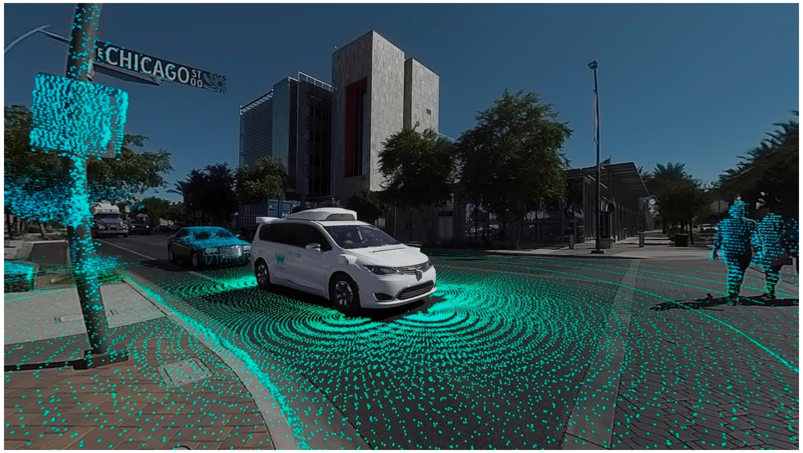
Data-enabled Predictive Control

Regularization and Robustness

Jeremy Coulson (joint work with Florian Dörfler & John Lygeros)

IFAC 2023

Living in a data-rich world



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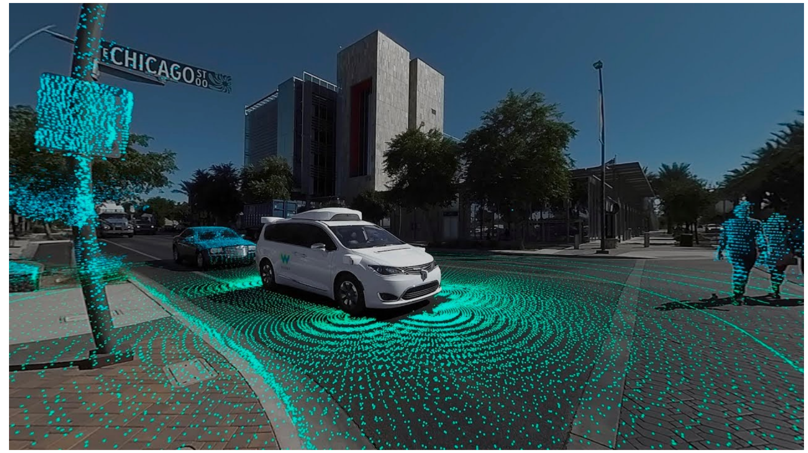

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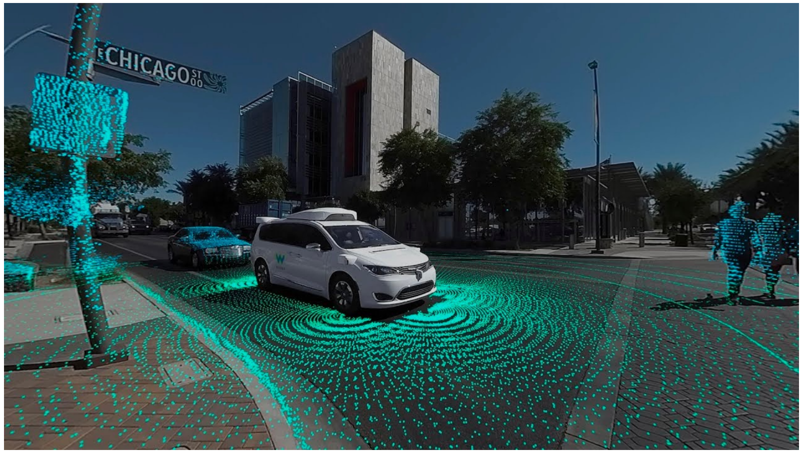
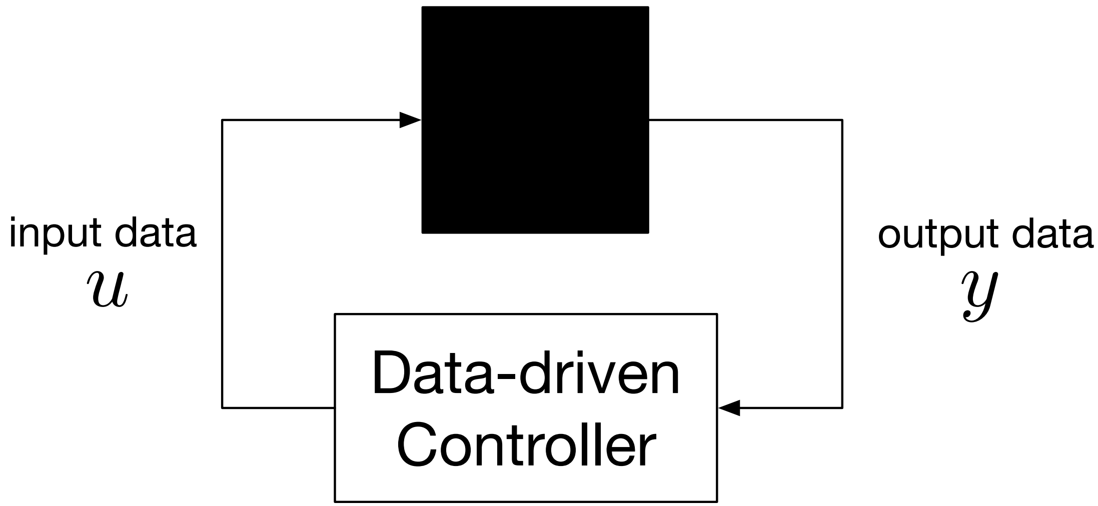


safety, robustness, reliability

Data-driven Control



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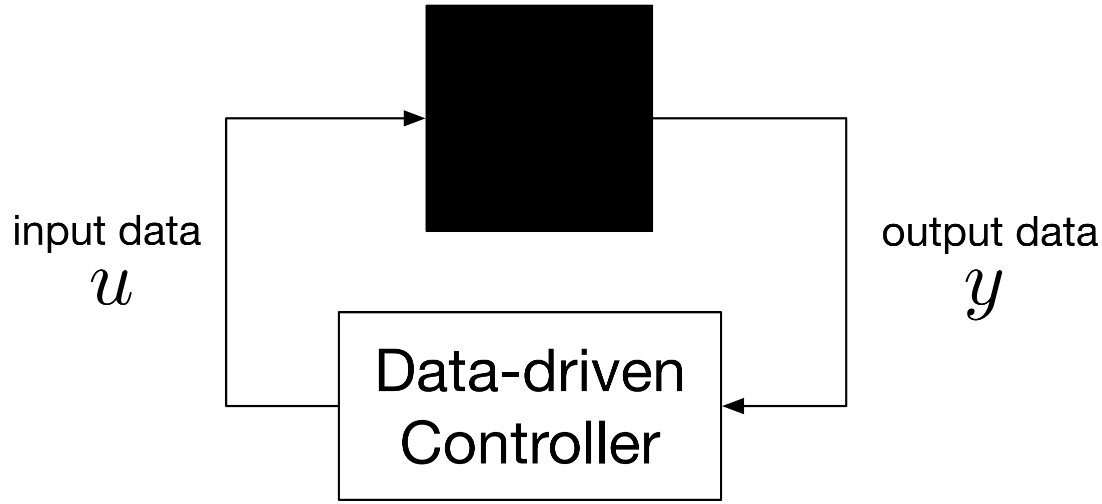
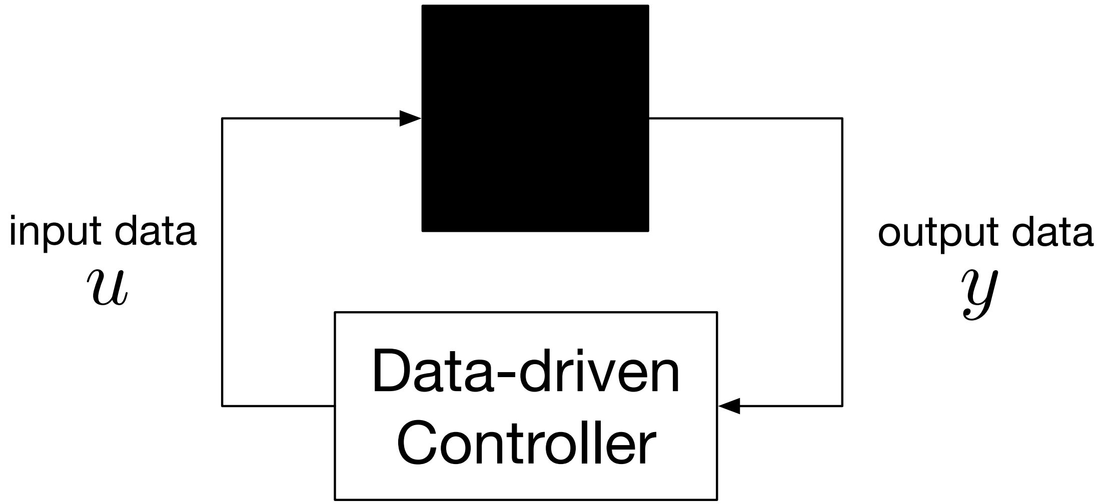
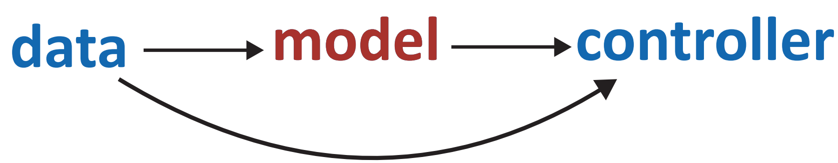


