Homework B004

Prepare answer as a .pdf file plus a .py file
Return to `despres@ann.jussieu.fr`
Deadline is Friday the 16th of November 2018

1 Semi-group estimates

Let $\sigma \in \mathbb{R}$ be a given coefficient. Consider the linear wave system in dimension one

$$\begin{cases} \partial_t p + \partial_x v = \sigma(v - p) \\ \partial_t v + \partial_x p = \sigma(p - v), \end{cases}$$

where the unknown is $U(x,t) = (p(x,t), v(x,t))$ for $t \geq 0$. The initial data is written $U_0 = (p_0, u_0)$.

a) Determine an explicit representation of $U(x,t)$ in function of the initial data.

Write explicitly the operator A such that one has the representation $U(t) = e^{At}U_0$ for $t \geq 0$.

b) Let $Y_1 = L^1(\mathbb{R})^2$ with the norm $\|(a,b)\|_1 = \|a\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})}$.

Show the inequality $\left\|e^{tA}\right\|_{\mathcal{L}(Y_p)} \leq \left(1 + e^{-2\sigma t}\right)$ for $t \geq 0$.

c) Let $Y_2 = L^2(\mathbb{R})^2$ with the norm $\|(a,b)\|_2 = \sqrt{\|a\|^2_{L^2(\mathbb{R})} + \|b\|^2_{L^2(\mathbb{R})}}$.

Show the inequality $\left\|e^{tA}\right\|_{\mathcal{L}(Y_2)} \leq \max\left(1, e^{-2\sigma t}\right)$ which has a better constant.

d) Let $Y_\infty = L^\infty(\mathbb{R})^2$ with the norm $\|(a,b)\|_\infty = \max\left(\|a\|_{L^\infty(\mathbb{R})}, \|b\|_{L^\infty(\mathbb{R})}\right)$. Take $\sigma = 0$ and $t > 0$. Show first that $\left\|e^{tA}\right\|_{\mathcal{L}(Y_\infty)} \leq 2$.

By taking a particular initial condition, show that $\left\|e^{tA}\right\|_{\mathcal{L}(Y_\infty)} = 2$.

2 Numerical methods

Let u be the solution of

$$\begin{cases} \partial_t u - \partial_{xx} u = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

The initial data and the solution are 1-periodic, that is $u(t, x+1) = u(t,x)$ for all $\forall t \geq 0$ and $x \in \mathbb{R}$. A Finite Difference mesh is defined by $t_n = n\Delta t$ and $x_j = j\Delta x$ with $\frac{1}{\Delta x} \in \mathbb{N}^*$, and we consider the explicit Finite Difference discretization

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}-\frac{4}{3}\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\Delta x^2}+\frac{u_{j+2}^n-2u_j^n+u_{j-2}^n}{12\Delta x^2} \tag{1}$$
$$-\frac{\Delta t}{2}\frac{u_{j+2}^n-4u_{j+1}^n+6u_j^n-4u_{j-1}^n+u_{j-2}^n}{\Delta x^4}=0.$$

a) Determine the symbol of the scheme.

b) Show the scheme is a consistent at order 2 in time and 4 in space.

c) Show the scheme is stable in the quadratic norm under a CFL condition that will be determined, and prove convergence.

e) Prove that the 3-points scheme

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}-\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\Delta x^2}=0$$

is fourth order (in space) for $\frac{\Delta t}{\Delta x^2}=1/6$.

f) Implement the scheme (1) in Python in a bounded periodic interval.

g) Consider the initial condition

$$u_0^2(x)=\alpha\sin(2\pi x)+\beta\cos(2\pi x),\quad \alpha,\beta\in\mathbb{R}. \tag{2}$$

Implement this test case: on a same graphic window, plot the discrete and the exact solution (that you need to calculate analytically).

In L^2 norm, draw a convergence table and check the measured numerical order of convergence is asymptotically equal to the theoretical one.

g) What can you say about the maximum principle for the scheme (1)?