Physics 623 Homework 3

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4.2.1. **Lienard-Wiechert potentials**

Consider a point charge e that moves on a given trajectory $\boldsymbol{X}(t)$ with velocity $\boldsymbol{v}(t) = \dot{\boldsymbol{X}}(t)$ which results in charge and current densities

$$\rho(\boldsymbol{x}, t) = e\delta(\boldsymbol{x} - \boldsymbol{X}(t)) \ \ , \ \boldsymbol{j}(\boldsymbol{x}, t) = e\boldsymbol{v}(t)\delta(\boldsymbol{x} - \boldsymbol{X}(t)) \tag{1}$$

The retarded potentials are given by

$$\begin{aligned} \varphi(\boldsymbol{x}, t) &= \int d\boldsymbol{y} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} \rho\Big(\boldsymbol{y}, t - \frac{1}{c}|\boldsymbol{x} - \boldsymbol{y}|\Big) \\ \boldsymbol{A}(\boldsymbol{x}, t) &= \frac{1}{c} \int d\boldsymbol{y} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} \boldsymbol{j}\Big(\boldsymbol{y}, t - \frac{1}{c}|\boldsymbol{x} - \boldsymbol{y}|\Big) \end{aligned} \tag{2}$$

Plugging (1) into (2), we have

$$\varphi(\boldsymbol{x}, t) = e \int d\boldsymbol{y} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} \delta\Big(\boldsymbol{y} - \boldsymbol{X}\Big(t - \frac{1}{c}|\boldsymbol{x} - \boldsymbol{y}|\Big)\Big) \tag{3}$$

In order to solve (3), we must determine the point(s) at which the argument of the delta function is zero, which requires that we solve a recursive equation for $\boldsymbol{y}$, or we must eliminate the recursive relation in the delta function. We can do this by introducing a new delta function, $\delta(t' - t + \frac{1}{c}|\boldsymbol{x} - \boldsymbol{y}|)$, and integrating over t'. [1]

$$\begin{aligned} \varphi(\boldsymbol{x}, t) &= e \int\int d\boldsymbol{y} dt' \frac{\delta\left(\boldsymbol{y} - \boldsymbol{X}\left(t'\right)\right)}{|\boldsymbol{x} - \boldsymbol{y}|} \delta\left(t' - t + \frac{1}{c}|\boldsymbol{x} - \boldsymbol{y}|\right) \\ \varphi(\boldsymbol{x}, t) &= e \int\int dt' d\boldsymbol{y} \frac{\delta\left(\boldsymbol{y} - \boldsymbol{X}\left(t'\right)\right)}{|\boldsymbol{x} - \boldsymbol{y}|} \delta\left(t' - t + \frac{1}{c}|\boldsymbol{x} - \boldsymbol{y}|\right) \\ \varphi(\boldsymbol{x}, t) &= e \int dt' \frac{\delta\left(t' - t + \frac{1}{c}|\boldsymbol{x} - \boldsymbol{X}\left(t'\right)|\right)}{|\boldsymbol{x} - \boldsymbol{X}\left(t'\right)|} \end{aligned} \tag{4}$$

Now let $g(t') = t' - t + \frac{1}{c}|\boldsymbol{x} - \boldsymbol{X}\left(t'\right)|$, and recall the following property of the delta function

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|} \tag{5}$$

where i indexes the set $\{x : g(x) = 0\}$. We claim that $g(t') = 0$ has exactly one solution, t_-.

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[1] This solution follows closely the solution in Landau and Lifshitz.

Fix $A = (ct, \boldsymbol{x})$ and, without loss of generality, define $\boldsymbol{X}(0) = \boldsymbol{0}$. Since we need only consider events A which can be causally related to $O = (0, \boldsymbol{X}(0))$, we may assume A occurs "above" the light cone whose origin is O. Therefore, $c^2t^2 > \boldsymbol{x}^2$, which implies that $g(0) = -t + \frac{1}{c}|\boldsymbol{x}| < 0$. Observe that $g(t) = \frac{1}{c}|\boldsymbol{x} - \boldsymbol{X}(t)| > 0$. Thus, by the Intermediate Value Theorem, there exists $t_- \in (0, t)$ such that $g(t_-) = t_- - t + \frac{1}{c}|\boldsymbol{x} - \boldsymbol{X}(t_-)| = 0$.

