X a sample space, P probability

f is a random variable

$$f : X \to \mathbb{R}.$$

Def: $E[f] = \int_X f(x)\, dP(x)$

- X is discrete

$$E[f] = \sum_{x \in X} f(x) P(x)$$

Bernoulli: parameter p $\quad X = \{H, T\}$ $\quad f(H) = 1$, $f(T) = 0$

$$E(f) = f(H)p + f(T)(1-p)$$
$$= p$$

$$E[f] = p.$$

~~Non~~ Continuous $X = \mathbb{R}$ $\quad P(U \subseteq \mathbb{R}) = \frac{1}{\sigma\sqrt{2\pi}} \int_U e^{-x^2/(2\sigma^2)}\, dx$

$U = [-\delta, \delta]$

$$P([-\delta, \delta]) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\delta}^{\delta} e^{-x^2/(2\sigma^2)}\, dx$$

$x : X \to \mathbb{R}$

$$E[x] = \frac{1}{\sigma\sqrt{2\pi}} \int_{X=\mathbb{R}} x e^{-x^2/2\sigma^2}\, dx \longleftarrow E[x] = 0$$

$$E[f] = \frac{1}{\sigma\sqrt{2\pi}} \int f(x) e^{-x^2/2\sigma^2}\, dx$$

<u>Prop</u>: $E[af+bg] = aE[f] + bE[g]$

f, g random variables, a, b constants

$$E[af+bg] = \sum_{x \in X} [af(x) + bg(x)] P(x) = a \sum_{x \in X} f(x)P(x) + b \sum_{X} g(x)P(x).$$

Continuous case

$$E[af+bg] = \int_X (af(x) + bg(x)) \underbrace{p(x)}_{\text{density}} dx = aE[f] + bE[g].$$

<u>Def</u>: Recall that A, B are independent if $P(A \cap B) = P(A)P(B)$.

<u>Discrete</u> f and g are independent random variables if

$$P(f = a \text{ and } g = b) = P(f=a)P(g=b) \quad \text{for all values } a, b.$$

$X = \{H, T\}^N$ $\quad f_1$ is 1 if first flip is Heads otherwise 0 $\quad f_2 = 1$ if 2^{nd} flip is heads 0 otherwise.

$$P(f_1 = a \text{ and } f_2 = b) = P(f_1 = a) P_2(f = b)$$

$$E[fg] = E[f]E[g].$$

What is $E[f]$ if f is binomial with parameters N, p.

N Bernoulli's with parameter

$$f = f_1 + \dots + f_N$$

$$E[f] = N E[f_i] = Np.$$

<u>Variance</u>: f is a random variable.

$$E[(f - E[f])^2] = \sigma^2, \text{ the variance.}$$

Lemma $\quad \sigma^2(f) = E[f^2] - E[f]^2$

$$(f - E[f])^2 = f^2 - \underline{2fE[f]} + E[f]^2$$

$$E[(f - E[f])^2] = E[f^2] - 2E[f]^2 + E[f]^2$$
$$= E[f^2] - E[f]^2.$$

Bernoulli:

$f^2 = f \qquad f^2(H) = 1 = f(H)$

$\qquad\qquad f^2(T) = 0 = f(T)$

$E[f^2] = E[f] = p$ Maximum

$\sigma^2 = p - p^2 = p(1-p).$ σ^2 at $p = 1/2$.

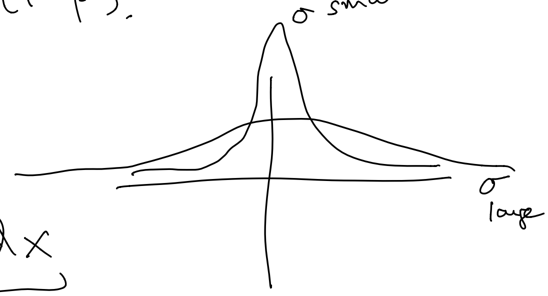
Binomial.

$$f = (f_1 + \cdots + f_N)$$
$$f^2 = f_1^2 + \cdots f_N^2 + \sum^{N(N-1)} f_i f_j$$
$$E(f^2) = N E[f_i^2] = Np + E\left[\sum f_i f_j\right]$$
$$= Np + N(N-1)p^2$$
$$E[f]^2 = N^2 p^2$$
$$\cancel{E}\ \sigma^2(f) = Np + N^2p^2 - Np^2 - N^2p^2$$
$$= Np(1-p).$$

Continuous case

X normal random variable

$E[X] = 0.$



function

$$E[X^2] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \underbrace{e^{-x^2/(2\sigma^2)}\,dx}_{p(x)dx}$$

$$\frac{d}{d\sigma}\,\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\,dx = \frac{d}{d\sigma} 1$$

... get answers ...

$$\sigma^2(x) = \sigma^2$$