3. Group morphisms and group actions

The isomorphism theorems

Theorem: (See DF Theorem 3.16) Let $f : G \to K$ be a homomorphism of groups, let N be the kernel of f, and let $\pi : G \to G/N$ be the canonical projection. Then there is a unique *injective* homomorphism $\overline{f} : G/N \to K$ such that $\overline{f} \circ \pi = f$.

$$\begin{tikzcd}
G \arrow[d, "\pi"] \arrow[dr, "f"] & \\
G/N \arrow[r, "\overline{f}"'] & K
\end{tikzcd}$$

We sometimes say that "f factors through π" or "f factors through G/N".

Theorem: (See DF Theorem 3.18) Suppose that G is a group and A and B are subgroups of G. Suppose further that A is a subgroup of $N_G(B)$ so that AB is a subgroup of G. Then 1. B is normal in AB. 2. $A \cap B$ is normal in A. 3. AB/B is isomorphic to $A/(A \cap B)$.

$$\begin{tikzcd}
& G & \\
& AB \arrow[u] & \\
A \arrow[ur] & & B \arrow[ul, "="'] \\
& A \cap B \arrow[ul, "="] \arrow[ur] & \\
& 1 \arrow[u] &
\end{tikzcd}$$

The arrows marked with "=" are inclusions of normal subgroups, and the corresponding quotients are isomorphic.

Theorem: (See DF, THeorem 3.19) Let G be a group, and suppose that H and K are normal subgroups of G and H is normal in K. Then K/H is normal in G/H and $(G/H)/(K/H)$ is isomorphic to G/K.

Theorem: (See DF, Theorem 3.20) Let G be a group and N be a normal subgroup of G. Then map $A \mapsto A/N$ is a bijection between the set of all subgroups of G/N and the set of subgroups of G containing N. Furthermore, if A and B are subgroups of G containing N, then 1. $A \subset B$ if and only if $A/N \subset B/N$ 2. If $A \subset B$ then $[A : B] = [A/N : B/N]$ 3. $\langle A, B\rangle/N = \langle A/N, B/N\rangle$ 4. $(A \cap B)/N = (A/N \cap B/N)$ 5. A is normal in G if and only if A/N is normal in G/N.

In other words, the lattice of subgroups of G/N is exactly the sublattice of the lattice of subgroups of G containing N.