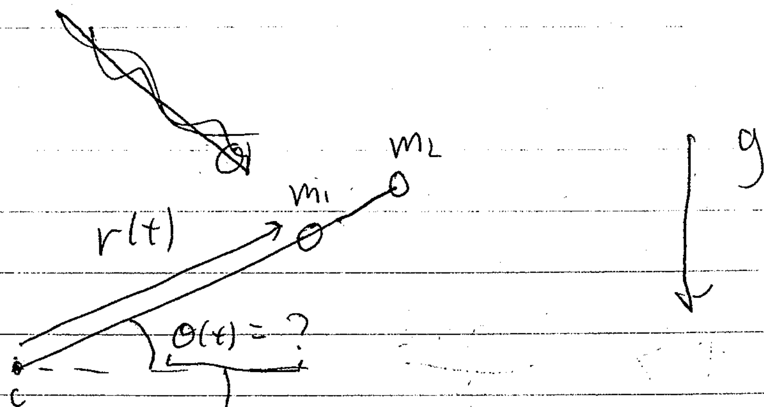
Bead on wire, wire not driven at

ω, but free: Still vertical, still gravity



Important point:

$v^2 =$

$\Rightarrow T =$

$U(r,\theta) =$

Two equations of motion:

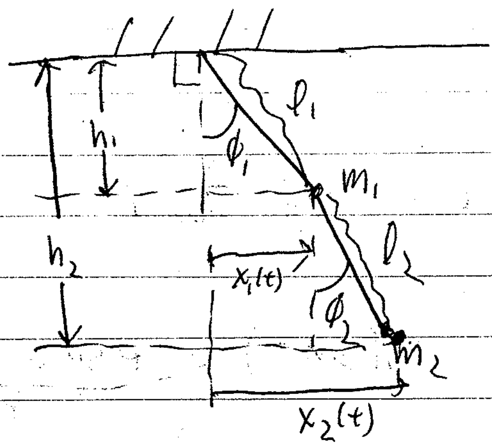
r: $\dfrac{\partial \mathcal{L}}{\partial r} =$

$\dfrac{\partial \mathcal{L}}{\partial \dot{r}} =$

Landau + Lifshitz, Problem 1

Double pendulum, vertical plane

$\Downarrow$
gravity



$\downarrow g$

$$U(\phi_1, \phi_2) = -m_1 g h_1(\phi_1,\phi_2) - m_2 g h_2(\phi_1, \phi_2)$$

$h_1(\phi_1, \phi_2) =$

$h_2(\phi_1, \phi_2) =$

} Just Geometry

$$T(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2) = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$v_1 =$

$\cancel{v_2 =}$

} Also just geometry

v_2 : a little trickier :

$$v_2^2 = \dot{x}_2^2 + \dot{h}_2^2 ,$$

$$x_2(\phi_1, \phi_2) = l_1 \sin\phi_1 + l_2 \sin\phi_2$$

$$\Rightarrow \dot{x}_2 = l_1 \cos\phi_1 \, \dot{\phi}_1 + l_2 \cos\phi_2 \, \dot{\phi}_2$$

$$\Rightarrow \dot{x}_2^{\,2} = l_1^2 \cos^2\phi_1 \, \dot{\phi}_1^2 + l_2^2 \cos^2\phi_2 \, \dot{\phi}_2^2 + 2 l_1 l_2 \cos\phi_1 \cos\phi_2 \, \dot{\phi}_1 \dot{\phi}_2$$

$$\dot{h}_2 = -\left(l_1 \sin\phi_1 \, \dot{\phi}_1 + l_2 \sin\phi_2 \, \dot{\phi}_2\right)$$

$$\Rightarrow \dot{h}_2^{\,2} = l_1^2 \sin^2\phi_1 \, \dot{\phi}_1^2 + l_2^2 \sin^2\phi_2 \, \dot{\phi}_2^2 + 2 l_1 l_2 \sin\phi_1 \sin\phi_2 \, \dot{\phi}_1 \dot{\phi}_2$$

