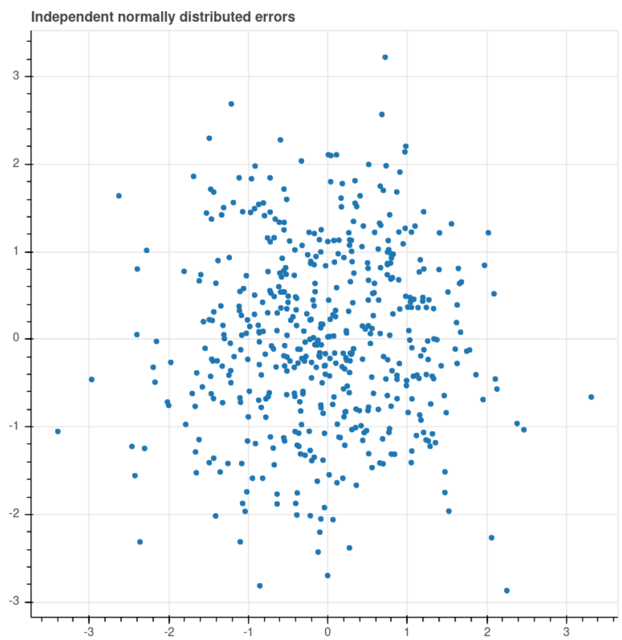
Random Variable: Continuous Case

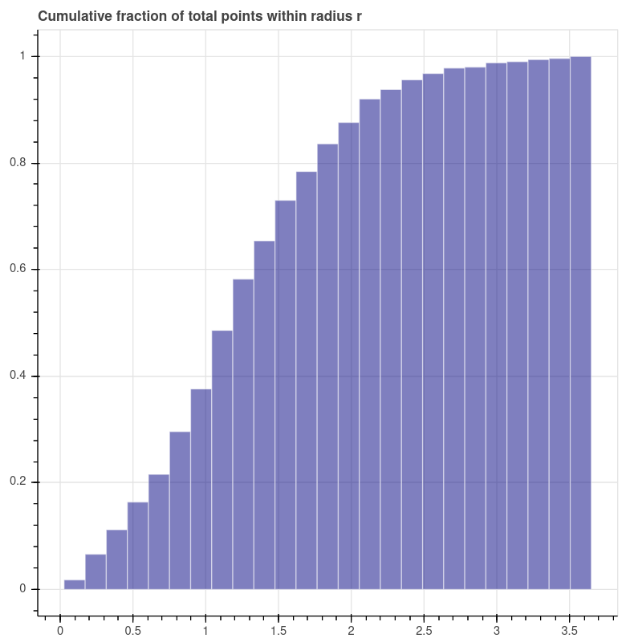
- We make two independent measurements of temperature t, where the true temperature is t_0 and the errors $x = t - t_0$ are independent normal random variables with variance 1.
- The sample space $X = \mathbf{R}^2$ and the probability density is the multivariate gaussian

$$p(x) = \frac{1}{2\pi} e^{(-x_1^2 - x_2^2)/2} = \frac{1}{2\pi} e^{-\|x\|^2/2}$$



Distribution of norms

- How is $\|x\| = \sqrt{x_1^2 + x_2^2}$ distributed? $\|x\|$ is a random variable on X.
- What is the probability $P(\|x\| < r)$?
- Here is a histogram using the sample data above showing the distribution of the distances. Notice that as r increases, more and more of the points lie within distance r of the origin.



Distribution of norms (continued)

- By definition,

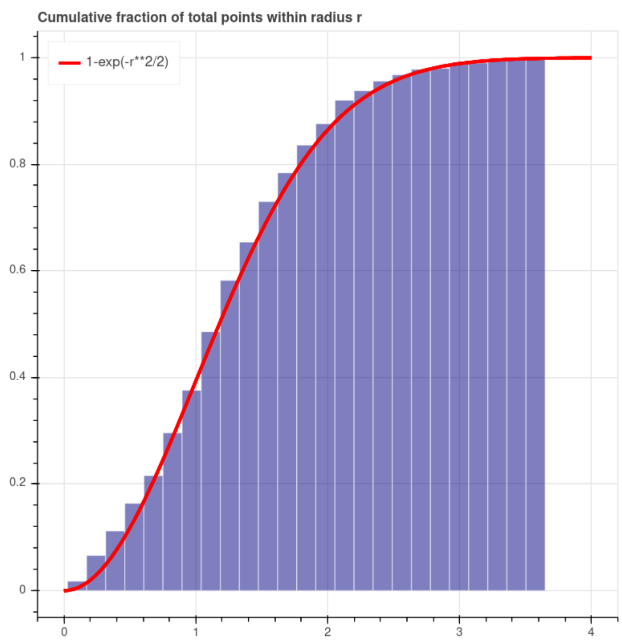
$$P(\|x\| < r) = P(\{(x_1, x_2) : x_1^2 + x_2^2 < r^2\}) = \frac{1}{2\pi} \int_{\|x\|<r} e^{-\|x\|^2/2}$$

- This is a doable integral using polar coordinates.

$$\begin{aligned}
\frac{1}{2\pi} \int_{\|x\|<r} e^{-\|x\|^2/2} &= \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{r} e^{-\rho^2/2} \rho d\rho d\theta \\
&= \int_{\rho=0}^{r} \rho e^{-\rho^2/2} d\rho \\
&= 1 - e^{-r^2/2}
\end{aligned}$$

Distribution of norms continued

- Here we superimpose our calculated cumulative density with the experimental data to see that they match.



Expected Value

- For the same of completeness, what is the expected value of the distance from the origin?

- By definition,

$$E[\|x\|] = \frac{1}{2\pi}\int_{\mathbf{R}^2} \|x\| e^{-\|x\|^2/2} dx_1 dx_2.$$

- In polar coordinates, this becomes

$$\begin{aligned} E[\|x\|] &= \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{\infty} \rho^2 e^{-\rho^2/2} d\rho d\theta \\ &= \int_{\rho=0}^{\infty} \rho^2 e^{-\rho^2/2} d\rho \end{aligned}$$

- This is a difficult integral but is in fact equal to $\sqrt{\pi/2}$, so this is the mean distance to the origin.