A Quantitative Notion of Persistency of Excitation and the Robust Fundamental Lemma

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Collaborators

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John Lygeros

Florian Dörfler

Today's menu

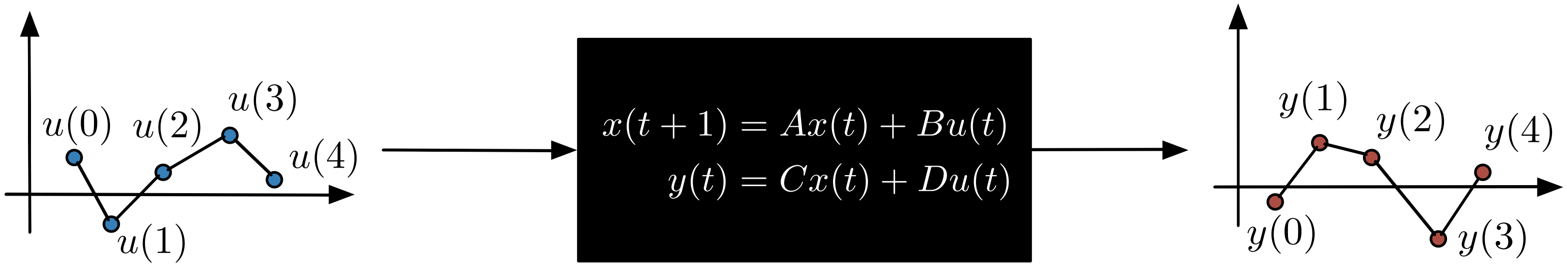
Fundamental lemma

Motivating example

Robust fundamental lemma

Data-driven control case study

Fundamental Lemma [Willems et al. '05]



Input: $u(t) \in \mathbb{R}^m$

Output: $y(t) \in \mathbb{R}^p$

Given data $(u_{[0,T-1]}, y_{[0,T-1]})$ and parameters n, ℓ, L:

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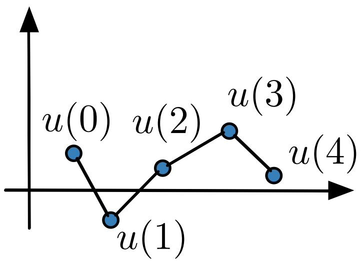
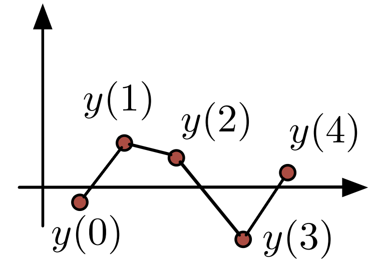


$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu(t)\\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$



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$$
\text{im} \begin{bmatrix}
u(0) & u(1) & \cdots & u(T-L) \\
\vdots & \vdots & \ddots & \vdots \\
u(L-1) & u(L) & \cdots & u(T-1) \\
y(0) & y(1) & \cdots & y(T-L) \\
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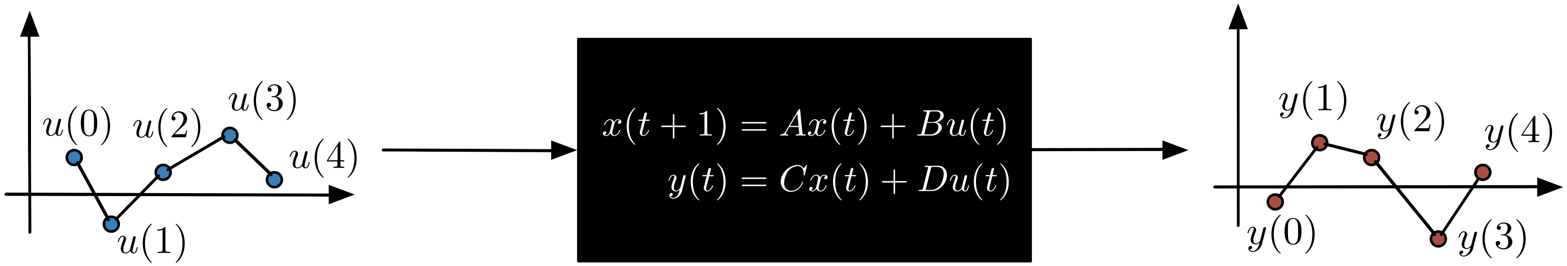

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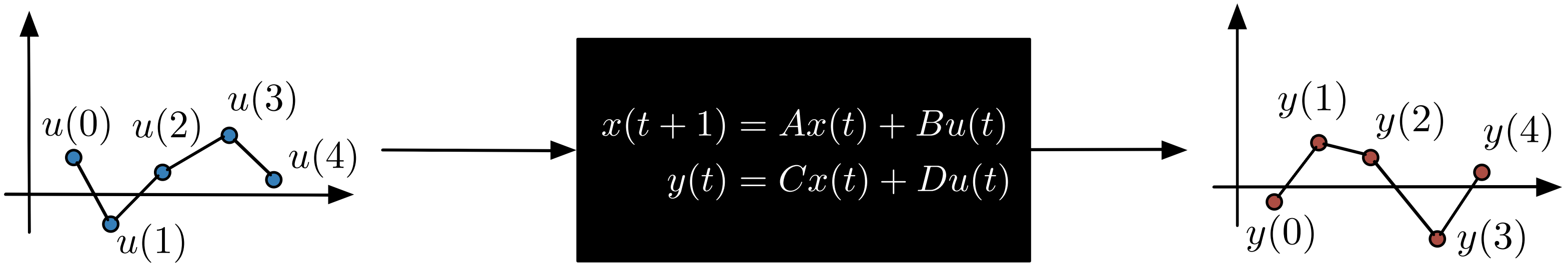

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if and only if trajectory matrix has rank $mL + n$ for $L \geq \ell$.

