

p-adic Fourier Theory and applications

CTNT 2022

Jeremy Teitelbaum

Overview

1. Brief look at classical fourier analysis for motivation
2. Continuous functions on \mathbb{Z}_p and Mahler's Theorem
3. Distributions (integrals), characters, and the Iwasawa algebra
4. A look ahead

Classical Fourier Analysis

Let G be an compact topological group that is abelian.

- ▶ G is a topological space
- ▶ G is an abelian group
- ▶ the group operations are continuous

Example:

- ▶ $G = \{z \in \mathbb{C} : |z| = 1\}$, so G is the circle group \mathbb{T} .

Function Spaces

In analysis, we think about spaces of functions on our group G . These are *topological vector spaces*.

Two important examples:

1. $C(G, \mathbb{C})$ - the space of *continuous functions* on G with $\|f\| = \sup_{x \in G} \|f(x)\|$.
2. $L^2(G)$ - the Hilbert space of square-integrable functions on G with inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f \bar{g} d\theta.$$

Distributions

A *continuous distribution* on G is a continuous linear map

$$\lambda : C(G, \mathbb{C}) \rightarrow \mathbb{C}.$$

An L^2 distribution on G is a continuous linear map $\lambda : L^2(G) \rightarrow \mathbb{C}$.
Any such map is of the form

$$\lambda(f) = \langle f, g_\lambda \rangle$$

for some $g_\lambda \in L^2(G)$.

Distributions and integrals/measures

A distribution λ converts a function f into a number $\lambda(f)$.

Think of it as an integral:

$$\lambda(f) = \int_G f d\lambda.$$

(The Riesz representation theorem makes this precise).

Compact groups have a continuous “invariant distribution of mass one” (for \mathbb{T} , this is $\frac{1}{2\pi} d\theta$).
Characters and the dual group

A **character** of $G = \mathbb{T}$ is a continuous homomorphism $\phi : G \rightarrow \mathbb{T}$.

The set of characters $X(G)$ form an abelian topological group in their own right. This is called the **dual group** to G .

When $G = \mathbb{T}$, then the continuous characters are the functions

$$\phi_n(z) = z^n$$

where $n \in \mathbb{Z}$ and $X(\mathbb{T}) = \mathbb{Z}$.

Fourier transform

A distribution λ gives a function F_λ on the space of characters $X(G)$ by

$$F_\lambda(\phi) = \lambda(\phi) = \langle \phi, g_\lambda \rangle = \int_G \phi d\lambda$$

This function is called the Fourier transform of λ .

Examples

If $G = \mathbb{T}$, then:

- ▶ $X(G) = \{z^n = e^{in\theta} : n \in \mathbb{Z}\} = \mathbb{Z}$
- ▶ If λ is the distribution $\lambda(f) = \langle f, g_\lambda \rangle$ for a function g_λ , then

$$F_\lambda(n) = \langle z^n, g_\lambda \rangle$$

which is the (conjugate of) the n^{th} fourier coefficient of g_λ .

Key ingredients

- ▶ Compact abelian group G
- ▶ Function space $C(G)$ on G
- ▶ Distributions are continuous linear forms on $C(G)$.
- ▶ Continuous characters form another topological group $X(G)$.
- ▶ Distributions \rightarrow functions on character space $X(G)$.

p -adic Fourier Theory

Overview

In p -adic Fourier theory the ingredients are:

- ▶ $\mathbb{G} = \mathbb{Z}_p$
- ▶ The space of continuous functions $C(G, K)$ where K is a complete extension field of \mathbb{Q}_p .
- ▶ Distributions are continuous linear maps $C(G, K) \rightarrow K$.
- ▶ The space $X(G)$ of continuous characters $\phi : G \rightarrow K^*$.
- ▶ Fourier transform converts distributions to functions on $X(G)$ by $\lambda \mapsto F_\lambda$ where $F_\lambda(\phi) = \lambda(\phi)$.

Differences

- ▶ The group \mathbb{Z}_p is totally discontinuous (every open set is closed).
- ▶ No L^2 theory
- ▶ No invariant continuous distribution (no Haar measure).

Continuous functions on \mathbb{Z}_p

Let $G = \mathbb{Z}_p$ and let K be a complete extension field of \mathbb{Q}_p . Let $C(G, K)$ be the space of continuous functions from G to K .

Example elements of $C(G, K)$:

- ▶ locally constant functions such as f where $f(x) = 1$ if $x \in p\mathbb{Z}_p$ and 0 otherwise.
- ▶ characters of order p^n . Let $\zeta \in \mu_{p^n}$.

$$\zeta^x = \zeta^{a_0 + a_1p + a_2p^2 + \dots}$$

makes sense, is locally constant, and satisfies $\zeta^{x+y} = \zeta^x \zeta^y$.

