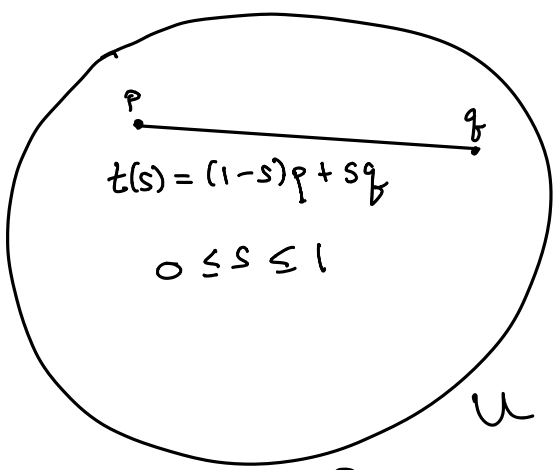
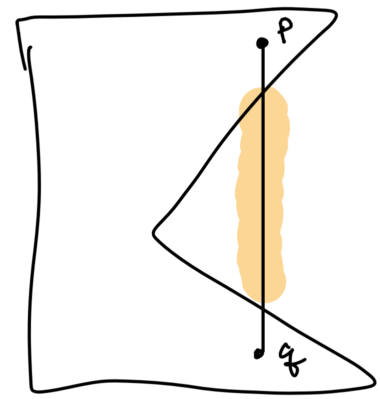
Convexity and Convex Hulls

Convex sets

Definition: A subset U of $\mathbb{R}^k$ is *convex* if, for any pair of points $p, q \in U$, the line segment joining p to q is in U. In vector terms, if $p, q \in U$, then for every $0 \leq s \leq 1$, $t(s) = (1-s)p + sq$ belongs to U.



Convex



Not convex

Proposition: The intersection of convex sets is convex.

U, V are convex.

$p, q \in U \cap V$

$t(s) = (1-s)p + sq \quad 0 \leq s \leq 1$

$t(s) \in U$ and $t(s) \in V$

so $t(s) \in U \cap V$

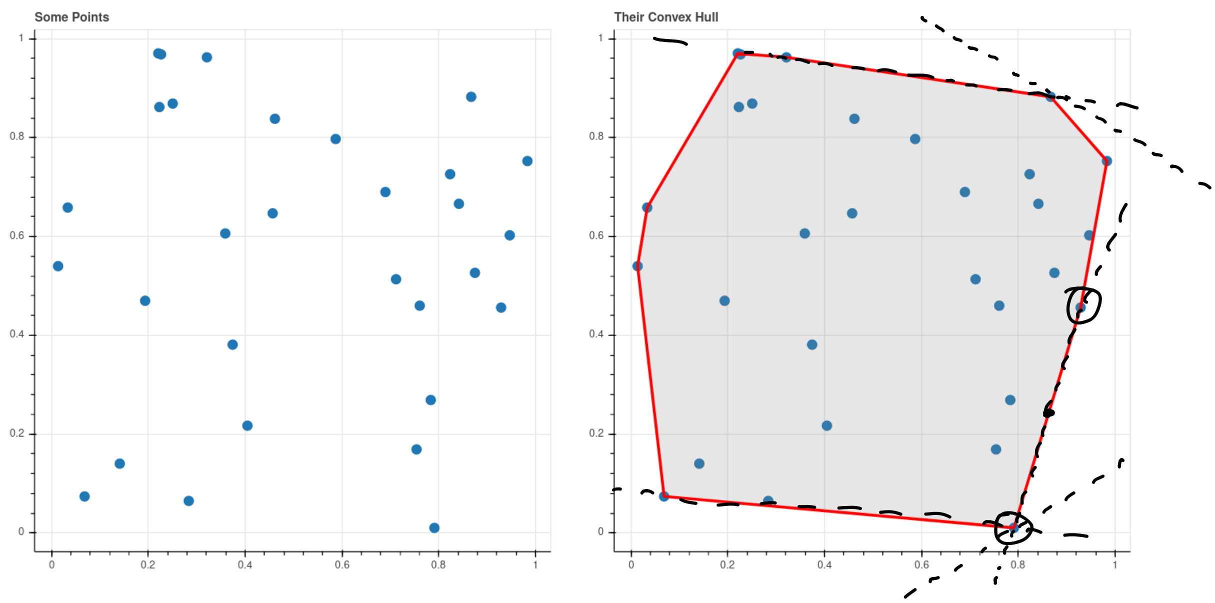
Convex Hulls

Definition: Let S be a finite set of points $\{q_1, \ldots, q_N\}$ be a finite set of points in $\mathbb{R}^k$. The *convex hull* $C(S)$ is the set of points

