Day 8

Applications of Sylow's theorems

Intersections of Sylow groups

Lemma: Let p and q be distinct primes and let P and Q be Sylow p and q subgroups of a finite group G. Then $P \cap Q$ is trivial.

Proof: An element of the intersection has order dividing a power of p and a power of q, so it must have order 1.

Abelian Groups

Proposition: Let G be a finite abelian group and let p be a prime dividing the order of G. Let P be the p-Sylow subgroup of G. Then P consists of the elements of G whose order is a power of p.

Proof: Since G is abelian, it's Sylow p-subgroup is unique. Clearly if $g \in P$ then the order of g is a power of p. Conversely if $g \in G$ has order a power of p, it generates a subgroup of p-power order, which must be contained in P.

Corollary: Let n be the order of G and suppose $n = p^k m$ where $(m, n) = 1$. Let $G[m]$ be the elements of G whose order divides m. Then:

1. $G[m]$ is a subgroup of G.
2. $G[m] \cap P$ is the trivial subgroup.
3. The map $G[m] \times P \to G$ given by $(a, b) \mapsto a + b$ is an isomorphism.

Proof:

1. If x and y have order h_1 and h_2, both dividing m, then $\mathrm{lcm}(h_1, h_2)(x+y) = 0$. Let $h = \mathrm{lcm}(h_1, h_2)$, which divides m. Then $h(x + y) = 0$ so $x + y$ has order dividing m as well.

2. This is the lemma above.

3. If (a, b) is in the kernel of the map, then since $a + b = 0$ we have $a = -b$ which means both a and b are in the intersection of $G[m]$ and P. By (2) this means $(a, b) = (0, 0)$ so the map is injective. Then by counting we conclude it is surjective.

Corollary: Any finite abelian group is isomorphic to the product of its Sylow p-subgroups.

This reduces the classification problem for finite abelian groups to the classification of finite abelian p-groups.

Groups of order pq.

Suppose G has order pq and suppose $p < q$. Let Q be a Sylow q subgroup. The number of such Sylow q subgroups must be one of $1, p, q, pq$ and must be congruent to 1 mod q. The only possibility is 1. Thus Q is a normal subgroup. Now let P be any Sylow p-subgroup. The intersection $P \cap Q$ is trivial so PQ is a group of order pq and must be G.

Suppose further that $q \not\equiv 1 \pmod{p}$. Then P must also be normal, because there cannot be q Sylow p-subgroups. G acts on P by conjugation, and in that action P acts trivially (since P is just a copy of $\mathbb{Z}/p\mathbb{Z}$ which is abelian). Therefore $G/P \cong Q$ acts on P by group automorphisms. But $\mathrm{Aut}(P) \cong (\mathbb{Z}/p\mathbb{Z})^*$ which has order $p-1$ and Q is a cyclic group of order q, so there is no nontrivial map $Q \to \mathrm{Aut}(P)$. This means Q commutes with P, so this group is abelian and in fact cyclic of order pq.

If $q \equiv 1 \pmod{p}$, then the group need not be abelian. For example S_3 or D_{2n} with n and odd prime are nonabelian groups of order pq with $q \equiv 1 \pmod{p}$.

Groups of order 30

Proposition: A group of order 30 has a normal subgroup of index 2 (which is necessarily cyclic of order 15 by the previous problem).

Lemma: Let P be a Sylow 3-subgroup and Q a Sylow 5 subgroup. If either of these groups is normal in G, then PQ is a subgroup of G of order 15 – which must be normal since it is of index 2. Now let $g \in G$. Then gQg^{-1} is contained in PQ and since Q is the Sylow 5-subgroup of PQ we have $gQg^{-1} = Q$. Therefore Q is normal in G; and similarly P is normal in G. So if either of the Sylow subgroups is normal, so is the other one, and we've found our subgroup.

Assume therefore that neither of them is normal. Then there must be at least 6 conjugates of the Sylow 5-subgroup and 10 of the Sylow 3 subgroup. This accounts for $6*4+10*2 = 44$ elements of G, which only has order 30.

Groups of order 12.

Proposition: Let G be a finite group of order 12. Then either G contains a normal subgroup of order 3 or G is isomorphic to A_4.

Proof: Let n_3 be the number of Sylow 3-subgroups. If $n_3 = 1$ we are done. The divisors of 12 are $1, 2, 3, 4, 6, 12$. If $n_3 > 1$ then $n_3 = 4$. Therefore the group G permutes the four Sylow 3-subgroups transitively and this gives us a map from

G to \$S_{4}. If P is one such Sylow subgroup, the stabilizer of P under this action is its normalizer, which contains P and thus has order at least 3. But its index must be 4 since that's the size of the orbit, so $N_G(P) = P$.

The map to S_4 given by the conjugation action is injective. To see this, suppose x is in the kernel of the action. This means $xPx^{-1} = P$, $x \in P$. But it also means $xgPg^{-1}x^{-1} = gPg^{-1}$ for one of the other conjugates of P. That means $g^{-1}xg$ is also in P, or $x \in gPg^{-1}$. Since gPg^{-1} is not P, the intersection of P with gPg^{-1} is a proper subgroup of P, and since P is cyclic of order 3 that must be the trivial subgroup.

So the image of the permutation map is a subgroup of S_4 of order 12. Furthermore, the image meets A_4 in at least 9 elements (the identity plus 8 elements of order 3). So it must be all of A_4.

A few more

Suppose G has order 45. The divisors of 45 are $1, 3, 5, 9, 15, 45$. The number of Sylow 3-subgroups has to be congruent to 1 mod 3, so it has to be 1. The number of Sylow 5 subgroups has to be one mod 5, so it has to be one. So there is a normal subgroup of order 9 and another of order 5. A group of order 9 must be abelian so the only possibilities are $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ or $\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

The smallest possible odd order for a nonabelian group is 21, and there is such a nonabelian group. It has 7 3-Sylow subgroups and a normal 7-Sylow subgroup.