Random Variables - continuous case

Random Variable: Continuous Case

- We make two independent measurements of temperature t, where the true temperature is t_0 and the errors $x = t - t_0$ are independent normal random variables with variance 1.

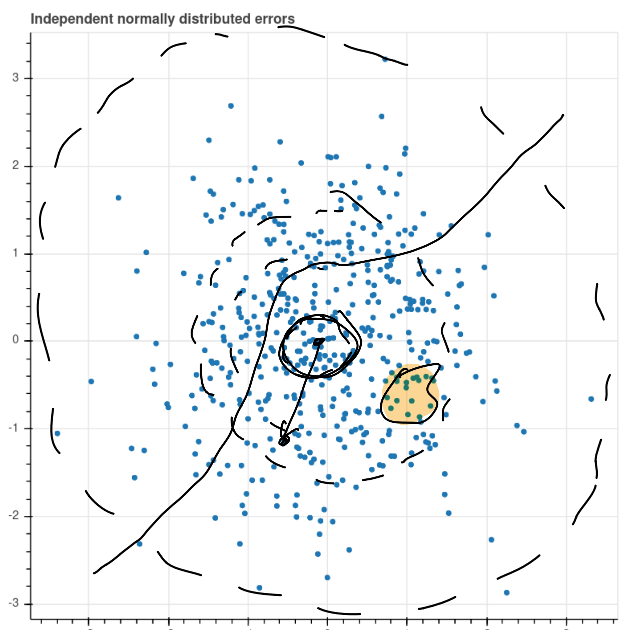
$\sigma = 1$

- The sample space $X = \mathbf{R}^2$ and the probability density is the multivariate gaussian

$$p(x) = \frac{1}{2\pi}e^{(-x_1^2-x_2^2)/2} = \frac{1}{2\pi}e^{-\|x\|^2/2}$$

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$\|x\|^2 = x_1^2 + x_2^2$



$p((x_1, x_2) \in U)$

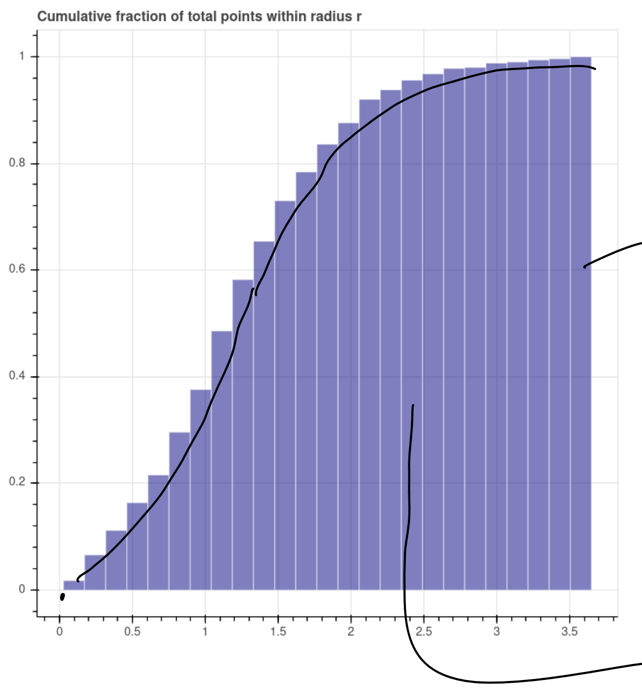
$= \int_U p(x)\,dx$

Distribution of norms

- How is $\|x\| = \sqrt{x_1^2 + x_2^2}$ distributed? $\|x\|$ is a random variable on X.

$\|x\|: \mathbb{R}^2 \to \mathbb{R}$

- What is the probability $P(\|x\| < r)$?
- Here is a histogram using the sample data above showing the distribution of the distances. Notice that as r increases, more and more of the points lie within distance r of the origin.



2.4 % of points lie within 2.4 of the origin

Cumulative histogram

$\|x\| < r$

$\|x\| \in [0, r)$

Distribution of norms (continued)

- By definition,

$$P(\|x\| < r) = P(\{(x_1, x_2) : x_1^2 + x_2^2 < r^2\}) = \boxed{\frac{1}{2\pi}\int_{\|x\|<r} \underbrace{e^{-\|x\|^2/2}}_{p(x)\,dx_1 dx_2} dx_1 dx_2}$$

- This is a doable integral using polar coordinates.

$\|x\| = \rho$

$dx_1 dx_2 = \rho\, d\rho\, d\theta$

$$\begin{aligned}\frac{1}{2\pi}\int_{\|x\|<r} e^{-\|x\|^2/2} &= \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{r} e^{-\rho^2/2}\rho\, d\rho\, d\theta \\ &= \int_{\rho=0}^{r} \rho e^{-\rho^2/2} d\rho \\ &= 1 - e^{-r^2/2}\end{aligned}$$

$P(\|x\| < r)$

Distribution of norms continued

- Here we superimpose our calculated cumulative density with the experimental data to see that they match.