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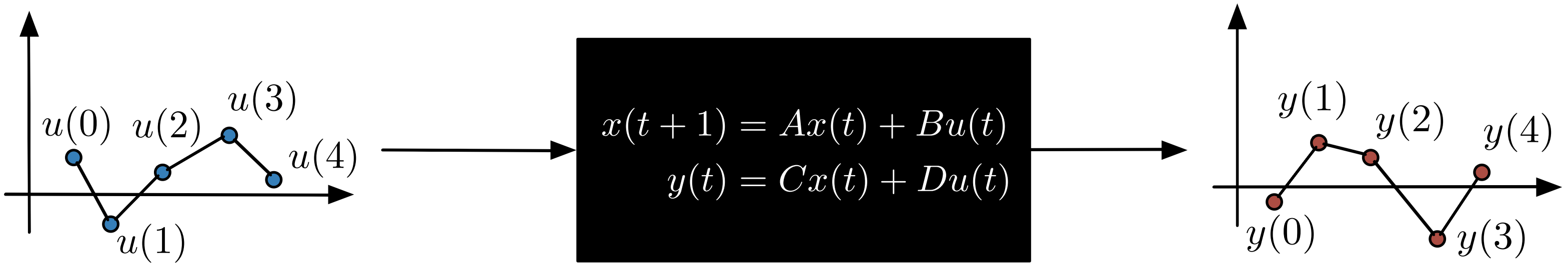

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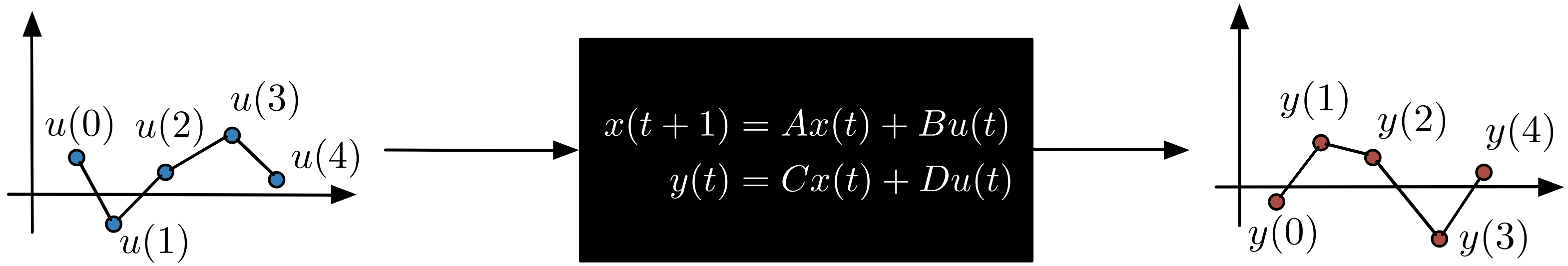
How can we ensure rank conditions?

Main ingredients for rank conditions



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Two main ingredients: (i) persistency of excitation, and (ii) controllability

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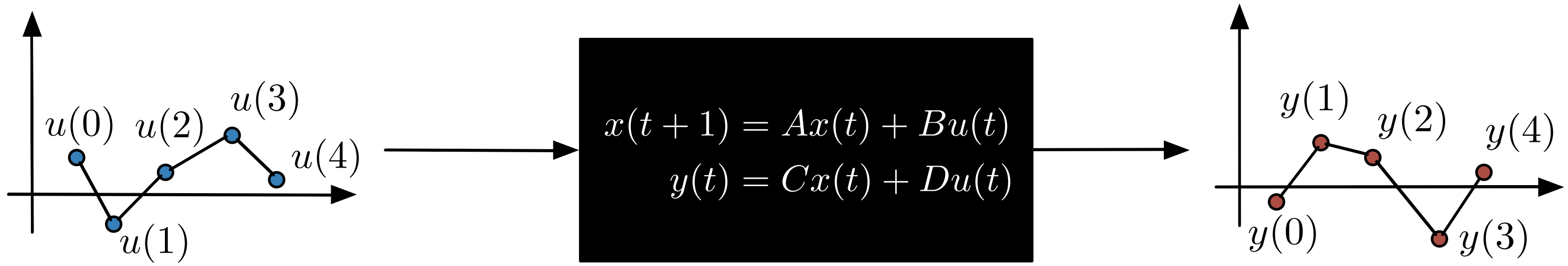

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Persistency of excitation

A sequence $u_{[0,T-1]} = (u(0), \ldots, u(T-1))$ is persistently exciting of order $k \in \mathbb{Z}_{>0}$ if

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has full row rank.

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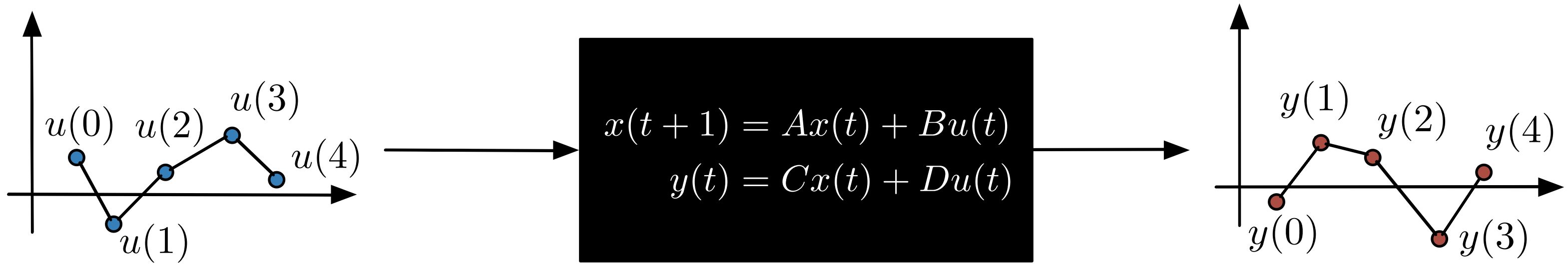

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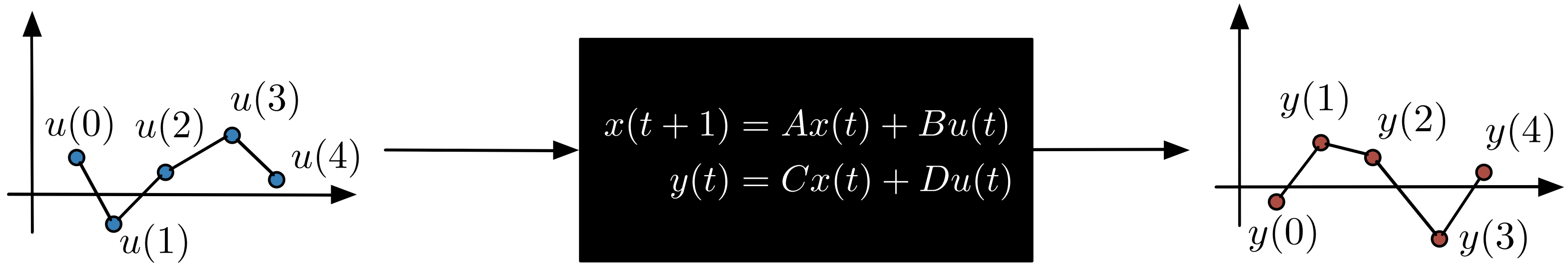
Sequence is sufficiently long and sufficiently rich

