1.1

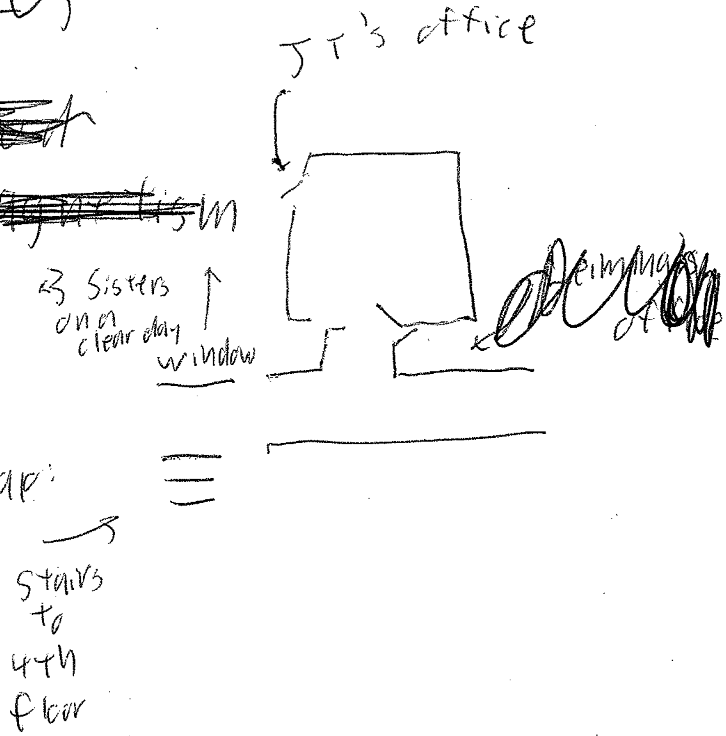
Physics 4~~12~~, 611, Theoretical Mechanics

~~[illegible]~~

~~Electricity and Magnetism~~

Prof.: John Toner

My office: Will. 446: Map:



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~~His Office: Will 441~~

~~" Phone: 6-5236~~

~~Office hrs: [illegible] 4PM~~

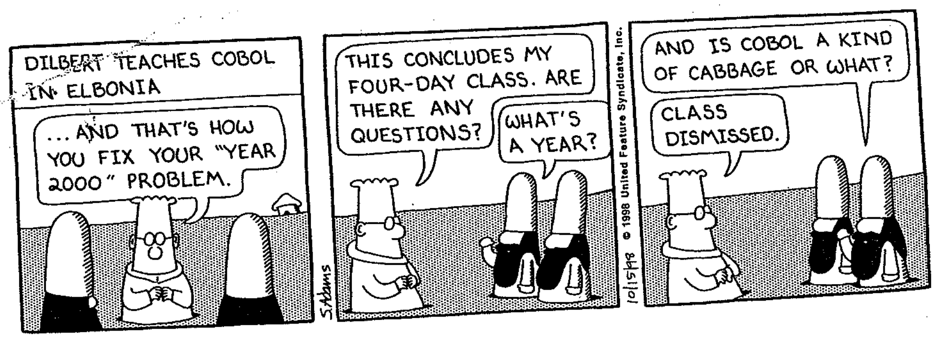
~~Prerequisites: 612, 613, 614 or equivalent (Stat Phys). Knowledge of Fourier transforms will help; bone up if you're not already comfortable with th~~

Grade:	Problem Sets	30%
	Midterm	30%
	Final	30%
	Class Participation:	10%

Note: You've heard it before, but I reiterate: ask questions, make comments, ask me to slow down, etc.

(e.g., "that sounds like bullshit to me, Toner")

The wrong way to ask questions:



The right way: ask a question as soon as you stop following the discussion, rather than waiting until you're totally lost.

