# Physics 633 Homework 5

Jeremy Welsh-Kavan

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1) We begin with the electromagnetic wave equation in an inhomogenous dielectric medium:

$$\left[\nabla^2 - \frac{n^2(\boldsymbol{r})}{c^2}\partial_t^2\right]\mathscr{E}(\boldsymbol{r},t) = 0 \tag{1}$$

where $n(\boldsymbol{r})$ is the medium's refractive index profile. We shall assume that $\mathscr{E}(\boldsymbol{r},t)$ can be expressed as a monochromatic plane wave

$$\mathscr{E}(\boldsymbol{r},t) = \mathscr{E}(\boldsymbol{r})e^{-i\omega t} \tag{2}$$

With $k = \omega/c$, the wave equation then becomes

$$\left[\nabla^2 + k^2 n^2(\boldsymbol{r})\right]\mathscr{E}(\boldsymbol{r}) = 0 \tag{3}$$

Define $V(\boldsymbol{r}) = k^2 - k^2 n^2(\boldsymbol{r})$. Then the wave equation can be rewritten in a form more reminiscent of the Schrodinger equation:

$$\begin{aligned} &\left[\nabla^2 + k^2 + V(\boldsymbol{r})\right]\mathscr{E}(\boldsymbol{r}) = 0 \\ \implies &\left[-\nabla^2 + V(\boldsymbol{r})\right]\mathscr{E}(\boldsymbol{r}) = k^2\mathscr{E}(\boldsymbol{r}) \end{aligned} \tag{4}$$

a) Now, setting $E = k^2$,

$$\left[-\nabla^2 + V(\boldsymbol{r})\right]\mathscr{E}(\boldsymbol{r}) = E\mathscr{E}(\boldsymbol{r}) \tag{5}$$

We can define a wave function, $\varphi$, by

$$\varphi(\boldsymbol{r},t) = \mathscr{E}(\boldsymbol{r}) e^{-iET} \tag{6}$$

In which case, $\varphi$ satisfies

$$\left[-\nabla^2 + V(\boldsymbol{r})\right] \varphi(\boldsymbol{r},t) = i\partial_T \varphi(\boldsymbol{r},t) \tag{7}$$

In general, solutions $\varphi(\boldsymbol{r},t)$ can be written as a superposition of plane waves

$$\varphi(\boldsymbol{r},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dE \, \tilde{\varphi}(\boldsymbol{r},E) \, e^{-iET} \tag{8}$$

In which case, we can also write

$$\mathscr{E}(\boldsymbol{r}) = \int_{-\infty}^{\infty} dT \, \varphi(\boldsymbol{r},T) \, e^{ik^2 T} \tag{9}$$

and we claim that this satisfies the wave equation in Eq. (3). To show this, we simply plug Eq. (5) into Eq. (3).

$$\begin{aligned} \left[\nabla^2 + k^2 n^2(\boldsymbol{r})\right] \mathscr{E}(\boldsymbol{r}) &= \int_{-\infty}^{\infty} dT \, \left[\nabla^2 + k^2 n^2(\boldsymbol{r})\right] \varphi(\boldsymbol{r},T) \, e^{ik^2 T} \\ &= \int_{-\infty}^{\infty} dT \, \left[\nabla^2 - V(\boldsymbol{r}) + i\partial_T\right] \varphi(\boldsymbol{r},T) \, e^{ik^2 T} \end{aligned} \tag{10}$$

But $\varphi(\boldsymbol{r},T)$ satisfies the Schrodinger equation, so we have

$$\left[\nabla^2 + k^2 n^2(\boldsymbol{r})\right] \mathscr{E}(\boldsymbol{r}) = 0 \tag{11}$$

b) The Feynman propagator can be written as

$$K(x,t;x_0,t_0) = \int Dx \exp\left[\frac{i}{\hbar}\int_{t_0}^{t} dt' L(x,\dot{x})\right] \tag{12}$$

