COHOMOLOGICAL INVARIANTS FOR QUADRATIC FORMS OVER LOCAL RINGS

JEREMY ALLEN JACOBSON

Abstract. Let A be local ring in which 2 is invertible and let n be a non-negative integer. We show that the nth cohomological invariant of quadratic forms is a well-defined homomorphism from the nth power of the fundamental ideal in the Witt ring of A to the degree n étale cohomology of A with mod 2 coefficients, which is surjective and has kernel the (n+1)th power of the fundamental ideal. This is obtained by proving the Gersten conjecture for Witt groups in an important mixed-characteristic case.

Introduction

Let A be a local ring with 2 invertible and let $W(A)$ denote the Witt ring of symmetric bilinear forms over A. Let $I(A)$ denote the fundamental ideal in the Witt ring of A, and let $I^n(A)$ denote its powers. Recall that a form $\langle 1, -a_1\rangle \otimes \langle 1, -a_2\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$, where the $a_i \in A^\times$ for all i, is called an n-fold Pfister form and is denoted by $\langle\langle a_1, a_2, \cdots, a_n\rangle\rangle$. When A is a field, a long standing problem in quadratic form theory had been to show that the assignment $\langle\langle a_1, a_2, \cdots, a_n\rangle\rangle \mapsto (a_1)\cup(a_2)\cup\cdots\cup(a_n)$ induces a well-defined homomorphism of groups, the so-called *nth cohomological invariant*

$$\mathrm{e_n}(A) : I^n(A) \to H^n_{\acute{e}t}(A, \mathbb{Z}/2)$$

and furthermore, to show it induces a bijection

$$\bar{\mathrm{e}}_\mathrm{n}(A) : I^n(A)/I^{n+1}(A) \to H^n_{\acute{e}t}(A, \mathbb{Z}/2)$$

When A is a field, the solution to this problem follows from the affirmation of Milnor's conjectures on the mod 2 Galois cohomology of fields and on quadratic forms. The Galois cohomology part is due to Vladimir Voevodsky [Voe03]. The quadratic forms part is due to Dmitri Orlov, Alexander Vishik, and Voevodsky: They proved the result for fields of characteristic zero in [OVV07] and this is sufficient[1]; There is another proof in a paper of Bruno Kahn and Ramdorai Sujatha [KS00, Remark 3.3] and Fabien Morel gave a proof for any field of characteristic not two [Mor05].[2] When A is a local ring (resp. semilocal ring) which contains

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[1]The result for fields of characteristic $p \neq 2$ is a consequence of the characteristic zero result by an easy lifting argument analogous to that of Milnor, [Mil70, p.344]. The author thanks Jean-Pierre Serre for sharing this insight.

[2]Morel also gave two other proofs: one in [Mor99] for fields of characteristic zero and another which uses only Sq^2 but is unpublished.

an infinite field of characteristic not two this was shown by Moritz Kerz and Stefan Müller-Stach [KMS07, Corollary 0.8] (resp. forthcoming work of Stefan Gille); in [Ker10, Proposition 16] Kerz removed the hypothesis that the field be infinite. When A is a henselian local ring this may be shown using the affirmation of the Milnor conjectures together with "rigidity" to reduce to the field case by passing to the residue field of A. When A is strictly henselian, $I(A) = 0$ as any element in $A^\times$ is a square, hence $\mathrm{e}_0 : W(A) \to H^0_{\acute{e}t}(A, \mathbb{Z}/2)$ is an isomorphism and $I^n(A) = 0$ for $n > 0$. For general local rings there are results up to e_2 [Bae79, Yuc86, Man75].

In this paper, Theorem 4.5 solves this problem in general, that is to say, for any local ring A (which may be singular and need not contain a field) with $2 \in A^\times$. There is also a globalization of these questions to any scheme with two invertible in its global sections, see Remark 4.6.

To remove the restriction from the previously cited work that the local ring contain a field it is sufficient to prove the Gersten conjecture for the Witt groups and purity for the powers of the fundamental ideal in the case of local rings essentially smooth over $\mathbb{Z}_{\langle p \rangle}$, where p is a prime integer $p \neq 2$ and $\mathbb{Z}_{\langle p \rangle}$ denotes the localization of $\mathbb{Z}$ at the prime ideal $\langle p \rangle$.

More generally, we prove in Theorem 2.8 the Gersten conjecture for the Witt groups of A in the unramified case, that is to say, for any regular local ring A of mixed-characteristic with residue field of characteristic $p \neq 2$ and A/pA a regular ring. The proof uses the global signature morphism to show that the Gersten conjecture for étale cohomology implies the Gersten conjecture for the Witt groups, and then concludes by using known results on the Gersten conjecture for étale cohomology. One may also prove the Gersten conjecture for Witt groups without relying on results for étale cohomology.[3]

In Theorem 3.2 we prove purity for the powers of the fundamental ideal in the Witt ring for local rings essentially smooth over $\mathbb{Z}_{\langle p \rangle}$ by following the argument given in the case of local rings essentially smooth over a field by M. Kerz and S. Mueller-Stach [KMS07, Corollary 0.5]. The Gersten conjecture result we proved in Theorem 2.8 is necessary here.

The results of this paper can likely be generalized to semi-local rings. We refrained from doing so, although the semi-local case may be important since locality is lost after passing to finite extensions.

Recall that the cohomological invariants e_n agree with the classical invariants dimension mod 2 (e_0), discriminant (e_1), Hasse invariant (e_2), and Arason invariant (e_3). In the last section we obtain hypotheses under which the cohomological invariants classify non-degenerate quadratic forms over A (Lemma 5.2 together with Corollary 4.7 B). For example, in Proposition 5.4 we prove that if A is a local ring with $2 \in A^\times$ that is essentially of finite type over the integers and if s denotes the least integer for which $H^n_{\acute{e}t}(A[i], \mathbb{Z}/2) = 0$ for all $n > s$ (such an integer always exists, see (4.8)), then: $\cap I^n(A) = 0$; the dimension, total signature (Definition 1.2), and the cohomological invariants $\mathrm{e}_0, \mathrm{e}_1, \cdots \mathrm{e}_\mathrm{s}$ classify non-degenerate quadratic forms over A; the level $s(A)$ and height $h(A)$ of A are bounded in terms of s; if A is additionally regular, then $I^n(A) \to I^n(K)$ is injective when $n > s$, where K is the fraction field of A.

[3]See the thesis of the author titled "On the Witt groups of schemes"

1. On the signature

In this section we use the global signature to prove Lemma 1.5.

Definition 1.1. Let A be a commutative ring with unit. The *real spectrum* of A is the topological space obtained by equipping the set

$$\operatorname{sper} A := \{(\mathrm{p}, P) | \mathrm{p} \in \operatorname{spec} A, P \text{ is an ordering on the residue field } k(\mathrm{p})\}$$

with the "Harrison-topology" [KS89, Kapitel III, §3, Definition 1]. For any scheme X, the *real spectrum of* X is the topological space denoted X_r obtained by glueing the real spectra $\operatorname{sper} A_\alpha$ of any open affine cover $X = \cup A_\alpha$ of X; it does not depend on the cover chosen [Sch94].

Definition 1.2. Let X be a scheme. The *global signature*, as defined for instance in [Mah82], is the ring homomorphism

$$\operatorname{sign} : W(X) \to \mathrm{C}(X_r, \mathbb{Z})$$

from the Witt ring of symmetric bilinear forms over X to the ring of continuous integer valued functions on X_r that assigns an isometry class $[\phi]$ of a symmetric bilinear form ϕ over X to the function on X_r defined by

$$\operatorname{sign}([\phi])(x, P) := \operatorname{sign}_P([i_x^* \phi])$$

where $i_x : x \to X$ is any point and P is any ordering on $k(x)$ and $\operatorname{sign}_P([i_x^*\phi])$ is the signature with respect to P [KS89, Kapitel I, §2, Satz 2]. See [Kne77] for an introduction to the Witt ring $W(X)$.
When $X = \operatorname{spec} A$ is a local ring or a field with $2 \in A^\times$ and ϕ is a non-degenerate quadratic form over A of dimension n, then diagonalizing ϕ we obtain $\phi \simeq \langle a_1 \rangle \perp \langle a_2 \rangle \perp \cdots \perp \langle a_n \rangle$ where $a_i \in A^\times$. Then, for any prime $\mathrm{p} \in \operatorname{spec} A$ and any ordering P on the residue field of p, one has

$$\operatorname{sign}([\phi])(\mathrm{p}, P) = \sum_{i=1}^{n} \operatorname{sgn}_P(\overline{a_i})$$

where $\overline{a_i} \in A_\mathrm{p}/\mathrm{p}A_\mathrm{p}$ and $\operatorname{sgn}_P(\overline{a_i}) = 1$ (resp. $\operatorname{sgn}_P(\overline{a_i}) = -1$) when $\overline{a_i} >_P 0$ (resp. $\overline{a_i} <_P 0$). In the field case, this map (which doesn't depend on the diagonalization chosen) is called the *total signature* (compare [KS89, Kapitel III, §8, Satz 1]), so in the local case we will also use this name instead of global signature.
Finally, let $\mathcal{W}$ denote the Zariski sheaf on X associated to the presheaf $U \mapsto W(U)$. Let $\operatorname{supp}_* \mathbb{Z}$ denote the sheaf $U \mapsto \mathrm{C}(U_r, \mathbb{Z})$, where $\mathrm{C}(U_r, \mathbb{Z})$ denotes the set of continuous $\mathbb{Z}$-valued functions on the topological space U_r. The global signature induces a morphism

$$\mathcal{S}\mathrm{ign} : \mathcal{W} \to \operatorname{supp}_* \mathbb{Z}$$

of Zariski sheaves on X.