data $\longrightarrow$ **model** $\longrightarrow$ **controller**

Data-driven Control





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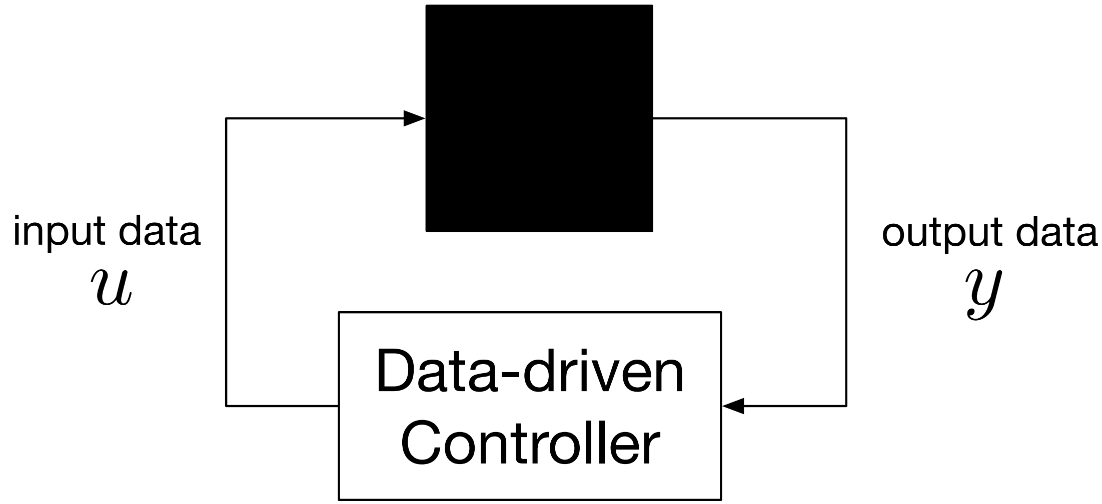
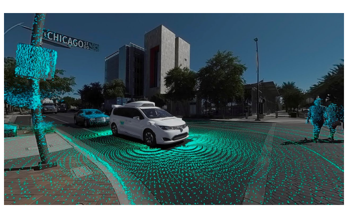


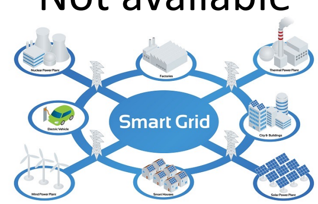
data → **model** → **controller**

Models not conceivable



Too complex

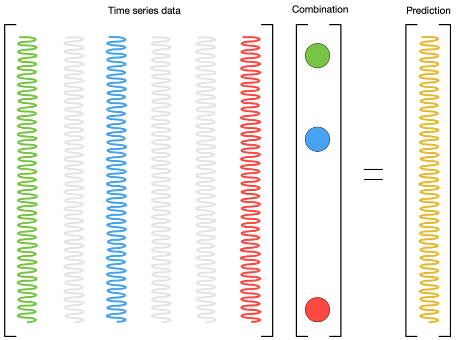
Not available



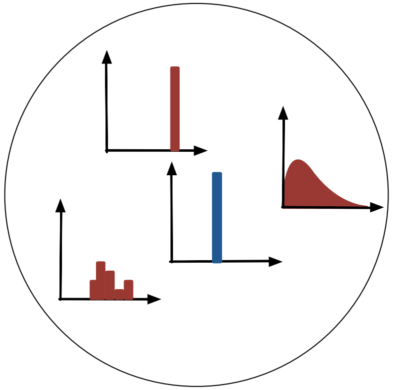
Too costly

Outline

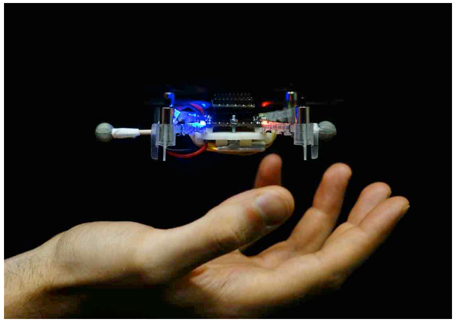
Data-enabled Predictive Control (DeePC)



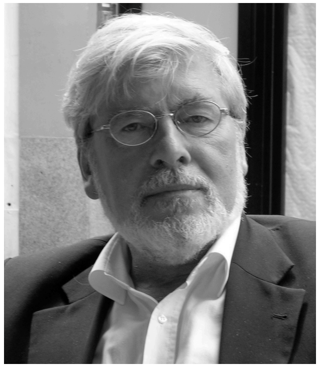
Robustification via Regularization



Applications in Robotics



Behavioral System Theory

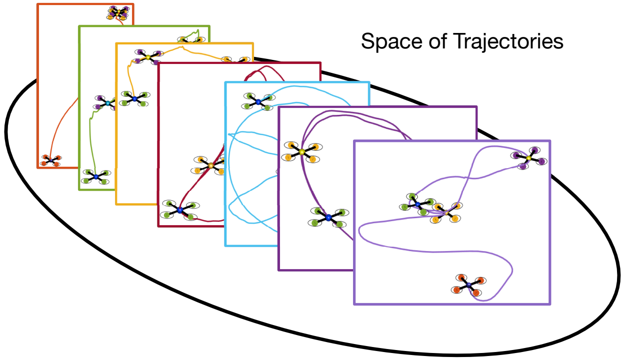


Jan Willems

Introduced behavioral system theory $\sim$ 1980s

Dynamical System

A discrete-time dynamical system is a 3-tuple $(\mathbb{Z}_{\geq 0}, \mathbb{R}^{m+p}, \mathscr{B})$ where $\mathscr{B} \subseteq (\mathbb{R}^{m+p})^{\mathbb{Z}_{\geq 0}}$ is the behavior (set of allowable trajectories).



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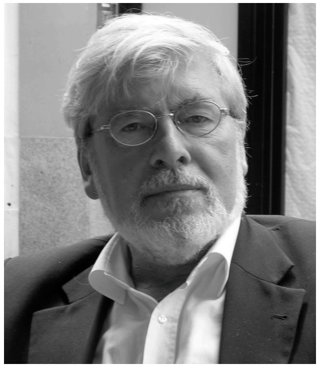

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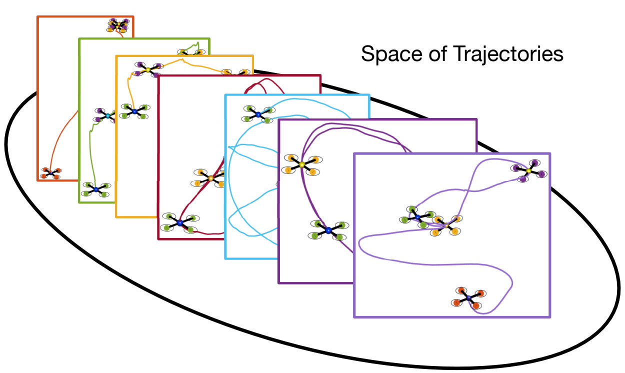


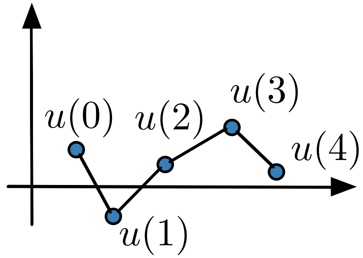
LTI system

=

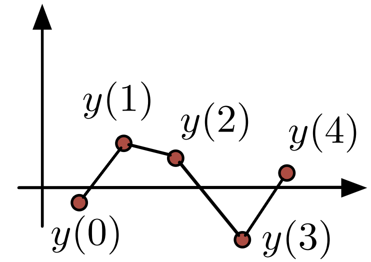
time-shift invariant subspace

Fundamental Lemma

[Willems et al. ‘05], [Markovsky, Dörfler ‘20]



$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu(t)\\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$



Given data $(u_{[0,T-1]}, y_{[0,T-1]})$ and parameters $\begin{cases} \text{state dimension } n \\ \text{lag } \ell \\ \text{trajectory length } L \end{cases}$

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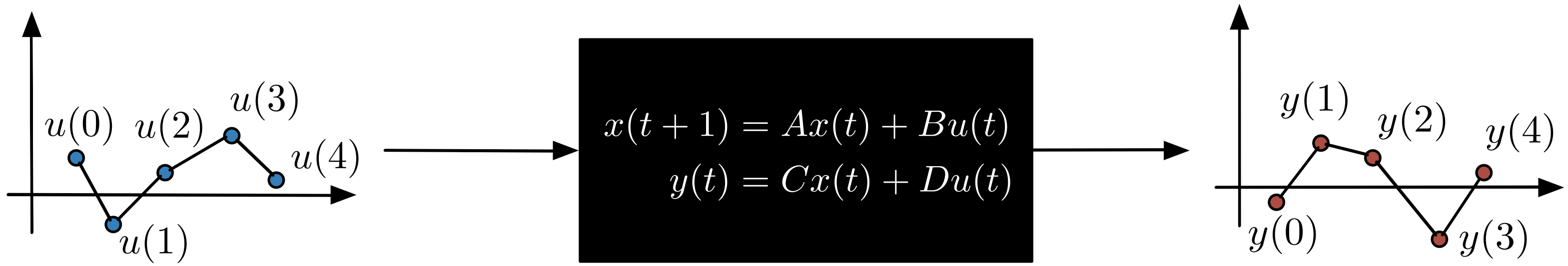

