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1)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\kappa \epsilon_{\mu\nu}{}^{\kappa\lambda} \tilde{A}_\lambda$$

$$\mathcal{L} = \frac{-1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} A_\mu J^\mu - \frac{1}{c} \tilde{A}_\mu \tilde{J}^\mu$$

a)

We know that $\mathcal{L}$ satisfies

$$\partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} = \frac{\partial \mathcal{L}}{\partial A_\beta}$$

and

$$\partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \tilde{A}_\beta)} = \frac{\partial \mathcal{L}}{\partial \tilde{A}_\beta} \qquad (*)$$

We first write $F^{\mu\nu}$ as

$$F^{\mu\nu} = g^{\alpha\mu} F_\alpha{}^\nu = g^{\alpha\mu} g^{\beta\nu} F_{\alpha\beta}$$

so

$$\mathcal{L} = \frac{-1}{16\pi} g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu} F_{\alpha\beta} - \frac{1}{c} A_\mu J^\mu - \frac{1}{c} \tilde{A}_\mu \tilde{J}^\mu$$

Now plugging this in to (*), we have

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} = -\frac{1}{16\pi}\left[4\left(\partial^\alpha A^\beta - \partial^\alpha A^\beta\right)\right]$$

$$= -\frac{1}{4\pi} F^{\alpha\beta}$$

(From the text)

$$\frac{\partial \mathcal{L}}{\partial A_\beta} = -\frac{1}{c} J^\beta$$

$$\rightarrow \quad \boxed{\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta}$$

I think using $\frac{\delta S}{\delta \tilde{A}}$ will be easier for this one

let $S = \int dx \, \mathcal{L}$

set $\frac{\delta S}{\delta \tilde{A}} = 0$

$$\frac{\delta S}{\delta \tilde{A}} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dx \Big[-\frac{1}{16\pi} g^{\alpha\mu} g^{\rho\nu} \big(\partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\kappa \epsilon_{\mu\nu}{}^{\kappa\lambda} (\tilde{A} + \epsilon\delta)_\lambda \big)$$

$$\big(\partial_\alpha A_\rho - \partial_\rho A_\alpha + \partial_\sigma \epsilon_{\alpha\rho}{}^{\sigma\tau} (\tilde{A} + \epsilon\delta)_\tau \big)$$

$$- \frac{1}{c} (\tilde{A} + \epsilon\delta)_\mu \tilde{J}^\mu$$

$$+ \frac{1}{16\pi} g^{\alpha\mu} g^{\rho\nu} F_{\mu\nu} F_{\alpha\rho} + \frac{1}{c} \tilde{A}_\mu \tilde{J}^\mu \Big]$$

omitting terms which cancel

$$\frac{\delta S}{\delta \tilde{A}} =$$

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dx \Big[-\frac{\epsilon}{16\pi} g^{\alpha\mu} g^{\rho\nu}$$

$$\times \big(\partial_\kappa \epsilon_{\mu\nu}{}^{\kappa\lambda} \delta_\lambda (\partial_\alpha A_\rho - \partial_\rho A_\alpha + \partial_\sigma \epsilon_{\alpha\rho}{}^{\sigma\tau} \tilde{A}_\tau)$$

$$+ \partial_\sigma \epsilon_{\alpha\rho}{}^{\sigma\tau} \delta_\tau (\partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\kappa \epsilon_{\mu\nu}{}^{\kappa\lambda} \tilde{A}_\lambda)$$

$$- \frac{\epsilon}{c}\, \delta_\mu \tilde{J}^\mu$$

(since terms of order ϵ^2 go to zero)

Now, integrating by parts to remove derivatives from δ's, we have

$$\frac{\delta S}{\delta \tilde{A}}$$

$$= \frac{1}{16\pi} g^{\alpha\mu} g^{\beta\nu} \Big(\partial_\kappa \epsilon_{\mu\nu}{}^{\kappa\lambda} (\partial_\alpha A_\beta - \partial_\beta A_\alpha + \partial_\sigma \epsilon_{\alpha\beta}{}^{\sigma\tau} \tilde{A}_\tau)$$

$$+ \partial_\sigma \epsilon_{\alpha\beta}{}^{\sigma\tau} (\partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\kappa \epsilon_{\mu\nu}{}^{\kappa\lambda} \tilde{A}_\lambda)$$

$$- \frac{1}{c} \tilde{J}^\mu$$

$$= \frac{1}{8\pi} \Big(\partial_\kappa \epsilon^{\alpha\beta\kappa\lambda} (\partial_\alpha A_\beta - \partial_\beta A_\alpha + \partial_\sigma \epsilon_{\alpha\beta}{}^{\sigma\tau} \tilde{A}_\tau)$$

$$- \frac{1}{c} \tilde{J}^\mu$$

$$= \frac{1}{8\pi} (\partial_\kappa \epsilon^{\alpha\beta\kappa\lambda} F_{\alpha\beta}) - \frac{1}{c} \tilde{J}^\mu$$

$$= \frac{1}{8\pi} (\partial_\kappa \epsilon^{\kappa\lambda\alpha\beta} F_{\alpha\beta}) - \frac{1}{c} \tilde{J}^\mu$$

$$= \frac{1}{4\pi} \partial_\kappa F^{\kappa\lambda} - \frac{1}{c} \tilde{J}^\mu = 0$$

I'm not sure what happened to my indices.

b)

let $F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$

Then, by prop 2,

$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$

implies $\nabla \cdot E = 4\pi \rho$

and $-\frac{1}{c}\partial_t E + \nabla \times B = \frac{4\pi}{c} j$

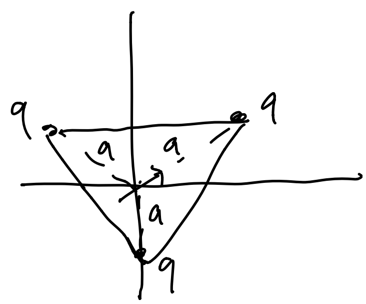
Where $J^\mu = (c\rho, j)$

and $\partial_\mu \tilde{F}^{\mu\nu} = \frac{4\pi}{c} \tilde{J}^\nu$

implies

$\nu = 0$: a bit of a long sum. and I have to do the other problem.

2)



As in the notes, the electrostatic interaction energy to dipole order is

$$U = \phi_0 Q - \vec{E} \cdot \vec{d} + \ldots$$

Where ϕ_0 and $\vec{E}$ are the potential and e-field due to the 3 charges and $\vec{d}$ is the dipole moment of the dipole.

at the origin,

$$\vec{E} = \frac{q}{a^2}\left(\cos\left(\frac{\pi}{6}\right), \sin\left(\frac{\pi}{6}\right), 0\right)$$
$$+ \frac{q}{a^2}\left(\cos\left(\frac{5\pi}{6}\right), \sin\left(\frac{5\pi}{6}\right), 0\right)$$

$$+ \frac{q}{a^2} \left(\cos\left(\frac{3\pi}{2}\right), \sin\left(\frac{3\pi}{2}\right), 0 \right)$$

$$\vec{E} = \vec{0}$$

That doesn't seem right but makes sense given the geometry. So unless the dipole has a total charge,

$$U(\phi) \equiv 0$$

which implies the system is in equilibrium for any angle ϕ.