11. Commutative Algebra and Geometry

Commutative Algebra and Algebraic Geometry

Throughout this section, rings R are commutative with unity.

Noetherian Rings

Definition: A ring is Noetherian if it satisfies either of the following equivalent conditions:

1. Every ideal is finitely generated.
2. Every ascending chain $I_0 \subset I_1 \subset \cdots$ of ideals eventually stabilizes.

Theorem: (Hilbert's Basis Theorem) If R is Noetherian, then so is $R[x]$.

Proof: The proof of Hilbert's theorem is given in detail in DF, Theorem 21 of Section 9.6. The idea is something like this. Suppose I is an ideal of $R[x]$. Then each element f of I is a polynomial of degree n: $f = a_n x^n + \ldots$ where $a_n \in R$ is not zero. Here a_n is called the leading term of f The ideal $\ell(I)$ of *leading terms* of I is the ideal of R generated by the leading terms of all elements of I. Since R is Noetherian, $\ell(I)$ is finitely generated. If $b_1, \ldots, b_N$ generate $\ell(I)$, we can find elements $f_1, \ldots, f_N$ so that $f_i = b_i x_i^k + \ldots$ in I.

Now let g be any element of I of degree m. Suppose m is larger then all of the k_i. Since $\ell(g)$ is in $\ell(I)$, we can find $r_i \in R$ so that

$$\ell(g) = \sum r_i b_i$$

and therefore

$$g - \sum r_i f_i x^{m-k_i} + \ldots$$

This g has degree smaller than m (since we killed off its leading term). It follows that the ideal generated by the f_i contains all elements of I of degree at least $N = \max\{k_i\}$.

To finish the proof, we consider all the leading terms of all elements of I of degree less than N and consider the ideal they generate. This again is finitely generated, by, say, $c_1, \ldots, c_M$ and there are corresponding elements h_j of $R[x]$ where $h_j = c_j x^r + \ldots$. An argument similar to what we used above says that we can use the h_j to systematically eliminate the leading terms of g until eventually we show that g is in the ideal generated by the h_j and the f_i. In other words, our original ideal I is finitely generated, and thus $R[x]$ is Noetherian.

Definition: If K is a field, a K-algebra S is finitely generated if there is a surjection $K[x_1, \ldots, x_n] \to S$ for some n.

Warning: Note the difference between being finitely generated as a K-module and as a K-algebra.

Affine space, points, and ideals

Affine n-space over a field k, written $\mathbb{A}^n_k$ is the set of points $(x_1, \ldots, x_n)$ with coordinates in k. The ring $k[x_1, \ldots, x_n]$ is the ring of polynomial functions on $\mathbb{A}^n_k$.

Sets and ideals:

- If $X \subset \mathbb{A}^n_k$ is a subset, then $I(X) = \{f : f(x) = 0 \forall x \in \mathbb{A}^n_k\|$.
- If $f_1, \ldots, f_k$ are a collection of elements in $k[x_1, \ldots, x_n]$ then $Z(f_1, \ldots, f_k) = \{x \in \mathbb{A}^n_k : f_1(x) = \cdots = f_k(x) = 0\}$. Notice that $Z(f_1, \ldots, f_k) = Z(I)$ where I is the ideal of functions generated by the f_i.

Both operations are inclusion reversing: $X \subset Y$ implies $I(Y) \subset I(X)$ and $J \subset L$ implies $Z(L) \subset Z(J)$.

The sets $Z(I)$ satisfy the axioms for the closed sets of a topology, called the *zariski topology.* A set X of the form $Z(I)$ is called an (affine) algebraic set.

Lemma: If $X \subset \mathbb{A}^n_k$, then $X \subset Z(I(X))$. If $J \subset k[x_1, \ldots, x_n]$ is an ideal, then $J \subset I(Z(J))$.

Also:

- $$Z(I(Z(J))) = Z(J)$$
  .

