

### 3. Group morphisms and group actions

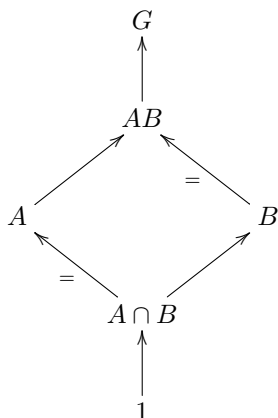
#### The isomorphism theorems

**Theorem:** (See DF Theorem 3.16) Let  $f : G \rightarrow K$  be a homomorphism of groups, let  $N$  be the kernel of  $f$ , and let  $\pi : G \rightarrow G/N$  be the canonical projection. Then there is a unique *injective* homomorphism  $\bar{f} : G/N \rightarrow K$  such that  $\bar{f} \circ \pi = f$ .

$$\begin{array}{ccc} G & & \\ \downarrow \pi & \searrow f & \\ G/N & \xrightarrow{\bar{f}} & K \end{array}$$

We sometimes say that “ $f$  factors through  $\pi$ ” or “ $f$  factors through  $G/N$ ”.

**Theorem:** (See DF Theorem 3.18) Suppose that  $G$  is a group and  $A$  and  $B$  are subgroups of  $G$ . Suppose further that  $A$  is a subgroup of  $N_G(B)$  so that  $AB$  is a subgroup of  $G$ . Then 1.  $B$  is normal in  $AB$ . 2.  $A \cap B$  is normal in  $A$ . 3.  $AB/B$  is isomorphic to  $A/(A \cap B)$ .



The arrows marked with “=” are inclusions of normal subgroups, and the corresponding quotients are isomorphic.

**Theorem:** (See DF, Theorem 3.19) Let  $G$  be a group, and suppose that  $H$  and  $K$  are normal subgroups of  $G$  and  $H$  is normal in  $K$ . Then  $K/H$  is normal in  $G/H$  and  $(G/H)/(K/H)$  is isomorphic to  $G/K$ .

**Theorem:** (See DF, Theorem 3.20) Let  $G$  be a group and  $N$  be a normal subgroup of  $G$ . Then map  $A \mapsto A/N$  is a bijection between the set of all subgroups of  $G/N$  and the set of subgroups of  $G$  containing  $N$ . Furthermore, if  $A$  and  $B$  are subgroups of  $G$  containing  $N$ , then 1.  $A \subset B$  if and only if  $A/N \subset B/N$  2. If  $A \subset B$  then  $[A : B] = [A/N : B/N]$  3.  $\langle A, B \rangle/N = \langle A/N, B/N \rangle$  4.  $(A \cap B)/N = (A/N \cap B/N)$  5.  $A$  is normal in  $G$  if and only if  $A/N$  is normal in  $G/N$ .

In other words, the lattice of subgroups of  $G/N$  is exactly the sublattice of the lattice of subgroups of  $G$  containing  $N$ .