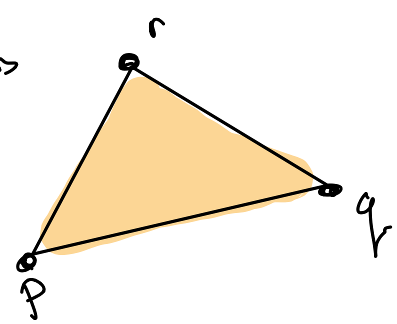
$$p = \sum_{i=1}^{N} \lambda_i q_i$$

as $(\lambda_1, \ldots, \lambda_N)$ runs over all N-tuples of real numbers such that

$$\sum_{i=1}^{N} \lambda_i = 1.$$



50 random points



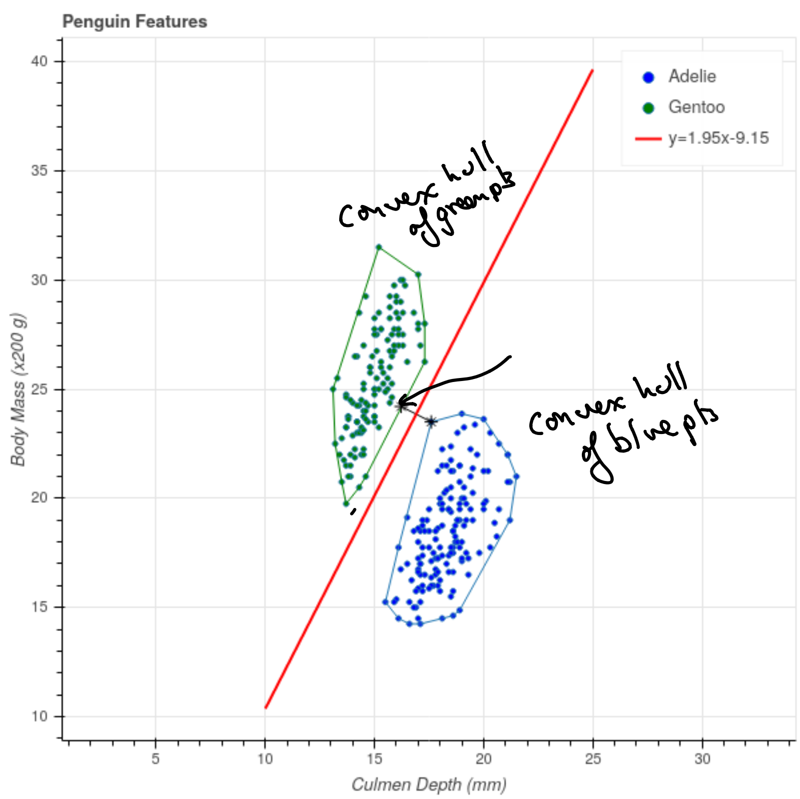
$\alpha p + \beta q + \gamma r$

$\alpha + \beta + \gamma = 1$

A Look Ahead

We care about convex hulls because of the following result that we will (eventually) prove.

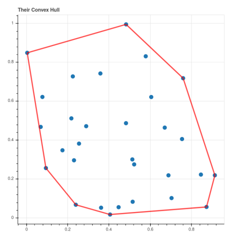
Proposition: The optimal margin between two linearly separable sets A^+ and A^- is equal to the closest distance between points in their convex hulls.



Solution

In addition, there is an iterative algorithm called "Sequential Minimal Optimization" that can find these closest points.

More on Convex Hulls



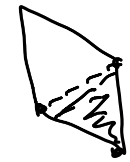
Proposition: $C(S)$ is convex.

$p, q \in C(S)$ then $(1-s)p + sq$ is also in $C(S)$?

$p = \sum \lambda_i q_i \qquad q = \sum \sigma_i q_i \qquad \left.\begin{array}{l} \sum \lambda_i = 1 \\ \sum \sigma_i = 1 \end{array}\right\} *$

$$
\begin{aligned}
(1-s)p + sq &= \sum (1-s)\lambda_i q_i + \sum s\sigma_i q_i \\
&= \underline{\sum \tau_i q_i} \\
&\in C(S)
\end{aligned}
$$

$$
\begin{aligned}
\tau_i &= (1-s)\lambda_i + s\sigma_i \\
\sum \tau_i &= (1-s)\underline{\sum \lambda_i} + s\underline{\sum \sigma_i} \\
&= (1-s) + s = \underline{1}
\end{aligned}
$$



The convex hull is the smallest containing convex set

Proposition: $C(S)$ is the smallest convex set containing S. In other words, if U is a convex set containing S, then $C(S) \subseteq U$.

