Exercise 1:

The vector $P|\psi_0, \gamma\rangle$ is not normalized in general because $\{|\psi_0, \gamma\rangle\}_{\gamma=1}^{g}$ and $\{|\psi, \gamma'\rangle\}_{\gamma'=1}^{g}$ don't span the same subspace. Therefore, projections between subspaces may not be length preserving. If they do span the same subspace the $P|\psi_0, \gamma\rangle$ is normalized.

Exercise 2:

$$(P_0 H P P_0)^\dagger = P_0^\dagger P^\dagger H^\dagger P_0^\dagger$$

$$= P_0 P H P_0$$

But since we can pick a basis in which H is diagonal and P projects orthogonally onto this basis (so P is also diagonal), we have $HP = PH$ so $P_0 H P P_0$ is Hermitian.

Exercise 3:

(i) Let $|\psi\rangle \in col(H)$. Suppose $|\psi\rangle$ is not in the orth. complement of P_0, $|\psi\rangle \notin P_0^{\perp}$.

Then $\qquad P_0|\psi\rangle \neq 0$

So $\qquad \langle\psi| P_0 P P_0 |\psi\rangle \neq 0$, since

there is $\qquad |\bar{\psi}\rangle \in col(P)$ such that

$P(P_0|\psi\rangle) = |\bar{\psi}\rangle$. So

$$\langle\psi| P_0 P P_0|\psi\rangle = \langle\psi| P_0 P^2 P_0 |\psi\rangle$$
$$= \langle\bar{\psi}|\bar{\psi}\rangle > 0.$$

(ii) If $|\psi\rangle \in P_0^{\perp}$ then

$\langle\psi| P_0 P P_0 |\psi\rangle = 0.$

Exercise 4:

$$\mathcal{U} = \sum_{\gamma} |\psi, \gamma\rangle \langle \bar{\psi}_0, \gamma|$$

(i)
$$\begin{aligned}
\mathcal{U} P_0 &= \sum_{\alpha} |\psi, \alpha\rangle \langle \bar{\psi}_0, \alpha| \sum_{\beta} |\psi_0, \beta\rangle \langle \psi_0, \beta| \\
&= \sum_{\alpha} |\psi, \alpha\rangle \langle \psi_0, \alpha| \langle \bar{\psi}_0, \alpha | \psi_0, \alpha\rangle \\
&= \sum_{\alpha} |\psi, \alpha\rangle \langle \bar{\psi}_0, \alpha| \\
&= \mathcal{U}
\end{aligned}$$

(ii)
$$\begin{aligned}
P\mathcal{U} &= \sum_{\alpha} |\psi, \alpha\rangle \langle \psi, \alpha| \sum_{\beta} |\psi, \beta\rangle \langle \bar{\psi}_0, \beta| \\
&= \sum_{\alpha} |\psi, \alpha\rangle \langle \bar{\psi}_0, \alpha| \\
&= \mathcal{U}
\end{aligned}$$

(iii)
$$\begin{aligned}
P_0 \mathcal{U} &= \sum_{\alpha} |\psi_0, \alpha\rangle \langle \psi_0, \alpha| \sum_{\beta} |\bar{\psi}, \beta\rangle \langle \psi_0, \beta| \\
&= \sum_{\beta} P_0 |\bar{\psi}, \beta\rangle \langle \psi_0, \beta| \\
&= \sum_{\beta} |\psi_0, \beta\rangle \langle \psi_0, \beta| \\
&= P_0
\end{aligned}$$

(iv) $$UP = \sum_{\alpha} |\psi,\alpha\rangle\langle\bar{\psi}_0,\alpha| \sum_{\beta} |\psi,\beta\rangle\langle\psi,\beta|$$

$$= \sum_{\alpha} |\psi,\alpha| \langle\bar{\psi}_0,\alpha| P$$

$$= \sum_{\alpha} |\psi,\alpha\rangle\langle\psi,\alpha|$$

$$= P$$

Exercise 5:

$$U_n = \sum_{\substack{k_1,\dots,k_n \\ k_1+\dots+k_n = n \\ k_1+\dots+k_j \geq j,\ \forall 0<j<n}} S_{k_1} \vee S_{k_2} \vee \dots \vee S_{k_n} \vee P_0$$

$$U_1 = \sum_{\substack{k_1 \\ k_1 = 1 \\ k_1 \geq 1}} S_{k_1} \vee P_0$$

$$= S_1 \vee P_0 = G_Q^1 \vee P_0$$

$$U_2 = \sum_{\substack{k_1, k_2 \\ k_1+k_2 = 2 \\ k_1 \geq 1}} S_{k_1} \vee S_{k_2} \vee P_0$$

$$= S_1 \vee S_1 \vee P_0 + S_2 \vee S_0 \vee P_0$$

$$= G_Q \vee G_Q \vee P_0 - G_Q^2 \vee P_0 \vee P_0$$

Exercise 6:

$$(E_0 + \lambda P_0 V P_0)|\psi_0\rangle = E|\psi_0\rangle$$

$$\rightarrow \quad \lambda P_0 V P_0 |\psi_0\rangle = (E - E_0)|\psi_0\rangle$$

$$\rightarrow \quad \lambda P_0 V P_0 |\psi_0\rangle = \delta E_1 |\psi_0\rangle$$