Remark 1.3. The most important result on the global signature is due to L. Mahé who proved if $X = \operatorname{spec} A$ is affine, then the cokernel of the global signature is 2-primary torsion [Mah82, Théorème 3.2]. Equivalently, after inverting 2, that is to say, localizing with respect to the multiplicative set $S = \{1, 2, 2^2, 2^3, \cdots\}$, the global signature induces a surjection

$$\operatorname{sign} : W(\operatorname{spec} A)[1/2] \twoheadrightarrow \mathrm{C}(\operatorname{sper} A, \mathbb{Z})[1/2] \tag{1.1}$$

of rings. For A a field this was well-known (for instance, [Lam77, p.34, Theorem 3.4]). When A is a connected ring and $\operatorname{sper} A \neq \emptyset$, the kernel of the global signature is the nilradical in the Witt ring [Mah82, Section 1.3]. When A is a connected local ring, using the prime ideal theory of the Witt ring one may show that either the nilradical is 2-primary torsion or the entire Witt ring $W(A)$ is 2-primary torsion [Kne81, Theorem 1.2]. In either case the kernel is 2-primary torsion, hence (1.1) is a bijection; bijectivity may also be shown using "cohomological" methods.[4] When A is a field, bijectivity of (1.1) is Pfister's local-global principle [Pfi66, Satz 22].

Proposition 1.4. *Let X be a scheme. The global signature morphism of sheaves induces an isomorphism of sheaves*

$$\varinjlim \mathcal{W} \to \varinjlim \operatorname{supp}_* \mathbb{Z}$$

where $\varinjlim \mathcal{W}$ denotes the colimit over the system of sheaves

$$\mathcal{W} \xrightarrow{2} \mathcal{W} \xrightarrow{2} \mathcal{W} \xrightarrow{2} \cdots$$

and similarly for $\varinjlim \operatorname{supp}_* \mathbb{Z}$.

Proof. Recall that direct limits of sheaves $\varinjlim \mathcal{F}_i$ exist and equal the sheaf associated to the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$. Also, a direct limit of a group over multiplication by 2 may be identified with the localization at the element 2, hence $\varinjlim \mathcal{W}(U) = \mathcal{W}(U)[\frac{1}{2}]$ and $\varinjlim \operatorname{supp}_*(U) \overset{def}{=} \varinjlim \mathrm{C}(U_r, \mathbb{Z}) = \mathrm{C}(U_r, \mathbb{Z})[\frac{1}{2}]$. In view of the results that were discussed in Remark 1.3, for any local ring $\mathcal{O}_{X,x}$ of X the map on stalks

$$W(\operatorname{spec} \mathcal{O}_{X,x})[1/2] \to \mathrm{C}(\operatorname{sper} \mathcal{O}_{X,x}, \mathbb{Z})[1/2]$$

is an isomorphism, where $\operatorname{sper} \mathcal{O}_{X,x}$ denotes the real spectrum of $\operatorname{spec} \mathcal{O}_{X,x}$. Thus the map of sheaves in the statement is an isomorphism. □

Lemma 1.5. *Let A be a noetherian regular excellent local ring with 2 invertible. If $f \in A$ is a regular parameter, then in positive degree*

$$H^*_{Zar}(\operatorname{spec} A_f, \varinjlim \mathcal{W}) = 0$$

where A_f denotes the localization of A at the element f, that is, A with f inverted.

Proof. From the previous proposition we obtain an isomorphism of Zariski cohomology groups

$$H^*_{Zar}(\operatorname{spec} A_f, \varinjlim \mathcal{W}) \overset{\text{sign}}{\simeq} H^*_{Zar}(\operatorname{spec} A_f, \varinjlim \operatorname{supp}_* \mathbb{Z})$$

The groups $H^*_{Zar}(\operatorname{spec} A_f, \varinjlim \operatorname{supp}_* \mathbb{Z})$ may be identified with the groups $\varinjlim H^*_{Zar}(\operatorname{spec} A_f, \operatorname{supp}_* \mathbb{Z})$ since $\operatorname{spec} A_f$ is noetherian, hence Zariski cohomology commutes with direct limits of sheaves [Tam94, Theorem 3.11.1]. The Zariski cohomology groups $H^*_{Zar}(\operatorname{spec} A_f, \operatorname{supp}_* \mathbb{Z})$ may be identified with the real cohomology groups $H^*(\operatorname{sper} A_f, \mathbb{Z})$, and the latter vanish in positive degree under the stated hypotheses on A by [Sch95, Corollary (1.10)] together with [Sch94, Proposition (19.2.1)]. □

[4] For instance, injectivity follows by using the field case together with the Gersten conjecture for the Witt groups $W(A)[\frac{1}{2}]$ with two inverted; The latter has been proved by the author for any regular excellent local ring with 2 invertible. See forthcoming work of the author titled *Real cohomology and the powers of the fundamental ideal* for more in this direction.

2. The Gersten conjecture for Witt groups in the mixed-characteristic case

In Theorem 2.8 of this section we prove the Gersten conjecture for the Witt groups of any unramified regular local ring (Definition 2.2). The method is to use Lemma 1.5, proved in the last section, to show that the Gersten conjecture for étale cohomology implies the Gersten conjecture for Witt groups.

Remark 2.1. Let A be a regular excellent local ring with $2 \in A^\times$ and let $X = \operatorname{spec} A$. Recall that the Gersten conjecture for étale cohomology with $\mathbb{Z}/2$-coefficients asserts that the so-called "Gersten" complexes that appear on the E_1-page of the coniveau spectral sequence are exact (see, for instance, [Jac12, Proposition 1.12] for more details on these complexes in this setting) and that the kernel of the first differential d_0 is isomorphic to to the étale cohomology of X, that is,

$$0 \to H^n_{\acute{e}t}(X, \mathbb{Z}/2) \to \bigoplus_{x \in X^{(0)}} H^n(k(x), \mathbb{Z}/2) \xrightarrow{d_0} \bigoplus_{x \in X^{(1)}} H^{n-1}(k(x), \mathbb{Z}/2) \to \cdots$$
$$\cdots \to \bigoplus_{x \in X^{(d-1)}} H^{n-(d-1)}(k(x), \mathbb{Z}/2) \to \bigoplus_{x \in X^{(d)}} H^{n-d}(k(x), \mathbb{Z}/2)$$

is an exact sequence of groups for any integer $n \geq 0$ (note, there is no differential leaving the last group in the complex, which is either $H^{n-d}(k(x), \mathbb{Z}/2)$ for $n-d > 0$ or $H^0(k(x), \mathbb{Z}/2)$). Here d denotes the Krull dimension of X, $X^{(i)}$ denotes the points of codimension i ($\dim \mathcal{O}_{X,x} = i$) and $H^n(k(x), \mathbb{Z}/2)$ the Galois cohomology of the residue field $k(x)$ with $\mathbb{Z}/2$-coefficients. Furthermore, recall that for étale cohomology with $\mathbb{Z}/2$-coefficients the Gersten conjecture is known for A in the following cases: A essentially smooth over a field [BO74], see also [CTHK97, Gab94b]; A containing a field, that is to say, equicharacteristic local rings [Pan03, Proved here for K-theory but the étale cohomology result is obtained from the essentially smooth case by the same argument.]; A essentially smooth over a discrete valuation ring (due to H. Gillet[5], also follows from Thomas Geisser's proof of the Gersten conjecture for motivic cohomology [Gei04, This is explicitly stated in the sentence after Theorem 1.2, because $R^n\epsilon_*\mu_2$ is the Zariski sheaf associated to the presheaf $U \mapsto H^n_{\acute{e}t}(U, \mu_2)$ and the affirmation of the Milnor conjecture allows one to identify the Gersten complex for motivic cohomology with the Gersten complex for étale cohomology.]).

