Logistic Regression

The logistic model

The log-odds of an event increase linearly with an independent variable.

$$\log \frac{p}{1-p} = ax + b$$

Example: The chance that a person buys a product depends on how many times they encounter advertising for that product.

The sigmoid function

$$\log \frac{p}{1-p} = ax + b$$

means that

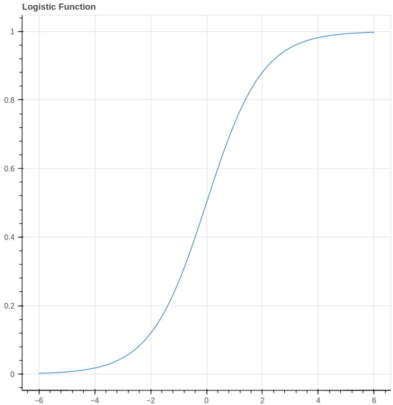
$$p(x) = \frac{1}{1 + e^{-ax-b}}$$

The logistic curve

The function

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

is called the logistic function.



Sample data

Likelihood of event increases with x. Out of 100 tries:

x	-3	-2	-1	0	1	2	3
Occurrences (out of 100)	10	18	38	50	69	78	86

Two points of view

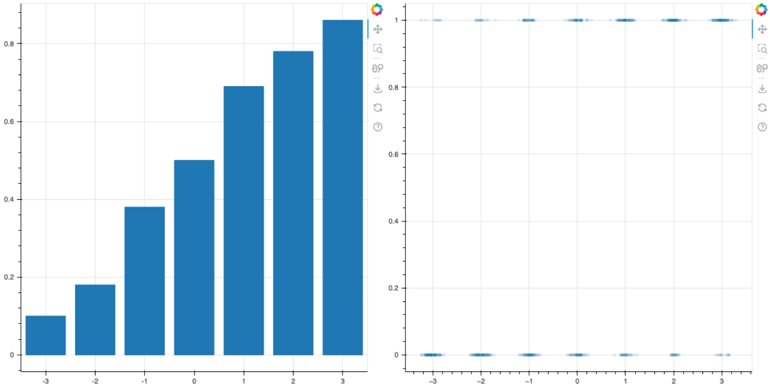


Figure 2: Logistic Data

The Likelihood

The parameters a and b are unknown. But if we knew them, then we could figure out how likely our results were. For example, the chance of getting 10 positive outcomes is

$$p(10+|x=-3, a, b) = C(\sigma(a(-3)+b)^{10}(1-\sigma(a(-3)+b))^{90}$$

where C is a constant (it's a binomial coefficient).

More on the likelihood

Assuming independence (given x, a, and b) the chance of our data is

$$P(\mathrm{data}|a,b) = CP(10+|x=-3)P(18+|x=-2)\cdots P(86+|x=3)$$

Still more

Here each term is

$$P(y + |x) = \sigma(ax + b)^y (1 - \sigma(ax + b)^{N(x) - y}$$

where $N(x)$ is the number of trials with that given x value. (this is a Binomial random variable).

The log likelihood

We want to find the a and b that make our observed data *most likely*. To do this we need to find a, b that maximize P or, more simply $\log P$.

$$\log P = \sum_{i=0}^{6} [y_i \log P(y_i|x_i) + (100 - y_i) \log(1 - p(y_i|x_i))]$$

We can drop the constant since it won't affect where the maximum occurs.

Vector/Regression Form

Our data matrix consists of N rows (and 1 column), one for each person viewing the ads. The entry in each row is the number of times they saw the add.

The target matrix consists of 0 and 1 depending on whether they made a purchase or not.

We want to “fit” an equation that gives 0 or 1 as a function of x, but we can’t do this exactly, only in probabilistic terms.

This is why it’s called “regression.”

More on Vector/Regression Form

For each row of our matrix, the chance that y_i is 1 is $p(x_i)$ (given by the sigmoid function with parameters a, b) and the chance that $y_i = 0$ is $(1 - p(x_i))$. So our likelihood is

$$L(a,b) = C \prod_{i=0}^{N-1} p(x_i)^{y_i} (1 - p(x_i))^{(1-y_i)}$$

and

$$\log L(a,b) = C' + \prod_{i=1}^{N-1} y_i \log p(x_i) + (1 - y_i) \log(1 - p(x_i)).$$

Ignoring irrelevant constants this is

$$\log L = Y \cdot \log p(X) + (1 - Y) \cdot \log(1 - p(X))$$

where for each row of X, $p(X)$ has $\sigma(ax_i + b)$ with (unknown)

The case of multiple features

In the case of multiple features, we have a set of k measurements for each sample (perhaps exposure to different types of ads) and a single outcome (buy/do not buy). This yields an $N \times k$ data matrix X. We seek a set of weights $m_1, \ldots, m_k$ and an "intercept" b so that

$$\log \frac{p}{1-p} = \sum m_i x_i + b$$

relates the log-odds of our event occurring with the values of the features.

