Exercise 1)

$$R(\zeta) = e^{\frac{-i\vec{J}\cdot\zeta}{\hbar}}$$

Let $|j\, m\rangle$ be a J^2, J_z eigenstate and suppose we want to rotate it into a superposition of $|j\, m'\rangle$ eigenstates.

then

$$R(\zeta)|j m\rangle = e^{\frac{-i\vec{J}\cdot\vec{\zeta}}{\hbar}}|j\, m\rangle$$

$$R(\zeta)|j\, m\rangle = \sum_{m'=-j}^{j} |j\, m'\rangle\langle j\, m'|\, e^{\frac{-i\vec{J}\cdot\vec{\zeta}}{\hbar}}|j\, m\rangle$$

$$R(\zeta)|j\, m\rangle = \sum_{m'=-j}^{j} d^{j}_{m'm}(\vec{\zeta})\,|j\, m'\rangle$$

Exercise 2)

We have a general rotation $R(\alpha,\beta,\gamma)$ about the three Euler angles α, β, then γ, which we can decompose into

$$R(\alpha,\beta,\gamma) = e^{-i\gamma \vec{J}_{z''}/\hbar}\, e^{-i\beta \vec{J}_{y'}/\hbar}\, e^{-i\alpha \vec{J}_{z}/\hbar}$$

Where the ' coordinate system is given by rotation is given by rotating by α about $\hat{z}$; and the " coordinate system is given by rotating $\hat{z}'$ by β into $\hat{x}'$.

So in terms of the original coordinate system,

$$e^{-i\beta \vec{J}_{y'}/\hbar} = R(\alpha)\, e^{-i\beta \vec{J}_{y}/\hbar} R^{\dagger}(\alpha)$$

$$e^{-i\beta \vec{J}_{y'}/\hbar} = e^{-\frac{i\alpha \vec{J}_z}{\hbar}} e^{-i\beta \vec{J}_y/\hbar} e^{i\alpha \vec{J}_z/\hbar}$$

and

$$e^{-i\frac{\gamma \vec{J}_{z''}}{\hbar}} = R'(\beta) R(\alpha) e^{-i\frac{\gamma J_z}{\hbar}} R^{\dagger}(\alpha) R^{\dagger\prime}(\beta)$$

but $\quad R'(\beta) = R(\alpha) e^{-i\beta \vec{J}_y/\hbar} R^{\dagger}(\alpha)$

so

$$e^{-i\gamma \vec{J}_{z''}/\hbar} = R(\alpha) e^{-i\beta \vec{J}_y/\hbar} e^{-i\frac{\gamma J_z}{\hbar}} e^{i\beta \vec{J}_y/\hbar} R^{\dagger}(\alpha)$$

so

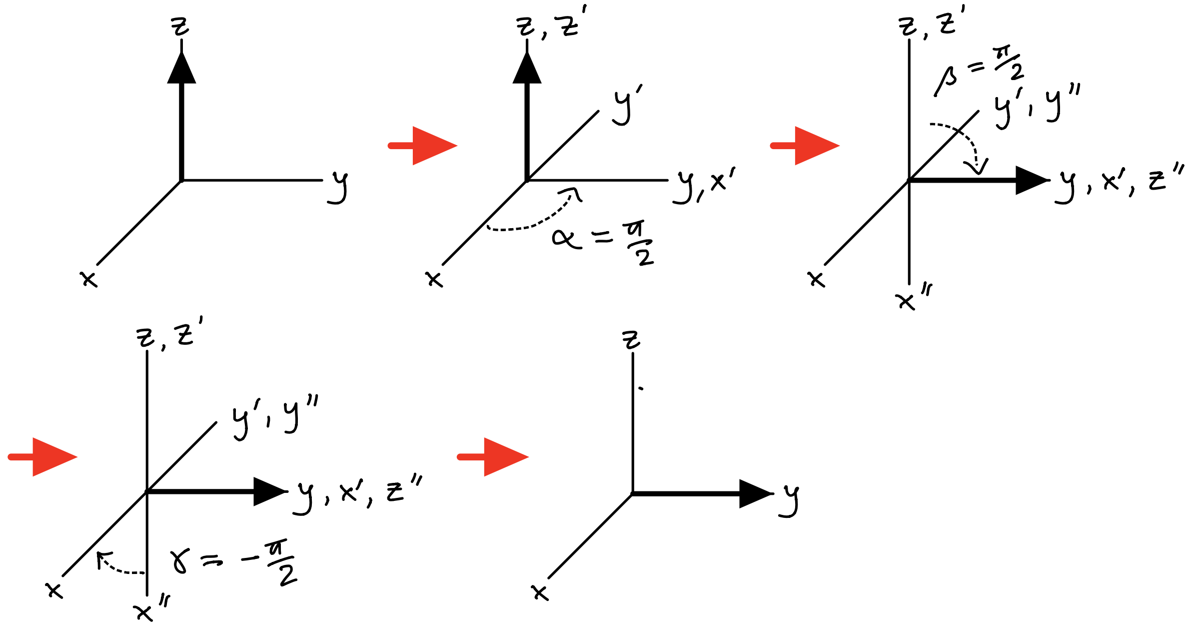
$$R(\alpha,\beta,\gamma) = e^{-i\gamma \vec{J}_{z''}/\hbar} e^{-i\beta \vec{J}_{y'}/\hbar} e^{-i\alpha \vec{J}_z/\hbar}$$

