Change of Basis and Lagrange Interpolation

Fix a positive integer n and a field F and consider the $n+1$ dimensional F-vector space

$$\mathrm{Pol}_n(F) = \{\sum_{i=0}^{n} a_i x^i\}$$

of polynomials of degree at most n over F.

This vector space has the "obvious" basis $A = \{1, x, \ldots, x^n\}$.

Now suppose we are given $n+1$ points $y_0, y_1, \ldots, y_n$ in F and $n+1$ corresponding elements $c_0, \ldots, c_n$ in F. We can construct a polynomial $f \in \mathrm{Pol}_n(F)$ that has the property that $f(y_i) = c_i$ for $i = 0, \ldots, n$.

A nice way to do this is to construct a new basis B for the polynomials consisting of $\{g_0, \ldots, g_n\}$ where

$$g_i(x) = \frac{(x-y_0)(x-y_1)\cdots\overbrace{(x-y_i)}^{\text{omit}}\cdots(x-y_n)}{(y_i-y_0)\cdots(y_i-y_n)}$$

The polynomials $g_i(x)$ have the property that

$$g_i(y_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

These polynomials are a basis for

$$\mathrm{Pol}_n(F)$$

because there are $n+1$ of them and they are linearly independent; if

$$\sum b_i g_i(x) = 0$$

then evaluating this at y_j forces $b_j = 0$ for $j = 0, \ldots, n$.

Then we can construct our polynomial with prescribed values c_i at y_i by setting

$$f(x) = \sum c_i g_i(x)$$

Let's work out the change of basis process between the bases A and B described above. For the sake of concreteness, let

$$D : \mathrm{Pol}_n(F) \to \mathrm{Pol}_n(F)$$

be the differentiation operator, the linear map

$$D(f)(x) = f'(x).$$

In terms of the standard basis A consisting of powers of x, we find the basis of D by computing

$$D(x^i) = ix^{i-1}.$$

Since the basis elements are

$$a_i = x^i$$

, this means

$$D(a_i) = ia_{i-1}$$

and so

$$[D]_A^A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Now in the notation of these notes our change of basis formula says that

$$[D]_A^A = [G^{-1}]_B^B [D]_B^B [G]_B^B$$

where G is the linear map that satisfies $G(x^i) = g_i$ and $G^{-1}(g_i) = x^i$. The matrix

$$[G^{-1}]_B^B$$

is the easiest to compute, because the *columns of this matrix are the powers of x written in terms of the g_i*. The interpolation formula gives us

$$x^i = \sum_{j=0}^{n} y_j^i g_i(x)$$

so $[G^{-1}]_B^B$ is what's called a Vandermonde matrix:

$$[G^{-1}]_B^B = \begin{bmatrix} 1 & y_0 & y_0^2 & \cdots & y_0^n \\ 1 & y_1 & y_1^2 & \cdots & y_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & y_n & y_n^2 & \cdots & y_n^n \end{bmatrix}$$

To lighten the notation a little bit, let's call this matrix M and we see that

$$[D]_A^A = [G^{-1}]_B^B [D]_B^B [G]_B^B = M^{-1} [D]_B^B M$$

and

$$[D]_B^B = M[D]_A^A M^{-1}$$

In the concrete case where $y_0 = -1, y_1 = 0, y_2 = 1$, we have

$$[D]_B^B = \begin{bmatrix} -\frac{3}{2} & 2 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -2 & \frac{3}{2} \end{bmatrix}$$

What this means is that $f(x)$ is a polynomial that satisfies $f(y_0) = c_0$, $f(y_1) = c_1$ and $f(y_2) = c_2$, then the corresponding values of f' at these three points are the entries in the vector

$$\begin{bmatrix} -\frac{3}{2} & 2 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -2 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -(3/2)c_0 + 2c_1 - (1/2)c_2 \\ (c_2 - c_0)/2 \\ (1/2)c_0 - 2c_1 + (3/2)c_2 \end{bmatrix}$$