3. Group morphisms and group actions

The isomorphism theorems

Theorem: (See DF Theorem 3.16) Let $f : G \to K$ be a homomorphism of groups, let N be the kernel of f, and let $\pi : G \to G/N$ be the canonical projection. Then there is a unique *injective* homomorphism $\overline{f} : G/N \to K$ such that $\overline{f} \circ \pi = f$.

$$\begin{tikzcd}
G \arrow[d, "\pi"] \arrow[dr, "f"] & \\
G/N \arrow[r, "\overline{f}"'] & K
\end{tikzcd}$$

We sometimes say that "f factors through π" or "f factors through G/N".

Theorem: (See DF Theorem 3.18) Suppose that G is a group and A and B are subgroups of G. Suppose further that A is a subgroup of $N_G(B)$ so that AB is a subgroup of G. Then 1. B is normal in AB. 2. $A \cap B$ is normal in A. 3. AB/B is isomorphic to $A/(A \cap B)$.

$$\begin{tikzcd}
& G & \\
& AB \arrow[u] & \\
A \arrow[ur] & & B \arrow[ul, "="'] \\
& A \cap B \arrow[ul, "="] \arrow[ur] & \\
& 1 \arrow[u] &
\end{tikzcd}$$

The arrows marked with "=" are inclusions of normal subgroups, and the corresponding quotients are isomorphic.

Theorem: (See DF, THeorem 3.19) Let G be a group, and suppose that H and K are normal subgroups of G and H is normal in K. Then K/H is normal in G/H and $(G/H)/(K/H)$ is isomorphic to G/K.

Theorem: (See DF, Theorem 3.20) Let G be a group and N be a normal subgroup of G. Then map $A \mapsto A/N$ is a bijection between the set of all subgroups of G/N and the set of subgroups of G containing N. Furthermore, if A and B are subgroups of G containing N, then 1. $A \subset B$ if and only if $A/N \subset B/N$ 2. If $A \subset B$ then $[A : B] = [A/N : B/N]$ 3. $\langle A, B\rangle/N = \langle A/N, B/N\rangle$ 4. $(A \cap B)/N = (A/N \cap B/N)$ 5. A is normal in G if and only if A/N is normal in G/N.

In other words, the lattice of subgroups of G/N is exactly the sublattice of the lattice of subgroups of G containing N.

Group Actions

Definition: Let G be a group and X be a set. A action of G on X is a map

$$a : G \times X \to X$$

that satisfies $a(e, x) = x$ for all x and $a(g, a(h, x)) = a(gh, x)$ for all $g, h \in G$ and $x \in X$. (*Remark:* We usually write gx or $g \cdot x$ instead of referring to the map a)

Equivalently, an action of G on X is a homomorphism $f : G \to S(X)$.

Note: Whenever we have a function $f : A \times B \to C$ we can think of it equivalently as a function $f : A \to \mathcal{F}(B, C)$ where $\mathcal{F}(B, C)$ is the set of all functions from B to C. The point is that we can take our function $f : A \times B \to C$, which is a function of two variables, and define $\tilde{f} : A \to \mathcal{F}(B, C)$ by definining $\tilde{f}(a)$ to be the function $\tilde{f}(a)(b) = f(a, b)$. Conversely, if $h : A \to \mathcal{F}(B, C)$ is a function, we can make a function $\overline{h} : A \times B \to C$ by setting $\overline{h}(a, b) = h(a)(b)$. These are mutually inverse constructions so $\mathcal{F}(A \times B, C) = \mathcal{F}(A, \mathcal{F}(B, C))$. This is a property of the cartesian product called *adjointness* or more specifically *left adjointness.*

Key Terminology

1. Let $x \in X$. The set $Gx = \{gx : g \in G\} \subset X$ is called the **orbit** of x. More generally, if H is a subgroup of G then Hx, defined similarly, is the orbit of x under H.
2. Let $x \in X$. The set $\mathrm{Stab}_G(x) = \{g : gx = x\} \subset G$ is called the **stabilizer** of x. It is a subgroup of G.
3. An action is called *transitive* if there is an $x \in X$ so that $X = Gx$.
4. The set of $g \in G$ such that $gx = x$ for all $x \in X$ is the *kernel* of the action.