$S = \{q_1, \ldots, q_N\}$

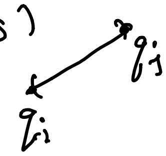
Proof: By induction.

- Let $C_n(S)$ be the set of points $\sum_{i=1}^{n} \lambda_i q_i$ where $\sum_{i=1}^{n} \lambda_i = 1$ and all λ_i are non-zero.

$C_1(S) = S$

- $C(S) = \bigcup_{i=1}^{\infty} C_n(S)$ $\quad C_1(S) \subseteq U$

$C_2(S)$



- U convex means $C_2(S) \subset U$.
- We show $C_n(S) \subset U \implies C_{n+1}(S) \subset U$. $\quad \Rightarrow$ all $C_n(S) \subseteq U$

$C(S) = \bigcup C_n(S) \subseteq U$

$p \in C_{n+1}(S)$

$$p = \sum_{i=1}^{n+1} \lambda_i q_i \qquad \sum_{i=1}^{n+1} \lambda_i = 1 \qquad \text{all } \lambda_i > 0.$$

$$T = \sum_{i=1}^{n} \lambda_i$$

$$p = \left[\sum_{i=1}^{n} \lambda_i q_i\right] + \lambda_{n+1} q_{n+1}$$

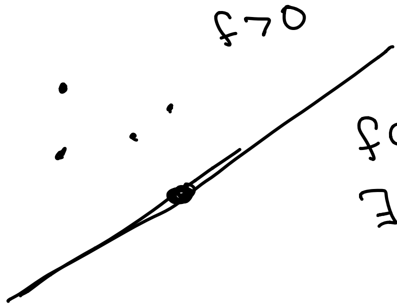
$$0 \le T < 1$$

$$= T\left[\sum \frac{\lambda_i}{T} q_i\right] + \lambda_{n+1} q_{n+1} \qquad \sum \frac{\lambda_i}{T} = 1$$

$\qquad\qquad\qquad\qquad \overset{\cap}{S} \subseteq U \qquad$ all $\frac{\lambda_i}{T} > 0.$

Conclude that $p \in U \Rightarrow C_{n+1}(S) \subseteq U \qquad \overset{\cap}{C_n(S)} \subseteq U \qquad T + \lambda_{n+1} = 1$

- By induction this shows that $C_n(S) \subset U$ for all n and therefore $C(S) \subset U$.



$f = w \cdot x + b = 0$

$f(s) \geq 0 \ \forall s \in A$

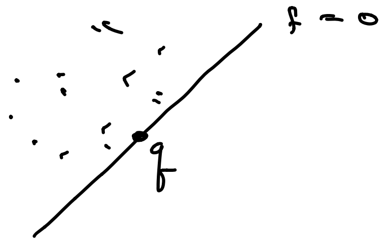
$\exists s \in A$ with $f(s) = 0$.

Convex Hulls and Supporting Hyperplanes

Proposition: S and $C(S)$ have the same supporting hyperplanes.

- Remember that $f(x) = w \cdot x + b = 0$ is a supporting hyperplane for a set A if $f(a) \geq 0$ for all $a \in A$ and $f(a) = 0$ for at least one $a \in A$.

- If $f = 0$ is a supporting hyperplane for S, then S is contained in the half plane $f \geq 0$ and $f(q) = 0$ for some $q \in S$. The halfplane is a convex set, so $C(S)$ is contained in it, and $q \in C(S)$ and $f(q) = 0$ so $f = 0$ is a supporting hyperplane for $C(S)$.

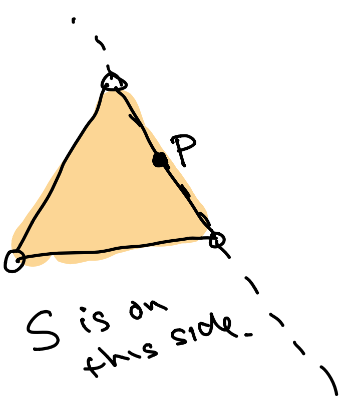


$S \subseteq \{p \mid f(p) \geq 0\}$ is convex.

$\Rightarrow C(S) \subseteq \{p \mid f(p) \geq 0\}$.