Fundamental Lemma [Willems et al. ‘05]



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Let $(u_{[0,T-1]}, y_{[0,T-1]})$ be a trajectory such that $u_{[0,T-1]}$ is persistently exciting of order $L + n$. Then

$$\text{rank} \begin{bmatrix} \mathcal{H}_L(u_{[0,T-1]}) \\ \mathcal{H}_L(y_{[0,T-1]}) \end{bmatrix} = mL + n.$$

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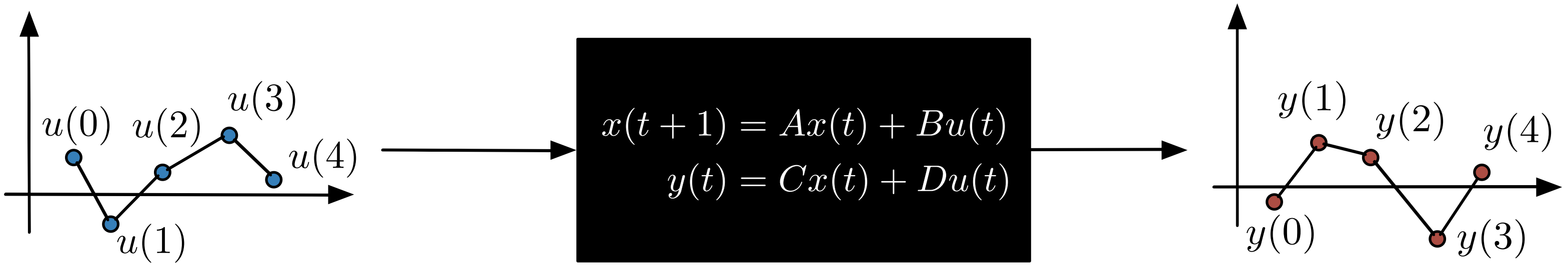

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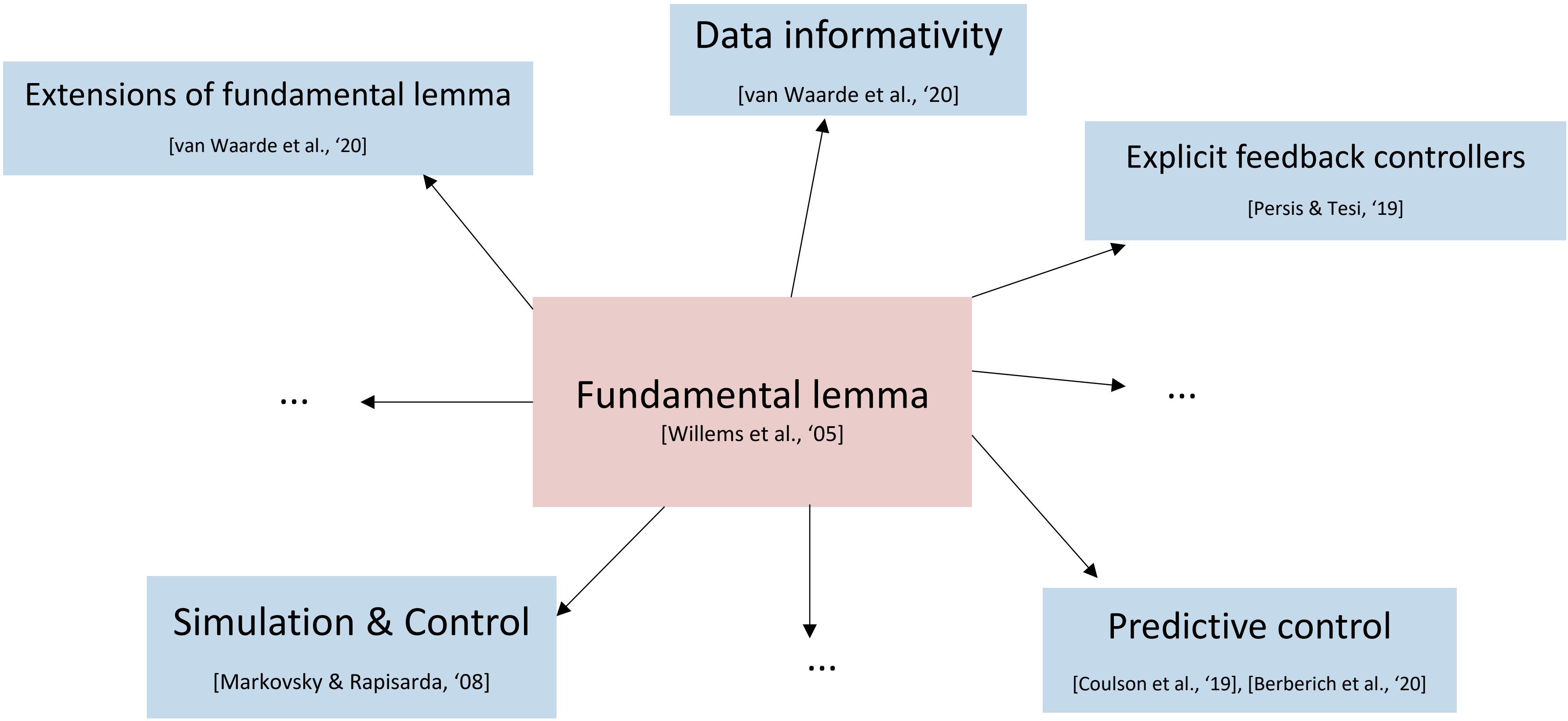
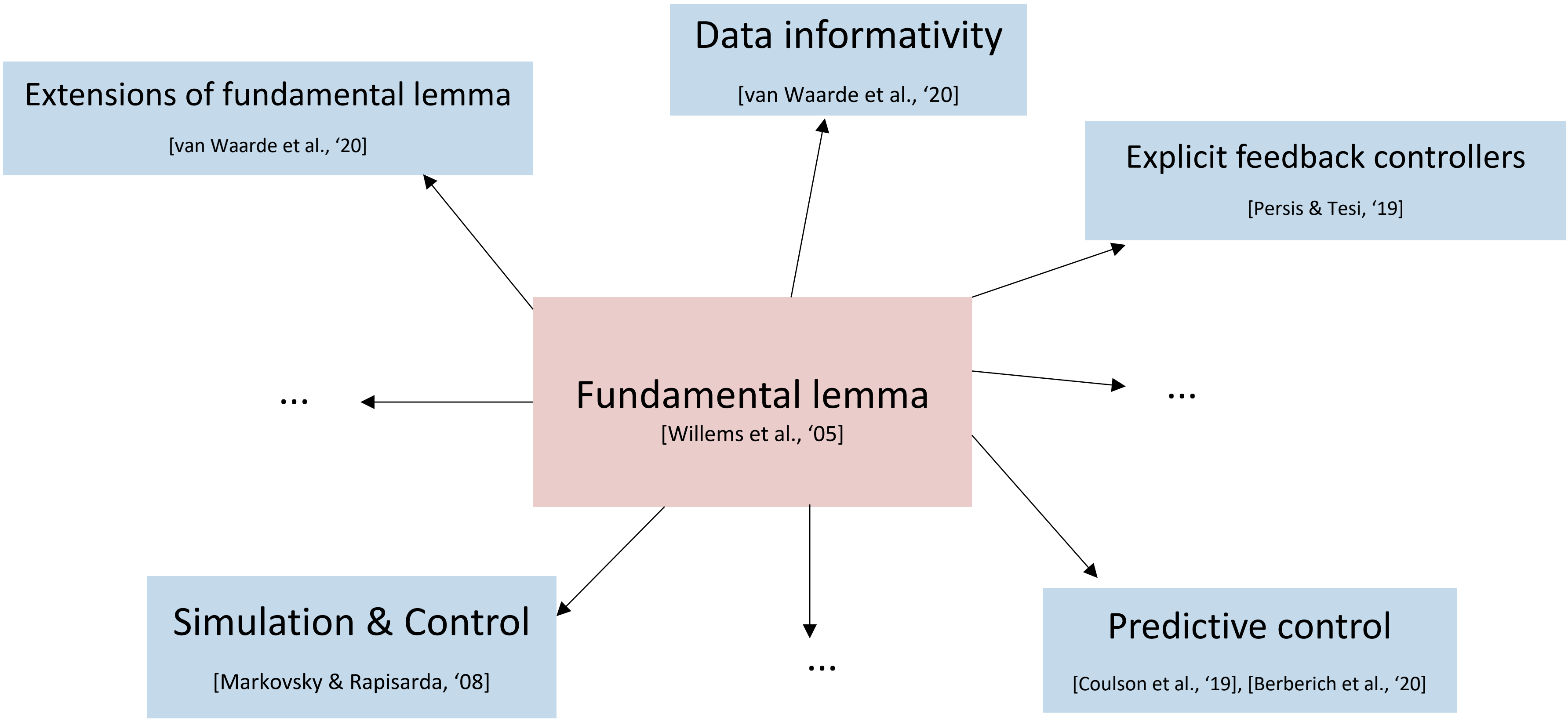
Data characterizes ALL trajectories!

