p-adic Fourier Theory and applications
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Overview

1. Brief look at classical fourier analysis for motivation
2. Continuous functions on $\mathbb{Z}_p$ and Mahler's Theorem
3. Distributions (integrals), characters, and the Iwasawa algebra
4. A look ahead

Classical Fourier Analysis

Let G be an compact topological group that is abelian.

- G is a topological space
- G is an abelian group
- the group operations are continuous

Example:

- $G = \{z \in \mathbb{C} : |z| = 1\}$, so G is the circle group $\mathbb{T}$.

Function Spaces

In analysis, we think about spaces of functions on our group G. These are *topological vector spaces.*

Two important examples:

1. $C(G, \mathbb{C})$ - the space of *continuous functions* on G with $\|f\| = \sup_{x \in G} \|f(x)\|$.
2. $L^2(G)$ - the Hilbert space of square-integrable functions on G with inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f \overline{g} d\theta.$$

Distributions

A *continuous distribution* on G is a continuous linear map

$$\lambda : C(G, \mathbb{C}) \to \mathbb{C}.$$

An L^2 distribution on G is a continuos linear map $\lambda : L^2(G) \to \mathbb{C}$. Any such map is of the form

$$\lambda(f) = \langle f, g_\lambda \rangle$$

for some $g_\lambda \in L^2(G)$.

Distributions and integrals/measures

A distribution λ converts a function f into a number $\lambda(f)$.

Think of it as an integral:

$$\lambda(f) = \int_G f d\lambda.$$

(The Riesz representation theorem makes this precise).

One example: the Dirac distribution δ_x at $x \in G$: $\delta_x(f) = f(x)$.

Compact groups have a continuous "invariant distribution of mass one" (for $\mathbb{T}$, this is $\frac{1}{2\pi} d\theta$).

Characters and the dual group

A **character** of $G = \mathbb{T}$ is a continuous homomorphism $\phi : G \to \mathbb{T}$.

The set of characters $X(G)$ form an abelian topological group in their own right. This is called the **dual group** to G.

When $G = \mathbb{T}$, then the continuous characters are the functions

$$\phi_n(z) = z^n$$

where $n \in \mathbb{Z}$ and $X(\mathbb{T}) = \mathbb{Z}$.

Fourier transform

A distribution λ gives a function F_λ on the space of characters $X(G)$ by

$$F_\lambda(\phi) = \lambda(\phi) =< \phi, g_\lambda >= \int_G \phi d\lambda$$

This function is called the Fourier transform of λ.

Examples

If $G = \mathbb{T}$, then:

- $X(G) = \{z^n = e^{in\theta} : n \in \mathbb{Z}\} = \mathbb{Z}$
- If λ is the distribution $\lambda(f) = \langle f, g_\lambda \rangle$ for a function g_λ, then

$$F_\lambda(n) = \langle z^n, g_\lambda \rangle$$

which is the (conjugage of) the n^{th} fourier coefficient of g_λ.

Key ingredients

- Compact abelian group G
- Function space $C(G)$ on G
- Distributions are continuous linear forms on $C(G)$.
- Continuous characters form another topological group $X(G)$.
- Distributions -$>$ functions on character space $X(G)$.

p-adic Fourier Theory

Overview

In p-adic Fourier theory the ingredients are:

- $\mathbb{G} = \mathbb{Z}_p$
- The space of continuous functions $C(G, K)$ where K is a complete extension field of $\mathbb{Q}_p$.
- Distributions are continuous linear maps $C(G, K) \to K$.
- The space $X(G)$ of continuous characters $\phi : G \to K^*$.
- Fourier transform converts distributions to functions on $X(G)$ by $\lambda \mapsto F_\lambda$ where $F_\lambda(\phi) = \lambda(\phi)$.

Differences

- The group $\mathbb{Z}_p$ is totally discontinuous (every open set is closed).
- No L^2 theory
- No invariant continuous distribution (no Haar measure).

Continuous functions on $\mathbb{Z}_p$

Let $G = \mathbb{Z}_p$ and let K be a complete extension field of $\mathbb{Q}_p$. Let $C(G, K)$ be the space of continuous functions from G to K.