$q \in C(S)$.

- Suppose $f = 0$ is a supporting hyperplane for $C(S)$. Let p be the point in $C(S)$ where $f(p) = 0$. Note that p need not be in S as far as we know. However, since $f \geq 0$ for all $a \in C(S)$, and $S \subset C(S)$, we have $f \geq 0$ for all $q \in S$. The question is whether there is $q \in S$ with $f(q) = 0$.

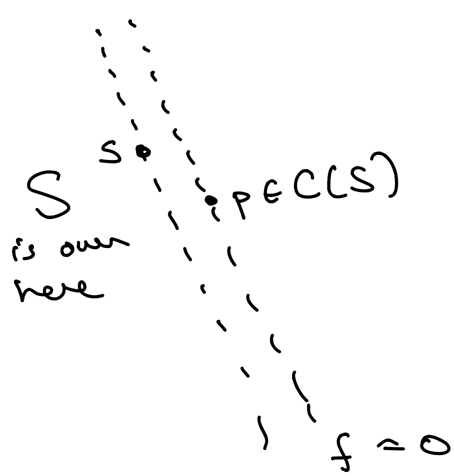


$f(x) \geq 0 \quad \forall x \in C(S)$

and $S \subseteq C(S)$ so $f(s) \geq 0$

for all $s \in S$.

- Let q be the point in S at which $f(q)$ is minimal. Then $g(x) = f(x) - f(q)$ is a hyperplane that is 0 at q and $g(x) \geq 0$ for all $x \in S$. Since the half space $g(x) \geq 0$ is convex and contains S, $C(S)$ is contained in that half space and so $g(x) \geq 0$ for all points in $C(S)$.



$f(s) \geq 0 \quad \forall s \in S$

Choose q in S so that $f(q)$ is minimal.

$g(x) = f(x) - f(q)$

$g(q) = 0,$

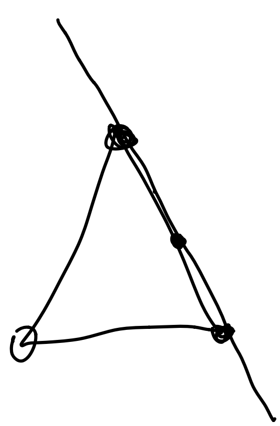
$g(s) \geq 0 \quad \forall s \in S.$

$g(S) \geq 0$ contains S and is convex so $C(S)$ is contained in set where $g(x) \geq 0$.

- Now $g(p) = f(p) - f(q) = 0 - f(q) \geq 0$ and $f(q) \geq 0$. Therefore $f(q) = 0$, and we found a point $q \in S$ where f vanishes.

$p \in C(S)$ so $g(p) = f(p) - f(q) \geq 0$

$$\left.\begin{array}{l} -f(q) \geq 0 \\ \text{but } f(q) \geq 0 \end{array}\right\} \Rightarrow f(q) = 0.$$



Convex Hulls and Supporting Hyperplanes

Proposition: Let K be the set of supporting hyperplanes $f(x) = w \cdot x + b = 0$ for S where $f(x) \geq 0$ for all $x \in S$. Then $C(S)$ is the intersection of all the positive half spaces for $f \in K$.

Proof:

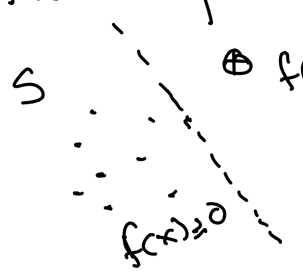
- $C(S)$ is contained in the intersection, since the intersection is convex and contains S.

if $f \in K$ then $\{x \mid f(x) \geq 0\}$ is convex and it contains S so $\Rightarrow$ $C(S)$ is contained in that half space

$$C(S) \subseteq \bigcap_{f \in K} \{x \mid f(x) \geq 0\}.$$

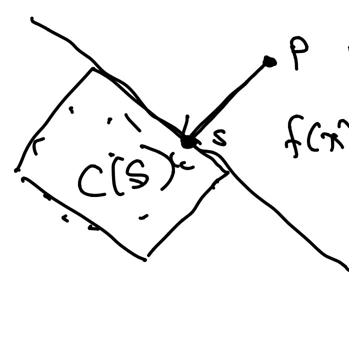
- Suppose that p is not in $C(S)$. Let s be the point in ~~S~~ $C(S)$ closest to p. Let $w = s - p$ and let $f(x) = w \cdot x - w \cdot s$. The hyperplane $f(x) = 0$ is perpendicular to the line joining p to s and passes through s. Also $f(p) < 0$ by construction.

