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1) Bremsstrahlung

a)

Let e be the particle's charge and let $\vec{v}(t)$ be the particle's velocity at time t.

Since the particle has no acceleration for $t<0$ and for $t>T$, and an acceleration of $-a$ for $0<t<T$, we have

$$a(t) = \dot{v}(t) = -a\,\theta(t(T-t))$$

which we can Fourier transform to get $\dot{v}(\omega)$

$$\dot{v}(\omega) = -a \int_{-\infty}^{\infty} \theta(t(T-t))\, e^{i\omega t}\, dt$$

$$\dot{v}(\omega) = -a\int_0^T e^{i\omega t}\, dt$$

$$\dot{v}(\omega) = -a\left[\frac{e^{i\omega t}}{i\omega}\right]_0^T$$

$$\dot{v}(\omega) = \frac{-a}{i\omega}\left[e^{i\omega T} - 1\right]$$

$$\dot{v}(\omega) = -\frac{2a e^{i\omega T/2}}{\omega}\left[\frac{e^{i\omega T/2} - e^{-i\omega T/2}}{2i}\right]$$

$$\dot{v}(\omega) = -2a\, e^{i\omega T/2}\, \frac{\sin(\omega T/2)}{\omega}$$

$$\boxed{\dot{v}(\omega) = -Ta\, e^{i\omega T/2}\, \mathrm{sinc}(\omega T/2)}$$

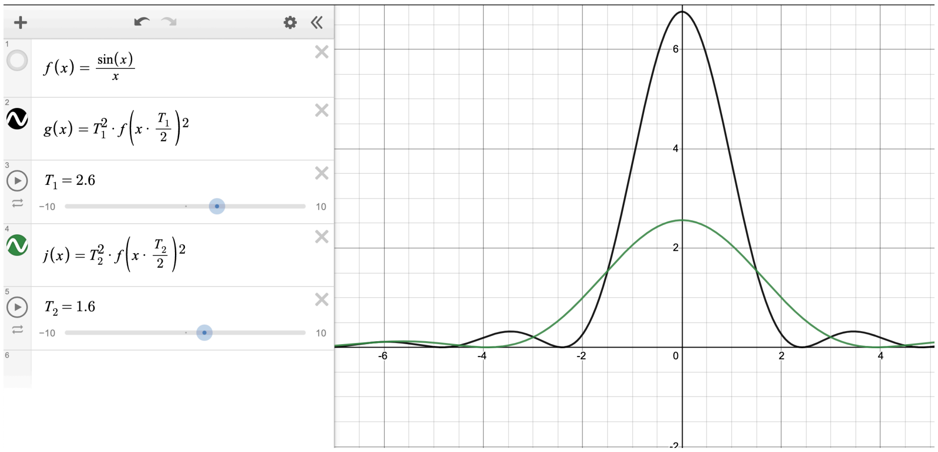
$$\rightarrow \quad |\dot{v}(\omega)|^2 = a^2 T^2\, \mathrm{sinc}^2(\omega T/2)$$

where $\quad \mathrm{sinc}(x) = \dfrac{\sin(x)}{x}$

So we have

$$\boxed{\frac{dU}{d\omega} = \frac{2e^2 a^2 T^2}{3\pi c^3}\, \mathrm{sinc}^2(\omega T/2)}$$

A sketch of the function is below



The radiated frequencies are concentrated around $\omega = 0$ and decay to roughly zero for $\omega > \frac{2\pi}{T}$. Thus, the spectrum is dominated by frequencies ω in $\boxed{-\frac{2\pi}{T} < \omega < \frac{2\pi}{T}}$.