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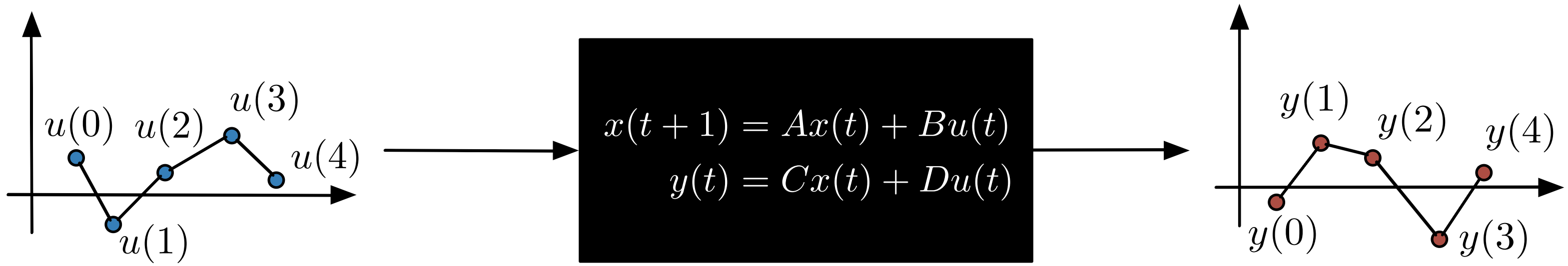
$$\text{im} \begin{bmatrix} u(0) & u(1) & \cdots & u(T-L) \\ \vdots & \vdots & \ddots & \vdots \\ u(L-1) & u(L) & \cdots & u(T-1) \\ y(0) & y(1) & \cdots & y(T-L) \\ \vdots & \vdots & \ddots & \vdots \\ y(L-1) & y(L) & \cdots & y(T-1) \end{bmatrix}$$

Fundamental Lemma [Willems et al. '05], [Markovsky, Dörfler '20]



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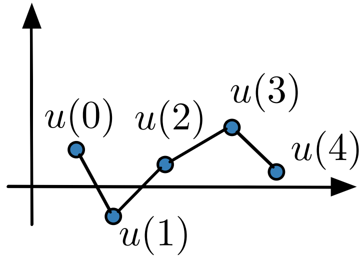
Fundamental Lemma [Willems et al. '05], [Markovsky, Dörfler '20]



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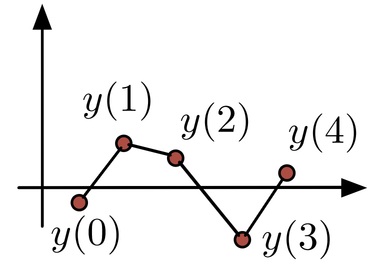

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$$
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\vdots & \vdots & \ddots & \vdots \\
y(L-1) & y(L) & \cdots & y(T-1)
\end{bmatrix} = \text{set of } \textbf{all } L\text{-length trajectories}
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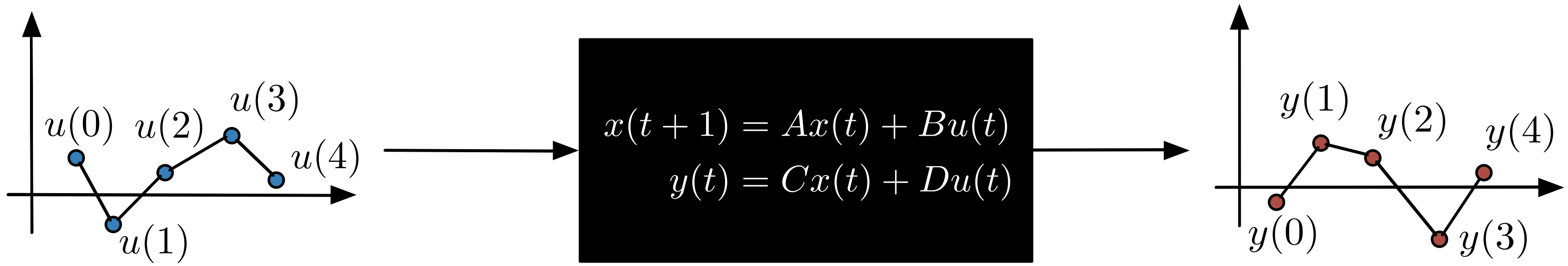
if and only if trajectory matrix has rank $mL + n$ for $L \geq \ell$.

Fundamental Lemma

[Willems et al. ‘05], [Markovsky, Dörfler ‘20]



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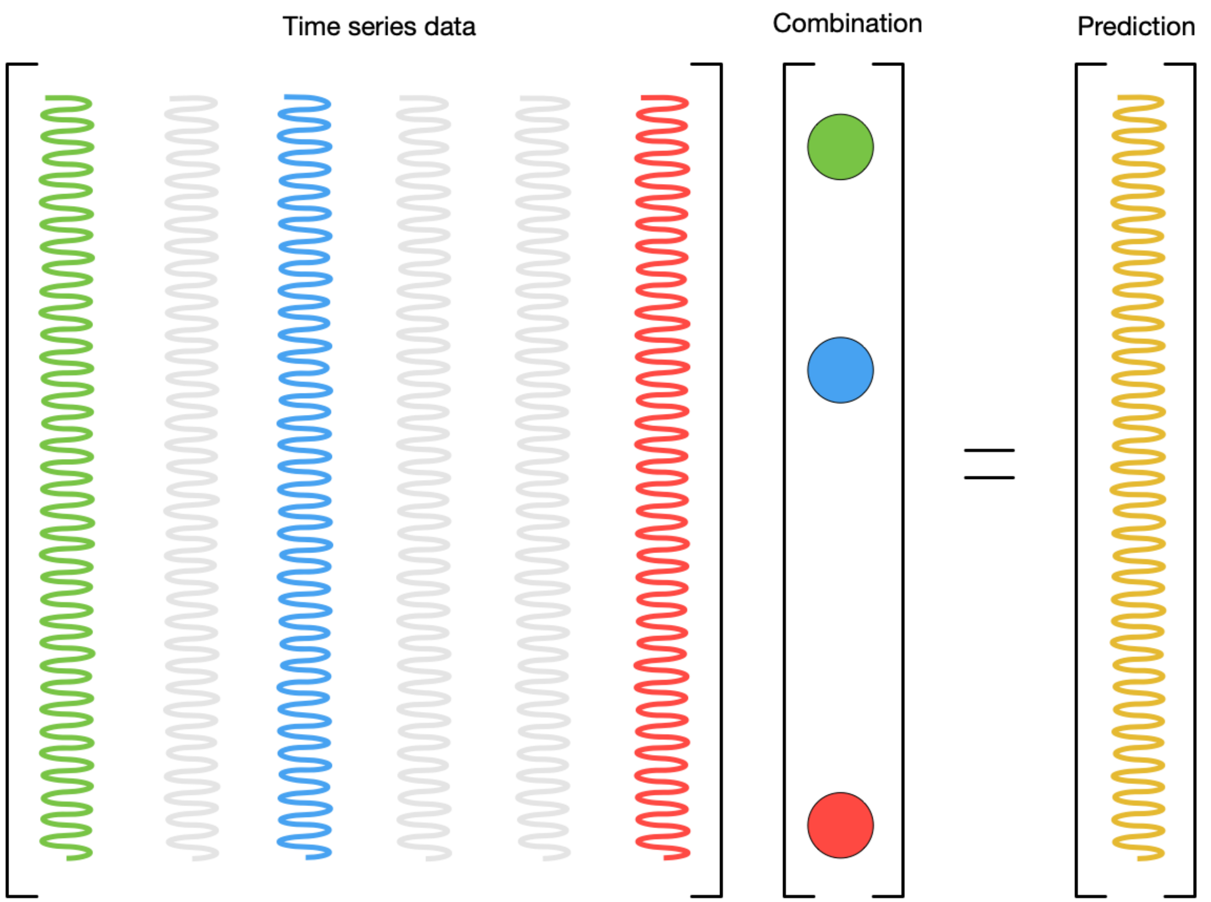
if and only if trajectory matrix has rank $mL + n$ for $L \geq \ell$.

Any trajectory can be generated from finitely many, sufficiently rich data trajectories

Trajectory Matrix: a predictive model

Idea: The trajectory matrix using raw data can serve as a non-parametric predictive model!

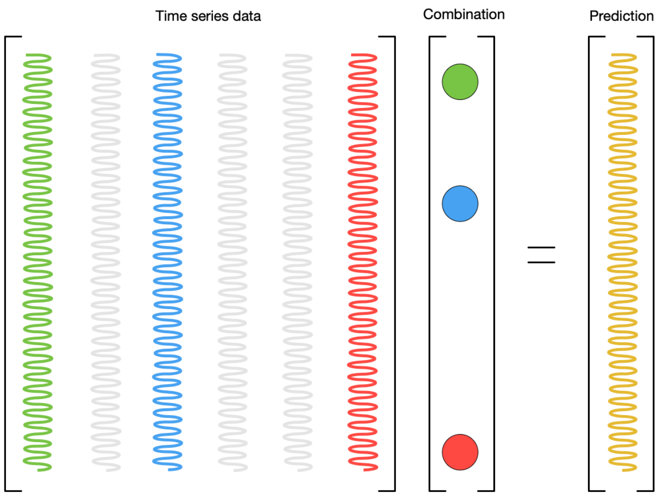


Trajectory Prediction [Markovsky, Rapisarda '08]

Given:

- Data $\left(u^{\mathrm{d}}_{[0,T-1]}, y^{\mathrm{d}}_{[0,T-1]}\right)$
- Initial trajectory $(u_{\mathrm{ini}}, y_{\mathrm{ini}}) \in \mathbb{R}^{(m+p)T_{\mathrm{ini}}}$
- Future input $u \in \mathbb{R}^{mT_{\mathrm{f}}}$

Predict the future output $y \in \mathbb{R}^{pT_{\mathrm{f}}}$.



