13.8 Series

You may recall from your calculus course that there is a big difference between a *sequence* and a *series*.

A **sequence** is an infinite list $a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ \cdots$.

But a **series** is an infinite *sum* $a_1+a_2+a_3+a_4+a_5+a_6+\cdots$.

We use the notation $\{a_n\}$ to denote the sequence $a_1, a_2, a_3, a_4, \ldots$, but we use sigma notation to denote a series:

$$\sum_{k=1}^{\infty} a_k \;=\; a_1+a_2+a_3+a_4+a_5+a_6+\cdots.$$

For example,

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \;=\; \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots.$$

You may have a sense that this series should sum to 1, whereas

$$\sum_{k=1}^{\infty} \frac{k+1}{k} \;=\; \frac{2}{1}+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\frac{6}{5}+\frac{7}{6}+\cdots$$

equals ∞ because every fraction in the infinite sum is greater than 1.

Series are significant in calculus because many complex functions can be expressed as series involving terms built from simple algebraic operations. For example, your calculus course may have developed the *Maclaurin series* for various functions, such as

$$\cos(x) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k} \;=\; 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\cdots.$$

But before we make any progress with series, it is essential that we clearly specify what it means to add up infinitely many numbers. We need to understand the situations in which this does and does not make sense.

The key to codifying whether or not a series

$$\sum_{k=1}^{\infty} a_k \;=\; a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8+a_9\cdots$$

adds up to a finite number is to terminate it at an arbitrary nth term:

$$\sum_{k=1}^{n} a_k \;=\; a_1+a_2+a_3+a_4+a_5+a_6+\cdots+a_n.$$

This is sum called the nth **partial sum** of the series, and is denoted as s_n.

The series has a partial sum s_n for each positive integer n:

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
s_4 &= a_1 + a_2 + a_3 + a_4 \\
s_5 &= a_1 + a_2 + a_3 + a_4 + a_5 \\
&\vdots \\
s_n &= a_1 + a_2 + a_3 + a_4 + a_5 + \cdots + a_n \;=\; \sum_{k=1}^{n} a_k. \\
&\vdots
\end{aligned}$$

If indeed the infinite sum $S = \sum_{k=1}^{\infty} a_k$ makes sense, then we expect that the partial sum $s_n = \sum_{k=1}^{n} a_k$ is a very good approximation to S when n is large. Moreover, the larger n gets, the closer s_n should be to S. In other words, the *sequence* $s_1, s_2, s_3, s_4, s_5, \ldots$ *of partial sums* should converge to S. This leads to our main definition. We say that an infinite series *converges* if its sequence of partial sums converges.

Definition 13.7 A series $\sum_{k=1}^{\infty} a_k$ **converges** to a real number S if its sequence of partial sums $\{s_n\}$ converges to S. In this case we say $\sum_{k=1}^{\infty} a_k = S$.

We say $\sum_{k=1}^{\infty} a_k$ **diverges** if the sequence $\{s_n\}$ diverges. In this case $\sum_{k=1}^{\infty} a_k$ does not make sense as a sum or does not sum to a finite number.

Example 13.10 Prove that $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$.

Proof. Consider the partial sum $s_n = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}$. We can get a neat formula for s_n by noting $s_n = 2s_n - s_n$. Then simplify and cancel like terms:

$$\begin{aligned}
s_n = 2s_n - s_n &= 2\left(\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n}\right) - \left(\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n}\right) \\
&= \left(\frac{2}{2^1} + \frac{2}{2^2} + \frac{2}{2^3} + \cdots + \frac{2}{2^{n-1}} + \frac{2}{2^n}\right) - \left(\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n}\right) \\
&= \left(1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}}\right) - \left(\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n}\right) = 1 - \frac{1}{2^n}.
\end{aligned}$$

Thus $s_n = 1 - \frac{1}{2^n}$, so the sequence of partial sums is $\{s_n\} = \{1 - \frac{1}{2^n}\}$, which converges to 1 by Exercise 13.7.4. Definition 13.7 yields $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$. ■

Despite the previous example, in practice definitions 13.7 and 13.5 are rarely used to prove that a particular sequence or series converges to a particular number. Instead we tend to use a multitude of convergence tests that are covered in a typical calculus course. Examples of such tests include the *comparison test*, the *ratio test*, the *root test* and the *alternating series test*. You learned how to use these tests and techniques in your calculus course, though that course may not have actually *proved* that the tests were valid. The point of our present discussion is that definitions 13.7 and 13.5 can be used to prove the tests. To underscore this point, this section's exercises ask you to prove several convergence tests.

By way of illustration, we close with a proof of a theorem that leads to a test for divergence.

Theorem 13.11 If $\sum_{k=1}^{\infty} a_k$ converges, then the sequence $\{a_n\}$ converges to 0.

