Math 2710

Oct 28-Nov 1

More on limit rules

Proposition: If a_n converges to L then ca_n converges to cL.

Definition: Divergence to infinity: A sequence $a(n)$ diverges to infinity if, for any B, there is an N so that $a(n) > B$ if $n \geq N$.

Remark on calculus trick for ratio of polynomials $P(n)/Q(n)$: look at highest degree terms and if they are the same degree take ratio of leading coefficients.

Follows from:

- Fact that the sequence $1/n$ converges to 0.
- Limit rules for sums, products, quotients.

Series

A *series* is an infinite sum, but it is really a shorthand for a sequence. The series

$$a_0 + a_1 + a_2 + \dots$$

is a short hand for the sequence of partial sums $(a_0, a_1 + a_0, a_2 + a_1 + a_0, \dots)$.

A series converges to a limit L means that the sequence of partial sums converges.

Key example is the geometric series $\sum_{n=0}^{\infty} ar^n$.

Geometric series

Proposition: Suppose $r \neq 1$. The finite geometric series has sum

$$a + ar + ar^2 + ar^3 + \cdots + ar^n = a\frac{r^{n+1} - 1}{r - 1}.$$

Proof: By induction. If $n = 1$ then we get $a + ar = a\frac{r^2-1}{r-1} = ar + r$ as desired. Suppose true for $n = k$ Then

$$a + ar + ar^2 + \cdots + ar^{k+1} = a\frac{r^{k+1} - 1}{r - 1} + ar^{k+1} = a\frac{r^{k+1} + r^{k+1}(r - 1)}{r - 1}$$

and

$$a\frac{r^{k+1}-1+r^{k+2}-r^{k+1}}{r-1}=a\frac{r^{k+2}-1}{r-1}.$$

infinite geometric seriens

Proposition: If $|r|<1$, the infinite geometric series

$$a+ar+ar^2+\cdots$$

converges to

$$\frac{a}{r-1}.$$

Proof: use the limit rules.

other examples

The harmonic series

$$1+1/2+1/3+\cdots$$

does not converge. For a proof, see problem 33 on page 105 of the Gilbert-Vanstone book where they ask you to show by induction that, for all n,

$$1+1/2+1/3+\cdots+1/2^n\geq 1+n/2$$

Therefore the sequence of partial sums diverges (slowly) to infinity.

Base ten decimals and the geometric series

An infinite decimal is shorthand notation for an infinite series (and thus a sequence).

$$.a_0a_1a_2\cdots=\sum_{i=0}^{\infty}a_i10^{-i}$$

An eventually repeating decimal is a decimal expansion such that there is an N and a k so that for all $i\geq N$ we have $a_{i+k}=a_i$. In other words there is a block of k digits $a_N,a_{N+1},\cdots,a_{N+k}$ that repeat over and over.

Proposition: An eventually repeating decimal converges to a rational number.

Base r expansions