12. Trace and Determinant

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Trace

Definition: The *trace* is the linear map

$$\mathrm{Tr} : M_n(F) \to F$$

given by the sum of the diagonal elements; namely, if $A = (a_{ij})$, then

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$$

In addition to linearity, the Trace map satisfies the following property.

Proposition: If A and B are two matrices in $M_n(F)$, then $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. More generally, given three matrices A, B, and C, we have

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CBA).$$

Definition: If $L : V \to V$ is a linear map, then the trace of L is the trace of the matrix $[L]_B^B$ where B is any basis of V.

This definition makes sense because any two matrix representations of the linear map L are related by conjugation:

$$[L]_B^B = C[L]_A^A C^{-1}.$$

Then

$$\mathrm{Tr}([L]_B^B) = \mathrm{Tr}(C[L]_A^A C^{-1}) = \mathrm{Tr}([L]_A^A C^{-1} C) = \mathrm{Tr}([L]_A^A)$$

so any two matrix representations have the same trace.

Multilinear maps

Definition: A map $F : V_1 \times V_2 \times V_k \to W$ is called multilinear if

$$

F(v_1,…, av_s,… v_k)=aF(v_1,…, v_s,…, v_k) $$

and

$$F(v_1, \ldots, v_s + v'_s, \ldots, v_k) = F(v_1, \ldots, v_s, \ldots, v_k) + F(v_1, \ldots, v'_s, \ldots, v_k)$$

for any scalar a, index s, and vectors v_s and v'_s in V_s. In other words, F is linear in each of its variables provided the other variables are held constant.

Definition: A multilinear linear map $F : V^k \to W$ is *alternating* if $F(v_1, \ldots, v_k) = 0$ whenever two of the v_i are the same.

Determinant

The *determinant* of a matrix $A = (a_{ij})$ in $M_n(F)$ can be given by the formula

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

For a two by two matrix this gives

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

.

Proposition: The determinant has the following properties.

- It is multiplicative, so that $\det(AB) = \det(A)\det(B)$.
- Viewed as a function of the columns of A (so as a map from $V^n \to F$), it is an alternating multilinear map.
- The determinant of the identity map is 1.

In fact, the determinant is the **unique alternating multilinear map from $M_n(F) \to F$ (taking the columns of the matrix as the independent variables) which takes the value 1 on the identity matrix.** Multiplicativity follows from this characterization.

Proposition: The determinant of the transpose of a matrix is the same as the determinant of the matrix.

Other important properties of the determinant (see DF pgs 438-440):

- One can (in principle, but not in practice) compute it recursively using the "cofactor" expansion yielding the determinant of a big matrix as a linear combination of determinants of submatrices.

- There is a (useless in practice) formula for the inverse of a matrix in terms of determinants of submatrices. ("Cramer's Rule")

Although Cramer's rule is useless in practice it has the theoretical consequence that a matrix M over an integral domain is invertible if and only if its determinant is a unit in R. It also says that given a matrix M over an integral domain, there is always a matrix N so that $MN = \det(M)I$.