Day 6

Quick review of examples of group actions

- Dihedral group D_{2n} acts on vertices of regular polygon with n sides. The stabilizer of a vertex is of order two; cosets are in bijection with n vertices. This group also acts on itself by conjugation. What are the orbits?
- $\mathrm{GL}_2(\mathbb{R})$ acts transitively on lines through the origin in $\mathbb{R}^2$. The stabilizer of the x-axis are the matrices of shape

  $$\begin{pmatrix} a & bf \\ 0 & d \end{pmatrix}$$

  The cosets are in bijection with points of the real projective line, with representatives

  $$\begin{bmatrix} u & 1 \end{bmatrix}$$

  and

  $$\begin{bmatrix} 1 & 0 \end{bmatrix}.$$

  The action on a coset representative is by linear fractional transformations in u.
- The symmetric group acts on the set $\{1, \ldots, \}n$ through its natural realization as bijections of this set to itself.
- The symmetric group acts on its conjugacy classes. What can you say about S_4 acting on its conjugacy classes?
- Let F be a finite graph. The automorphisms of F are the maps from F to itself that preserve the edges (so connected vertices stay connected). Examples?

Proposition: Suppose that G has order n and p is the smallest prime dividing n. Then any subgroup of G of index p is normal.

Proof: Consider the action of G on the p cosets of H. The kernel of this action is the set of $g \in G$ such that $gg_1H = g_1H$, which means that $gg_1 = g_1h$ for some $h \in H$. This in turn means that $g \in g_1Hg_1^{-1}$. Since g_1 is arbitrary, we must have g in the intersection of all conjugates $g_1Hg_1^{-1}$ as g_1 ranges over G. This is the largest normal subgroup of G contained in H – call it K.

The group G/K acts faithfully on the cosets G/H, so it is isomorphic to a subgroup of S_p. Therefore $[G : K]$ divides $p!$. But $[G : K] = [G : H][H : K] = p[H : K]$ and if this is a divisor of $p!$ then $[H : K]$ can only have prime divisors less than p. Since p is the smallest prime divisor of the order of G, this means $[H : K] = 1$ so $K = H$ and H is normal.

Remark: In the course of this we proved that, in general, the kernel of the action of G on G/H is the largest normal subgroup of G contained in H.

Class equation and applications

Lemma: The stabilizer of an element $g \in G$ under conjugation is the centralizer of $\{g\}$.

Theorem: Let G be a finite group. Let G act on itself by conjugation, yielding a partition of G into disjoint conjugacy classes $K_1, \ldots, K_g$. Choose a representative g_i for each class. Then

$$\mid G \mid = \sum_{i=1}^{g} \mid K_i \mid = \sum_{i=1}^{g} [G : C_G(g_i)].$$

Proposition: Let G be a group of prime power order. Then G has nontrivial center.

Corollary: If G has order p^2 for some prime p, then G is abelian.

Proof: If xZ generates the quotient group, then every element of G is of the form $x^i z^j$ with $z \in Z$. This forces G to be abelian.

Conjugacy in S_n

The conjugacy classes in S_n correspond to the cycle decompositions, and there is one class for each partition of n as a sum of positive integers.

The centralizer of a cycle are the permutations which fix the integers appearing in the cycle.

The normalizer of a cycle was computed in the homework, at least in one case.

A_5 is a simple group

The conjugacy classes in S_5 that contain even permutations are contained in A_5 but they might split up into multiple classes. - there is one conjugacy class of 3-cycles in A_5 (there are 20 of these) - there are two conjugacy classes of 5-cycles in A_5 (there are 4!=24 5 cycles, but they split into two groups of 12) - all elements of order 2 in A_5 are conjugate to (12)(34). There are 15 of these.

So the conjugacy classes in A_5 have orders 1, 12, 15, and 20.

If H were a normal subgroup, it would have to have order dividing 60, or 1,2,3,4,5,6,10,12,15,20,30,60. And it would have to be 1 plus a sum of some subset of 12,12,15,20. The only way that works is if it has order 1 or 60.