2. Subgroups and quotient groups

Subgroups and Quotient Groups

Basic Definitions

Suppose G is a group and H is a subgroup of G.

1. The **centralizer** $C_G(H)$ of H is the set of elements $g \in G$ such that $gh = hg$ for all $h \in H$.
2. The **normalizer** $N_G(H)$ of H is the set of elements $g \in G$ such that $gHg^{-1} = H$. In other words, $ghg^{-1} \in H$ for all $h \in H$.
3. H is a **normal subgroup** if $N_G(H) = G$.
4. The **center** $Z(G)$ of G is the the of elements z of G such that $zg = gz$ for all $g \in G$.
5. If $f : G \to H$ is a homomorphism, the **kernel** of f is the set of $g \in G$ such that $f(g) = e$.

Notice that: 1. $C_G(H) \subset N_G(H)$ 2. The center $Z(G)$ is a normal subgroup of G. 3. $H \subset N_G(H)$ and H is a normal subgroup of $N_G(H)$. 4. The kernel of any homomorphism is a normal subgroup. (In fact, the converse is true as well, as we will see later)

Subgroups from group actions

Suppose that X is a set and G acts on X. Remember that one way to think of this is that we have a homomorphism from G to $S(X)$. Another way is that we have a map $G \times X \to X$ satisfying $ex = x$ and $g(h(x)) = (gh)(x)$ for all $x \in X$ and all $g, h \in G$. Such an action yields subgroups of G as follows:

1. The *kernel* of the action is the set of $g \in G$ such that $gx = x$ for all $x \in X$. In other words, the kernel of the action is the kernel of the homomorphism from G to $S(X)$ corresponding to the action. The kernel of the action is therefore a normal subgroup of G.
2. If $x \in X$, the set of elements $g \in G$ such that $gx = x$ is a subgroup of G called the *stabilizer* of x.

Normalizers and centralizers via group actions

Let $\mathcal{P}(G)$ be the power set of G – that is, the set of subsets of G. If $S \subset \mathcal{P}(G)$ is a subset, define

$$g(S) = \{gsg^{-1} : s \in S\}.$$

This defines an *action* of G on $\mathcal{P}(S)$.

If we choose $S = H$, then by definition $N_G(H)$ is exactly the *stabilizer* of H for this action.

If we restrict the action of $N_G(H)$ to the set H, then $C_G(H)$ is exactly the subset of $N_G(H)$ that fixes H pointwise. In other words, $C_G(H)$ is the *kernel* of the conjugation action of $N_G(H)$ on H.

In general the operation $h \mapsto ghg^{-1}$ is called conjugation of h by g.