To Xerox [7.18

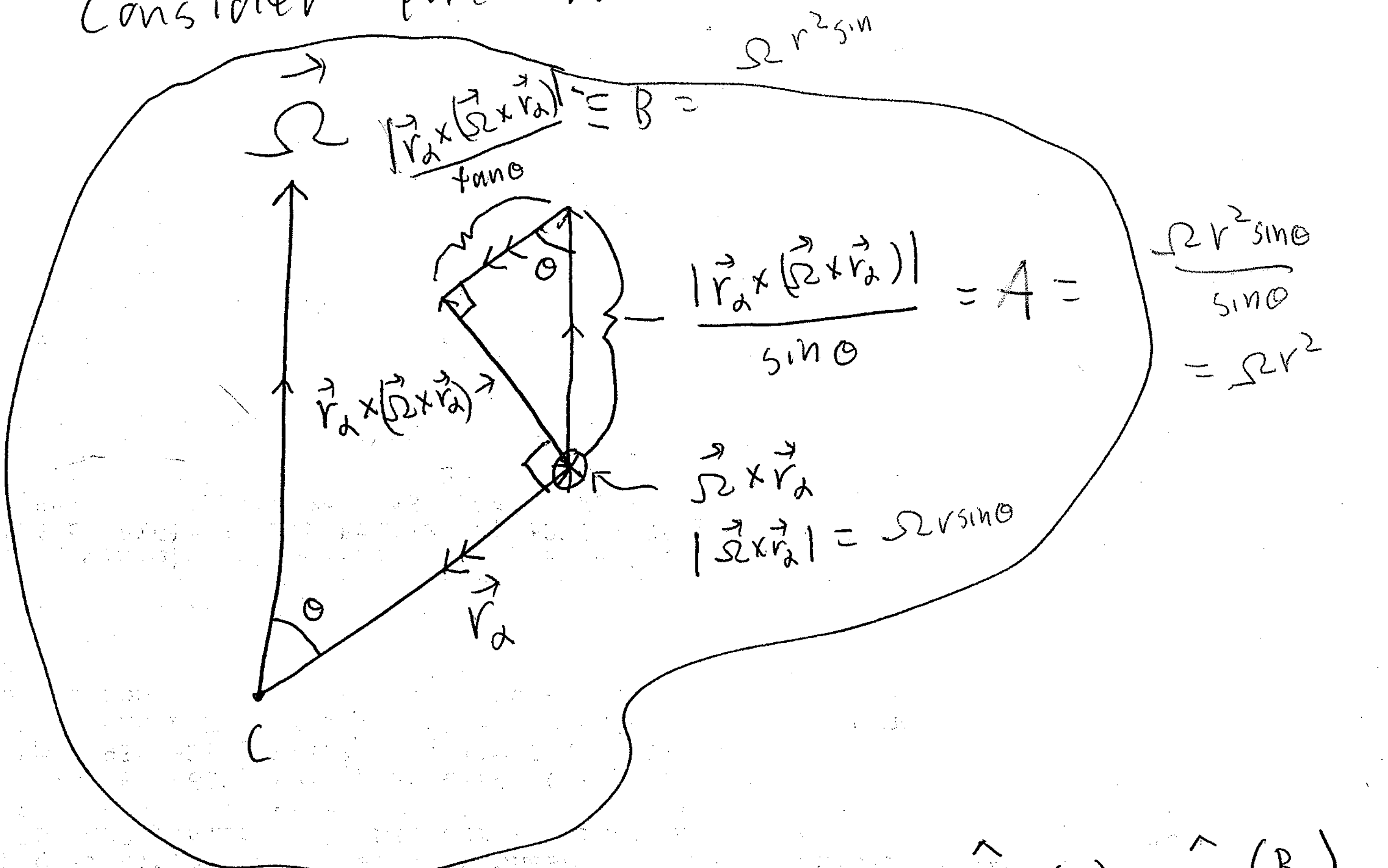
Angular momentum:

$$\vec{L} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha$$

Why important? Conserved if ~~rotation~~

$\mathcal{L}$ rotation invariant $\Leftrightarrow$ No external torques

Consider pure rotation about a center:



$$\vec{V}_\alpha = \vec{\Omega}\times\vec{r}_\alpha \quad \Rightarrow \quad \vec{r}_\alpha \times (\vec{\Omega}\times\vec{r}_\alpha) = \hat{\Omega}(A) + \hat{r}(B)$$

$$\Rightarrow \vec{r}_\alpha \times (\vec{\Omega} \times \vec{r}_\alpha) = \Omega r^2 \hat{\Omega} - \hat{r}(\vec{\Omega}\cdot\vec{r})r$$

$$= r^2\vec{\Omega} - \vec{r}(\vec{\Omega}\cdot\vec{r})$$

Cartesian components; tensor notation:

$$\Rightarrow \left[\vec{r}_\alpha \times (\vec{\Omega} \times \vec{r}_\alpha)\right]_i = r_\alpha^2 \Omega_i - r_i^\alpha r_j^\alpha \Omega_j$$

$$= (r_\alpha^2 \delta_{ij} - r_i^\alpha r_j^\alpha)\Omega_j$$

$$\Rightarrow L_i = \sum_\alpha m_\alpha (r_\alpha^2 \delta_{ij} - r_i^\alpha r_j^\alpha)\Omega_j = I_{ij}\Omega_j$$

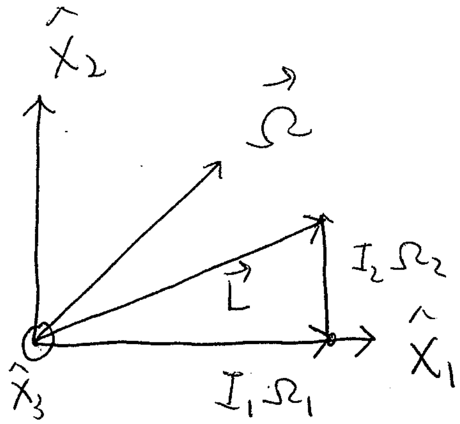
Moment of inertia tensor

$\Rightarrow$ ~~X momentum related~~

$$\Rightarrow \text{Given } \vec{\Omega};\ \underset{\sim}{I}\vec{\Omega} = \vec{L}$$

Note: $\vec{L} \not\parallel \vec{\Omega}$ in general

$\hat{x}_1 \quad \hat{x}_2 \quad \hat{x}_3$: Principal axes of $\underset{\sim}{I}$



In this co-ord. system,

$$\underset{\sim}{I} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array}$$

Suppose $I_1 > I_2 > I_3$:

$L_1 =$

$L_2 =$

$L_3 =$

When is $\vec{L} \parallel \vec{\Omega}$?

$\cancel{\vec{L} \parallel \vec{\Omega}}$

$\vec{L} \parallel \vec{\Omega}$ iff $\vec{\Omega}$ along principal axis.

Or:

If $\underset{\sim}{I}$ has special symmetry.

Example: Isotropic (e.g., sphere, symmetric Jack)

$$\underset{\sim}{I} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

$\Rightarrow \vec{L} = \underset{\sim}{I}\,\vec{\Omega} = I\vec{\Omega} \Rightarrow \vec{L}$ always $\parallel \vec{\Omega}$

$\Rightarrow$ For such a free object, $\vec{L} =$ constant $\Rightarrow \vec{\Omega} =$ constant

$\Rightarrow$ motion is simple (constant rotation)

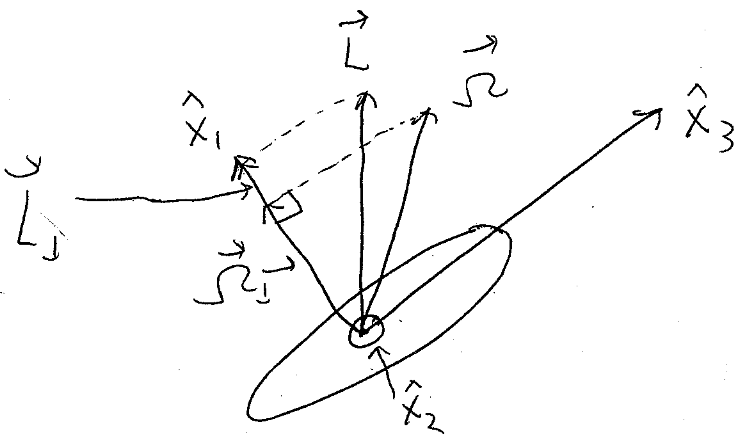
Now, lower symmetry: $I_1 = I_2 \neq I_3$

"symmetrical top" (= Football)
(American)

Theory of free motion of symmetrical top = " " "wounded duck"

$$\underset{\sim}{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

$I_3 < I_1$



$$\vec{L} = I_1 \underbrace{(\Omega_1 \hat{x}_1 + \Omega_2 \hat{x}_2)}_{\vec{\Omega}_\perp} + I_3 \Omega_3 \hat{x}_3$$

