# Day 9

## Some more work with semidirect products

### Groups of order 12

If $G$ has order 12, then it has a Sylow subgroup $V$ of order 4 and a Sylow subgroup $P$ of order 3. Either $V$ or $P$ is normal. To see this, note that $V \cap P$ is the identity. Suppose $P$ is not normal, so $n_3 = 4$. Then there are 8 elements of order 3 in $G$, and together with the identity this accounts for 9 elements of $G$. So there are at most 3 elements, plus the identity, available for $V$, so $n_2 = 1$.

Since either $V$ or $P$ is normal, $G = VP$.

If they are both normal, then $G$ is abelian, and the possibilities are:

- $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ (invariant factors are 6 and 2).
- $\mathbb{Z}/12\mathbb{Z}$

If $G$ is not abelian, it is a semidirect product. Also, $P$ is cyclic of order 3 and $V$ is either cyclic or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

1. $V$ is normal.

   a. $V = \mathbb{Z}/4\mathbb{Z}$. Then we need to look at maps $P \to \mathrm{Aut}(V)$. But $P$ has order 3 and $\mathrm{Aut}(V)$ has order 2 so there are no nontrivial homomorphisms, so in this case $G$ is cyclic.

   b. $V = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The automorphisms of $V$ are $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$. This group has order 6 and is isomorphic to $S_3$; it acts on $V$ by permuting the three non-zero elements. Therefore we can send the generator of $P$ to an automorphism that cyclically permutes those elements. The resulting semi-direct product is $A_4$.

2. $P$ is normal. The automorphism group of $P$ is $\mathbb{Z}/2\mathbb{Z}$, with non-trivial automorphism $x \to x^{-1}$ where $x$ is a generator of $P$.

   a. $V = \mathbb{Z}/4\mathbb{Z}$. We need a map from $V \to \mathrm{Aut}(P)$, and there's only one, which sends the generator $y \in V$ to the automorphism $x \mapsto x^{-1}$. This gives us a group with generators $x$ and $y$ such that $x^3 = y^4 = 1$ and $yxy^{-1} = x^{-1}$.

b. $V = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In this situation there are three nontrivial maps $V \to \mathrm{Aut}(P)$ (either $(1,0)$, $(0,1)$ or $(1,1)$ acts like $x \mapsto x^{-1}$.) The resulting group has generators $x, y, z$ where $x^3 = y^2 = z^2 = 1$, $y$ and $z$ commute with each other, $x$ and $z$ commute with each other, and $yxy^{-1} = x^{-1}$. (The actual choice of map $V \to \mathrm{Aut}(P)$ does not affect the isomorphism class of the result). This is the group $D_{12}$ of symmetries of the hexagon.

So there are five isomorphism types of groups of order 12, two abelian, three nonabelian: $D_{12}$, $A_4$, and $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$.