Math 2710

Nov 11-15

Functions

Definition: A function $f : A \to B$ is a subset $f \subset A \times B$ with the property that for every $a \in A$ there is exactly one b in B so that $(a, b) \in f$. We write $f : A \to B$ to mean that f is a function with domain A and co-domain B.

Compare this with the definition of a function as a “rule.” The associated “rule” says: to compute $f(a)$, find the (uniquely determined) pair (a, b) in f, and then set $f(a) = b$.

In some sense this definition replaces the notion of a function with its graph.

Domain, co-Domain, Range

A is called the **DOMAIN** of f and B is called the **CO-DOMAIN.**

The **range** of f is the subset of b that occur in a pair $(a, b) \in f$.

A function can be drawn as a graph or as a “mapping” between two sets (see figure 12.3 in Hammack).

Examples

Consider problem 1 from Hammack. $A = \{0, 1, 2, 3, 4\}$ and $B = \{2, 3, 5, 6\}$. Then $f = \{(0, 3), (1, 3), (2, 4), (3, 2), (4, 2)\}$ Verify that f is a function and then find its range.

Is the subset of pairs of integers (x, y) where $3x + y = 4$ a function from $\mathbf{Z}$ to $\mathbf{Z}$?

Terminology: injective and surjective

Definition: A function is *injective* if it has the property that $f(a) \neq f(b)$ implies $a \neq b$. (or $f(a) = f(b)$ implies $a = b$). A function is *surjective* if for every $b \in B$ there is an a with $(a, b) \in f$. A function is *bijective* if it is both injective and surjective.

- *Injective* is also called "one-to-one." *Surjective* is also called "onto."
- Surjectivity depends on the codomain (you can shrink the codomain to make the function surjective).

Some examples

- $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n, m) = 2n - 4m$.
- $f : \mathbb{Q} - \{2\} \to \mathbb{Q} - \{5\}$ given by $f(x) = (5x + 1)/(x - 2)$ is bijective.
- Consider maps from $\{0, 1, 2\}$ to $\{0, 1, 2\}$. how many are there? How many are injective? How many surjective? How many bijective?
- The function $f : \mathbb{Z}_m \to \mathbb{Z}_m$ given by $f(x) = ax$ is bijective if $\gcd(a, m) = 1$.

Composition of functions

Definition: Suppose that $f : A \to B$ is a function and $g : B \to C$ is a function. The *composition* $g \circ f$ is a function from $A \to C$ defined by $(g \circ f)(a) = g(f(a))$.

This makes sense because $f(a) \in B$ and $g : B \to C$.

Examples:

- $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$, and $g : \mathbf{R} \to \mathbf{R}$ with $g(x) = \sin(x)$. What are $f \circ g$ and $g \circ f$?
- $A = \{0, 1, 2, 3\}$, $B = \{1, 2, 3, 4\}$, $C = \{1, 3, 5, 6\}$. $f = \{(0, 1), (1, 2), (2, 3), (3, 3)\}$ in $A \times B$. $g = \{(1, 3), (2, 3), (3, 3), 4, 3)\}$ in $B \times C$. What is the composition?

Properties of composition

Theorem: Composition of functions is associative.

Theorem: If $f : A \to B$ and $g : B \to C$ are injective then $g \circ f$ is injective. Similarly if f and g are surjective then $g \circ f$ is surjective. And if both are bijective, so is the composition.