I) Point Particle Dynamics

- Conservative Forces $\Rightarrow$ No dissipation

A) "Derivation" of Lagrangian Description

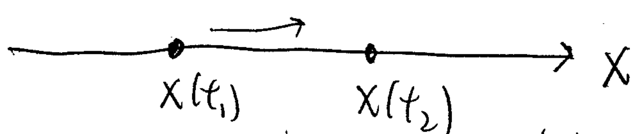
Note: My approach $\neq$ L+L approach

L+L: "Least action Principle" $\Rightarrow$ $F = ma$

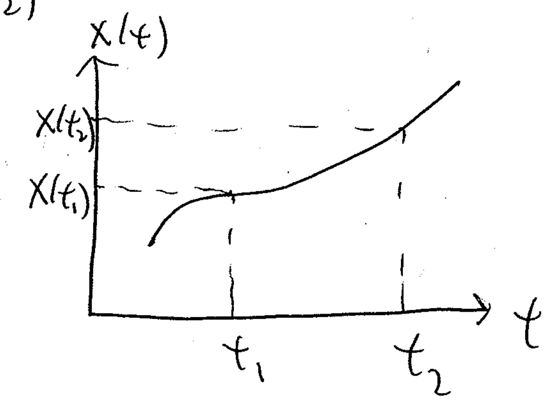
Me: $F = ma$ $\Rightarrow$ "Least action principle"

Start with simplest case:

- Single particle
- One dimensional motion:



We seek $x(t)$:



Freshman Physics:

What is "law of motion" that determines $x(t)$?

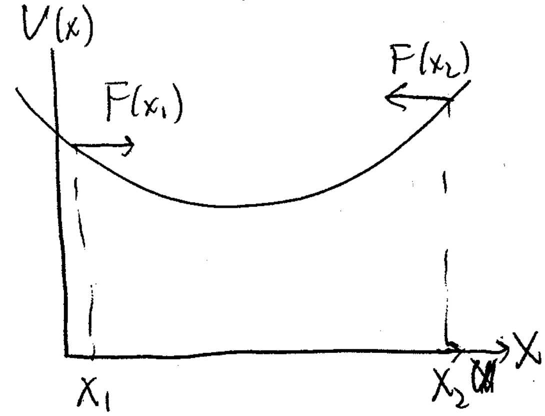
Ans: Newton's 3rd Law:

$F(t) = m\,a(t)$

What is $a(t)$ in terms of $x(t)$?

" " $F(t)$ " " " $x(t)$?

Conservative system $\Rightarrow$ Potential Energy $U(x)$



$$F = -\frac{dU(x)}{dx}$$

$$\Rightarrow \boxed{m\ddot{x} = -\frac{dU(x)}{dx} \qquad (1.1)}$$

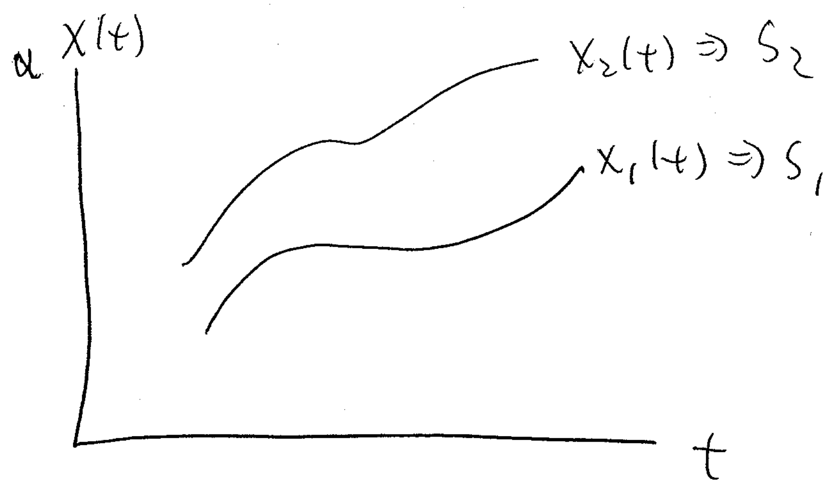
Newton's law of motion (1d, 1 particle, conservative system)

