Exercise 1:

$$H_{hfs} = -\mu_I \cdot \vec{B} = -\frac{\mu_B g_I b}{\hbar} \vec{I} \cdot \vec{J}$$
$$=: A_{hfs} \frac{\vec{I} \cdot \vec{J}}{\hbar^2}$$

$$\vec{F} = \vec{I} + \vec{J}$$

$$\vec{I} \cdot \vec{J} = \frac{1}{2}(F^2 - I^2 - J^2)$$

$$\vec{B}_L = \frac{-\hbar\alpha}{e m_e c r^3} \vec{L} = -\frac{\alpha^2 a_0}{e r^3} \vec{L}$$

$$\vec{B}_S = \frac{\mu_0}{4\pi r^3} \left[3(\vec{\mu}_S \cdot \hat{r})\hat{r} - \mu_S \right] + \frac{2\mu_0}{3} \vec{\mu}_S \delta^3(\vec{r})$$

$$\mu_0 \mu_B^2 = \frac{e^2 \hbar^2}{4 m_e^2 \epsilon_0 c^2} = \frac{\pi \alpha^2 a_0 \hbar^2}{m_e}$$

$$\mu_S = -g_S \mu_B \frac{\vec{S}}{\hbar}$$

$$H_{hfs} = -\vec{\mu}_I \cdot (\vec{B}_L + \vec{B}_S)$$
$$= -\frac{\mu_B g_I}{\hbar} \vec{I} \cdot \frac{\alpha^2 a_0}{e r^3} \vec{L}$$
$$+ \frac{\mu_B g_I}{\hbar} \vec{I} \cdot \frac{\mu_0}{4\pi r^3} \left(\frac{-g_S \mu_B}{\hbar} \right) \left((\vec{S} \cdot \hat{r})\hat{r} - \vec{S} \right)$$
$$- \frac{2\mu_0}{3} \vec{\mu}_I \cdot \vec{\mu}_S \, \delta^3(\vec{r})$$

$$H_{hfs} = -\frac{\mu_B g_I \alpha^2 a_0}{\hbar e r^3} \vec{I} \cdot \vec{L}$$
$$- \frac{\mu_0 \mu_B^2 g_I g_S}{\hbar^2 4\pi r^3} \left((\vec{S} \cdot \hat{r})(\vec{I} \cdot \hat{r}) - \vec{S} \cdot \vec{I} \right)$$

$$-\frac{2\cancel{\mu_0}\cancel{\mu_B}^2 g_S g_I}{3}\,\vec{I}\cdot\vec{S}\,\delta^3(r)$$

$$\boxed{\begin{aligned}
H_{hfs} = &-\frac{g_I\,\alpha^2 a_0}{2 m_e r^3}\,\vec{I}\cdot\vec{L} \\
&-\frac{g_I g_S\,\alpha^2 a_0}{4 m_e r^3}\left[(\vec{S}\cdot\hat{r})(\vec{I}\cdot\hat{r})-\vec{S}\cdot\vec{I}\right] \\
&-\frac{2\pi\,\alpha^2 a_0 \hbar^2}{3 m_e}\,\vec{I}\cdot\vec{S}\,\delta^3(r)
\end{aligned}}$$

Exercise 2:

Ignoring the electron spin, we have

$\vec{F} = \vec{L} + \vec{I}$

$\rightarrow \quad F^2 = L^2 + I^2 + 2\vec{L}\cdot\vec{I}$

$\vec{L}\cdot\vec{I} = \frac{1}{2}\left(F^2 - L^2 - I^2\right)$

But the 1S state has orbital angular momentum zero so

$\langle L^2 \rangle = 0$. Therefore,

$\langle \vec{L}\cdot\vec{I} \rangle = \frac{1}{2}\left(F^2 - I^2\right)$

But then the only contribution to F^2 is from I^2 so we must have

$\langle \vec{L}\cdot\vec{I} \rangle = 0.$

Exercise 3:

$$K = F(F+1) - I(I+1) - J(J+1)$$

$$I = \frac{1}{2} \quad , \quad J = \frac{1}{2}$$

$$0 = \left|\frac{1}{2} - \frac{1}{2}\right| \le F \le \frac{1}{2} + \frac{1}{2} = 1$$

$$F = 0 \quad \text{or} \quad F = 1$$

$$F=0 \rightarrow K = -\frac{3}{2}$$

$$F=1 \rightarrow K = \frac{1}{2}$$

$$\Delta E_{hfs}(1S) = -\frac{2\pi g_I g_s \alpha^2 a_0}{3 m_e} \langle \vec{S} \cdot \vec{I} \rangle |\psi_{100}|_{r=0}|^2$$

$$\langle \vec{J} \cdot \vec{I} \rangle = \hbar^2 \frac{K}{2} = \frac{\hbar^2}{2} \begin{cases} -\frac{3}{2} & (F=0) \\ \frac{1}{2} & (F=1) \end{cases}$$

$$\boxed{\Delta E_{hfs}(1S) = \frac{g_I g_s \alpha^2}{6 m_e a_0^2} \begin{cases} -3 & (F=0) \\ 1 & (F=1) \end{cases}}$$

since $|\psi_{100}(r=0)|^2 = \frac{1}{\pi a_0^3}$

Exercise 4:

We can apply the WE thm to each exp. value on the right hand side of (27) to get

$$\langle \alpha j m | \vec{J} \cdot \vec{A} | \alpha' j m \rangle = c_j \langle \alpha j \| \vec{A} \| \alpha' j \rangle$$

but since c_j includes a sum over CG coefficients, it is independent of m.

Zeeman Effect:

Exercise 1:

$$\langle \alpha j m | A_q | \alpha' j m' \rangle$$

$$= \frac{\langle \alpha j m | \vec{J} \cdot \vec{A} | \alpha' j m \rangle}{\hbar^2 j(j+1)} \langle \alpha j m | J_q | \alpha j m' \rangle$$

$$\rightarrow \quad J \cdot S = \tfrac{1}{2}(J^2 + S^2 - L^2)$$

$$\langle J m_J | S_z | J m_J \rangle$$

$$= \frac{\langle J m_J | \vec{J} \cdot \vec{S} | J m_J \rangle}{\hbar^2 J(J+1)} \hbar m_J$$

$$= \frac{\langle J m_J | J^2 + S^2 - L^2 | J m_J \rangle}{\hbar J(J+1)} m_J$$

$$\boxed{= \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} \hbar m_J}$$

Exercise 2:

$$\vec{I}\cdot\vec{J} = I_x J_x + I_y J_y + I_z J_z$$

$$= \left(\frac{I_+ + I_-}{2}\right)\left(\frac{J_+ + J_-}{2}\right)$$

$$- \frac{(I_+ - I_-)(J_+ - J_-)}{4} + I_z J_z$$

$$= \frac{1}{4}\Big(\cancel{I_+ J_+} + \cancel{I_- J_-} + I_- J_+$$

$$+ I_+ J_- - \cancel{I_+ J_+} - \cancel{I_- J_-}$$

$$+ I_+ J_- + I_- J_+ \Big) + I_z J_z$$

$$\boxed{= \frac{1}{2}\left(I_+ J_- + I_- J_+ \right) + I_z J_z}$$