Physics 623 Homework 7

Jeremy Welsh-Kavan

4.6.1. **Properties of Bessel functions**

a) The Bessel function J_n can be written in terms of an integral as

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} d\phi\ \cos(x\sin\phi - n\phi) \tag{1}$$

We claim that $J_{2n}(x)$ can be written as

$$J_{2n}(x) = \frac{1}{\pi}\int_0^{\pi} d\phi\ \cos(x\sin(\phi/2))\cos(n\phi) \tag{2}$$

Starting with the representation in Eq. (1), we have

$$\begin{aligned}
J_{2n}(x) &= \frac{1}{\pi}\int_0^{\pi} d\phi\ \cos(x\sin\phi - 2n\phi) \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\phi\ \cos(x\sin(\phi/2) - n\phi) \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\phi\ \cos(x\sin(\phi/2))\cos(n\phi) + \sin(x\sin(\phi/2))\sin(n\phi)
\end{aligned} \tag{3}$$

Observe that $\sin(\phi/2)$ is even with respect to $\phi = \pi$, so for all $x \in \mathbb{R}$, $\sin(x\sin(\phi/2))$ is even with respect to $\phi = \pi$. We also know that $\sin(n\phi)$ is odd with respect to $\phi = \pi$. Therefore, $\sin(x\sin(\phi/2))\sin(n\phi)$ is odd with respect to $\phi = \pi$ so the second integral in the last line of Eq. (3) is zero.

$$J_{2n}(x) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\ \cos(x\sin(\phi/2))\cos(n\phi) \tag{4}$$

We also have that $\cos(x\sin(\phi/2))$ is even with respect to $\phi = \pi$, and since $\cos(n\phi)$ is even with respect to $\phi = \pi$, we have can divide the integral in 4 to get

$$
\begin{aligned}
J_{2n}(x) &= \frac{1}{2\pi}\int_0^{2\pi} d\phi\ \cos(x\sin(\phi/2))\cos(n\phi) \\
&= \frac{2}{2\pi}\int_0^{\pi} d\phi\ \cos(x\sin(\phi/2))\cos(n\phi) \\
\implies J_{2n}(x) &= \frac{1}{\pi}\int_0^{\pi} d\phi\ \cos(x\sin(\phi/2))\cos(n\phi)
\end{aligned} \tag{5}
$$

as desired.

4.6.2. **Synchrotron radiation**

a) We can begin with the expression for the power per unit frequency per solid angle stated after Lemma 3 of ch. 5 § 6.2. Simplifying the dot products, we have

$$
\begin{aligned}
\frac{d^2\mathscr{P}(T)}{d\Omega\, d\omega} &= \frac{\omega^2 e^2}{4\pi^2 c^3}\int d\tau\, e^{i\omega\tau}\left[\boldsymbol{v}\left(T+\frac{\tau}{2}\right)\cdot\boldsymbol{v}\left(T-\frac{\tau}{2}\right)-c^2\right]\, e^{-i\frac{\omega}{c}\hat{\boldsymbol{x}}\cdot\left[\boldsymbol{y}\left(T+\frac{\tau}{2}\right)-\boldsymbol{y}\left(T-\frac{\tau}{2}\right)\right]} \\
&= \frac{\omega^2 e^2}{4\pi^2 c}\int d\tau\, e^{i\omega\tau}\left[\frac{v^2}{c^2}\cos\omega_0\tau-1\right]\, e^{-i\frac{\omega}{c}\hat{\boldsymbol{x}}\cdot\left[\boldsymbol{y}\left(T+\frac{\tau}{2}\right)-\boldsymbol{y}\left(T-\frac{\tau}{2}\right)\right]}
\end{aligned} \tag{6}
$$

We now compute the dot products in the exponential in a coordinate system in which $\hat{\boldsymbol{x}}$ lies along the z-direction.

$$
\hat{\boldsymbol{x}}\cdot\left[\boldsymbol{y}\left(T+\frac{\tau}{2}\right)-\boldsymbol{y}\left(T-\frac{\tau}{2}\right)\right] = \left|\boldsymbol{y}\left(T+\frac{\tau}{2}\right)-\boldsymbol{y}\left(T-\frac{\tau}{2}\right)\right|\cos\theta \tag{7}
$$

