Physics 623 Homework 6

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4.4.1. **Radiation by an accelerated point particle**

To check consistency between the energy spectrum and power of a non-relativistic charged particle, we can write the total energy radiated in terms of both the energy per unit frequency and in terms of the power.

$$
\begin{aligned}
U &= \int_0^\infty d\omega \, \frac{dU}{d\omega} \\
U &= \int_0^\infty dt \, \mathscr{P}(t)
\end{aligned}
\tag{1}
$$

Plugging in the expressions derived in class for $\frac{dU}{d\omega}$, we have

$$
\begin{aligned}
U &= \frac{2}{3}\frac{e^2}{\pi c^3} \int_0^\infty d\omega \, |\dot{\boldsymbol{v}}(\omega)|^2 \\
U &= \frac{2}{3}\frac{e^2}{\pi c^3} \int_0^\infty d\omega \, \left(\dot{\boldsymbol{v}}(\omega)\right)\left(\dot{\boldsymbol{v}}(\omega)^*\right) \\
U &= \frac{2}{3}\frac{e^2}{\pi c^3} \int_0^\infty d\omega \, \left[\int dt_1 e^{i\omega t_1} \dot{\boldsymbol{v}}(t_1)\right] \cdot \left[\int dt_2 e^{-i\omega t_2} \dot{\boldsymbol{v}}(t_2)\right] \\
U &= \frac{2}{3}\frac{e^2}{\pi c^3} \int\int dt_1 dt_2 \, \dot{\boldsymbol{v}}(t_1) \cdot \dot{\boldsymbol{v}}(t_2) \int_0^\infty d\omega \, e^{i\omega(t_1 - t_2)} \\
U &= \frac{2}{3}\frac{e^2}{\pi c^3} \int\int dt_1 dt_2 \, \dot{\boldsymbol{v}}(t_1) \cdot \dot{\boldsymbol{v}}(t_2) \left(\pi \delta(t_1 - t_2)\right)
\end{aligned}
\tag{2}
$$

The last step is justified because the delta function is an even function of its argument. Therefore, it can be written as an inverse Fourier transform as follows:

$$
\begin{aligned}
\delta(t_1 - t_2) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ e^{i\omega(t_1-t_2)} \\
&= \delta(t_2 - t_1) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ e^{-i\omega(t_1-t_2)} \\
\implies\ 2\pi\delta(t_1 - t_2) &= \int_{0}^{\infty} d\omega\ e^{i\omega(t_1-t_2)} + \int_{-\infty}^{0} d\omega\ e^{i\omega(t_1-t_2)} \\
2\pi\delta(t_1 - t_2) &= 2\int_{0}^{\infty} d\omega\ e^{i\omega(t_1-t_2)}
\end{aligned}
\tag{3}
$$

Thus, we can simplify the integral in Eq. (3) to read

$$
\begin{aligned}
U &= \frac{2}{3}\frac{e^2}{\pi c^3}\int\int dt_1 dt_2\ \dot{\boldsymbol{v}}(t_1)\cdot\dot{\boldsymbol{v}}(t_2)\left(\pi\delta(t_1 - t_2)\right) \\
U &= \frac{2}{3}\frac{e^2}{c^3}\int dt_1\ \dot{\boldsymbol{v}}(t_1)\cdot\dot{\boldsymbol{v}}(t_1) \\
U &= \frac{2}{3}\frac{e^2}{c^3}\int dt\ (\dot{\boldsymbol{v}}(t))^2 \\
U &= \int dt\ \mathscr{P}(t)
\end{aligned}
\tag{4}
$$

So the two formulae are consistent.

4.4.2. **Classical model of an atom**

We consider a classical model of a radiating atom as a damped harmonic oscillator with charge e obeying

$$
\begin{aligned}
\ddot{x} + \gamma\dot{x} + \omega_0^2 x &= 0 \\
x(t=0) &= a \\
\dot{x}(t=0) &= 0
\end{aligned}
\tag{5}
$$

a) The approximate result from ch.4 § 4.5 states that $\gamma = \frac{4}{3}\frac{e^2\omega_0^2}{mc^3}$. Overdamping occurs when $\gamma > 2\omega_0$. However, for an electron whose radiation spectrum is peaked somewhere in the range of visible light, we have

$$
\begin{aligned}
\frac{\gamma}{2\omega_0} &= \frac{2}{3}\frac{e^2\omega_0}{mc^3} \\
\frac{\gamma}{2\omega_0} &= \frac{4\pi}{3}\frac{e^2}{\lambda mc^2}
\end{aligned} \tag{6}
$$

In the visible range, the spectrum of λ is roughly $300\text{nm} \leq \lambda \leq 1100\text{nm}$. So for $\gamma/2\omega_0$ we have

$$
\begin{aligned}
\frac{4\pi}{3}\frac{e^2}{(1100\text{nm})mc^2} \leq \frac{\gamma}{2\omega_0} \leq \frac{4\pi}{3}\frac{e^2}{(300\text{nm})mc^2} \\
\leadsto 1 \times 10^{-8} \leq \frac{\gamma}{2\omega_0} \leq 4 \times 10^{-8} \ll 1
\end{aligned} \tag{7}
$$

So the case of the oscillator being overdamped is out of the question.