To find $x(t)$: Merely (!) solve (1.1)

Idea of Lagrangian dynamics:

Derive Newton's laws from a "minimum principle"

I.e., have some number $S(\{x(t)\})$ that depends on whole path $x(t)$:



(Jargon: $S(\{x(t)\})$ is a functional of $x(t)$)

(Not a function, because functions depend on 1 numbers; $x(t)$ is an ∞ # of #'s)

Since S ~~dep~~ (the "action") depends on whole path, must be integral over whole path

$$S = \int_{t_1}^{t_2} \mathcal{L}(x(t), \dot{x}(t); t)\, dt$$

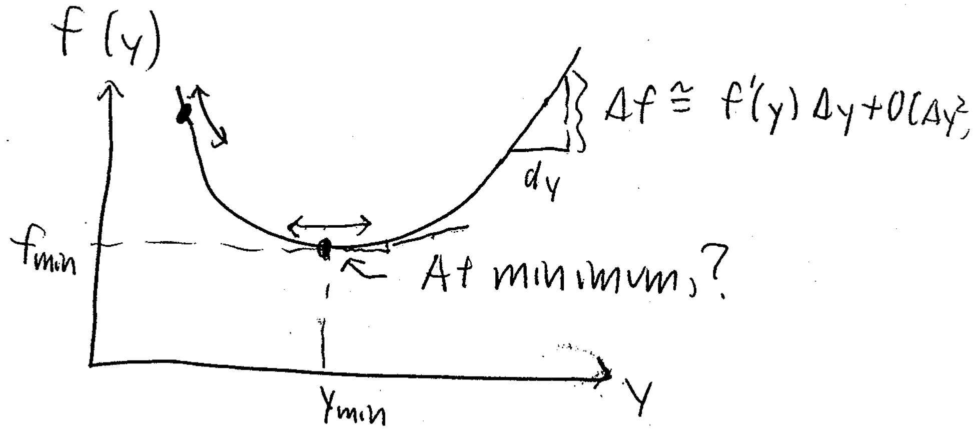
"Action" $\qquad$ "Lagrangian"

~~Now t~~

Minimum principle: To find actual path $x_a(t)$, minimize S over all possible paths $x(t)$.

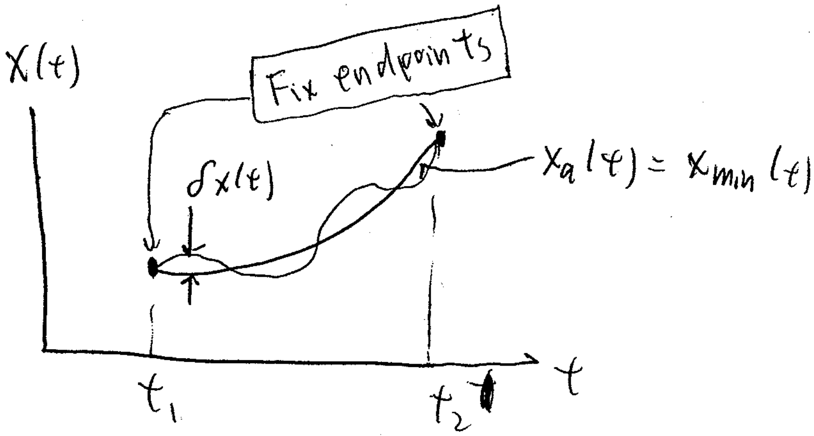
Note: Much harder than calculus

Calculus: Minimizing function



"Calculus of variations"

Minimizing functional



Condition on $x_{min}(t)$?

