HW Set 4

Homework Set 4

Instructions: These problems are due November 20.

Problem 1.

In $\mathbb{Z}[\sqrt{5}]$, the ideal $\mathbf{p} = (2, 1+\sqrt{5})$ is maximal (and thus prime), and $\mathbb{Z}[\sqrt{5}]/\mathbf{p}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

1. Show that $X^3 - \sqrt{5}X + 1 - 2\sqrt{5}$ is irreducible mod $\mathbf{p}$ in $\mathbb{Z}[\sqrt{5}]$.
2. Show that $\mathbf{p}^2 = \{a + b\sqrt{5} : a, b \text{ both even}\}$.
3. Show that $X^4 + 2(1-\sqrt{5})X^2 - 6X + 3 + \sqrt{5}$ is an Eisenstein polynomial in $\mathbb{Z}[\sqrt{5}][X]$ at the prime $\mathbf{p}$.
4. Show that $X^n - (7 + 3\sqrt{5})$ is Eisenstein at $\mathbf{p}$ for every $n \geq 1$.
5. If $n \geq 2$ and $d \geq 1$, prove that $X_1^d + X_2^d + \cdots X_n^d - 1$ is irreducible in $\mathbb{Q}[X_1, \ldots, X_n]$. (Hint: For fixed d, use induction on n and the Eisenstein Criterion in its more general form as in DF Proposition 13 pg. 309.)

Problem 2.

Compute the matrix representation of the following linear maps with respect to the indicated ordered basis. In each case the field is $\mathbb{R}$.

1. $V = \mathbb{C}$; $m : V \to V$ is $m(x) = (2-i)x$; $B = \{1+i, 3i\}$.
2. $V = \mathbb{R}[x]/(x^3 - 1)$; $m : V \to V$ is $m(v) = xv$; $B = \{1, x, x^2\}$.
3. V is the space of solutions to the differential equation $y'' + y' + y = 0$; $m : V \to V$ is $m(y) = y'$; you choose your basis.

Problem 3.

DF Problem 12 on page 302. This problem constructs an integral domain R which has the property that any *finitely generated* ideal in R is principal; but R has some ideals which are not finitely generated. Such a ring is called a Bezout ring. In such a ring, any two elements a and b have a gcd d such that $ax + by = d$, but you don't have unique factorization because the other requirement – that every element is a *finite* product of irreducibles – fails. In the ring you construct in this problem, there is an element x which is not a unit but has roots of arbitrarily high order.

See Theorem 14 in DF on page 287 to see how, in the case when R is a PID (in which *every* ideal is principal, not just the finitely generated ones), one has to use Zorn's lemma to prove that every non-zero non-unit in R is a *finite* product of irreducibles.

Problem 4.

For $n \geq 1$, Let $\mathrm{Pol}_n(\mathbb{R})$ be the space of polynomials with real coefficients and degree *at most* n. So Pol_n is a vector space of dimension n over $\mathbb{R}$. Let $D : \mathrm{Pol}_n(R) \to \mathrm{Pol}_n(\mathbb{R})$ be the map $\frac{d}{dx}$, let D^j be the j^{th} derivative, and let $\mathcal{D}_n(\mathbb{R})$ be the vector space of linear differential operators in one variable with constant coefficients over $\mathbb{R}$. So an element of $\mathcal{D}_n(\mathbb{R})$ is a polynomial in D of degree at most n. Note for the sake of clarity that D^0 is the identity map, so

$$(2 - D + D^2)(x^3) = 2x^3 - 3x^2 + 6x$$

with the initial 2 acting on a polynomial just by multiplication.

1. For $f = a_0 + a_1 X + \cdots + a_n X^n$, and $L = b_0 + b_1 D + \cdots + b_n D^n$, compute $(Lf)(0)$ in terms of the coefficients of f and L.
2. For $L \in \mathcal{D}_n(\mathbb{R})$, define $L_0 : \mathrm{Pol}_n(\mathbb{R}) \to \mathbb{R}$ by $L_0(f) = (Lf)(0)$. Show that the map $L \to L_0$ is an isomorphism of vector spaces from $\mathcal{D}_n$ to the dual space $\mathrm{Pol}_n(R)^\vee$. (Check the dimension of $\mathcal{D}_n(\mathbb{R})$; prove the map is linear; check its kernel.)
3. Find the basis of $\mathcal{D}_n(\mathbb{R})$ dual to the standard basis $1, X, \ldots, X^n$ of $\mathrm{Pol}_n(\mathbb{R})$.
4. Let $H(f) = f(1)$ and $G(f) = \int_0^1 f(X)dX$. Both H and G are elements of the dual space to $\mathrm{Pol}_n(\mathbb{R})$ and therefore correspond to elements of $\mathcal{D}_n(\mathbb{R})$. What are those elements?

Problem 5.

Let V be a vector space of dimension n over a field F. A *complete flag* in V is a sequence of subspaces

$$Z : W_0 = (0) \subset W_1 \subset W_2 \subset \cdots \subset W_{n-1} \subset W_n = V$$

where W_i has dimension i. So for example, in $\mathbb{R}^3$, one could choose W_1 to be the span of the vector $\mathbf{i}$ (the x-axis) and W_2 to be the span of the x and y axes (the xy-plane).

The group $\mathrm{GL}(V)$ acts on the flags in V by acting on the subspaces:

$$gZ = gW_0 = (0) \subset gW_1 \subset \cdots \subset gW_{n-1} \subset W_n = V.$$

1. Prove that the action of $\mathrm{GL}(V)$ on the flags is transitive.
2. Fix a basis $a_1, \ldots, a_n$ for V and let Z be the standard flag where $W_0 = 0$ and $W_i = \mathrm{span}(a_1, \ldots, a_i)$ for $i = 1, \ldots, n$. Prove that $g \in \mathrm{GL}(V)$ stabilizes Z if and only if g is upper triangular in the matrix representation coming from the choice of basis $\{a_i\}$.
3. Use the orbit stabilizer theorem for $\mathrm{GL}(V)$ to give a formula for the number of flags in a vector space of dimension n over a field with q elements.
4. Find the number of flags in the three dimensional vector space over $\mathbb{Z}/2\mathbb{Z}$.