Suppose there exists another value, $t_+ \in (-\infty, \infty)$, such that $g(t_+) = 0$. Without loss of generality, assume that $t_- < t_+$. [2] Then, by the Mean Value Theorem, there exists a point $t^* \in (t_-, t_+)$ such that $g'(t^*)(t_+ - t_-) = g(t_+) - g(t_-)$. But then we would have

$$\begin{aligned} g'(t^*)(t_+ - t_-) &= g(t_+) - g(t_-) \\ \left(1 - \frac{1}{c}\frac{(\boldsymbol{x} - \boldsymbol{X}(t^*)) \cdot \boldsymbol{X}'(t^*)}{|\boldsymbol{x} - \boldsymbol{X}(t^*)|}\right)(t_+ - t_-) &= 0 \\ c|\boldsymbol{x} - \boldsymbol{X}(t^*)| + (\boldsymbol{X}(t^*) - \boldsymbol{x}) \cdot \boldsymbol{v}(t^*) &= 0 \\ |\boldsymbol{X}'(t^*)| \geq |\boldsymbol{v}(t^*)||\cos(\vartheta)| &= c \\ \implies |\boldsymbol{v}(t^*)| &\geq c \end{aligned} \tag{6}$$

where $\vartheta = \arctan((\boldsymbol{X}(t^*) - \boldsymbol{x}) \cdot \boldsymbol{X}'(t^*)/|\boldsymbol{X}'(t^*)||\boldsymbol{x} - \boldsymbol{X}(t^*)|)$. But we know *a priori* that the particle cannot travel faster than the speed of light, so this is a contradiction. Therefore, there exists one and only one value, t_-, such that $g(t_-) = 0$.

Therefore, we can rewrite the integral in (4) in terms of t_- using (5) as follows

$$\begin{aligned} \varphi(\boldsymbol{x}, t) &= e \int dt' \frac{\delta\left(t' - t + \frac{1}{c}|\boldsymbol{x} - \boldsymbol{X}(t')|\right)}{|\boldsymbol{x} - \boldsymbol{X}(t')|} \\ \varphi(\boldsymbol{x}, t) &= e \frac{1}{|1 - \frac{1}{c}\frac{(\boldsymbol{x} - \boldsymbol{X}(t_-)) \cdot \boldsymbol{v}(t_-)}{|\boldsymbol{x} - \boldsymbol{X}(t_-)|}|} \int dt' \frac{\delta(t' - t_-)}{|\boldsymbol{x} - \boldsymbol{X}(t')|} \\ \varphi(\boldsymbol{x}, t) &= e \frac{1}{1 - \frac{1}{c}\frac{(\boldsymbol{x} - \boldsymbol{X}(t_-)) \cdot \boldsymbol{v}(t_-)}{|\boldsymbol{x} - \boldsymbol{X}(t_-)|}} \frac{1}{|\boldsymbol{x} - \boldsymbol{X}(t_-)|} \\ \implies \varphi(\boldsymbol{x}, t) &= \frac{e}{|\boldsymbol{x} - \boldsymbol{X}(t_-)| - \boldsymbol{v}(t_-) \cdot (\boldsymbol{x} - \boldsymbol{X}(t_-))/c} \end{aligned} \tag{7}$$

Note that $g'(t_-) > 0$ so the absolute value signs in the second line of (7) can be eliminated.

In the same manner, we can evaluate the integral for $\boldsymbol{A}(\boldsymbol{x}, t)$.

