TP 2: jeudi 04/10/2018
Equation de transport à coefficients variables

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1 Preliminary materials

For a given real function $a \in C^1(\mathbb{R})$ such that $|a|$ and $|a'|$ are bounded on $\mathbb{R}$ by some $A > 0$, we consider the following linear transport equation in one dimension

$$\begin{cases} \partial_t u + a(x)\ \partial_x u = 0, & \forall (x,t) \in \mathbb{R} \times \mathbb{R}_*^+, \\ u(x,0) = u_0(x), & \forall x \in \mathbb{R}, \end{cases} \tag{1}$$

with $u_0 \in C_c^1(\mathbb{R}) = C_0^1(\mathbb{R})$ has a compact support. This equation is not in divergence form, it is called the **non conservative equation**.
Denote by $y(X,t)$ the solution of

$$\begin{cases} \partial_t y(X,t) = a(y(X,t)), & \forall (X,t) \in \mathbb{R} \times \mathbb{R}, \\ y(X,0) = X, & \forall X \in \mathbb{R}. \end{cases}$$

1. Show that $u(x,t) = u_0(X)$ with $x = y(X,t)$. What could mean the formula $u(x,t) = u_0(Y(x,t))$?

2. Next we consider
$$\begin{cases} \partial_t v + \partial_x (a(x) v) = 0, & \forall (x,t) \in \mathbb{R} \times \mathbb{R}_*^+, \\ v(x,0) = v_0(x), & \forall x \in \mathbb{R}, \end{cases} \tag{2}$$
This equation is in divergence form, it is called the **conservative equation**.
Show that $v(x,t) = J(X,t) v_0(X)$ with $J(X,t) = e^{-\int_0^t \partial_x a(y(X,s))ds} > 0$.

3. For $a(x) = x$, show that $v(x,t) = e^{-t} v_0(X)$.

2 Characteristics

The aim of this exercise is to get familiar with the characteristics for various velocity a.

1. Download `https://www.ljll.math.upmc.fr/despres/BD_fichiers/caracteristics.py` and run it. It is a Python file which plots the characteristics
$$x' = a(x) \qquad \text{with } a(x) = x.$$

2. Plot the characteristics in the cases, $a(x) = \pm x$, $a(x) = \sin(2\pi x)$.

3. Now we consider the case where the velocity/celerity is not Lipshitz continuous.
Plot the characteristics for $a(x) = \pm\sqrt{|x|}$.

3 Schemes

The aim is to define numerical scheme for solving (1) and (2). For $a \in \mathbb{R}$, we define $a^+ = \max(a, 0)$ and $a^- = \max(-a, 0)$.

1. Download and run the Python file `https://www.ljll.math.upmc.fr/despres/BD_fichiers/transport.py`. It solves (1) with the scheme

$$\Delta x \frac{u_j^{n+1} - u_j^n}{\Delta t} + a_j^- (u_j^n - u_{j+1}^n) - a_{j-1}^+ (u_{j-1}^n - u_j^n) \ = \ 0.$$

   It is implemented in the case $a(x) = x$ and for a non-regular initial condition (hat function).

2. Write in the file the analytical solution and compare it graphically with the discrete solution.

3. Implement in the same file

$$\Delta x \frac{u_j^{n+1} - u_j^n}{\Delta t} + (a_j^+ u_j^n - a_j^- u_{j+1}^n) - (a_{j-1}^+ u_{j-1}^n - a_{j-1}^- u_j^n) \ = \ 0$$

   to solve (2).

4. Write in the file the new analytical solution and compare it graphically with the discrete solution.

5. Try to observe some of the scheme properties as the L^∞ stability, the discrete mass conservation and the discrete maximum principal.

6. Do the same work in the case $a(x) = -x$ and for a better understanding you can plot the characteristics on an other window.

7. With numerical simulations, show the first order convergence of the schemes.