Proof. We use direct proof. Suppose $\sum_{k=1}^{\infty} a_k$ converges, and say $\sum_{k=1}^{\infty} a_k = S$. Then by Definition 13.7, the sequence of partial sums $\{s_n\}$ converges to S. From this, Definition 13.5 says that for any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ for which $n > N$ implies $|s_n - S| < \varepsilon$. Thus also $n-1 > N$ implies $|s_{n-1} - S| < \varepsilon$.

We need to show that $\{a_n\}$ converges to 0. So take $\varepsilon > 0$. By the previous paragraph, there is an $N' \in \mathbb{N}$ for which $n > N'$ implies $|s_n - S| < \frac{\varepsilon}{2}$ and $|s_{n-1} - S| < \frac{\varepsilon}{2}$. Notice that $a_n = s_n - s_{n-1}$ for any $n > 2$. So if $n > N'$ we have

$$\begin{aligned} |a_n - 0| = |s_n - s_{n-1}| &= \big|(s_n - S) - (s_{n-1} - S)\big| \\ &\leq |s_n - S| + |s_{n-1} - S| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, by Definition 13.5, the sequence $\{a_n\}$ converges to 0. ■

The contrapositive of this theorem is a convenient test for divergence:

Corollary 13.1 (Divergence test) If $\{a_n\}$ diverges, or if it converges to a non-zero number, then $\sum_{k=1}^{\infty} a_k$ diverges.

For example, according to the divergence test, the series $\sum_{k=1}^{\infty} \left(1 - \frac{1}{k}\right)$ diverges, because the sequence $\left\{1 - \frac{1}{n}\right\}$ converges to 1. Also, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k+1)}{k}$ diverges because $\left\{\frac{(-1)^{n+1}(n+1)}{n}\right\}$ diverges. (See Example 13.9 on page 263.)

The divergence test gives only a criterion for deciding if a series diverges. It says nothing about convergence. If $\{a_n\}$ converges to 0, then $\sum_{k=1}^{\infty} a_k$ may or may not converge, depending on the particular series. Certainly if $\sum_{k=1}^{\infty} a_k$ converges, then $\{a_n\}$ converges to 0, by Theorem 13.11. But $\{a_n\}$ converging to 0 does not necessarily mean that $\sum_{k=1}^{\infty} a_k$ converges. A significant example of this is the so-called **harmonic series**:

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots.$$

According to Exercise 21 in Chapter 10, if we go out as far as 2^n terms, then the 2^nth partial sum satisfies

$$s_{2^n} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2^n-1}+\frac{1}{2^n} \geq 1+\frac{n}{2}.$$

Because $1+\frac{n}{2}$ grows arbitrarily large as n increases, the sequence of partial sums diverges to ∞. Consequently the harmonic series diverges.

Exercises for Section 13.8

Use Definition 13.7 (and Definition 13.5, as needed) to prove the following results. Solutions for these exercises are not included in the back of the book, for they can be found in most good calculus texts. In the exercises we abbreviate $\sum_{k=1}^{\infty} a_k$ as $\sum a_k$.

1. A *geometric series* is one having the form $a+ar+ar^2+ar^3\cdots$, where $a,r \in \mathbb{R}$. (The first term in the sum is a, and beyond that, the kth term is r times the previous term.) Prove that if $|r|<1$, then the series converges to $\frac{a}{1-r}$. Also, if $a \neq 0$ and $|r| \geq 1$, then the series diverges. (If you need guidance, you may draw inspiration from Example 13.10, which concerns a geometric series with $a=r=\frac{1}{2}$.)

2. Prove the *comparison test*: Suppose $\sum a_k$ and $\sum b_k$ are series. If $0 \leq a_k \leq b_k$ for each k, and $\sum b_k$ converges, then $\sum a_k$ converges. Also, if $0 \leq b_k \leq a_k$ for each k, and $\sum b_k$ diverges, then $\sum a_k$ diverges.

3. Prove the *limit comparison test*: Suppose $\sum a_k$ and $\sum b_k$ are series for which $a_k, b_k > 0$ for each k. If $\lim_{n\to\infty} \frac{a_k}{b_k} = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges. (Your proof may use any of the above exercises.)

4. Prove the *absolute convergence test*: Let $\sum a_k$ be a series. If $\sum |a_k|$ converges, then $\sum a_k$ converges. (Your proof may use any of the above exercises.)

5. Prove the *ratio test*: Given a series $\sum a_k$ with each a_k positive, if $\lim_{n\to\infty} \frac{a_{k+1}}{a_k} = L < 1$, then $\sum a_k$ converges. Also, if $L > 1$, then $\sum a_k$ diverges. (Your proof may use any of the above exercises.)