- ▶ polynomial functions
- ▶ locally polynomial functions

Continuous functions on \mathbb{Z}_p

The space $C(G, K)$ is a K -vector space with a norm given by

$$\|f\| = \sup_{x \in G} |f(x)|.$$

Proposition: $C(G, K)$ is complete (it is a “Banach Space”).

Proof: It is a general fact that if X is a compact metric space and Y is a complete metric space, then the continuous functions $C(X, Y)$ is complete under the sup norm. This is an exercise in uniform convergence.

Binomial polynomials

For $n \in \mathbb{Z}$, let $\binom{x}{n}$ be the binomial coefficient viewed as a polynomial in x . For example

$$\binom{x}{3} = \frac{x(x-1)(x-2)}{3!}.$$

Proposition: The polynomials $\binom{x}{n}$ have the property that

$$x \in \mathbb{Z}_p \implies \binom{x}{n} \in \mathbb{Z}_p.$$

This follows from the fact that $\binom{x}{n}$ takes integer values when x is a positive integer, and these are dense in \mathbb{Z}_p .

Mahler's Theorem

Theorem: Any continuous function $f : G \rightarrow K$ has a unique expansion

$$f(x) = \sum_{n=0}^{\infty} a(n) \binom{x}{n}$$

where $a(n) \in K$ for all n and $\lim_{n \rightarrow \infty} |a(n)|_p = 0$. Furthermore,

$$\|f\| = \sup_{n=0}^{\infty} |a(n)|_p.$$

Since the $a(n)$ go to zero, such an f converges pointwise. A uniform convergence argument shows that $f(x)$ is continuous.

Corollary: The space $C(G, K)$ is isomorphic (as a Banach space) to the space of sequences $(a(n))_{n=0}^{\infty}$ where $a(n) \in K$, under pointwise addition and scalar multiplication and with norm given by the sup-norm.

Comments on Mahler's Theorem

Let Δ be the finite difference operator $\Delta(f) = f(x+1) - f(x)$.

The finite difference calculus says that, if $f : \mathbb{Z} \rightarrow \mathbb{Z}$, then

$$f(x) = \sum_{n=0}^{\infty} \Delta^n(f)_{x=0} \binom{x}{n}.$$

If $f : \mathbb{Z}_p \rightarrow K$ one can still construct this series and if it makes sense it will agree with f on the integers, and then by continuity on all of \mathbb{Z}_p .

To prove convergence you need to show that the coefficients $\Delta^n(f)_{x=0}$ go to zero p -adically. A detailed proof is given by Conrad – see the references.

Characters

A continuous character is a continuous function $f : \mathbb{Z}_p \rightarrow K^*$. Such a function is determined by continuity and by $f(1)$:

- ▶ $f(1) = a$
- ▶ $f(n) = a^n$ for $n \in \mathbb{Z} \subset \mathbb{Z}_p$.
- ▶ $f(\sum_{i=0}^{\infty} b_i p^i) = \lim_{N \rightarrow \infty} a^{\sum_{i=0}^N b_i p^i}$

In particular, $a^{p^n} \rightarrow 1$ as $n \rightarrow \infty$, which forces $|a - 1| < 1$.

If $a = 1 + z$ where $|z| < 1$ then Mahler's theorem tells us that

$$(1 + z)^x = \sum_{i=0}^{\infty} \binom{x}{i} z^i$$

is a continuous function of $x \in \mathbb{Z}_p$ and the uniqueness tells us that

$$(1 + z)^{(x+y)} = (1 + z)^x (1 + z)^y.$$

The space of (p -adic) characters

Proposition: The space of continuous K^* valued characters of \mathbb{Z}_p is the set

$$X(\mathbb{Z}_p)(K) = \{1 + z \in K : |z| < 1\}.$$

That is, the character space is an “open p -adic disk of radius one centered at 1”. Given z with $|z| < 1$, we write

$$\phi_z(x) = (1 + z)^x = \sum \binom{x}{n} z^n.$$

Distributions

A linear map $\lambda : C(G, K) \rightarrow K$ is continuous if and only if it is bounded:

$$\sup_f \|\lambda(f)\|_p < C\|f\|$$

for some constant C . It is enough to check this on the set of binomial polynomials $\binom{x}{n}$, which have sup-norm 1.