In the case of $H = -\nabla^2 + V(\boldsymbol{r})$ with $i\partial_T = H$, the time evolution operator can be rewritten as $U(t+\delta t, t) = e^{-iH\delta t}$. Thus, identifying $m \to 1/2$, and $\hbar \to 1$, we can rewrite the usual propagator for the Schrodinger equation as

$$\begin{aligned} K(x,T;x_0,0) &= \int Dx \exp\left[\frac{i}{\hbar}\int_0^T d\tau \left(\frac{1}{2}m\dot{x}^2 - V(x)\right)\right] \\ K(x,T;x_0,0) &= \int Dx \exp\left[i\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 - V(x)\right)\right] \end{aligned} \tag{13}$$

c) We have that the energy-space Green's function is given by

$$G^+(\boldsymbol{r},\boldsymbol{r}',E) = \frac{1}{i\hbar}\int_0^\infty d\tau\, e^{i(E+i0^+)\tau/\hbar} G^+(\boldsymbol{r},\boldsymbol{r}';\tau) \tag{14}$$

which can be rewritten under the identifications defined in b), and with $E = k^2$, as

$$\begin{aligned} G^+(\boldsymbol{r},\boldsymbol{r}',E) &= \frac{1}{i}\int_0^\infty dT\, e^{ik^2T} K^+(x,T;x_0,0) \\ &= -i\int_0^\infty dT\, e^{ik^2T}\int Dx \exp\left[i\int_0^T d\tau\left(\frac{\dot{x}^2}{4} - V(x)\right)\right] \end{aligned} \tag{15}$$

d) Starting with the reduced action,

$$S_{\text{reduced}}\left[\boldsymbol{x}\right] = \int_0^s \sqrt{k^2 - V(\boldsymbol{x})}ds' \tag{16}$$

we wish to derive Fermat's principle. Recall that $V(\boldsymbol{r}) = k^2 - k^2n^2(\boldsymbol{r})$. Substituting this into the integral gives

$$
\begin{aligned}
S_{\text{reduced}}\left[\boldsymbol{x}\right] &= \int_0^s \sqrt{k^2 n^2(\boldsymbol{r})} ds' \\
S_{\text{reduced}}\left[\boldsymbol{x}\right] &= k \int_0^s n(\boldsymbol{r}) ds'
\end{aligned} \tag{17}
$$

But requiring that this action is stationary is precisely Fermat's principle, since the constant $k$ does not change the functional minimization, so the derivation is complete.

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2) We claim that

$$
\lim_{\epsilon \to 0^+} \int_0^\infty d\tau\, e^{-\epsilon\tau} \cos(\omega\tau) = \pi\delta(\omega) \tag{18}
$$

and

$$
\lim_{\epsilon \to 0^+} \int_0^\infty d\tau\, e^{-\epsilon\tau} \sin(\omega\tau) = \mathscr{P}\left(\frac{1}{\omega}\right) \tag{19}
$$

To show this, consider the following integral

$$
j(\omega) = \lim_{\epsilon \to 0^+} \int_0^\infty d\tau\, e^{-\epsilon\tau} e^{-i\omega\tau} \tag{20}
$$

This integral is easily computed and yields

$$
\begin{aligned}
j(\omega) &= \lim_{\epsilon\to 0^+}\int_0^\infty d\tau\, e^{-\epsilon\tau}e^{-i\omega\tau} \\
&= \lim_{\epsilon\to 0^+}\left[\frac{e^{-(\epsilon+i\omega)\tau}}{-\epsilon-i\omega}\right]_0^\infty \\
&= \lim_{\epsilon\to 0^+}\frac{1}{\epsilon+i\omega} \\
&= \lim_{\epsilon\to 0^+}\frac{1}{i(\omega-i\epsilon)} \\
&= -i\left[\mathscr{P}\left(\frac{1}{\omega}\right)+i\pi\delta(\omega)\right] \\
&= \pi\delta(\omega)-i\mathscr{P}\left(\frac{1}{\omega}\right)
\end{aligned} \tag{21}
$$

So $\mathrm{Re}[j(\omega)] = \pi\delta(\omega)$ and $\mathrm{Im}[j(\omega)] = -\mathscr{P}\left(\frac{1}{\omega}\right)$. But we also have $\mathrm{Re}[e^{-i\omega\tau}] = \cos(\omega\tau)]$ and $\mathrm{Im}[e^{-i\omega\tau}] = -\sin(\omega\tau)]$. Thus, the claim is true.

