Bayesian Inference

Introduction

- Elements of Bayesian inference
  - a statistical model with parameters
  - a "prior distribution" on the parameters representing your state of knowledge about them
  - data arising from an experiment
  - an update to your prior distribution based on the experiment, leading to a "posterior distribution"
- Rough example
  - you have a thermometer that reports the true temperature up to a normally distributed error. This is your statistical model.
  - you have a prior sense that the external temperature is around 30 degrees, based on the time of day and the time of year. This is your prior distribution.
  - you make several independent measurements using your thermometer, and it reports temperatures scattered around 40 degrees.
  - You conclude that the temperature is probably closer to 40 than 30 based on this data.

Bayes Theorem and Bayesian Inference

Suppose that t is the temperature and D is the data that is the result of our experiment. The heart of Bayesian inference is Bayes theorem:

$$P(t|D) = \frac{P(D|t)P(t)}{P(D)}$$

- $P(t|D)$ is the distribution of the temperature *given* the observed data.
- $P(D|t)$ is the probability that we would have observed the data, *given* what we know about the temperature.
- $P(t)$ is the *prior* distribution on the temperature.
- $P(D)$ is the probability of the data given all possible temperatures. Often it amounts to a constant that we can ignore.

More details in a specific case

We will use temperature measurements. There are two parameters: the true temperature t_* and the variance σ^2 of the errors in measurement. The probability density for our temperature measurements is the normal distribution

$$p(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(t-t_*)/(2\sigma^2)} dt$$

We don't know either the true temperature t_* or the variance σ^2.

We conduct an experiment and obtain temperature values $\mathbf{t}_0 = (t_1, \ldots, t_n)$.

In this situation Bayes Theorem takes the following form

$$P(t_*, \sigma^2 | \mathbf{t} = \mathbf{t}_0) = \frac{P(\mathbf{t} = \mathbf{t}_0 | t_*, \sigma^2) P(t_*, \sigma^2)}{P(\mathbf{t} = \mathbf{t}_0}$$

- The left hand side $P(t_*, \sigma^2 | \mathbf{t} = \mathbf{t}_0)$ is the *posterior distribution* and it is the distribution on t^* and σ^2 *given the results of our experiment.*
- The probability $P(\mathbf{t} = \mathbf{t}_0 | t_*, \sigma^2)$ is the *likelihood* that we would have obtained the data we got depending on the values of t_* and σ^2.
- The probability $P(t_*, \sigma^2)$ is the *prior distribution* that reflects our initial impression of the value of these parameters.
- The denominator $P(\mathbf{t} = \mathbf{t}_0)$ is the total probability of the results of the experiment:

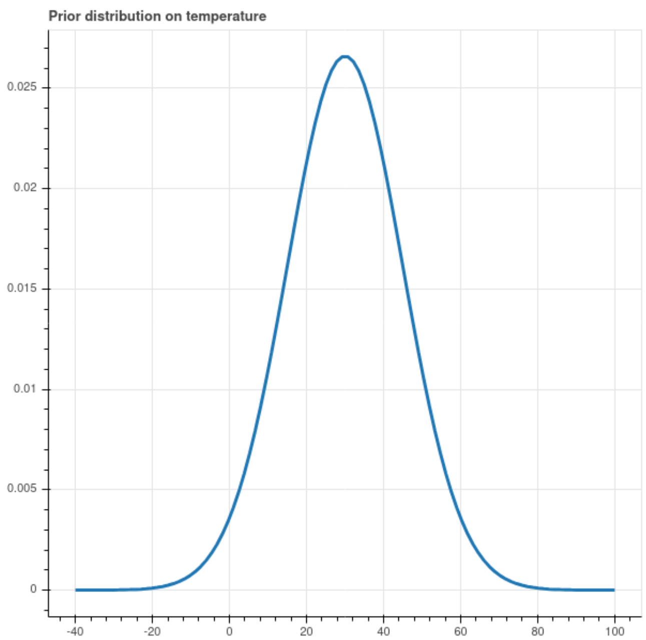
  $$P(\mathbf{t} = \mathbf{t}_0) = \int_{t_*, \sigma^2} P(\mathbf{t} = \mathbf{t}_0 | t_*, \sigma^2) P(t_*, \sigma^2)$$

  It functions as a normalizing constant and often we can get by without computing it at all.

Now we will do a worked example. We simplify the situation so that the variance in our thermometer $\sigma_*^2 = 1$.
Therefore the only unkown parameter is the true temperature t_*.

Prior Distribution

For our prior distribution, we assume that the average temperature is 30 degrees and the variance of is 15 degrees. That yields the following prior distribution.