Rapidly expanding literature

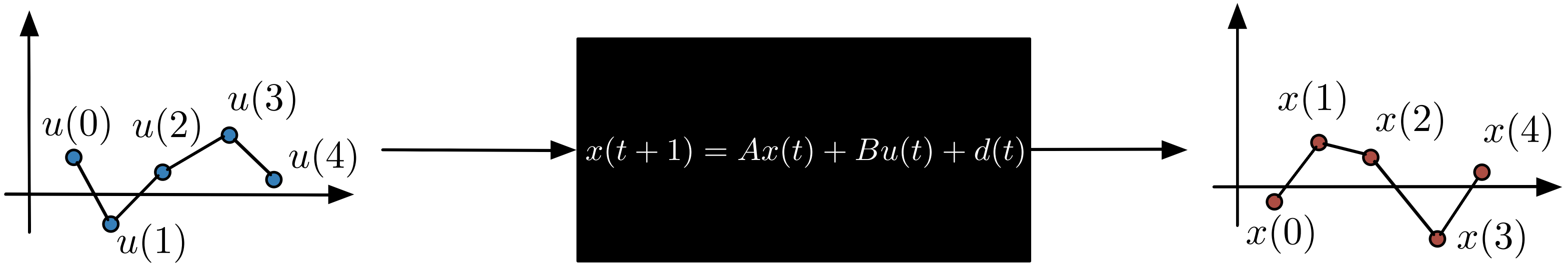


Rapidly expanding literature



Drawback: Fundamental lemma only holds for noise-free data.

Motivating example



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State: $x(t) \in \mathbb{R}^n$

Goal: Estimate (A, B) from data

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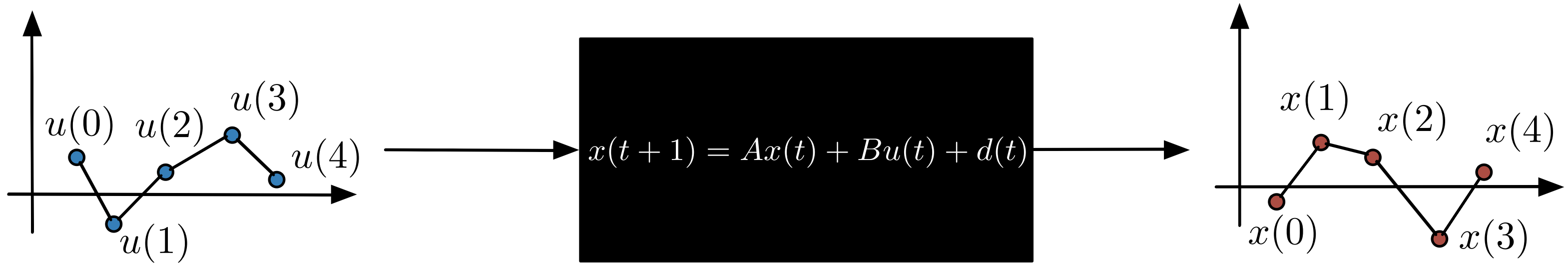

