Comments on HW set 3

Problem 2

We want to show that there is exactly one group G of order 75 up to isomorphism.

If G has order 75, then it has a Sylow 5-subgroup H of order 25 and a Sylow 3 subgroup K of order 3. Notice that $H \cap K$ must be trivial since H and K have coprime orders.

The number n_5 of Sylow 5-subgroups must both divide 3 and also be congruent to one mod 5, so $n_5 = 1$. This means H is normal. Since all groups of order p^2, where p is prime, are abelian, H must be abelian. Therefore H is either $\mathbb{Z}/25\mathbb{Z}$ or $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

Meanwhile, the number n_3 of Sylow 3-subgroups must divide 25 and be congruent to 1 mod 3. Therefore either $n_3 = 1$ or $n_3 = 25$. If $n_3 = 1$, then K is normal. In this case, HK is abelian and so G is either $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$ or $\mathbb{Z}/75\mathbb{Z}$.

If $n_3 = 25$, then K is one of 25 conjugate subgroups of G of order 3 and $G = HK$ is the semidirect product of H with K arising from a map $K \to \mathrm{Aut}(H)$.

If $H = \mathbb{Z}/25\mathbb{Z}$, then $\mathrm{Aut}(H)$ is of order $\phi(25) = 20$, and there are no non-trivial maps from K to $\mathrm{Aut}(H)$, so in this case the only group is the abelian group $\mathbb{Z}/75\mathbb{Z}$.

If $H = \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$, then $\mathrm{Aut}(H)$ is $\mathrm{GL}_2(\mathbb{Z}/5\mathbb{Z})$ which has order 480. By Cauchy's theorem, $\mathrm{Aut}(H)$ has an element σ of order 3. We can fix an isomorphism of K with $\mathbb{Z}/3\mathbb{Z}$ and define a map $\mathbb{Z}/3\mathbb{Z} \to \mathrm{Aut}(H)$ sending 1 to σ. The resulting semidirect product is nonabelian.

Now we need to prove that this semi-direct product is unique up to isomorphism. We can use problem 1 for this, but it is a little subtle.

Looking at the construction of the semi-direct product, we needed to choose an element σ of order 3 in $\mathrm{GL}_2(\mathbb{Z}/5\mathbb{Z})$ and then we defined

$$\phi : \mathbb{Z}/3\mathbb{Z} \to \mathrm{GL}_2(\mathbb{Z}/5\mathbb{Z})$$

by sending $\phi(1) = \sigma$. The element σ generates a Sylow 3-subgroup in $\mathrm{GL}_2(\mathbb{Z}/5\mathbb{Z})$; call this $A(\sigma) \subset \mathrm{GL}_2(\mathbb{Z}/5\mathbb{Z})$.

The first thing to notice is that, by Sylow's theorem (which is stronger than Cauchy's Theorem), every Sylow 3-subgroup of is of order 3 and they are all conjugate. This is because 480 is divisible by 3 but not 9, so these elements of order 3 actually generate Sylow 3-subgroups, all of which are conjugate.

Suppose we had chosen a *different* element of order 3 in our construction, say σ'. Then the subgroup generated by σ' would be conjugate to $A(\sigma)$. This means we can find an automorphism g in $\mathrm{GL}_2(\mathbb{Z}/5\mathbb{Z})$ that conjugates $A(\sigma')$ into $A(\sigma)$. In other words, $gA(\sigma')g^{-2} = A(\sigma)$. *However,* it could happen that when we do this conjugation, that $g\sigma' g^{-1} = \sigma^{-1}$ because both σ and σ^{-1} generate $A(\sigma)$. However, we have an automorphism $f : A(\sigma) \to A(\sigma)$ such that $f(\sigma^{-1}) = \sigma$.

In other words, any two nontrivial maps $\phi, \tau : \mathbb{Z}/3\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z})$ can be changed into each other by finding $g \in \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z})$ and $f \in \mathrm{Aut}(\mathbb{Z}/3\mathbb{Z})$ so that

$$\phi = \gamma_g \circ \tau \circ f$$

and therefore by problem 1 the corresponding groups are isomorphic.