Calculate $\delta S(\{\delta x(t)\})$; make it $= 0(+O(\delta x^2))$ for all $\delta x(t)$ (small)

$$\overset{\text{trial}}{\downarrow} \qquad\qquad \overset{\text{minimum we're looking for}}{\downarrow}$$

$$x_T(t) = x(t) + \delta x(t)$$

$$S(\{x_T(t)\}) = \int_{t_0}^{t_f} \underbrace{\mathcal{L}(x(t) + \delta x(t), \dot{x}(t) + \delta\dot{x}(t); t)}\, dt$$

Since δx small, expand in δx

$$\mathcal{L}(x_T(t), \dot{x}_T(t); t) = \mathcal{L}(x(t), \dot{x}(t); t) +$$

$$\Rightarrow \delta S \equiv S(\{x_T(t)\}) - S(\{x(t)\})$$

$$= \int_{t_0}^{t_f} \left[\frac{\partial \mathcal{L}}{\partial x} \delta x(t) + \frac{\partial \mathcal{L}}{\partial \dot{x}} \delta \dot{x}(t) \right] dt$$

Look at 2nd term:

$$\int_{t_0}^{t_f} \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{x}}(t)}_{\text{Note: function of } t \text{ Why?}} \delta \dot{x}(t)\, dt$$

$$\int \text{by parts} = \frac{\partial \mathcal{L}}{\partial \dot{x}}(t)\, \cancel{\delta x(t)}\Big]_{t_0}^{t_f} - \int_{t_0}^{t_f} \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\right] \delta x(t)\, dt$$

0 why?

Note: same as coefficient of 1st term

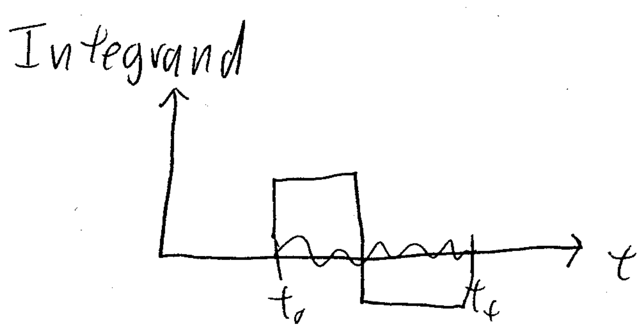
$$\Rightarrow \delta S = \int_{t_0}^{t_f} \left[\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\right] \delta x(t)\, dt$$

$$= 0$$

minimum condition

Tricky condition: $\int = 0$

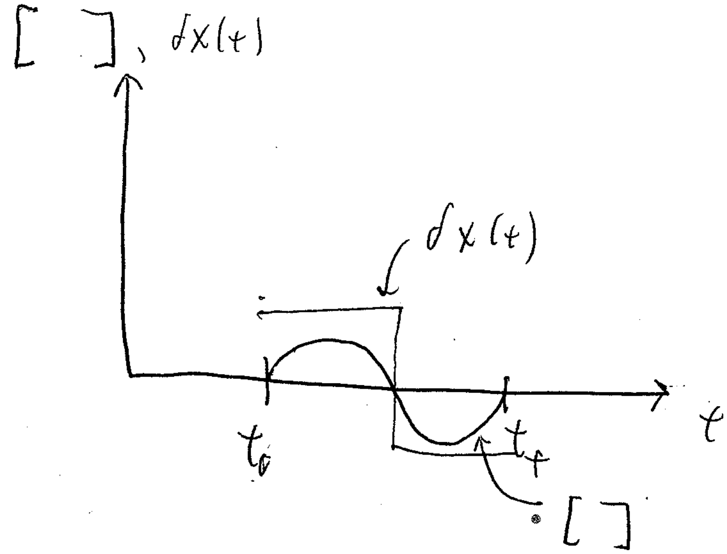
Seems like ∞'y many ways to accomplish:



But: $\delta x(t)$ <u>arbitrary</u>

$\int = 0$ for <u>any</u> $\delta x(t)$

So, suppose, e.g.:



Devil's advocate: I can <u>choose</u> $\delta x(t)$ to <u>always</u> have same sign as [ ]

$\Rightarrow [\ ]\, \delta x(t) > 0$ always

$\Rightarrow \int > 0$

Only way to beat the devil: $[\ ] = 0$ <u>everywhere</u>

(Turns $\int$ condition into local condition)

So

$$[\;] = \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0$$

$$\Rightarrow \boxed{(1.2)} \quad \boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x}}$$

: "Euler-Lagrange Equation"

($f'(y)=0$ of calculus of variations)

General result: minimizes <u>any</u>

$$S = \int_{t_0}^{t_f} \mathcal{L}(x(t), \dot{x}(t); t)\, dt$$

Now, back to physics:

Seek an $\mathcal{L}(x(t), \dot{x}(t); t)$ such that E-LG eqn

(1.2) $\Rightarrow$ $F=ma$ (1.1)

Comparison:

$$F = ma: \quad m\ddot{x}(t) = -\frac{dU(x)}{dx}$$

$$\qquad\qquad \| \qquad\qquad \| \leftarrow \text{let's try}$$

$$E\text{-}LG: \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

$$RHS: \quad \frac{\partial \mathcal{L}(x,\dot{x})}{\partial x} = -\frac{dU(x)}{dx} \Rightarrow$$

$$LHS: \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m\ddot{x} \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{x}} =$$

combine $\Rightarrow$ $\boxed{\mathcal{L} = \frac{1}{2}m\dot{x}^2 - U(x)}$

General expression for Lagrangian:

$$\boxed{\mathcal{L} = T - U}$$

$T = \frac{1}{2}m\dot{x}^2$ = kinetic energy

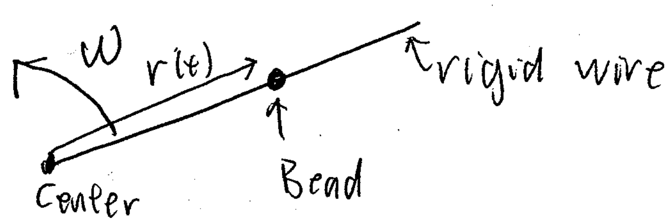
U = Potential energy

Why bother? ~~And~~ Since we already know $F=ma$?

Ans: Minimum principle works even for <u>constrained</u> systems.

Can write down $\mathcal{L}$, then turn Euler-Lagrange crank, without <u>ever</u> determining constraining forces.

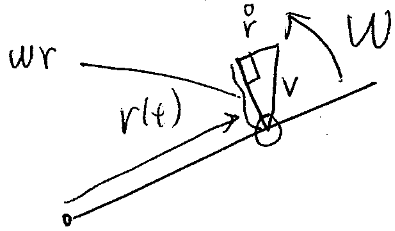
Example: Bead on spinning wire:



$U(r) = ?$

$T(r, \dot{r}) =$

Bead on wire:



$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (\omega^2 r^2 + \dot r^2)$$

$$U = 0$$

$$\Rightarrow \mathcal{L} = T - U = T = \frac{1}{2} m (\omega^2 r^2 + \dot r^2)$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial r} = m \omega^2 r$$

$$\frac{\partial \mathcal{L}}{\partial \dot r} = m \dot r$$

$$\cancel{\Rightarrow \frac{d}{dt}(m \dot r) = m \ddot r = m \omega^2 r}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot r}\right) = m \ddot r = \frac{\partial \mathcal{L}}{\partial r} = m \omega^2 r$$

$$\Rightarrow \boxed{\ddot r = \omega^2 r} \quad (2.1)$$

$\Rightarrow$ bead accelerates <u>out</u>
"centrifugal force"

Comments:

1) I Derived $\mathcal{L}$ using the fact that

T was independent of x.

Here, T <u>depends</u> on r.