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Goal: Estimate (A, B) from data $\min\limits_{A,B} \left\| \mathcal{H}_1(x_{[1,T]}) - [A\ B] \begin{bmatrix} \mathcal{H}_1(x_{[0,T-1]}) \\ \mathcal{H}_1(u_{[0,T-1]}) \end{bmatrix} \right\|_F$

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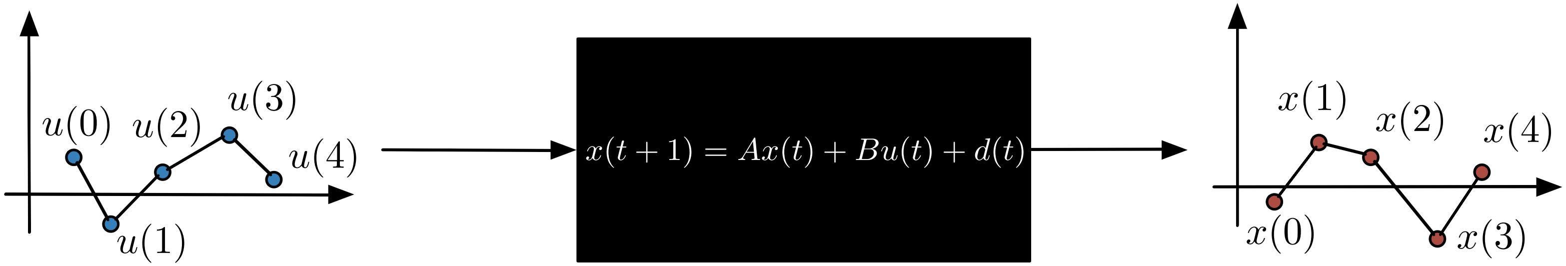

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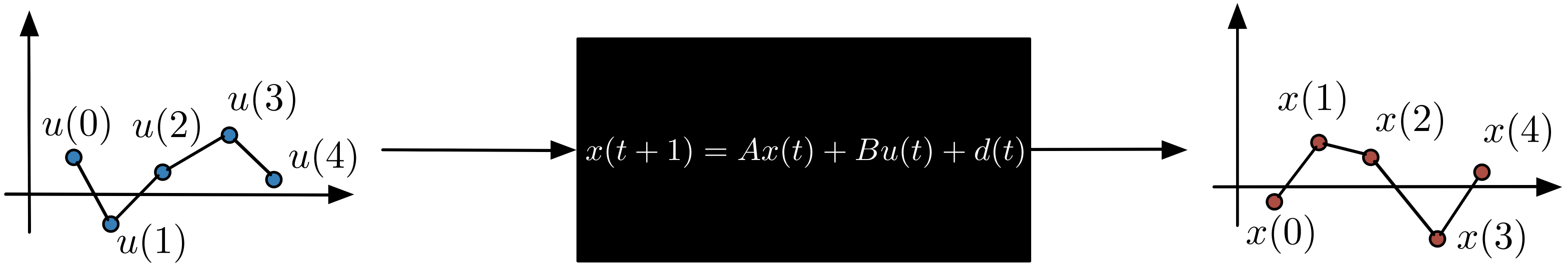

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Error:

$$\left\| \begin{bmatrix} \hat{A} & \hat{B} \end{bmatrix} - \begin{bmatrix} A & B \end{bmatrix} \right\| \leq \frac{\sigma_1\left(\mathcal{H}_1(d_{[0,T-1]})\right)}{\sigma_{n+m}\left(\begin{bmatrix} \mathcal{H}_1(x_{[0,T-1]}) \\ \mathcal{H}_1(u_{[0,T-1]}) \end{bmatrix}\right)}$$

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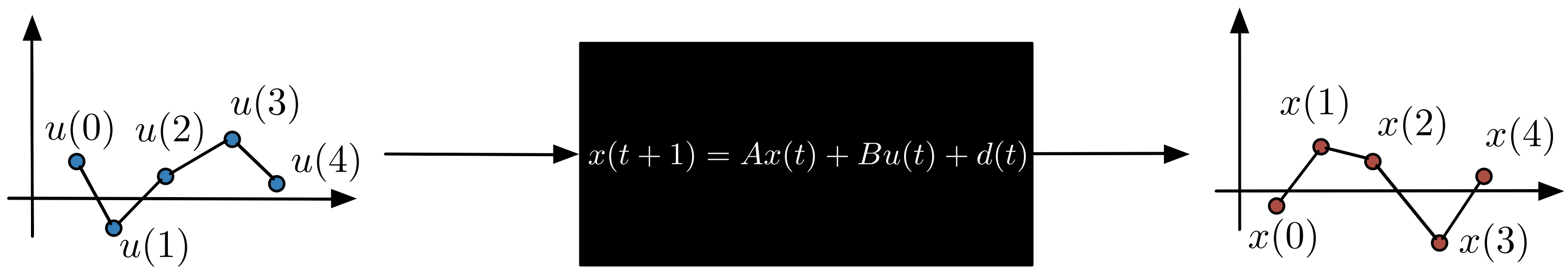

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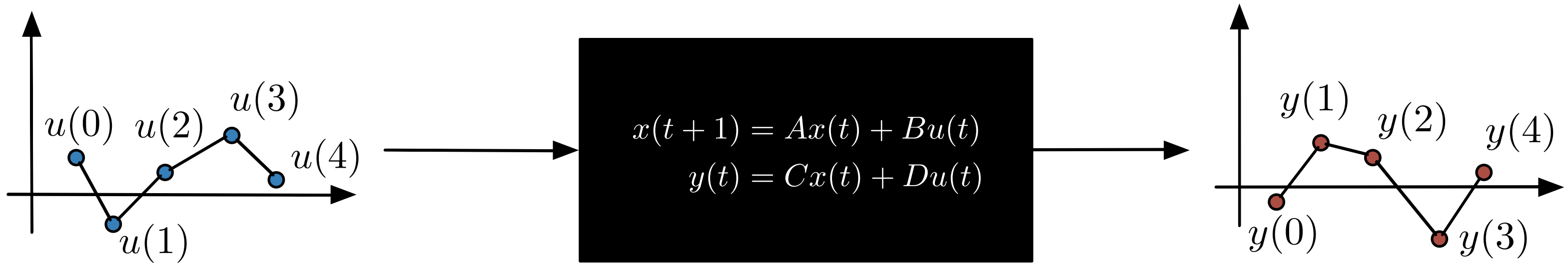
Full row rank is not enough!

