14. Additional topics in linear algebra

The Singular Value Decomposition

Suppose that $T : V \to W$ is a linear map between inner product spaces. The adjoint T^* of T is the linear map $T^* : W \to V$ satisfying

$$\langle w, Tv \rangle = \langle T^* w, v \rangle.$$

To see that T^* exists, fix w and consider the linear map $\phi_w : v \mapsto \langle w, Tv \rangle$. Since ϕ_w belongs to $\mathrm{Hom}(V, F)$, there is a unique $v' \in V$ such that $\phi_w(v) = \langle v', v \rangle$. Then $v' = T^* w$ by definition. The adjoint has the properties:

- $(S + T)^* = S^* + T^*$ and $(aS)^* = \overline{a} S^*$
- $(ST)^* = T^* S^*$.
- $(T^*)^* = T$
- $(T^{-1})^* = (T^*)^{-1}$
- The identity is its own adjoint.

The spectral theorem (over $\mathbb{R}$) applies to self-adjoint operators $T : V \to V$ where V is an inner product space. The singular value decomposition is a generalization to operators $T : V \to W$. So let V and W be real inner product spaces and $T : V \to W$ a linear map.

Let $Q = T^* T$. Then Q is a map from $V \to V$:

- Q is self-adjoint.
- $\langle Qv, v \rangle \geq 0$ for all $v \in V$. A self adjoint operator with this property is called *positive*.
- The null space of Q is the same as the null space of T.
- The range (column space) of Q is the same as the column space of T^*.
- The ranks of T, *, and Q all coincide.

Proof:

The key point is that $\langle Qv, v \rangle = \langle T^* T v, v \rangle = \langle Tv, Tv \rangle$. So $\langle Qv, v \rangle$ is greater than or equal to zero; and if it equals zero then $Tv = 0$ and conversely. This proves that $\mathrm{null}(Q) = (T)$.

In general:

- null space of T^* is orthogonal to the range of T.

- range of T^* is orthogonal to the null space of T.
- null space of T is orthogonal to the range of T^*.
- range of T is orthogonal to the nullspace of T^{ast}.

So since the null space of Q and the null space of T coincide, and Q is self-adjoint, we have:

$$\text{range} Q = (\text{null} Q)^{\perp} = (\text{null}(T))^{\perp} = \text{range}(T^*).$$

Also

$$\dim \text{range}(T) = \dim \text{null}(T^*)^{\perp} = \dim W - \dim \text{null}(T^*) = \dim \text{range}(T^*).$$

Definition: If $T : V \to W$ is a linear operator, then $Q = T^*T$ is diagonalizable. Let Λ be diagonal the matrix of Q in an orthonomormal basis given by the spectral theorem, with eigenvalues listed in decreasing order. The singular values of T are the entries in $\Lambda^{1/2}$ – the square roots of the eigenvalues of Q.

Proposition: (Singular value decomposition) Let $T : V \to W$ be a linear map of inner product spaces with singular values $s_1, \ldots, s_n$. Then there are orthonormal basis $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ for V and W such that

$$T(v) = \sum_{i=1}^{n} s_i < v, e_i > f_i$$

for all $v \in V$. In matrix terms this basically means that T is "diagonal" in the bases given by the e's and f's (although T needn't be square).

This is usually written like this for matrices.

Theorem: (SVD) Let A be an $m \times n$ matrix over $\mathbb{R}$. Then there are orthogonal matrices P and Q such that

$$A = PDQ$$

where P is $m \times m$, Q is $n \times n$, and D is $m \times n$. The "diagonal" entries of D are the singular values of A.

This is the matrix version of the previous statement.

Proof:

There's an orthonormal basis of V so that $Qe_i = \lambda_i e_i$ where the λ_i are the eigenvalues of the self-adjoint operator Q. For the nonzero eigenvectors e_i for $i = 1, \ldots, r$, the vectors

$$f_i = \frac{Te_i}{\sqrt{\lambda_i}}$$

are orthonormal. We can complete the set of f_i (if needed) to a basis of W. Any $v \in V$ can be written

$$v = \sum_{i=1}^{n} \langle v, e_i \rangle e_i.$$

The sought-after formula comes from applying T to this.