12. Localization

Localization

Ideals and Localization

The ideals of a quotient R/I are the ideals of R containing I. We sill see that, if P is a prime ideal, the ideals of R_P are the ideals of R contained in P. So localization allows us to focus on a limited set of primes in a ring.

Proposition: Let J be an ideal of $D^{-1}R$. Then

1. $\pi^{-1}J$ is an ideal of R, and $(\pi^{-1}J)D^{-1}R = J$. If two ideals I and I' of $D^{-1}R$ have $I \cap R = I' \cap R$, then $I = I'$.

It's always the case that the inverse image of an ideal is an ideal; and since $\pi(\pi^{-1}(J) \subset J)$ we know that $\pi^{-1}JD^{-1}R$ is contained in J. So suppose $(a, d) \in J \subset D^{-1}R$. Then $(a, 1) = d(a, d) \in J$, so $a \in \pi^{-1}J \subset R$. But then $(a, 1)$ is in $\pi^{-1}JD^{-1}J$ so $(a, 1)(1, d) = (a, d)$ is in the extended ideal.

- If J is an ideal of R, then $(\pi(I)D^{-1}R) \cap R$ consists of all elements of R such that $dx \in J$ for some $d \in D$.

If $dx \in J$, then in $D^{-1}R$ we have $x = (1/d)y$ where $y \in J$ so x is in the extended ideal, and then in intersection back to R. Conversely

Localization of modules

Let M be a module over the ring R and let D be a multiplicatively closed subset of R.

Definition: $D^{-1}M$ is the module $M \times D/\sim$ where the equivalence relation $\sim$ is given by $(m,d) \sim (m',d')$ if there is an $x \in D$ so that $x(md' - dm') = 0$. There is a natural map $M \to D^{-1}M$ sending $m \to (m,1)$. $D^{-1}M$ is a $D^{-1}R$ module via the action $(r,d)(m,d') = (rm, dd')$.

As in the case of rings above, the kernel of the map $M \to D^{-1}M$ is the subset of M such that $dm = 0$ for some $d \in D$.

Given a map $f : M \to N$, there is a map $f : D^{-1}M \to D^{-1}N$ defined by $f(m,d) = (f(m),d)$.

Proposition: $D^{-1}M$ is isomorphic to $D^{-1}R \otimes_R M$.

Proof: The map $D^{-1}R \times M \to D^{-1}M$ given by $((r,d),m) \mapsto (rm,d)$ is bilinear and so yields a map from the tensor product to $D^{-1}M$. The inverse map sends $(m,d) \to (1,d) \otimes m$. (If (m,d)=(m',d') then $\mu(md' - dm') = 0$ for some $\mu \in D$. But (m,d)

Local Rings

Local Rings

Definition: A commutative ring with unity that has a unique maximal ideal is called a *local ring.*

Proposition: TFAE:

- R is local with maximal ideal M
- The units of R are exactly the elements of R outside M.
- there is a maximal ideal M of R wuch that $1 + m$ is a unity for $m \in M$.

Proposition: Let R be a commutative ring with 1 and let R_P be the localization of R at P.

- R_P is local with maximal ideal $P^e = PR_P$. The map $R \to R_P$ induces an injection $R/P \to R_P/PR_P$. R_P/PR_P is a field equal to the quotient field of R/P.

- If R is an integral domain, so is R_P. The map $R \to R_P$ is injective.

- The prime ideals of R_P are in bijective correspondence with the