Example elements of $C(G, K)$:

- locally constant functions such as f where $f(x) = 1$ if $x \in p\mathbb{Z}_p$ and 0 otherwise.
- characters of order p^n. Let $\zeta \in \mu_{p^n}$.

$$\zeta^x = \zeta^{a_0 + a_1 p + a_2 p^2 + \cdots}$$

  makes sense, is locally constant, and satisfies $\zeta^{x+y} = \zeta^x \zeta^y$.
- polynomial functions

Continuous functions on $\mathbb{Z}_p$

The space $C(G, K)$ is a K-vector space with a norm given by

$$\|f\| = \sup_{x \in G} |f(x)|.$$

Proposition: $C(G, K)$ is complete (it is a "Banach Space").

Proof: It is a general fact that if X is a compact metric space and Y is a complete metric space, then the continuous functions $C(X, Y)$ is complete under the sup norm. This is an exercise in uniform convergence.

Binomial polynomials

For $n \in \mathbb{Z}$, let $\binom{x}{n}$ be the binomial coefficient viewed as a polynomial in x. For example

$$\binom{x}{3} = \frac{x(x-1)(x-2)}{3!}.$$

Proposition: The polynomials $\binom{x}{n}$ have the property that

$$x \in \mathbb{Z}_p \implies \binom{x}{n} \in \mathbb{Z}_p.$$

This follows from the fact that $\binom{x}{n}$ takes integer values when x is a positive integer, and these are dense in $\mathbb{Z}_p$.

Mahler's Theorem

Theorem: Any continuous function $f : G \to K$ has a unique expansion

$$f(x) = \sum_{n=0}^{\infty} a(n) \binom{x}{n}$$

where $a(n) \in K$ for all n and $\lim_{n\to\infty} |a(n)|_p = 0$. Furthermore,

$$\|f\| = \sup_{n=0}^{\infty} |a(n)|_p.$$

Since the $a(n)$ go to zero, such an f converges pointwise. A uniform convergence argument shows that $f(x)$ is continuous.

Corollary: The space $C(G, K)$ is isomorphic (as a Banach space) to the space of sequences $(a(n))_{n=0}^{\infty}$ where $a(n) \in K$, under pointwise addition and scalar multiplication and with norm given by the sup-norm.

Comments on Mahler's Theorem

Let Δ be the finite difference operator $\Delta(f) = f(x+1) - f(x)$.

The finite difference calculus says that, if $f : \mathbb{Z} \to \mathbb{Z}$, then

$$f(x) = \sum_{n=0}^{\infty} \Delta^i(f)_{x=0} \binom{x}{i}.$$

If $f : \mathbb{Z}_p \to K$ one can still construct this series and if it makes sense it will agree with f on the integers, and then by continuity on all of $\mathbb{Z}_p$.

To prove convergence you need to show that the coefficients $\Delta^i(f)_{x=0}$ go to zero p-adically. A detailed proof is given by Conrad – see the references.

Characters

A continuous character is a continuous function $f : \mathbb{Z}_p \to K^*$. Such a function is determined by continuity and by $f(1)$:

- $f(1) = a$
- $f(n) = a^n$ for $n \in \mathbb{Z} \subset \mathbb{Z}_p$.
- $f(\sum_{i=0}^{\infty} b_i p^i) = \lim_{N\to\infty} a^{\sum_{i=0}^{N} b_i p^i}$

In particular, $a^{p^n} \to 1$ as $n \to \infty$, which forces $|a - 1| < 1$.

Characters

If $a = 1 + z$ where $|z| < 1$ then Mahler's theorem tells us that

$$(1 + z)^x = \sum_{i=0}^{\infty} \binom{x}{i} z^i$$

is a continuous function of $x \in \mathbb{Z}_p$ and the uniqueness tells us that

$$(1 + z)^{(x+y)} = (1 + z)^x (1 + z)^y.$$

The space of (p-adic) characters

Proposition: The space of continuous K^* valued characters of $\mathbb{Z}_p$ is the set

$$X(\mathbb{Z}_p)(K) = \{1 + z \in K : |z| < 1\}.$$

That is, the character space is an "open p-adic disk of radius one centered at 1". Given z with $|z| < 1$, we write

$$\phi_z(x) = (1+z)^x = \sum \binom{x}{n} z^n.$$

Distributions

A linear map $\lambda : C(G, K) \to K$ is continuous if and only if it is bounded:

$$\sup_{f \neq 0} \frac{\|\lambda(f)\|_p}{\|f\|} < C$$

for some constant C.

