First midterm exam

Instructions

You may refer to the book or your notes, or consult other sources, but you must cite any materials you use outside of the text and the course notes. Please do not consult with other students.

Problem 1

Suppose that a group G acts transitively on a set X, and that $H = \mathrm{Stab}_G(x)$ is the stabilizer of a element of X. The orbit-stabilizer theorem says that there is a bijective map π from the homogeneous space G/H (meaning the cosets of H viewed as a set, not necessarily as a group) to X given by $\pi(gH) = gx$ which has the property that $\pi(h \cdot gH) = h \cdot \pi(gH)$. In particular, if X is finite, then the order of X is the same as $[G : H]$.

Give three interesting applications of the orbit-stabilizer theorem with explanation of why they are interesting. Your answer should run no more than 2 pages.

Problem 2

1. Describe the automorphism group $\mathrm{Aut}(\mathbb{Z}/8\mathbb{Z})$ and show that it is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
2. Show that there are four distinct homomorphisms from $\mathbb{Z}/2\mathbb{Z}$ into $\mathrm{Aut}(\mathbb{Z}/8\mathbb{Z})$ (including the trivial one). This yields 4 semidirect products $\mathbb{Z}/8\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$.
3. Match these 4 semidirect products with the groups $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, D_{16}, and the two groups QD_{16} described in DF, exercise 11, on page 71 and M described in DF, exercise 14 on page 72.

Problem 3

Let A be a commutative ring with unity.

1. An element $a \in A$ is called *nilpotent* if there is an integer $n \geq 1$ such that $a^n = 0$. Prove that the set of all nilpotent elements in A form an ideal in A.
2. If $A = \mathbb{Z}/m\mathbb{Z}$ where $m = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, find the ideal I of nilpotent elements in A.

Problem 4

Let R be a commutative ring with unity. Let a and b be nonzero elements of R. Define a *least common multiple* of a and b to be an element $x \in R$ such that:

1. a divides x and b divides x.
2. If a divides y and b divides y then x divides y.

Prove that:

a. a least common multiple of a and b, if it exists, is a generator for the unique largest principal ideal contained in $(a) \cap (b)$.

b. any two nonzero elements in a Principal Ideal Domain have a least common multiple that is unique up to multiplication by a unit.

c. in a principal ideal domain, we have $ab = \operatorname{lcm}(a, b) \gcd(a, b)$ up to a unit.

Note: refer to the definitions in DF, on page 274, for divisibility in commutative rings and Proposition 6 on page 280 for gcd in PIDs.