If we wanted to produce radiation of higher frequency we would decrease T.

b)

The total radiated energy is given by

$$U = \int_0^\infty d\omega \, \frac{dU}{d\omega} = \int_0^\infty dt \, P(t)$$

$$U = \frac{2e^2 a^2 T^2}{3\pi c^3} \int_0^\infty d\omega \, \text{sinc}^2(\omega T/2)$$

$$U = \frac{2e^2 a^2 T^2}{3\pi c^3} \int_0^\infty d\omega \, \frac{\sin^2(\omega \frac{T}{2})}{\omega^2 T^2/2^2}$$

$$x = \omega \frac{T}{2}, \quad dx = \frac{T}{2} \, d\omega$$

$$U = \frac{2e^2 a^2 T^2}{3\pi c^3} \cdot \frac{2}{T} \int_0^\infty dx \, \frac{\sin^2(x)}{x^2}$$

$$U = \frac{4e^2 a^2 T}{3\pi c^3} \cdot \frac{\pi}{2}$$

$$U = \frac{2e^2 a^2}{3c^3} T$$

But since $\dot{v}(t)$ is zero for $t > T$, we have

$$\mathcal{U} = \int_0^{\infty} dt \; P(t) = \int_0^{T} dt \; P(t)$$

$$\rightarrow \quad \mathcal{U} = \frac{2e^2 a^2}{3c^3} T = \int_0^{T} dt \; P(t)$$

The radiated energy upto a time t is given by

$$\mathcal{U}(t) = \int_0^{t} dt' \; P(t')$$

Therefore, we have

$\frac{d\mathcal{U}}{dT} = P$, the total power radiated.

$$\boxed{P = \frac{d\mathcal{U}}{dT} = \frac{2e\,a^2}{3c^3}}$$

which is precisely the Larmor formula.

2) Pulsar

$T = 7.5\,s, \quad \dot{T} = 8 \times 10^{-11}$

$M = 3 \times 10^{33}\,g, \quad R = 10^{6}\,cm$

a) $P = \frac{2}{3c^3}\left(\ddot{\vec{m}}(t)\right)^2$

If we assume the star loses energy due purely to magnetic dipole radiation, then

$-\frac{dU}{dt} = P$ where $P = \frac{2}{3c^3}\,\ddot{\vec{m}}(t)^2$

and $U = \frac{I\,\Omega^2}{2}$, $I = \frac{2MR^2}{5}$

Since $\vec{m}(t)$ lies in the xy-plane we can parameterize it by

$\vec{m}(t) = m\left(\cos\left(\frac{2\pi}{T(t)}t\right), \sin\left(\frac{2\pi}{T(t)}t\right), 0\right)$

Since $\dot{T} \ll 1$, we can assume $\dddot{m} \approx -\Omega^2 \dot{m}$ so that

$$P = \frac{2}{3c^3} \ddot{m}^2 = \frac{2}{3c^3} \Omega^4 m^2$$

and $\Omega(t) = \dfrac{2\pi}{T + \dot{T} t}$

Next, we have

$$U(t) = \frac{MR^2}{5} \Omega^2$$

so $$-\frac{dU}{dt} = 2\frac{MR^2}{5} \Omega(t) \dot{\Omega}(t)$$

Now setting $P = -\dfrac{dU}{dt}$, we have

$$\frac{2}{3c^3} \Omega^4 m^2 = -\frac{2}{5} M R^2 \Omega(t) \dot{\Omega}(t)$$

$$\rightarrow \quad m^2 = -\frac{3}{5} M R^2 c^3 \frac{\dot{\Omega}(t)}{\Omega^3}$$

$$m^2 = \frac{3}{5} M R^2 c^3 \frac{\dot{T}\, T(t)}{4\pi^2}$$

$$\rightarrow \boxed{m^2 \approx \frac{3}{20\pi^2} M R^2 c^3 \dot{T} T}$$

b)

$$\vec{B}(\vec{x}) = \frac{3(\hat{x}\cdot\vec{m})\hat{x} - \vec{m}}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right)$$

since this quantity is maximized for $\hat{x} \parallel \vec{m}$, we have

$$B_{max}(R) = \frac{2m}{R^3}$$

$$B_{max}^2 = \frac{4m^2}{R^6}$$

$$B_{max} = \sqrt{\frac{3}{5\pi^2}\frac{M^2}{R^4} c^3 \dot{T} T}$$

which seems quite wrong since I can't get the units to work out.