We have, in addition, that

$$
\left|\boldsymbol{y}\left(T+\frac{\tau}{2}\right)-\boldsymbol{y}\left(T-\frac{\tau}{2}\right)\right| = 2R\sin\left(\omega_0\frac{\tau}{2}\right) \tag{8}
$$

this was done in Mathematica. We can plug these into Eq. (6) and integrate over Ω to get

$$
\begin{aligned}
\frac{d\mathscr{P}(T)}{d\omega} &= \frac{\omega^2 e^2}{4\pi^2 c} \int d\tau\, e^{i\omega\tau} \left[\frac{v^2}{c^2}\cos\omega_0\tau - 1\right] \int d\Omega\, e^{-i\frac{\omega}{c}2R\sin(\omega_0\tau/2)\cos\theta} \\
&= \frac{\omega^2 e^2}{4\pi^2 c} \int d\tau\, e^{i\omega\tau} \left[\frac{v^2}{c^2}\cos\omega_0\tau - 1\right] \int_0^\pi \int_0^{2\pi} d\theta\, d\phi\, \sin\theta\, e^{-i\frac{\omega}{c}2R\sin(\omega_0\tau/2)\cos\theta}
\end{aligned}
\tag{9}
$$

We can compute this easily using a substitution to get

$$
\begin{aligned}
\frac{d\mathscr{P}(T)}{d\omega} &= \frac{\omega^2 e^2}{4\pi^2 c} \int d\tau\, e^{i\omega\tau} \left[\frac{v^2}{c^2}\cos\omega_0\tau - 1\right] \frac{4\pi\sin\left(\frac{\omega}{c}2R\sin(\omega_0\tau/2)\right)}{\frac{\omega}{c}2R\sin(\omega_0\tau/2)} \\
&= \frac{\omega\, e^2}{2\pi R} \int d\tau\, e^{i\omega\tau} \left[\frac{v^2}{c^2}\cos\omega_0\tau - 1\right] \frac{\sin\left(\frac{\omega}{c}2R\sin(\omega_0\tau/2)\right)}{\sin(\omega_0\tau/2)}
\end{aligned}
\tag{10}
$$

Thus, we may write the power spectrum of synchrotron radiation as

$$
\frac{d\mathscr{P}(T)}{d\omega} = \frac{\omega\, e^2}{2\pi R} \int d\tau\, e^{i\omega\tau} f\left(\omega_0\tau\right) \tag{11}
$$

where

$$
f\left(t\right) = \left[\frac{v^2}{c^2}\cos t - 1\right] \frac{\sin\left(\frac{\omega}{c}2R\sin(t/2)\right)}{\sin(t/2)} \tag{12}
$$

It is clear that $\frac{v^2}{c^2}\cos t - 1$ is 2π periodic so we need only show that $\mathrm{sinc}(\frac{\omega}{c}2R\sin(t/2))$ is also 2π periodic. Observe that $\sin((t+2\pi n)/2) = (-1)^n \sin(t/2)$ for $n \in \mathbb{Z}$. So we have

$$
\begin{aligned}
\frac{\sin\left(\frac{\omega}{c}2R\sin((t+2\pi n)/2)\right)}{\sin((t+2\pi n)/2)} &= \frac{\sin\left(\frac{\omega}{c}2R(-1)^n\sin(t/2)\right)}{(-1)^n\sin(t/2)} \\
&= \frac{(-1)^n\sin\left(\frac{\omega}{c}2R\sin(t/2)\right)}{(-1)^n\sin(t/2)} \\
&= \frac{\sin\left(\frac{\omega}{c}2R\sin(t/2)\right)}{\sin(t/2)}
\end{aligned}
\tag{13}
$$

Therefore, $\mathrm{sinc}(\frac{\omega}{c}2R\sin(t/2))$ is also 2π periodic. Thus, $f(t)$ is the product of two 2π periodic functions and is therefore also 2π periodic.

b) Didn't make it here :(

♣