Math 2710

Aug 26-28

Course Info

Key links

- Syllabus
- Tests
- Homework
- Piazza

Grading

- Two midterms (25 points) tentatively Sep 30 and Nov 5.
  - Notify me by Sep 20 if you need an alternate date for the first exam because of Rosh Hashanah.
- Final Exam (40 points)
- Homework + participation (including Piazza) (10 points)

Homework

- daily assignments
- periodically collected and graded with short lead time
- assorted short quizzes or other assignments from time to time

1.1 What is this course about?

Mathematics as a discipline

This course is about

- *how mathematics is done*
- *how mathematics is communicated*.

The actual mathematics we will learn in this course is less important than the approach

A very simple example

Assertion: The sum of two even numbers is an even number.

Goal: find a mathematical proof of this fact.

Mathematical Proof

A *mathematical proof* of this assertion is an argument that starts from known facts and definitions and establishes the the truth of the assertion using the tools of logic.

A proof in *formal logic* starts from explicit hypotheses or axioms and applies the rules of deductive logic to reach a conclusion. Proofs of even simple facts in formal logic are extremely long and mostly not readable by humans.

In principle, a mathematical proof contains enough information to produce a formal logical proof.

Good Mathematical Proofs

A good mathematical proof is

- *rigorous*, meaning it gives a complete logical argument,
- *informative*, meaning that it provides enough information to explain why the assertion is true
- *efficient*, meaning that it is as short as possible while still being rigorous and informative.

Example, continued

To construct a proof of this assertion, we need:

- to know exactly what the terms mean (what is an even integer?)
- to establish in our own minds that the assertion IS true, and figure out why
- communicate our understanding of why the assertion is true rigorously and efficiently.

Discussion

- Define *even number*.
- Explain why the assertion about even numbers is true, as rigorously and efficiently as you can.

Key Vocabulary

- theorem
- lemma
- proposition
- corollary
- example
- algorithm
- definition
- proof

- statement
- proposition
- converse
- contrapositive
- conditional statement

1.2 Logic

Statements

A statement is a sentence that is either **True** or **False**

Compound statements are built up using logical operators **AND**, **OR**, **NOT**, and others.

The truth of a compound statement depends on the individual statements and the properties of the operators.

AND, OR, NOT

P	Q	**AND**
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	**OR**
T	T	T
T	F	T
F	T	T
F	F	F

P	**NOT**
T	F
F	T

Implications/Conditionals

A great deal of reasoning is about *conditional statments*.

- If I get a vaccine for flu today, then I will not get the flu this year.
- If there is a recession in the next six months, President Trump will not be re-elected.
- If the nucleotide at a certain genomic position is switched from A to T, the affected individual will have a certain genetic disease.

The truth of an implication depends on the Truth/Falsehood of BOTH components. But if the first clause is FALSE, the statement is TRUE. So the interesting cases are when the first clause is TRUE.

For discussion: Compare how other disciplines think about implications such as those above with how mathematicians do. Which of the statements above might be susceptible to proof in "real life"?

Truth Tables for conditionals

P	Q	$\Longrightarrow$
T	T	T
T	F	F
F	T	T
F	F	T

Alternate formulations

- P implies Q
- If P, then Q
- Q if P
- P only if Q
- P suffices for Q, or is sufficient for Q
- Q is necessary for P.

Equivalence

The claim that two statements are *equivalent* is the claim that they are either both True or both False.

- P is equivalent to Q
- P if and only if Q
- P is necessary and sufficient for Q.

P	Q	$\iff$
T	T	T
T	F	F
F	T	F
F	F	T

Example Computation

(X and (Y or Z)) $\iff$ ((X and Y) or (X and Z))

X	Y	Z	Y or Z	X and (Y or Z)	X and Y	X and Z	((X and Y
T	T	T	T	T	T	T	
T	T	F	T	T	T	F	
T	F	T	T	T	F	T	
T	F	F	F	F	F	F	
F	T	T	T	F	F	F	
...							

(X and (Y or Z)) have the same truth values as ((X and Y) or (X and Z)) and so the statements are equivalent.

Discussion

- Give some examples (in English) of statements where P implies Q (meaning P implies Q is TRUE), but Q does not imply P.
- Give some examples of statements that are equivalent.
- Are their statements P and Q so that neither P implies Q nor Q implies P are True?
- P and Q are equivalent means $P \iff Q$ or "P if and only if Q".
- Show: $P \implies Q$ is equivalent to (Q OR NOT P).
- Exclusive OR is the operator that is TRUE when one of two statements is True, but not both. Express it in terms of AND, OR, and NOT.
- If my basement is wet, then it is either very rainy or a pipe has broken. Express this using the various operators and test its truth under different conditions.

Problems Originally posted to piazza

Definition: An integer x is 5-ish if there is an integer y so that $x=5y$.

1. Write the definition for a number that is NOT 5-ish.
2. Is 37 a 5-ish number? How do you know?

Definition: An integer x is "purple" if there is a integer y so that $x=5y+1$.

Which of the following statements are true?

1. If x is purple, then x is not 5-ish.
2. If x is 5-ish, then x is not purple.
3. There is a number z that is neither purple nor 5-ish.

1.3 Sets

Sets

We rely on a "naive" notion of set, meaning a collection of objects. For example:

- the set of integers
- the set of words in the English language
- the set of people in the world
- the empty set

For a discussion of why this is naive, see Russell's paradox.

Subsets

We can construct sets by selecting elements of another set, yielding a *subset* of the original set.

Explicit specification

$A = \{1, 3, 5, 8, 9\}$, a subset of the integers.

Selection by a property

Suppose P is the set of people. Then

$$\{p \in P : p \text{ is a legal resident of Chicago}\}$$

is shorthand for the set consisting of people p for which the statement "p is a legal resident of Chicago" is True.

Set operations

If A and B are both subsets of some huge (and usually unmentioned) set U, then:

- $A \subset B$ means that every element of A is also an element of B or $x \in A \implies x \in B$.
- $A = B$ means that A and B have the same elements, or $x \in A \iff x \in B$.

More set operations

- $A \cup B$, the union of A and B, is $\{x \in U : x \in A \text{ OR } x \in B\}$.
- $A \cap B$, the intersection of A and B, is $\{x \in U : x \in A \text{ AND } x \in B\}$.
- $A \times B$, the product of A and B, is the set of all ordered pairs (x, y) where $x \in A$ and $y \in B$.

Discussion 1

Suppose A and B are two sets contained in some big set U. Prove the following by a truth table:

- $((A \cap B) = A)$ implies $A \subset B$.

Hint: Start with the statements $X = (x \in A)$ and $Y = (x \in B)$. Then $A \subset B$ is $X \implies Y$. Express the left hand side similarly and work out the truth table.

Discussion 2

What is the truth table associated to the proposition about any three sets A, B, C:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Is the proposition true?