Countability Examples

Eventually zero sequences

Proposition: Let S be the set of *eventually zero* sequences of non-negative integers $a_1, a_2, \ldots$; eventually zero means that for each sequence s there is an N so that $a_i = 0$ for $i \geq N$. (Informally, the sequence can have anything at the beginning but eventually it becomes $0, 0, 0, \ldots$.) Then S is countable.

This is interesting because we know on the one hand that, for any k, the set of k-tuples of non-negative integers $(a_1, \ldots, a_k)$ is countable because it is a finite product of countable sets; and we know that the set of *all* sequences of non-negative integers $(a_1, a_2, \ldots)$ is uncountable by the diagonalization argument. So S is a kind of intermediate case, but it turns out to be countable as well.

Proof: First, let $\mathbb{W}$ be the set of non-negative integers and let $A(n)$ be

$$A(n) = \overbrace{W \times W \times W \times \cdots \times W}^{(n-1)} \times \mathbb{N}$$

For each n, $A(n)$ is countable because it is a finite cartesian product of countable sets. For each $n > 0$ let

$$f_n : \mathbb{N} \to A(n)$$

be a bijection, and let f_0 send 0 to the zero sequence.

Now, let $S(n) \subset S$ be the subset of sequences whose last non-zero entry is in position n (and let $S(0)$ be the zero sequence). Now each element of S belongs to exactly one $S(n)$ – just choose the n corresponding to the last non-zero entry in the sequence, or zero if the sequence is all zero. It suffices to prove that S^*, which is S with the zero sequence deleted, is countable.

To construct a bijection from $\mathbb{N} \to S^*$, first construct a bijection $h : \mathbb{N} \times \mathbb{N} \to S$ by sending (x, y) to $f_x(y)$. This is a bijection, since every $s \in S$ belongs to exactly one $S(n)$ and $f_n : \mathbb{N} \to S(n)$ is bijective by construction.

Finally, since $\mathbb{N} \times \mathbb{N}$ is countable, S is countable.