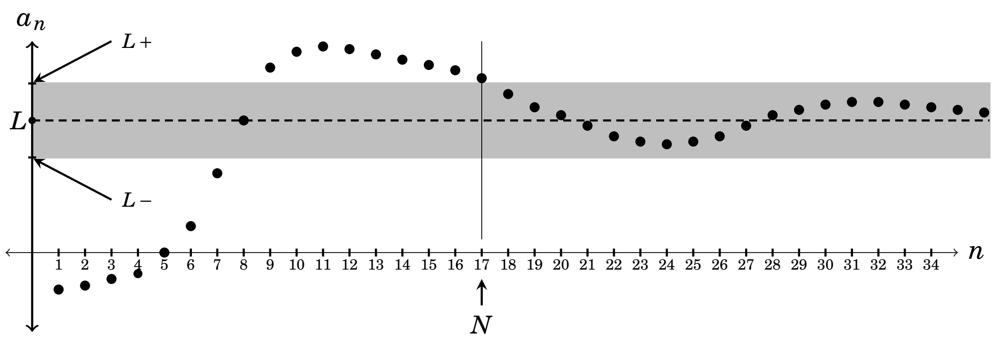
Roughly speaking, we say a sequence $\{a_n\}$ *converges* to a number L if the numbers a_n get closer and closer to L as n gets bigger and bigger.

For example, the sequence $\{1-\frac{1}{n}\}$ from the previous page converges to $L = 1$, because as n gets big, the number $1-\frac{1}{n}$ approaches 1.

In general, proving facts about convergence requires a precise definition. For this, we can adapt the definition of a limit at infinity from Section 13.6. The sequence $\{a_n\}$ converges to L if a_n can be made arbitrarily close to L by choosing n sufficiently large. Here is the exact definition.

> **Definition 13.5** A sequence $\{a_n\}$ **converges** to a number $L \in \mathbb{R}$ provided that for any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ for which $n > N$ implies $|a_n - L| < \varepsilon$.
>
> If $\{a_n\}$ converges to L, we denote this state of affairs as $\lim\limits_{n\to\infty} a_n = L$.
>
> If $\{a_n\}$ does not converge to any number L, then we say it **diverges**.

Definition 13.5 is illustrated below. For any $\varepsilon > 0$ (no matter how small), there is an integer N for which the terms a_n of the sequence lie between $L-\varepsilon$ and $L+\varepsilon$ provided $n > N$. Smaller values of ε require larger values of N. But no matter how small ε is, there is a (possibly quite large) number N for which a_n is within ε units from L when $n > N$.



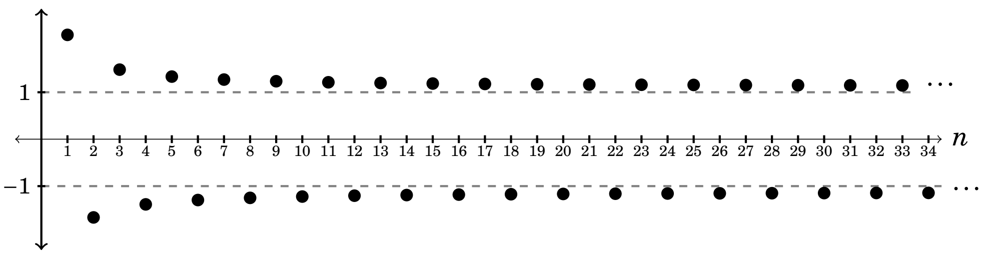
For our first example, let's return to the sequence $\{1-\frac{1}{n}\}$, which is graphed on the previous page (page 261). Notice that as n gets large, $\frac{1}{n}$ approaches 0, and $1-\frac{1}{n}$ approaches 1. So we can see that the sequence converges to 1. But let's prove this, in order to illustrate Definition 13.5.

Example 13.8 Prove that the sequence $\{1-\frac{1}{n}\}$ converges to 1.

Proof Suppose $\varepsilon > 0$. Choose an integer $N > \frac{1}{\varepsilon}$, so that $\frac{1}{N} < \varepsilon$. Then if $n > N$ we have $|a_n - 1| = \left|\left(1-\frac{1}{n}\right)-1\right| = \frac{1}{n} < \frac{1}{N} < \varepsilon$. By Definition 13.5 the sequence $\{1-\frac{1}{n}\}$ converges to 1. ■

Example 13.9 Investigate the sequence $\left\{\frac{(-1)^{n+1}(n+1)}{n}\right\}$.

The rst few terms of this sequence are $2, -\frac{3}{2}, \frac{4}{3}, -\frac{5}{4}, \frac{6}{5}, -\frac{7}{6}, \cdots$. The terms alternate between positive and negative, with the odd terms positive and the even terms negative. Here is a graph of the sequence.