Note: $\vec{L}_\perp \parallel \vec{\Omega}_\perp \Rightarrow \vec{L}, \vec{\Omega}, \hat{x}_3$ always co-planar

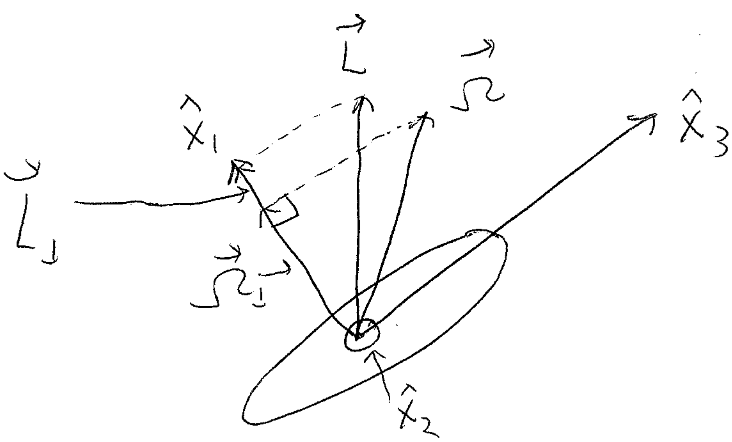
Now, lower symmetry: $I_1 = I_2 \neq I_3$

"symmetrical top" (= Football)
(American)

Theory of free motion of symmetrical top = " " "wounded duck"

$$\underset{\sim}{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

$I_3 < I_1$



$$\vec{L} = I_1 \underbrace{(\Omega_1 \hat{x}_1 + \Omega_2 \hat{x}_2)}_{\vec{\Omega}_\perp} + I_3 \Omega_3 \hat{x}_3$$

Note: $\vec{L}_\perp \parallel \vec{\Omega}_\perp \Rightarrow \vec{L}, \vec{\Omega}, \hat{x}_3$ always co-planar

What's motion of object?

Note: $\hat{x}_1(t)$, $\hat{x}_2(t)$, $\hat{x}_3(t)$ all change with time (object is rotating)

So does $\vec{\Omega}(t)$

What stays constant?

$$\vec{L} = I_1 \vec{\Omega} + (I_3 - I_1)\, \Omega_3 \hat{x}_3$$

$$\Delta I \equiv I_1 - I_3$$

$$= I_1 \vec{\Omega} - \Delta I \, (\vec{\Omega} \cdot \hat{x}_3)\, \hat{x}_3$$

$$\frac{d\vec{L}}{dt} = I_1 \frac{d\vec{\Omega}}{dt} - \Delta I \left[\frac{d\hat{x}_3}{dt} (\vec{\Omega} \cdot \hat{x}_3) + \hat{x}_3 \left(\frac{d\vec{\Omega}}{dt} \cdot \hat{x}_3 + \frac{d\hat{x}_3}{dt} \cdot \vec{\Omega} \right) \right]$$

$$= 0$$

$$\Rightarrow \boxed{\frac{d\vec{\Omega}}{dt} = \frac{\Delta I}{I_1} \left(\frac{d\hat{x}_3}{dt} (\vec{\Omega} \cdot \hat{x}_3) + \hat{x}_3 \left(\frac{d\vec{\Omega}}{dt} \cdot \hat{x}_3 + \frac{d\hat{x}_3}{dt} \cdot \vec{\Omega} \right) \right)}$$

What's $\frac{d\hat{x}_3}{dt}$?

~~what's~~

$$\frac{d\hat{x}_3}{dt} = \vec{\Omega} \times \hat{x}_3$$

$$\Rightarrow \frac{d\hat{x}_3}{dt} \cdot \vec{\Omega} = ?$$

$$\Rightarrow \frac{d\vec{\Omega}}{dt} = \frac{\Delta I}{I_1}\left(\vec{\Omega} \times \hat{x}_3 \, (\vec{\Omega} \cdot \hat{x}_3) + \hat{x}_3\left(\frac{d\vec{\Omega}}{dt} \cdot \hat{x}_3\right)\right)$$

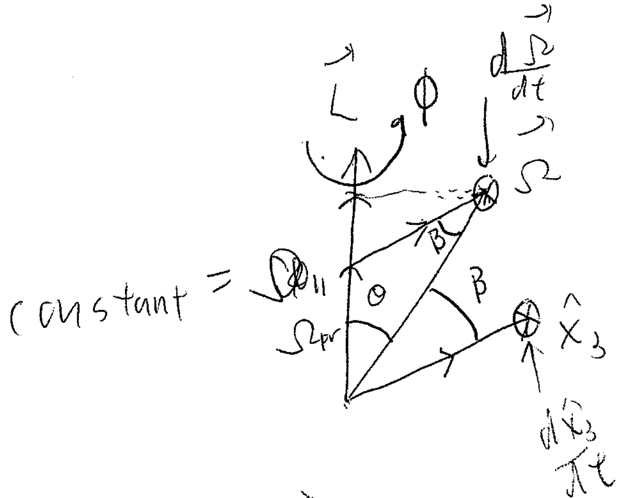
~~$\Rightarrow$ x_3~~ Dot both sides with $\hat{x}_3$, to calculate

$$\Rightarrow \hat{x}_3 \cdot \frac{d\vec{\Omega}}{dt} = \frac{\Delta I}{I_1}\left(\underbrace{(\vec{\Omega} \times \hat{x}_3) \cdot \hat{x}_3}_{?} \, (\vec{\Omega} \cdot \hat{x}_3) + \frac{d\vec{\Omega}}{dt} \cdot \hat{x}_3\right)$$

$$\Rightarrow \hat{x}_3 \cdot \frac{d\vec{\Omega}}{dt}\left(1 - \frac{\Delta I}{I_1}\right) = \frac{I_3}{I_1}\,\hat{x}_3 \cdot \frac{d\vec{\Omega}}{dt} = 0$$

$$\Rightarrow \boxed{\hat{x}_3 \cdot \frac{d\vec{\Omega}}{dt} = 0}$$

$$\Rightarrow \boxed{\frac{d\vec{\Omega}}{dt} = \frac{\Delta I}{I_1}\left(\vec{\Omega} \cdot \hat{x}_3\right) \vec{\Omega} \times \hat{x}_3}$$



$$\Rightarrow \quad \vec{L} \cdot \frac{d\vec{\Omega}}{dt} = 0$$

$$\Rightarrow \quad \frac{d}{dt}(\vec{L}\cdot\vec{\Omega}) = \underbrace{\vec{L}\cdot\cancel{\frac{d\vec{\Omega}}{dt}}}_{0} + \underbrace{\vec{\Omega}\cdot\cancel{\frac{d\vec{L}}{dt}}}_{0\ (\text{why?})} = 0$$

$$\cancel{\vec{\Omega} = \Omega_{pr}\hat{L} + (\vec{\Omega}\cdot\hat{x}_3)\hat{x}_3}$$

$$\cancel{\Rightarrow \hat{L}\frac{d\Omega_{pr}}{dt} +}$$

$$\vec{\Omega}\cdot\frac{d\vec{\Omega}}{dt} = 0 \Rightarrow |\vec{\Omega}|^2 = \text{constant}$$

$$\vec{L}\cdot\vec{\Omega} = |\vec{L}|\,|\vec{\Omega}|\cos\theta = \text{constant} \Rightarrow \theta = \text{constant}$$

$$\Rightarrow \Omega_{pr} = \text{constant.} \quad \Rightarrow \quad \frac{d\hat{x}_3}{dt} = \Omega_{pr}\hat{L}\times\hat{x}_3$$

$\Rightarrow$ $\hat{x}_3$ precesses uniformly about $\vec{L}$

7.26

$$\frac{d}{dt}\left(\hat{x}_3 \cdot \vec{\Omega}\right) = \hat{x}_3 \cdot \cancel{\frac{d\vec{\Omega}}{dt}} + \vec{\Omega} \cdot \cancel{\frac{d\hat{x}_3}{dt}} = 0$$

0 (eq. 7.24) $\longleftrightarrow$ 0 (

$\Rightarrow \beta = \text{constant} \Rightarrow \vec{\Omega}$ always makes same (initial) $\measuredangle$ with $\hat{x}_3$

$\Rightarrow \measuredangle$ between $\hat{x}_3$ and $\vec{L}$ $= \theta + \beta = \text{constant}$

