Start : 15:45 Jeremy Welsh-Kavan

1) Bosonic Canonical ensemble

Two states $|\phi_1\rangle$, $|\phi_2\rangle$, with energies ϵ_1, ϵ_2 respect. and $\epsilon_1 \neq \epsilon_2$.

$N = 2$ identical bosons

(i) The energy eigenstates $|\psi_\alpha\rangle$ of the two particle system are symmetrizing sums over equivalent permutations of the 2 particles occupying products of single particle states.

The energy eigenstates have energies:

$$\boxed{E_\alpha = \{2\epsilon_1 , \epsilon_1 + \epsilon_2 , 2\epsilon_2\}}$$

Let $\alpha = (i,j)$ where i and j are the number of particles in states $|\phi_1\rangle$ and $|\phi_2\rangle$ respectively.

Then the energy eigenstates are given by the following symmetrized product states:

$$\boxed{\begin{aligned} \psi_{2,0}(\vec{r}_1,\vec{r}_2) &= \phi_1(\vec{r}_1)\phi_1(\vec{r}_2) \\ \psi_{1,1}(\vec{r}_1,\vec{r}_2) &= \frac{1}{\sqrt{2}}\left(\phi_1(\vec{r}_1)\phi_2(\vec{r}_2)+\phi_1(\vec{r}_2)\phi_2(\vec{r}_1)\right) \\ \psi_{0,2}(\vec{r}_1,\vec{r}_2) &= \phi_2(\vec{r}_1)\phi_2(\vec{r}_2) \end{aligned}}$$

(ii)

In the anonical ensemble the density matrix is given by

$$\rho = \sum_\alpha \omega_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$$

where $\omega_\alpha = \dfrac{e^{-\beta E_\alpha}}{Z}$

and $Z = \sum_\alpha e^{-\beta E_\alpha}$

In this case

$$Z = e^{-2\beta\epsilon_1} + e^{-\beta(\epsilon_1+\epsilon_2)} + e^{-2\beta\epsilon_2}$$

So we have

$$\boxed{\rho = \frac{1}{Z}\Big[e^{-2\beta\epsilon_1} |\psi_{2,0}\rangle\langle\psi_{2,0}| + e^{-\beta(\epsilon_1+\epsilon_2)} |\psi_{1,1}\rangle\langle\psi_{1,1}| + e^{-2\beta\epsilon_2} |\psi_{0,2}\rangle\langle\psi_{0,2}| \Big]}$$

with $\beta = \dfrac{1}{k_B T}$

(iii)

We can read this directly from ρ. The probability of finding ϕ_1 occupied at temp T is given by the relative weights of $|\Psi_{1,1}\rangle$ and $|\Psi_{2,0}\rangle$ in the ensemble

$$\boxed{P(\phi_1 \text{ occupied}) = \frac{e^{-2\beta\epsilon_1} + e^{-\beta(\epsilon_1+\epsilon_2)}}{Z}}$$

2) Entropy of The BE condensate

$$E = TS - PV + \mu N \qquad (*)$$

For the BEC, the equation of state is

$$\frac{PV}{N k_B T} = \frac{g_{5/2}(1)}{\rho \lambda^3} - \frac{1}{N} \ln(1-z)$$

We also know that the energy of BEC is related to PV by

$$E = \frac{3}{2} PV$$

so (*) becomes

$$\frac{3}{2} PV = TS - PV + \mu N$$

$$\rightarrow \frac{5}{2} PV = TS + \mu N$$

$$\rightarrow S = \frac{5}{2} \frac{PV}{T} - \mu \frac{N}{T}$$

$$\frac{S}{k_B} = \frac{5}{2} \frac{PV}{k_B T} - \frac{\mu N}{k_B T}$$

$$\frac{S}{k_B} = \frac{5}{2} \frac{g_{5/2}(1) N}{\rho \lambda^3} - \ln(1-z)$$

$$\frac{S}{k_B} = \frac{5}{2} N \frac{g_{\frac{5}{2}}(1)}{g_{\frac{3}{2}}(1)} - \ln(1-z)$$

$$\frac{S}{N k_B} = \frac{5}{2} \frac{g_{\frac{5}{2}}(1)}{g_{\frac{3}{2}}(1)} - O\left(\frac{\ln(n_0)}{N}\right)$$

$$\boxed{\frac{S}{N k_B} = \frac{5}{2} \frac{g_{\frac{5}{2}}(1)}{g_{\frac{3}{2}}(1)}}$$

This is the entropy per particle in units of k_B... but I get the sense that this is not precisely what I'm supposed to find...

End: 17:38

3) Blackbody Radiation in 2D

The energy of eigenstates is

$$\epsilon(k) = \hbar\omega_{\vec{k}} = \hbar c k$$

The density of states is given by

$$\sum_{k}\sum_{pol} = 2\sum_{\vec{k}} \approx 2\int_0^\infty 2\pi k\,dk\,\frac{L^2}{(2\pi)^2}$$

from 2D polar integration

$$\rightarrow \; = 2\int_0^\infty \frac{d\omega}{(d\omega/dk)}\,2\pi\frac{\omega}{c}\,\frac{L^2}{(2\pi)^2}$$

$$= \int_0^\infty d\omega\,\frac{L^2}{\pi c^2}\,\omega$$

So the dens. of states is

$$g(\omega) = \frac{L^2}{\pi c^2}\,\omega$$

in this case the average energy is

$$\langle E\rangle = \frac{L^2}{\pi c^2}\int_0^\infty d\omega\,\frac{\hbar\omega^2}{e^{\beta\hbar\omega}-1}$$

$$= L^2\int_0^\infty d\omega\,u(\omega)$$

So the spectral energy density in 2D is

$$\boxed{u(\omega) = \frac{\hbar}{\pi c^2} \frac{\omega^2}{e^{\beta \hbar \omega} - 1}}$$