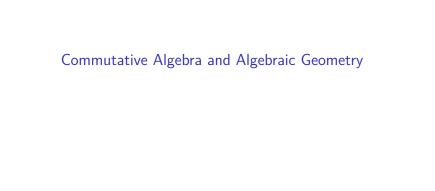
# 11. Commutative Algebra and Geometry



## Noetherian Rings

**Definition:** A ring is Noetherian if it satisfies either of the following equivalent conditions:

- 1. Every ideal is finitely generated.
- 2. Every ascending chain  $I_0 \subset I_1 \subset \cdots$  of ideals eventually stabilizes.

**Theorem:** (Hilbert's Basis Theorem) If R is Noetherian, then so is R[x].

**Proof:** The proof of Hilbert's theorem is given in detail in DF, Theorem 21 of Section 9.6. The idea is something like this. Suppose I is an ideal of R[x]. Then each element f of I is a polynomial of degree n:  $f = a_n x^n + \ldots$  where  $a_n \in R$  is not zero. Here  $a_n$  is called the leading term of f The ideal  $\ell(I)$  of leading terms of I is the ideal of R generated by the leading terms of all elements of I. Since R is Noetherian,  $\ell(I)$  is finitely generated. If  $b_1, \ldots, b_N$  generate  $\ell(I)$ , we can find elements  $f_1, \ldots, f_N$  so that  $f_i = b_i x_i^k + \ldots$  in I.

Now let g be any element of I of degree m. Suppose m is larger

## Affine space, points, and ideals

Affine *n*-space over a field k, written  $\mathbb{A}^n_k$  is the set of points  $(x_1, \ldots, x_n)$  with coordinates in k. The ring  $k[x_1, \ldots, x_n]$  is the ring of polynomial functions on  $\mathbb{A}^n_k$ .

Sets and ideals:

- ▶ If  $X \subset \mathbb{A}^n_k$  is a subset, then  $I(X) = \{f : f(x) = 0 \forall x \in \mathbb{A}^n_k \|.$
- ▶ If  $f_1, ..., f_k$  are a collection of elements in  $k[x_1, ..., x_n]$  then  $Z(f_1, ..., f_k) = \{x \in \mathbb{A}^n_k : f_1(x) = \cdots = f_k(x) = 0\}$ . Notice that  $Z(f_1, ..., f_k) = Z(I)$  where I is the ideal of functions generated by the  $f_i$ .

Both operations are inclusion reversing:  $X \subset Y$  implies  $I(Y) \subset I(X)$  and  $J \subset L$  implies  $Z(L) \subset Z(J)$ .

The sets Z(I) satisfy the axioms for the closed sets of a topology, called the *zariski topology*. A set X of the form Z(I) is called an (affine) algebraic set.

**Lemma:** If  $X \subset \mathbb{A}_k^n$ , then  $X \subset Z(I(X))$ . If  $J \subset k[x_1, \dots, x_n]$  is an

### Coordinate rings and morphisms

If X is an algebraic set, then the ring  $k[X] = k[x_1, \ldots, x_n]/I(X)$  is called the coordinate ring of X. Notice that elements of this ring give well-defined functions on X; they are called "regular functions" on X.

**Definition:** A map  $f: X \to Y$  between algebraic sets is called a morphism if there are polynomials so that the map f is given by

$$f(x_1,\ldots,x_n)=(\phi_1(x_1,\ldots,x_n),\cdots,\phi_m(x_1,\ldots,x_n))$$

A morphism whose inverse is also a morphism is called an isomorphism of algebraic sets.

In general, if  $f: x \to Y$  is a morphism, and  $h \in k[Y]$  is a regular function on Y then the composition  $h \circ f$  is a regular function on X. This operation is called *pullback*.

The pullback induces a welldefined algebra homomorphism  $f^*: k[Y] \to k[X]$ .

### Universal property

**Theorem:** Suppose  $\phi: X \to Y$  is an arbitrary map of algebraic sets. Then  $\phi$  is a morphism if and only if, for every  $f \in k[Y]$ , the composition  $f \circ \phi$  is in k[X]. If  $\phi$  is a morphism, then  $\phi(v) = w$  if and only if  $\tilde{\phi}^{-1}(I(v)) = I(w)$  where  $\tilde{\phi}$  is the pullback of  $\phi$ .

**Proof:** The second part first. If  $\phi(v) = w$ , and  $f \in I(w)$ , then  $f(\phi(v)) = 0$  so  $\tilde{\phi}(I(w)) \subset I(v)$  and  $(\tilde{\phi})^{-1}I(v) \supset I(w)$ . Since I(w) is maximal, we have  $(\tilde{\phi})^{-1}(I(v)) = I(w)$ . Conversely, if  $(\tilde{\phi})^{-1}(I(v)) = I(w)$ , then  $f(\phi(v)) = 0$  if and only if f(w) = 0 so  $I(\phi(v)) = I(w)$  and therefore  $\phi(v) = w$ .