Day 7

Automorphisms

$\mathrm{Aut}(G)$ is the group of isomorphisms from G to G under composition.

For $\mathbb{Z}/n\mathbb{Z}$, the automorphism group is $\mathbb{Z}/n\mathbb{Z}^\times$, the multiplicative group of elements relatively prime to n.

For $\mathbb{Z}/n\mathbb{Z}^k$ the automorphism group is $\mathrm{GL}_n(\mathbb{Z}/n\mathbb{Z})$, the invertible $n \times n$ matrices with entries in $\mathbb{Z}/n\mathbb{Z}$.

The *inner automorphisms* of G are the conjugations $f_g : G \to G$ given by $f_g(h) = ghg^{-1}$. The inner automorphisms form a normal subgroup of the automorphism group. The quotient group is called the group of *outer automorphisms.*

S_n has the weird property that all of its automorphisms are inner unless $n = 6$.

Conjugacy in S_n

The conjugacy classes in S_n correspond to the cycle decompositions, and there is one class for each partition of n as a sum of positive integers.

The centralizer of a cycle are the permutations which fix the integers appearing in the cycle.

The normalizer of a cycle was computed in the homework, at least in one case.

A_5 is a simple group

The conjugacy classes in S_5 that contain even permutations are contained in A_5 but they might split up into multiple classes. - there is one conjugacy class of 3-cycles in A_5 (there are 20 of these) - there are two conjugacy classes of 5-cycles in A_5 (there are 4!=24 5 cycles, but they split into two groups of 12) - all elements of order 2 in A_5 are conjugate to (12)(34). There are 15 of these.

So the conjugacy classes in A_5 have orders 1, 12, 15, and 20.

If H were a normal subgroup, it would have to have order dividing 60, or 1,2,3,4,5,6,10,12,15,20,30,60. And it would have to be 1 plus a sum of some subset of 12,12,15,20. The only way that works is if it has order 1 or 60.

Proof of the Sylow theorems

See the main page for this section.