Note: Just as with linear regression, we can create a "fake" feature that is all 1, and then extend our data matrix to $N \times (k+1)$. Then $b = m_{k+1}$ and we can write

$$\log \frac{P}{1-P} = XM$$

The probability

From this we get the matrix equation

$$P = \sigma(XM)$$

The matrix P has the probability of getting a positive outcome for each sample given the features.

A geometric remark

One way to think of this is that if the features (a row of X), thought of as a vector, points “more in the direction of the weight vector” M, then the probability of getting a positive outcome increases. If it’s perpendicular, you get even odds. If it points oppositve the weight vector, you’re unlikely to get what you want.

The target

We have a vector Y which records when our event happened, and when it didn't.

The log-likelihood

$$L(M) = Y^{\intercal} \log(\sigma(XM)) + (1 - Y^{\intercal})(1 - \log(\sigma(XM)))$$

Problem: Given X and Y, find M that maximizes this.

Credit card default

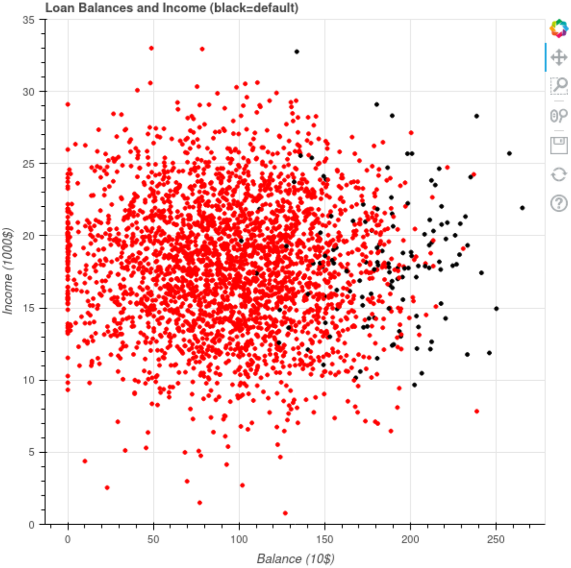


Figure 3: Default

Default with logistic line

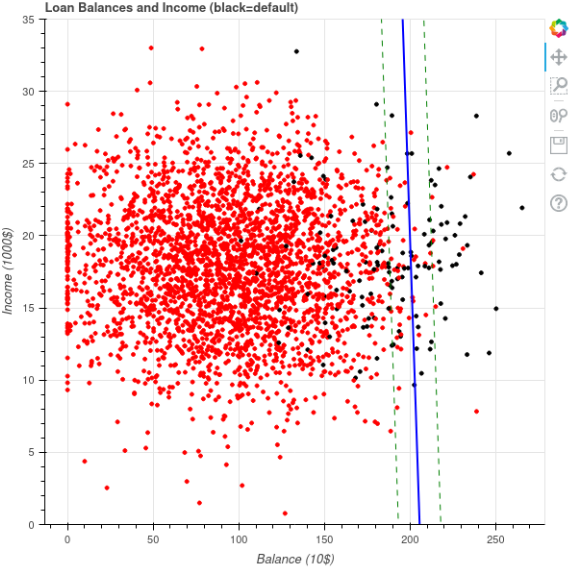


Figure 4: Default with line

Logistic regression for classification

We can use logistic regression for classification by fitting the logistic model and then saying that a point should be classified as 1 if the probability p given by the model says it is 1 with greater than .5 probability.

We could also set a more stringent requirement.

The "decision surface"

In the logistic regression model, for a given sample x with features x_i ($i = 1, \ldots, k+1$), the log odds of that sample yielding a "positive" result is

$$\log \frac{P}{1-P} = \sum x_i m_i$$

where the m_i are the weights. Notice that the equation $f(x) = \sum x_i m_i$ is a linear function of the features. The equation $f(x) = 0$ defines a "hyperplane" in feature space. (In the graph above, this is the blue line on the default data). On that line, it's even odds if the target is 1 or 0.

If $f(x) > 0$, the odds are better than even that the target value for that point is 1; and if $f(x) < 0$ the odds are less than even.

If you are trying to classify points, you could say points where $f(x) > 0$ should be classified as 1 (because, more likely than not, the model says that they are a 1).

An example of classification

The sklearn digits dataset consists of a large number of 8×8 bitmap images together with labels from 0 to 9. For example:

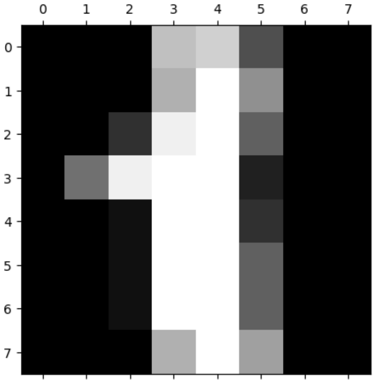


Figure 5: digit