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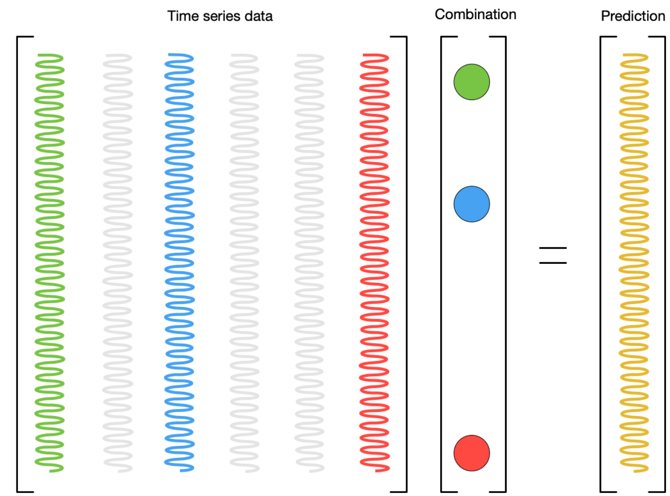

$$\underbrace{\left[\begin{array}{cccc}
u^{\mathsf{d}}(0) & u^{\mathsf{d}}(1) & \cdots & u^{\mathsf{d}}(T-T_{\mathsf{ini}}-T_{\mathsf{f}}) \\
\vdots & \vdots & \ddots & \vdots \\
u^{\mathsf{d}}(T_{\mathsf{ini}}-1) & u^{\mathsf{d}}(T_{\mathsf{ini}}) & \cdots & u^{\mathsf{d}}(T-T_{\mathsf{f}}-1) \\
\hline
y^{\mathsf{d}}(0) & y^{\mathsf{d}}(1) & \cdots & y^{\mathsf{d}}(T-T_{\mathsf{ini}}-T_{\mathsf{f}}) \\
\vdots & \vdots & \ddots & \vdots \\
y^{\mathsf{d}}(T_{\mathsf{ini}}-1) & y^{\mathsf{d}}(T_{\mathsf{ini}}) & \cdots & y^{\mathsf{d}}(T-T_{\mathsf{f}}-1) \\
\hline
u^{\mathsf{d}}(T_{\mathsf{ini}}) & u^{\mathsf{d}}(T_{\mathsf{ini}}+1) & \cdots & u^{\mathsf{d}}(T-T_{\mathsf{f}}) \\
\vdots & \vdots & \ddots & \vdots \\
u^{\mathsf{d}}(T_{\mathsf{ini}}+T_{\mathsf{f}}-1) & u^{\mathsf{d}}(T_{\mathsf{ini}}+T_{\mathsf{f}}) & \cdots & u^{\mathsf{d}}(T-1) \\
\hline
y^{\mathsf{d}}(T_{\mathsf{ini}}) & y^{\mathsf{d}}(T_{\mathsf{ini}}+1) & \cdots & y^{\mathsf{d}}(T-T_{\mathsf{f}}) \\
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\end{array}\right]}_{=:\begin{bmatrix} U_{\mathsf{p}} \\ Y_{\mathsf{p}} \\ U_{\mathsf{f}} \\ Y_{\mathsf{f}} \end{bmatrix}}$$

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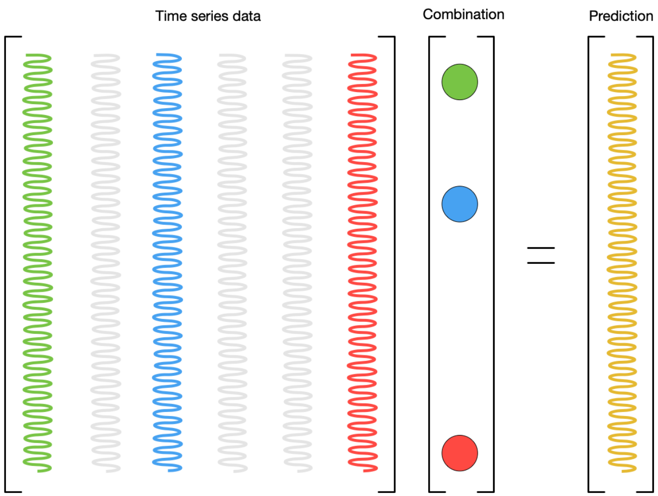

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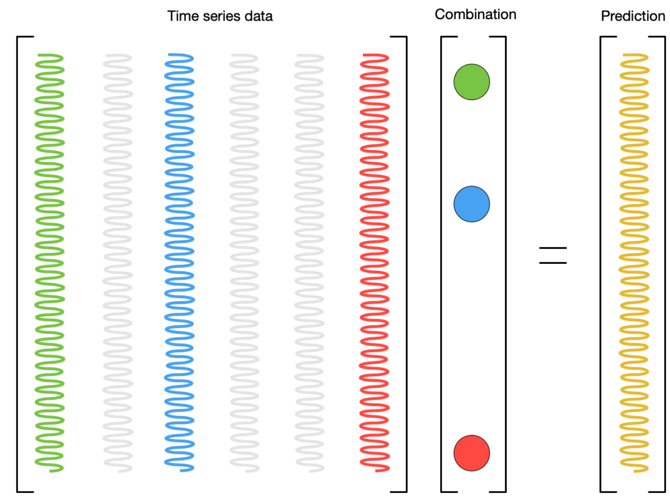

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Predict the future output $\color{red}{y} \in \mathbb{R}^{pT_{\mathsf{f}}}$.

$$\underbrace{\left[\begin{array}{cccc}
u^{\mathsf{d}}(0) & u^{\mathsf{d}}(1) & \cdots & u^{\mathsf{d}}(T-T_{\mathsf{ini}}-T_{\mathsf{f}}) \\
\vdots & \vdots & \ddots & \vdots \\
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\begin{array}{l} \left.\vphantom{\begin{matrix} a \\ a \end{matrix}}\right\} \text{set initial condition} \\ \left.\vphantom{\begin{matrix} a \\ a \end{matrix}}\right\} \text{prediction} \end{array}$$

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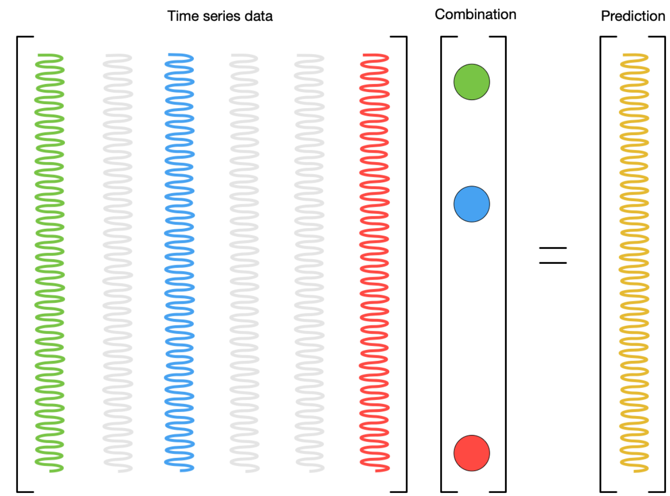

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Prediction unique if $T_{\mathsf{ini}} \geq \ell$

Data-enabled Predictive Control

Goal

Using data, design a controller to:

- track reference output trajectory
- satisfy safety constraints $u(t) \in \mathcal{U},\ y(t) \in \mathcal{Y}$

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DeePC:

$$\min_{g,u,y} \quad \sum_{k=0}^{T_{\mathsf{f}}-1} \|y_k - r_{t+k}\|_Q^2 + \|u_k\|_R^2$$

performance

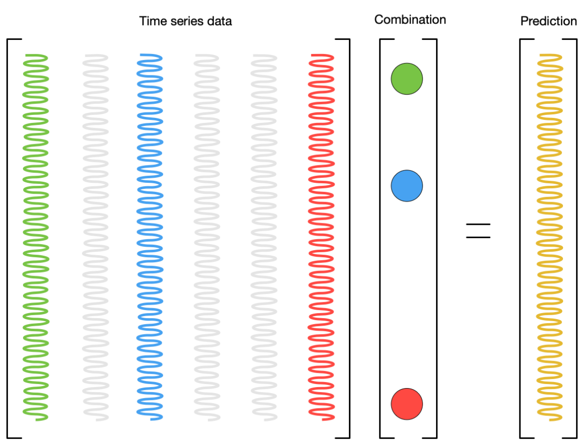
$$\text{s.t.} \quad \begin{bmatrix} U_{\mathsf{p}} \\ Y_{\mathsf{p}} \\ U_{\mathsf{f}} \\ Y_{\mathsf{f}} \end{bmatrix} g = \begin{bmatrix} u_{\mathsf{ini}} \\ y_{\mathsf{ini}} \\ u \\ y \end{bmatrix}$$

initial condition

prediction

$$u_k \in \mathcal{U},\ \forall k \in \{0, \ldots, T_{\mathsf{f}} - 1\}$$
$$y_k \in \mathcal{Y},\ \forall k \in \{0, \ldots, T_{\mathsf{f}} - 1\}$$

safety



Models vs data

MPC:

$$\min_{u,x,y} \quad \sum_{k=0}^{T_{\mathsf{f}}-1} \|y_k - r_{t+k}\|_Q^2 + \|u_k\|_R^2$$

s.t.

