# Math 2710

Aug 26-28

# Course Info

# Key links

- Syllabus
- Tests
- Homework
- Piazza

# Grading

- Two midterms (25 points) tentatively Sep 30 and Nov 5.
  - Notify me by Sep 20 if you need an alternate date for the first exam because of Rosh Hashanah.
- Final Exam (40 points)
- Homework + participation (including Piazza) (10 points)

# Homework

- daily assignments
- periodically collected and graded with short lead time
- assorted short quizzes or other assignments from time to time

## 1.1 What is this course about?

# Mathematics as a discipline

This course is about

- *how mathematics is done*
- *how mathematics is communicated*.

The actual mathematics we will learn in this course is less important than the approach

# A very simple example

**Assertion:** The sum of two even numbers is an even number.

Goal: find a mathematical proof of this fact.

# Mathematical Proof

A *mathematical proof* of this assertion is an argument that starts from known facts and definitions and establishes the the truth of the assertion using the tools of logic.

A proof in *formal logic* starts from explicit hypotheses or axioms and applies the rules of deductive logic to reach a conclusion. Proofs of even simple facts in formal logic are extremely long and mostly not readable by humans.

In principle, a mathematical proof contains enough information to produce a formal logical proof.

# Good Mathematical Proofs

A good mathematical proof is

- *rigorous*, meaning it gives a complete logical argument,
- *informative*, meaning that it provides enough information to explain why the assertion is true
- *efficient*, meaning that it is as short as possible while still being rigorous and informative.

# Example, continued

To construct a proof of this assertion, we need:

- to know exactly what the terms mean (what is an even integer?)
- to establish in our own minds that the assertion IS true, and figure out why
- communicate our understanding of why the assertion is true rigorously and efficiently.

# Discussion

- Define *even number*.
- Explain why the assertion about even numbers is true, as rigorously and efficiently as you can.

# Key Vocabulary

- theorem
- lemma
- proposition
- corollary
- example
- algorithm
- definition
- proof

- statement
- proposition
- converse
- contrapositive
- conditional statement

# 1.2 Logic

# Statements

A statement is a sentence that is either **True** or **False**

Compound statements are built up using logical operators **AND**, **OR**, **NOT**, and others.

The truth of a compound statement depends on the individual statements and the properties of the operators.

# AND, OR, NOT

| P | Q | **AND** |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

| P | Q | **OR** |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

| P | **NOT** |
|---|---|
| T | F |
| F | T |

# Implications/Conditionals

A great deal of reasoning is about *conditional statments*.

- If I get a vaccine for flu today, then I will not get the flu this year.
- If there is a recession in the next six months, President Trump will not be re-elected.
- If the nucleotide at a certain genomic position is switched from A to T, the affected individual will have a certain genetic disease.

The truth of an implication depends on the Truth/Falsehood of BOTH components. But if the first clause is FALSE, the statement is TRUE. So the interesting cases are when the first clause is TRUE.

**For discussion:** Compare how other disciplines think about implications such as those above with how mathematicians do. Which of the statements above might be susceptible to proof in "real life"?

# Truth Tables for conditionals

| P | Q | $\implies$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

# Alternate formulations

- P implies Q
- If P, then Q
- Q if P
- P only if Q
- P suffices for Q, or is sufficient for Q
- Q is necessary for P.

# Equivalence

The claim that two statements are *equivalent* is the claim that they are either both True or both False.

- P is equivalent to Q
- P if and only if Q
- P is necessary and sufficient for Q.

| P | Q | $\iff$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

# Discussion

- Give some examples (in English) of statements where P implies Q (meaning P implies Q is TRUE), but Q does not imply P.
- Give some examples of statements that are equivalent.
- Are their statements P and Q so that neither P implies Q nor Q implies P are True?
- $P$ and $Q$ are equivalent means $P \iff Q$ or "$P$ if and only if $Q$".
- Show: $P \implies Q$ is equivalent to ($Q$ OR NOT $P$).
- Exclusive OR is the operator that is TRUE when one of two statements is True, but not both. Express it in terms of AND, OR, and NOT.
- If my basement is wet, then it is either very rainy or a pipe has broken. Express this using the various operators and test its truth under different conditions.

# 1.3 Sets

# Sets

We rely on a "naive" notion of set, meaning a collection of objects.
For example:

- the set of integers
- the set of words in the English language
- the set of people in the world
- the empty set

# Subsets

We can construct sets by selecting elements of another set, yielding a *subset* of the original set.

# Explicit specification

$A = \{1, 3, 5, 8, 9\}$, a subset of the integers.

# Selection by a property

Suppose $P$ is the set of people. Then

$$\{p \in P : p \text{ is a legal resident of Chicago}\}$$

is shorthand for the set consisting of people $p$ for which the statement "$p$ is a legal resident of Chicago" is True.

# Set operations

If $A$ and $B$ are both subsets of some huge (and usually unmentioned) set $U$, then:

- $A = B$ means that $A$ and $B$ have the same elements.
- $A \subset B$ means that every element of $A$ is also an element of $B$.

# More set operations

- $A \cup B$, the union of $A$ and $B$, is $\{x \in U : x \in A \text{ OR } x \in B\}$
- $A \cap B$, the intersection of $A$ and $B$, is $\{x \in U : x \in A \text{ AND } x \in B\}$
- $A \times B$, the product of $A$ and $B$, is the set of all ordered pairs $(x, y)$ where $x \in A$ and $y \in B$.

# Discussion

Suppose $A$ and $B$ are two sets contained in some big set $U$. Prove the following; truth tables may be helpful.

- $A = B$ if and only if $A \subset B$ and $B \subset A$.
- $((A \cap B) = A)$ implies $A \subset B$.