

p-adic Fourier Theory and applications

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Overview

1. Brief look at classical fourier analysis for motivation
2. Continuous functions on \mathbb{Z}_p and Mahler's Theorem
3. Distributions, characters, and the Iwasawa algebra
4. The Kubota-Leopoldt p-adic zeta function.

Classical Fourier Analysis

Let G be an compact topological group that is abelian.

- ▶ G is a topological space
- ▶ G is an abelian group
- ▶ the group operations are continuous

Examples:

- ▶ $G = \{z \in \mathbb{C} : |z| = 1\}$, so G is the circle group \mathbb{T} .
- ▶ G is any finite abelian group.
- ▶ G is the additive group of p -adic integers \mathbb{Z}_p
- ▶ G is the multiplicative group of p -adic units \mathbb{Z}_p^*

Haar Measure

A compact abelian group G has a unique invariant measure μ such that $\mu(G) = 1$.

“Invariant” means $\mu(gX) = \mu(X)$ for subsets X of G .

On \mathbb{T} this is Lebesgue measure.

On a finite group it is the measure that assigns mass $|G|^{-1}$ to each point.

On \mathbb{Z}_p , we have

$$\mu(a + p^n\mathbb{Z}_p) = 1/p^n$$

because

$$\mathbb{Z}_p = \coprod_{a=0}^{p^n-1} (a + p^n\mathbb{Z}_p).$$

Characters and the dual group

A **character** of a compact abelian group G is a continuous homomorphism $\phi : G \rightarrow \mathbb{T}$.

The set of characters $X(G)$ form a topological group in their own right. This is called the **dual group** to G .

When $G = \mathbb{T}$, then the continuous characters are the functions

$$\phi_n(z) = z^n$$

where $n \in \mathbb{Z}$ and $X(\mathbb{T}) = \mathbb{Z}$.

When G is finite, the character group $X(G)$ is a finite group isomorphic to G . To see this:

- ▶ If $G = \mathbb{Z}/n\mathbb{Z}$, then $X(G)$ is isomorphic to the cyclic group μ_n of n^{th} -roots of unity via $\phi \mapsto \phi(1)$.
- ▶ Now use the fundamental theorem of abelian groups.

Distributions

$L^2(G)$ is the Hilbert space of complex valued functions on G with inner product

$$\langle f, g \rangle = \int_G f \bar{g} d\mu$$

where $d\mu$ is the normalized Haar measure on G .

A distribution on G is a continuous linear map $\lambda : L^2(G) \rightarrow \mathbb{C}$.

By Hilbert space theory, any such λ is of the form $\lambda(f) = \langle f, g_\lambda \rangle$ for some $g_\lambda \in L^2(G)$.

Fourier transform

A distribution λ gives a function F_λ on the space of characters $X(G)$ by

$$F_\lambda(\phi) = \lambda(\phi) = \langle \phi, g_\lambda \rangle .$$

This function is called the Fourier transform of λ .

Examples

If $G = \mathbb{T}$, then:

- ▶ $X(G) = \mathbb{Z}$
- ▶ the function $F_\lambda(n)$ gives the Fourier coefficients of g_λ .

If G is finite:

- ▶ the distributions on G are the group algebra $\mathbb{C}[G]$.
- ▶ If $\lambda = \sum_g a(g)g$ and $\phi \in X(G)$ then

$$F_\lambda(\phi) = \sum_g a(g)\phi(g).$$

- ▶ this is the *discrete fourier transform*.

Key ingredients

- ▶ Compact abelian group G
- ▶ Function space $C(G)$ on G
- ▶ Distributions are continuous linear forms on $C(G)$.
- ▶ Continuous characters form another topological group $X(G)$.
- ▶ Distributions \rightarrow functions on character space $X(G)$.

p -adic Fourier Theory

Overview

In p -adic Fourier theory the ingredients are:

- ▶ $\mathbb{G} = \mathbb{Z}_p$ or $G = \mathbb{Z}_p^*$.
- ▶ The space of continuous functions $C(G, K)$ where K is a complete extension field of \mathbb{Q}_p .
- ▶ The space $X(G)$ of continuous characters $\phi : G \rightarrow K^*$.
- ▶ Distributions are continuous linear maps $C(G, K) \rightarrow K$.
- ▶ Fourier transform converts distributions to functions on $X(G)$ by $\lambda \mapsto F_\lambda$ where $F_\lambda(\phi) = \lambda(\phi)$.

Different because:

- ▶ continuous functions aren't Haar integrable (measures of small sets get p -adically large)
- ▶ there are many more continuous functions from \mathbb{Z}_p to K than from \mathbb{Z}_p to \mathbb{C} .