$$x_{k+1} = Ax_k + Bu_k, \ \forall k \in \{0, \dots, T_{\mathsf{f}} - 1\}$$
$$y_k = Cx_k + Du_k, \ \forall k \in \{0, \dots, T_{\mathsf{f}} - 1\}$$
$$x_{k+1} = Ax_k + Bu_k, \ \forall k \in \{-T_{\mathsf{ini}}, \dots, -1\}$$
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$$y_k \in \mathcal{Y}, \ \forall k \in \{0, \dots, T_{\mathsf{f}} - 1\}$$

DeePC:

$$\min_{g,u,y} \quad \sum_{k=0}^{T_{\mathsf{f}}-1} \|y_k - r_{t+k}\|_Q^2 + \|u_k\|_R^2$$

s.t.

$$\begin{bmatrix} U_{\mathsf{p}} \\ Y_{\mathsf{p}} \\ U_{\mathsf{f}} \\ Y_{\mathsf{f}} \end{bmatrix} g = \begin{bmatrix} u_{\mathsf{ini}} \\ y_{\mathsf{ini}} \\ u \\ y \end{bmatrix}$$
$$u_k \in \mathcal{U}, \ \forall k \in \{0, \dots, T_{\mathsf{f}} - 1\}$$
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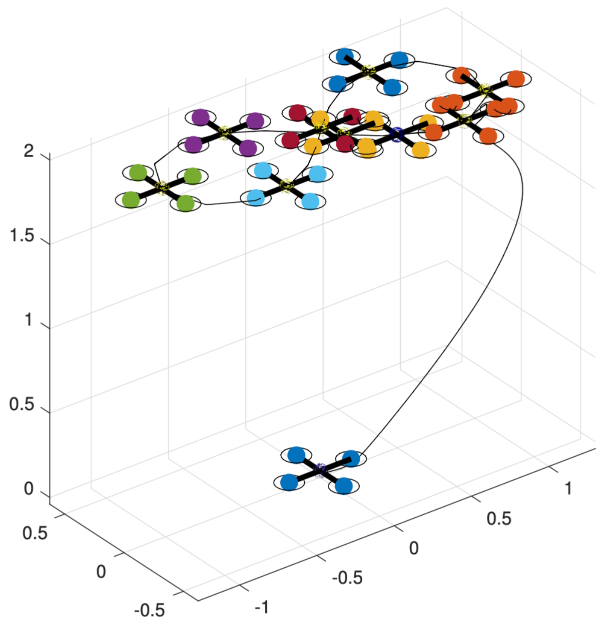
Predictive model and state estimation in MPC is replaced by <u>raw data</u> in DeePC!

Consistent for deterministic LTI systems

Theorem [Coulson et al. '19]

MPC and DeePC have equivalent closed-loop behavior.



Stability and recursive feasibility proven by [Berberich et al. '19]

Beyond Deterministic LTI

What about noisy data?
…Nonlinear systems?

Beyond Deterministic LTI

What about noisy data?
…Nonlinear systems?

We need a robustified approach!

Robustify against what?

Infeasibility

Online data $(u_{\text{ini}}, y_{\text{ini}})$ may be inconsistent with data in the trajectory matrix

$$\text{No } g \text{ such that } \begin{bmatrix} U_{\text{p}} \\ Y_{\text{p}} \\ U_{\text{f}} \\ Y_{\text{f}} \end{bmatrix} g = \begin{bmatrix} u_{\text{ini}} \\ y_{\text{ini}} \\ u \\ y \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} u_{\text{ini}} \\ y_{\text{ini}} \end{matrix}}\right\} \text{initial condition} \\ \vphantom{\begin{matrix} u \\ y \end{matrix}} \end{matrix}$$

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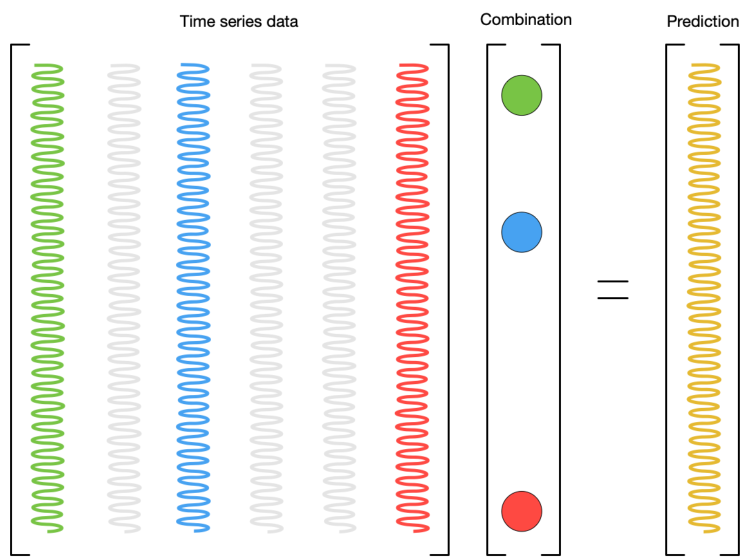
Over-optimism

Trajectory matrix contains noisy data $\Longrightarrow$ Trajectory selection over-optimistic

No longer low rank $\rightarrow$



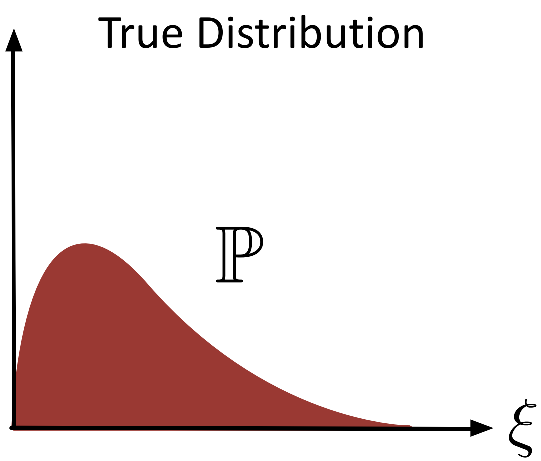
$\leftarrow$ Can predict trajectories outside of subspace of allowable trajectories robustification via regularization

Distributionally Robust Optimization

[Bertsimas et al. '04], [Ben-Tal et al. '09], [Kuhn et al. '19]

Objective: $\underset{g \in G}{\text{minimize}} \quad \mathbb{E}_{\mathbb{P}}[c(\xi, g)]$

ξ is a random variable distributed according to unknown distribution $\mathbb{P}$



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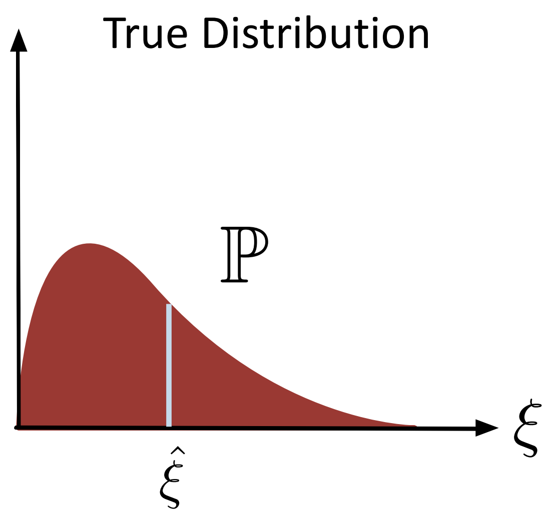

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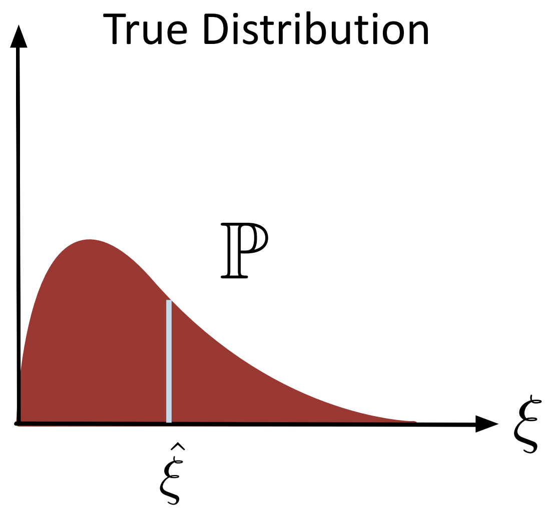

