Exercise 1)

If $R(t)$ is diagonal in the H basis then

$\langle \psi_n(t) | R(t) | \psi_n(t) \rangle$

$= R_{mn}(t) = \delta_{mn} A_n, \quad A_n \in \mathbb{C}$

But since $R(t)$ is unitary

$(R R^{\dagger})_{mn} = A_n A_n^* \delta_{mn} = \delta_{mn}$

and if $A_n A_n^* = |A_n| = 1$

then $A_n \in \mathbb{T}$

So A_n can be written

$A_n = e^{-i\phi_n}$ for some $\phi_n \in \mathbb{R}$

Exercise 2:

We have that

$$i\hbar \partial_t \tilde{U}(t) = \tilde{H}(t)\tilde{U}(t)$$

writing $i\hbar\partial_t \tilde{U}(t)$ as $i\hbar \dot{\tilde{U}}(t)$,

we have

$$i\hbar \dot{\tilde{U}}(t) = \tilde{H}(t)\tilde{U}(t)$$

since $\tilde{H}(t) = RHR^{\dagger} + i\hbar \dot{R}R^{\dagger}$,

and since $[H(t), R(t)] = 0$

$$\tilde{H}(t) = H + i\hbar \dot{R}R^{\dagger}$$

So

$$i\hbar \dot{\tilde{U}}(t) = H\tilde{U} + i\hbar \dot{R}R^{\dagger}\tilde{U}.$$

$$\rightarrow \quad \tilde{U}^{\dagger}\dot{\tilde{U}} = -\frac{i}{\hbar}\tilde{U}^{\dagger}H\tilde{U} + \tilde{U}^{\dagger}\dot{R}R^{\dagger}\tilde{U}$$

$$\rightarrow \langle \psi_n(0) | \tilde{U}^\dagger \dot{\tilde{U}} | \psi_n(0) \rangle = \frac{-i}{\hbar} \langle \psi_n(0) | \tilde{U}^\dagger H \tilde{U} | \psi_n(0) \rangle$$
$$+ \langle \psi_n(0) | \tilde{U}^\dagger \dot{R} R^\dagger \tilde{U} | \psi_n(0) \rangle$$

We would like

$$\tilde{U}(t) | \psi_n(0) \rangle = \tilde{U}(t) | \tilde{\psi}_n(0) \rangle = | \tilde{\psi}_n(t) \rangle$$

Exercise 3:

By eq (12)

$$R_{mn}(t) = \delta_{mn} e^{-i\varphi_n(t)}$$

So $R^\dagger_{mn}(t) = \delta_{mn} e^{i\varphi_n(t)}$

$$\dot{R}_{mn}(t) = -i \dot{\varphi}_n(t) R_{mn}(t)$$

So

$$(\dot{R} R^\dagger)_{mn} = -i \dot{\varphi}_n(t) \delta_{mn}$$

So

$$\langle \psi_n(0) | \tilde{U}^\dagger \dot{R} R^\dagger \tilde{U} | \psi_n(0) \rangle$$

$$= -i \dot{\varphi}_n(t) \langle \psi_n(0) | \tilde{U}^\dagger \tilde{U} | \psi_n(0) \rangle$$

$$= -i \dot{\varphi}_n(t) .$$ ✓

Exercise 4:

simultaneously setting

$$\psi \to \psi e^{-iq\chi(r)/\hbar}$$

$$A \to A - \nabla \chi ,$$

$$i\hbar \dot{\psi} = \frac{1}{2m} \left(\frac{\hbar}{i} \nabla - q A(r) \right)^2 \psi + q\phi(r) \psi \quad (28)$$

becomes

$$i\hbar \dot{\psi} e^{-iq\chi(r)/\hbar}$$

$$= \frac{1}{2m} \left(\frac{\hbar}{i} \nabla - q A(r) + q \nabla \chi \right)^2 \psi e^{-iq\chi(r)/\hbar}$$

$$+ q\phi(r) \psi e^{-iq\chi(r)/\hbar}$$

but

$$\left(\frac{\hbar}{i}\nabla - q\mathcal{A}(r) + q\nabla\chi\right)\psi e^{-iq\chi(r)/\hbar}$$

$$= \frac{\hbar}{i}\left(\nabla\psi - \frac{iq\nabla\chi}{\hbar}\psi\right)e^{-iq\chi(r)/\hbar}$$

$$+ q\nabla\chi\psi e^{-iq\chi(r)/\hbar}$$

$$- q\mathcal{A}(r)\psi e^{-iq\chi(r)/\hbar}$$

$$= \left(\frac{\hbar}{i}\nabla\psi - q\mathcal{A}(r)\psi\right)e^{-iq\chi(r)/\hbar}$$

$$= \left(\frac{\hbar}{i}\nabla - q\mathcal{A}(r)\right)\psi e^{-iq\chi(r)/\hbar}$$

so $\vec{p}$ is invariant under

$$\psi \rightarrow \psi e^{-iq\chi(r)/\hbar}$$

$$\mathcal{A} \rightarrow \mathcal{A} - \nabla\chi,$$

Therefore

$$i\hbar\dot{\psi} e^{-iq\chi(r)/\hbar} = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathcal{A}(r)\right)^2\psi e^{-iq\chi(r)/\hbar}$$

$$+ q\phi(r)\psi e^{-iq\chi(r)/\hbar}$$

and we recover

$$i\hbar \dot{\psi} = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathcal{A}(r)\right)^2 \psi + q\phi(r)\psi.$$

Exercise 5:

$$0 = \int d^3r \, \nabla_R |\psi_0(r-R)|^2$$

$$= \int d^3r \, \nabla_R \psi_0^*(r-R)\, \psi_0(r-R)$$

$$+ \int d^3r \, \psi_0^*(r-R) \cdot \nabla_R \psi_0(r-R)$$

Which is only true if

$$\int d^3r \, \nabla_R \psi_0^*(r-R)\, \psi_0(r-R)$$

$$= \int d^3r \, \psi_0^*(r-R) \cdot \nabla_R \psi_0(r-R)$$