Spectral Theorem for real symmetric matrices.

Suppose D is a real $K \times K$ symmetric matrix.

Then

1. All of the k eigenvalues of D are real. Furthermore if $u^T D u \geq 0$ for all vectors u then all eigenvalues are non-negative. $\lambda_1 \geq \cdots \geq \lambda_K \geq 0$. If $u^T D u \geq 0$ for all u, we say that D is positive semi-definite..

2. If u and v are eigenvectors for D with eigenvalues λ and λ', $\lambda \neq \lambda'$, then $u \cdot v = 0$.

3. There is an orthonormal basis $u_1, \ldots, u_K$ for $\mathbb{R}^K$ consisting of eigenvectors of D with eigenvalues $\lambda_1, \ldots, \lambda_K$.

4. Let

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_K \end{pmatrix}$$

let

$$P = \begin{pmatrix} u_1 & u_2 & \cdots & u_K \\ & & & \end{pmatrix}$$

$\|u_i\|^2 = 1$.

Then $D = P \Lambda P^T$.

$P P^T = $ identity

$P^T = P^{-1}$

P orthogonal matrix

Apply to D = covariance matrix.

There are $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_K \geq 0$. eigenvalues

with orthonormal eigenvectors

$u_1, \ldots \ldots, u_K$ $\qquad \|u_i\|^2 = 1$

$u_i \cdot u_j = 0$ if $i \neq j$.

Since $u^T D_0 u = \sigma_u^2 \geq 0 \Rightarrow$ all $\lambda_i \geq 0$.

u_i are called the <u>principal directions</u>

Suppose S is a score $\Leftrightarrow$ given by a vector in $\mathbb{R}^K$.

$$S = (S \cdot u_1) u_1 + \ldots + (S \cdot u_K) u_K$$

let $a_i = S \cdot u_i$

$$S = \sum a_i u_i$$

$$\sigma_S^2 = S^T D S = \left(\sum a_i u_i\right)^T D_0 \left(\sum a_i u_i\right)$$

$$= \left(\sum a_i u_i\right)^T \left(\sum a_i D_0 u_i\right)$$

$$= \left(\sum a_i u_i\right)^T \left(\sum a_i \underline{\lambda_i} u_i\right)$$

$$= \sum_{(i,j)} a_i a_j \lambda_j \underline{(u_i^T u_j)}$$ 0 unless $i = j$, 1 if $i = j$

$$= \sum_{i=1}^{K} a_i^2 \lambda_i$$

If we assume $\|S\|^2 = 1$

Maximize $\sigma_S^2 \Leftrightarrow$ maximize $\underline{\sum a_i^2 \lambda_i}$ $\qquad \lambda_1$

$\underline{a_1^2 + \ldots + a_K^2 = 1}$.

max happens when $a_1 = 1$, all others $= 0$ $\quad S = u_1 \quad \sigma_{u_1}^2 = \lambda_1$.