The picture suggests that as n increases, the terms bounce back and forth between values that are alternately close to 1 and −1. This is also evident by inspection of the general term $a_n = \frac{(-1)^{n+1}(n+1)}{n}$, because $\frac{n+1}{n}$ approaches 1 as n grows toward in nity, while the power of −1 alternates the sign. Because the general term does not approach any single number, it appears that this sequence diverges. Now let's set out to prove this. Our proof formalizes the idea that if the sequence *did* converge to a number L, then L would have to be within units of *both* −1 and 1, and this is impossible if < 1.

Proof Suppose for the sake of contradiction that the sequence $\left\{\frac{(-1)^{n+1}(n+1)}{n}\right\}$ converges to a real number L. Let $= 1$. By De nition 13.5 there is an $N \in \mathbb{N}$ for which $n > N$ implies $\left|\frac{(-1)^{n+1}(n+1)}{n} - L\right| < 1$.

If n is odd, then the nth term of the sequence is $a_n = \frac{(-1)^{n+1}(n+1)}{n} = \frac{n+1}{n} > 1$. For n even, the nth term of the sequence is is $a_n = \frac{(-1)^{n+1}(n+1)}{n} = -\frac{n+1}{n} < -1$. Take an odd number $m > N$ and an even number $n > N$. The above three lines yield

$$
\begin{aligned}
2 = 1-(-1) &< a_m - a_n && \text{(because } 1 < a_m \text{ and } 1 < -a_n\text{)} \\
&= \left|a_m - a_n\right| && (a_m - a_n \text{ is positive}) \\
&= \left|(a_m - L) - (a_n - L)\right| && (\text{add } 0 = L - L \text{ to } a_m - a_n) \\
&\le \left|a_m - L\right| + \left|a_n - L\right| && (\text{using } |x-y| < |x| + |y|) \\
&< 1 + 1 = 2. && (\text{because } \left|a_n - L\right| < 1 \text{ when } n > N)
\end{aligned}
$$

Thus $2 < 2$, which is a contradiction. Consequently the series diverges. ■

For another example of a sequence that diverges, consider $1,4,9,16,25,\ldots$ whose nth term is $a_n = n^2$. Clearly this diverges, because $\lim_{n\to\infty} n^2 = \infty$, which is not a number. In such a case we say that the sequence *diverges to* ∞.

Definition 13.6 (Divergence to infinity)

1. We say a sequence $\{a_n\}$ **diverges to** ∞ if $\lim_{n\to\infty} a_n = \infty$. This means that for any $L > 0$, there is a positive N for which $n > N$ implies $a_n > L$.
2. We say a sequence $\{a_n\}$ **diverges to** $-\infty$ if $\lim_{n\to\infty} a_n = -\infty$. This means that for any $L < 0$, there is a positive N for which $n > N$ implies $a_n < L$.

This definition spells out a condition called *divergence to* ∞. But we haven't yet proved that a sequence meeting this condition actually diverges in the sense of Definition 13.5. Exercise 7 below asks you to do this.

Exercises for Section 13.7

1. Prove that $\left\{\frac{2^n}{n!}\right\}$ converges to 0.

2. Prove that $\left\{5+\frac{2}{n^2}\right\}$ converges to 5.

3. Prove that $\left\{\frac{2n^2+1}{3n-1}\right\}$ diverges to ∞.

4. Prove that $\{1-\frac{1}{2^n}\}$ converges to 1.

5. Prove that $\{\frac{2n+1}{3n-1}\}$ converges to $\frac{2}{3}$.

6. Prove that $\left\{\frac{5n^2+n+1}{4n^2+2}\right\}$ converges to $\frac{5}{4}$.

7. Prove that if a sequence diverges to infinity, then it diverges.

8. Prove that the **constant sequence** $c,c,c,c,\ldots$ converges to c, for any $c \in \mathbb{R}$.

9. Prove that if $\{a_n\}$ converges to L, and $c \in \mathbb{R}$, then the sequence $\{ca_n\}$ converges to cL.

10. Prove that if $\{a_n\}$ converges to L and $\{b_n\}$ converges to M, then the sequence $\{a_n + b_n\}$ converges to $L+M$.

11. Prove that if $\{a_n\}$ converges to L and $\{b_n\}$ converges to M, then the sequence $\{a_n b_n\}$ converges to LM.

12. Prove that if $\{a_n\}$ converges to L and $\{b_n\}$ converges to $M \neq 0$, then the sequence $\{\frac{a_n}{b_n}\}$ converges to $\frac{L}{M}$. (You may assume $b_n \neq 0$ for each $n \in \mathbb{N}$.)

13. For any sequence $\{a_n\}$, there is a corresponding sequence $\{|a_n|\}$. Prove that if $\{|a_n|\}$ converges to 0, then $\{a_n\}$ converges to 0. Give an example of a sequence $\{a_n\}$ for which $\{|a_n|\}$ converges to a number $L \neq 0$, but $\{a_n\}$ diverges.

14. Suppose that $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences for which $a_n \le b_n \le c_n$ for all sufficiently large n. (That is, $a_n \le b_n \le c_n$ for all $n > M$ for some integer M.) Prove that if $\{a_n\}$ and $\{c_n\}$ converge to L, then $\{b_n\}$ also converges to L.