First midterm exam

Instructions

Problem 1

Suppose that a group G acts transitively on a set X, and that $H = \mathrm{Stab}_G(x)$ is the stabilizer of a element of X. The orbit-stabilizer theorem says that there is a bijective map π from the homogeneous space G/H (meaning the cosets of H viewed as a set, not necessarily as a group) to X given by $\pi(gH) = gx$ which has the property that $\pi(h \cdot gH) = h \cdot \pi(gH)$. In particular, if X is finite, then the order of X is the same as $[G : H]$.

Give three interesting applications of the orbit-stabilizer theorem with explanation of why they are interesting. Your answer should run no more than 2 pages.

Problem 2

1. Describe the automorphism group $\mathrm{Aut}(\mathbb{Z}/8\mathbb{Z})$ and show that it is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
2. Show that there are four distinct homomorphisms from $\mathbb{Z}/2\mathbb{Z}$ into $\mathrm{Aut}(\mathbb{Z}/8\mathbb{Z})$ (including the trivial one). This yields 4 semidirect products $\mathbb{Z}/8\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$.
3. Match these 4 semidirect products with the groups $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, D_{16}, and the two groups QD_{16} described in DF, exercise 11, on page 71 and M described in DF, exercise 14 on page 72.

Problem 3

Let A be a commutative ring with unity.

1. An element $a \in A$ is called *nilpotent* if there is an integer $n \geq 1$ such that $a^n = 0$. Prove that the set of all nilpotent elements in A form an ideal in A.
2. If $A = \mathbb{Z}/m\mathbb{Z}$ where $m = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, find the ideal I of nilpotent elements in A.

Problem 3

Let R be a commutative ring with unity. Let a and b be nonzero elements of R. Define a *least common multiple* of a and b to be an element $x \in R$ such that:

1. a divides x and b divides x.
2. If a divides y and b divides y then x divides y.

Prove that:

a. a least common multiple of a and b, if it exists, is a generator for the unique largest principal ideal contained in $(a) \cap (b)$.

b. any two nonzero elements in a Principal Ideal domain have a least common multiple that is unique up to multiplication by a unit.

c. in a principal ideal domain, we have $ab = \operatorname{lcm}(a, b) \gcd(a, b)$ up to a unit.

Note: refer to the definitions in DF, on page 274, for divisibility in commutative rings.