Minimum singular value of data matrix important!

Quantitative bound on singular value



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Problem statement

Let $\delta > 0$. Design an input sequence such that the data matrix satisfies

$$\sigma_{mL+n}\left(\begin{bmatrix} \mathcal{H}_L(u_{[0,T-1]}) \\ \mathcal{H}_L(y_{[0,T-1]}) \end{bmatrix}\right) \geq \delta.$$

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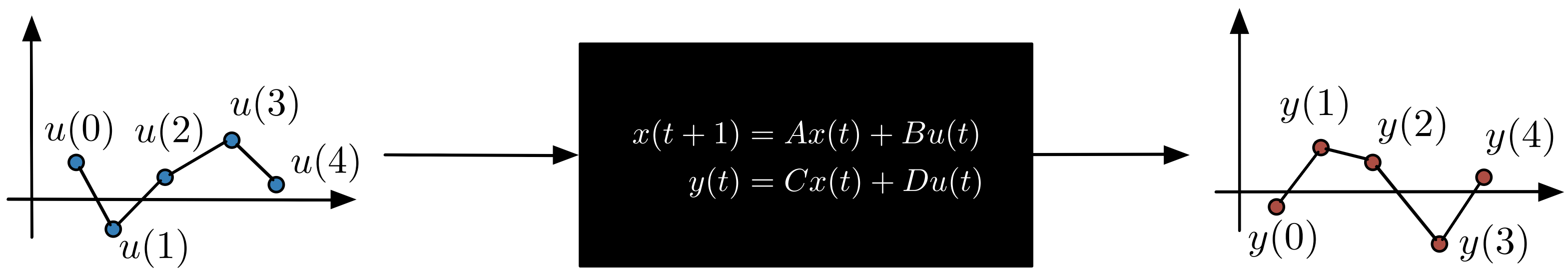

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Need: (i) quantitative persistency of excitation

(ii) quantitative controllability

Quantitative persistency of excitation

α-persistently exciting

Let $\alpha > 0$, $k \in \mathbb{Z}_{>0}$, $T \geq k(m+1) - 1$. A sequence $u_{[0,T-1]}$ is called α-persistently exciting of order k if

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Generalization of persistency of excitation

Quantitative controllability

Truncated controllability matrix: $\Theta_z = \begin{bmatrix} z & M^\top z & \cdots & (M^\top)^n z \end{bmatrix}$

$$M = \begin{bmatrix} A & B & 0 \\ 0 & 0 & I_{m(L+n-1)} \\ 0 & 0 & 0 \end{bmatrix},\ z = \begin{bmatrix} \xi \\ \eta \\ 0_{nm} \end{bmatrix},\ \xi \in \mathbb{R}^n,\ \eta \in \mathbb{R}^{mL}$$

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Quantitative controllability assumption

There exists $\rho > 0$ such that

$$\sigma_{n+1}(\Theta_z) \geq \rho,$$

for all z with $\|z\| = 1$.

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for all z with $\|z\| = 1$.

Fact: This assumption holds for any controllable (A, B)

Quantitative observability

$$\mathcal{O}_k = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix}, \ \mathcal{T}_k = \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{k-2}B & CA^{k-3}B & \cdots & D \end{bmatrix}$$

Quantitative observability assumption

There exists $\beta > 0$ such that

$$\sigma_{mL+n}\left(\begin{bmatrix} 0 & I_{mL} \\ \mathcal{O}_L & \mathcal{T}_L \end{bmatrix}\right) \geq \beta.$$

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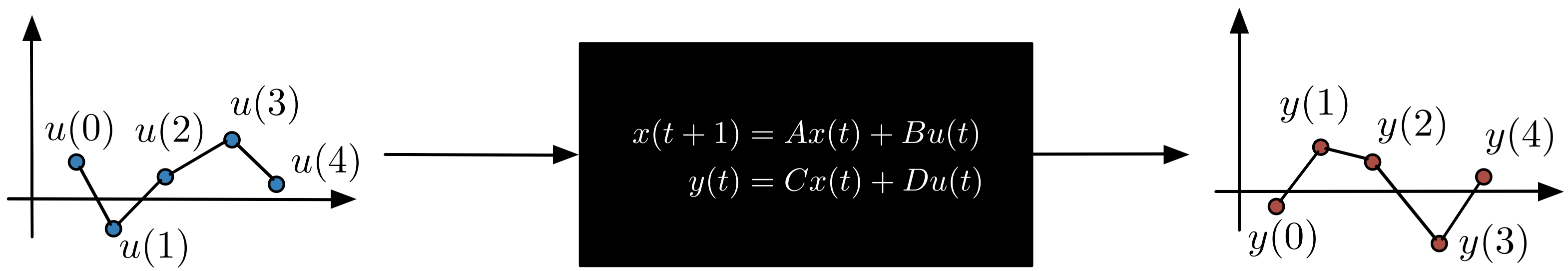
Fact: This assumption holds for any observable (A, C)

Robust fundamental lemma



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Let $\delta > 0$. Let $(u_{[0,T-1]}, y_{[0,T-1]})$ be an input/output trajectory.

Let $u_{[0,T-1]}$ be $\delta\frac{\sqrt{n+1}}{\rho\beta}$-persistently exciting of order $L+n$. Then

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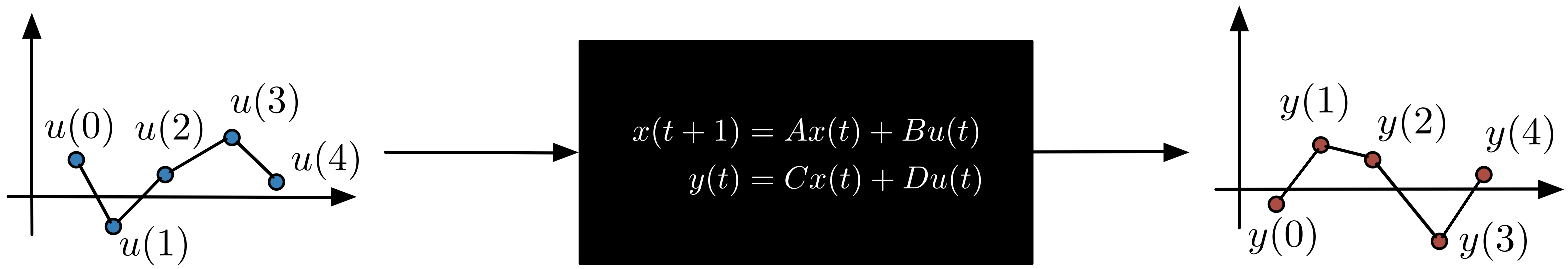