Proof: We know that $J \subset I(Z(J))$ so $Z(I(Z(J))) \subset Z(J)$. On the other hand, if $x \in Z(J)$, then $f(x) = 0$ for all $f \in I(Z(J))$. That means that x is a common zero for $I(Z(J))$, so x is in $Z(I(Z(J)))$. Therefore $Z(J) \subset Z(I(Z(J)))$.

- $$I(Z(I(X))) = I(X)$$
  .

Proof: We know that $X \subset Z(I(X))$ since if $x \in X$, every $f \in I(X)$ satisfies $f(x) = 0$, and therefore $x \in Z(I(X))$. So $I(Z(I(X))) \subset I(X)$. On the other hand, if $f \in I(X)$, then $f(x) = 0$ for all $x \in Z(I(X))$. This in turn means that f is in $I(Z(I(X)))$ so $I(x) \subset I(Z(I(X)))$.

Coordinate rings and morphisms

If X is an algebraic set, then the ring $k[X] = k[x_1, \dots, x_n]/I(X)$ is called the coordinate ring of X. Notice that elements of this ring give well-defined functions on X; they are called "regular functions" on X.

Definition: A map $f : X \to Y$ between algebraic sets is called a morphism if there are polynomials so that the map f is given by

$$f(x_1, \dots, x_n) = (\phi_1(x_1, \dots, x_n), \cdots, \phi_m(x_1, \dots, x_n))$$

A morphism whose inverse is also a morphism is called an isomorphism of algebraic sets.

In general, if $f : x \to Y$ is a morphism, and $h \in k[Y]$ is a regular function on Y then the composition $h \circ f$ is a regular function on X. This operation is called *pullback*.

- The pullback induces a welldefined algebra homomorphism $f^* : k[Y] \to k[X]$.
- Conversely, any algebra homomorphism $\psi : K[Y] \to K[X]$ yields a morphism $X \to Y$. Choose generators $x_1, \dots, x_m$ for $K[Y]$ and let $F_i = \psi(x_i)$. Let F be the corresponding map of algebraic sets. We need to check that the morphism given by F carries X to Y. The key is that if H is in $I(Y)$ then $H(F_1, \dots, F_m)$ is in $I(X)$. So every polynomial in $I(Y)$ vanishes on the image of F; and since $Y = Z(I(Y))$ it follows that the image of F is in Y.

It follows that there is a bijective correspondence between k-algebra homomorphisms $k[X] \to k[Y]$ and regular maps $Y \to X$. This corresondence respects composition of functions and preserves isomorphisms.

Universal property

Theorem: Suppose $\phi : X \to Y$ is an arbitrary map of algebraic sets. Then ϕ is a morphism if and only if, for every $f \in k[Y]$, the composition $f \circ \phi$ is in $k[X]$. If ϕ is a morphism, then $\phi(v) = w$ if and only if $\tilde{\phi}^{-1}(I(v)) = I(w)$ where $\tilde{\phi}$ is the pullback of ϕ.

Proof: The second part first. If $\phi(v) = w$, and $f \in I(w)$, then $f(\phi(v)) = 0$ so $\tilde{\phi}(I(w)) \subset I(v)$ and $(\tilde{\phi})^{-1}I(v) \supset I(w)$. Since $I(w)$ is maximal, we have $(\tilde{\phi})^{-1}(I(v)) = I(w)$. Conversely, if $(\tilde{\phi})^{-1}(I(v)) = I(w)$, then $f(\phi(v)) = 0$ if and only if $f(w) = 0$ so $I(\phi(v)) = I(w)$ and therefore $\phi(v) = w$. For the first part, pullback automatically respects addition and multiplication, so if it happens to land in $k[X]$ then it will be an algebra homomorphism $\tilde{\phi} : k[Y] \to k[X]$.

Consequently there is a morphism from X to Y coming from $\tilde{\phi}$; call it $\tilde{\Phi}$ where Φ is a morphism from $X \to Y$. However, since the algebra maps are the same, their behavior on the ideals $I(v)$ and $I(w)$ are the same, so Φ and ϕ agree with one another.