$p \notin C(S) \Rightarrow$ then there is a supporting hyperplane that separates p from S.



⊕ $f(p) < 0$.

$$\Rightarrow p \notin \bigcap_{f \in K} \{x \mid f(x) \geq 0\}.$$

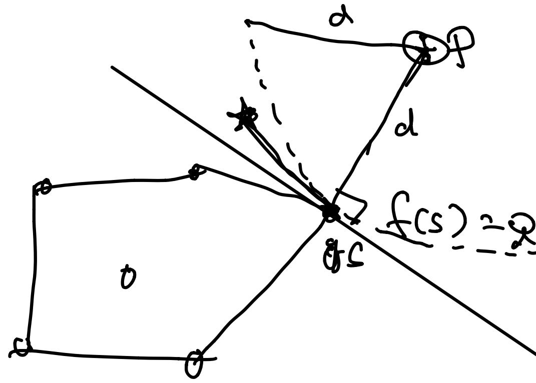


$w = s - p$

$f(x) = w \cdot x - w \cdot s$

$f(p) < 0$.

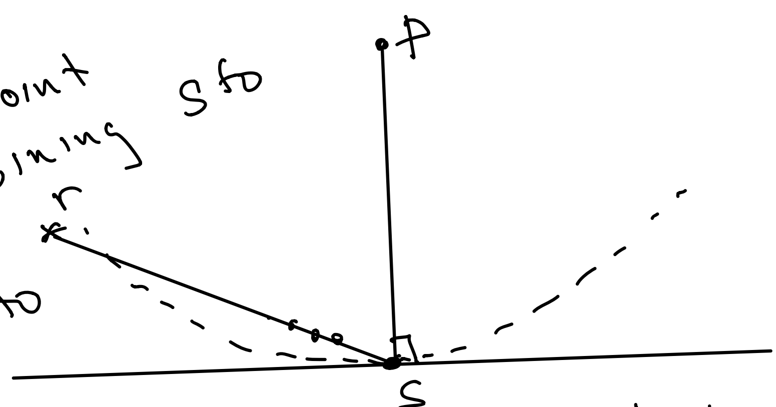
- We claim that $f(x) = 0$ is a supporting hyperplane for S. In other words, $f(x) \geq 0$ for all $x \in S$. Thus p is not in the intersection of the half spaces, which proves the proposition. To see this we draw a picture.



$f(p) < 0$

Suppose $r \in C(S)$
$f(r) < 0$.
line joining r to s is in $C(S)$.

$f(q) \geq 0$ for all $q \in C(S)$.



there is a point on line joining s to r that is closer to p than s

Contradicts fact that s is the closest pt!

$f(x) \geq 0 \quad \forall x \in C(S)$

so $f = 0$ is a supporting hyperplane.

$C(S) = \bigcap_{f \in K} \{x \mid f(x) \geq 0\}$

Convex Hulls of finite point sets are compact

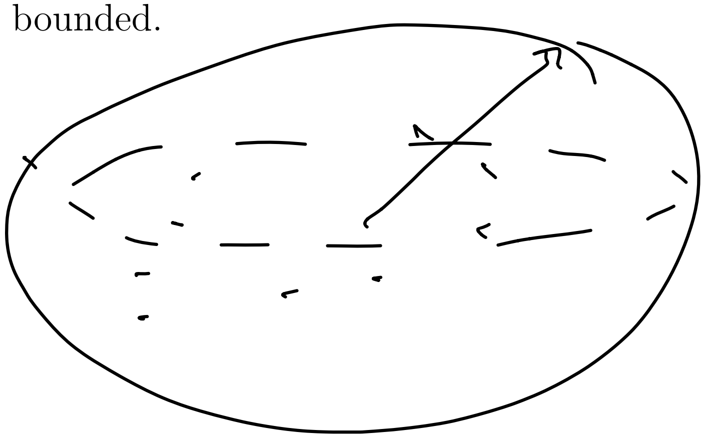
Proposition: $C(S)$ is compact.

Proof:

- It is an intersection of closed sets, therefore closed.

$f(x) \geq 0 \iff$ generalization of an interval $[a, \infty)$

- It is bounded.



$S \subseteq$ sufficiently large sphere.

$C(S) \subseteq$ large sphere

$\Rightarrow C(S)$ bounded.

Any continuous function on $C(S)$ attains a maximum and a minimum value.