Problem Set 2

Instructions: Write up your solutions using LaTeX and submit them on HuskyCT by September 25, 2022.

Problem 1: Let M and N be normal subgroups of a group G such that $G = MN$.

1. Prove that
$$G/(M \cap N) \cong (G/M \times G/N).$$
In the case where $M \cap N$ is trivial, we say that G is the *internal direct product* of M and N.
2. Prove inductively that $\mathbb{Z}/m\mathbb{Z}$ is the internal direct product of the subgroups $\mathbb{Z}/p_i^{e_i}\mathbb{Z}$ for $i = 1, \ldots, k$ where $m = \prod_{i=1}^{k} p_i^{e_i}$ is the factorization of m into powers of distinct primes.

Problem 2: DF Problems 19-20 on Page 131. The conclusion of these problems is that, if $K(\sigma)$ is the conjugacy class of σ in S_n, then $K(\sigma)$ is a conjugacy class in A_n if and only if σ commutes with an odd permutation. Otherwise $K(\sigma)$ breaks up into two conjugacy classes.

Problem 3: (with thanks to Keith Conrad) The group $G = \mathrm{SL}_2(\mathbb{Z})$ is the group of 2×2 integer matrices with determinant 1. It acts on $\mathbb{Z} \times \mathbb{Z}$ viewed as column vectors with integer entries by matrix multiplication.

1. Show that if $m \neq n$ in $\mathbb{Z}$ are positive, then $\begin{bmatrix} m \\ 0 \end{bmatrix}$ and $\begin{bmatrix} n \\ 0 \end{bmatrix}$ are in different G-orbits.
2. Show that the G orbit of $\begin{bmatrix} x \\ y \end{bmatrix}$ contains $\begin{bmatrix} gcd(x,y) \\ 0 \end{bmatrix}$. Conclude that the orbits consist of the vectors having the same gcd.
3. Show that the stabilizer of $\begin{bmatrix} m \\ 0 \end{bmatrix}$ is the subgroup N of upper triangular matrices with 1 on the diagonal.
4. Conclude that the stabilizer in G of a nonzero vector in $\mathbb{Z}^2$ is conjugate to N. Make this explicit by finding an $A \in G$ so that ANA^{-1} stabilizes $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

Problem 4: DF, Problem 18, p. 138. This problem shows that if $n \neq 6$ then every automorphism of S_n is inner. As it happens, S_6 has an outer automorphism, which is described in the Wikipedia article Automorphisms of the symmetric and alternating groups.

Problem 5: (another Keith Conrad problem) Let G be a group of order $1683 = 9 \times 11 \times 17$.

1. Prove $G = HK$ where H is a cyclic normal subgroup of order 187 and K is a subgroup of order 9.

2. Prove that H and K commute with each other.
3. Conclude that groups of order 1683 are abelian (and up to isomorphism there are two of them).