[2] Note that the algebra is identical if we assume $t_+ < t_-$

$$
\begin{aligned}
\boldsymbol{A}(\boldsymbol{x},t) &= \frac{1}{c}\int d\boldsymbol{y}\,\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\boldsymbol{j}\Big(\boldsymbol{y},t-\frac{1}{c}|\boldsymbol{x}-\boldsymbol{y}|\Big)\\
\boldsymbol{A}(\boldsymbol{x},t) &= \frac{1}{c}\int d\boldsymbol{y}\,\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\boldsymbol{v}(t-\frac{1}{c}|\boldsymbol{x}-\boldsymbol{y}|)\rho\Big(\boldsymbol{y},t-\frac{1}{c}|\boldsymbol{x}-\boldsymbol{y}|\Big)\\
\boldsymbol{A}(\boldsymbol{x},t) &= \frac{e}{c}\int dt'\,\frac{\delta\left(t'-t+\frac{1}{c}|\boldsymbol{x}-\boldsymbol{X}\left(t'\right)|\right)}{|\boldsymbol{x}-\boldsymbol{X}\left(t'\right)|}\boldsymbol{v}(t')\\
\implies \boldsymbol{A}(\boldsymbol{x},t) &= \frac{1}{c}\boldsymbol{v}(t_-)\varphi(\boldsymbol{x},t)
\end{aligned}
\tag{8}
$$

4.2.2. **Potential of a uniformly moving charge**

Consider a charge e moving uniformly along the x-axis with velocity v and let $\boldsymbol{X}(t) = (vt, 0, 0)$ parameterize its position as a function of time. To determine the Lienard-Wiechert potentials, we need only to find t_- which is the solution to $t_- = t - \frac{1}{c}|\boldsymbol{x} - \boldsymbol{X}(t_-)|$.

$$
\begin{aligned}
t_- &= t-\frac{1}{c}|\boldsymbol{x}-\boldsymbol{X}(t_-)|\\
c^2(t-t_-)^2 &= (x-vt_-)^2+y^2+z^2\\
y^2+z^2 &= (c^2t^2-x^2)+2(xv-c^2t)t_-+(c^2-v^2)t_-^2\\
0 &= (c^2-v^2)t_-^2+2(xv-c^2t)t_-+(c^2t^2-x^2)-y^2-z^2\\
0 &= At_-^2+Bt_-+C\\
t_- &= \frac{-B-\sqrt{B^2-4AC}}{2A}\\
t_- &= \frac{-(xv-c^2t)-\sqrt{(xv-c^2t)^2-(c^2-v^2)((c^2t^2-x^2)-y^2-z^2)}}{(c^2-v^2)}
\end{aligned}
\tag{9}
$$

where we have chosen the negative root since we must have $t_- < t$. Plugging this into equations (7) and (8) we get

$$
\begin{aligned}
\varphi(\boldsymbol{x},t) &= \frac{e}{\sqrt{(x-vt_-)^2+y^2+z^2}-\frac{v}{c}(x-vt_-)} \\
\varphi(\boldsymbol{x},t) &= \frac{e}{c(t-t_-)-\frac{v}{c}(x-vt_-)} \\
\varphi(\boldsymbol{x},t) &= \frac{e}{\sqrt{(x-vt)^2+y^2+z^2-\frac{v^2}{c^2}(y^2+z^2)}} \\
\varphi(\boldsymbol{x},t) &= \frac{e}{\sqrt{(x-vt)^2+\left(1-\frac{v^2}{c^2}\right)(y^2+z^2)}} \\
\Longrightarrow\ \varphi(\boldsymbol{x},t) &= \frac{e}{R^*} \ \ , \ \ R^* := \sqrt{(x-vt)^2+\left(1-\frac{v^2}{c^2}\right)(y^2+z^2)} \\
\boldsymbol{A}(\boldsymbol{x},t) &= \frac{1}{c}\boldsymbol{v}(t_-)\varphi(\boldsymbol{x},t) \\
\boldsymbol{A}(\boldsymbol{x},t) &= \frac{e}{c}\frac{\boldsymbol{v}}{R^*} \ \ , \ \ \boldsymbol{v} := (v,0,0)
\end{aligned}
\tag{10}
$$

both of which agree with the potentials found in ch. 3 §3.4.

♣