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3) We compute the following Fourier transform:

$$
j(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\int_{-\infty}^{t} dt'\, \dot{q}(t')\Gamma(t-t') \tag{22}
$$

where

$$
\Gamma(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \frac{2J(\omega)}{M\omega}e^{-i\omega t} \tag{23}
$$

We can insert a $\Theta(t)$ function to eliminate the time dependence of the integral to obtain

$$
\begin{aligned}
j(\omega) &= \int_{-\infty}^{\infty} dt\, e^{i\omega t} \int_{-\infty}^{\infty} dt'\, \dot{q}(t')\Gamma(t-t')\Theta(t-t') \\
&= \int_{-\infty}^{\infty} dt'\, \dot{q}(t') \int_{-\infty}^{\infty} dt\, \Gamma(t-t')\Theta(t-t') e^{i\omega t} \\
&= \int_{-\infty}^{\infty} dt'\, \dot{q}(t') \int_{-\infty}^{\infty} dt\, \Gamma(t)\Theta(t) e^{i\omega(t+t')} \\
&= \int_{-\infty}^{\infty} dt'\, \dot{q}(t') e^{i\omega t'} \int_{-\infty}^{\infty} dt\, \Gamma(t)\Theta(t) e^{i\omega t}
\end{aligned}
\tag{24}
$$

Next, we perform an integration by parts on the first integral to extract $\tilde{q}(\omega)$, and insert the definition of $\Gamma(t)$.

$$
\begin{aligned}
j(\omega) &= -i\omega \int_{-\infty}^{\infty} dt'\, q(t') e^{i\omega t'} \int_{-\infty}^{\infty} dt\, \Gamma(t)\Theta(t) e^{i\omega t} \\
&= -i\omega\, \tilde{q}(\omega) \int_{-\infty}^{\infty} dt\, \Gamma(t)\Theta(t) e^{i\omega t} \\
&= -i\omega\, \tilde{q}(\omega) \int_{0}^{\infty} dt\, \Gamma(t) e^{i\omega t} \\
&= -i\omega \frac{1}{\pi M} \tilde{q}(\omega) \int_{0}^{\infty} dt \int_{-\infty}^{\infty} d\omega'\, \frac{J(\omega')}{\omega'} e^{-i\omega' t} e^{i\omega t} \\
&= -i\omega \frac{1}{\pi M} \tilde{q}(\omega) \int_{-\infty}^{\infty} d\omega'\, \frac{J(\omega')}{\omega'} \int_{0}^{\infty} dt\, e^{-i(\omega'-\omega)t}
\end{aligned}
\tag{25}
$$

The last integral can be rewritten as $\lim_{\epsilon\to 0} \int_0^\infty dt\; e^{-\epsilon t} e^{-i(\omega'-\omega)t}$, which we know from the previous problem is given by $\pi\delta(\omega'-\omega) - i\mathscr{P}\left(\frac{1}{\omega'-\omega}\right)$. So we have

$$
\begin{aligned}
j(\omega) &= -i\omega \frac{1}{\pi M} \tilde{q}(\omega) \int_{-\infty}^{\infty} d\omega'\, \frac{J(\omega')}{\omega'} \left(\pi\delta(\omega'-\omega) - i\mathscr{P}\left(\frac{1}{\omega'-\omega}\right)\right) \\
&= \frac{-i}{M} J(\omega)\tilde{q}(\omega) - i \int_{-\infty}^{\infty} d\omega'\, \frac{J(\omega')}{\omega'} \mathscr{P}\left(\frac{1}{\omega'-\omega}\right) \\
&= \frac{-i}{M} J(\omega)\tilde{q}(\omega) - i \lim_{\epsilon\to 0^+} \int_{\mathbb{R}\setminus(\omega-\epsilon,\omega+\epsilon)} d\omega'\, \frac{J(\omega')}{\omega'(\omega'-\omega)}
\end{aligned}
\tag{26}
$$

But, in an ohmic reservoir, $J(\omega') \propto \omega'$, so the principal value integral is zero. Therefore,

$$\int_{-\infty}^{\infty} dt\, e^{i\omega t} \int_{-\infty}^{t} dt'\ \dot{q}(t')\Gamma(t-t') = \frac{-i}{M} J(\omega)\tilde{q}(\omega) \tag{27}$$

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