Recommended Problems

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Problem 1 (DF, problem 6, p. 375)

Let R be an integral domain with field of fractions Q. Prove that $(Q/R) \otimes_R (Q/R) = 0$. In particular $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z} = 0$.

Problem 2 (DF, problem 5, p. 375)

Let G be a finite abelian group and let p^k be the highest power of a prime p dividing $|G|$. Prove that $Z/p^k\mathbb{Z} \otimes G$ is the Sylow p-subgroup of G.

Problem 3 (DF, Problem 8, p. 375ff)

Suppose R is an integral domain with quotient field Q and let N be any R module. Let $U = R^*$ be the nonzero elements in R and let $U^{-1}N$ be the set of equivalence classes of pairs (u, n) with $(u, n) \sim (u', n')$ if and only if $u'n = un'$.

- Show that UN^{-1} is an abelian group with $(u_1, n_1) + (u_2, n_2) = (u_1u_2, u_2n_1 + u_1n_2)$ and that the operation $r(u, n) = (u, rn)$ makes $U^{-1}N$ into an R-module.
- Show that the map $Q \times N \to U^{-1}N$ sending $(a/b, n) \to (b, an)$ is R-balanced and so yields a unique map $Q \otimes N \to U^{-1}N$. Show that the map $(u, n) \to u^{-1} \otimes n$ is a well-defined inverse, and so $Q \otimes N$ is isomorphic to $U^{-1}N$.
- Show that $(1/d) \otimes n = 0$ if and only if there is $r \in R$ so that $rn = 0$.
- Show that $\mathbb{Q} \otimes A = 0$ (where A is an abelian group) if and only if A is torsion.

Problem 4 (DF Problem 16 pg. 376)

Let R be commutative and let I and J be ideals. Prove that every element of $R/I \otimes R/J$ has the form $1 \otimes r$ for $r \in R$. Then show that $R/I \otimes R/J$ is isomorphic to $R/(I + J)$ via the map sending $r \otimes r'$ to rr'.

Problem 5 (DF, Problem 17 and 19, pg. 37

Let $R = \mathbb{Z}[x]$ and let $I = (2, x)$. Show that the map $\phi : I \times I \to \mathbb{Z}/2\mathbb{Z}$ given by

$$\phi(fg) = f(0)g'(0)/2 \pmod 2$$

is R-bilinear. Thus it gives a map $I \otimes I \to \mathbb{Z}/2\mathbb{Z}$. Conclude that $2 \otimes x$ and $x \otimes 2$ are not equal in $I \otimes I$.