$\Rightarrow$ $\hat{x}_3$ always lies on same cone

also, $\Omega_{pr} = \text{constant}$

$\Rightarrow$

$\cancel{\frac{d\hat{x}_3}{dt} = \dot{\phi}}$

same constant

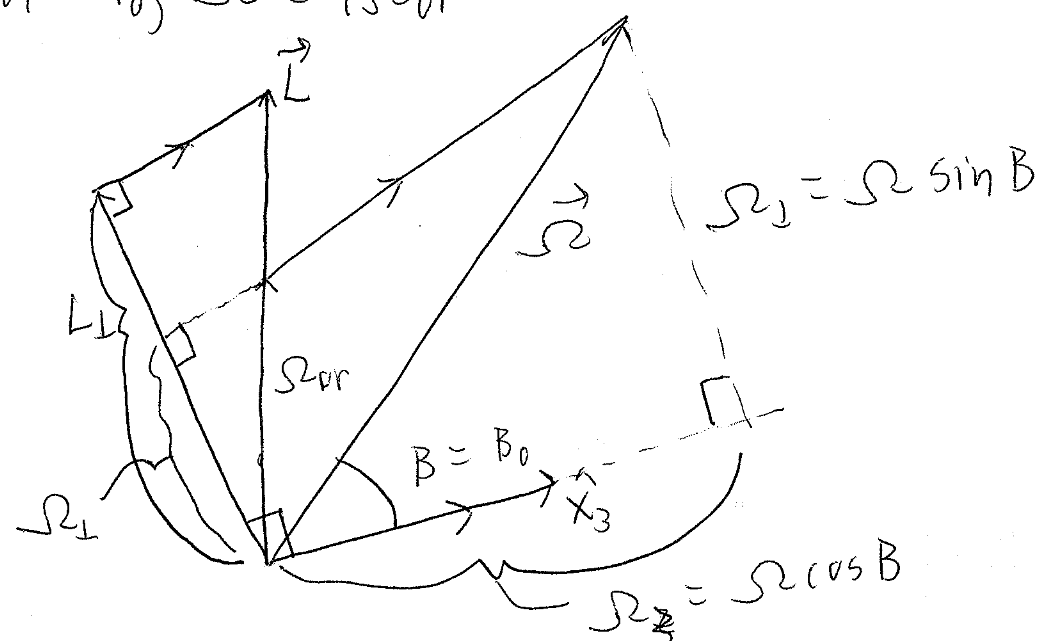
$$\frac{d\hat{x}_{3B}}{dt} = \left(\vec{\Omega} \times \hat{x}_3\right) = \left(\Omega_{pr}\,\hat{L} + C\,\hat{x}_3\right) \times \hat{x}_3$$

$$= \Omega_{pr}\,\hat{L} \times \hat{x}_3$$

$$\Rightarrow \left|\frac{d\hat{x}_3}{dt}\right| = \dot{\phi}\sin(\theta + \beta) = \Omega_{pr}\sin(\theta+\beta)$$

$$\Rightarrow \dot{\phi} = \Omega_{pr} = \text{constant}$$

Given B_0, $\Omega \equiv |\vec{\Omega}_0|$



$$\Rightarrow \quad \frac{\Omega_{pr}}{\Omega_\perp} = \frac{L}{L_\perp} \Rightarrow \Omega_{pr} = \frac{L}{L_\perp}\Omega_\perp = \frac{L\,\Omega_\perp}{I_1 \Omega_\perp}$$

$$\Rightarrow \boxed{\Omega_{pr} = \frac{L}{I_1}}$$

$$\vec{L} = I_1 \vec{\Omega}_\perp + I_3 \Omega_3 \hat{x}_3$$

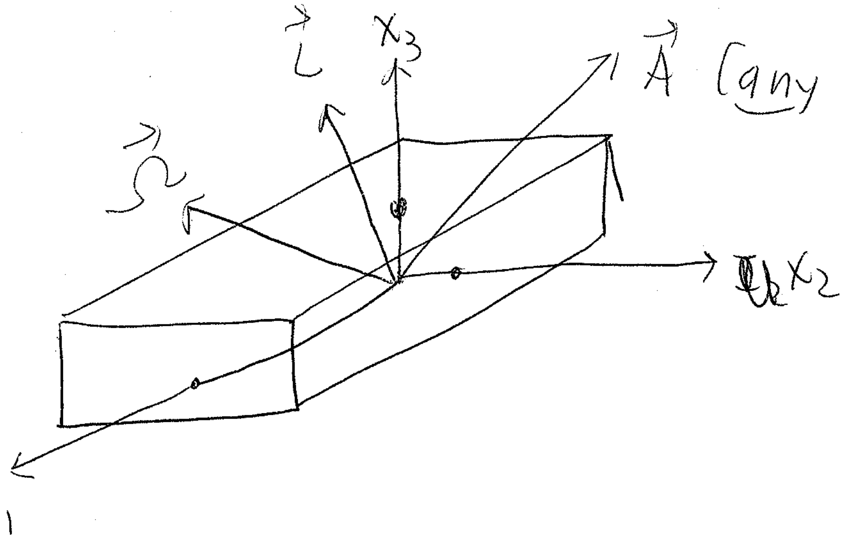
$$\Rightarrow |L| = \sqrt{I_1^2 \Omega_\perp^2 + I_3^2 \Omega_3^2} = \sqrt{I_1^2 \sin^2 B + I_3^2 \cos^2 B}$$

$$\Rightarrow \boxed{\Omega_{pr} = \Omega \sqrt{\sin^2 B + \left(\frac{I_3}{I_1}\right)^2 \cos^2 B}} \ll \Omega \text{ if } B \ll 1, \; I_3 \ll I_1$$

Now, nasty problem: Asymmetrical top (book)

$I_1 < I_2 < I_3$.

Approach: Look in co-rotating co-ordinate system:



Rate of apparent change of <u>any</u> vector $\vec{A}$ in rotating co-ord system:

In co-rotating co-ords: $\vec{A}'$

In lab (unrotating) co-ords: $\vec{A}$

$$\underbrace{\frac{d\vec{A}}{dt}}_{\text{Lab}} = \underbrace{\frac{d\vec{A}'}{dt}}_{\text{co-rotating}} + \vec{\Omega} \times \vec{A}'$$

$$\Rightarrow \boxed{\underbrace{\frac{d\vec{A}'}{dt}}_{\text{co-rotating}} = \underbrace{\frac{d\vec{A}}{dt}}_{\text{lab}} - \vec{\Omega} \times \vec{A}}$$

Now, apply this for freely rotating object, to $\vec{L}$:

$$\underbrace{\frac{d\vec{L}}{dt}}_{\text{lab}} = ?$$

$$\Rightarrow \boxed{\underbrace{\frac{d\vec{L}'}{dt}}_{\text{co-rotating}} = -\vec{\Omega} \times \vec{L}'} \quad (1)$$

$\Rightarrow$ $\vec{L}$ appears to change in co-rotating frame

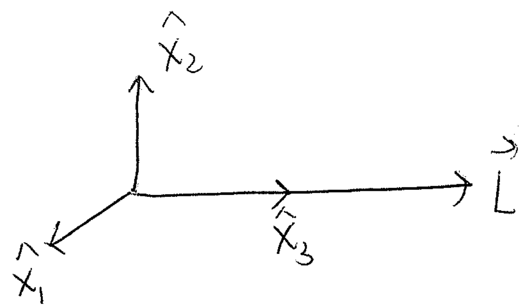
Unless $\vec{L}' \parallel \vec{\Omega}$.

When does that happen?

$\vec{\Omega} \parallel$ principal axis

If that happens, $\vec{L}$ looks constant, even in co-rotating frame

Say, e.g., $\vec{\Omega} \parallel \hat{x}_3 \Rightarrow \vec{L} \parallel \hat{x}_3$



$\Rightarrow$ rotation must be purely around $\hat{x}_3$ always

$$\Omega_3 = \frac{L_3}{I_3} = \Omega_{30} = \text{constant}$$

$\Rightarrow$ simple rotation.

Now, suppose $\vec{\Omega} \nparallel$ any principle axis

$\Rightarrow \vec{L} \nparallel \vec{\Omega} \Rightarrow \vec{L}$ moves in co-rotating frame

$\Rightarrow$ motion of object <u>not</u> simple rotation.

How to analyse? ~~in general?~~

Linearize EOM's for $\vec{L}$!

1st, write out (1) component by component

$$\frac{dL_x}{dt} = -(\Omega_y L_z - \Omega_z L_y)$$

$$\frac{dL_y}{dt} = -(\Omega_z L_x - \Omega_x L_z)$$

$$\frac{dL_z}{dt} = -(\Omega_x L_y - \Omega_y L_x)$$

What are Ω_x, Ω_y, Ω_z in terms of L_x, L_y, L_z?

$\Omega_x =$

$\Omega_y =$

$\Omega_z =$

$$\Rightarrow \frac{dL_x}{dt} = \left(\frac{1}{I_z} - \frac{1}{I_y}\right) L_y L_z$$

$$\frac{dL_y}{dt} = \left(\frac{1}{I_x} - \frac{1}{I_z}\right) L_x L_z$$

$$\frac{dL_z}{dt} = \left(\frac{1}{I_y} - \frac{1}{I_x}\right) L_x L_y$$

Nasty, non-linear. How to solve?

Linearize!

