Day 12

Ring homomorphisms

Some example computations with ideals and quotient rings

- ideals of $\mathbb{Z}$ and quotients
- ideals $(x^2+1)R$ of $R=\mathbb{Q}[x]$, $R=\mathbb{R}[x]$, and $R=\mathbb{C}[x]$
- Ring of functions $X \to A$ and the evaluation map at points of X.
- Evaluation map on polynomials $R[x] \to S$ extending $R \to S$ given by evaluation at $s \in S$.
- $\mathbb{Z}/p\mathbb{Z}[x]/(f(x))$
- $\mathbb{Z}[i]/2$, $\mathbb{Z}[i]/3$, $\mathbb{Z}[i]/5$
- two sided ideals of $M_n(R)$
- examples of left ideals of $M_n(R)$.

Isomorphism theorems

1. R a ring, A a subring, and B and ideal of R. Let $A+B=\{a+b|a \in A, b \in B\}$. Then $A \cap B$ is an ideal of A and $(A+B)/B \cong A/(A \cap B)$.
2. If I and J are ideals of R and $I \subset J$, then J/I is an ideal of R/I and $(R/I)/(J/I) \cong R/J$.
3. Let $I \subset R$ be an ideal. There is a bijective correspondence $A \to A/I$ between subrings of R containing I and subrings of R/I. This correspondence respects ideals, so A/I is an ideal of R/I if and only if A is an ideal of R.

Sums and products of ideals

1. The sum $I+J$ of two ideals is the collection of sums of elements of I and J; it is an ideal.
2. The product IJ is the subring generated by products ab with $a \in I$ and $b \in J$; it is an ideal.
3. I^n is the product of I with itself n times. it is an ideal.

Prime and maximal ideals

Suppose that R has an identity element 1 (and that 1 is not zero, so the ring is not trivial).

- An ideal $I = R$ if and only if I contains a unit.
- R is a field iff its only ideals are zero and R.
- Any homomorphism from a field F to another ring R is either zero or injective.