Definition 2.2. Let (A, m) be a regular local ring of mixed $(0,p)$-characteristic, that is to say, $\mathrm{char} K = 0$ where $K := \mathrm{Frac} A$ and $\mathrm{char} A/\mathrm{m} = p$ where $p \neq 2$. When the local ring A/pA is again regular then we will say that A is *unramified.*

Lemma 2.3. *Let A be essentially smooth over $\mathbb{Z}_{\langle p \rangle}$ for some prime integer $p \neq 2$ and let A_p denote the localization of A at the element p (that is, A with p inverted). Then, there exists an integer $N \geq 0$ for which the map of sheaves on* $\operatorname{spec} A_p$

$$\mathcal{I}^n \xrightarrow{2} \mathcal{I}^{n+1}$$

is an isomorphism whenever $n \geq N$, where $\mathcal{I}^n$ denotes the Zariski sheaf on $\operatorname{spec} A_p$ *associated to the presheaf $U \mapsto I^n(U)$, U any open in* $\operatorname{spec} A_p$.

[5] Manuscript notes titled "Bloch-Ogus for the étale cohomology of certain arithmetic schemes" distributed at the 1997 Seattle algebraic K-theory conference.

Proof. To prove the assertion of the lemma, it is sufficient to show that the induced map on stalks is an isomorphism. Let $\mathrm{p} \in \operatorname{spec} A_p$. Since A is essentially smooth over $\mathbb{Z}_{\langle p\rangle}$, there exists a bound on the transcendence degree of the residue fields of the local ring $(A_p)_{\mathrm{p}}$ over their prime fields. Consequently, there exists an integer $N \geq 0$ such that for any residue field $k(x)$ of $(A_p)_{\mathrm{p}}$, including the fraction field, the map $I^n(k(x)) \xrightarrow{2} I^{n+1}(k(x))$ is an isomorphism of groups whenever $n \geq N$ [AE01, Lemma 2.1]. Let $Y = \operatorname{spec}(A_p)_{\mathrm{p}}$ and let K denote the field of fractions of Y. The rows in the diagram below

$$\begin{tikzcd}
0 \arrow[r] & I^n(Y) \arrow[r] \arrow[d, "2"] & I^n(K) \arrow[r, "\partial"] \arrow[d, "2"] & \bigoplus_{x\in Y^{(1)}} I^{n-1}(k(x)) \arrow[d, "2"] \\
0 \arrow[r] & I^{n+1}(Y) \arrow[r] & I^{n+1}(K) \arrow[r, "\partial"] & \bigoplus_{x\in Y^{(1)}} I^{n}(k(x))
\end{tikzcd}$$

are exact as a consequence of purity for the powers of the fundamental ideal [Ker10, Proposition 16 (1)] and the Gersten Conjecture for the Witt groups [BGPW02, Theorem 6.1]. As usual, $I^{n-1}(k(x))$ indicates $W(k(x))$ whenever $n-1 \leq 0$, similarly for $I^n(Y)$ and $I^n(K)$. Hence, we get from the diagram an isomorphism of kernels

$$I^n(Y) \overset{2}{\simeq} I^{n+1}(Y)$$

whenever $n \geq N$, finishing the proof of the lemma. □

Lemma 2.4. *Let A be essentially smooth over $\mathbb{Z}_{\langle p\rangle}$, then there is a sequence of sheaves on A_p*

$$0 \to \mathcal{I}^{n+1} \to \mathcal{I}^n \to \mathcal{H}^n \to 0 \tag{2.1}$$

and this sequence is exact.

Proof. First we describe how to obtain the maps in the sequence. Let U be any open subscheme of $\operatorname{spec} A_p$, and let K denote the fraction field of U. It is well-known that the lower square in the diagram below commutes as a consequence of a theorem of J. Arason on the second residue homomorphism [Ara75, Satz 4.11]. For details see [Jac12, Theorem 2.5].

$$\begin{tikzcd}
 & 0 \arrow[d] \\
I^n(U) \arrow[r] \arrow[d] & H^0_{Zar}(U, \mathcal{H}^n) \arrow[d] \\
I^n(K) \arrow[r] \arrow[d] & H^n(K, \mathbb{Z}/2) \arrow[d] \\
\bigoplus_{x\in U^{(1)}} I^{n-1}(k(x)) \arrow[r] & \bigoplus_{x\in U^{(1)}} H^{n-1}(k(x), \mathbb{Z}/2)
\end{tikzcd} \tag{2.2}$$

In the diagram above we used the fact that the kernel of the residue

$$H^n(K, \mathbb{Z}/2) \to \bigoplus_{x\in U^{(1)}} H^{n-1}(k(x), \mathbb{Z}/2)$$

equals $H^0_{Zar}(U, \mathcal{H}^n)$ as a consequence of the Gersten conjecture for étale cohomology (Remark 2.1). The top horizontal map in Diagram (2.2) induces a morphism

$$\mathcal{I}^n \to \mathcal{H}^n$$

of sheaves on A_p. The sequence (2.1) is exact on stalks using [Ker10, Remark preceding Proposition 16] together with [KMS07, Corollary 0.8], which finishes the proof. □

Lemma 2.5. *Let A be essentially smooth over $\mathbb{Z}_{\langle p \rangle}$, $p \neq 2$. Then*

$$H^*_{Zar}(\operatorname{spec} A_p, \mathcal{H}^n) = 0$$

in positive degree.

Proof. The Gersten conjecture for étale cohomology with $\mathbb{Z}/2$-coefficients is known in the case of local rings essentially smooth over a field or over a discrete valuation ring (Remark 2.1). Hence we have that the Gersten complex for A/pA and for A is exact in positive degree. The Gersten conjecture is known for the local rings of A_p as these are essentially smooth over $\mathbb{Q}$, hence the cohomology of the Gersten complex of A_p agrees with the Zariski cohomology groups $H^*_{Zar}(\operatorname{spec} A_p, \mathcal{H}^n)$. Finally, we have that $H^*_{Zar}(\operatorname{spec} A_p, \mathcal{H}^n) = 0$ since the cohomology of the Gersten complex for A_p lives in a long exact sequence with the cohomology of the Gersten complexes for A/pA and for A (see, for example, [Ros96, Ch. 5]). □

Lemma 2.6. *Let A be essentially smooth over $\mathbb{Z}_{\langle p \rangle}$, $p \neq 2$. There exists an integer $N \geq 0$ such that*

$$H^m_{Zar}(\operatorname{spec} A_p, \mathcal{I}^N) = 0$$

whenever $m \geq 2$.

Proof. Using Lemma 2.3 we obtain an integer $N \geq 0$ such that the leftmost vertical map in the commutative diagram below of sheaves on $\operatorname{spec} A_p$ is an isomorphism.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}^N \arrow[r, "2"] \arrow[d, "2"] & \mathcal{I}^N \arrow[r] \arrow[d] & \mathcal{I}^N/2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}^{N+1} \arrow[r] & \mathcal{I}^N \arrow[r] & \overline{\mathcal{I}^N} \arrow[r] & 0
\end{tikzcd}$$

where $\overline{\mathcal{I}^N}$ denotes the Zariski sheaf associated to the presheaf $U \mapsto \mathcal{I}^N(U)/\mathcal{I}^{N+1}(U)$. It follows that $\mathcal{I}^N/2 \to \overline{\mathcal{I}^N}$ is an isomorphism of sheaves. Using Lemma 2.4 one may obtain an isomorphism of sheaves $\overline{\mathcal{I}^N} \simeq \mathcal{H}^N$, thus from Lemma 2.5 we obtain that $H^*(\operatorname{spec} A_p, \mathcal{I}^N/2) = 0$ in positive degree. From this it follows that, for $m \geq 2$, multiplication by 2 induces an isomorphism in cohomology

$$H^m_{Zar}(\operatorname{spec} A_p, \mathcal{I}^N) \overset{2}{\simeq} H^m_{Zar}(\operatorname{spec} A_p, \mathcal{I}^N)$$

and consequently, for $m \geq 2$,

$$H^m_{Zar}(\operatorname{spec} A_p, \mathcal{I}^N) \simeq \varinjlim H^m_{Zar}(\operatorname{spec} A_p, \mathcal{I}^N)$$

where $\varinjlim H^m_{Zar}(\operatorname{spec} A_p, \mathcal{I}^N)$ denotes the colimit over the system

$$H^m_{Zar}(\operatorname{spec} A_p, \mathcal{I}^N) \xrightarrow{2} H^m_{Zar}(\operatorname{spec} A_p, \mathcal{I}^N) \xrightarrow{2} H^m_{Zar}(\operatorname{spec} A_p, \mathcal{I}^N) \xrightarrow{2} \cdots$$

