Day 20

More on subspace and dimension

Some counting

Let F be a finite field with $q = p^d$ elements. Let W be a k-dimensional vector space. Then

1. The number of distinct bases of W is $(q^k - 1)(q^k - q)(q^k - q^2) \cdots (q^k - q^{k-1})$.
2. The number of subspaces of dimension k is

$$\frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - (q^{k-1})}$$

3. The group $\mathrm{Aut}(V)$ has the same order as in part 1. (To see this, fix a basis of V. Given another basis, there is a bijective linear map from the fixed basis to this new basis. So the number of linear maps is the same as the number of different bases of V)

Proposition: If $W \subset V$ is a subspace, then the abelian group V/W is a vector space with $a(v+W) = av+W$ being the scalar multiplication. The "isomorphism theorem" for abelian groups holds for vector spaces as well.

$$\begin{tikzcd} V \arrow[d, "\pi"] \arrow[dr, "f"] & \\ V/W \arrow[r, "\overline{f}"'] & K \end{tikzcd}$$

We have $\dim(V) = \dim W + \dim(V/W)$. A linear map $f : V \to K$ is equivalent to an injective linear map $V/\ker(f) \to K$, and identifies the quotient with a subspace of K.

Proposition: If V and W are of the same finite dimension, and $f : V \to W$ is a linear map, then the following are equivalent: 1. f is injective. 2. f is surjective. 3. f is bijective. 4. If $v_1, \ldots, v_n$ is a basis of V, then $f(v_1), f(v_2), \ldots, f(v_n)$ is a basis of W.