Day 3

Generators

A group is *finitely generated* if there is a finite set A which generates it. - finite groups are finitely generated - $\mathbb{Z}$ is finitely generated - $\mathbb{Q}$ is *not* finitely generated.

Special subgroups

If H is normal in G and G is normal in K, is H normal in K?

If H is normal in G, and $f : G \to K$ is a homomorphism, is $f(H)$ normal in K?

Let G be the dihedral group D_{2n} of symmetries of a regular n-gon. - Show that the subgroup of rotations is normal. - Let H be the subgroup generated by a reflection. What is $N_G(H)$? What is $C_G(H)$? - What is the center of G? - Draw the lattice of subgroups of D_{2n}.

Let Q be the quaternion group with 8 elements. Draw its lattice of subgroups. (The quaternion group has 8 elements $\{\pm 1, \pm i, \pm j, \pm k\}$ where each of i, j, k satisfy $x^2 = -1$ and $ijk = -1$. Compare Q with D_8.

Show that Q is isomorphic to the subgroup of $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ generated by the matrices $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Show that the center of S_n is trivial if $n \geq 3$.

What is the normalizer of the rotation group in $\mathrm{GL}_2(\mathbb{R})$? What is its centralizer? Interpret this in terms of group actions.

The affine group $\mathrm{Aff}(\mathbb{R}^2)$ of the plane is the subgroup of $\mathrm{GL}_3(\mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}.$$

It acts on points (u, v) viewed as column vectors

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

by matrix multiplication.

Show that $\mathrm{GL}_2(\mathbb{R})$ viewed as the upper triangular block is a normal subgroup of the affine group.

The additive group $\mathbb{R}^2$ is a subgroup of the affine group; what is its normalizer?

The subgroup of the affine group where the upper triangular matrix is a rotation matrix (an "orthogonal matrix") is called the Euclidean group.

Every group is a permutation group

Theorem (Cauchy): Any group is a subgroup of a permutation group.

Proof: Given $g \in G$, let $f_g : G \to G$ be the map $f_g(h) = gh$. This gives a map from G to $S(G)$. We have $f_e(h) = eh = h$ so f_e is the identity map. Also

$$f_{ab}(h) = abh = (f_a \circ f_b)(h)$$

for any $h \in G$. Since composition of functions is the group operation in $S(G)$, the map $g \mapsto f_g$ is a homomorphism from $G \to S(G)$.

Finally, the map f_h is the identity map only when $h = e$. So this map is injective.

Make this explicit for the Dihedral group.