Distributions continued

Proposition: The space $D(G, K)$ of continuous linear maps $C(G, K) \rightarrow K$ is isomorphic (as a Banach space) to the space of bounded sequences $(b(n))_{n=0}^{\infty}$ with entries in K , where

$$b(n) = \lambda\left(\binom{x}{n}\right).$$

For reasons we will explain later we package these sequences up into formal power series with p -adically bounded coefficients.

$$\lambda \mapsto F_{\lambda} = \sum_{n=0}^{\infty} \lambda\left(\binom{x}{n}\right) T^n.$$

Duality

If

$$g(T) = \sum_{n=0}^{\infty} c(n) T^n \in D(G, K)$$

and

$$f \in C(G, K) = \sum a(n) \binom{x}{n},$$

then the duality pairing $D(G, K) \times C(G, K) \rightarrow K$ is given by

$$\langle \sum c(n) T^n, \sum a(n) \binom{x}{n} \rangle = \sum_{n=0}^{\infty} c(n) a(n)$$

which converges since the $c(n)$ are bounded and the $a(n)$ go to zero.

The Fourier transform

Theorem: The Fourier transform of the distribution λ is a power series $F_\lambda(T)$ whose value at z is the distribution evaluated at the character ϕ_z where

$$\phi_z(x) = (1 + z)^x.$$

That is,

$$F_\lambda(z) = \lambda(\phi_z).$$

Remember that the Fourier transform converts a distribution λ into a function on the character space by the rule $F_\lambda(\phi) = \lambda(\phi)$.

If $F_\lambda(T) = \sum c(n)T^n$ and ϕ_z is the character where $\phi_z(x) = (1 + z)^x$ then

$$\lambda(\phi_z) = \langle \sum c(n)T^n, \sum \binom{x}{n} z^n \rangle = \sum c(n)z^n = F_\lambda(z).$$

This converges because the $c(n)$ are bounded and $|z| < 1$ so $z^n \rightarrow 0$.

The Iwasawa algebra

Let's look at $D(G, \mathbb{Q}_p)$, which is the space of power series with bounded \mathbb{Q}_p coefficients.

Inside this ring is the subring

$$\Lambda = \mathbb{Z}_p[[T]]$$

of power series with integral coefficients. This important algebra is called the Iwasawa algebra.

Proposition: The Iwasawa algebra is the space of distributions λ on $C(G, \mathbb{Q}_p)$ with the property that $|\lambda(f)| \leq \|f\|$ for all f ; that is, it is the unit ball in the Banach space $D(G, \mathbb{Q}_p)$.

Some special “integrals”

Let $F(T) \in D(G, K)$ be a distribution. We know that

$$\langle F(T), \phi_z \rangle = F(z).$$

In other words, the values of F give the integrals of characters.

Let $x \in \mathbb{Z}_p$ be fixed and let $F(T) = (1 + T)^x = \sum_{n=0}^{\infty} \binom{x}{n} T^n$. Then

$$\langle F(T), \phi_z \rangle = \sum \binom{x}{n} z^n = \phi_z(x).$$

In other words, the power series $(1 + T)^x$ with x fixed is the Dirac distribution at x .

More special integrals

Proposition:

$$\langle F(T), x^n \rangle = \partial^n F(T)|_{T=0}$$

where $\partial F = (1 + T) \frac{d}{dT} F$.

To prove this, notice that

$$x \binom{x}{n} = (n+1) \binom{x}{n+1} + n \binom{x}{n}.$$

Then compare $\langle T^n, x \binom{x}{j} \rangle$ with $\langle \partial T^n, \binom{x}{j} \rangle$ and see that they are the same for all n and j .

By linearity (and continuity) we conclude that $\langle \partial F, h \rangle = \langle F, xh \rangle$ for any h and so

$$\partial^n F(T)|_{T=0} = \langle \partial^n F(T), 1 \rangle = \langle F(T), x^n \rangle$$

Conclusions

Conclusions

The first important application of Fourier theory is the construction of the Kubota-Leopoldt p -adic zeta function. The first step uses our result on “integrals” of x^n .

Lemma: The power series

$$F_a(T) = \frac{1}{T} - \frac{a}{(1+T)^a - 1}$$

has coefficients in \mathbb{Z}_p provided a is an integer coprime to p .

Therefore $F_a(T)$ can be viewed as a distribution on \mathbb{Z}_p . It follows from the theory of the zeta function that

$$\langle F_a(T), x^k \rangle = \partial^k F_a(T)|_{T=0} = (-1)^k (1 - a^{1+k}) \zeta(-k).$$

We'd like this transform to be a p -adic function of k which involves comparing Fourier theory of \mathbb{Z}_p^* and \mathbb{Z}_p . See the references to learn how this works.