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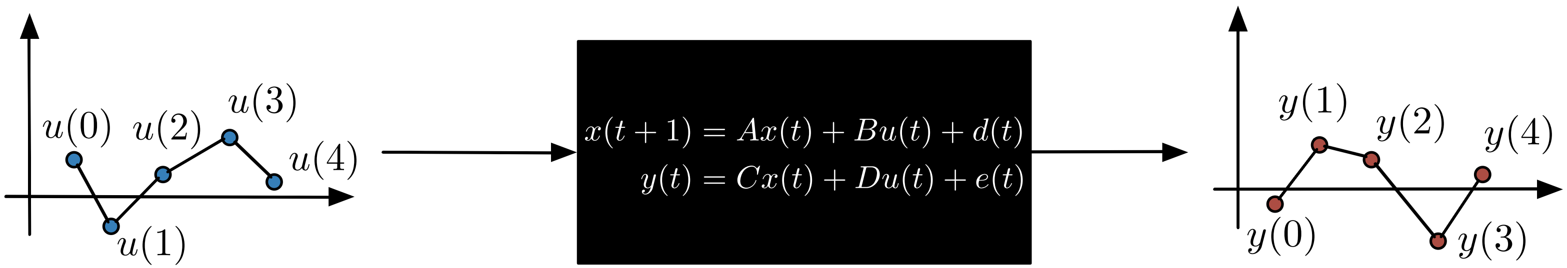
How does noise affect this?

Robust fundamental lemma



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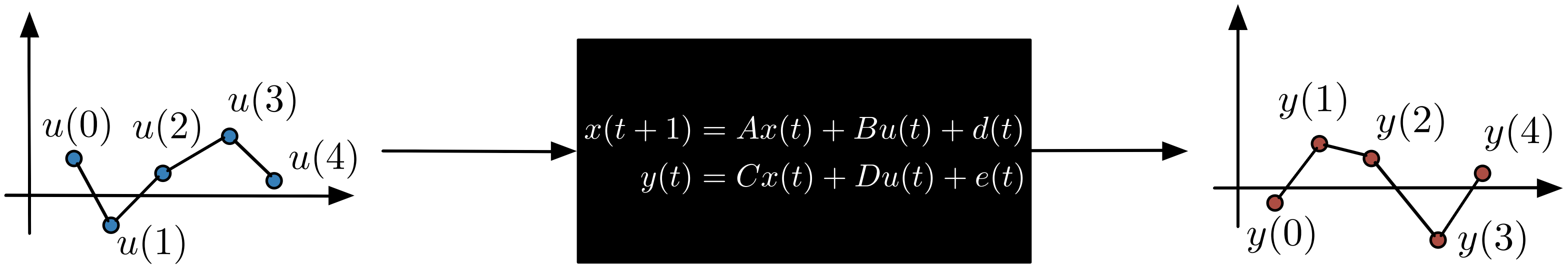
$$\sigma_{mL+n}\left(\begin{bmatrix}\mathcal{H}_L(u_{[0,T-1]})\\ \mathcal{H}_L(y_{[0,T-1]})\end{bmatrix}\right) \geq \delta - K\|\mathcal{H}_L(d_{[0,T-1]})\| - \|\mathcal{H}_L(e_{[0,T-1]})\|.$$

Case study: data-enabled predictive control



Input: $u(t) \in \mathbb{R}^m$

Output: $y(t) \in \mathbb{R}^p$

$$A = \begin{bmatrix} 1.5 & 1 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D = 0$$

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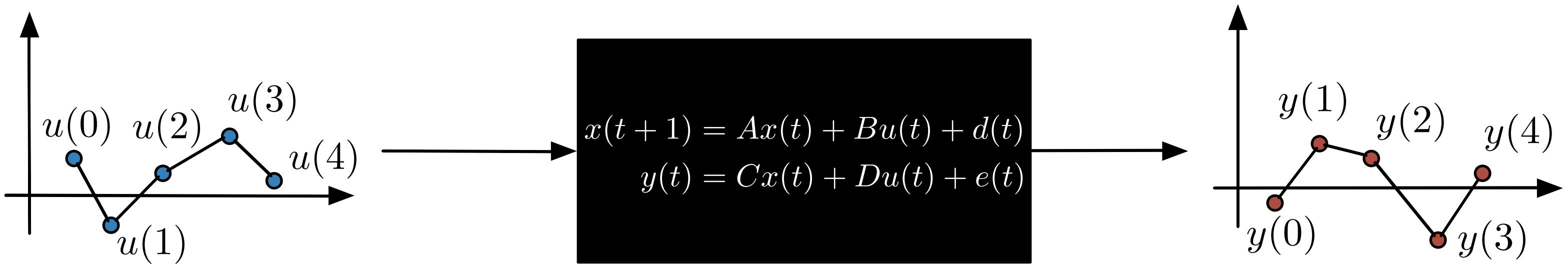

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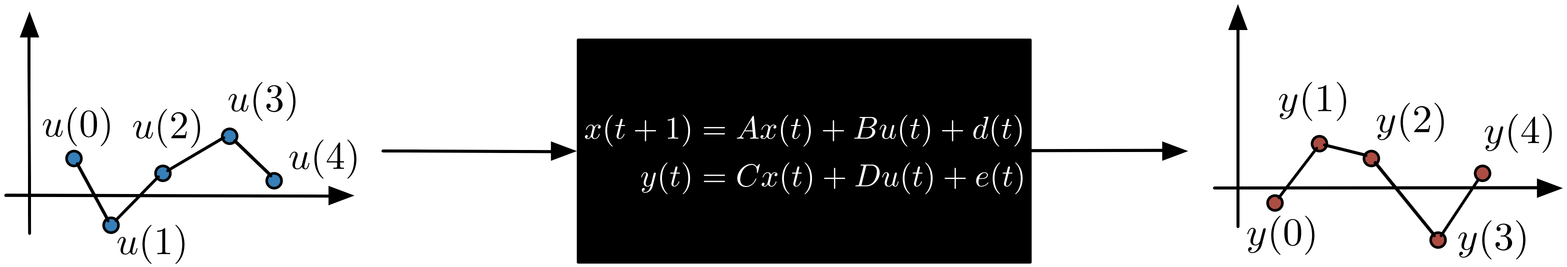
Collect data: 3 data sets $(u^{(i)}_{[0,T-1]}, y^{(i)}_{[0,T-1]})$ with

