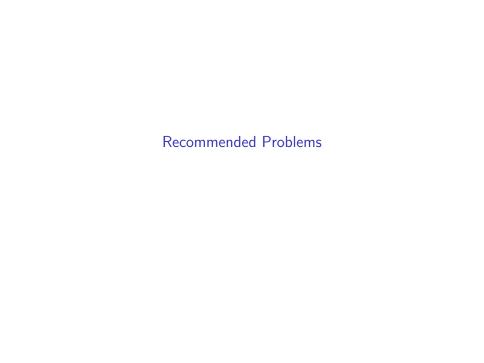
# Recommended Problems



Problem 1 (DF, problem 6, p. 375)

Let R be an integral domain with field of fractions Q. Prove that  $(Q/R) \otimes_R (Q/R) = 0$ . In particular  $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z} = 0$ .

### Problem 2 (DF, problem 5, p. 375)

Let G be a finite abelian group and let  $p^k$  be the highest power of a prime p dividing |G|. Prove that  $Z/p^k\mathbb{Z}\otimes G$  is the Sylow p-subgroup of G.

#### Problem 3 (DF, Problem 8, p. 375ff)

Suppose R is an integral domain with quotient field Q and let N be any R module. Let  $U=R^*$  be the nonzero elements in R and let  $U^{-1}N$  be the set of equivalence classes of pairs (u,n) with  $(u,n) \sim (u',n')$  if and only if u'n=un'.

- Show that  $UN^{-1}$  is an abelian group with  $(u_1, n_1) + (u_2, n_2) = (u_1u_2, u_2n_1 + u_1n_2)$  and that the operation r(u, n) = (u, rn) makes  $U^{-1}N$  into an R-module.
- Show that the map  $Q \times N \to U^{-1}N$  sending  $(a/b, n) \to (b, an)$  is R-balanced and so yields a unique map  $Q \otimes N \to U^{-1}N$ . Show that the map  $(u, n) \to u^{-1} \otimes n$  is a well-defined inverse, and so  $Q \otimes N$  is isomorphic to  $U^{-1}N$ .
- Show that  $(1/d) \otimes n = 0$  if and only if there is  $r \in R$  so that rn = 0.
- ▶ Show that  $\mathbb{Q} \otimes A = 0$  (where A is an abelian group) if and only if A is torsion.

## Problem 4 (DF Problem 16 pg. 376)

Let R be commutative and let I and J be ideals. Prove that every element of  $R/I \otimes R/J$  has the form  $1 \otimes r$  for  $r \in R$ . Then show that  $R/I \otimes R/J$  is isomorphic to R/(I+J) via the map sending  $r \otimes r'$  to rr'.

## Problem 5 (DF, Problem 17 and 19, pg. 37

Let  $R = \mathbb{Z}[x]$  and let I = (2, x). Show that the map  $\phi: I \times I \to \mathbb{Z}/2\mathbb{Z}$  given by

$$\phi(fg) = f(0)g'(0)/2 \pmod{2}$$

is *R*-bilinear. Thus it gives a map  $I \otimes I \to \mathbb{Z}/2\mathbb{Z}$ . Conclude that  $2 \otimes x$  and  $x \otimes 2$  are not equal in  $I \otimes I$ .