It is enough to check this on the set of binomial polynomials $\binom{x}{n}$, which have sup-norm 1.

Distributions continued

Proposition: The space $D(G, K)$ of continuous linear maps $C(G, K) \to K$ is isomorphic (as a Banach space) to the space of bounded sequences $(b(n))_{n=0}^{\infty}$ with entries in K, where

$$b(n) = \lambda(\binom{x}{n}).$$

For reasons we will explain later we package these sequences up into formal power series with p-adically bounded coefficients.

$$\lambda \mapsto F_\lambda = \sum_{n=0}^{\infty} \lambda(\binom{x}{n}) T^n.$$

Duality

If

$$g(T) = \sum_{n=0}^{\infty} c(n) T^n \in D(G, K)$$

and

$$f \in C(G, K) = \sum a(n) \binom{x}{n},$$

then the duality pairing $D(G, K) \times C(G, K) \to K$ is given by

$$\langle \sum c(n) T^n, \sum a(n) \binom{x}{n} \rangle = \sum_{n=0}^{\infty} c(n) a(n)$$

which converges since the $c(n)$ are bounded and the $a(n)$ go to zero.

The Fourier transform

Theorem: The Fourier transform of the distribution λ is a power series $F_\lambda(T)$ whose value at z is the distribution evaluated at the character ϕ_z where

$$\phi_z(x) = (1+z)^x.$$

That is,

$$F_\lambda(z) = \lambda(\phi_z).$$

The Fourier transform as a function on $X(\mathbb{Z})$

Remember that the Fourier transform converts a distribution λ into a function on the character space by the rule $F_\lambda(\phi) = \lambda(\phi)$.

If $F_\lambda(T) = \sum c(n)T^n$ and ϕ_z is the character where $\phi_z(x) = (1+z)^x$ then

$$\lambda(\phi_z) = \langle \sum c(n)T^n, \sum \binom{x}{n} z^n \rangle = \sum c(n)z^n = F_\lambda(z).$$

This converges because the $c(n)$ are bounded and $|z| < 1$ so $z^n \to 0$.

The Iwasawa algebra

Let's look at $D(G, \mathbb{Q}_p)$, which is the space of power series with bounded $\mathbb{Q}_p$ coefficients.

Inside this ring is the subring

$$\Lambda = \mathbb{Z}_p[[T]]$$

of power series with integral coefficients. This important algebra is called the Iwasawa algebra.

Proposition: The Iwasawa algebra is the space of distributions λ on $C(G, \mathbb{Q}_p)$ with the property that $|\lambda(f)| \leq \|f\|$ for all f; that is, it is the unit ball in the Banach space $D(G, \mathbb{Q}_p)$.

Some special "integrals"

Let $F(T) \in D(G, K)$ be a distribution. We know that

$$\langle F(T), \phi_z \rangle = F(z).$$

In other words, the values of F give the integrals of characters.

Let $x \in \mathbb{Z}_p$ be fixed and let $F(T) = (1+T)^x = \sum_{n=0}^{\infty} \binom{x}{n} T^n$. Then

$$\langle F(T), \phi_z \rangle = \sum \binom{x}{n} z^n = \phi_z(x).$$

In other words, the power series $(1+T)^x$ with x fixed is the Dirac distribution at x.

More special integrals

Proposition:

$$\langle F(T), x^n \rangle = \partial^n F(T)|_{T=0}$$

where $\partial F = (1+T)\frac{d}{dT}F$.

To prove this, notice that

$$x\binom{x}{n} = (n+1)\binom{x}{n+1} + n\binom{x}{n}.$$

Then compare $\langle T^n, x\binom{x}{j} \rangle$ with $\langle \partial T^n, \binom{x}{j} \rangle$ and see that they are the same for all n and j.

