Day 3

Generators

A group is *finitely generated* if there is a finite set A which generates it.

- finite groups are finitely generated
- $\mathbb{Z}$ is finitely generated
- $\mathbb{Q}$ is *not* finitely generated.

Special subgroups

If H is normal in G and G is normal in K, is H normal in K?

If H is normal in G, and $f : G \to K$ is a homomorphism, is $f(H)$ normal in K? Show $f(G)$ is contained in $N_K(f(H))$.

Let G be the dihedral group D_{2n} of symmetries of a regular n-gon.

- Show that the subgroup R of rotations is normal.
- Let H be the subgroup generated by a reflection. What is $N_G(H)$? What is $C_G(H)$?
- What is the center of G?
- Draw the lattice of subgroups of D_8. (See DF, p. 69)

Consider this prelim problem from the August, 2022 prelim.

3. (**10 pts**) Let $n \geq 3$ and let r and s be the standard generators of the dihedral group of order $2n$: r has order n, s has order 2, and $srs^{-1} = r^{-1}$ (equivalently, $sr = r^{-1}s$).

   (a) (**5 pts**) For an *automorphism* f of this dihedral group $\langle r, s \rangle$, show $f(r) = r^a$ for some integer a where $(a, n) = 1$ and $f(s) = r^b s$ for some integer b.

   (b) (**5 pts**) Conversely, given integers a and b such that $(a, n) = 1$, show there is a unique automorphism f of $\langle r, s \rangle$ such that $f(r) = r^a$ and $f(s) = r^b s$.

Figure 1: Prelim Problem

Let Q be the quaternion group with 8 elements. Draw its lattice of subgroups. (The quaternion group has 8 elements $\{\pm 1, \pm i, \pm j, \pm k\}$ where each of i, j, k satisfy $x^2 = -1$ and $ijk = -1$. Compare Q with D_8.

Show that Q is isomorphic to the subgroup of $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ generated by the matrices $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Show that the center of S_n is trivial if $n \geq 3$.

What is the normalizer of the rotation group in $\mathrm{GL}_2(\mathbb{R})$? What is its centralizer? Interpret this in terms of group actions.

The affine group $\mathrm{Aff}(\mathbb{R}^2)$ of the plane is the subgroup of $\mathrm{GL}_3(\mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}.$$

It acts on points (u, v) viewed as column vectors

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

by matrix multiplication.

Show that $\mathrm{GL}_2(\mathbb{R})$ viewed as the upper triangular block is a normal subgroup of the affine group.

The additive group $\mathbb{R}^2$ is a subgroup of the affine group; what is its normalizer?

The subgroup of the affine group where the upper triangular matrix is a rotation matrix (an "orthogonal matrix") is called the Euclidean group.

Every group is a permutation group

Theorem (Cauchy): Any group is a subgroup of a permutation group.

Proof: Given $g \in G$, let $f_g : G \to G$ be the map $f_g(h) = gh$. This gives a map from G to $S(G)$. We have $f_e(h) = eh = h$ so f_e is the identity map. Also

$$f_{ab}(h) = abh = (f_a \circ f_b)(h)$$

for any $h \in G$. Since composition of functions is the group operation in $S(G)$, the map $g \mapsto f_g$ is a homomorphism from $G \to S(G)$.

Finally, the map f_h is the identity map only when $h = e$. So this map is injective.

Make this explicit for S_3.