Problem Set 1

Instructions: Write up your solutions using LaTeX and submit them on HuskyCT by September 11, 2022.

Problem 1: Let $\sigma = (13)(1235)(4567)$ and $\tau = (24)(35)(245)$.

1. Find $\sigma\tau$.
2. Find disjoint cycle decompositions of both σ and τ.
3. Write each of σ and τ as products of transpositions.
4. Find the sign of σ and τ.

Problem 2: Define an ordering $\lesssim$ on the positive integers by saying that $n \lesssim m$ if there is an injective homomorphism from $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$.

1. Show that this relation is reflexive ($n \lesssim n$ for all n), antisymmetric ($n \lesssim m$ and $m \lesssim n$ implies $n = m$), and transitive ($n \lesssim m$ and $m \lesssim k$ implies $n \lesssim k$.) These axioms mean that the positive integers are a *partially ordered set* under this order relation.

2. The *meet* g of two elements a and b in a partially ordered set is an element that satisfies these two conditions:

   - $g \lesssim a$ and $g \lesssim b$.
   - If h is any element satisfying $h \lesssim a$ and $h \lesssim b$, then $h \lesssim g$.

Prove that any two elements m and n have a *meet*. 3. Describe all of this fancy stuff in a simpler way. 4. (Extra) What happens if, instead of considering injective homomorphisms from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$, we consider surjective ones?

Problem 3: If H is a subgroup of G, then the normalizer $N_G(H) = \{g \in G : gHg^{-1} = H\}$ and the centralizer $C_G(H) = \{g \in G : gh = hg \quad \forall h \in H\}$.

1. Let $G = \mathrm{GL}_2(\mathbb{R})$ and let $H = \mathrm{Aff}(\mathbb{R})$ be the affine group consisting of two by two matrices of the form $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ where $x \neq 0$. Show that the centralizer of H in G consists of matrices aI where $a \neq 0$ and I is the identity matrix, and the normalizer consists of the upper triangular matrices with nonzero diagonal entries. What about the same question for $\mathrm{GL}_2(\mathbb{Q})$? Does the field matter for this?
2. Let $G = \mathrm{GL}_2(\mathbb{R})$ and H be the subgroup of diagonal matrices. Show that H is its own centralizer, and its normalizer consists of diagonal matrices and "anti-diagonal" matrices $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ in G.
3. View S_n as $\mathrm{Sym}(\mathbb{Z}/n\mathbb{Z})$. Given integers a and b, with $\gcd(a, n) = 1$, let $\sigma_{a,b} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the map $\sigma_{a,b}(x) = ax + b \pmod{n}$. Show that $\sigma_{a,b}$ is in the normalizer of the cyclic subgroup H generated by $(123 \ldots n)$ and, conversely, any element of the normalizer of this subroup is $\sigma_{a,b}$ for some a and b. Conclude that the normalizer in S_n of H is the affine group of $\mathbb{Z}/n\mathbb{Z}$.

Problem 4: Let M be the group with two generators u and v satisfying $u^2 = v^8$ and $vu = uv^5$. This group has order 16 (can you verify this?). M has three subgroups of order 8: $\langle u, v^2 \rangle$, $\langle v \rangle$, and $\langle uv \rangle$. Every proper subgroup is contained in one of these three groups.

1. Draw the lattice of subgroups of M.
2. Prove that the group generated by v^4 is normal in M.
3. Find the lattice of subgroups of $M/\langle v^4 \rangle$ inside that of M using the lattice isomorphism theorem.