\mathbb{Z}_p^* and \mathbb{Z} are almost the same ($p > 2$)

Proposition: The group \mathbb{Z}_p^* is isomorphic to the product

$$\mu_{p-1} \times (1 + p\mathbb{Z}_p)$$

where μ_{p-1} is the finite group of $(p-1)^{st}$ roots of 1.

Proof:

- ▶ Hensel's lemma applied to $x^{p-1} - 1 = 0$ gives $\mu_{p-1} \subset \mathbb{Z}_p^*$.
- ▶ There's one element of μ_{p-1} in each congruence class. So every element $a \in \mathbb{Z}_p^*$ can be written uniquely as

$$a = \omega(a)\langle a \rangle$$

where $\omega(a) \in \mu_{p-1}$ and $\langle a \rangle \in 1 + p\mathbb{Z}_p$.

\mathbb{Z}_p^* and \mathbb{Z}_p are almost the same ($p > 2$)

The power series

$$\frac{1}{p} \log(1+x) = \frac{1}{p} \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right)$$

converges for any $1+x \in 1+p\mathbb{Z}_p$ and satisfies $\log((1+x)(1+y)) = \log(1+x) + \log(1+y)$.

The power series

$$\exp(px) = 1 + px + \frac{(px)^2}{2!} + \frac{(px)^3}{3!} + \dots$$

converges for $x \in \mathbb{Z}_p$ and satisfies $\exp(p(x+y)) = \exp(px) \exp(py)$.

These maps give mutually inverse isomorphisms between $1+p\mathbb{Z}_p$ and \mathbb{Z}_p .

So $1+p\mathbb{Z}_p$ (under multiplication) is isomorphic to \mathbb{Z}_p (under addition).

Continuous functions on \mathbb{Z}_p

Let $G = \mathbb{Z}_p$ and let K be a complete extension field of \mathbb{Q}_p . Let $C(G, K)$ be the space of continuous functions from G to K .

Example elements of $C(G, K)$:

- ▶ locally constant functions such as f where $f(x) = 1$ if $x \in p\mathbb{Z}_p$ and 0 otherwise.
- ▶ characters of order p^n . Let $\zeta \in \mu_{p^n}$.

$$\zeta^x = \zeta^{a_0 + a_1p + a_2p^2 + \dots}$$

makes sense, is locally constant, and satisfies $\zeta^{x+y} = \zeta^x \zeta^y$.

- ▶ polynomial functions
- ▶ locally polynomial functions

Continuous functions on \mathbb{Z}_p

The space $C(G, K)$ is a K -vector space with a norm given by

$$\|f\| = \sup_{x \in G} |f(x)|.$$

Proposition: $C(G, K)$ is complete (it is a “Banach Space”).

Proof: It is a general fact that if X is a compact metric space and Y is a complete metric space, then the continuous functions $C(X, Y)$ is complete under the sup norm. This is an exercise in uniform convergence.

Binomial polynomials

For $n \in \mathbb{Z}$, let $\binom{x}{n}$ be the binomial coefficient viewed as a polynomial in x . For example

$$\binom{x}{3} = \frac{x(x-1)(x-2)}{3!}.$$

Proposition: The polynomials $\binom{x}{n}$ have the property that

$$x \in \mathbb{Z}_p \implies \binom{x}{n} \in \mathbb{Z}_p.$$

This follows from the fact that $\binom{x}{n}$ takes integer values when x is a positive integer, and these are dense in \mathbb{Z}_p .

Mahler's Theorem

Theorem: Any continuous function $f : G \rightarrow K$ has a unique expansion

$$f(x) = \sum_{n=0}^{\infty} a(n) \binom{x}{n}$$

where $a(n) \in K$ for all n and $\lim_{n \rightarrow \infty} |a(n)| = 0$. Furthermore,

$$\|f\| = \sup_{n=0}^{\infty} |a(n)|.$$

Since the $a(n)$ go to zero, such an f converges pointwise. A uniform convergence argument shows that $f(x)$ is continuous.

Corollary: The space $C(G, K)$ is isomorphic (as a Banach space) to the space of sequences $(a(n))_{n=0}^{\infty}$ where $a(n) \in K$, under pointwise addition and scalar multiplication and with norm given by the sup-norm.

Comments on Mahler's Theorem

Let Δ be the finite difference operator $\Delta(f) = f(x+1) - f(x)$.

The finite difference calculus says that, if $f : \mathbb{Z} \rightarrow \mathbb{Z}$, then

$$f(x) = \sum_{n=0}^{\infty} \Delta^n(f)_{x=0} \binom{x}{n}.$$

If $f : \mathbb{Z}_p \rightarrow K$ one can still construct this series and if it makes sense it will agree with f on the integers, and then by continuity on all of \mathbb{Z}_p .