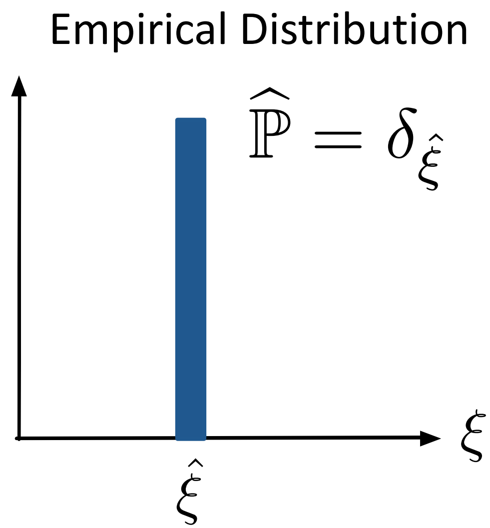

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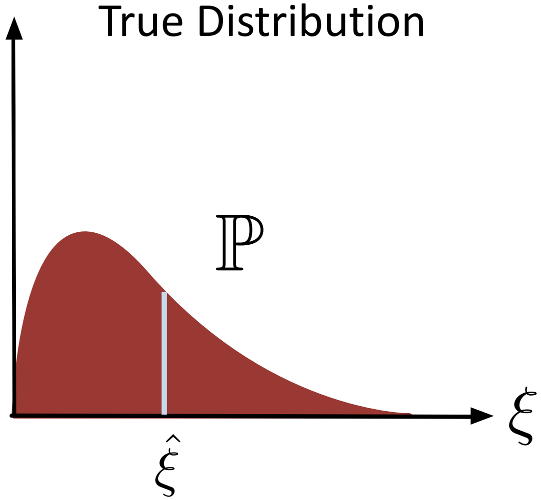
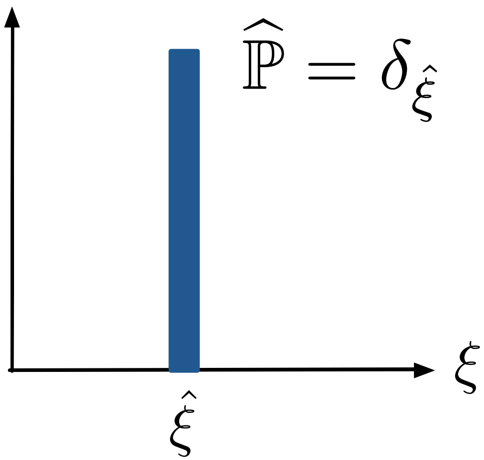


$\neq$

Empirical Distribution



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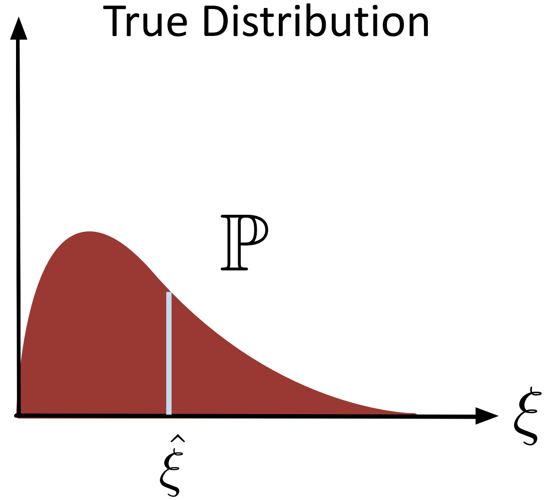

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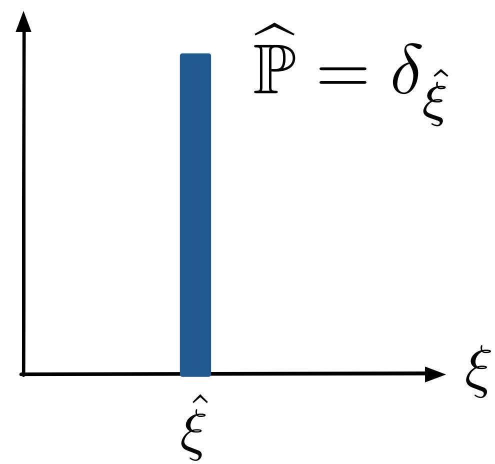

Solve sample average approximation problem

$$g^\star \in \underset{g \in G}{\text{argmin}} \quad \mathbb{E}_{\widehat{\mathbb{P}}}[c(\xi, g)]$$

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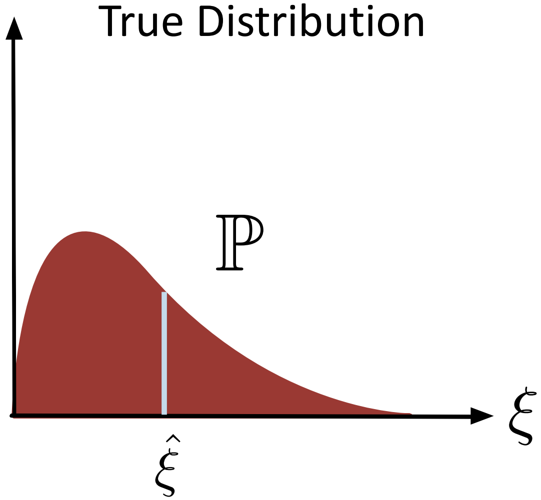

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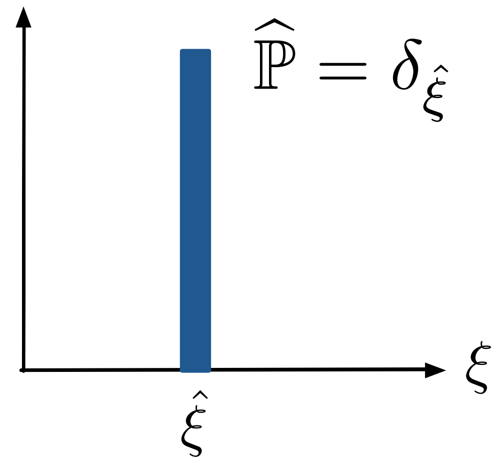


Solve sample average approximation problem

$$g^\star \in \underset{g \in G}{\text{argmin}} \quad \mathbb{E}_{\widehat{\mathbb{P}}}[c(\xi, g)] \longrightarrow$$

Poor true expected cost

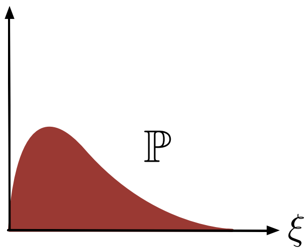
$$\mathbb{E}_{\mathbb{P}}[c(\xi, g^\star)]$$

Wasserstein Ball

[Kuhn et al. '19]

Robustify against the fact that:

True Distribution



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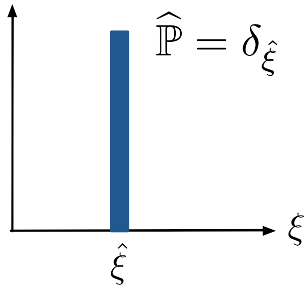

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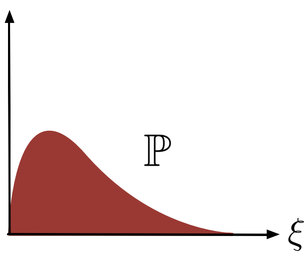

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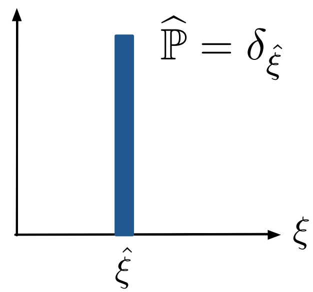

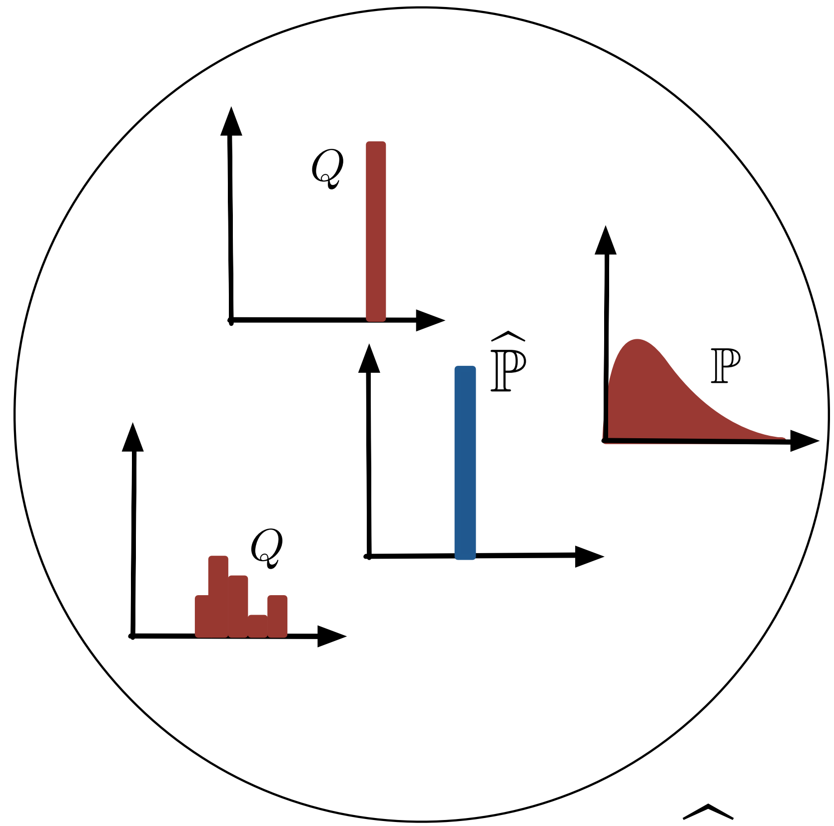

$B_\epsilon(\widehat{\mathbb{P}})$

Distributionally Robust Optimization

$$\inf_{g \in G} \sup_{Q \in B_\epsilon(\widehat{\mathbb{P}})} \mathbb{E}_Q[c(\xi, g)]$$

where $B_\epsilon(\widehat{\mathbb{P}}) = \left\{ Q \middle| \int_\Xi \|\xi - \hat{\xi}\| Q(d\xi) \leq \epsilon \right\}$

is the Wasserstein ball.

