Problem Set 3

Instructions: This set is due October 16th.

Problem 1: Let H and K be groups and let $\phi : K \to \mathrm{Aut}(H)$ be a homomorphism.

a. Given an automorphism $f : K \to K$, the map $\phi \circ f$ is a homomorphism from K to $\mathrm{Aut}(H)$. Show that the groups $H \rtimes_\phi K$ and $H \rtimes_{\phi \circ f} K$ are isomorphic.

b. Suppose $f : H \to H$ is an automorphism. If $\sigma : H \to H$ is an automorphism, so is the conjugate $f \circ \sigma \circ f^{-1}$. Show that $\gamma_f(\sigma) = f \circ \sigma \circ f^{-1}$ is an automorphism of $\mathrm{Aut}(H)$, and prove that $H \rtimes_\phi K$ and $H \rtimes_{\gamma_f \circ \phi} K$ are isomorphic.

Problem 2: Construct a non-abelian group of order 75 and (using the previous problem) show that it is unique up to isomorphism. Is it always the case that there is only one nonabelian group of order pq^2 when $q > p$?

Problem 3: DF p. 231–232 Problems 15, 21 and 22 about Boolean rings.

Problem 4: DF p. 250, problem 34. Note that R is not assumed commutative, and “ideal” means two-sided ideal.

Problem 5: Classify the following ideals in $\mathbb{Z}[x]$ as prime but not maximal, maximal, or not prime. (Hint: look at the quotient rings).

a. $(10) = 10\mathbb{Z}[x]$.
b. $(5, x + 7)$
c. $(x^2 - 3)$
d. $\{f \in \mathbb{Z}[x] : f(2) = 0\}$