Math 2710

Oct 28-Nov 1

More on limit rules

Proposition: If a_n converges to L then ca_n converges to cL.

Definition: Divergence to infinity: A sequence $a(n)$ diverges to infinity if, for any B, there is an N so that $a(n) > B$ if $n \geq N$.

Remark on calculus trick for ratio of polynomials $P(n)/Q(n)$: look at highest degree terms and if they are the same degree take ratio of leading coefficients.

Follows from:

- Fact that the sequence $1/n$ converges to 0.
- Limit rules for sums, products, quotients.

Series

A *series* is an infinite sum, but it is really a shorthand for a sequence. The series

$$a_0 + a_1 + a_2 + \dots$$

is a short hand for the sequence of partial sums $(a_0, a_1 + a_0, a_2 + a_1 + a_0, \dots)$.

A series converges to a limit L means that the sequence of partial sums converges.

Key example is the geometric series $\sum_{n=0}^{\infty} ar^n$.

Geometric series

Proposition: Suppose $r \neq 1$. The finite geometric series has sum

$$a + ar + ar^2 + ar^3 + \cdots + ar^n = a\frac{r^{n+1} - 1}{r - 1}.$$

Proof: By induction. If $n = 1$ then we get $a + ar = a\frac{r^2-1}{r-1} = ar + r$ as desired. Suppose true for $n = k$ Then

$$a + ar + ar^2 + \cdots + ar^{k+1} = a\frac{r^{k+1} - 1}{r - 1} + ar^{k+1} = a\frac{r^{k+1} + r^{k+1}(r - 1)}{r - 1}$$

and

$$a\frac{r^{k+1}-1+r^{k+2}-r^{k+1}}{r-1}=a\frac{r^{k+2}-1}{r-1}.$$

Infinite geometric seriens

Proposition: If $|r|<1$, the infinite geometric series

$$a+ar+ar^2+\cdots$$

converges to

$$\frac{a}{1-r}.$$

Proof: The partial sums are $s(n)=a\frac{r^{n+1}-1}{r-1}$. Since $\frac{a}{1-r}$ is a constant we can write this as

$$s(n)=\frac{a}{1-r}(1-r^{n+1}).$$

Now the sequence r^{n+1} converges to zero when $r<1$ (proof?) and so $1-r^{n+1}$ converges to 1 (proof) and so $s(n)$ converges to $a/(1-r)$.

Other examples

The harmonic series

$$1+1/2+1/3+\cdots$$

does not converge. For a proof, see problem 33 on page 105 of the Gilbert-Vanstone book where they ask you to show by induction that, for all n,

$$1+1/2+1/3+\cdots+1/2^n\geq 1+n/2$$

Therefore the sequence of partial sums diverges (slowly) to infinity.

Base ten decimals and the geometric series

An infinite decimal is shorthand notation for an infinite series (and thus a sequence).

$$.a_0a_1a_2\cdots=\sum_{i=0}^{\infty}a_i10^{-i}$$

An eventually repeating decimal is a decimal expansion such that there is an N and a k so that for all $i\geq N$ we have $a_{i+k}=a_i$. In other words there is a block of k digits $a_N,a_{N+1},\cdots,a_{N+k}$ that repeat over and over.

Repeating decimals converge to rational numbers

Proposition: An eventually repeating decimal converges to a rational number.

Proof: First suppose our decimal begins repeating right at the decimal point, so it looks like $a_1a_2\cdots a_ka_1a_2\cdots$. Let A be the integer $a_1a_2\cdots a_k$. Then the decimal corresponds to the series

$$\frac{A}{10^k}(1+\frac{1}{10^k}+\cdots)$$

This is a geometric series which we know converges to

$$\frac{A}{10^k}\frac{1}{1-(1/10)^k}$$

which is a rational number.

Repeating decimals cont'd

If the decimal x has an initial, non repeating part, of length N, with digits $a_1a_2\ldots a_N$ followed by blocks of k digits $b_1b_2\ldots b_k$, we can split out the leading part and write x as the series:

$$x=\frac{A}{10^N}+\frac{1}{10^{N+k}}(B+B/10^k+\cdots)$$

where A and B are the integers made up of the initial digits and the repeating block respectively. The second part of this series converges to a rational number by the earlier work, and so the sum is a rational number.

Base r

There was nothing special about base 10 and the same would be true for repeating "decimals" in any base.

Every rational has a repeating decimal expansion

For the converse we make an inductive argument. We are given a fraction a/b and we want to construct a decimal expansion and prove that it is repeating. First, we use the division algorithm to write $a=qb+r$ and see that $a/b=q+r/b$. Now we proceed as follows. We multiply r by 10 and divide again by b using the division algorithm:

$$10r=q_1b+r_1.$$

Repeating decimals continued

This tells us that $r/b = q_1/10 + r_1/10b$ Since $r_1 < b$, the fraction $r_1/10b$ is less than 1/10. Also, since $r_1 < b$, $10r_1 < 10b$ and so $q_1 < 10$. Therefore q_1 is the first decimal digit of r/b. Now we repeat the process with r_1:

$$10r_1 = q_2 b + r_2$$

which yields $r_1/b = q_2/10 + r_2/(10b)$ or

$$r/b = q_1/10 + q_2/100 + r_2/(100b).$$

Again, $q_2 < 10$ and $r_2/100b < 1/100$.

Repeating decimals continued

Now we proceed by induction. Suppose we have

$$r/b = q_1/10 + q_2/100 + \cdots + q_n/10^n + \frac{r_n}{10^n b}$$

with each $q_i < 10$ and $r_n < b$ so that $r_n/(10^n b) < 1/10^n$. Define r_{n+1} by dividing by $10r_n$ by b:

$$10r_n = q_{n+1} b + r_{n+1}.$$

Again, $q_{n+1} < 10$ and $r_{n+1} < b$. Also

$$r_n/b = \frac{q_{n+1}}{10} + \frac{r_{n+1}}{10b}$$

so

$$r/b = \frac{q_1}{10} + \frac{q_2}{100} + \cdots + \frac{q_n}{10^n} + \frac{q_{n+1}}{10^{n+1}} + \frac{r_{n+1}}{10^{n+1} b}.$$

Finishing up

The key final observation in the proof is that the sequence of remainders r_{n+1} have two important properties:

- First, they are all between 0 and b, and second
- Knowing r_n determines r_{n+1}.

These two facts force the sequence of remainders to ultimately repeat, because after considering $b+1$ of them there have to be two of them which are equal; say $r_k = r_j$ for some pair k, j with $j > k$. But then $r_{k+1} = r_{j+1}$, $r_{k+2} = r_{j+2}$, and so on, and in fact $r_{i+(j-k)} = r_i$ for any $i \geq k$. Thus the sequence of r's, and therefore the sequence of q's, eventually becomes periodic.

Other bases: Notice that there was nothing special about 10 in the proof above, so rational numbers have periodic expansions in any base.