To prove convergence you need to show that the coefficients $\Delta^n(f)_{x=0}$ go to zero. A detailed proof is given by Conrad (google "conrad mahlerexpansions")

Characters

A continuous character is a continuous function $f : \mathbb{Z}_p \rightarrow K^*$. Such a function is determined by continuity and by $f(1)$:

- ▶ $f(1) = a$
- ▶ $f(n) = a^n$ for $n \in \mathbb{Z} \subset \mathbb{Z}_p$.
- ▶ $f(\sum_{i=0}^{\infty} b_i p^i) = \lim_{N \rightarrow \infty} a^{\sum_{i=0}^N b_i p^i}$

In particular, $a^{p^n} \rightarrow 1$ as $n \rightarrow \infty$, which forces $|a - 1| < 1$.

If $a = 1 + z$ where $|z| < 1$ then Mahler's theorem tells us that

$$(1 + z)^x = \sum_{i=0}^{\infty} \binom{x}{i} z^i$$

is a continuous function of $x \in \mathbb{Z}_p$ and the uniqueness tells us that

$$(1 + z)^{(x+y)} = (1 + z)^x (1 + z)^y.$$

The space of (p -adic) characters

Proposition: The space of continuous K^* valued characters of \mathbb{Z}_p is the set

$$X(\mathbb{Z}_p)(K) = \{1 + z \in K : |z| < 1\}.$$

That is, the character space is an “open p -adic disk of radius one centered at 1”. Given z with $|z| < 1$, we write

$$\phi_z(x) = (1 + z)^x = \sum \binom{x}{n} z^n.$$

Distributions

Proposition: The space $D(G, K)$ of continuous linear maps $C(G, K) \rightarrow K$ is isomorphic (as a Banach space) to the space of bounded power series with coefficients in K via

$$\lambda \mapsto \sum_{n=0}^{\infty} \lambda\left(\binom{x}{n}\right) T^n.$$

If $g(T) = \sum_{n=0}^{\infty} c(n) T^n$, then the duality pairing $D(G, K) \times C(G, K) \rightarrow K$ is given by

$$\left\langle \sum c(n) T^n, \sum a(n) \binom{x}{n} \right\rangle = \sum_{n=0}^{\infty} c(n) a(n)$$

The Fourier transform

Remember that the Fourier transform converts a distribution λ into a function on the character space by the rule $F_\lambda(\phi) = \lambda(\phi)$.

In our situation, if $\lambda = \sum c(n) T^n$ and $\phi = \phi_z$ then

$$F_\lambda(\phi) = \langle \sum c(n) T^n, \sum \binom{x}{n} z^n \rangle = \sum c(n) z^n.$$

In other words, the Fourier transform of the distribution λ is a power series whose value at z is the distribution evaluated at ϕ_z .

The Iwasawa algebra

Let's look at $D(G, \mathbb{Q}_p)$, which is the space of power series with bounded \mathbb{Q}_p coefficients.

Inside this ring is the subring

$$\Lambda = \mathbb{Z}_p[[T]]$$

of power series with integral coefficients. This important algebra is called the Iwasawa algebra.

Proposition: The Iwasawa algebra is the space of distributions λ on $C(G, \mathbb{Q}_p)$ with the property that $|\lambda(f)| \leq \|f\|$ for all f ; that is, it is the unit ball in the Banach space $D(G, \mathbb{Q}_p)$.

Some special “integrals”

Let $F(T) \in D(G, K)$ be a distribution. We know that

$$\langle F(T), \phi_z \rangle = F(z).$$

In other words, the values of F give the integrals of characters.

Let $x \in \mathbb{Z}_p$ be fixed and let $F(T) = (1 + T)^x = \sum_{n=0}^{\infty} \binom{x}{n} T^n$. Then

$$\langle F(T), \phi_z \rangle = \sum \binom{x}{n} z^n = \phi_z(x).$$

In other words, the power series $(1 + T)^x$ with x fixed is the Dirac distribution at x .

More special integrals

Proposition:

$$\langle F(T), x^n \rangle = \partial^n F(T)|_{T=0}$$

where $\partial F = (1 + T) \frac{d}{dT} F$.

To prove this, notice that

$$x \binom{x}{n} = (n+1) \binom{x}{n+1} + n \binom{x}{n}.$$

Then compare $\langle T^n, x \binom{x}{j} \rangle$ with $\langle \partial T^n, \binom{x}{j} \rangle$ and see that they are the same for all n and j .

By linearity (and continuity) we conclude that $\langle \partial F, h \rangle = \langle F, xh \rangle$ for any h and so

$$\partial^n F(T)|_{T=0} = \langle \partial^n F(T), 1 \rangle = \langle F(T), x^n \rangle$$