(1)

Calcul de F_{ox} :

$F_{ox}(i,j,k)$ correspond à la force interne pour une fonction test $w^x = \begin{pmatrix}1\\0\\0\end{pmatrix} \cdot w_{ijk} = \begin{pmatrix}1\\0\\0\end{pmatrix} \underbrace{l_i(\xi)\, l_j(\eta) \cdot l_k(\zeta)}_{w_{ijk}}$.

$$\int_{\Omega^e} \bar{\bar{\sigma}} : \nabla w^x \, d\Omega = \int_{\Omega^e} \bar{\bar{\sigma}} : \begin{pmatrix} \frac{\partial w_{ijk}}{\partial x} & \frac{\partial w_{ijk}}{\partial y} & \frac{\partial w_{ijk}}{\partial z} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\Omega .$$

$$F_{ox}(i,j,k) = \int_{\Omega^e} \left[\sigma_{xx} \cdot \frac{\partial w_{ijk}}{\partial x} + \sigma_{xy} \frac{\partial w_{ijk}}{\partial y} + \sigma_{xz} \frac{\partial w_{ijk}}{\partial z} \right] d\Omega .$$

changement de repère / variables

$$= \int_{\square} \left(\sigma_{xx} \cdot \frac{\partial w_{ijk}}{\partial x} + \sigma_{xy} \frac{\partial w_{ijk}}{\partial y} + \sigma_{xz} \frac{\partial w_{ijk}}{\partial z} \right) |J| \, d\xi \, d\eta \, d\zeta$$

chain-rule

$$= \int_{\square} \Bigg[\sigma_{xx} \left(\frac{\partial w_{ijk}}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial w_{ijk}}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} + \frac{\partial w_{ijk}}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial x} \right)$$
$$+ \sigma_{xy} \left(\frac{\partial w_{ijk}}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial w_{ijk}}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} + \frac{\partial w_{ijk}}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial y} \right)$$
$$+ \sigma_{xz} \left(\frac{\partial w_{ijk}}{\partial \xi} \cdot \frac{\partial \xi}{\partial z} + \frac{\partial w_{ijk}}{\partial \eta} \cdot \frac{\partial \eta}{\partial z} + \frac{\partial w_{ijk}}{\partial \zeta} \frac{\partial \zeta}{\partial z} \right) \Bigg] \cdot |J| \cdot d\xi \, d\eta \, d\zeta .$$

réarrangement et regroupement selon les facteurs de $\frac{\partial w_{ijk}}{\partial \xi}$, $\frac{\partial w_{ijk}}{\partial \eta}$ et $\frac{\partial w_{ijk}}{\partial \zeta}$.

$$= \int_{\square} \Bigg[\frac{\partial w_{ijk}}{\partial \xi} \left(\sigma_{xx} \frac{\partial \xi}{\partial x} + \sigma_{xy} \frac{\partial \xi}{\partial y} + \sigma_{xz} \frac{\partial \xi}{\partial z} \right)$$
$$+ \frac{\partial w_{ijk}}{\partial \eta} \left(\sigma_{xx} \frac{\partial \eta}{\partial x} + \sigma_{xy} \frac{\partial \eta}{\partial y} + \sigma_{xz} \frac{\partial \eta}{\partial z} \right)$$
$$+ \frac{\partial w_{ijk}}{\partial \zeta} \left(\sigma_{xx} \frac{\partial \zeta}{\partial x} + \sigma_{xy} \frac{\partial \zeta}{\partial y} + \sigma_{xz} \frac{\partial \zeta}{\partial z} \right) \Bigg] \cdot |J| \cdot d\xi \cdot d\eta \cdot d\zeta$$

quadrature

$$= \sum_{r,s,t=0}^{ngll-1} \Bigg[\frac{\partial w_{ijk}}{\partial \xi}(\xi_r, \eta_s, \zeta_t) \cdot \left(\sigma_{xx} \frac{\partial \xi}{\partial x} + \sigma_{xy} \frac{\partial \xi}{\partial y} + \sigma_{xz} \frac{\partial \xi}{\partial z} \right)(\xi_r, \eta_s, \zeta_t)$$
$$+ \frac{\partial w_{ijk}}{\partial \eta}(\xi_r, \eta_s, \zeta_t) \cdot \left(\sigma_{xx} \frac{\partial \eta}{\partial x} + \sigma_{xy} \frac{\partial \eta}{\partial y} + \sigma_{xz} \frac{\partial \eta}{\partial z} \right)(\xi_r, \eta_s, \zeta_t)$$
$$+ \frac{\partial w_{ijk}}{\partial \zeta}(\xi_r, \eta_s, \zeta_t) \cdot \left(\sigma_{xx} \frac{\partial \zeta}{\partial x} + \sigma_{xy} \frac{\partial \zeta}{\partial y} + \sigma_{xz} \frac{\partial \zeta}{\partial z} \right)(\xi_r, \eta_s, \zeta_t) \Bigg] \cdot |J(\xi_r, \eta_s, \zeta_t)| \cdot \omega_r \omega_s \cdot \omega_t$$

On écrit ensuite $w_{ijk} = l_i(\xi) \cdot l_j(\eta) \cdot l_k(\zeta)$.

et on remplace dans l'expression ci-dessus.