Since $\operatorname{spec} A_p$ is noetherian, $\varinjlim H^m_{Zar}(\operatorname{spec} A_p, \mathcal{I}^N) = H^m_{Zar}(\operatorname{spec} A_p, \varinjlim \mathcal{I}^N)$ [Tam94, Theorem 3.11.1]. For any open U in $\operatorname{spec} A_p$, $\mathcal{W}(U)/\mathcal{I}^N(U) \xrightarrow{2^N} \mathcal{W}(U)/\mathcal{I}^N(U)$ is zero. It follows from this that the exact sequence of sheaves

$$0 \to \mathcal{I}^N \to \mathcal{W} \to \mathcal{W}/\mathcal{I}^N \to 0$$

degenerates after taking direct limits (over multiplication by 2) to an isomorphism of sheaves

$$\varinjlim \mathcal{I}^N \xrightarrow{\simeq} \varinjlim \mathcal{W}$$

and thus $H^m_{Zar}(\operatorname{spec} A_p, \varinjlim \mathcal{I}^N) \simeq H^m_{Zar}(\operatorname{spec} A_p, \varinjlim \mathcal{W})$. Then, apply Lemma 1.5 to obtain the desired vanishing statement. □

Lemma 2.7. *If A is unramified (Definition 2.2), then*

$$H^m_{Zar}(\operatorname{spec} A_p, \mathcal{W}) = 0$$

for $m \geq 2$.

Proof. Let A be an unramified regular local ring. Since A/p is regular and contains $\mathbb{Z}/p$, the inclusion $\mathbb{Z}/p \to A/p$ is geometrically regular. It follows that the inclusion $\mathbb{Z}_{\langle p \rangle} \to A$ is flat with geometrically regular fibers, that is to say, it is a regular morphism. From Popescu's Theorem [Pop86, Theorem 1.8], one may obtain that A is a filtered colimit of local rings essentially smooth over $\mathbb{Z}_{\langle p \rangle}$. As Zariski cohomology commutes with such colimits, it suffices to prove the vanishing for the essentially smooth case. Therefore, for the remainder of the proof, A will denote a local ring essentially smooth over $\mathbb{Z}_{\langle p \rangle}$, $p \neq 2$. Consider the long exact sequence in cohomology

$$\cdots \to H^m_{Zar}(\operatorname{spec} A_p, \mathcal{I}^{n+1}) \to H^m_{Zar}(\operatorname{spec} A_p, \mathcal{I}^n) \to H^m_{Zar}(\operatorname{spec} A_p, \mathcal{H}^n) \to \cdots$$

associated to the short exact sequence of sheaves of Lemma 2.4. Using Lemma 2.5 and the exact sequence above, for any integer $n \geq 0$, we obtain isomorphisms in cohomology

$$H^m_{Zar}(\operatorname{spec} A_p, \mathcal{I}^{n+1}) \xrightarrow{\simeq} H^m_{Zar}(\operatorname{spec} A_p, \mathcal{I}^n)$$

for all $m \geq 2$ and $n \geq 0$. Using Lemma 2.6 we obtain the desired vanishing statement. □

Theorem 2.8. *Let A be a regular local ring with $2 \in A^\times$. If A is unramified, then A satisfies the Gersten conjecture for Witt groups.*

Proof. Let A be a mixed-characteristic unramified regular local ring with residue field of characteristic $p \neq 2$. As $p \in A$ is a regular parameter, one may obtain the long exact sequence in the cohomology of the Gersten complexes below [BGPW02, Lemma 3.3 and proof of Theorem 4.4].

$$\cdots \to H^*(C(A, W)) \to H^*(C(A_p, W)) \to H^*(C(A/pA, W)) \to \cdots$$

Since the Gersten conjecture is known for any local ring containing a field [BGPW02, Theorem 6.1], the groups $H^*(C(A/pA, W))$ vanish in positive degree and

$$H^*(C(A_p, W)) \simeq H^*_{Zar}(\operatorname{spec} A_p, \mathcal{W})$$

in all degrees. From Lemma 2.7 we have that $H^*_{Zar}(\operatorname{spec} A_p, \mathcal{W}) = 0$ for $m \geq 2$. Thus $H^*(C(A, W)) = 0$ in degrees two and higher, and from this it follows that the Gersten conjecture holds for A [BGPW02, Lemma 3.2 (2)]. □

Remark 2.9. Although we don't use it in this article, one can show that the Gersten conjecture for the Witt groups holds for any regular local ring A with $2 \in A^\times$ that is equipped with a regular morphism $\Lambda \to A$, where Λ is a regular local ring having Krull dimension $\dim A \leq 1$. Indeed, this is known when Λ has Krull dimension one, that is to say, when Λ is a discrete valuation ring[6] and it is known when Λ is a field [BGPW02, Theorem 6.1].

3. Purity for powers of the fundamental ideal

Proposition 3.1. *Let A be a local ring essentially smooth over $\mathbb{Z}_{\langle p \rangle}$, $p \neq 2$. Then the sequence*

$$I^n(A)/I^{n+1}(A) \xrightarrow{i^*} I^n(K)/I^{n+1}(K) \xrightarrow{\partial} \bigoplus_{x \in X^{(1)}} I^n(k(x))/I^{n+1}(k(x))$$

where i^* denotes the map induced by $i : \operatorname{spec} K \to \operatorname{spec} A$, is exact.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
 & K_n^M(A)/2 \arrow[r] \arrow[d] & K_n^M(K)/2 \arrow[r] \arrow[d] & \bigoplus_{x \in X^{(1)}} K_n^M(k(x))/2 \arrow[d] \\
0 \arrow[r] & H^n_{\acute{e}t}(A) \arrow[r] & H^n_{\acute{e}t}(K, \mathbb{Z}/2) \arrow[r] & \bigoplus_{x \in X^{(1)}} H^n_{\acute{e}t}(k(x), \mathbb{Z}/2)
\end{tikzcd}$$

The lower row is exact (Remark 2.1). Furthermore, for A essentially smooth over a discrete valuation ring, the Galois symbol $K_n^M(A)/2 \to H^n_{\acute{e}t}(A, \mathbb{Z}/2)$ is surjective[7], compare [Kah02, p.114, surjectivity of the Galois symbol]. It follows that the upper row is exact in the middle using the affirmation of the Milnor conjecture (as cited in the Introduction) to obtain that the middle vertical map is an isomorphism. Hence, in the commutative diagram

$$\begin{tikzcd}
K_n^M(A)/2 \arrow[r] \arrow[d] & K_n^M(K)/2 \arrow[r] \arrow[d] & \bigoplus_{x \in X^{(1)}} K_n^M(k(x))/2 \arrow[d] \\
I^n(A)/I^{n+1}(A) \arrow[r] & I^n(K)/I^{n+1}(K) \arrow[r] & \bigoplus_{x \in X^{(1)}} I^n(k(x))/I^{n+1}(k(x))
\end{tikzcd}$$

we have that the lower row is exact in the middle by again using the Milnor conjecture to obtain that the middle and rightmost vertical arrows are isomorphisms. $\square$

The following result was proved for local ring essentially smooth over a field by M. Kerz and S. Mueller-Stach [KMS07, Corollary 0.5]. The proof of the next theorem is obtained by following their proof and inputting the Gersten conjecture results for Witt groups proved in Theorem 2.8.

Theorem 3.2. *Let A be essentially smooth over $\mathbb{Z}_{\langle p \rangle}$, $p \neq 2$, and n a non-negative integer. Let $I^n(K)_{unr}$ denote the intersection of the kernel of the residue morphism ∂ and $I^n(K)$. Let $i^* : W(A) \to W(K)$ denote the map on Witt groups induced by*

[6] Thesis of the author, Theorem 4.28. One can use a transfer argument to remove the restriction that the residue field of the discrete valuation ring be infinite.