$u^{(i)}_{[0,T-1]}$ $\alpha^{(i)}$-persistently exciting of order $L+n$, $\begin{cases} \alpha^{(1)} = 12.4 \\ \alpha^{(2)} = 3.7 \\ \alpha^{(3)} = 2.5 \end{cases}$

Case study: data-enabled predictive control



Input: $u(t) \in \mathbb{R}^m$ Output: $y(t) \in \mathbb{R}^p$

$$A = \begin{bmatrix} 1.5 & 1 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; D = 0$$

Collect data: 3 data sets $(u^{(i)}_{[0,T-1]}, y^{(i)}_{[0,T-1]})$ with

$u^{(i)}_{[0,T-1]}$ $\alpha^{(i)}$-persistently exciting of order $L+n$, $\begin{cases} \alpha^{(1)} = 12.4 \\ \alpha^{(2)} = 3.7 \\ \alpha^{(3)} = 2.5 \end{cases}$

Build trajectory matrices:

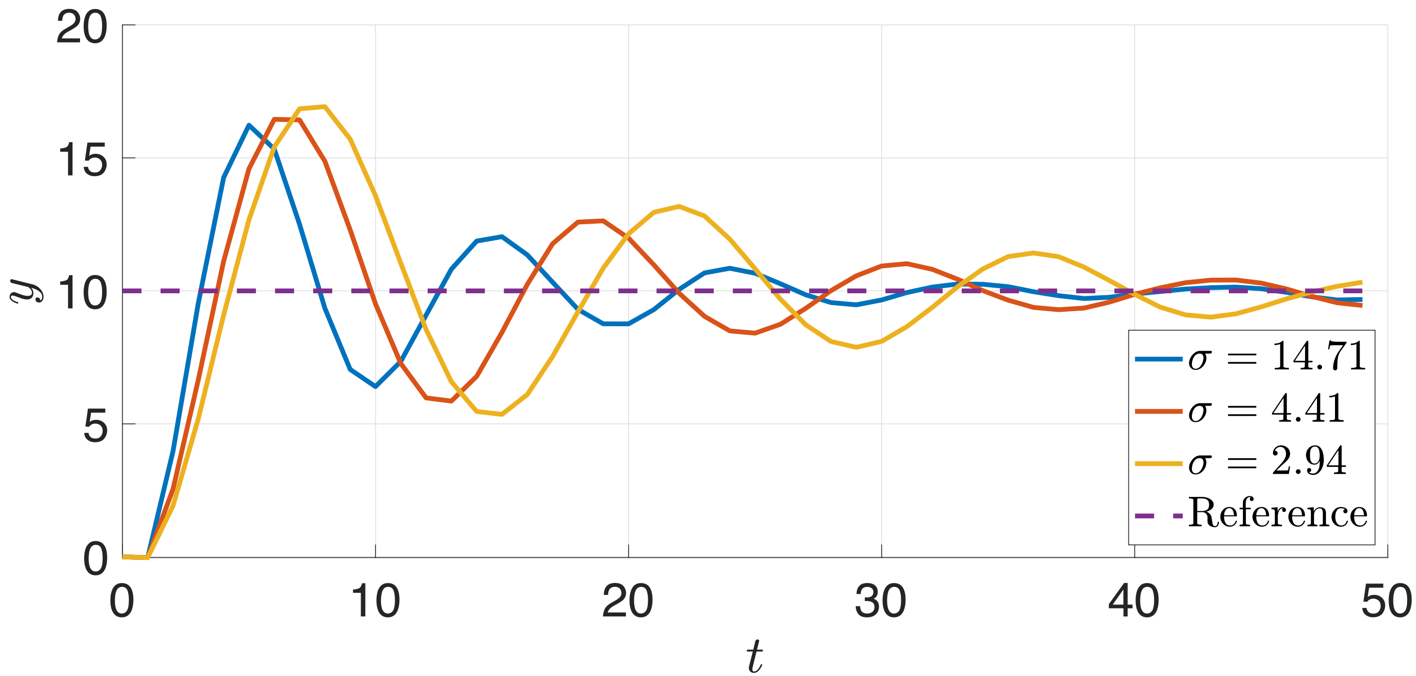
$$\begin{bmatrix} \mathcal{H}_L\left(u^{(i)}_{[0,T-1]}\right) \\ \mathcal{H}_L\left(y^{(i)}_{[0,T-1]}\right) \end{bmatrix}$$

Case study: data-enabled predictive control

Receding horizon control:

$$\min_{g,u,y} \quad \|y-r\|^2 + \|u\|^2 + \lambda_g \|g\|^2$$

$$\text{s.t.} \quad \begin{bmatrix} \mathcal{H}_L\left(u^{(i)}_{[0,T-1]}\right) \\ \mathcal{H}_L\left(y^{(i)}_{[0,T-1]}\right) \end{bmatrix} g = \begin{bmatrix} u_{\text{ini}} \\ u \\ y_{\text{ini}} \\ y \end{bmatrix}$$



Summary

- Studied quantitative persistency of excitation in the context of the fundamental lemma
- Informs input selection so that data matrix has lower bounded singular values
- Characterized the effect of noise
- Illustrated performance with a simulation case study

Future work:

- Online input selection

Thanks!

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Proof sketch

Study singular vectors of the data matrix:

$$[\xi^\top \ \eta^\top] \begin{bmatrix} \mathcal{H}_1(x_{[0,T-1]}) \\ \mathcal{H}_1(u_{[0,T-1]}) \end{bmatrix} = \sigma v^\top.$$

Leverage dynamics through "data equation":

$$\Theta_z \begin{bmatrix} \mathcal{H}_1(x_{[0,T-n-1]}) \\ \mathcal{H}_{n+1}(u_{[0,T-1]}) \end{bmatrix} = \sigma \mathcal{H}_{n+1}(v_{[0,T-1]}).$$

Use singular value interlacing and min-max formulations

Leverage quantitative persistency of excitation and controllability