· Rotation <u>nearly</u> about, say, x-axis

$\Rightarrow L_x >> L_y, L_z$

$\Rightarrow$ Linearize ~~about~~ in L_y, L_z

EOM's become: ?

$$\frac{dL_x}{dt} = 0 \Rightarrow L_x = \text{constant} \cong I_x \Omega_0$$

$$\frac{dL_y}{dt} = \left(\frac{1}{I_x} - \frac{1}{I_z}\right) L_x L_z = \left(1 - \frac{I_x}{I_z}\right) \Omega_0 L_z$$

$$\frac{dL_z}{dt} = \left(\frac{1}{I_y} - \frac{1}{I_x}\right) L_x L_y = \left(\frac{I_x}{I_y} - 1\right) \Omega_0 L_y$$

2 coupled linear equations. Solution?

Suppose $I_x < I_y < I_z \Rightarrow$ stable or unstable ?

" $I_x > I_y > I_z \Rightarrow$?

" $I_y > I_x > I_z$ or $I_z > I_y > I_x$:

Solution:

$$\begin{bmatrix} L_y \\ L_z \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} e^{\lambda t}$$

$$\lambda a = \left(1 - \frac{I_y}{I_z}\right)\Omega_0 b$$

$$\lambda b = \left(\frac{I_x}{I_y} - 1\right)\Omega_0 a$$

$$\Rightarrow \begin{pmatrix} \lambda & \left(1-\frac{I_x}{I_z}\right)\Omega_0 \\ \left(\frac{I_x}{I_y}-1\right)\Omega_0 & \lambda \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \lambda & \left(1-\frac{I_x}{I_z}\right)\Omega_0 \\ \left(\frac{I_x}{I_y}-1\right)\Omega_0 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - \Omega_0^2\left(1-\frac{I_x}{I_z}\right)\left(\frac{I_x}{I_y}-1\right) = 0$$

$$\Rightarrow \lambda^2 = \Omega_0^2\left(1-\frac{I_x}{I_y}\right)\left(\frac{I_x}{I_y}-1\right) \begin{cases} < 0 & \text{if } I_x > I_y > I_z \Rightarrow I_x \text{ biggest} \\ < 0 & \text{if } I_x < I_y < I_z \Rightarrow I_x \text{ smallest} \\ > 0 & \text{if } I_y > I_x > I_z \text{ or } I_z < I_x < I_y \end{cases}$$

stable

$\Downarrow$ unstable

How to analyse?

Conservation laws: $I_x < I_y < I_z$

What's conserved?

$$E = \frac{1}{2} I_{ij} \Omega_i \Omega_j = \frac{1}{2}\left(I_x \Omega_x^2 + I_y \Omega_y^2 + I_z \Omega_z^2\right)$$

$$\Omega_x = \frac{L_x}{I_x}, \quad \Omega_y = \frac{L_y}{I_y}, \quad \Omega_z = \frac{L_z}{I_z}$$

$$\Rightarrow \boxed{E = \frac{1}{2}\left(\frac{L_x^2}{I_x} + \frac{L_y^2}{I_y} + \frac{L_z^2}{I_z}\right) = \text{constant}} \quad (2)$$

Ellipsoid

what else (co-rotating frame)

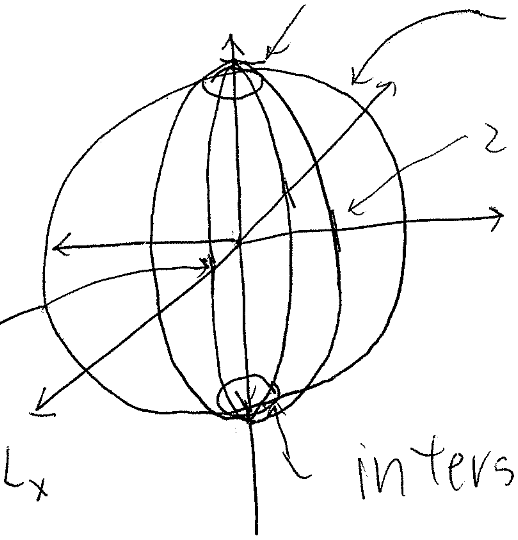
$$\vec{L} \cdot \frac{d\vec{L}}{dt} = ?$$

