12. Localization

Localization

Localization in commutative algebra is the algebraic version of restricting functions to open sets. Throughout this section, R is a commutative ring with unity, and ring maps send 1 to 1.

Definition: A subset $D \subset R$ is multiplicatively closed if $1 \in D$ and $x, y \in D$ implies $xy \in D$. If D is multiplicatively closed, then $D^{-1}R$ is the quotient of $R \times D$ by the relation $(x, y) \sim (x', y')$ if there is a $d \in D$ so that $d(xy - x'y) = 0$.

Proposition: $D^{-1}R$ is a commutative ring with unity. The operations treat (x, y) as if it was a fraction x/y, so that

$$(x/y)(x'/y') = (xx'/yy')$$

and

$$(x/y) + (x'/y') = (xy' + x'y)/(yy').$$

There is a natural map $u : R \to D^{-1}R$ that sends $x \to (x, 1)$. If $x \in D$, then $(x, 1)$ is a unit in $D^{-1}R$, with inverse $(1, x)$.

The ring $D^{-1}R$ satsifies a universal property. Given any map $f : R \to S$ such that the images $f(d)$ for $d \in D$ are units in S, there is a unique map $\phi : D^{-1}R \to S$ such that $\phi \circ u = f$.

Examples:

If R is an integral domain, and D consists of all nonzero elements of R, then $D^{-1}R$ is the field of fractions of R.

Suppose that $R = \mathbb{Z}/6\mathbb{Z}$ and $D = \{1, 2, 4\}$, the powers of 2. Then $(3, 1) = (0, 1)$ in $D^{-1}R$ because $2(3 - 0) = 0$. However, $(2, 1)$ is not zero. So that map $R \to D^{-1}R$ is not injective. Its kernel is the set $(a, 1)$ where $2a = 0$, and therefore $D^{-1}R = \mathbb{Z}/3\mathbb{Z}$.

More generally, suppose we are in the general situation and $(x, 1)$ is in the kernel of the map $R \to D^{-1}R$. Then there is an element $d \in D$ such that $dx = 0$. If

$x \neq 0$, this means that D contains a zero divisor (or zero). If $0 \in D$, then $D^{-1}R$ is the zero ring.

If R is a comm. ring with 1 and $f \in R$, then R_f is another notation for $D^{-1}R$ where D consists of 1 and all powers of f. If f is not a zero divisor, this ring contains a copy of R and also $1/f$. In fact $R_f = R[x]/(fx-1)$.

If P is a prime ideal, then the complement D of P is multiplicatively closed. We write R_P for $D^{-1}R$ in this case. Elements of R_P are rational functions that are defined on the algebraic variety arising from P

If $R = k[x_1, \ldots, x_n]$ and $M = (x_1 - a_1, \ldots, x_n - a_n)$ is a maximal ideal, then the localization R_M is the ring of rational functions that are defined at that point corresponding to M.

Ideals and Localization

The ideals of a quotient R/I are the ideals of R containing I. We sill see that, if P is a prime ideal, the ideals of R_P are the ideals of R contained in P. So localization allows us to focus on a limited set of primes in a ring.

Proposition: Let J be an ideal of $D^{-1}R$. Then

1. $\pi^{-1}J$ is an ideal of R, and $(\pi^{-1}J)D^{-1}R = J$. If two ideals I and I' of $D^{-1}R$ have $I \cap R = I' \cap R$, then $I = I'$.

It's always the case that the inverse image of an ideal is an ideal; and since $\pi(\pi^{-1}(J) \subset J)$ we know that $\pi^{-1}JD^{-1}R$ is contained in J. So suppose $(a,d) \in J \subset D^{-1}R$. Then $(a,1) = d(a,d) \in J$, so $a \in \pi^{-1}J \subset R$. But then $(a,1)$ is in $\pi^{-1}JD^{-1}J$ so $(a,1)(1,d) = (a,d)$ is in the extended ideal.

- If J is an ideal of R, then $(\pi(I)D^{-1}R) \cap R$ consists of all elements of R such that $dx \in J$ for some $d \in D$.

