Day 4

Cosets and index

Proposition: If K is a subgroup of H and H is a subgroup of G then $[G:K] = [G:H][H:K]$.

Proposition: If H and K are subgroups of G, and $HK = \{hk : h \in H, k \in K\}$ then 1. $\mid HK \mid = \frac{|H||K|}{|H \cap K|}$ 2. HK is a subgroup if and only if $HK = KH$. This holds if H is a subgroup of $N_G(K)$, and *a fortiori* if K is normal in G. 3. If m and n are $[G:H]$ and $[G:K]$ respectively, then the index of $H \cap K$ in G is between $\mathrm{lcm}(m,n)$ and mn.

Coset examples

Example 1 Let $G = S_n$. The permutation group S_k is a subgroup of S_n. What are its cosets? There are $n!/k!$ of them.

The coset representatives can be viewed as maps from the set $\{k+1, \ldots, n\}$ to the set $\{1, \ldots, n\}$.

Example 2 Let $G = S_n$. The permutation group $S_k \times S_{n-k}$ is a subgroup of S_n where S_{n-k} is viewed as the permutations of the set $\{k+1, \ldots, \}$. What are the cosets of this subgroup?

Example 3 Let $G = \mathrm{GL}_2(\mathbb{R})$. Let P be the subgroup of upper triangular matrices. Show that the cosets G/P are in bijection with equivalence classes of vectors $[x, y]$ where $xy \neq 0$ and $[x, y] \sim [x', y']$ whenever there is a nonzero constant a so that $ax = x'$ and $ay = y'$.

Equivalently show that the cosets are in bijection with the lines through the origin in $\mathbb{R}^2$.

Example 4 Let $G = D_{2n}$ and let H be a subgroup generated by a reflection fixing a vertex. Show that the cosets of H are in bijection with the n vertices of the polygon on which D_{2n} acts in its usual representation.

Example 5 Let G be the affine group of $\mathbb{R}^2$. Let H be the copy of $\mathrm{GL}_2(\mathbb{R})$ inside G. Show that the cosets are in bijection with the points of $\mathbb{R}^2$.

Quotients

Show that every subgroup of the quaternion group Q_8 is normal. What are the quotient groups?

Show that every quotient group of a cyclic group is cyclic.

A group is *simple* if it has no nontrivial normal subgroups, hence no nontrivial quotients. $\mathbb{Z}/p\mathbb{Z}$, for p prime, are the simple cyclic groups.

$\mathrm{SL}_n(F)$ is the kernel of the determinant map $\mathrm{GL}_n(F) \to F^\times$.

$\mathrm{PGL}_n(F)$ is the quotient of $\mathrm{GL}_n(F)$ by the normal subgroup of matrices aI_n for $a \in F^\times$.

Universal property

1. Every homomorphism $f : G \to H$ makes a quotient of G into a subgroup of H.
2. Every surjective homomorphism $f : G \to H$ is an isomorphism from a quotient of G to H.
3. Every injective homomorphism $f : G \to H$ makes G into a subgroup of H.
4. If G is simple, every homomorphism $f : G \to H$ is either trivial or injective.

The alternating group

The alternating group A_n is the subgroup of S_n consisting of even permutations.