$$\Rightarrow |\vec{L}|^2 = \text{constant}$$

$$\Rightarrow \boxed{L_x^2 + L_y^2 + L_z^2 = L^2 \qquad (3)}$$

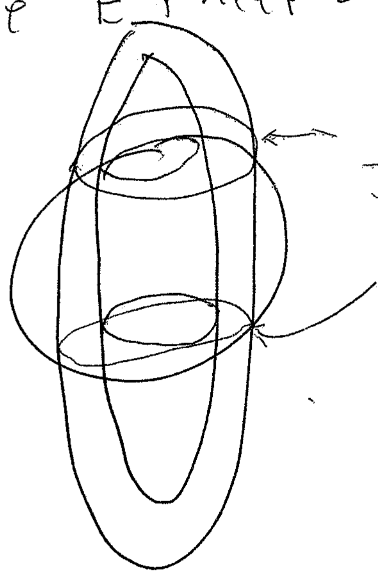
sphere

Plot these conditions in $\vec{L}$ space



$\Rightarrow$ start near z, stay near z

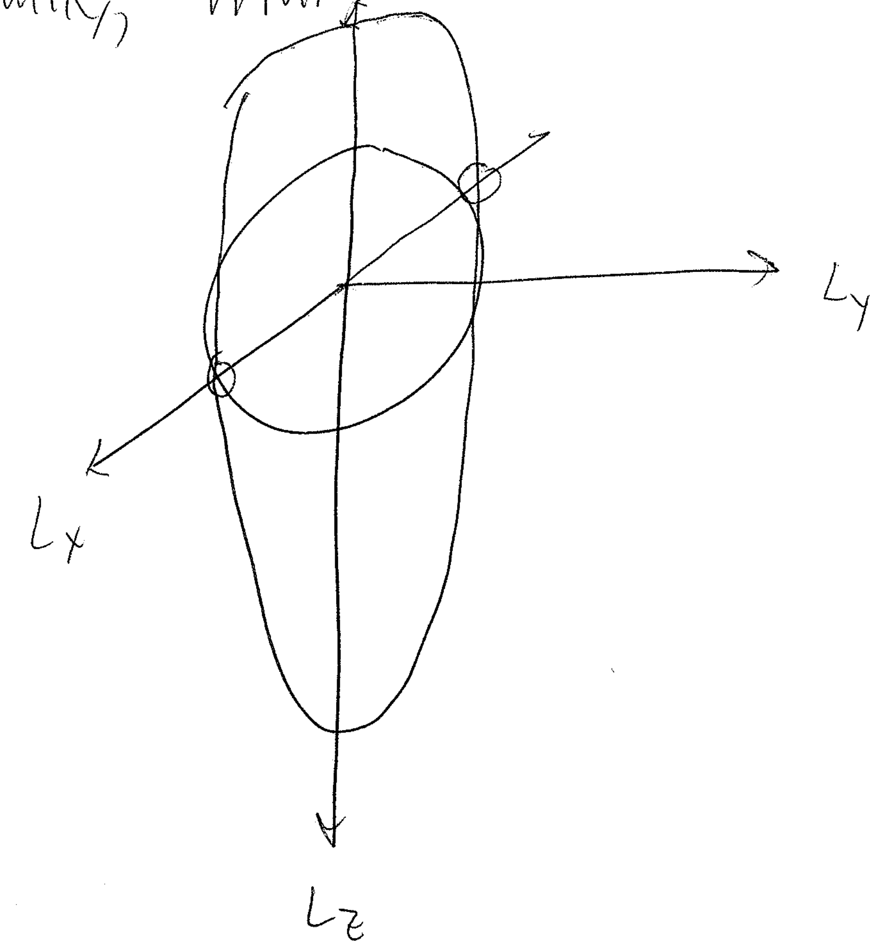
Now, increase E, keep L fixed



Intersections grow

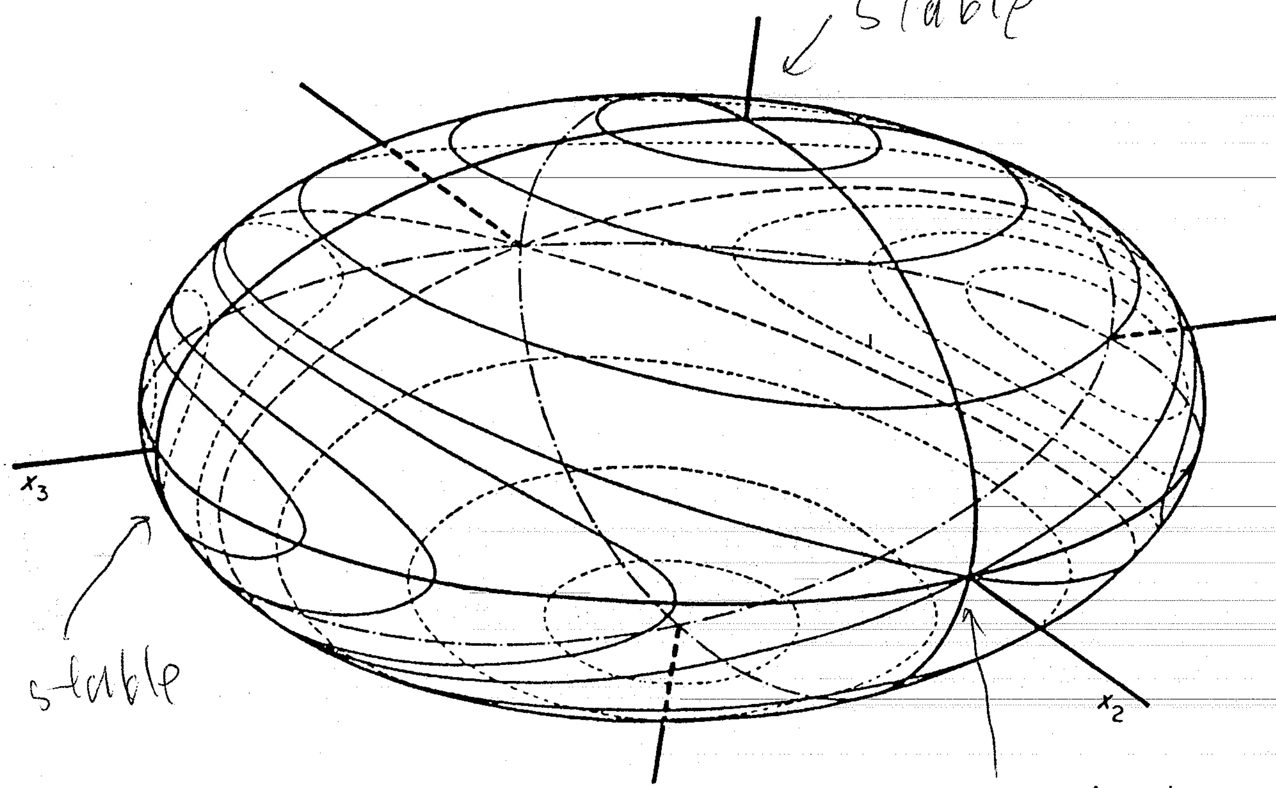
$\Rightarrow$ wander further

Eventually, maximize E



7.3 #8

Better picture



PS #2: due date postponed at least until 2/7 (wednesday)

7.39

$$\dot{\vec{L}} = \frac{d}{dt}\left(\sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha\right)$$

0 (why?)

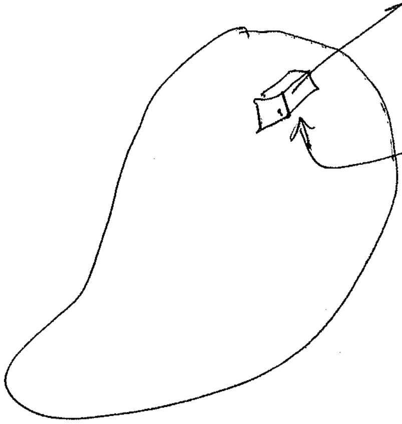
$$= \sum_\alpha \left(\cancel{\dot{\vec{r}}_\alpha \times \vec{p}_\alpha} + \vec{r}_\alpha \times \dot{\vec{p}}_\alpha\right)$$

$$= \sum_\alpha \vec{r}_\alpha \times \vec{f}_\alpha \equiv \vec{\tau} = \text{"torque"}$$

"local Force density" = force/unit volume

Continuous bodies: $d\vec{f} \equiv \vec{f}(\vec{r})\, dV$

Force on dV



volume $dV = dx\, dy\, dz$

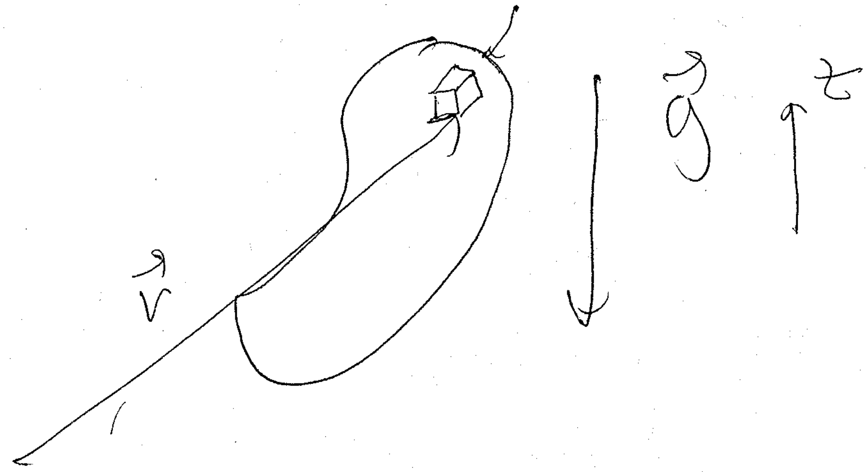
$$\Rightarrow \boxed{\vec{\tau} = \sum_\alpha \vec{r}_\alpha \times \vec{f}_\alpha = \sum_{dV\text{'s}} \vec{r}_\alpha \times \vec{f}(\vec{r})\, dV = \int dx\, dy\, dz \left(\vec{r} \times \vec{f}(\vec{r})\right)}$$

with similar results for thin rods, planes, etc.

~~$\vec{\tau}$ changes if we change origin~~

Special case: gravity (uniform):

$\vec{f}(\vec{r}) = ?$



$\vec{f}(\vec{r})\, dV =$

$$\Rightarrow \vec{\tau} = \cancel{0} - g \int \rho(\vec{r})\, \vec{r} \times \hat{z}\, d^3 r$$

$$= -g \left(\underbrace{\int \rho(\vec{r})\, \vec{r}\, d^3 r}_{= \, ?} \right) \times \hat{z}$$

~~Cheat sheet~~ (7.4)

$$\vec{f}(\vec{r})\,dV = -dm\,g\,\hat{z} = -\rho(\vec{r})\,g\,\hat{z}\,dV$$

$$\Rightarrow \boxed{\vec{f}(\vec{r}) = -\rho(\vec{r})\,g\,\hat{z}}$$

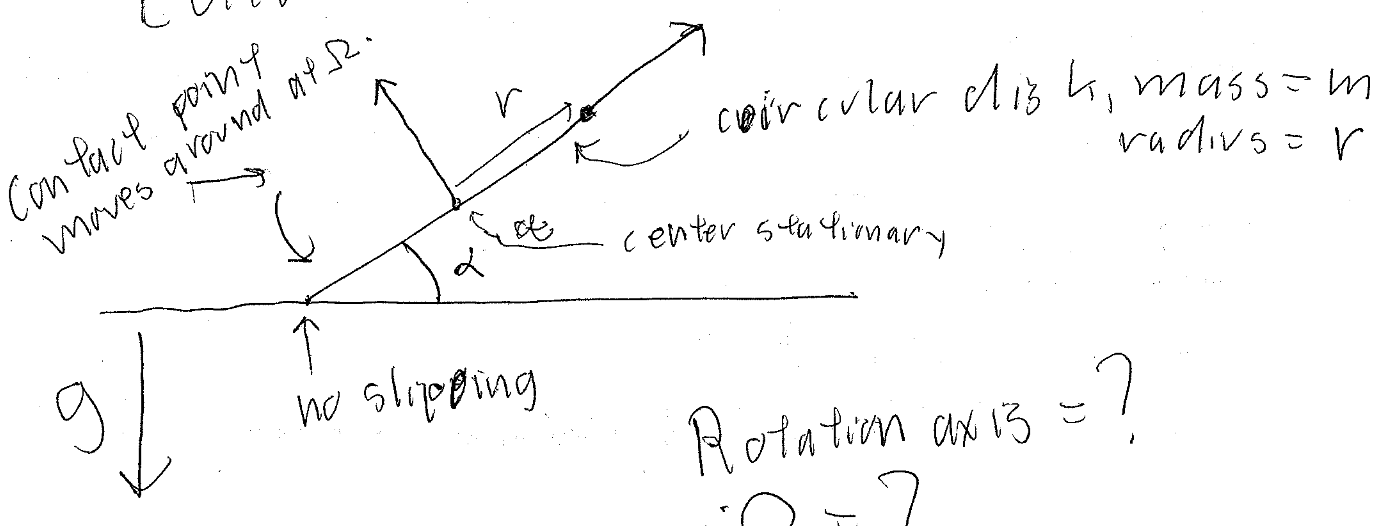
~~Put~~

$$\int \rho(\vec{r})\,\vec{r}\,d^3r = M\vec{R}_{cm}$$

$$\Rightarrow \boxed{\vec{\tau} = -Mg\,\vec{R}_{cm}\times\hat{z}}$$

~~Example~~: Using this:

