Physics 623 Homework 8

Jeremy Welsh-Kavan

4.6.2. **Synchrotron radiation** (continued)

c) We know from problem 4.6.1 that the Bessel function $J_{2m}(x)$ can be written

$$J_{2m}(x) = \frac{1}{\pi} \int_0^\pi d\phi \; \cos(x \sin(\phi/2)) \cos(m\phi) \tag{1}$$

By part b), P_m can be expressed as

$$P_m = \frac{e^2\omega_0}{R} \frac{m}{\pi} \int_0^\pi d\phi \; \cos(m\phi) \left[\beta^2 - 1 - 2\beta^2 \sin^2(\phi/2)\right] \frac{\sin(2m\beta \sin(\phi/2))}{\sin(\phi/2)} \tag{2}$$

where $\omega_0 R/c = \beta$. We now rewrite this as a sum of integrals. Define $A := e^2\omega_0 m/R$. Then we have

$$\begin{aligned}
P_m &= \frac{A}{\pi} \int_0^\pi d\phi \; \cos(m\phi) \left[\beta^2 - 1 - 2\beta^2 \sin^2(\phi/2)\right] \frac{\sin(2m\beta \sin(\phi/2))}{\sin(\phi/2)} \\
\frac{P_m}{A} &= (\beta^2 - 1)\frac{1}{\pi} \int_0^\pi d\phi \; \cos(m\phi) \frac{\sin(2m\beta \sin(\phi/2))}{\sin(\phi/2)} - 2\beta^2 \frac{1}{\pi} \int_0^\pi d\phi \; \sin^2(\phi/2) \frac{\sin(2m\beta \sin(\phi/2))}{\sin(\phi/2)} \\
\frac{P_m}{A} &= (\beta^2 - 1)\frac{1}{\pi} \int_0^\pi d\phi \; \cos(m\phi) \frac{\sin(2m\beta \sin(\phi/2))}{\sin(\phi/2)} - 2\beta^2 \frac{1}{\pi} \int_0^\pi d\phi \; \cos(m\phi) \sin(\phi/2) \sin(2m\beta \sin(\phi/2))
\end{aligned} \tag{3}$$

Now with $\xi = 2m\beta$, we can write these in terms of the Bessel functions as

$$\begin{aligned}
\frac{P_m}{A} &= (\beta^2 - 1)\frac{1}{\pi} \int_0^\pi d\phi \; \frac{\partial^{-1}}{\partial \xi^{-1}} \cos(m\phi) \cos(\xi \sin(\phi/2)) + 2\beta^2 \frac{1}{\pi} \int_0^\pi d\phi \; \frac{\partial}{\partial \xi} \cos(m\phi) \cos(\xi \sin(\phi/2)) \\
&= (\beta^2 - 1) \frac{\partial^{-1}}{\partial \xi^{-1}} J_{2m}(\xi) + 2\beta^2 \frac{\partial}{\partial \xi} J_{2m}(\xi)
\end{aligned} \tag{4}$$

Replacing the constants, we have

$$P_m = \frac{e^2 \omega_0 m}{R} \left[2\beta^2 J'_{2m}(2m\beta) - (1-\beta^2) \int_0^{2m\beta} dx\, J_{2m}(x) \right] \tag{5}$$

as desired.

d) For $\beta \approx 1$, the second term vanishes and the primary contribution comes from the first term. Therefore, for $\beta \approx 1$ we have

$$P_m \approx \frac{e^2 \omega_0 m}{R} 2\beta^2 J'_{2m}(2m\beta) \tag{6}$$

By the asymptotic form of $J'_{2m}(2m\beta)$ derived in the last homework, we have for $2m \gg 1$ and $\beta \approx 1$,

$$\begin{aligned} P_m &\approx \frac{e^2 \omega_0 \beta^2}{R} 2m \begin{cases} \frac{2^{2/3}}{3^{1/3}\Gamma(1/3)} (2m)^{-2/3} & \text{for } 1 \ll m \ll \gamma^3 \\ \frac{1}{\sqrt{2\pi}} \sqrt{\gamma/2m} e^{-2m/3\gamma^3} & \text{for } m \gg \gamma^3 \end{cases} \\ P_m &\approx \frac{e^2 \omega_0 \beta^2}{R} \begin{cases} \frac{2}{3^{1/3}\Gamma(1/3)} m^{1/3} & \text{for } 1 \ll m \ll \gamma^3 \\ \frac{1}{\sqrt{2\pi}} \sqrt{2m\gamma} e^{-2m/3\gamma^3} & \text{for } m \gg \gamma^3 \end{cases} \end{aligned} \tag{7}$$

Since P_m is an increasing function of m in the first case and decreasing in the second case, P_m must reach a maximum, $m_{\max}$, when m is of order γ^3. Therefore, $m_{\max} \propto \gamma^3$.

e) For the Advanced Light Source, electrons with $T = 1.9$ GeV of energy travel in a circle of radius roughly $R = 20$ m. From the previous problem, the power spectrum is peaked around $\omega_0 m$ where m is of order γ^3. The energy yields a velocity, and hence an angular frequency, given by

$$\begin{aligned} T &= \gamma mc^2 = \frac{1}{\sqrt{1 - v^2/c^2}} mc^2 \\ v &= c\sqrt{1 - \frac{m^2c^4}{T^2}} \\ \omega_0 &= \frac{v}{R} \end{aligned} \tag{8}$$

Therefore, the peak frequency for the Advanced Light Source is roughly

$$
\begin{aligned}
\omega &= m\omega_0 \\
&\approx \gamma^3 \omega_0 \\
&= \gamma^3 \frac{c}{R}\sqrt{1 - \frac{m^2c^4}{T^2}} \\
&= \frac{c}{R}\sqrt{\frac{T^2}{m^2c^4} - 1} \\
&\approx 55.7 \text{rad /s} \\
&\approx 8.87 \text{GHz}
\end{aligned} \tag{9}
$$

with associated wavelength $\lambda \approx 0.0338$ m... which I think should be in the X-ray range but is not.

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