10. Multilinear Algebra

The tensor algebra

The tensor algebra - definition

Suppose that R is a commutative ring with unity. Let M be an R module.

If $T^j(M) = \overbrace{M \otimes_R \cdots \otimes_R M}^{j}$

then we can define a (non-commutative) product map

$$T^j(M) \times T^k(M) \to T^{j+k}(M)$$

by defining

$$(a_1 \otimes \cdots \otimes a_j)(b_1 \otimes \cdots \otimes b_k) = (a_1 \otimes \cdots \otimes a_j \otimes b_1 \otimes \cdots \otimes b_k)$$

This allows us to view the infinite direct sum

$$T(M) = R \oplus M \oplus T^2(M) \oplus T^3(M) \cdots$$

as a (non-commutative) R-algebra. This ring is called the *tensor algebra of* M.

Universal property

If A is any R-algebra and M is any R module, then given a map $f : M \to A$ there is a unique map $T(M) \to A$ which, when restricted to M gives f.

Graded rings

The tensor algebra $T(M)$ is an example of a *graded ring*. A general graded ring is any ring S that is a direct sum of additive subgroups S_i where $S_iS_j \subset S_{i+j}$. Each S_j is called the subset of homogeneous elements of degree j.

A graded ideal I is an ideal of S that is the direct sum of its homogeneous components $I \cap S_j$.

A map $f : S \to T$ between graded rings is a homomorphism of graded rings if $f(S_j) \subset T_j$ for every j.

Example: The polynomial ring $F[x_1, \ldots, x_n]$ is graded where the homogeneous components are spanned by the monomials of a given degree. The ideal generated by $x_1, \ldots, x_n$ is graded because any element of this ideal – that is, any polynomiial with zero constant term – can be written as a sum of monomials of a fixed degree in a unique way.

The ring $R = \mathbb{Z}[x]$ is graded by degree, and the ideal generated by $J = (1 + x)$ is not graded because 1 and x are not in J.

An ideal generated by homogeneous elements in a graded ring is graded.

Quotients of graded rings

If S is a graded ring and I is a graded ideal, then S/I is graded with homogeneous components S_j/I_j.

The symmetric algebra

The symmetric algebra - definition

A bilinear map $f : M \times M \to L$ is symmetric if $f(m_1, m_2) = f(m_2, m_1)$ for all pairs (m_1, m_2). The symmetric tensor product $S^2(M)$ has the universal property that any symmetric bilinar map $f : M \times M \to L$ "factors through" $S^2(M)$ in a unique way. $S^2(M)$ is constructed from $M \otimes M$ by imposing the relation $m \otimes n - n \otimes m = 0$ for all pairs (m, n).

The symmetric *algebra* $S(M)$ is obtained from the tensor algebra by modding out by the ideal $C(M)$ generated by all $m \otimes n - n \otimes m$ in $T(M)$. The map

$$T(M) \to S(M)$$

is a map of graded rings since $C(M)$ is a graded ideal, so $S(M)$ is a graded ring. $S^0(M) = R$ and $S^1(M) = M$.

More on the symmetric algebra

In $S(M)$, the degree k terms are spanned by tensors $m_1 \otimes \cdots \otimes m_k$ modulo the relation which allows you to freely permute the terms.

Any k-symmetric multilinear map $M \times \cdots \times M \to L$ factors through $S^k(M)$ in a unique way.

If A is any commutative R algebra, and $f : M \to A$ is a module map, then there is a unique R-algebra map $S(M) \to A$ which restricts to f on M.

If R is a field and V is a vector space of dimension k then $S(V)$ is the (graded) polynomial ring in k variables over R. If R is a commutative ring and V is a free module of rank k then $S(V)$ is the (graded) polynomial ring over R in k variables.

The exterior algebra

Alternating maps and the wedge product

A bilinear map $f : M \times M \to L$ is *alternating* if $f(m,m) = 0$ for all $m \in M$. The exterior product (or wedge product) $M \wedge M$ has the property that there is a map $M \times M \to M \wedge M$ such that if $f : M \times M \to L$ is an alternating map, then there is a unique map $M \wedge M \to L$ making the usual triangle commute.

The map $M \otimes M \to M \wedge M$ is written $(m_1, m_2) \to m_1 \wedge m_2$ and $M \wedge M$ is spanned by all $m_1 \wedge m_2$. We have $m_1 \wedge m_2 = -m_2 \wedge m_1$ for elementary tensors where $m_1, m_2 \in M$.

$M \wedge M$ is the quotient of $M \otimes M$ by the elementary tensors $m \otimes m$.

The exterior algebra

The quotient of the tensor algebra by the ideal generated by all tensors of the form $m \otimes m$ is called the *exterior algebra*, written $\bigwedge M$. It is graded so that that the degree k terms consist of linear combinations of the "wedge" of k elements of M.

If A is an R-algebra such that $a^2 = 0$ for all $a \in A$, and $f : M \to A$ is a module homomorphism, then there is a unique map $\bigwedge M \to A$ which restricts to f.

If V is a finite dimensional vector space over a field F, then $\bigwedge V$ is finite dimensional. This holds for any free, finite rank module over a ring R.

If $R = \mathbb{Z}[x, y]$, if $M = R$ then $\bigwedge M$ is spanned by $1 \wedge 1 = 0$ so $\bigwedge M = 0$. If $I = (x, y)$, then there is an alternating map from I to $\mathbb{Z}$ defined by

$$f(ax + by, cx + dy) = ad - bc \pmod{(x, y)}$$

so $x \wedge y$ is not zero.

The cross product in $\mathbb{R}^3$.

The coefficients of

$$(ae_0 + be_1 + ce_2) \wedge (xe_0 + ye_1 + ze_2)$$

in terms of the two-forms $e_i \wedge e_j$ give the cross product in $\mathbb{R}^3$.

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