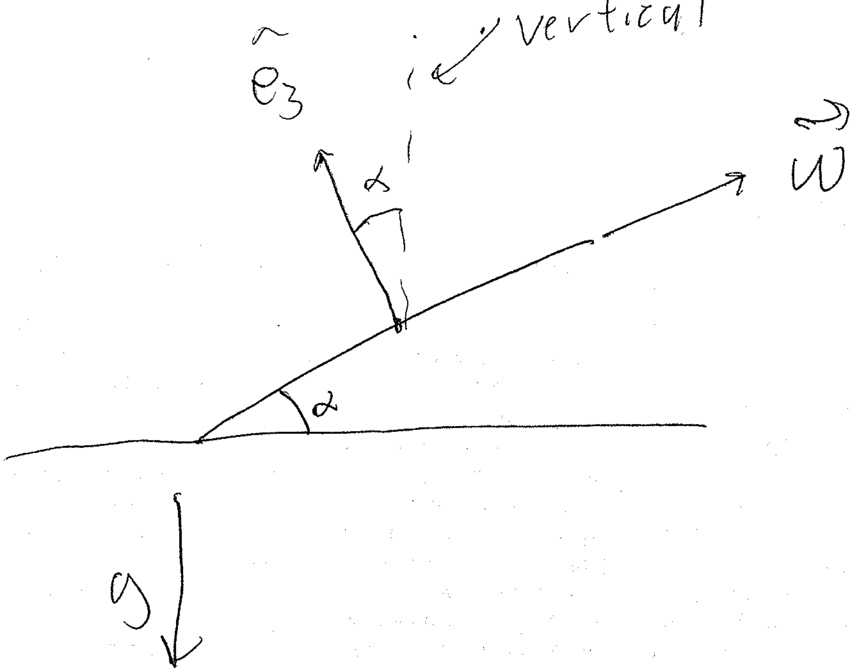
Euler disk:



Rotation axis = ?

$\Omega = ?$

7.42



$$\frac{d\hat{e}_3}{dt} =$$

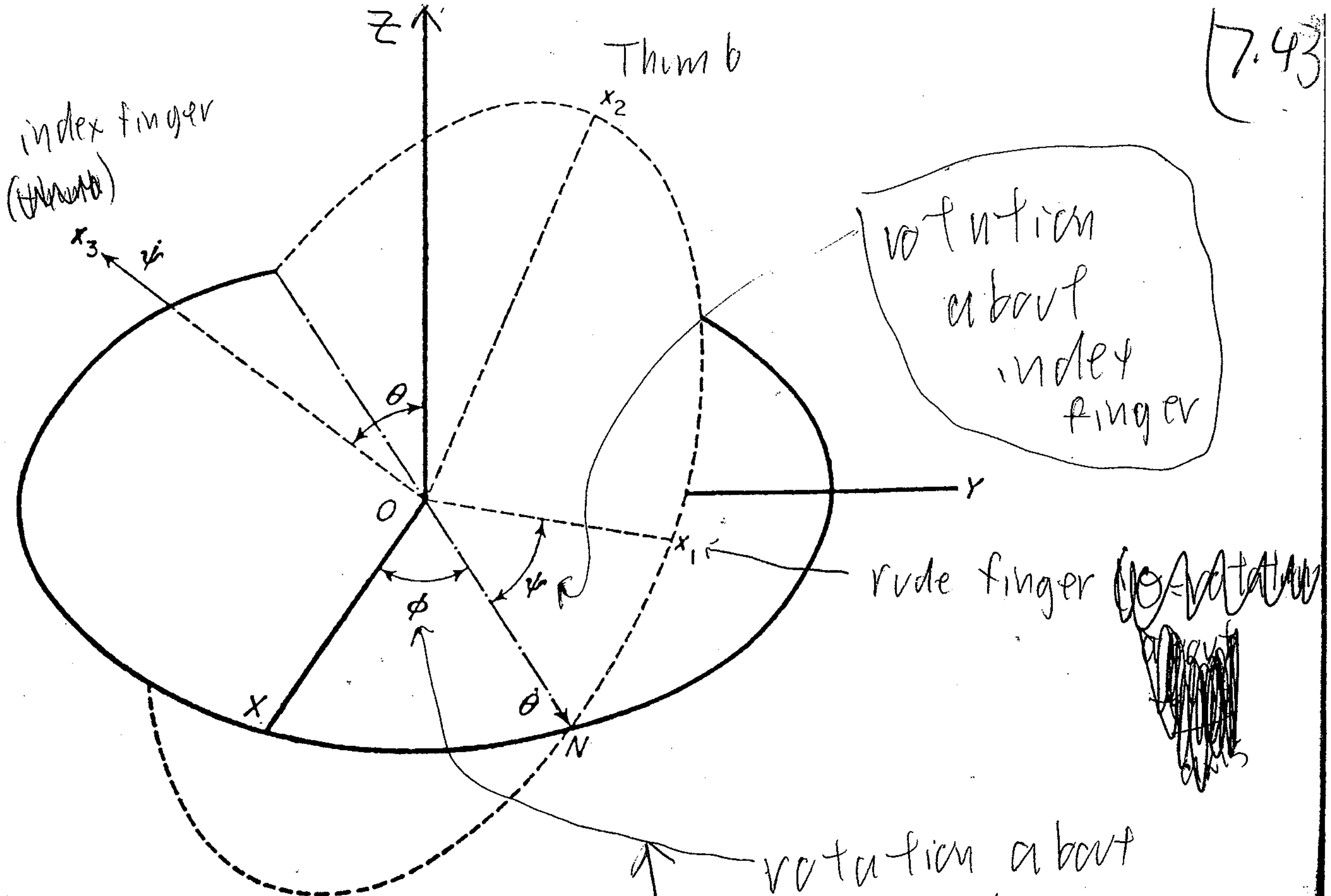


FIG. 47

Better pic from L+L

Euler & Lagrangian:

Getting T (kinetic E)

7.44

Let Lagrangian for this:

$$\mathcal{L} = T - U$$

$$\cancel{T = \frac{1}{2}\left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega\right.}$$

$$T = \frac{1}{2} I_{ij} \Omega_i \Omega_j = \frac{1}{2}\left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2\right)$$

So: what are $\Omega_1, \Omega_2, \Omega_3$ in terms of $\theta, \phi, \psi, \dot{\theta}, \dot{\phi}, \dot{\psi}$?

~~Suppose only~~ $\cancel{\dot{\theta} \neq 0}$

$\cancel{\Omega_1 =}$

$\cancel{|\vec{\Omega}| = \dot{\theta}}$

Suppose only $\dot{\theta} \neq 0$

$|\vec{\Omega}| = ?$

direction of $\vec{\Omega} = ?$

$$\Rightarrow \boxed{\vec{\Omega}(\dot{\phi} = \dot{\psi} = 0) = \dot{\theta}\,\hat{N}}$$

Now, suppose ~~$\dot{\theta} = 0$~~, $\dot{\phi} \neq 0$, $\dot{\theta} = \dot{\psi} = 0$

$|\vec{\Omega}| = ?$

direction of $\vec{\Omega} = ?$

$$\Rightarrow \vec{\Omega}(\dot{\theta} = \dot{\psi} = 0) = \dot{\phi}\,\hat{z}$$

Finally, $\dot{\theta} = \dot{\phi} = 0$, $\dot{\psi} \neq 0$

$|\vec{\Omega}| = ?$

direction of $\vec{\Omega} = ?$

$$\vec{\Omega}(\dot{\theta}=\dot{\phi}=0)=\dot{\psi}\hat{x}_3$$

Now, what if all 3 $(\dot{\theta},\dot{\phi},\dot{\psi})\neq 0$?

Just add above results:

$$\Rightarrow \boxed{\vec{\Omega}=\dot{\theta}\hat{N}+\dot{\phi}\hat{z}+\dot{\psi}\hat{x}_3}$$

So now, what are $(\Omega_1, \Omega_2, \Omega_3)$?