References and further reading

References and further reading

1. Conrad, Keith. Mahler Expansions. Mahler Expansions
2. Jacinto, J. and Williams, An Introduction to p -adic L -functions
3. Washington, Lawrence. Introduction to Cyclotomic Fields. Graduate Texts in Mathematics Volume 83. See especially Chapter 13.
4. De Shalit, E. Mahler bases and elementary p -adic analysis
5. Bhargava, M. and Kedlaya, K. Continuous functions on compact subsets of local fields
6. Schneider, P. and Teitelbaum, J.
<https://www.math.uni-bielefeld.de/documenta/vol-06/18.pdf>

Additional Material

The Mellin transform

Remember that we showed that \mathbb{Z}_p^* is isomorphic to $\mu_{p-1} \times \mathbb{Z}_p$ via the map sending x to the pair $(\omega(x), \langle x \rangle)$.

If $z \in 1 + p\mathbb{Z}_p$ and $0 \leq i < p - 1$ then we have a character

$$\psi = \omega^i \psi_z$$

where $\psi_z(x) = (1 + z)^{\log(x)/p}$. The character $x \mapsto \langle x \rangle^s$ corresponds to ψ_z where $z = \exp(ps) - 1$ for $s \in \mathbb{Z}$.

If λ is a distribution on \mathbb{Z}_p^* , its “Mellin transform” is

$$M_\lambda(z, \omega^i) = \lambda(\omega^i \psi_z).$$

M_λ is a vector of $q - 1$ power series – it’s a Fourier transform for the multiplicative group.

Mellin and Fourier

Suppose λ is a distribution on \mathbb{Z}_p which vanishes on functions with support in $p\mathbb{Z}_p$. Therefore λ is also a distribution on \mathbb{Z}_p^* .

Let M_λ be its Mellin transform and F_λ be its Fourier transform.

if $n \equiv i \pmod{p-1}$, we have

$$M_\lambda(\exp(pn) - 1, \omega^i) = \lambda(x^n) = \partial^n F_\lambda(T)|_{T=0}$$

In addition, $M_\lambda(\exp(ps) - 1, \omega^i)$ is an analytic function in s – that is, given by a power series in s for $s \in \mathbb{Z}_p$.

From \mathbb{Z}_p to \mathbb{Z}_p^*

To complete the construction of the K-L zeta function, we need to adjust our power series F_a so that it vanishes on functions supported on $p\mathbb{Z}_p$.

For this we use the function

$$H(x) = 1 - \frac{1}{p} \sum_{\zeta} \zeta^x$$

where ζ runs over the p^{th} roots of unity.

$H(x) = 1$ if x is not divisible by p and 0 if it is.

If F is a power series corresponding to a distribution, let F^* be the function

$$F^*(T) = F(T) - \frac{1}{p} \sum_{\zeta} F(\zeta - 1 + \zeta T).$$

One can show that this is still a power series with bounded coefficients, and is therefore still a distribution.

From \mathbb{Z}_p to \mathbb{Z}_p^*

Proposition: If f is a continuous function on \mathbb{Z}_p , then

$$\langle F^*, f \rangle = \langle F, Hf \rangle.$$

Check this on characters. We have

$$\langle F^*, (1+z)^x \rangle = \langle F, (1+z)^x \rangle - \frac{1}{p} \sum_{\zeta} \langle F(\zeta - 1 + \zeta T), (1+z)^x \rangle.$$

But

$$\langle F(\zeta - 1 + \zeta T), (1+z)^x \rangle = \langle F(T), (\zeta(1+z))^x \rangle$$

and combining the terms gives you what you want.

Corollary: F^* vanishes on functions supported on $p\mathbb{Z}_p$ and is therefore a distribution on \mathbb{Z}_p^* .

Wrapping up.

To finish, apply the operation above to the power series

$$F_a(T) = \frac{1}{T} - \frac{a}{(1+T)^a - 1}.$$

It turns out that

$$F_a^*(T) = F_a(T) - F_a((1+T)^p - 1).$$

With some calculations using this you obtain the analytic function $M(\exp(pn) - 1, \omega^i)$ satisfying

$$M(\exp(pn) - 1, \omega^i) = (-1)^k (1 - a^{k+1})(1 - p^{k+1})\zeta(-k)$$

With a bit more work one can get rid of the $1 - a^{k+1}$ term, but the p -Euler factor has to be “removed” for the p -adic construction to work.