Day 25

Proof of the real spectral theorem

Orthogonal bases and Gram-Schmidt

Proposition: (The Gram-Schmidt process) Let V be a real inner product space of dimension n, and let $v_1, \ldots, v_k$ be a linearly independent set in V. Then there is a set $w_1, \ldots, w_k$ of vectors such that

- $\langle w_i, w_j \rangle = 0$ if $i \neq j$.
- the span of $w_1, \ldots, w_l$ is the same as the span of $v_1, \ldots, v_l$ for $l \leq k$.

Proof: Let $w_1 = v_1$ and

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1.$$

Then the span of w_2 and w_1 is the same as that of v_2 and v_1, and $\langle w_2, w_1 \rangle = 0$ by construction. Now suppose we have constructed $w_1, \ldots, w_l$ with the desired property. Set

$$w_{l+1} = v_{l+1} - \sum_{i=1}^{l} \frac{\langle v_{l+1}, w_i \rangle}{\|w_i\|^2} w_i.$$

Then $\langle w_{l+1}, w_i \rangle = 0$ and the span property is preserved.

Orthogonal complements

If W is a subspace of V, define $W^\perp = \{v : \langle v, w \rangle = 0\}$.

Proposition: $W^\perp$ is a subspace of V. Furthermore:

- $\dim W + \dim W^\perp = \dim V$ (so $W \cap W^\perp = 0$.)
- $(W^\perp)^\perp = W$.
- if $U \subset W$, then $W^\perp \subset U^\perp$.

Proof: Suppose $\dim W = k$ and $\dim V = n$. Use Gram-Schmidt to construct an orthogonal basis $v_1, \ldots, v_n$ for V whose first k elements are an orthogonal basis for W. A vector

$$v = \sum a_i v_i$$

is in $W^{\perp}$ if and only if $a_i = 0$ for $i = 1, \ldots, k$.

Proposition: Suppose that A is a self adjoint operator and $AW \subset W$. Then $AW^{\perp} \subset W^{\perp}$.

Proof: Suppose $z \in W^{\perp}$ and $w \in W$. Then

$$\langle Az, w \rangle = \langle z, A^* w \rangle = \langle z, Aw \rangle = 0$$

since $Aw \in W$.

Proof of the (real) spectral theorem

We have a self-adjoint map $A : V \to V$. Pick a basis for V and use Gram-Schmidt to construct an orthonormal basis (an orthogonal basis where the elements all have norm 1).

The Q-matrix for this basis is the identity, and so the inner product is just the dot product.

If $[A]$ is the matrix representation of A in this basis, then the matrix representation of A^* is the transpose of $[A]$. So since A is self-adjoint, $[A]$ is symmetric.

We know that a symmetric matrix has a real eigenvalue λ_1 with eigenvector v_1.

Let V_1 be the orthogonal complement to the one-dimensional space W spanned by v_1. Since v_1 is an eigenvector, $AW \subset W$. Therefore $AV_1 \subset V_1$. Furthermore, if $x, y \in V_1$, then

$$\langle Ax, y \rangle = \langle x, Ay \rangle$$

so A is self-adjoint as a linear map from V_1 to itself. Thus we can continue by induction to construct an orthogonal basis of eigenvectors for A.

Orthogonal matrices

Let $Q \in M_n(\mathbb{R})$ be a symmetric matrix. As such it is a self adjoint map from $\mathbb{R}^n$ to itself with respect to the usual dot product. Therefore there is a basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$ consisting of orthonormal eigenvectors for the dot product Q-eigenvalues $\lambda_1, \ldots, \lambda_n$.

Let P be the matrix whose columns are the vectors v_i written in the standard basis of $\mathbb{R}^n$. Since the v_i are an orthonormal basis, the matrix P satisfies $P^T P = I$.

