# Scores

## Scores

- A (linear) "score" is an artificially created feature that is a linear combination of existing features.

$$S_j = \sum_{i=1}^{K} a_i x_{ji}$$

$X$

N samples
K features

- A score is defined by $k$ weights

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$$

and the values of a score on the samples is computed by

$$\begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \end{bmatrix} = X_0 \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$$

$$\underset{N \times K}{\left( \quad \right)} \underset{K \times 1}{\begin{bmatrix} a_1 \\ \vdots \\ a_K \end{bmatrix}}$$

$N \times 1$

$$\begin{bmatrix} S_1 \\ \vdots \\ S_N \end{bmatrix} = a_1 X_0[:,1] + a_2 X_0[:,2] + \cdots + a_K X_0[:,K]$$

N students
we have totals on 2 MIDS and 1 final

midterms count 30%
Final counts 40%

$$\begin{array}{c c} & \begin{matrix} \text{MID}_1 & \text{MID}_2 & \text{FINAL} \end{matrix} \\ N & \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ \vdots & \vdots & \vdots \\ x_{N1} & x_{N2} & x_{N3} \end{pmatrix} \end{array} \times \begin{bmatrix} .3 \\ .3 \\ .4 \end{bmatrix} = \begin{bmatrix} \text{score}_1 \\ \vdots \\ \text{score}_N \end{bmatrix}$$

### Mean of scores

A linear combination of features with mean zero has mean zero.

$X_0$ has columns with mean zero.

$$S = X_0 \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$$

**Prop:** $\mu_S = 0$ $\quad \frac{1}{N}\sum_{i=1}^{N} S_i = 0.$

Pf

$$S = a_1 X_0[:,1] + a_2 X_0[:,2] + \dots + a_k X_0[:,k].$$

$$\sum_{i=1}^{N} S = a_1 \sum (X_0)_{i1} + a_2 \sum (X_0)_{i2} + \dots + a_k \sum (X_0)_{ik}$$

these are all zero since they are the column sums of $X_0$

$$S = \begin{bmatrix} S_1 \\ \vdots \\ S_N \end{bmatrix} \quad \sum_{i=1}^{N} S_i = 0.$$

## Variance and Covariance of scores

If $S$ is a score corresponding to a weight vector $a$, then

$$\sigma_S^2 = a^\intercal D_0 a.$$

If $S$ and $T$ are two scores corresponding to weights $a$ and $b$, then the covariance of $S$ and $T$ is given by

$$\sigma_{ST} = b^\intercal D_0 a.$$

$$S = \begin{bmatrix} S_1 \\ \vdots \\ S_N \end{bmatrix} \qquad \mu_S = 0$$

$$\sigma_S^2 = \frac{1}{N}\sum S_i^2 = \frac{1}{N}\begin{bmatrix} S_1 \\ \vdots \\ S_N \end{bmatrix} \cdot \begin{bmatrix} S_1 \\ \vdots \\ S_N \end{bmatrix}$$

$$a = \begin{bmatrix} a_1 \\ \vdots \\ a_K \end{bmatrix}$$

$$= \frac{1}{N}\begin{bmatrix} S_1 & \cdots & S_N \end{bmatrix} \cdot \begin{bmatrix} S_1 \\ \vdots \\ S_N \end{bmatrix}$$

$$= \underline{\frac{1}{N} S^T S}$$

$$\underline{S = X_0 a}$$

$$\boxed{(AB)^T = B^T A^T}$$

$$\sigma_S^2 = \frac{1}{N}(X_0 a)^T (X_0 a)$$

$$= \frac{1}{N} a^T X_0^T X_0 a$$

$$= a^T D_0 a = \sigma_S^2$$

$$S = X_0 \begin{bmatrix} a_1 \\ \vdots \\ a_K \end{bmatrix} = X_0 a$$

$$T = X_0 \begin{bmatrix} b_1 \\ \vdots \\ b_K \end{bmatrix} = X_0 b$$

$$\sigma_{ST} = \frac{1}{N}(b^T X_0^T X_0 a) = \boxed{b^T D_0 a}$$

**Prop:** $S$ is a scale with weights $\begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} = a$

$T$ " " " " " $\begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix} = b$

$$\sigma_S^2 = a^T D_0 a$$

$$\sigma_{ST} = b^T D_0 a = a^T D_0^T b = a^T D_0 b$$