Day 1

Groups and Subgroups

For discussion:

Products

1. If G and H are groups, then $G \times H$ is a group where $(a, b)(c, d) = (ac, bd)$.
2. The product of abelian groups is abelian.
3. If G and H are groups, then $G \times H$ is isomorphic to $H \times G$.
4. Let $G = \mathbb{Z}/2\mathbb{Z}$ and $H = \mathbb{Z}/3\mathbb{Z}$. Prove that $G \times G$ has 4 elements but is not isomorphic to $\mathbb{Z}/4\mathbb{Z}$, and $G \times H$ has 6 elements and *is* isomorphic to $\mathbb{Z}/6\mathbb{Z}$.

Subgroups

1. The intersection of (arbitrarily many) subgroups of G is a subgroup of G.
2. If G is abelian and $n > 1$ is an integer, then the set $G(n) =$ $x^n : x \in G$ is a subgroup of G. What if G is not abelian?
3. If A is a subgroup of G and B is a subgroup of H then $A \times B$ is a subgroup of $G \times H$.
4. G is isomorphic to the set of diagonal elements (g, g) in $G \times G$.
5. Using the notation in (2), let $G = \mathbb{Z}/10\mathbb{Z}$. What is $G(3)$? What is $G(2)$?

Dihedral and Symmetric Groups

1. If there is a bijection from X to Y, prove that $S(X)$ is isomorphic to $S(Y)$.
2. Review the rules for cycle decomposition and multiplication of permutations in S_n. Let $\sigma = (4\ 8\ 6)(4\ 3\ 8)(1\ 3\ 5\ 2)$.
   - Write σ as a product of disjoint cycles.
   - Write σ as a product of transpositions. What is its sign?
3. The group of symmetries of the square D_8 “comes with” an action on the 4 corners of the square. Labelling the four corners with $X =$ $1, 2, 3, 4$, this action associates a permutation of X to each element of D_8. Show that this is an injective homomorphism from D_8 to S_4 and compute it for each element of D_8. What can you say about the general case?

Matrix Groups

1. Let $G = \mathrm{GL}_2(\mathbb{R})$. Describe some proper subgroups of G. Include some finite and some infinite examples.

2. Let $G = \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ be the set of two by two matrices with entries in $\mathbb{Z}/2\mathbb{Z}$. Verify that G is a group of order 6. It acts on the set $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by matrix multiplication. Show that it preserves the set of 3 vectors (a, b) where not both a and b are zero, and that G is isomorphic to S_3.