Day 23

Trace and Determinant

Definition: If $A = (a_{ij})$ is a matrix in $M_n(F)$, then the trace of A is

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Proposition: The trace is linear in the matrix A. Also

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB).$$

In particular $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

Proof:

Let $X = AB$. Then

$$x_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}.$$

Now if $Y = XC$ then

$$y_{rs} = \sum_{t=1}^{n} x_{rt} c_{ts} = \sum_{t=1}^{n} \sum_{j=1}^{n} a_{rj} b_{jt} c_{ts}$$

and then $\mathrm{Tr}(Y)$ is

$$\mathrm{Tr}(ABC) = \sum_{r=1}^{n} y_{rr} = \sum_{r=1}^{n} \sum_{t=1}^{n} \sum_{j=1}^{n} a_{rj} b_{jt} c_{tr}.$$

On the other hand

$$\mathrm{Tr}(BCA) = \sum_{j=1}^{n} \sum_{r=1}^{n} \sum_{t=1}^{n} b_{jt} c_{tr} a_{rj}$$

which is just a rearrangement of the sum; and similarly for $\mathrm{Tr}(CAB)$.

Trace of a linear map:

This allows us to define the trace of a linear map as the trace of any matrix representing it – different matrices differ by conjugation by the change of basis matrix.

Notice also that trace is a conjugacy class invariant in $\mathrm{GL}_n(F)$. Two conjugate matrices have the same trace.

Multilinear Functions

Definition: A function $H : V_1 \times V_2 \times \cdots \times V_k \to W$ is multilinear if it is linear as a function of each variable, with the other variables held fixed.

A function $H : \overbrace{V \times \cdots \times V}^{n} \to F$ is called an n-multilinear form.

If $A = \{a_1, \ldots, a_n\}$ is a basis for V, then an n-multilinear form is determined by its values $H(x_1, \ldots, x_n)$ where each x_i is chosen from A. There are n^n such values.

For example if $\dim V = 2$ with basis a_1, a_2 then

$$H(x_{11}a_1+x_{12}a_2, x_{21}a_1+x_{22}a_2) = x_{11}x_{21}f(a_1, a_1)+x_{11}x_{22}F(a_1, a_2)+x_{12}x_{21}F(a_2, a_1)+x_{12}x_{22}F(a_2, a_2)$$

The "dot product" is a 2-linear form (a "bilinear" form) on $\mathbb{R}^n$ or more generally on F^n.

If we think of the trace as a function of the column vectors of a matrix, it is a multilinear form.

Symmetric and Alternating forms

A multilinear form $H : V^n \to F$ is called *alternating* if $H(v_1, \ldots, v_n) = 0$ whenever two adjacent v_i are equal to each other. It is called *symmetric* if $H(v_1, \ldots, v_n)$ stays the same under rearrangement of the v_i.

The dot product is a symmetric bilinear form since $H(v, w) = H(w, v)$.

Lemma: If H is an alternating multilinear form, then $H(v_1, \ldots, v_n) = 0$ whenever two of the v_i coincide; and $H(v_1, \ldots, v_n)$ changes sign whenever two of the v_i are interchanged. More generally,

$$H(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)}) = \mathrm{sgn}(\sigma)H(v_1, \ldots, v_n)$$

where σ is a permutation in S_n and $\mathrm{sgn}(\sigma)$ is the sign character.

Proof: Suppose $H_{i,i+1}(x, y)$ is the function H with fixed entries in all positions except i and $i+1$. Then $H_{i,i+1}(x+y, x+y) = 0$ by the alternating property; but

$$H_{i,i+1}(x+y, x+y) = H_{i,i+1}(x, x) + H_{i,i+1}(x, y) + H_{i,i+1}(y, x) + H_{i,i+1}(y, y)$$

by multilinearity. Since the outer terms are zero by the alternating property, we get

$$H_{i,i+1}(x, y) = -H_{i,i+1}(y, x).$$

Now if $H_{i,j}(x, y)$ is H with all positions fixed except i and j, notice that we can progressively swap adjacent values until y is in position $i+1$, changing signs each time. Therefore $H_{i,j}(x, y) = \pm H_{i,i+1}(x, y)$. In particular $H_{i,i+1}(x, x) = 0 = H_{i,j}(x, x)$.

Therefore $H(v_1, \ldots, v_n) = 0$ whenever any two of the v_i coincide; and repeating the argument we used for adjacent entries we get that $H(v_1, \ldots, v_n)$ changes sign when we swap any two variables.

Since an arbitrary permutation is a product of transpositions, and the sign character is defined as $(-1)^k$ where k is the number of such transpositions, we get the formula for a general permutation.

Remark: Why not define alternating to mean $H(v_1, \ldots, v_n)$ changes sign if we swap adjacent entries? Look at characteristic two.

Corollary: If H is alternating, and $w_i = v_i$ except that $w_j = v_j + av_k$ for some j, then $H(w_1, \ldots, w_n) = H(v_1, \ldots, v_n)$. Use linearity in the j slot to see this.

An alternating multilinear form is defined by its values $H(a_1, \ldots, a_n)$ where the a_i are chosen from a basis of V, but these elements of F have to satisfy the permutation property *and* must vanish if any basis elements are repeated.

If V is n-dimensional and $a_1, \ldots, a_n$ is a basis, then an n-multilinear form is determined by a single value $H(a_1, a_2, \ldots, a_n)$ and if $v_i = \sum x_{ji} a_j$ then

$$H(v_1, \ldots, v_n) = H(a_1, \ldots, a_n) \sum_{\sigma} \operatorname{sgn}(\sigma) x_{\sigma(1)1} x_{\sigma(2)2} \cdots x_{\sigma(n)n}.$$

Definition: The determinant is the unique alternating multilinear map $\det : M_n(F) \to F$ such that $\det(I) = 1$. Here det is viewed as a function of the columns of a matrix A.

Lemma: The determinant of A and its transpose are the same.

Proposition: $\det(AB) = \det(A)\det(B)$.

Proof: Let $C = AB$. Then the columns of C are linear combinations of the columns of A. In fact

$$C_j = \sum_{i=1}^{n} b_{ij} A_i$$

where C_j and A_i are the corresponding columns of C and A. So

$$\det C = \sum (\operatorname{sgn}(\sigma) b_{\sigma(1)1} b_{\sigma(2)2} \cdots b_{\sigma(n)n}) \det(A_1, \dots A_n) = \det(B) \det(A)$$