At the same time,

$$QP = P\Lambda$$

where Λ is the diagonal matrix with entries λ_i. Since the v_i are linearly independent, the matrix P is invertible and Q is diagonalizable:

$$P^{-1}QP = \Lambda$$

The bilinear map $\langle v, w\rangle$ defined by

$$\langle v, w\rangle = v^T Q w$$

is an inner product provided that $\langle v, v\rangle \geq 0$ with equality only one $v = 0$. If we write $v \neq 0$ in terms of the orthogonal basis $v_1, \ldots, v_n$:

$$v = \sum a_i v_i$$

then we get

$$\langle v, v\rangle = (\sum a_i v_i^T)Q(\sum a_i v_i) = \sum a_i v_i^T \lambda_i v_i = \sum a_i^2 \lambda_i v_i^T v_i$$

which will be positive provided that all $\lambda_i > 0$.

Example

Let Q be the symmetric matrix

$$Q = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

Its eigenvalues are 2 and 4 with eigenvectors

$$\begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

and

$$\begin{bmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{bmatrix}.$$

The norm of a vector in the inner product given by Q is

$$\|(x, y)\| = 3x^2 + 2xy + 3y^2$$

The level curves of this are ellipses, and the eigenvectors point in the directions of the major and minor axes of the ellipse.

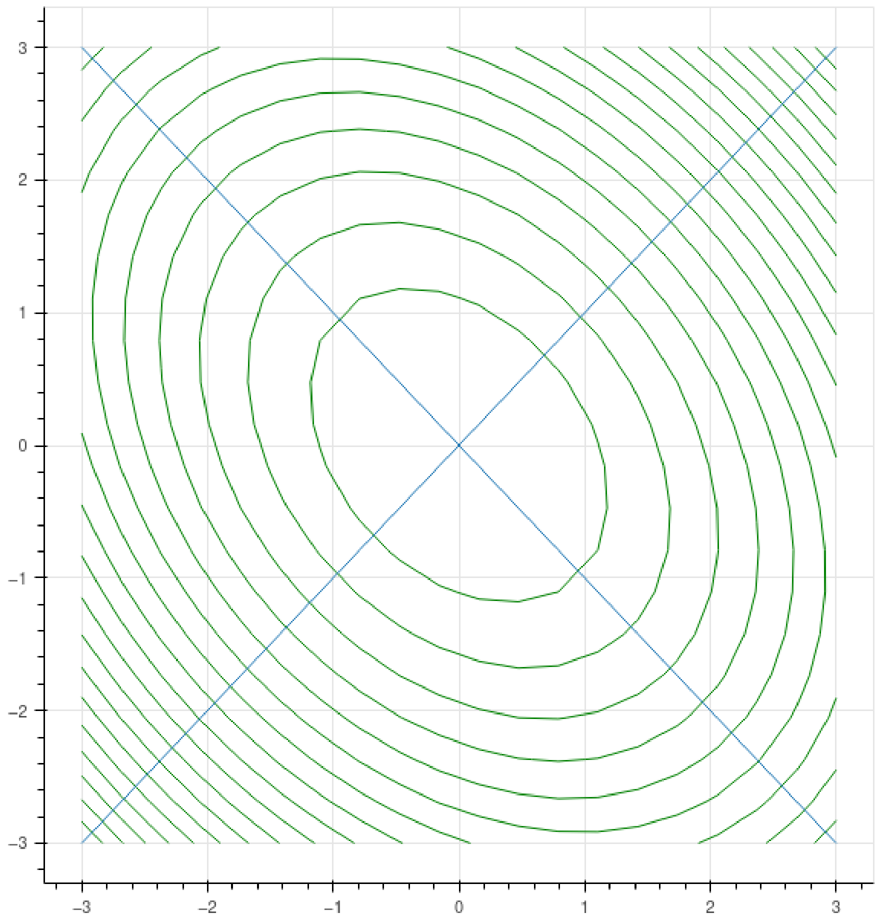

Figure 1: img

These ellipses are the family

$$2(x-y)^2 + 4(x+y)^2 = C$$