5. An action is *faithful* if its kernel is trivial; in other words, if the corresponding map $G \to S(X)$ is injective.
6. If G acts on X and Y, a map $f : X \to Y$ is called a morphism of actions if $f(gx) = gf(x)$. If f is bijective then it is an isomorphism of actions.

Key formalities

1. If G acts on X, the action partitions X into a disjoint union of orbits. These can be seen as the equivalence classes for the equivalence relation $x \sim y \iff x = gy$ for some $g \in G$.
2. The action of G on each orbit is transitive (by definition).
3. Given $x \in X$, the map
$$\pi : gH \mapsto gx$$
gives a well-defined bijection between the cosets of $H = G/\mathrm{Stab}_G(x)$ and the orbit Gx.
This bijection is an isomorphism of group actions, since if $H = \mathrm{Stab}_G(x)$, then $k\pi(gH) = kgx = \pi(kgH)$.

If G is finite, the size of each orbit is a divisor of the order of G.

Key examples

1. If G is a group, and H is a subgroup, let X be the set of left cosets of H in G (regardless of whether H is normal). Then G acts on X via $g \cdot kH = gkH$. The set X is called a *homogeneous space* for G and is sometimes written G/H even when H isn't normal. Property 3 under "formalities" says that *every orbit in a group action is isomorphic to a homogeneous space for the group.* Notice that if H is the trivial subgroup, then this is the action of G on itself by left multiplication; this is called the (left) *regular* action.+-
2. If G is a group, then G acts on itself via conjugation: $g \cdot h = ghg^{-1}$. The orbits are called *conjugacy classes.* The stabilizer of an element g under conjugation is the centralizer $C_G(\{g\})$ and the index of this stabilizer is the size of the conjugacy class of g.
3. If $g \in Z(G)$ is an element of the center of G, then it forms a one-element conjugacy class and its centralizer is all of G.

The class equation

Theorem: Let G be a finite group. Let G act on itself by conjugation, yielding a partition of G into disjoint conjugacy classes $K_1, \ldots, K_g$. Choose a representative g_i for each class. Then

$$\mid G \mid = \sum_{i=1}^{g} \mid K_i \mid = \sum_{i=1}^{g} [G : C_G(g_i)].$$

Grouping the conjugacy classes of size one together, we can rewrite this as

$$| G |=| Z(G) | + \sum_{\{i:|K_i|>1\}} [G : C_G(g_i)]$$

This is called **the class equation**.

Automorphisms

If G is a group, the automorphism group $\mathrm{Aut}(G)$ of G is the set of isomorphisms $G \to G$, with group operation given by composition of functions.

If $G = \mathbb{Z}/n\mathbb{Z}$ then $\mathrm{Aut}(G)$ is $(\mathbb{Z}/n\mathbb{Z})^*$, the multiplicative group of elements mod n that are relatively prime to n.

If $G = \mathbb{Z}/n\mathbb{Z}^k$, then $\mathrm{Aut}(G)$ is $\mathrm{GL}_n(\mathbb{Z}/n\mathbb{Z})$, the group of $n \times n$ matrices with entries in $\mathbb{Z}/n\mathbb{Z}$ that are invertible (meaning their determinant is relatively prime to n).

For $g \in G$, conjugation by g is an automorphism of G. This gives a homomorphism $G \to \mathrm{Aut}(G)$. THe kernel of this map is the center of G. The image is called the group of *inner automorphisms.* The inner automorphisms form a normal subgroup of the automorphism group.

A group G acts on a normal subgroup H by conjugation. The centralizer of H is the kernel of the action. Therefore $G/C_G(H)$ is a subgroup of $\mathrm{Aut}(H)$. And $G/Z(G)$ is a subgroup of $\mathrm{Aut}(H)$.

Definition: A subgroup H of G is called *characteristic* if it is fixed by *every* automorphism of G, not just the inner ones.

Weird fact: Every automorphism of S_n is inner, *except* S_6 has an outer automorphism.