b) Now we attempt to solve the equation of motion exactly. Let $\omega_* = \sqrt{\omega_0^2 - \gamma^2/4}$ and let $\omega_\pm = -\gamma/2 \pm i\omega_*$. Then the differential equation in Eq. (5) is solved, in general, by

$$
x(t) = c_1 e^{-\gamma t/2}\cos(\omega_* t) + c_2 e^{-\gamma t/2}\sin(\omega_* t) \tag{8}
$$

Plugging in the initial conditions gives

$$
x(t) = a e^{-\gamma t/2}\cos(\omega_* t) + \frac{a\gamma}{2\omega_*} e^{-\gamma t/2}\sin(\omega_* t) \tag{9}
$$

So we have

$$
v(t) = -a\left(1 + \frac{\gamma^2}{4\omega_*^2}\right) e^{-\gamma t/2}\sin(\omega_* t) \tag{10}
$$

And we can define $\xi = 1 + \frac{\gamma^2}{4\omega_*^2}$ to write

$$
v(t) = -a\xi e^{-\gamma t/2}\sin(\omega_* t) \tag{11}
$$

We now Fourier transform Eq. (11) to get

$$
\begin{aligned}
v(\omega) &= -a\xi \int dt\; e^{-i\omega t-\gamma t/2}\sin(\omega_* t) \\
v(\omega) &= \frac{ia\xi}{2}\int dt\; \left(e^{-i(\omega-\omega_*+i\gamma/2)t} - e^{-i(\omega+\omega_*+i\gamma/2)t}\right) \\
v(\omega) &= -\frac{a\xi}{2}\left(\frac{1}{\omega-\omega_*+i\gamma/2} - \frac{1}{\omega+\omega_*+i\gamma/2}\right)
\end{aligned} \tag{12}
$$

For the energy radiated per unit frequency, we have

$$
\frac{dU}{d\omega} = \frac{2e^2}{3\pi c^3}|\dot{v}(\omega)|^2 \tag{13}
$$

Since $\dot{v}(\omega) = -i\omega v(\omega)$, we can rewrite Eq. (13) in terms of (12) to get the radiated energy per unit frequency.

$$
\begin{aligned}
\frac{dU}{d\omega} &= \frac{2\omega^2 e^2}{3\pi c^3}|v(\omega)|^2 \\
\frac{dU}{d\omega} &= \frac{a^2\xi^2\omega^2 e^2}{6\pi c^3}\left|\frac{1}{\omega-\omega_*+i\gamma/2} - \frac{1}{\omega+\omega_*+i\gamma/2}\right|^2 \\
\frac{dU}{d\omega} &= \frac{a^2\xi^2\omega^2 e^2}{6\pi c^3}\left(\frac{4\omega_*^2}{((\omega-\omega_*)^2+\gamma^2/4)\,((\omega+\omega_*)^2+\gamma^2/4)}\right) \\
\frac{dU}{d\omega} &= \frac{2a^2\xi^2\omega_*^2 e^2}{3\pi c^3}\frac{\omega^2}{((\omega-\omega_*)^2+\gamma^2/4)\,((\omega+\omega_*)^2+\gamma^2/4)}
\end{aligned} \tag{14}
$$

To get the total radiated energy we need only integrate this over ω. This yields

$$
\begin{aligned}
U &= \int_0^\infty d\omega \frac{dU}{d\omega} \\
U &= \frac{2a^2\xi^2\omega_*^2 e^2}{3\pi c^3}\int_{-\infty}^{\infty} d\omega \frac{\omega^2}{((\omega-\omega_*)^2+\gamma^2/4)\,((\omega+\omega_*)^2+\gamma^2/4)} \\
U &= \frac{2a^2\xi^2\omega_*^2 e^2}{4\pi c^3}\int_{-\infty}^{\infty} d\omega \frac{(\omega/\omega_*)^2}{\omega_*^2\,((\omega/\omega_*-1)^2+\gamma^2/4\omega_*^2)\,((\omega/\omega_*+1)^2+\gamma^2/4\omega_*^2)} \\
U &= \frac{2a^2\xi^2 e^2}{3\pi c^3}\omega_*\int_{-\infty}^{\infty} dz \frac{z^2}{((z-1)^2+\gamma^2/4\omega_*^2)\,((z+1)^2+\gamma^2/4\omega_*^2)} \\
U &= \frac{2a^2\xi^2 e^2}{3\pi c^3}\omega_*\frac{\pi\omega_*}{\gamma} \\
U &= \frac{2a^2\xi^2\omega_*^2 e^2}{3\gamma c^3}
\end{aligned} \tag{15}
$$

where $\xi = 1 + \frac{\gamma^2}{4\omega_*^2}$ and $\omega_* = \sqrt{\omega_0^2 - \gamma^2/4}$. The integral above was computed in Mathematica.

If γ is small compared to $2\omega_0$ then $\xi^2\omega_*^2 = \omega_*^4\left(1 + \frac{\gamma^2}{4\omega_*^2}\right)^2 \approx \omega_0^4\left(1 + \frac{\gamma^2}{4\omega_0^2}\right)^2 \approx \omega_0^4$. So in this limit we have

$$
\begin{aligned}
U &= \frac{2a^2\xi^2\omega_*^2 e^2}{3\gamma c^3} \\
U &\approx \frac{2a^2\omega_0^4 e^2}{3\gamma c^3}
\end{aligned}
\tag{16}
$$

So the underdamped case is recovered.

4.5.1. **Čerenkov radiation**

This one didn't make it :(

♣