Day 2

Euclid's Algorithm

1. Well-ordering of the integers. Every nonempty set of positive integers has a least element.
2. Let a and b be nonzero integers. Consider the set

$$X = \{ax + by : x, y \in \mathbb{Z}\}$$

1. X contains positive elements.
2. X contains a smallest positive element, call it d. So $ax + by = d$.
3. Note that X contains every (positive and negative) multiple of d.
4. Take any other positive element z of X. Then $z = qd + r$ but also $z = ax' + by'$. Suppose $r > 0$. So $qd + r = ax' + by'$ and therefore $r = a(x' - qx) + b(y' - qy)$. This means that r is in X; but since r is less than d, this cannot happen. It follows that $r = 0$ and every element of X is a multiple of d.
5. We conclude that $X = d\mathbb{Z}$.
6. Since X contains both a and b, we see that d is a common divisor of a and b.
7. If g is any other common divisor of a and b, then g divides d since $d = ax + by$.
8. Therefore d is the *greatest* common divisor of a and b, and any other common divisor of a and b is a divisor of d.

Theorem: Given nonzero integers x and y, the equation $ax + by = z$ has a solution if and only if z is a multiple of the greatest common divisor of a and b.

Congruences

Theorem: The congruence equation

$$ax \equiv b \pmod{n}$$

has solutions if and only if $d = \gcd(a, n)$ divides b. In that case it has n/d solutions modulo n.

Proof: Solving the congruence equation

$$ax \equiv b \pmod{n}$$

is equivalent to solving the equation

$$ax + ny = b.$$

Euclid's algorithm tells us this equation has a solution if and only if $d = \gcd(a, n)$ divides b. When this holds, swe have a solution to our congruence x to our congruence. Notice that $x + k\frac{n}{d}$ is *also* a solution to this equation for $k = 0, \ldots, d-1$, so we actually have d solutions. If x and x' are any two solutions to this equation, then subtracting $ax + ny = b$ from $ax' + ny' = b$ yields

$$a(x - x') + n(y - y') = 0$$

and so, since n/d and a/d have gcd equal to one we conclude that $x - x'$ is divisible by n/d. Therefore we have found all solutions.

Cyclic Groups

Key Facts about Cyclic Groups

1. Any cyclic group of infinite order is isomorphic to $\mathbb{Z}$. Any cyclic group of finite order n is isomoprhic to $\mathbb{Z}/n\mathbb{Z}$. A group G is cyclic if and only if there is a surjective homomorphism from $\mathbb{Z}$ to G.
2. An infinite cyclic group has two generators.

3. If g is a generator of a finite cyclic group G of order n, then g^a has order $n/\gcd(n, a)$. Thus g^a generates G if and only if $\gcd(n, a) = 1$.
4. The Euler function $\phi(n)$ is (by definition) the number of generators of a cyclic group of order n.
5. Every subgroup of $\mathbb{Z}/n\mathbb{Z}$ is cyclic, and there is a unique such subgroup for every $d\|n$.
6. If H is the cyclic subgroup of $\mathbb{Z}/n\mathbb{Z}$ of order d where $d\|n$, then the elements of H are the multiples of n/d. The generators of H are the multiples kn/d where $\gcd(k, d) = 1$.
7. If $\gcd(n, m) = 1$ then $\mathbb{Z}/nm\mathbb{Z}$ is isomorphic to $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.

8. A pair (a, b) generates $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ if and only if a and b generate $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ respectively. Therefore $\phi(nm) = \phi(n)\phi(m)$ when n and m are relatively prime.
9. If $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ where the p_i are distinct primes then

$$\phi(n) = \prod_{i=1}^{k} (p^{e_i} - p^{e_i - 1}) = n \prod_{p|n} (1 - \frac{1}{p})$$

For discussion

1. We know that $\mathbb{Z}/6\mathbb{Z}$ is a subgroup of $\mathbb{Z}/24\mathbb{Z}$. Find all injective maps $\mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$.

2. "Reduction mod 6" gives a surjective homomorphism $\mathbb{Z}/24\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$. Find the inverse image of 5 under this map.
3. Find all surjective homomorphisms $\mathbb{Z}/24\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$.
4. Prove that $(\mathbb{Z}/11\mathbb{Z})^\times$ is cyclic. In fact $(\mathbb{Z}/p\mathbb{Z})^\times$ is always cyclic, we'll prove this later.