Day 21

Change of Basis

Given vector spaces V and W of dimension n and m respectively, and bases $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ for V and W, a linear map $L : V \to W$ has an $m \times n$ matrix representation

$$[L]_A^B = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

where the c_{ij} are defined by

$$L(a_i) = \sum_{j=1}^{m} c_{ji} b_j.$$

If $v = \sum x_i a_i \in V$, define an $n \times 1$ matrix

$$[v]_A = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

We can view this as an isomorphism from V to F^n if we write our elements of F^n as column vectors. We can do the same construction for W and F^m, yielding $[w]_B$.

Then

$$[Lv]_B = [L]_A^B [v]_A$$

meaning that the matrix representation turns the map into matrix multiplication.

More generally if $L : V \to W$ and $H : W \to K$ are linear maps, and A, B, C are bases for V, W, K then

$$[H \circ L]_A^C = [H]_B^C [L]_A^B.$$

Now suppose we choose different bases A' and B' for V and W.

There is a unique invertible linear map $G : V \to V$ which satisfies $G(a_i') = a_i$ for $i = 1, \ldots, n$. This means that if

$$v = \sum x_i a_i'$$

then

$$G(v) = \sum x_i a_i.$$

Important: In this convention, the *inverse* of the linear map G carries a_i to a_i', so if you look at its matrix $[G^{-1}]_A^A$ in the basis A, its columns give the coordinates of the new basis in terms of the old. This is in some ways the more natural thing to consider.

Now

$$[Gv]_A = [v]_{A'}.$$

Since $[Gv]_A = [G]_A^A [v]_A$ we see that

$$[v]_{A'} = [G]_A^A [v]_A.$$

Now given a linear map $L : V \to W$ and basis A' and B' for V and W, with:

- G the map carrying a_i' to a_i, (note the convention here. The columns of G^{-1} express the new basis in terms of the old one.)
- H the map carrying b_i' to b_i.

If $L(v) = w$ we know that the matrix for L is characterized by

$$[L]_{A'}^{B'} [v]_{A'} = [w]_{B'}.$$

Then

$$[L]_{A'}^{B'} [G]_A^A [v]_A = [H]_B^B [w]_B$$

so

$$[L]_{A'}^{B'} = (H_B^B)^{-1} [L]_A^B G_A^A.$$

So a change of basis on the source and target modifies the matrix of the linear map by left- and right- multiplication by invertible matrices.

Example: Lagrange interpolation

Given $n+1$ points $x_0, \ldots, x_n$ in $\mathbb{R}$, there is a polynomial of degree n with prescribed values $f(x_i) = a_i$

Let

$$f_i(x) = \frac{(x-x_0)(x-x_1)\cdots\widehat{(x-x_i)}\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_n)}$$

This polynomial vanishes on all the x's except x_i, where it takes value 1. These are linearly independent and

$$f = \sum a_i f_i$$

is the desired expression.

Now x^k takes the value x_i^k at x_i, so

$$x^k = \sum x_i^k f_i$$

So the basis consisting of the powers of x, expressed in terms of the x_i, is the matrix whose k^{th} column, i^{th} row is x_i^k. Call this matrix G.

Let D be the matrix giving the derivative operator on polynomials of degree n in the standard basis $1, x, \ldots, x^n$.

Then $G^{-1}DG$ expresses the derivative operator in terms of the basis f_i. In practice this tells you have who to compute derivatives from values of polynomials at chosen points.

See *Inverses of Vandermonde Matrices*, by N. Macon and A. Spitzbart, American Math Monthly 1958 vol 65 number 2.

Duality

Linear forms

A *linear form* or a *linear functional* on a vector space V over a field F is a linear map $h : V \to F$.

- The space $\mathrm{Hom}(V, F)$ of linear forms is a vector space called the **dual vector space** to V. DF use the notation V^* for $\mathrm{Hom}(V, F)$.