$$= e^{-i\alpha \vec{J}_z/\hbar} e^{-i\beta \vec{J}_y/\hbar} e^{-i\gamma \vec{J}_z/\hbar} \cancel{e^{i\beta \vec{J}_y/\hbar}} \cancel{e^{i\alpha \vec{J}_z/\hbar}}$$

$$\cdot \cancel{e^{-i\alpha \vec{J}_z/\hbar}} \cancel{e^{-i\beta \vec{J}_y/\hbar}} \cancel{e^{i\alpha \vec{J}_z/\hbar} e^{-i\alpha \vec{J}_z/\hbar}}$$

$$\boxed{R(\alpha,\beta,\gamma) = e^{-i\alpha \vec{J}_z/\hbar} e^{-i\beta \vec{J}_y/\hbar} e^{-i\gamma \vec{J}_z/\hbar}}$$

Exercise 3:



Exercise 4:

Consider a rotation of a state with $j = \frac{3}{2}$ by $\alpha = 2\pi, \beta = 0, \gamma = 0$.

Then

$$d^{(\frac{3}{2})}_{\frac{3}{2},\frac{3}{2}}(\alpha = 2\pi, \beta = 0, \gamma = 0) = e^{-i\frac{3}{2}\cdot 2\pi} = -1$$

So this rotation matrix cannot be the identity.

Exercise 5:

$$\vec{A} = \sum_q \hat{e}_q^* A_q$$

$$= \hat{e}_{-1}^* A_{-1} + \hat{e}_0^* A_0 + \hat{e}_1^* A_1$$

$$= \frac{1}{2}(\hat{x} + i\hat{y})(A_x - iA_y) + A_z \hat{z}$$

$$+ \frac{1}{2}(\hat{x} - i\hat{y})(A_x + iA_y)$$

$$= \frac{1}{2}A_x \hat{x} + \frac{1}{2}A_y \hat{y} - \cancel{\frac{i}{2}A_y \hat{x}} + \cancel{\frac{i}{2}A_x \hat{y}} + A_z \hat{z}$$

$$+ \frac{1}{2}A_x \hat{x} + \frac{1}{2}A_y \hat{y} - \cancel{\frac{i}{2}A_x \hat{y}} + \cancel{\frac{i}{2}A_y \hat{x}}$$

$$= A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

$$= \vec{A} \quad \checkmark$$

Exercise 6:

$$\vec{A}^* = \sum_q \hat{e}_q^* A_q^*$$

$$\vec{B} = \sum_p \hat{e}_p^* B_p$$

$$\vec{A}^* \cdot \vec{B} = \sum_p \sum_q \hat{e}_q^* \cdot \hat{e}_p^* A_q^* B_p$$

$$\vec{A}^* \cdot \vec{B} = \sum_p \sum_q \delta_{qp} A_q^* B_p$$

$$\vec{A}^* \cdot \vec{B} = \sum_q A_q^* B_p$$