Day 21

Change of Basis

Given vector spaces V and W of dimension n and m respectively, and bases $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ for V and W, a linear map $L : V \to W$ has an $m \times n$ matrix representation

$$[L]_A^B = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

where the c_{ij} are defined by

$$L(a_i) = \sum_{j=1}^{m} c_{ji} b_j.$$

If $v = \sum x_i a_i \in V$, define an $n \times 1$ matrix

$$[v]_A = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

We can view this as an isomorphism from V to F^n if we write our elements of F^n as column vectors. We can do the same construction for W and F^m, yielding $[w]_B$.

Then

$$[Lv]_B = [L]_A^B [v]_A$$

meaning that the matrix representation turns the map into matrix multiplication.

More generally if $L : V \to W$ and $H : W \to K$ are linear maps, and A, B, C are bases for V, W, K then

$$[H \circ L]_A^C = [H]_B^C [L]_A^B.$$

Now suppose we choose different bases A' and B' for V and W.

There is a unique invertible linear map $G : V \to V$ which satisfies $G(a_i') = a_i$ for $i = 1, \ldots, n$. This means that if

$$v = \sum x_i a_i'$$

then

$$G(v) = \sum x_i a_i.$$

Important: In this convention, the *inverse* of the linear map G carries a_i to a_i', so if you look at its matrix $[G^{-1}]_A^A$ in the basis A, its columns give the coordinates of the new basis in terms of the old. This is in some ways the more natural thing to consider.

Now

$$[Gv]_A = [v]_{A'}.$$

Since

$$[Gv]_A = [G]_A^A [v]_A$$

we see that

$$[v]_{A'} = [G]_A^A [v]_A.$$

Now given a linear map $L : V \to W$ and basis A' and B' for V and W, with:

- G the map carrying a_i' to a_i, (note the convention here. The columns of G^{-1} express the new basis in terms of the old one.)
- H the map carrying b_i' to b_i.

If $L(v) = w$ we know that the matrix for L is characterized by

$$[L]_{A'}^{B'} [v]_{A'} = [w]_{B'}.$$

Then

$$[L]_{A'}^{B'} [G]_A^A [v]_A = [H]_B^B [w]_B$$

so

$$[L]_{A'}^{B'} = (H_B^B)^{-1}[L]_A^B G_A^A.$$

So a change of basis on the source and target modifies the matrix of the linear map by left- and right- multiplication by invertible matrices.

Example: Lagrange interpolation

Given $n+1$ points $x_0, \ldots, x_n$ in $\mathbb{R}$, there is a polynomial of degree n with prescribed values $f(x_i) = a_i$

Let

$$f_i(x) = \frac{(x-x_0)(x-x_1)\cdots\widehat{(x-x_i)}\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_n)}$$

This polynomial vanishes on all the x's except x_i, where it takes value 1. These are linearly independent and

$$f = \sum a_i f_i$$

is the desired expression.

Now x^k takes the value x_i^k at x_i, so

$$x^k = \sum x_i^k f_i$$

So the basis consisting of the powers of x, expressed in terms of the x_i, is the matrix whose k^{th} column, i^{th} row is x_i^k. Call this matrix G.

Let D be the matrix giving the derivative operator on polynomials of degree n in the standard basis $1, x, \ldots, x^n$.

Then $G^{-1}DG$ expresses the derivative operator in terms of the basis f_i. In practice this tells you have who to compute derivatives from values of polynomials at chosen points.

See *Inverses of Vandermonde Matrices*, by N. Macon and A. Spitzbart, American Math Monthly 1958 vol 65 number 2.