[7] In a correspondence with the author B. Kahn explained that the passage from surjectivity in the case of local rings essentially smooth over a field to this case is easy and goes back to Lichtenbaum, if you grant Gillet's Gersten conjecture for étale cohomology. Surjectivity in the case of local rings essentially smooth over a field is known [Ker09, Ker10].

the inclusion of A into the fraction field K of A. Then purity holds for the nth power of the fundamental ideal, that is, for $i^*(q) \in I^n(K)$ we have that $i^*(q) \in I^n(K)_{unr}$ if and only if $q \in I^n(A)$.

Proof. Let A be a local ring essentially smooth over $\mathbb{Z}_{\langle p\rangle}$, $p \neq 2$, and n a non-negative integer. The one direction, $q \in I^n(A)$ implies $i^*(q) \in I^n(K)_{unr}$ is clear. To prove the other direction, let $q \in W(A)$ such that $i^*(q) \in I^n_{unr}(K)$. Using Proposition 3.1 we may obtain $q_n \in I^n(A)$ satisfying $i_*(q - q_n) \in I^{n+1}(K)_{unr}$. Repeating this argument, one finds $i_*(q - q_n - q_{n+1} \cdots - q_{n+j}) \in I^{n+j+1}(K)_{unr}$ for any $j \geq 0$. Since K is of finite transcendence degree over its prime field, for some integer s sufficiently large, $I^{n+s+1}(K) = 2^n I^{s+1}(K)$ [EKM08, Lemma 41.1]. Using the fact that for any residue field of A one has a similar result (with an s less than or equal to the previous s used for K), it follows that $I^{n+s+1}(K)_{unr} = 2^n I^{s+1}(K)_{unr}$. Using this equality together with Theorem 2.8, we obtain that $i_*(q - q_n - q_{n+1} \cdots - q_{n+s}) = 2^n i_*(q')$, where $i_*(q') \in I^{s+1}(K)_{unr}$ and $q' \in W(A)$. Thus, $i_*(q) = i_*(2^n q' + q_n + q_{n+1} \cdots + q_{n+s})$, where $2^n q' + q_n + q_{n+1} \cdots + q_{n+s} \in I^n(A)$. As $W(A) \to W(K)$ is injective (Theorem 2.8), $q = 2^n q' + q_n + q_{n+1} \cdots + q_{n+s} \in I^n(A)$, completing the proof. □

Corollary 3.3. *Let A be an regular local ring with $2 \in A^\times$. If A is unramified, then $\mathrm{e_n} : I^n(A) \to H^n_{ét}(A, \mathbb{Z}/2)$ is well-defined and the sequence*

$$0 \to I^{n+1}(A) \to I^n(A) \xrightarrow{\mathrm{e_n}} H^n_{ét}(A, \mathbb{Z}/2) \to 0$$

is exact.

Proof. Let A be a local ring essentially smooth over $\mathbb{Z}_{\langle p\rangle}$. The sequence of "unramified" groups, that is, kernels of the corresponding residue homomorphisms

$$0 \to I^{n+1}_{unr}(K) \to I^n_{unr}(K) \to H^n_{unr}(K, \mathbb{Z}/2) \tag{3.1}$$

is exact as there is an exact sequence of Gersten complexes $0 \to C(A, I^{n+1}) \to C(A, I^n) \to C(A, H^n) \to 0$ and the residue is the first differential in the Gersten complex. Using Proposition 3.1 we have that $I^n(A)/I^{n+1}(A)$ surjects onto $(I^n(K)/I^{n+1}(K))_{unr}$. Using the affirmation of the Milnor conjecture (as cited in the Introduction), one has that $(I^n(K)/I^{n+1}(K))_{unr} \simeq H^n_{unr}(K, \mathbb{Z}/2)$. As purity holds for the étale cohomology of A (Remark 2.1) it follows that $I^n(A) \xrightarrow{\mathrm{e_n}} H^n_{ét}(A, \mathbb{Z}/2)$ is well-defined and surjective. So the sequence below is exact on the right

$$0 \to I^{n+1}(A) \to I^n(A) \xrightarrow{\mathrm{e_n}} H^n_{ét}(A, \mathbb{Z}/2) \to 0 \tag{3.2}$$

Using purity (Theorem 3.2) together with the exact sequence (3.1) we obtain that it is exact everywhere. As both $I^n(A)$ and $H^n_{ét}(A, \mathbb{Z}/2)$ commute with filtered colimits of rings, and filtered colimits are exact, we have that the sequence (3.2) is exact when A is any filtered colimit of local rings essentially smooth over $\mathbb{Z}_{\langle p\rangle}$. Using Popescu's Theorem [Pop86, Theorem 1.8], one has that this is the case when A is an unramified regular local ring with $2 \in A^\times$. □

4. On the nth cohomological invariant

Definition 4.1. Recall that a *henselian pair* is a pair (B, I) where B is a local ring and I is an ideal in B satisfying the following: for any monic $f \in B[T]$ and any factorization $\overline{f} = g_0 h_0$ in $B/I[T]$ where g_0 and h_0 are both monic and both generate the unit ideal in $B/I[T]$, there exists a factorization $f = gh$ where g and h are monic polynomials in $B[T]$ and $\overline{g} = g_0$, $\overline{h} = h_0$.

Lemma 4.2. *If B is a local ring and (B, I) a henselian pair (Definition 4.1) such that 2 is invertible in B, then for all integers $n \geq 0$, the homomorphisms of groups*

$$I^n(B) \to I^n(B/I)$$

and

$$I^n(B)/I^{n+1}(B) \to I^n(B/I)/I^{n+1}(B/I)$$

induced by the surjection $B \to B/I$ are bijections.

Proof. Let B a local ring and (B, I) a henselian pair such that 2 is invertible in both B and B/I. Considering the diagram

$$\begin{tikzcd}
0 \arrow[r] & I^{n+1}(B) \arrow[r] \arrow[d] & I^n(B) \arrow[r] \arrow[d] & I^n(B)/I^{n+1}(B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^{n+1}(B/I) \arrow[r] & I^n(B/I) \arrow[r] & I^n(B/I)/I^{n+1}(B/I) \arrow[r] & 0
\end{tikzcd}$$

we see that, by the two out of three lemma, it is sufficient to prove that $I^n(B) \to I^n(B/I)$ is a bijection for all $n \geq 0$. To prove injectivity for all $n \geq 0$, note that as $I^n(B)$ is contained in $W(B)$, it suffices to prove that $W(B) \to W(B/I)$ is injective.

We claim that the assignment $b+I \mapsto b$ determines a well-defined map $(B/I)^\times/(B/I)^{\times 2} \to B^\times/B^{\times 2}$. This claim follows from rigidity for étale cohomology due to Strano and Gabber independently but one may also prove it directly from the definition of Henselian pair[8]: let $b_1, b_2 \in B^\times$ such that $b_1 + I = b_2 + I$; the polynomial $T^2 - \frac{b_1}{b_2}$ has image $T^2 - 1$ in $B/I[T]$; as (B, I) is a henselian pair, from the factorization $T^2 - 1 = (T-1)(T+1)$ in $B/I[T]$ we obtain a factorization $T^2 - \frac{b_1}{b_2} = (T-a)(T+a)$ in $B[T]$, hence $b_1 = a^2 b_2$ for some $a \in B^\times$, that is, $b_1 = b_2$ in $B^\times/(B^\times)^2$. The claim follows. Next recall that for any semilocal ring A, the Witt group $W(A)$ is a quotient of the group ring $\mathbb{Z}[A^\times/A^{\times 2}]$ modulo the set of relations R additively generated by $[1] - [-1]$ and all elements

$$\sum_{i=1}^{h} [a_i] - \sum_{i=1}^{h} [b_i]$$

satisfying

$$\perp_{i=1}^{h} \langle a_i \rangle \simeq \perp_{i=1}^{h} \langle b_i \rangle$$

with $h = 4$ [Kne77, §4, Theorem 2]. Hence, in the commutative diagram below

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] & \mathbb{Z}[B^\times/B^{\times 2}] \arrow[r] & W(B) \arrow[r] & 0 \\
0 \arrow[r] & R \arrow[r] \arrow[u] & \mathbb{Z}[(B/I)^\times/(B/I)^{\times 2}] \arrow[r] \arrow[u] & W(B/I) \arrow[r] & 0
\end{tikzcd}$$

the rows are exact. Thus we obtain a well-defined map of cokernels $W(B/I) \to W(B)$ such that the composition $W(B) \to W(B/I) \to W(B)$ is the identity. This proves the desired injectivity. The composition $W(B/I) \to W(B) \to W(B/I)$