Wasserstein Ball

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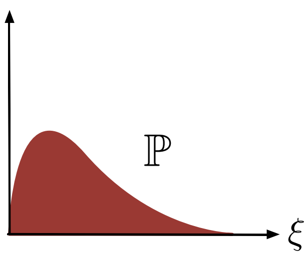

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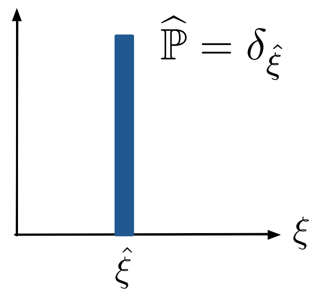

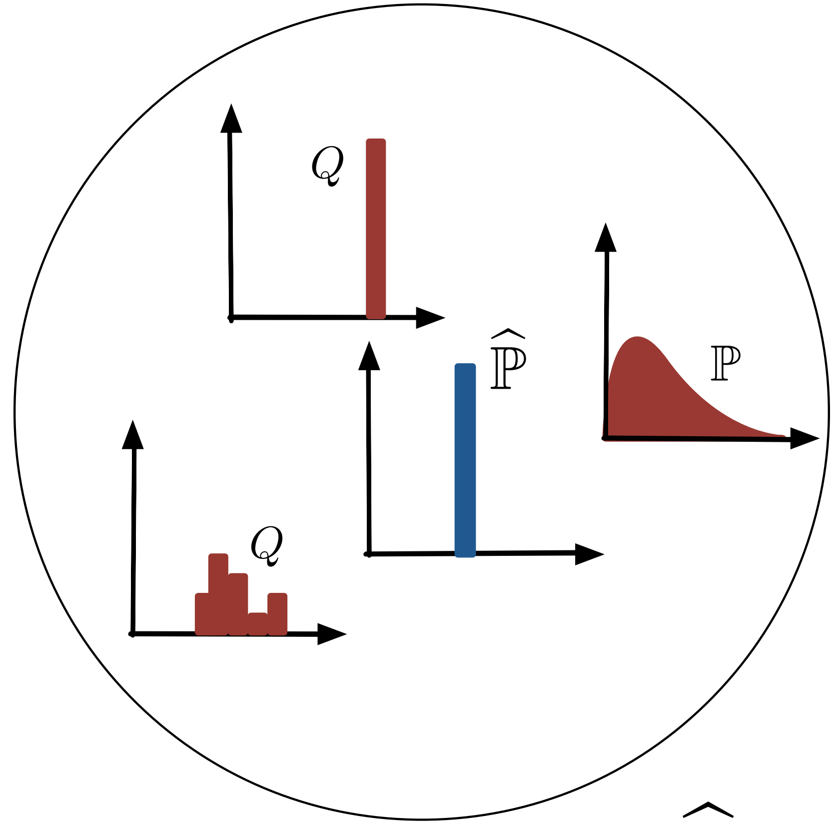

$B_\epsilon(\widehat{\mathbb{P}})$

Contains distributions that could have generated the data

Distributionally Robust Optimization

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Abstracted DeePC

DeePC + slack

$$\min_{g,u,y} \quad \sum_{k=0}^{T_{\mathsf{f}}-1} f(u_k, y_k) + h(\sigma_y)$$

$$\text{s.t.} \quad \begin{bmatrix} U_{\mathsf{p}} \\ Y_{\mathsf{p}} \\ U_{\mathsf{f}} \\ Y_{\mathsf{f}} \end{bmatrix} g = \begin{bmatrix} u_{\mathsf{ini}} \\ y_{\mathsf{ini}} + \sigma_y \\ u \\ y \end{bmatrix}$$

$$u_k \in \mathcal{U}, \ \forall k \in \{0, \ldots, T_{\mathsf{f}} - 1\}$$

Abstracted DeePC

DeePC + slack

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with $\hat{\xi} = \begin{bmatrix} U_{\mathsf{p}}^\top & Y_{\mathsf{p}}^\top & U_{\mathsf{f}}^\top & Y_{\mathsf{f}}^\top \end{bmatrix}^\top$

$$G = \left\{ g \;\middle|\; \begin{bmatrix} U_{\mathsf{p}} \\ U_{\mathsf{f}} \end{bmatrix} g = \begin{bmatrix} u_{\mathsf{ini}} \\ u \end{bmatrix}, u \in \mathcal{U}^{T_{\mathsf{f}}} \right\}$$

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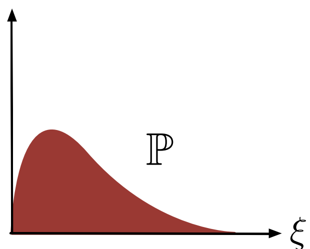

True Distribution

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DeePC + slack

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What we care about:

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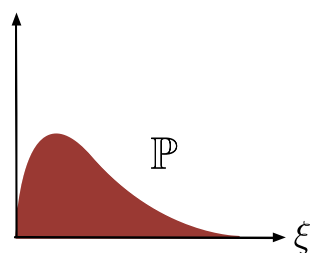
What DeePC is solving:

$$\min_{g \in G} \ c(\hat{\xi}, g)$$



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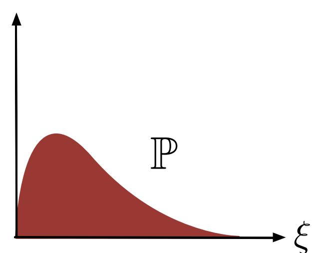

True Distribution

Abstracted DeePC

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$$
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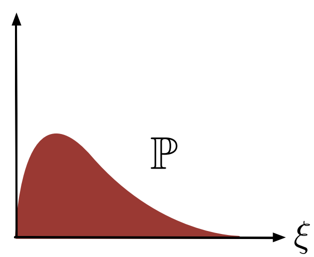

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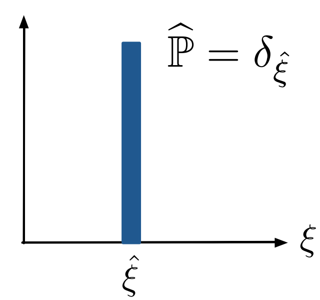


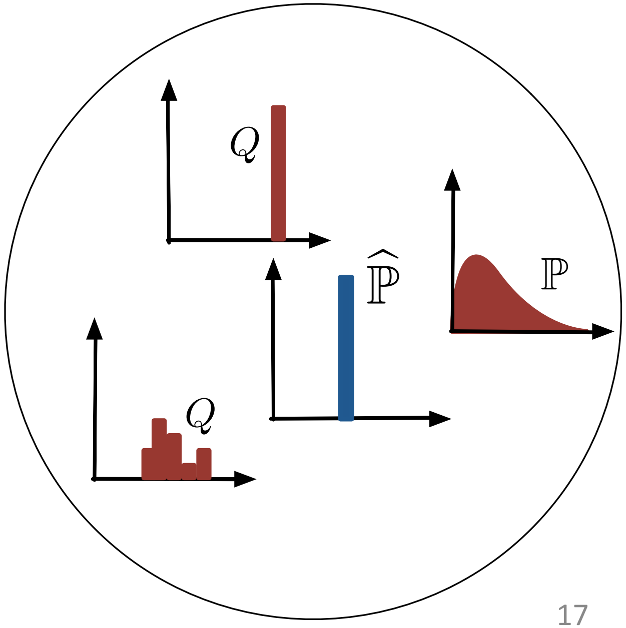
Empirical Distribution

Distributionally Robust DeePC

Theorem [Coulson et al. '21]

Under minor technical conditions

$$\inf_{g \in G} \sup_{Q \in B_\epsilon(\widehat{\mathbb{P}})} \mathbb{E}_Q[c(\xi, g)]$$



Proof uses methods from [Mohajerin Esfahani, Kuhn '18]

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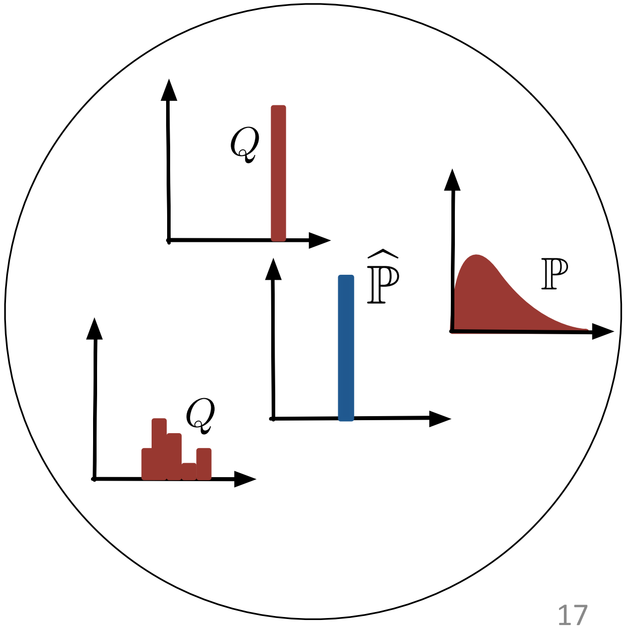