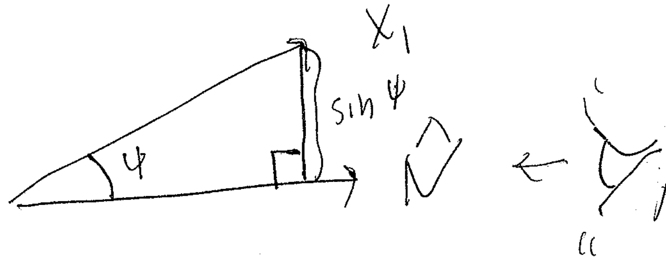
$$\Omega_1=\vec{\Omega}\cdot\hat{x}_1=\dot{\theta}\hat{N}\cdot\hat{x}_1+\dot{\phi}\hat{z}\cdot\hat{x}_1+\dot{\psi}\hat{x}_3\cdot\hat{x}_1$$

$\hat{x}_3\cdot\hat{x}_1=?$

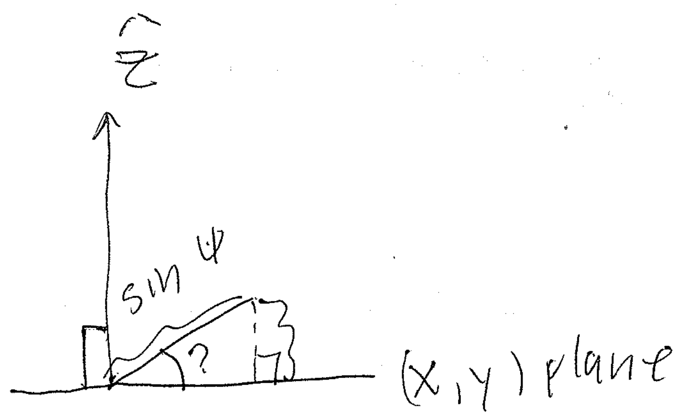
$\hat{N}\cdot\hat{x}_1=?$

$\hat{z}\cdot\hat{x}_1=?$

Look down along $\hat{x}_3$



Now look down along $\hat{N}$:



$\Rightarrow \hat{x}_3 \cdot \hat{z} = \sin\psi \sin\theta$

$$\Rightarrow \boxed{\Omega_1 = \dot{\theta}\cos\psi + \dot{\phi}\sin\psi\sin\theta}$$

$$\Omega_2 = \vec{\Omega} \cdot \hat{x}_2$$

$$= \dot{\theta}\, \hat{N} \cdot \hat{x}_2 + \dot{\phi}\, \hat{z} \cdot \hat{x}_2 + \dot{\psi}\, \hat{x}_3 \cdot \hat{x}_2$$

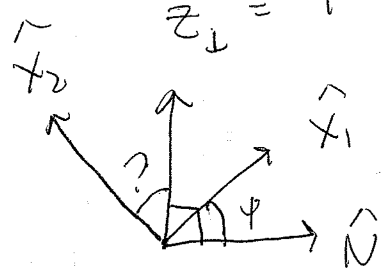
$\hat{N} \cdot \hat{x}_2 = ?$

$\hat{z} \cdot \hat{x}_2 = ?$

$\hat{x}_3 \cdot \hat{x}_2 = ?$

Look down $\hat{x}_3$ again:

$\vec{z}_\perp \equiv$ proj. of $\hat{z}$ $\perp$ to $\hat{x}_3$



$$\Rightarrow \hat{z} \cdot \hat{x}_2 = \vec{z}_\perp \cdot \hat{x}_2 = \cancel{\cos}\, |\vec{z}_\perp| \cos\psi$$

$$|\vec{z}_\perp| = \sin(\theta)$$

$$\Rightarrow \hat{z} \cdot \hat{x}_2 = \sin\theta \cos\psi$$

$$\Rightarrow \boxed{\Omega_2 = -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi}$$

Finally,

$$\Omega_3 = \vec{\Omega}\cdot\hat{X}_3 = \dot{\theta}\,\hat{N}\cdot\hat{X}_3 + \dot{\phi}\,\hat{z}\cdot\hat{X}_3 + \dot{\psi}\,\hat{X}_3\cdot\hat{X}_3$$

$$\hat{N}\cdot\hat{X}_3 = ?$$

$$\hat{z}\cdot\hat{X}_3 = ?$$

$$\hat{X}_3\cdot\hat{X}_3 = ?$$

$$\Rightarrow \boxed{\Omega_3 = \dot{\phi}\cos\theta + \dot{\psi}}$$

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$$\Omega_3 = \hat{x}_3 \cdot \vec{\Omega}$$

$$\cancel{= \cos\theta\,\dot{\phi}}$$

$$\Rightarrow \boxed{\Omega_3 = \cos\theta\,\dot{\phi} + \dot{\psi}}$$

$$\boxed{\Omega_1 = \dot{\theta}\cos\psi + \dot{\phi}\sin\psi\sin\theta}$$

$$\boxed{\Omega_2 = -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi}$$

$$T = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

$$= \frac{1}{2}\Big\{\dot{\theta}^2\left[I_1\cos^2\psi + I_2\sin^2\psi\right]$$

$$+ \dot{\phi}^2\left[I_1\sin^2\psi\sin^2\theta + I_2\sin^2\theta\cos^2\psi + I_3\cos^2\theta\right]$$

$$+ \dot{\psi}^2 I_3$$

$$+ 2\dot{\theta}\dot{\phi}\sin\psi\cos\psi\sin\theta\left[I_1 - I_2\right]$$

$$+ 2\dot{\phi}\dot{\psi}\cos I_3\Big\}$$

7.51

Symmetrical top:

$$I_1 = I_2 \neq I_3$$

$$\Rightarrow T = \frac{1}{2}\left\{ I_1\left(\dot\theta^2 + \dot\phi^2 \sin^2\theta\right) + I_3\left(\dot\phi^2\cos^2\theta + 2\dot\phi\dot\psi\cos\theta + \dot\psi^2\right)\right\}$$

Note: Independent of ϕ, ψ!

$$\hookrightarrow = \frac{1}{2}\left[I_1\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) + I_3\left(\dot\phi\cos\theta + \dot\psi\right)^2\right]$$

Consider Free top: $U(\theta, \phi, \psi) = 0 \Rightarrow \mathcal{L} = T$

$$p_\phi \equiv \frac{\partial \mathcal{L}}{\partial \dot\phi} = I_1 \sin^2\theta\,\dot\phi + I_3\left(\dot\phi\cos\theta + \dot\psi\right)\cos\theta = \text{constant}$$

because $\mathcal{L}$ independent of ϕ

What is this, physically?

$$\vec{L} = \underset{\sim}{I}\,\vec\Omega = I_1\left(\Omega_1 \hat{x}_1 + \Omega_2\hat{x}_2\right) + I_3\Omega_3\hat{x}_3$$

$$\Rightarrow L_z = \vec{L}\cdot\hat{z} = I_1\left(\Omega_1\hat{x}_1\cdot\hat{z} + \Omega_2\hat{x}_2\cdot\hat{z}\right) + I_3\Omega_3\hat{x}_3\cdot\hat{z}$$

$$\cancel{= I_1\left(\dot\theta\cos\psi + \dot\phi\sin\psi\sin\theta - \dot\theta\sin\psi\right.}$$

~~Stable~~

7.52

$$L_z = I_1 \Big[\big(\cancel{\dot\theta \cos\psi} + \dot\phi \sin\psi \sin\theta \big) \sin\theta \sin\psi + \big(\cancel{-\dot\theta \sin\psi} + \dot\phi \sin\theta \cos\psi \big) \sin\theta \cos\psi \Big]$$

$$+ I_3 (\cos\theta\, \dot\phi + \dot\psi) \cos\theta$$

$$= I_1 \dot\phi \sin^2\theta + I_3 (\dot\phi \cos\theta + \dot\psi) \cos\theta$$

$$\Rightarrow p_\phi = L_z = \text{constant}$$

Other conserved quantities:

$$\frac{\partial \mathcal{L}}{\partial \dot\psi} = I_3 (\dot\phi \cos\theta + \dot\psi) = \text{const.}$$

because $\mathcal{L}$ independent of ψ

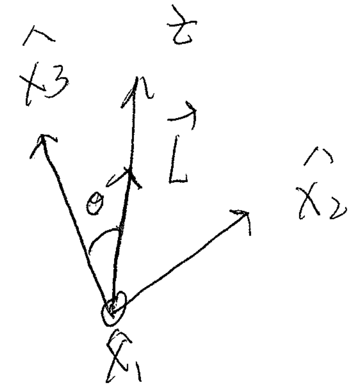
$$L_3 = \hat{x}_3 \cdot \vec{L} = I_3 \Omega_3 = I_3 (\dot\phi \cos\theta + \dot\psi) = \text{const}$$

One other conserved quantity:

$$E = T = \frac{1}{2} \Big[I_1 (\dot\theta^2 + \dot\phi^2 \sin^2\theta) + I_3 (\dot\phi \cos\theta + \dot\psi)^2 \Big]$$

$$= \text{constant}$$

$$I_1 \dot{\phi} \sin^2\theta + L_3 \cos\theta = L_z^0 = \text{const.}$$



$$\vec{L} = I_1 \vec{\Omega}_\perp + I_3 \Omega_3 \hat{x}_3$$

$$\Rightarrow \vec{L}_\perp = I_1 \vec{\Omega}_\perp = L \sin\theta \, \hat{x}_2$$

$$\Rightarrow \vec{\Omega}_\perp \overset{t}{=} \frac{L \sin\theta}{I_2} \hat{x}_2$$