[8]The author learned this from a recent preprint of Stefan Gille titled *On quadratic forms over semilocal rings.*

is the identity hence $W(B) \to W(B/I)$ is surjective. To prove surjectivity of $I^n(B) \to I^n(B/I)$ for all $n \geq 0$, recall that $I^n(B/I)$ is additively generated by Pfister forms $\langle\langle \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n \rangle\rangle$ where $\bar{b}_1, \bar{b}_2, \ldots \bar{b}_n$ are units in B/I [Bae78, Ch. V, Section 1, Remark 1.3]. For any Pfister form $\langle\langle \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n \rangle\rangle$ we may lift the $\bar{b}_i$ to units b_i of B to obtain an element $\langle\langle b_1, b_2, \ldots, b_n \rangle\rangle \in I^n(B)$ mapping to it, proving surjectivity of $I^n(B) \to I^n(B/I)$ and finishing the proof of the lemma. □

4.3 (Construction of a Henselian pair)**.** Let $R = \mathbb{Z}[T_1, T_2, \cdots, T_n]/I$ for some ideal I and integer $n \geq 0$. Let $A := R_{\mathrm{p}}$ denote the localization of R with respect to a prime ideal $\mathrm{p} \in \operatorname{spec} R$. Then, one may obtain a *henselian pair* (B, I) *for* A as follows: let s denote the quotient map $\mathbb{Z}[T_1, T_2, \cdots, T_n] \to R$ and let $B_0 := \mathbb{Z}[T_1, T_2, \cdots, T_n]_{s^{-1}(\mathrm{p})}$ and similarly $I_0 := I_{s^{-1}(\mathrm{p})}$; let B denote the henselization of B_0 along I_0 and $I := I_0 B$. Recall, the henselization along I_0 is obtained by taking the colimit over the directed category consisting of those étale B_0-algebras C having the property that $B_0/I_0 \to C/I_0C$ is an isomorphism. The map that $B_0 \to B$ induces on quotients $A = B_0/I_0 \to B/I$ is an isomorphism of local rings.

Lemma 4.4. *Let $R = \mathbb{Z}[T_1, T_2, \cdots, T_n]/I$ for some ideal I, some integer $n \geq 0$. Let $A := R_{\mathrm{p}}$ denote the localization of R with respect to a prime ideal $\mathrm{p} \in \operatorname{spec} R$. Let (B, I) denote the henselian pair for A constructed in (4.3). Then B is a regular local ring satisfying exactly one of the following two conditions:*

(1) *B contains $\mathbb{Q}$;*
(2) *B contains $\mathbb{Z}_{\langle p \rangle}$ for some integer $p \in \mathbb{Z}$ and B/p is regular.*

Proof. Pulling back the maximal ideal of the local ring B over $\mathbb{Z} \to B$ induces a map of local rings $\mathbb{Z}_{\langle p \rangle} \to B$, where $\langle p \rangle \in \operatorname{spec} \mathbb{Z}$. We claim that $\mathbb{Z}_{\langle p \rangle} \to B$ is a *regular morphism*, that is to say, a flat morphism having geometrically regular fibers. It follows from this claim that: B is regular, since $\mathbb{Z}_{\langle p \rangle}$ is regular and the morphism $\mathbb{Z}_{\langle p \rangle} \to B$ is regular; $\mathbb{Z}_{\langle p \rangle} \to B$ is injective, since it is a flat morphism; when $p = 0$ we find that B contains $\mathbb{Q}$ and when $p \neq 0$ we find that B contains $\mathbb{Z}_{\langle p \rangle}$; if $p \neq 0$, then B/p is regular as it is the closed fiber $\mathbb{Z}_{\langle p \rangle}/p \to B/p$ of a regular morphism. To finish the proof of the lemma we prove the claim. First, note that $\mathbb{Z}_{\langle p \rangle} \to B_0$ is a regular morphism. As B is a filtered colimit of étale B_0-algebras, it follows that $B_0 \to B$ is also a regular morphism. Finally, recall that the composition of regular morphisms is again regular. □

Theorem 4.5. *Let A be any local ring with 2 invertible and let $\overline{I^n}(A) := I^n(A)/I^{n+1}(A)$.*

(1) *The assignment*
$$\langle\langle a_1, a_2, \cdots, a_n \rangle\rangle \mapsto (a_1) \cup (a_2) \cup \cdots \cup (a_n)$$
determines a well-defined group homomorphism
$$\mathrm{e_n} : I^n(A) \to H^n_{\acute{e}t}(A, \mathbb{Z}/2)$$
(2) *The map $\mathrm{e_n}$ is surjective with kernel equal to $I^{n+1}(A)$, that is to say, the induced map*
$$\bar{\mathrm{e}}_{\mathrm{n}} : \overline{I^n}(A) \to H^n_{\acute{e}t}(A, \mathbb{Z}/2)$$
is a bijection.

Proof. Any local ring A may be written as a union of its finitely generated subrings A_α. Localizing the A_α with respect to the pullback p_α over $A_\alpha \to A$ of the maximal ideal of A, one obtains local rings $(A_\alpha)_{\mathrm{p}_\alpha}$, and A is the filtered colimit of the $(A_\alpha)_{\mathrm{p}_\alpha}$.

As both groups $I^n(A)$ and $H^n_{\acute{e}t}(A, \mathbb{Z}/2)$ respect filtered colimits, and filtered colimits are exact, to prove (1) and (2) it is sufficient to prove that there is a sequence

$$0 \to I^{n+1}(A_\alpha) \to I^n(A_\alpha) \xrightarrow{\mathrm{e_n}} H^n_{\acute{e}t}(A_\alpha, \mathbb{Z}/2) \to 0$$

which is exact for any α. To demonstrate this, now let $A = R_p$ be the localization of $R = \mathbb{Z}[T_1, T_2, \cdots, T_n]/I$ for some ideal I and some integer $n \geq 0$. We will show that $\mathrm{e_n} : I^n(A) \to H^n_{\acute{e}t}(A, \mathbb{Z}/2)$ is well-defined and surjective with kernel $I^{n+1}(A)$. Let (B, I) be a henselian pair for A as constructed in (4.3). By Lemma 4.4 we have that B is a regular local ring which either contains $\mathbb{Q}$, or, contains $\mathbb{Z}_{\langle p \rangle}$ and is unramified. When B contains $\mathbb{Q}$ (resp. is unramified), it follows from [Ker10, Proposition 16] (resp. Corollary 3.3) that the map $\mathrm{e_n}$ in the commutative diagram below is well-defined and that the upper horizontal sequence is exact

$$\begin{tikzcd}
0 \arrow[r] & I^{n+1}(B) \arrow[r] \arrow[d, "\simeq"] & I^n(B) \arrow[r, "\mathrm{e_n}"] \arrow[d, "\simeq"] & H^n_{\acute{e}t}(B, \mathbb{Z}/2) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & I^{n+1}(A) \arrow[r] & I^n(A) & H^n_{\acute{e}t}(A, \mathbb{Z}/2) \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are induced by the surjection $B \to B/I \simeq A$. The vertical maps are isomorphisms by a theorem of O. Gabber [Gab94a] and independently, R. Strano [Str84] for étale cohomology, and by Lemma 4.2 for the powers of the fundamental ideal. It follows that $\mathrm{e_n} : I^n(A) \to H^n_{\acute{e}t}(A, \mathbb{Z}/2)$ is well-defined and surjective with kernel $I^{n+1}(A)$, finishing the proof of the theorem. $\square$

Remark 4.6. Let X be a scheme with 2 invertible in its global sections and let $I^n(X)$ denote the nth power of the fundamental ideal $I(X)$, where $I(X)$ denotes the kernel of the rank homomorphism $W(X) \to H^0_{\acute{e}t}(X, \mathbb{Z}/2)$ [Kne77, Chapter 1, §7]. Let $\mathcal{I}^n$ denote the Zariski sheafification of the presheaf $U \mapsto I^n(U)$, and let $\mathcal{H}^n$ denote the Zariski sheafification of the presheaf $U \mapsto H^n_{\acute{e}t}(U, \mathbb{Z}/2)$. Then, one may ask a similar question, does $\mathrm{e_n}$ globalize to a morphism of sheaves

$$e_n : \mathcal{I}^n \to \mathcal{H}^n$$

and furthermore, does this morphism induce an isomorphism

$$\bar{e}_n : \overline{\mathcal{I}^n} \to \mathcal{H}^n$$

of sheaves? Here $\overline{\mathcal{I}^n}$ denotes the sheafification of the presheaf

$$U \mapsto \mathcal{I}^n(U)/\mathcal{I}^{n+1}(U)$$

To see that the answer is yes, it is sufficient, by Theorem 4.5, to define a map of sheaves $\mathcal{I}^n \to \mathcal{H}^n$. We don't use this result in this paper, so we will only sketch an argument by which such a map may be obtained. Follow the argument of [Kah02, proof of Proposition 2.5] mutatis mutandis, which is to say, one has such a map whenever the Gersten conjecture is known for the local rings of X, then one obtains a map for any X with two invertible by reducing to X of finite type over $\mathbb{Z}$ and henselizing along $X \hookrightarrow \mathbb{A}^n_{\mathbb{Z}}$ and proceeding as in the local case.

