2. Subgroups and quotient groups

Subgroups and Quotient Groups

Basic Definitions

Generating sets

Definition: Suppose G is a group and A is a subset of G (not necessarily a subgroup, just a bunch of elements). The *subgroup* $\langle A \rangle$ *of* G *generated by* A is

$$\langle A \rangle = \bigcap H$$

where the intersection is over all subgroups H of G that contain the set A.

Some special types of subgroups

Suppose G is a group and H is a subgroup of G.

1. The **centralizer** $C_G(H)$ of H is the set of elements $g \in G$ such that $gh = hg$ for all $h \in H$.
2. The **normalizer** $N_G(H)$ of H is the set of elements $g \in G$ such that $gHg^{-1} = H$. In other words, $ghg^{-1} \in H$ for all $h \in H$.
3. H is a **normal subgroup** if $N_G(H) = G$.
4. The **center** $Z(G)$ of G is the the of elements z of G such that $zg = gz$ for all $g \in G$.
5. If $f : G \to H$ is a homomorphism, the **kernel** of f is the set of $g \in G$ such that $f(g) = e$.

Notice that: 1. $C_G(H) \subset N_G(H)$ 2. The center $Z(G)$ is a normal subgroup of G. 3. $H \subset N_G(H)$ and H is a normal subgroup of $N_G(H)$. 4. The kernel of any homomorphism is a normal subgroup. (In fact, the converse is true as well, as we will see later).

Subgroups from group actions

Suppose that X is a set and G acts on X. Remember that one way to think of this is that we have a homomorphism from G to $S(X)$. Another way is that we have a map $G \times X \to X$ satisfying $ex = x$ and $g(h(x)) = (gh)(x)$ for all $x \in X$ and all $g, h \in G$. Such an action yields subgroups of G as follows:

1. The *kernel* of the action is the set of $g \in G$ such that $gx = x$ for all $x \in X$. In other words, the kernel of the action is the kernel of the homomorphism from G to $S(X)$ corresponding to the action. The kernel of the action is therefore a normal subgroup of G.
2. If $x \in X$, the set of elements $g \in G$ such that $gx = x$ is a subgroup of G called the *stabilizer* of x.

Normalizers and centralizers via group actions

One way to think of the normalizer of H in G as being the largest subgroup of G in which H is normal. Alternatively one can think of them in terms of group actions.

Let $\mathcal{P}(G)$ be the power set of G – that is, the set of subsets of G. If $S \subset \mathcal{P}(G)$ is a subset, define

$$g(S) = \{gsg^{-1} : s \in S\}.$$

This defines an *action* of G on $\mathcal{P}(S)$. In general the operation $h \mapsto ghg^{-1}$ is called conjugation of h by g.

If we choose $S = H$, then by definition $N_G(H)$ is exactly the *stabilizer* of H for this action.

If we restrict the action of $N_G(H)$ to the set H, then $C_G(H)$ is exactly the subset of $N_G(H)$ that fixes H pointwise. In other words, $C_G(H)$ is the *kernel* of the conjugation action of $N_G(H)$ on H.

Cosets

Definition: Let H be a subgroup of a group G. A set $gH =$
$gh : h \in H$ is called a *left coset* of H. The corresponding set Hg is called a right coset.

Basics

1. $gH = H$ if and only if $g \in H$.
2. Any two left (right) cosets are in bijection with each other, so if H is finite all cosets of H have the same number of elements as H.
3. Two left (right) cosets are either equal or disjoint.

4. There is a bijection between the set of left cosets and the set of right cosets of H in G given by $f(gH) = Hg^{-1}$.
5. Together, the left (right) cosets of G form a partition of G into disjoint sets.
6. The *index* of H in G, written $[G : H]$, is the number of left (right) cosets, if that number is finite; otherwise we say H has infinite index in G.
7. H is normal if and only if $gH = Hg$ for every $g \in G$.

One way to obtain the key properties of cosets is to observe that the relation $x \sim y$ defined by $x = yh$ for some $h \in H$ – or, expressed another way, that $x \in yH$ – is an equivalence relation.

Theorem: (Lagrange) If G is finite and H is a subgroup of G then $| H | [G : H] =| G |$.

Corollary: In a finite group, the order of an element divdes the order of the group.

Quotient Group

If H is a normal subgroup, then the set of left (right) cosets of G form a group called the quotient group G/H. The group law is $(aH)(bH) = (ab)H$.

The key ingredient of this definition is that the product is well defined. In other words, if $aH = a'H$ and $bH = b'H$ then $(ab)H = (a'b')H$. Since H is normal, $xH = Hx$ for any $x \in G$. We have $a = a'h$ and $b = b'k$ for elements h, k in H. Then $a'hb'k = a'b'h'k$ since $Hb = bH$ by normality of H. This shows that $a'b'H = abH$.

Universal property

Let $\pi_H : G \to G/H$ be the "canonical map" that sends $g \mapsto gH$. - The kernel of this map is H. - Let K be any group and let $f : G \to K$ be a homomorphism such that H is contained in the kernel of f. Then there is a unique homomorphism $\overline{f} : G/H \to K$ such that $f = \overline{f}\pi_G$.

The map $\overline{f}$ is defined by $\overline{f}(aH) = f(a)$. This is well defined since $f(h) = e$ for all $h \in H$.

Corollary: A subgroup H is normal if and only if it is the kernel of a homomorphism.

In fact if H is normal then H is the kernel of the homomorphism π_G.