$\Rightarrow \vec{\Omega}$ in $\hat{x}_2, \hat{x}_3$ plane

$$\Rightarrow \cancel{\vec{\Omega} \times \hat{x}_3} \; \left.\frac{d\hat{x}_3}{dt}\right|_{t=0} = \vec{\Omega} \times \hat{x}_3 \quad \perp \text{ to plane}$$

$$\Rightarrow \dot{\theta}(t=0) = 0$$

7.54

$$\Rightarrow \cancel{E = \frac{1}{2}\left[I_1 \dot{\phi}_0^2 \sin^2\theta_0 + I_3(\dot{\phi}_0 \cos\theta_0 + \dot{\psi}_0)^2\right]}$$

$$\cancel{L_3 = I_3(\dot{\phi}_0 \cos\theta_0 + \dot{\psi}_0)}$$

$$\cancel{L_z = I_1 \dot{\phi}_0 \sin^2\theta_0 + L_3 \cos\theta}$$

$$E = \frac{1}{2}\left[I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{L_3^2}{I_3}\right]$$

$$I_1 \dot{\phi} \sin^2\theta + L_3 \cos\theta = L_z$$

$$\Rightarrow \boxed{\dot{\phi} = \frac{L_z - L_3 \cos\theta}{I_1 \sin^2\theta}}$$

$$\Rightarrow \boxed{E = \frac{1}{2}\left[I_1 \dot{\theta}^2 + \frac{(L_z - L_3\cos\theta)^2}{I_1 \sin^2\theta} + \frac{L_3^2}{I_3}\right]}$$

choose z axis along $\vec{L}(t=0)$

$$\Rightarrow L_z = L \ , \ L_3 = L\cos\theta_0$$

$$\cancel{\Rightarrow L_z - L_3\cos\theta = L(1-\cos^2\theta) = L\sin^2\theta}$$

7.55

$$E = \frac{1}{2}\left[\frac{L^2(1-\cos^2\theta_0)^2}{I_1\sin^2\theta_0} + \frac{L_3^2}{I_3}\right]$$

$$= \frac{1}{2}\left(\frac{L^2\sin^2\theta_0}{I_1} + \frac{L_3^2}{I_3}\right)$$

$$= \frac{1}{2}I_1\dot{\theta}^2 + \frac{L^2(1-\cos\theta_0\cos\theta)^2}{2I_1\sin^2\theta} + \frac{L_3^2}{2I_3}$$

$$\Rightarrow \frac{I_1}{\cancel{L^2}}\dot{\theta}^2 = \frac{L^2}{I_1}\underbrace{\left[\sin^2\theta_0 - \frac{(1-\cos\theta_0\cos\theta)^2}{\sin^2\theta}\right]}_{U_e(\theta)}$$

$$\cancel{U_e(\theta) = \sin^2\theta_0 - \frac{1-\cos}{\sin^2\theta}}$$

$$U_e(\theta) = \frac{\sin^2\theta_0\sin^2\theta - 1 + 2\cos\theta_0\cos\theta - \cos^2\theta_0\cos^2\theta}{\sin^2\theta}$$

7.56

Num

$$= \sin^2\theta\,(\sin^2\theta_0 + \cos^2\theta_0) - 1 + 2\cos\theta_0\cos\theta$$

$$- \cos^2\theta_0$$

$$= \sin^2\theta + 2\cos\theta_0\cos\theta - 1 - \cos^2\theta_0$$

$$= 2\cos\theta_0\cos\theta - \cos^2\theta - \cos^2\theta_0$$

$X \equiv \cos\theta$

$$\text{Num} = 2\cos\theta_0\, X - X^2 - \cos^2\theta_0 \equiv f(X)$$

$$f'(X) = 2\cos\theta_0 - 2X = 0 \Rightarrow \boxed{X_m = \cos\theta_0}$$

$$\Rightarrow \theta_m = \theta_0$$

is this a min, or a max?

$$f(1) = 2\cos\theta_0 - 1 - \cos^2\theta_0$$

$$= -(1 - 2\cos\theta_0 + \cos^2\theta_0)$$

$$= -(1-\cos\theta_0)^2 < 0$$

$\Rightarrow$ ~~this is~~ $X_m = X_m = \cos\theta_0$ is a max

$$\Rightarrow U_e^{max} = 0 \;\cancel{[\sin^2\theta]} \quad \Rightarrow \dot{\theta}^2_{max} = 0$$

$$\Rightarrow \dot{\theta} = 0, \text{ always}$$

$\theta = \theta_0$, always

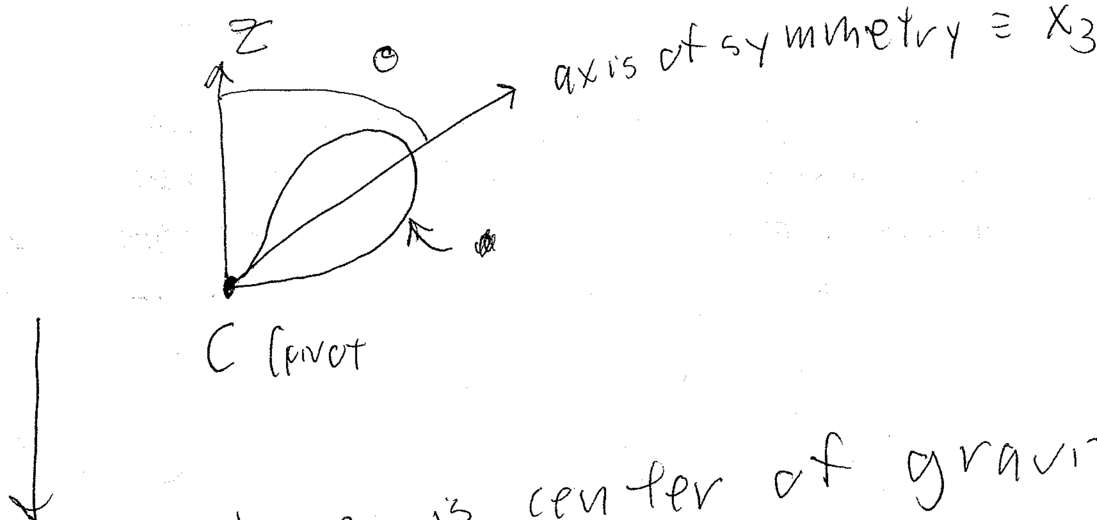
7.57

$\Rightarrow \dot{\phi}$ = constant , $\dot{\psi}$ = constant

$\Downarrow$ constant precession rate

$\Downarrow$ constant & rotation velocity about $\hat{x}_3$.

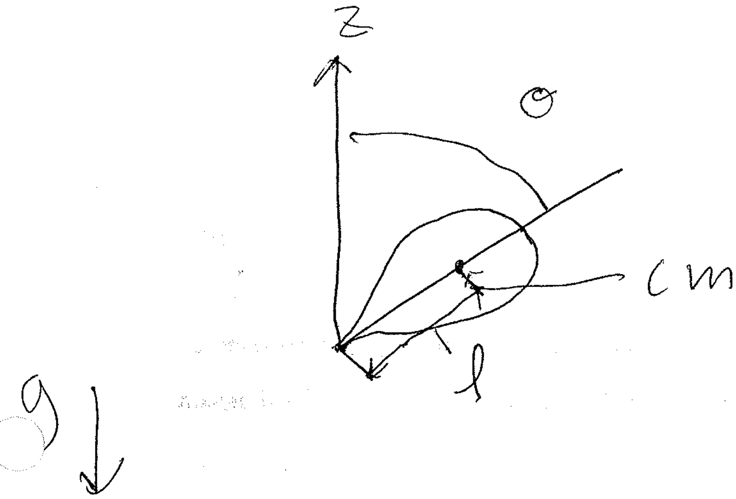
Now, symmetrical top in gravity:



where is center of gravity?

$\Rightarrow U_{grav}(\theta, \phi, \psi) = ?$

7.58



$$U_g(\theta, \phi, \psi) = mg z_{cm} = mgl\cos\theta$$

Note: Independent of ϕ, ψ!

$\Rightarrow \mathcal{L} = T - U$ <u>also</u> independent of ϕ, ψ!

$$\Rightarrow p_\phi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = I_1 \sin^2\theta\, \dot\phi + I_3(\dot\phi\cos\theta + \dot\psi)\cos\theta = \text{constant}$$

$$= L_z$$

$$p_\psi = L_3 = I_3(\dot\phi\cos\theta + \dot\psi) = \text{constant.}$$

Third conserved quantity: Energy; which now has $U_g(\theta)$ piece:

(0) $$\boxed{E = T + U = \frac{1}{2}\left[I_1(\dot\theta^2 + \dot\phi^2\sin^2\theta) + I_3(\dot\phi\cos\theta + \dot\psi)^2\right] + mgl\cos\theta = \text{constant.}}$$

Can use these 3 conservation laws to get "effective potential" for θ. Just like "effective potential" for central force motion.

(1) $$\boxed{\dot\phi\cos\theta + \dot\psi = \frac{L_3}{I_3}}$$

(2) $$\boxed{I_1\sin^2\theta\,\dot\phi + I_3(\dot\phi\cos\theta + \dot\psi)\cos\theta = L_z}$$