In particular, for any scheme with 2 invertible in its global sections, one may obtain an isomorphism in cohomology

$$H^m_{Zar}(X, \overline{\mathcal{I}^n}) \xrightarrow{\simeq} H^m_{Zar}(X, \mathcal{H}^n)$$

for all $m \geq 0$ and all $n \geq 0$.

Furthermore, if one sheafifies with respect to the étale topology to obtain a sheaf $\mathcal{W}_{\acute{e}t}$, then the rank homomorphism induces an isomorphism of étale sheaves $\mathcal{W}_{\acute{e}t} \xrightarrow{\simeq} (\mathbb{Z}/2)_{\acute{e}t}$: on stalks one finds the bijection $\mathrm{e}_0 : W(A) \to H^0_{\acute{e}t}(A, \mathbb{Z}/2) \simeq \mathbb{Z}/2$ since A is strictly henselian (see Introduction). In particular, one obtains the isomorphism in (étale) cohomology

$$H^m_{\acute{e}t}(X, \mathcal{W}_{\acute{e}t}) \xrightarrow{\simeq} H^m_{\acute{e}t}(X, \mathbb{Z}/2)$$

for all $m \geq 0$.

For a local ring with $2 \in A^\times$, Corollary C. below generalizes to any integer s a result of R. Baeza [Bae79, §2, Theorem] ($s = 1$) and J. Yucas [Yuc86, Theorem 2.10] ($s = 2$), while D. generalizes a result due to K. Mandelberg [Man75, Theorem 1.1] ($s = 2$).

Corollary 4.7. *Let A be a local ring with $2 \in A^\times$, let E and F be two non-degenerate quadratic forms over A, and let $s \geq 0$ be an integer. Consider the following assertions:*

I. *E is isometric to F, $E \simeq F$;*
II. *E and F have the same dimension and the same cohomological invariants $\mathrm{e_n}(E) = \mathrm{e_n}(F)$ for all $n \geq 0$;*
III. *E and F have the same dimension, total signature $\mathrm{sign}\,(E) = \mathrm{sign}\,(F)$, and the same cohomological invariants up to s, $\mathrm{e_n}(E) = \mathrm{e_n}(F)$ for $0 \leq n \leq s$;*
IV. *E and F have the same dimension and the same cohomological invariants up to s, $\mathrm{e_n}(E) = \mathrm{e_n}(F)$ for $0 \leq n \leq s$;*
V. *E and F have the same total signature $\mathrm{sign}\,(E) = \mathrm{sign}\,(F)$.*

Then, the following implications hold:

A. $\mathrm{II} \Rightarrow \mathrm{V}$*;*
B. *If $\cap I^n(A) = 0$, then* $\mathrm{I} \iff \mathrm{II}$*;*
C. *If $I^{s+1}(A)$ is torsion free, then* $\mathrm{I} \iff \mathrm{II} \iff \mathrm{III}$*;*
D. *If $I^{s+1}(A) = 0$, then* $\mathrm{I} \iff \mathrm{II} \iff \mathrm{III} \iff \mathrm{IV}$*.*

Proof. To prove A. use Theorem 4.5 to obtain that if $\mathrm{e_n}(E) = \mathrm{e_n}(F)$ for all $n \geq 0$, then $[E] - [F] \in \cap I^n(A)$. Note that $\cap I^n(A)$ is in the kernel of the signature morphism $\mathrm{sign} : W(A) \to \mathrm{C}(\mathrm{sper}\, A, \mathbb{Z})$ because sign maps $I^n(A)$ into $\mathrm{C}(\mathrm{sper}\, A, 2^n\mathbb{Z})$, hence $\cap I^n(A)$ maps into $\mathrm{C}(\mathrm{sper}\, A, \cap 2^n\mathbb{Z}) = 0$. To prove B. only implication $\mathrm{II} \Rightarrow \mathrm{I}$ needs proof. Use Theorem 4.5 to obtain that II implies $[E] - [F] \in \cap I^n(A)$, so $\cap I^n(A) = 0$ implies that $[E] - [F]$ is trivial in the Witt ring $W(A)$. As E and F have the same dimension and same class in the Witt group $[E] = [F]$, it follows using cancellation [Bae79, Ch. III, Corollary 4.3] that E and F are isometric. To prove C. we will only demonstrate that $\mathrm{III} \Rightarrow \mathrm{I}$ as $\mathrm{I} \iff \mathrm{II}$ using B. and clearly $\mathrm{I} \Rightarrow \mathrm{III}$. Using Theorem 4.5 we find that III implies that $[E] - [F] \in I^{s+1}(A)$ and that $[E] - [F]$ is in the kernel of the signature. Since the kernel of the signature is torsion, $I^{s+1}(A)$ torsion free implies $[E] - [F]$ is trivial in the Witt ring. As argued previously, it follows that E and F are isometric. To prove D. use that IV together with $I^{s+1}(A) = 0$ imply $[E] - [F]$ is trivial in the Witt ring, hence $\mathrm{IV} \Rightarrow \mathrm{I}$. □

4.8. Let X be a scheme with 2 invertible. The *étale cohomological 2-dimension* $\mathrm{cd}_2\,(X)$ is defined to be the largest integer n for which there is a 2-primary torsion sheaf F on X with $H^n_{\acute{e}t}(X, F) \neq 0$. If no such n exists we write $\mathrm{cd}_2\,(X) = \infty$. The *virtual étale cohomological two-dimension* $\mathrm{vcd}_2\,(X)$ is defined to be the étale cohomological two-dimension $\mathrm{cd}_2\,(X[i])$ of $X[i]$, where $i = \sqrt{-1}$. Recall that when

2 is invertible on X that $\mathrm{cd}_2(X) < \infty$ if and only if the real spectrum of X is empty (for instance if -1 is a sum of squares in all residue fields of X) and $\mathrm{vcd}_2(X) < \infty$ [Sch94, Remark 7.5 together with Proposition 7.2 c)]. If X is a finite type $\mathbb{Z}$-scheme of Krull dimension d, then $\mathrm{vcd}_2(X) \leq 2d+1$ [SGA73, X.6.2]. It follows that any local ring A essentially of finite type over $\operatorname{spec}\mathbb{Z}[1/2]$ has $\mathrm{vcd}_2(A) < \infty$.

Next we apply Theorem 4.5 to demonstrate that the virtual etale cohomological two-dimension of A bounds the n for which $I^n(A)$ is torsion free.

Corollary 4.9. *Let A be a local ring with $2 \in A^\times$. If $\cap I^n(A) = 0$ and $\mathrm{vcd}_2(A) = s$, then $I^{s+1}(A)$ is torsion free and $I^n(A) \xrightarrow{2} I^{n+1}(A)$ is a bijection whenever $n > s$.*

Proof. Apply Theorem 4.5 in order to obtain the exact rows in the commutative diagram below.

$$\begin{tikzcd}
0 \arrow[r] & I^{n+1}(A) \arrow[r] \arrow[d, "\otimes\langle\langle -1\rangle\rangle"] & I^n(A) \arrow[r, "\mathrm{e_n}"] \arrow[d, "\otimes\langle\langle -1\rangle\rangle"] & H^n_{\acute{e}t}(A, \mathbb{Z}/2) \arrow[r] \arrow[d, "\cup(-1)"] & 0 \\
0 \arrow[r] & I^{n+2}(A) \arrow[r] & I^{n+1}(A) \arrow[r, "\mathrm{e_n}"] & H^n_{\acute{e}t}(A, \mathbb{Z}/2) \arrow[r] & 0
\end{tikzcd}$$

Note that in the diagram above we wrote $\langle\langle -1\rangle\rangle$, but this is by definition $\langle 1, 1\rangle = 2$. As $\mathrm{vcd}_2(A) = s$, we may apply [Sch94, Corollary 7.20, confer with diagram of paragraph (7.20.2)] to obtain that the vertical map on the right is a bijection whenever $n > s$. It follows by a diagram chase that the kernel of $I^n(A) \overset{\otimes\langle\langle -1\rangle\rangle}{\to} I^{n+1}(A)$ is contained in $\cap_{n>s} I^n(A)$, and the latter is trivial by hypothesis. Therefore the map $I^n(A) \xrightarrow{2} I^{n+1}(A)$ is a bijection for all $n > s$. Since the only torsion in $W(A)$ is two-primary, we find that $I^{s+1}(A)$ is torsion free. □

When the local ring is regular, we obtain the following stronger corollary.

