

## Comments on HW set 3

### Problem 2

We want to show that there is exactly one group  $G$  of order 75 up to isomorphism.

If  $G$  has order 75, then it has a Sylow 5-subgroup  $H$  of order 25 and a Sylow 3 subgroup  $K$  of order 3. Notice that  $H \cap K$  must be trivial since  $H$  and  $K$  have coprime orders.

The number  $n_5$  of Sylow 5-subgroups must both divide 3 and also be congruent to one mod 5, so  $n_5 = 1$ . This means  $H$  is normal. Since all groups of order  $p^2$ , where  $p$  is prime, are abelian,  $H$  must be abelian. Therefore  $H$  is either  $\mathbb{Z}/25\mathbb{Z}$  or  $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ .

Meanwhile, the number  $n_3$  of Sylow 3-subgroups must divide 25 and be congruent to 1 mod 3. Therefore either  $n_3 = 1$  or  $n_3 = 25$ . If  $n_3 = 1$ , then  $K$  is normal. In this case,  $HK$  is abelian and so  $G$  is either  $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$  or  $\mathbb{Z}/75\mathbb{Z}$ .

If  $n_3 = 25$ , then  $K$  is one of 25 conjugate subgroups of  $G$  of order 3 and  $G = HK$  is the semidirect product of  $H$  with  $K$  arising from a map  $K \rightarrow \text{Aut}(H)$ .

If  $H = \mathbb{Z}/25\mathbb{Z}$ , then  $\text{Aut}(H)$  is of order  $\phi(25) = 20$ , and there are no non-trivial maps from  $K$  to  $\text{Aut}(H)$ , so in this case the only group is the abelian group  $\mathbb{Z}/75\mathbb{Z}$ .

If  $H = \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ , then  $\text{Aut}(H)$  is  $\text{GL}_2(\mathbb{Z}/5\mathbb{Z})$  which has order 480. By Cauchy's theorem,  $\text{Aut}(H)$  has an element  $\sigma$  of order 3. We can fix an isomorphism of  $K$  with  $\mathbb{Z}/3\mathbb{Z}$  and define a map  $\mathbb{Z}/3\mathbb{Z} \rightarrow \text{Aut}(H)$  sending 1 to  $\sigma$ . The resulting semidirect product is nonabelian.

Now we need to prove that this semi-direct product is unique up to isomorphism. We can use problem 1 for this, but it is a little subtle.

Looking at the construction of the semi-direct product, we needed to choose an element  $\sigma$  of order 3 in  $\text{GL}_2(\mathbb{Z}/5\mathbb{Z})$  and then we defined

$$\phi : \mathbb{Z}/3\mathbb{Z} \rightarrow \text{GL}_2(\mathbb{Z}/5\mathbb{Z})$$

by sending  $\phi(1) = \sigma$ . The element  $\sigma$  generates a Sylow 3-subgroup in  $\text{GL}_2(\mathbb{Z}/5\mathbb{Z})$ ; call this  $A(\sigma) \subset \text{GL}_2(\mathbb{Z}/5\mathbb{Z})$ .

The first thing to notice is that, by Sylow's theorem (which is stronger than Cauchy's Theorem), every Sylow 3-subgroup is of order 3 and they are all conjugate. This is because 480 is divisible by 3 but not 9, so these elements of order 3 actually generate Sylow 3-subgroups, all of which are conjugate.

Suppose we had chosen a *different* element of order 3 in our construction, say  $\sigma'$ . Then the subgroup generated by  $\sigma'$  would be conjugate to  $A(\sigma)$ . This means we can find an automorphism  $g$  in  $\text{GL}_2(\mathbb{Z}/5\mathbb{Z})$  that conjugates  $A(\sigma')$  into  $A(\sigma)$ . In other words,  $gA(\sigma')g^{-2} = A(\sigma)$ . *However*, it could happen that when we do

this conjugation, that  $g\sigma'g^{-1} = \sigma^{-1}$  because both  $\sigma$  and  $\sigma^{-1}$  generate  $A(\sigma)$ . However, we have an automorphism  $f : A(\sigma) \rightarrow A(\sigma)$  such that  $f(\sigma^{-1}) = \sigma$ .

In other words, any two nontrivial maps  $\phi, \tau : \mathbb{Z}/3\mathbb{Z} \rightarrow \text{Aut}(\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z})$  can be changed into each other by finding  $g \in \text{Aut}(\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z})$  and  $f \in \text{Aut}(\mathbb{Z}/3\mathbb{Z})$  so that

$$\phi = \gamma_g \circ \tau \circ f$$

and therefore by problem 1 the corresponding groups are isomorphic.