②

Calcul de F_{0x}

$$
\begin{aligned}
F_{0x}(i,j,k) = \sum_{r,s,t=0}^{ngll-1} \Big[& \ell'_i(\xi_r)\,\overbrace{\ell_j(\eta_s)}^{\delta_{js}}\,\overbrace{\ell_k(\zeta_t)}^{\delta_{kt}} \cdot \left(\left(\sigma_{xx}\frac{\partial \xi}{\partial x} + \sigma_{xy}\frac{\partial \xi}{\partial y} + \sigma_{xz}\frac{\partial \xi}{\partial z}\right)|J|\right)(\xi_r,\eta_s,\zeta_t)\ .\ \omega_r\,\omega_s\,\omega_t \\
&+ \overbrace{\ell_i(\xi_r)}^{\delta_{ir}}\,\ell'_j(\eta_s)\,\overbrace{\ell_k(\zeta_t)}^{\delta_{kt}} \cdot \left(\left(\sigma_{xx}\frac{\partial \eta}{\partial x} + \sigma_{xy}\frac{\partial \eta}{\partial y} + \sigma_{xz}\frac{\partial \eta}{\partial z}\right)\cdot|J|\right)(\xi_r,\eta_s,\zeta_t)\ .\ \omega_r\,\omega_s\,\omega_t \\
&+ \overbrace{\ell_i(\xi_r)}^{\delta_{ir}}\,\overbrace{\ell_j(\eta_s)}^{\delta_{js}}\,\ell'_k(\zeta_t) \cdot \left(\left(\sigma_{xx}\frac{\partial \zeta}{\partial x} + \sigma_{xy}\frac{\partial \zeta}{\partial y} + \sigma_{xz}\frac{\partial \zeta}{\partial z}\right)\cdot|J|\right)(\xi_r,\eta_s,\zeta_t)\ \omega_r\,\omega_s\,\omega_t\ . \Big]
\end{aligned}
$$

$$
\begin{aligned}
= \sum_{r=0}^{ngll-1} & \Bigg[\overset{dx}{\ell'_i(\xi_r)} \times \overbrace{\left(\underbrace{\left(\overset{sxx}{\sigma_{xx}}\,\overset{xi1}{\frac{\partial \xi}{\partial x}} + \overset{sxy}{\sigma_{xy}}\,\overset{xi2}{\frac{\partial \xi}{\partial y}} + \overset{sxz}{\sigma_{xz}}\,\overset{xi3}{\frac{\partial \xi}{\partial z}}\right)}_{xt1} \cdot \underbrace{\overset{Jac}{|J|}\Big)(\xi_r,\eta_j,\zeta_k)\cdot \overset{poids\,x}{\omega_r}}_{t4} \right)}^{t1} \Bigg]\ \overset{poids\,y}{\omega_j}\,\overset{poids\,z}{\omega_k} \\
+ \sum_{s=0}^{ngll-1} & \Bigg[\overset{dy}{\ell'_j(\eta_s)} \times \underbrace{\left(\underbrace{\left(\overset{sxx}{\sigma_{xx}}\,\overset{et1}{\frac{\partial \eta}{\partial x}} + \overset{sxy}{\sigma_{xy}}\,\overset{et2}{\frac{\partial \eta}{\partial y}} + \overset{sxz}{\sigma_{xz}}\,\overset{et3}{\frac{\partial \eta}{\partial z}}\right)}_{xt2} \cdot \underbrace{\overset{Jac}{|J|}\Big)(\xi_i,\eta_s,\zeta_k)\cdot \overset{poids\,y}{\omega_s}}_{t4} \right)}_{t2} \Bigg]\ \overset{poids\,x}{\omega_i}\,\overset{poids\,z}{\omega_k} \\
+ \sum_{t=0}^{ngll-1} & \Bigg[\overset{dz}{\ell'_k(\zeta_t)} \cdot \underbrace{\left(\underbrace{\left(\overset{sxx}{\sigma_{xx}}\,\overset{ga1}{\frac{\partial \zeta}{\partial x}} + \overset{sxy}{\sigma_{xy}}\,\overset{ga2}{\frac{\partial \zeta}{\partial y}} + \overset{sxz}{\sigma_{xz}}\,\overset{ga3}{\frac{\partial \zeta}{\partial z}}\right)}_{xt3} \underbrace{\overset{Jac}{|J|}\Big)(\xi_i,\eta_j,\zeta_t)\cdot \overset{poids\,z}{\omega_t}}_{t4} \right)}_{t3} \Bigg]\ \overset{poids\,x}{\omega_i}\,\overset{poids\,y}{\omega_j}\ .
\end{aligned}
$$