$$\Rightarrow v_2^2 = \dot{x}_2^{\,2} + \dot{h}_2^{\,2} = l_1^2 \dot{\phi}_1^2 \underbrace{(\sin^2\phi_1 + \cos^2\phi_1)}_{1} + l_2^2 \dot{\phi}_2^2 \underbrace{(\sin^2\phi_2 + \cos^2\phi_2)}_{1}$$

$$+ 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \underbrace{(\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2)}$$

$$= \cos(\phi_1 - \phi_2) \quad \text{(trigonometry)}$$

So...

$$\boxed{\begin{aligned} \mathcal{L} = T - V = & \tfrac{1}{2} \cancel{m_1} l_1^2 \dot{\phi}_1^2 (m_1 + m_2) + \tfrac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 \\ & + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \\ & + (m_1 + m_2) g l_1 \cos\phi_1 + m_2 g l_2 \cos\phi_2 \end{aligned}}$$

So, what's the equation of motion? Don't turn the page!

1st: ϕ_1 Equation of motion (vary ϕ_1 in E-LG) (2.10

$$\frac{\partial \mathcal{L}}{\partial \phi_1} = -m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \sin(\phi_1 - \phi_2) - (m_1 + m_2) g\, l_1 \sin\phi_1$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} = (m_1 + m_2) l_1^2 \dot{\phi}_1 \cancel{\Rightarrow \frac{d}{dt}\partial\mathcal{L}} + m_2 l_1 l_2 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$

$\Rightarrow$ ~~1st Equation of motion (ϕ_1 EOM)~~

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1}\right) = (m_1 + m_2) l_1^2 \ddot{\phi}_1 + m_2 l_1 l_2 \ddot{\phi}_2 \cos(\phi_1 - \phi_2)$$
$$- m_2 l_1 l_2 \dot{\phi}_2 (\dot{\phi}_1 - \dot{\phi}_2) \sin(\phi_1 - \phi_2)$$

1st Equation of motion

$\boxed{\text{Note: Involves } \underline{\text{both}}\ \ddot{\phi}_1 \text{ and } \ddot{\phi}_2}$

2.
$$\boxed{(m_1 + m_2) l_1^2 \ddot{\phi}_1 + m_2 l_1 l_2 \left[\ddot{\phi}_2 \cos(\phi_1 - \phi_2) + (\dot{\phi}_2^2 - \cancel{\dot{\phi}_1 \dot{\phi}_2}) \sin(\phi_1 - \phi_2)\right] = -m_2 l_1 l_1 \cancel{\dot{\phi}_1 \dot{\phi}_2 \sin(\phi_1 - \phi_2)} - (m_1 + m_2) g\, l_1 \sin\phi_1}$$

cancel

2nd Equation of motion:

$$\frac{\partial \mathcal{L}}{\partial \phi_2} = + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \sin(\phi_1 - \phi_2) - m_2 g\, l_2 \sin\phi_2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}_2} = m_2 l_2^2 \dot{\phi}_2 + m_2 l_1 l_2 \dot{\phi}_1 \cos(\phi_1 - \phi_2)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_2}\right) = m_2 l_2^2 \ddot{\phi}_2 + m_2 l_1 l_2 \ddot{\phi}_1 \cos(\phi_1 - \phi_2) - m_2 l_1 l_2 \dot{\phi}_1 (\dot{\phi}_1 - \dot{\phi}_2) \sin(\phi_1 - \phi_2)$$

Conservation laws

Helps to find

"Conserved quantity"
or
"Constants of motion"

Some $\boxed{(3.1)\ f(\{q_i, \dot{q}_i\}) = \text{constant}}$ for entire motion

obeying Lagrangian E.O.M.

$\Downarrow$ Given (3.1),

Solve (in principle) for

$\dot{q}_i = f(\{q_j, \dot{q}_j\})$

1st order equation (instead of $F=ma$, which is <u>2nd</u> order) $\Rightarrow$ Easier to solve.