Why's this still work?

Ans: I <u>could</u> have used $x(t), y(t)$ as variables

Derive $\mathcal{L} = \frac{1}{2} m v^2$

Then I know

~~$S = \int \mathcal{L}\, dt$~~

$S = \int \mathcal{L}(x, y, \dot{x}, \dot{y}, t)\, dt$ is minimized

<u>Subject</u> to constraint (bead stays on wire)

But: If it's minimized ~~for~~ over $x(t), y(t)$,

I can change variables to $r = \sqrt{x^2+y^2}$,

and it's <u>still</u> a minimum over new variable i.e.,

$$S = \int \mathcal{L}(r(t), \dot{r}(t); t)\, dt$$

∇ Minimum over new variable still determined by E-L G eqn:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{\partial \mathcal{L}}{\partial r}$$

since this did not depend on the form of $\mathcal{L}$ (only that it only depended on $(r, \dot{r})$).

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2) # of independent variables ~~# of dimension~~

= # of dimensions × # of particles − # of constraints

- # of dimensions = 2 for bead on wire
- # of particles = 1 for bead
- # of constraints = 1 here (stays on wire)

= 1 here (namely r)

In general, call variables $\{q_i\}$, $i = 1, 2, \ldots$ # of independent variables

3) Choice of independent variable

<u>arbitrary</u> (like picking variable of integration in calculus)

e.g., for bead, could have used $x(t)$, or $y(t)$,

or $\dfrac{x^{1.8}(t) + y^{3.9}(t)}{\text{Grandma's age}}$.

"Best" choice: one that gives easiest equation to solve.

Here, $r(t)$ was best, because (2.1) was <u>linear</u> (easiest $F=ma$ to solve)

How, in general, do you find "best" choice?

Ans #1: By guess and by gosh

" #2: Sometimes, you don't.

[redacted]

So far, derived ~~$\mathcal{L}$ for 1 inde~~ E-LG for $\mathcal{L}(q_1(t), \dot{q}(t))$ with 1 q.

What about many variable system? $\{q_i\}$

Simple: Just do prior E-LG derivation for each q_i separately!

$$q_1 \to q_1 + \delta q_1(t),\ q_2 \to q_2,\ q_3 \to q_3, \cdots$$

$$\Rightarrow \frac{\partial \mathcal{L}(q_1, \overbrace{q_2, q_3, \cdots q_N}^{\text{all held fixed}})}{\partial q_1} = \frac{d}{dt}\left[\frac{\partial \mathcal{L}(q_1, \overbrace{q_2, q_3, \cdots q_N}^{\text{all held fixed}}, \dot{q}_1, \overbrace{\dot{q}_2, \cdots \dot{q}_N}^{\text{all held fixed}})}{\partial \dot{q}_1}\right]$$

Same for q_2

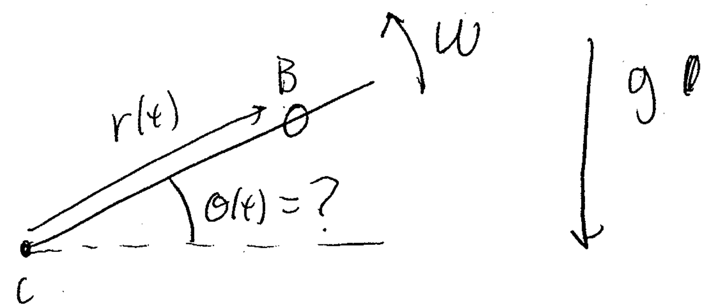
$$\frac{\partial \mathcal{L}}{\partial q_2} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right)$$

In general,

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$$: How many equations?
" " unknowns?

Now, some examples:

First, put bead on wire in <u>vertical</u> plane, with gravity:



$T = ?$

$U = ?$

$\mathcal{L} = T - U =$

Note: explicit dependence on t

Equations of motion (E-LG equations)