If $dx \in J$, then in $D^{-1}R$ we have $x = (1/d)y$ where $y \in J$ so x is in the extended ideal, and then in intersection back to R. Conversely, if x is in the intersection back to R then $x = a/d$ where $a \in J$ and $d \in D$. This means that there is an $e \in D$ so that $e(xd - a) = 0$ or $exd = ea$. Now ed is in D, and ea is in J, so x has the property that a D-multiple of it lands in J.

- There is a bijective correspondence between the primes of R disjoint from D and the primes of $D^{-1}R$.

If Q is prime in $D^{-1}R$, the $Q \cap R$ is prime. So the map in that direction takes prime ideals to prime ideals. Now suppose P is a prime of R and suppose that $(x/d)(y/d')$ is in $PD^{-1}R$. Then $xy/dd' = c/d$ where $c \in P$ and $d \in D$. Then there is a $u \in D$ with $u(dxy - cdd') = 0$. This means that $(ud)(xy)$ is in P. Since P does not meet D, that means xy is in P. That in turn means either x or y is in P, so either x/d or x'/d' is in $PD^{-1}R$, so that ideal is prime.

- If R is Noetherian, so is $D^{-1}R$.

This follows from (1) since an ascending chain of ideals of $D^{-1}R$ contracts to one in R, which then terminates; and the extension of that termination gives a termination of the original chain.

Localization of modules

Let M be a module over the ring R and let D be a multiplicatively closed subset of R.

Definition: $D^{-1}M$ is the module $M \times D/\sim$ where the equivalence relation $\sim$ is given by $(m,d) \sim (m',d')$ if there is an $x \in D$ so that $x(md' - dm') = 0$. There is a natural map $M \to D^{-1}M$ sending $m \to (m,1)$. $D^{-1}M$ is a $D^{-1}R$ module via the action $(r,d)(m,d') = (rm, dd')$.

As in the case of rings above, the kernel of the map $M \to D^{-1}M$ is the subset of M such that $dm = 0$ for some $d \in D$.

Given a map $f : M \to N$, there is a map $f : D^{-1}M \to D^{-1}N$ defined by $f(m,d) = (f(m),d)$.

Proposition: $D^{-1}M$ is isomorphic to $D^{-1}R \otimes_R M$.

Proof: The map $D^{-1}R \times M \to D^{-1}M$ given by $((r,d),m) \mapsto (rm,d)$ is bilinear and so yields a map from the tensor product to $D^{-1}M$. The inverse map sends $(m,d) \to (1,d) \otimes m$. (If (m,d)=(m',d') then $u(md' - dm') = 0$ for some $u \in D$. But (m,d) maps to $(1,d) \otimes m$ and (m',d') maps to $(1,d') \otimes m'$. So $1/d \otimes m = 1/udd' \otimes ud'm = 1/udd' \otimes udm' = 1/d' \otimes m'$). These maps are $D^{-1}R$ module homomorphisms.

Functorial properties of localization:

1. Localization commutes with finite sums and intersections of ideals.
2. Localization commutes with radicals: the radical of $D^{-1}I$ is $D^{-1}(\mathrm{rad} I)$.
3. Localization commutes with finite sums, intersections, and quotients of modules.
4. Localization commutes with finite direct sums.
5. Localization is flat.

Local Rings

Local Rings

Definition: A commutative ring with unity that has a unique maximal ideal is called a *local ring.*

Proposition: TFAE:

- R is local with maximal ideal M
- The units of R are exactly the elements of R outside M.

- there is a maximal ideal M of R wuch that $1 + m$ is a unity for $m \in M$.

Proposition: Let R be a commutative ring with 1 and let R_P be the localization of R at P.

- R_P is local with maximal ideal $P^e = PR_P$. The map $R \to R_P$ induces an injection $R/P \to R_P/PR_P$. R_P/PR_P is a field equal to the quotient field of R/P.
- If R is an integral domain, so is R_P. The map $R \to R_P$ is injective.
- The prime ideals of R_P are in bijective correspondence with the prime ideals of R contained in P.
- If P is a maximal ideal then R/M is isomorphic to R_M/MR_M.

Lemma: Let M be an R module. Then TFAE:

- $M = 0$
- $M_P = 0$ for all primes P of R
- $M_m = 0$ for all maximal ideals m of R.