Example: Conservation of "Energy"

1st, 1 d, <u>unconstrained</u> motion

$$(3.2) \quad m\ddot{x} = -\frac{dU}{dx}$$: LHS = total time derivative

How can we turn RHS into total time derivative?

Ans: multiply by $\frac{dy}{dt} = \dot{x}$

$$\Rightarrow m\dot{x}\ddot{x} = -\frac{dU}{dx}\frac{dx}{dt} = \frac{d}{dt}(? \quad)$$

Is LHS still total time derivative?

If so, of what?

So, $\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + U(x)\right) = 0$

$\Rightarrow \frac{1}{2}m\dot{x}^2 + U(x) = \text{constant} \equiv \text{Energy}$

Conservation of energy (follows directly from $F = ma$ for conservative forces ($F = -\frac{dU}{dx}$))

Note: ~~Not Lagran~~

$$E \neq \mathcal{L}$$

$$T + U \neq T - U$$

Deriving conservation of "energy" from Lagrangian formalism:

Is $\mathcal{L}$ itself conserved?

Let's calculate (for 1 variable q)

$$\frac{d\mathcal{L}}{dt} = \frac{d}{dt}\mathcal{L}(q(t), \dot{q}(t); t)$$

$$= ?$$

Now, use E.O.M. (E-LG equation)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q}$$

$$\Rightarrow \frac{d\mathcal{L}}{dt} = \dot{q}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) + \frac{\partial \mathcal{L}}{\partial \dot{q}}\ddot{q} = \frac{d}{dt}\left(\dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)$$

$$\Rightarrow \frac{d}{dt}\left[\dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L}\right] = 0$$

$$\Rightarrow \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} = \text{constant} \equiv \text{"Energy"}$$

"First integral of motion"

Example: 1d unconstrained motion:

$$q = x$$

$$\mathcal{L} = T - U = \tfrac{1}{2} m \dot{x}^2 - U(x)$$

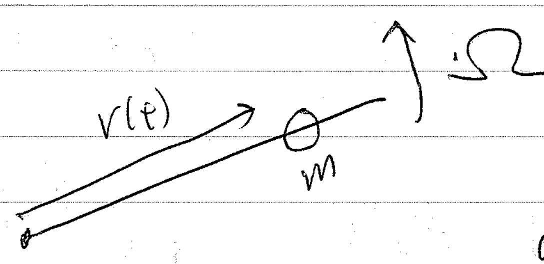
$$\dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} =$$

$$= \text{Energy}$$

Warning: "1st integral of motion" $\neq$ energy

if <u>constraints</u> depend on time

Example: Bead on rotating wire



$$q = r$$

What's true energy for this system?

" 1st integral of motion

Are they equal?

Where does energy come from?

<u>Fictitious</u> "centrifugal potential"