Corollary 4.10. *Let A be a regular local ring with $2 \in A^\times$ and let $K = \mathrm{Frac}(A)$. If $\cap I^n(A) = 0$ and $\mathrm{vcd}_2(A) = s$, then $I^n(A) \to I^n(K)$ injective for all $n > s$.*

Proof. For any regular semi-local ring A the kernel of $W(A) \to W(K)$ is two-primary torsion [CT79, 2.3.4] and by Corollary 4.9 we have that $I^n(A)$ has no two-primary torsion whenever $n > s$, so $I^n(A) \to I^n(K)$ is injective for all $n > s$. □

Next we note that the etale cohomological two-dimension of A bounds the n for which $I^n(A) = 0$.

Corollary 4.11. *Let A be a local ring with $2 \in A^\times$. If $\cap I^n(A) = 0$ and $H^n_{\acute{e}t}(A, \mathbb{Z}/2) = 0$ for all $n > s$, then $I^{s+1}(A) = 0$.*

Proof. Apply Theorem 4.5 in order to obtain the short exact sequence

$$0 \to I^{n+1}(A) \to I^n(A) \to H^n_{\acute{e}t}(A, \mathbb{Z}/2) \to 0$$

and using that $H^n_{\acute{e}t}(A, \mathbb{Z}/2) = 0$ for all $n > s$, we obtain that $I^{n+1}(A) \to I^n(A)$ is an isomorphism for all $n > s$. Thus $I^{s+1}(A) = \cap_{n>s} I^n(A) = \cap_{n\geq 0} I^n(A)$. The latter intersection is trivial by hypothesis, proving $I^{s+1}(A) = 0$. □

Definition 4.12. Let A be a local ring. Recall [Bae79, Appendix A] that one defines the *level* of A to be the number

$$s(A) := \min\{r \mid -1 = a_1^2 + a_2^2 + \cdots + a_r^2,\ a_i \in A\}$$

if -1 is a sum of squares in A, and $s(A) = \infty$ otherwise. Similarly, define the *height* of A to be the number

$$h(A) := \min\{2^r \mid 2^r W(A)_{tors} = 0\}$$

if it exists, and $h(A) = \infty$ otherwise. When $2 \in A^\times$ and $s(A) < \infty$, recall that $h(A) = 2s(A)$ [Bae79, Appendix A, Remark A.21].

Using Theorem 4.5 we bound the level and height of a local ring A in terms of its étale cohomological dimension.

Corollary 4.13. *Let A be a local ring with $2 \in A^\times$ satisfying $\cap I^n(A) = 0$ and $\operatorname{vcd}_2 A = s$. Then:*

A. *$h(A) \leq 2^{s+1}$;*
B. *If $\operatorname{sper} A = \emptyset$, then $s(A) \leq 2^s$.*

Proof. For A. note that $2^{s+1}W(A)_{tors} \subset I^{s+1}$ and then use Corollary 4.9 to obtain that I^{s+1} is torsion free, hence $h(A) \leq 2^{s+1}$. For B., if $\operatorname{sper} A = \emptyset$, then $\operatorname{vcd}_2(A) = \operatorname{cd}_2(A)$ [Sch94, Corollary 7.21], so $I^{s+1} = 0$ (Corollary 4.11), and this implies that -1 is a sum of 2^{s+1} squares [Bae79, Lemma 10.11]. Thus $s(A) < \infty$, therefore $h(A) = 2s(A)$ [Bae79, Appendix A, Remark A.21], that is, $s(A) \leq 2^s$ as desired. □

5. On the vanishing of the intersection

5.1. Recall that when A is a field, it follows from the Arason-Pfister Haupsatz that $\cap I^n(A) = 0$ [AP71, Korollar 1]. We do not know if $\cap I^n(A) = 0$ for any local ring A with $2 \in A^\times$. In the next Lemma we list hypotheses on A under which $\cap I^n(A) = 0$.

Lemma 5.2. *Let A be a local ring with $2 \in A^\times$. Suppose that A satisfies at least one of the following hypotheses:*

I. *A is regular and the Gersten conjecture for the Witt groups holds for A;*
II. *A is regular and $1 + \mathrm{m} \subset A^2$, where m is the maximal ideal in A;*
III. *A is henselian;*
IV. *A is essentially of finite type over $\mathbb{Z}$, that is to say, $A = R_{\mathrm{p}}$ for some prime ideal $\mathrm{p} \in \operatorname{spec} R$, where $R = \mathbb{Z}[T_1, T_2, \cdots, T_n]/I$ for some ideal I.*

Then $\cap I^n(A) = 0$.

Proof. If A satisfies the Gersten conjecture, then in particular $W(A) \to W(K)$ is injective, hence one may use that $\cap I^n(K) = 0$ for any field K [AP71, Korollar 1] in order to conclude that $\cap I^n(A) \subset \cap I^n(K) = 0$. Similarly, if A is regular and $1 + \mathrm{m} \subset A^2$, then $W(A) \to W(K)$ is injective by a classic result of T. Craven, A. Rosenberg, and R. Ware [CRW75, Corollary 2.3]. If A is henselian, then one may use Lemma 4.2 to obtain that $I^n(A) \simeq I^n(A/\mathrm{m})$ for all $n \geq 0$. Then $\cap I^n(A) \simeq \cap I^n(A/\mathrm{m}) \simeq 0$, and $\cap I^n(A/\mathrm{m}) = 0$ [AP71, Korollar 1], hence $\cap I^n(A) = 0$. Now let A be a local ring essentially of finite type over $\mathbb{Z}$. We construct a henselian pair (B, I) for A as in (4.3). Applying Lemma 4.4 we have that B satisfies hypotheses under which the Gersten conjecture is known, hence $\cap I^n(B) = 0$. By Lemma 4.2 we have an isomorphism $I^n(B) \xrightarrow{\simeq} I^n(A)$ for all $n \geq 0$, where (B, I) is a henselian pair with $B/I \simeq A$, hence $0 = \cap I^n(B) \simeq \cap I^n(A)$. □

Remark 5.3. More generally, one may obtain that $\cap I^n(A) = 0$ whenever A is essentially of finite type over D, where D is a regular ring having Krull dimension $\dim D \leq 1$. As we do not use this in this paper, we will only sketch the argument.

Mutatis mutandis, one constructs a henselian pair (B, I) for A as in 4.3 and follows the proof of Lemma 4.4 in order to obtain that the B constructed will be equipped with a regular morphism $D_{\langle \mathrm{p} \rangle} \to B$, for some prime ideal $\mathrm{p} \in \operatorname{spec} B$. In view of Remark 2.9, the Gersten conjecture for the Witt groups of B is known in this case, hence we may apply Lemma 4.2 to obtain that $0 = \cap I^n(B) \simeq \cap I^n(A)$.

Finally, we conclude by writing down a proposition which summarizes what one obtains from the corollaries in an interesting case.

Proposition 5.4. *Let A be a local ring with $2 \in A^\times$ that is essentially of finite type over $\mathbb{Z}$, let s denote the least integer for which $H^n_{\acute{e}t}(A[i], \mathbb{Z}/2) = 0$ for all $n > s$ (such an integer always exists, see (4.8)). Then:*

A. $\cap I^n(A) = 0$;
B. *The dimension, total signature, and the cohomological invariants* $\mathrm{e}_0, \mathrm{e}_1, \cdots \mathrm{e}_\mathrm{s}$ *classify non-degenerate quadratic forms over A.*
C. $h(A) \leq 2^{s+1}$;
D. *If* $\operatorname{sper} A = \emptyset$, *then* $s(A) \leq 2^s$;
E. *If A is regular, then $I^n(A) \to I^n(K)$ is injective when $n > s$, where* $\operatorname{Frac}(K)$ *is the fraction field of A.*

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Emory University, 532 Kilgo Circle, Office 403, Atlanta, GA 30322, USA.
E-mail address: jeremy.a.jacobson@emory.edu