Part I

Exercise 1:

$$|\psi\rangle_I = U_0(0,t)|\psi\rangle_S = e^{iH_0 t/\hbar}|\psi\rangle_S$$

$$A_I(t) = U_0(0,t) A_S U_0(t,0)$$
$$= e^{iH_0 t/\hbar} A_S e^{-iH_0 t/\hbar}$$

$$\partial_t |\psi\rangle_I = \partial_t U_0(0,t)|\psi\rangle_S + U_0(0,t)\partial_t|\psi\rangle_S$$

$$\partial_t |\psi\rangle_I = \frac{i}{\hbar} H_0 |\psi\rangle_I - \frac{i}{\hbar} U_0(0,t) H |\psi\rangle_S$$

$$\partial_t |\psi\rangle_I = \frac{-i}{\hbar}\left(U_0(0,t) H U_0(t,0) - H_0 \right)|\psi\rangle_I$$

$$\partial_t |\psi\rangle_I = \frac{-i}{\hbar} U_0(0,t)\left(H - H_0\right) U_0(t,0)|\psi\rangle_I$$

$$\boxed{\partial_t |\psi\rangle_I = \frac{-i}{\hbar} V_I |\psi\rangle_I}$$

$$\partial_t A_I = \partial_t \left(e^{iH_0 t/\hbar} A_S e^{-iH_0 t/\hbar} \right)$$

$$\partial_t A_I = \frac{i}{\hbar} H_0 e^{iH_0 t/\hbar} A_S e^{-iH_0 t/\hbar} - \frac{i}{\hbar} e^{iH_0 t/\hbar} A_S e^{-iH_0 t/\hbar} H_0$$

$$\partial_t A_I = -\frac{i}{\hbar} \left(A_I H_0 - H_0 A_I \right)$$

$$\boxed{\partial_t A_I = -\frac{i}{\hbar} [A_I, H_0]}$$

Exercise 2:

$$G^+(E) = -\lim_{\delta \to 0^+} \frac{i}{\hbar} \int_0^\infty d\tau \, e^{i(E-H)\tau/\hbar} e^{-\tau\delta/\hbar}$$

$$G^+(E) = -\lim_{\delta \to 0^+} \frac{i}{\hbar} \left[\frac{e^{i(E-H)\tau/\hbar} e^{-\tau\delta/\hbar}}{\left(\frac{i}{\hbar}(E-H) - \frac{\delta}{\hbar}\right)} \right]_0^\infty$$

$$G^+(E) = \lim_{\delta \to 0^+} \left[\frac{1}{E - H + i\delta} \right]$$

$$\boxed{G^+(E) = G(E + i0^+)}$$

Exercise 3:

$$G^-(E) = \lim_{\delta \to 0^+} \frac{i}{\hbar} \int_{-\infty}^{0} d\tau \, e^{i(E-H)\tau/\hbar} e^{\tau\delta/\hbar}$$

$$G^-(E) = \lim_{\delta \to 0^+} \frac{i}{\hbar} \left[\frac{e^{i(E-H)\tau/\hbar} \, e^{\tau\delta/\hbar}}{\frac{i}{\hbar}(E-H) + \frac{\delta}{\hbar}} \right]_{-\infty}^{0}$$

$$G^-(E) = \lim_{\delta \to 0^+} \left[\frac{1}{E - H - i\delta} \right]$$

$$\boxed{G^-(E) = G(E - i0^+)}$$

Exercise 4:

$$i\hbar\,\partial_t G^{\pm}(t,t_0) = \pm i\hbar\,\partial_t\left(U(t,t_0)\,\theta(\pm(t-t_0))\right)$$

$$i\hbar\,\partial_t G^{\pm}(t,t_0) = \pm i\hbar\,\theta(\pm(t-t_0))\,\partial_t U(t,t_0) \pm i\hbar\,U(t,t_0)\,\partial_t\,\theta(\pm(t-t_0))$$

$$i\hbar\,\partial_t G^{\pm}(t,t_0) = \pm\,\theta(\pm(t-t_0))\,H\,U(t,t_0) + i\hbar\,U(t,t_0)\,\delta(t-t_0)$$

$$\boxed{i\hbar\,\partial_t G^{\pm}(t,t_0) = H\,G^{\pm}(t,t_0) + i\hbar\,\delta(t-t_0)}$$

Part II

Exercise 1:

$$\delta_t(E_{fi}) = \frac{1}{2\pi\hbar}\int_{-t/2}^{t/2} dt' \, e^{iE_{fi}t'/\hbar}$$

$$\delta_t(E_{fi}) = \frac{1}{2\pi\hbar}\left[\frac{e^{iE_{fi}t/2} - e^{-iE_{fi}t/2}}{iE_{fi}/\hbar}\right]$$

$$\delta_t(E_{fi}) = \frac{t}{2\pi\hbar}\left[\frac{e^{iE_{fi}t/2\hbar} - e^{-iE_{fi}t/2\hbar}}{2itE_{fi}/2\hbar}\right]$$

$$\delta_t(E_{fi}) = \frac{t}{2\pi\hbar}\,\frac{\sin(E_{fi}t/2\hbar)}{E_{fi}t/2\hbar}$$

$$\boxed{\delta_t(E_{fi}) = \frac{t}{2\pi\hbar}\,\mathrm{sinc}(E_{fi}t/2\hbar)}$$

Exercise 2:

$$\tilde{K}_{fi}(t) = -\frac{1}{\hbar^2}\sum_K V_{fK}V_{Ki}\int_0^t dt_2\int_0^{t_2} dt_1\, e^{iE_{fK}t_2/\hbar}\, e^{iE_{Ki}t_1/\hbar}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} dE\ \frac{e^{-iE(t_2-t_1)/\hbar}}{E+i0^+-E_K} = -e^{iE_K(t_2-t_1)/\hbar}\,\theta(t_2-t_1)$$

$$\rightarrow \tilde{K}_{fi}(t) = \frac{1}{2\pi i\hbar^2}\sum_K V_{fK}V_{Ki}\int_0^t dt_2\int_0^t dt_1\int_{-\infty}^{\infty} dE\ \frac{e^{i(E_f-E)t_2/\hbar}\, e^{i(E_i-E)t_1/\hbar}}{E+i0^+-E_K}$$

$$\rightarrow \tilde{K}_{fi}(t) = -2\pi i\sum_K V_{fK}V_{Ki}\int_{-\infty}^{\infty} dE\ \delta_t(E_f-E)\delta_t(E-E_i)\cdot\frac{e^{i(E_f-E)t/2\hbar}\, e^{i(E-E_i)t/2\hbar}}{E+i0^+-E_K}$$

$$\tilde{K}_{fi}(t) \approx -2\pi i\sum_K \frac{V_{fK}V_{Ki}}{E_i-E_K+i0^+}\int_{-\infty}^{\infty} dE\ \delta_t(E_f-E)\delta_t(E-E_i)e^{iE_{fi}t/2\hbar}$$

$$\boxed{\tilde{K}_{fi}(t) \approx -2\pi i\sum_K \frac{V_{fK}V_{Ki}}{E_i-E_K+i0^+}\, e^{iE_{fi}t/2\hbar}\,\delta_t(E_{fi})}$$