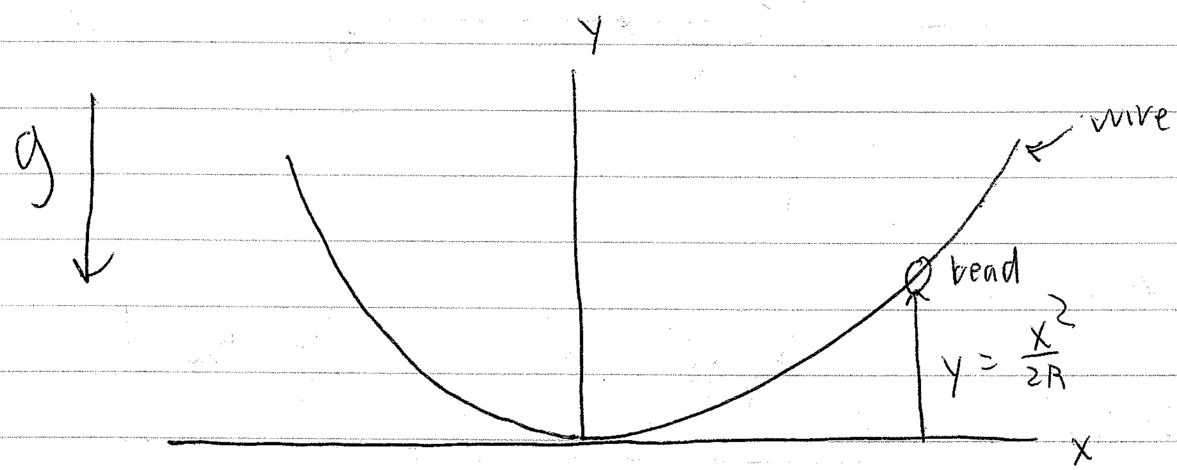
(3.~~4~~5

But ~~Note~~: This ~~one~~ "Energy" does = true energy $= T+U$

if constraints independent of time.

~~Exam~~

Example: Bead on stationary parabolic wire:



~~Take~~

Variable choice: $q = x$

$T = \frac{1}{2} m v^2 =$

$\dot{y} = \frac{dy}{dt} =$

$\Rightarrow T = \frac{1}{2} m \left(1 + \left(\frac{x}{R}\right)^2\right) \dot{x}^2$

$U(x) =$

$$\Rightarrow \mathcal{L} = \frac{1}{2} m \left(1 + \left(\frac{x}{R}\right)^2\right) \dot{x}^2 - \frac{mgx^3}{2R}$$

1st $\int$ of motion $= \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} = \underbrace{\dot{x} \frac{\partial \mathcal{L}}{\partial \dot{x}}} - \mathcal{L}$

$= \underbrace{\qquad\qquad}_{=2T} - (T - U)$

$= T + U =$ true energy

What would happen if I'd chosen $q = y$ instead?

$\dot{x} = \frac{dx}{dt} =$

$$\Rightarrow \mathcal{L} = \frac{1}{2} m \dot{y}^2 \left(1 + \frac{R}{2y}\right) - mgy$$

$\dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} = \underbrace{\qquad\qquad}_{=2T} - (T - U) = T + U =$ true enery

Why did this happen?

In all cases (and, indeed, in general if constraints constant in time)

$V_{i\alpha} = g_{\alpha}(q)\, \dot{q}$ $\Big($ $q = x$ case, $f_x =$

$f_y =$

any component (here, $i = x, y$)

$\Rightarrow \quad |\vec{v}|^2 = \sum_\alpha v_i^2 \doteq \left(\sum_\alpha g_\alpha^2(q)\right)\dot{q}^2$ ~~$\alpha f(q)\dot{q}$~~

Note $T \propto \dot{q}^2$

$\Rightarrow \quad T = \frac{1}{2}m|\vec{v}|^2 = \left(\frac{1}{2}\left(m \sum_\alpha g_\alpha^2(q)\right)\right)\dot{q}^2 \equiv f(q)\dot{q}^2$

Note: only depends on q

~~Gi~~ $U(q)$ independent of $\dot{q}$

$\Rightarrow \quad \dot{q}\,\frac{\partial \mathcal{L}}{\partial \dot{q}} = \qquad\qquad = 2T$

$\Rightarrow$ 1st $\int$ of motion $= 2T - (T - U) = T + U$

Now, more than 1 variable.

1st, derive 1st $\int$ of motion again

~~$\frac{d\mathcal{L}}{dt}$~~ $\mathcal{L}(\{q_i, \dot{q}_i\})$, $i = 1, 2, \ldots N = \#$ of d.o.f.

