Day 5

Remarks on the isomorphism theorems

In the second theorem, we have AB/B isomorphic to $A/A \cap B$ assuming A is contained in $N_G(B)$. B is normal in AB because $abB = aB = Ba$ since $aBa^{-1} = B$.

This isomorphism comes from the maps

$$A \to AB \to AB/B$$

where the first is inclusion and the second is projection.

THe kernel of the composition is $A \cap B$, and the composition is surjective because every coset abB is of the form aB. This proves that $A \cap B$ is normal in A, since it's the kernel of a homomorphism.

In the Third theorem, we want to understand $(G/H)/(K/H) = G/K$ if H and K are normal in G and H is normal in K. The key is to use the map

$$G/H \to G/K$$

given by $aH \mapsto aK$. This is well defined since K contains H, and surjective since gK is the image of gH. The kernel of this map is the set of gH such that $g \in K$, which is exactly K/H.

Some applications of the isomorphism theorems

Proposition: Every subgroup of a cyclic group of order n is cyclic of order a divisor of n, and there is a unique such subgroup for every divisor d of n.

Suppose G is cyclic of order n. This means that G is isomorphic to the quotient group $\mathbb{Z}/n\mathbb{Z}$. If H is a subgroup of G, then H must be of the form $\tilde{H}/n\mathbb{Z}$ where $\tilde{H}$ is a subgroup of $\mathbb{Z}$ containing $n\mathbb{Z}$. This means that $H = d\mathbb{Z}$ where d is a divisor of n, and $H/n\mathbb{Z} = d\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $\mathbb{Z}/n/d\mathbb{Z}$.

Conversely, suppose d divides n. Then $d\mathbb{Z}$ contains $n\mathbb{Z}$, so $d\mathbb{Z}/n\mathbb{Z}$ is a subgroup of $\mathbb{Z}/n\mathbb{Z}$ of order n/d.

Thus every subgroup of $\mathbb{Z}/n\mathbb{Z}$ is cyclic of order dividing n, and there is a subgroup of each such order.

Finally suppose H_1 and H_2 are two subgroups of order d dividing n. Then

$$\tilde{H}_1$$

and $\tilde{H}_2$ are subgroups of $\mathbb{Z}$ of index n/d; since there is only one such subgroup, they are the same. Therefore there is a unique subgroup of G of each order d dividing n.

Note that, in the proof above, we never mention any elements of the group.

The commutator subgroup

Suppose that G is an arbitrary group. When is there a homomorphism from G to an abelian group?

Suppose A is abelian and $f : G \to A$ is a surjective homomorphism. Then every element of G of the form $xyx^{-1}y^{-1}$, where x and y are in G, is in the kernel of f.

Therefore the subgroup of G generated by these elements is in the kernel of f.

Note: Not every product of commutators is a commutator, so you really need $[G, G]$ to be the subgroup *generated* by the commutators. In fact the matrix $-I_2$ in $\mathrm{SL}_2(\mathbb{R})$ is not a commutator.

Definition: Let G be a group. If $x, y \in G$, let $[x, y] = xyx^{-1}y^{-1}$. This is called the commutator of x and y. Let $[G, G]$ be the subgroup of G generated by all such commutators. This is called the commutator subgroup of G.

Example: in the Dihedral group D_{2n}, the commutator subgroup is generated by r^2, so depending on whether n is even or odd the commutator subroup is R or its subgroup of index 2.

Proposition: Let G be a group. Then the commutator subgroup of G is normal. If $f : G \to A$ is a homomorphism where A is abelian, then there is a commutative diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"] & A \\ G/[G,G] \arrow[ur, "\overline{f}"] & \end{tikzcd}$$

where $\pi : G \to [G, G]$ is the quotient map. In other words, every homomorphism from G to an abelian group A “comes from” a homomorphism from $G/[G, G]$ to A.

This is the first isomorphism theorem once we observe that $G/[G, G]$ is abelian and $[G, G]$ is in the kernel of every map to an abelian group.

The commutator subgroup of $\mathrm{GL}_2(\mathbb{R})$ is $\mathrm{SL}_2(R)$. (This is proved by somewhat painful calculations.)