$$P_0 = \sum_{\gamma'} |\psi_0, \gamma'\rangle \langle \psi_0, \gamma'|$$

$$\lambda P_0 V \sum_{\gamma'} |\psi_0, \gamma'\rangle \langle \psi_0, \gamma' | |\psi_0\rangle = \delta E_1 |\psi_0\rangle$$

$$\rightarrow \quad \sum_{\gamma'} \lambda P_0 \langle \psi_0, \gamma | V | \psi_0, \gamma'\rangle \langle \psi_0, \gamma' | \psi_0\rangle$$

$$= \langle \psi_0, \gamma | \delta E_1 | \psi_0\rangle$$

$$\delta E_1 \psi_{0,\gamma} = \sum_{\gamma'} \lambda P_0 V_{\gamma\gamma'} \psi_{0,\gamma'}$$

Exercise 7:

$$(E_0 + \lambda P_0 V P_0 + \lambda^2 P_0 V G_Q V P_0)|\psi_0\rangle$$
$$= E|\psi_0\rangle$$
$$E = E_0 + \delta E_1 + \delta E_2$$

$$\rightarrow \quad \lambda^2 P_0 V G_Q V P_0 |\psi_0\rangle = \delta E_2 |\psi_0\rangle$$

$$\lambda^2 \langle \psi_{0,\gamma} | P_0 V \sum_{\alpha \neq 0} \frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}} V P_0 |\psi_0\rangle = \delta E_2 \psi_{0,\gamma}$$

$$\lambda^2 \langle \psi_{0,\gamma} | V \sum_{\alpha \neq 0} \frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}} V \sum_{\gamma'} |\psi_{0,\gamma'}\rangle \psi_{0,\gamma'}$$
$$= \delta E_2 \psi_{0,\gamma}$$

$$\lambda^2 \sum_{\gamma', \alpha \neq 0} \frac{V_{\gamma\alpha} V_{\alpha\gamma'}}{E_{0\alpha}} \psi_{0,\gamma'} = \delta E_2 \psi_{0,\gamma}$$

Fine structure of Hydrogen

Exercise 1: $\vec{F} = -e\vec{E} = -\hat{r}\,\partial_r V(r)$

$$\vec{E} = \frac{\hat{r}}{e}\partial_r V(r)$$

$$\vec{B} = -\frac{\partial_r V(r)}{e m_e c^2}\,\vec{p}\times\hat{r}$$

$$\vec{p}\times\hat{r} = \vec{p}\times\vec{r}\cdot\frac{1}{r}$$

$$(\vec{p}\times\vec{r})_i = \epsilon_{ijk}\,p_j\,r_k = \epsilon_{ijk}\,r_k\,p_j$$

Since $\boxed{[p_j, r_k] = i\hbar\,\delta_{jk}}$

so

$$\vec{p}\times\vec{r}\cdot\frac{1}{r} = -\vec{r}\times\vec{p}\cdot\frac{1}{r} = -\vec{L}\cdot\frac{1}{r} = -\frac{\vec{L}}{r}$$

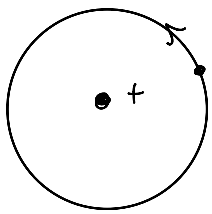
$$\vec{B} = -\frac{\partial_r V(r)}{e m_e c^2}\,\vec{p}\times\hat{r}$$

$$= \frac{\partial_r V(r)}{e m_e c^2 r}\,\vec{r}\times\vec{p}$$

$$= \frac{\partial_r V(r)}{e m_e c^2 r}\,\vec{L}$$

$$H_{fs} = -\vec{\mu}_s\cdot\vec{B} = \frac{g_s\mu_B\,\partial_r V(r)}{e m_e c^2 \hbar r}\,\vec{L}\cdot\vec{S}$$

Exercise 2:



Since the perturbation hamiltonian contains powers of p no greater than 2, we can treat the system classically.

Exercise 3:

$$H_{FS} = \frac{(g_s-1)\alpha^2 a_0}{2m_e r^3} \vec{L}\cdot\vec{S}$$

$$= \frac{(g_s-1)\alpha^2 a_0}{4m_e r^3}(J^2 - L^2 - S^2)$$

$$\langle r^{-3}\rangle = \frac{1}{L(L+\frac{1}{2})(L+1)n^3 a_0^3}$$

$$\langle L S J m_J | H_{FS} | L S J m_J \rangle$$

$$= \frac{(g_s-1)\alpha^2 a_0}{4m_e n^3 a_0^3} \frac{\hbar^2(J(J+1) - L(L+1) - S(S+1))}{L(L+\frac{1}{2})(L+1)}$$

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \longrightarrow \frac{(g_s-1)\alpha^2\hbar^2}{4m_e n^3 a_0^2} = \frac{(g_s-1)\alpha^2\hbar^2}{4m_e n^3}\frac{\alpha^2 m_e^2 c^2}{\hbar^2}$$

$$E_n = -\frac{m_e c^2 \alpha^2}{2n^2}$$

$$= \frac{(g_s-1)\alpha^4 m_e c^2}{4n^3}$$

$$= \frac{(g_s-1)E_n^2 n}{m_e c^2}$$