The formula for this density is

$$P(t_*) = \frac{1}{\sqrt{30\pi}} e^{-(t_* - 30)^2/30}$$

Likelihood of the data

We make independent measurements $\mathbf{t}_0 = (t_1, \ldots, t_n)$.

The errors are $t_i - t_*$ where t_* is the true temperature. We have fixed the measurement variance at $\sigma^2 = 1$. Therefore

$$P(\mathbf{t} = \mathbf{t}_0 | t_*) = (\frac{1}{\sqrt{2\pi}})^n e^{-\|\mathbf{t}_0 - t_* \mathbf{e}\|^2/2}$$

where $\mathbf{e} = (1, 1, 1, \ldots, 1)$.

The total probability

The total probability is the integral

$$P(\mathbf{t}_0) = \int_{t_*} P(\mathbf{t} = \mathbf{t}_0 | t_*) P(t_*)$$

Let's just call this T and avoid it for the moment.

Bayes Theorem

If we combine up all the constants in Bayes Theorem and call them A, we have

$$P(t_*|\mathbf{t} = \mathbf{t}_0) = Ae^{-\|\mathbf{t}-t_*\mathbf{e}\|^2/2-(t_*-30)^3/30}$$

The exponent in the exponential is

$$\begin{aligned} Q &= \|\mathbf{t} - t_*\mathbf{e}\|^2/2 + (t^* - 30)^2/30 \\ &= (t_* - 30)^2/30 + \sum_i (t_i - t_*)^2/2 \end{aligned}$$

By expanding this out and completing the square, you can show that

$$Q = (t_* - U)^2/2V + K$$

where K is a constant that doesn't involve t_*,

$$U = \frac{2 + \sum_i t_i}{\frac{1}{15} + n}$$

and

$$V = \frac{1}{\frac{1}{15} + n}$$

The posterior density

The previous calculation shows that the posterior density (up to multiplicative constants B) has the form

$$P(t_*|\mathbf{t}=\mathbf{t}_0) = Be^{-(t_*-U)^2/2V}$$

In other words, *it is a normal distribution centered at U with variance V.*

Suppose we measured temperatures

$$40, 41, 39, 37, 44.$$

Then $n = 5$, the mean of these observations is 40.2 and the variance is 5.4

We have

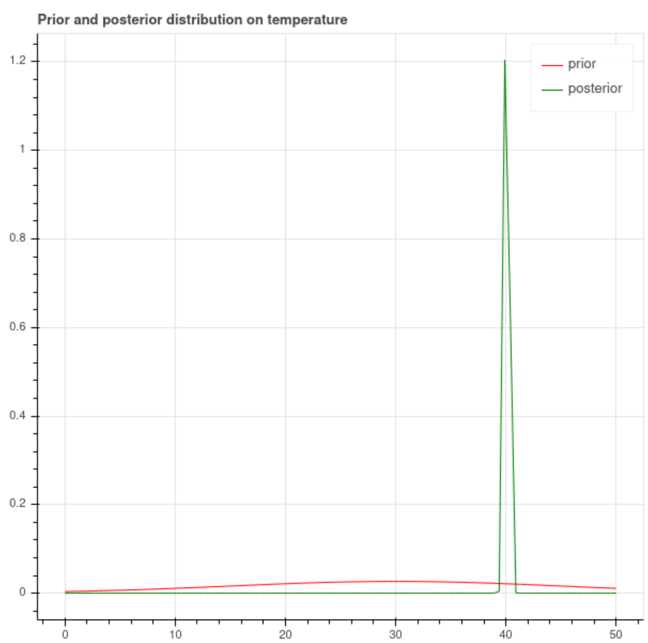
$$U = 40.1$$

and

$$V = 0.2$$

. The posterior mean is a bit less than the observed mean because our prior pulls it towards 30.

Notice in the formula for U and V that as $N \to \infty$ the posterior mean U approaches the sample mean $\frac{1}{n}\sum t_i$ and the variance approaches 0.



General Result

Proposition: Suppose that our statistical model for an experiment proposes that the measurements are normally distributed around an unknown mean value of μ with a fixed, known variance of σ^2. Suppose that our prior distribution on μ is also normal with mean μ_0 and variance τ^2. Finally imagine that we make measurements

$$y_1, \ldots, y_n.$$

The posterior distribution on μ is again normal, with posterior variance

$$\tau'^2 = \frac{1}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}$$

and posterior mean

$$\mu' = \frac{\frac{\mu_0}{\tau^2} + \frac{n}{\sigma^2}\overline{y}}{\tau'^2}$$

So the posterior mean is a weighted average of the sample mean and the prior mean, and as $n \to \infty$, the posterior mean approaches the sample mean and the prior has less and less influence on the interpretation of the experiment.