Suppose that S is a basis for V (not necessarily finite). For each $s \in S$, define a linear map $s^* : V \to F$ by saying that $s^*(s) = 1$ and $s^*(x) = 0$ for any other

$x \in S$; then extending s^* by linearity to all of V. *Note that the linear map s^* depends on all of S, not just on s itself.*

If V is finite dimensional, and $s_1, \ldots, s_n$ is a basis for V, then $s_1^*, \ldots, s_n^*$ is a basis for V^* called the dual basis. To show that it spans V^*, let f be any linear form and compute $f(s_i)$. Then

$$f = \sum f(s_i) s_i^*$$

since the right side agrees with f on the basis s_i. The s_i^* are linearly independent since if

$$\sum a_i s_i^* = 0$$

Then

$$(\sum a_i s_i^*)(s_i) = 0 = a_i$$

for all i.

If V is infinite dimensional and S is a basis, you can still construct elements s^* dual to the elements of S and they are linearly independent. However they won't span.

Dual transformations

Suppose $L : V \to W$ is a linear map. Then there is a "dual map" $L^* : W^* \to V^*$ defined abstractly by setting $((L^*)(f))(v) = f(L(v))$.

Proposition: If A is a finite basis for V and B is a finite basis for W then let A^* and B^* be the corresponding dual bases. Then $[L^*]_{B^*}^{A^*}$ is the transpose of $[L]_A^B$.

Proof: Consider b_i^* in B^*. Then

$$L^*(b_j^*)(a_i) = b_j^*(L(a_i))$$

which is the coefficient of b_j in the expansion of $L(a_i)$. This is by definition the entry x_{ji} in the matrix of L relative to the bases A and B.

On the other hand, if we write

$$L^*(b_j^*) = \sum y_{ij} a_i^*$$

where y_{ij} are the matrix entries of L^* relative to the dual bases, then we see that $x_{ji} = y_{ij}$. In other words, the matrix entries for L^* are those for L, but with rows and columns interchanged.

Let $H \subset V$ be a subspace. Then any linear form on V restricts to one on H, so there is a map $V^* \to H^*$. This map is surjective since any linear form on H extends to one on V. It's kernel is the set of linear forms on V that vanish on H; this is called the "annihilator of H".

Corollary: The row and column ranks of a matrix coincide.

Proof: Let $L : V \to W$ and $L^* : W^* \to V^*$ be a linear map and its dual. L gives an isomorphism from V/K to the image H of L in W where K is the kernel of L. So we can view $L : V/K \to H$.

The dual transform L^* takes a linear form on W and makes it one on V by the formula $L^*(f)(v) = f(L(v))$. Since $L(v) \in H \subset W$, $L^*(f)$ is determined by its values on H and the kernel of L^* is the annihilator of H. Therefore the image of L^* is H^*. But H^* and H have the same dimension, so the rank of L^* and the rank of L are the same. However, the rank of L^* is the column rank of its matrix representation, which is the row rank of the matrix of L.

Some remarks on analysis

In very rough terms, the Riesz Representation theorem says that if X is a compact hausdorff space and C(X) are the continuous functions on X, then any(positive) *continuous* linear form $a : C(X) \to R$ there is a unique Borel measure μ satisfying some extra properties such that $a(f) = \int f(x)d\mu$.

Roughly speaking, *continuous* linear forms are the same as measures.

Continuity is essential here; the space of continuous linear forms is MUCH SMALLER than the space of all linear forms. Here the topology on $C(X)$ is the metric topology given by the sup norm.

"Functional Analysis" is the study of possible topologies on vector spaces and their relationship to spaces of linear forms.

The double dual

The "Double dual" space $V^{**} = \mathrm{Hom}(\mathrm{Hom}(V, F), F)$ is another vector space. Notice that if V is of dimension n then V^{**} is also of dimension n. But the relationship is closer than that.

Given v in V, define e_v in V^{**} by $e_v(f) = f(v)$. This is called the "evaluation map".

If V is finite dimensional, e is an isomorphism.

In general, the evaluation map is injective but far from surjective.