By linearity (and continuity) we conclude that $\langle \partial F, h \rangle = \langle F, xh \rangle$ for any h and so

$$\partial^n F(T)|_{T=0} = \langle \partial^n F(T), 1 \rangle = \langle F(T), x^n \rangle$$

Conclusions

References and further reading

Additional Material

The Mellin transform

Remember that we showed that $\mathbb{Z}_p^*$ is isomorphic to $\mu_{p-1} \times \mathbb{Z}_p$ via the map sending x to the pair $(\omega(x), \langle x \rangle)$.

If $z \in 1 + p\mathbb{Z}_p$ and $0 \leq i < p - 1$ then we have a character

$$\psi = \omega^i \psi_z$$

where $\psi_z(x) = (1 + z)^{\log(x)/p}$. The character $x \mapsto \langle x \rangle^s$ corresponds to ψ_z where $z = \exp(ps) - 1$ for $s \in \mathbb{Z}$.

If λ is a distribution on $\mathbb{Z}_p^*$, its "Mellin transform" is

$$M_\lambda(z, \omega^i) = \lambda(\omega^i \psi_z).$$

M_λ is a vector of $q - 1$ power series – it's a Fourier transform for the multiplicative group.

Mellin and Fourier

Suppose λ is a distribution on $\mathbb{Z}_p$ which vanishes on functions with support in $p\mathbb{Z}_p$. Therefore λ is also a distribution on $\mathbb{Z}_p^*$.

Let M_λ be its Mellin transform and F_λ be its Fourier transform.

if $n \equiv i \pmod{p-1}$, we have

$$M_\lambda(\exp(pn) - 1, \omega^i) = \lambda(x^n) = \partial^n F_\lambda(T)|_{T=0}$$

In addition, $M_\lambda(\exp(ps) - 1, \omega^i)$ is an analytic function in s – that is, given by a power series in s for $s \in \mathbb{Z}_p$.

From $\mathbb{Z}_p$ to $\mathbb{Z}_p^*$

To complete the construction of the K-L zeta function, we need to adjust our power series F_a so that it vanishes on functions supported on $p\mathbb{Z}_p$.

For this we use the function

$$H(x) = 1 - \frac{1}{p}\sum_{\zeta} \zeta^x$$

where ζ runs over the p^{th} roots of unity.

$H(x) = 1$ if x is not divisible by p and 0 if it is.

If F is a power series corresponding to a distribution, let F^* be the function

$$F^*(T) = F(T) - \frac{1}{p}\sum_{\zeta} F(\zeta - 1 + \zeta T).$$

One can show that this is still a power series with bounded coefficients, and is therefore still a distribution.

From $\mathbb{Z}_p$ to $\mathbb{Z}_p^*$

Proposition: If f is a continuous function on $\mathbb{Z}_p$, then

$$\langle F^*, f\rangle = \langle F, Hf\rangle.$$

Check this on characters. We have

$$\langle F^*, (1+z)^x\rangle = \langle F, (1+z)^x\rangle - \frac{1}{p}\sum_{\zeta}\langle F(\zeta - 1 + \zeta T), (1+z)^x\rangle.$$

But

$$\langle F(\zeta - 1 + \zeta T), (1+z)^x\rangle = \langle F(T), (\zeta(1+z))^x\rangle$$

and combining the terms gives you what you want.

Corollary: F^* vanishes on functions supported on $p\mathbb{Z}_p$ and is therefore a distribution on $\mathbb{Z}_p^*$.

Wrapping up.

To finish, apply the operation above to the power series

$$F_a(T) = \frac{1}{T} - \frac{a}{(1+T)^a - 1}.$$

It turns out that

$$F_a^*(T) = F_a(T) - F_a((1+T)^p - 1).$$

With some calculations using this you obtain the analytic function $M(\exp(ps) - 1, \omega^i)$ satisfying

$$M(\exp(pn) - 1, \omega^i) = (-1)^k(1 - a^{k+1})(1 - p^{k+1})\zeta(-k)$$

With a bit more work one can get rid of the $1 - a^{k+1}$ term, but the p-Euler factor has to be "removed" for the p-adic construction to work.