

# Problem Set 1

**Instructions:** Write up your solutions using LaTeX and submit them on HuskyCT by September 11, 2022.

**Problem 1:** Let  $\sigma = (13)(1235)(4567)$  and  $\tau = (24)(35)(245)$ .

1. Find  $\sigma\tau$ .
2. Find disjoint cycle decompositions of both  $\sigma$  and  $\tau$ .
3. Write each of  $\sigma$  and  $\tau$  as products of transpositions.
4. Find the sign of  $\sigma$  and  $\tau$ .

**Problem 2:** Define an ordering  $\lesssim$  on the positive integers *greater than one* by saying that  $n \lesssim m$  if there is an injective homomorphism from  $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ .

1. Show that this relation is reflexive ( $n \lesssim n$  for all  $n$ ), antisymmetric ( $n \lesssim m$  and  $m \lesssim n$  implies  $n = m$ ), and transitive ( $n \lesssim m$  and  $m \lesssim k$  implies  $n \lesssim k$ .) These axioms mean that the positive integers are a *partially ordered set* under this order relation.
2. An element  $n$  of this set is *minimal* if  $m \lesssim n$  implies  $m = n$ . What are the minimal elements of this partial order?
3. The *meet*  $g$  of two elements  $a$  and  $b$  in a partially ordered set is an element that satisfies these two conditions:
  - $g \lesssim a$  and  $g \lesssim b$ .
  - If  $h$  is any element satisfying  $h \lesssim a$  and  $h \lesssim b$ , then  $h \lesssim g$ .

Prove that any two elements  $m$  and  $n$  have a *meet*. 4. Describe all of this fancy stuff in a simpler way. 5. (Extra) What happens if, instead of considering injective homomorphisms from  $\mathbb{Z}/n\mathbb{Z}$  to  $\mathbb{Z}/m\mathbb{Z}$ , we consider surjective ones?

**Problem 3:** If  $H$  is a subgroup of  $G$ , then the normalizer  $N_G(H) = \{g \in G : gHg^{-1} = H\}$  and the centralizer  $C_G(H) = \{g \in G : gh = hg \ \forall h \in H\}$ .

1. Let  $G = \text{GL}_2(\mathbb{R})$  and let  $H = \text{Aff}(\mathbb{R})$  be the affine group consisting of two by two matrices of the form  $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$  where  $x \neq 0$ . Show that the centralizer of  $H$  in  $G$  consists of matrices  $aI$  where  $a \neq 0$  and  $I$  is the identity matrix, and the normalizer consists of the upper triangular matrices with nonzero diagonal entries. What about the same question for  $\text{GL}_2(\mathbb{Q})$ ? Does the field matter for this?
2. Let  $G = \text{GL}_2(\mathbb{R})$  and  $H$  be the subgroup of diagonal matrices. Show that  $H$  is its own centralizer, and its normalizer consists of diagonal matrices and “anti-diagonal” matrices  $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$  in  $G$ .
3. View  $S_n$  as  $\text{Sym}(\mathbb{Z}/n\mathbb{Z})$ . Given integers  $a$  and  $b$ , with  $\gcd(a, n) = 1$ , let  $\sigma_{a,b} : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$  be the map  $\sigma_{a,b}(x) = ax + b \pmod{n}$ . Show that  $\sigma_{a,b}$  is in the normalizer of the cyclic subgroup  $H$  generated by  $(123 \dots n)$  and, conversely, any element of the normalizer of this subgroup is  $\sigma_{a,b}$  for

some  $a$  and  $b$ . Conclude that the normalizer in  $S_n$  of  $H$  is the affine group of  $\mathbb{Z}/n\mathbb{Z}$ .

**Problem 4:** Let  $M$  be the group with two generators  $u$  and  $v$  satisfying  $u^2 = v^8$  and  $vu = uv^5$ . This group has order 16 (can you verify this?).  $M$  has three subgroups of order 8:  $\langle u, v^2 \rangle$ ,  $\langle v \rangle$ , and  $\langle uv \rangle$ . Every proper subgroup is contained in one of these three groups.

1. Draw the lattice of subgroups of  $M$ .
2. Prove that the group generated by  $v^4$  is normal in  $M$ .
3. Find the lattice of subgroups of  $M/\langle v^4 \rangle$  inside that of  $M$  using the lattice isomorphism theorem.