13.1 The Triangle Inequality

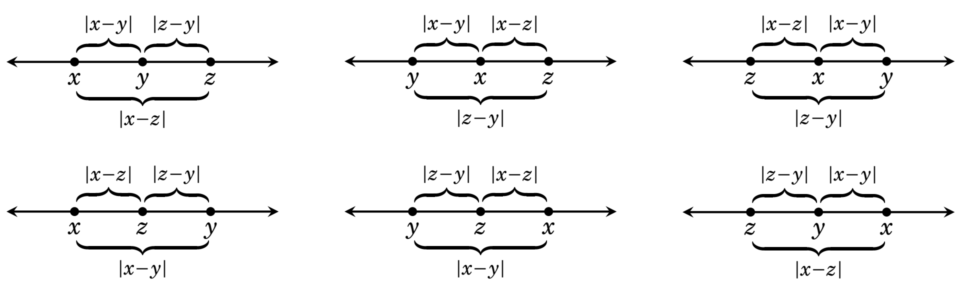
Definitions in calculus and analysis use absolute value extensively. As you know, the absolute value of a real number x is the non-negative number

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Fundamental properties of absolute value include $|xy| = |x| \cdot |y|$ and $x \leq |x|$. Another property—used often in proofs—is the *triangle inequality*:

Theorem 13.1 (Triangle inequality) If $x, y, z \in \mathbb{R}$, then $|x-y| \leq |x-z| + |z-y|$.

Proof The name *triangle inequality* comes from the fact that the theorem can be interpreted as asserting that for any "triangle" on the number line, the length of any side never exceeds the sum of the lengths of the other two sides. Indeed, the distance between any two numbers $a, b \in \mathbb{R}$ is $|a-b|$. With this in mind, observe in the diagrams below that regardless of the order of x, y, z on the number line, the inequality $|x-y| \leq |x-z| + |z-y|$ holds.



(These diagrams show x, y, z as distinct points. If $x = y$, $x = z$ or $y = z$, then $|x-y| \leq |x-z| + |z-y|$ holds automatically.) ■

The triangle inequality says the shortest route from x to y avoids z unless z lies between x and y. Several useful results flow from it. Put $z = 0$ to get

$$|x-y| \leq |x| + |y| \quad \text{for any } x, y \in \mathbb{R}. \tag{13.1}$$

Using the triangle inequality, $|x+y| = |x-(-y)| \leq |x-0| + |0-(-y)| = |x| + |y|$, so

$$|x+y| \leq |x| + |y| \quad \text{for any } x, y \in \mathbb{R}. \tag{13.2}$$

Also by the triangle inequality, $|x-0| \leq |x-(-y)| + |-y-0|$, which yields

$$|x| - |y| \leq |x+y| \quad \text{for any } x, y \in \mathbb{R}. \tag{13.3}$$

The three inequalities (13.1), (13.2) and (13.3) are very useful in proofs.