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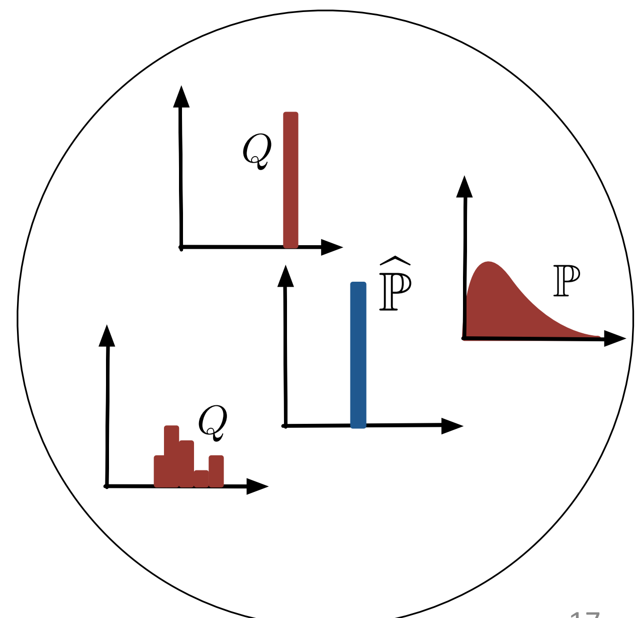
p-norm robustness $\iff$ q-norm regularization

$$\frac{1}{p} + \frac{1}{q} = 1$$



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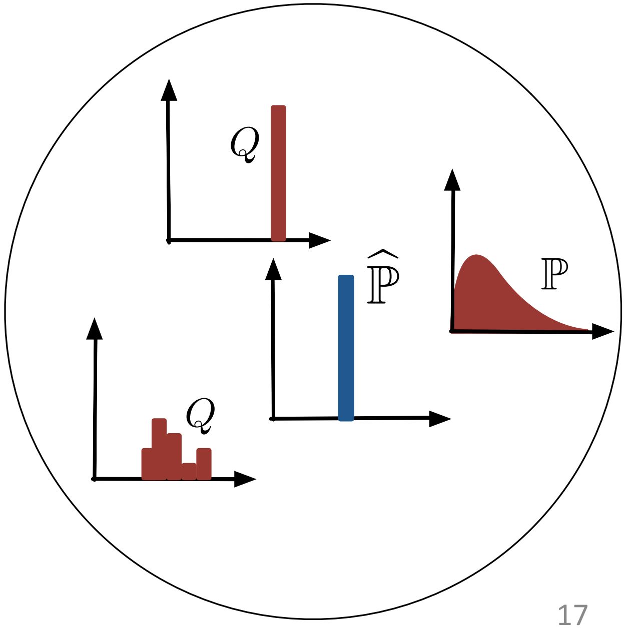
p-norm robustness $\Longleftrightarrow$ q-norm regularization

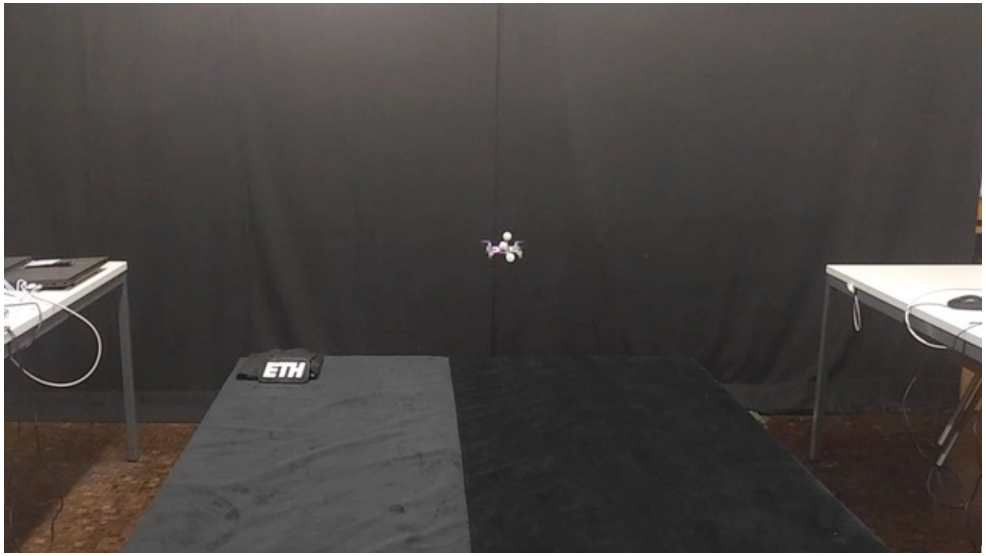
$$\frac{1}{p}+\frac{1}{q}=1$$

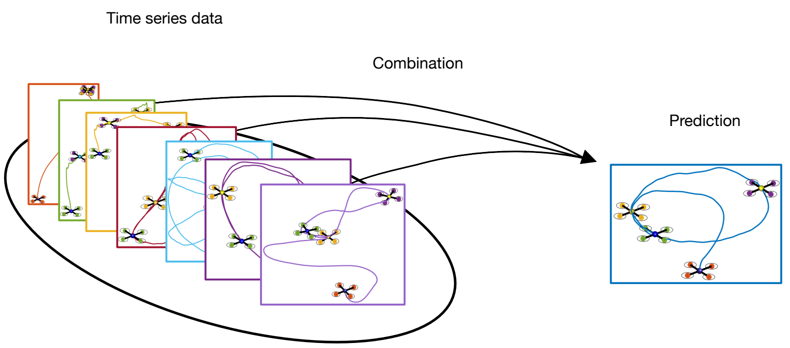
The Wasserstein ball contains more than just LTI systems with additive noise

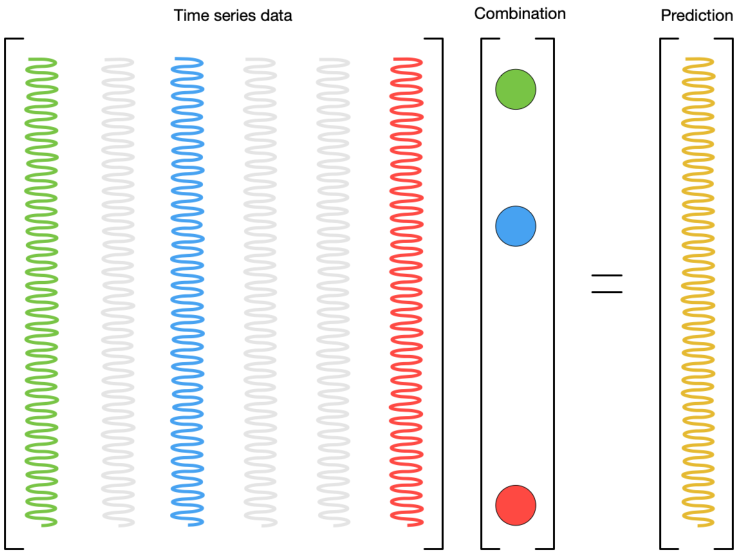


Proof uses methods from [Mohajerin Esfahani, Kuhn '18]

Into the real world [Elokda, Coulson et al. '21]



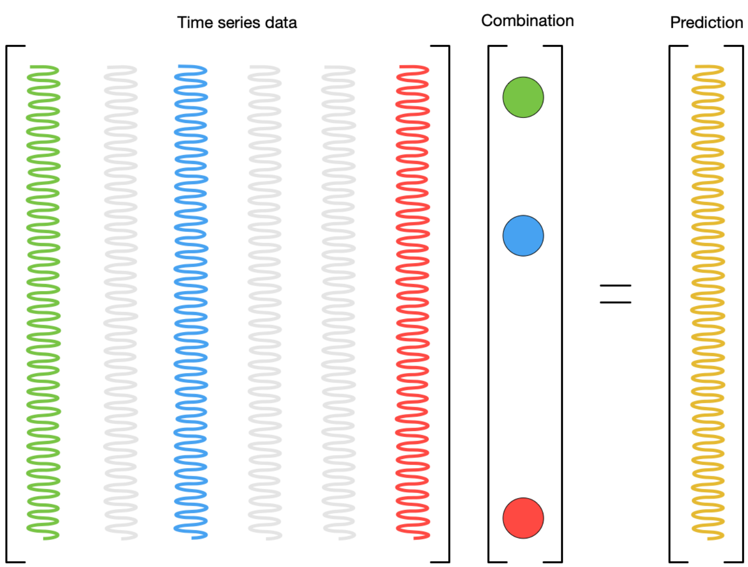


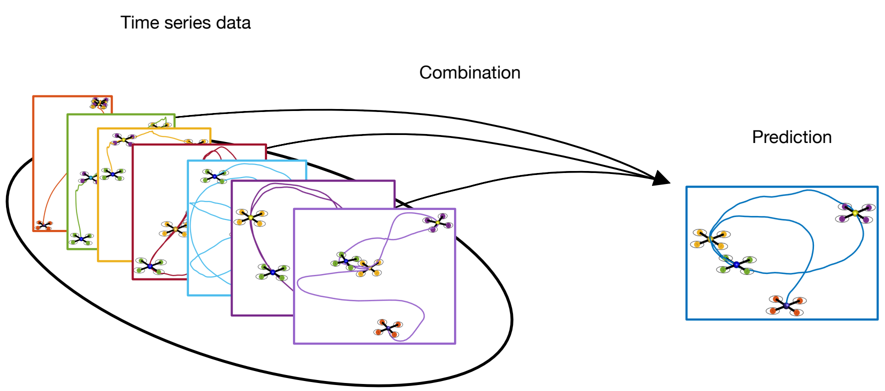


Summary

- Matrix of time-series data is a predictive model
- DeePC equivalent to MPC for deterministic LTI
- Distributionally robustify DeePC through regularizations to extend beyond deterministic LTI
- Successful deployment on many experiments and simulations

Future work: online + adaptive guarantees





Thanks!

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