$\frac{d\mathcal{L}}{dt} = ?$

$$\frac{d\mathcal{L}}{dt} = \sum_{i=1}^{N}\left(\frac{\partial\mathcal{L}}{\partial q_i}\dot{q}_i + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\ddot{q}_i\right) + \cancel{\frac{\partial\mathcal{L}}{\partial t}} \rightarrow 0$$

0 if $\mathcal{L}$ does not depend explicitly on time

E-LG equation (one of N)

$$\frac{\partial\mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right)$$

$$\Rightarrow \frac{d\mathcal{L}}{dt} = \sum_{i=1}^{N}\underbrace{\left(\dot{q}_i\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right) + \ddot{q}_i\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right)}_{=\frac{d}{dt}(?\quad)}$$

$$\Rightarrow \frac{d}{dt}\left[\left(\sum_i \dot{q}_i\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right) - \mathcal{L}\right] = 0$$

$\Rightarrow$ 1st $\int$ of motion $= \sum_i \dot{q}_i\frac{\partial\mathcal{L}}{\partial\dot{q}_i} - \mathcal{L} =$ constant

Now, I'll show that if constraints are time independent, 1st $\int = T+U =$ Energy

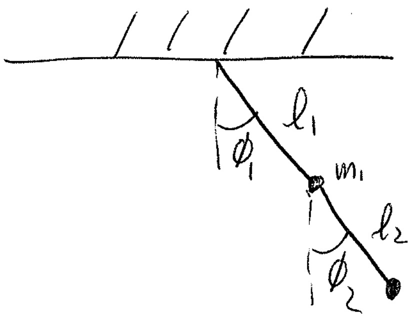
Just like 1 variable case, with 1 wrinkle:

α'th component of k'th particle

$$\dot{V}_{\alpha k} = \sum_i f^{k}_{i\alpha}(\{q_i\})\, \dot{q}_i$$

Vake

Example: Double pendulum:



$$V_{1x} = l_1 \cos\phi_1\, \dot\phi_1 \Rightarrow f^1_{1x} = l_1 \cos\phi_1, \; f^1_{2x} =$$

$$V_{1y} = l_1 \sin\phi_1\, \dot\phi_1 \Rightarrow f^1_{1y} = \quad , f^1_{2y} =$$

$$v_{2x} = l_1 \cos\phi_1\, \dot\phi_1 + l_2 \cos\phi_2\, \dot\phi_2 \Rightarrow f^2_{1x} = \quad , f^2_{2x} =$$

$$v_{2y} = -(l_1 \sin\phi_1\, \dot\phi_1 + l_2 \sin\phi_2\, \dot\phi_2) = f^2_{1y} = \quad , f^2_{2y} =$$

$$\Rightarrow |\vec{V}_k|^2 = \sum_\alpha v^2_{k\alpha} = \sum_\alpha \Big(\sum_j f^k_{j\alpha}\, \dot q_j\Big)\Big(\sum_l f^k_{l\alpha}\, \dot q_l\Big)$$

$$= \sum_{j,l} \dot q_j \dot q_l \underbrace{\Big(\sum_\alpha f^k_{j\alpha} f^k_{l\alpha}\Big)}$$

sum of product = product of sums

note: depends on $\{q_i\}$, but not $\dot q_i$ (consequence of constraints independent of time)

$$\Rightarrow T = \frac{1}{2}\sum_k m_k |\vec{v}_k|^2 = \sum_{j,l} \dot{q}_j \dot{q}_l \underbrace{\left[\sum_k \frac{m_k}{2} \sum_\alpha f^k_{j\alpha}(\{q_i\}) f^k_{l\alpha}(\{q_i\})\right]}_{\equiv a_{jl}(\{q_i\})}$$

<u>Only</u> Two important properties of a_{jl}:

1) Independent of $\dot{q}_i$'s
2) $a_{jl} = a_{lj}$ "symmetric"

Example: Double pendulum:

$$T = [\text{mess}(\phi_1, \phi_2)]\,\dot{\phi}_1^2 + (\text{mess}(\phi_1, \phi_2))\,\dot{\phi}_2^2 + \text{mess}(\phi_1, \phi_2)\,\dot{\phi}_1 \dot{\phi}_2$$

$$\equiv a_{11}\dot{\phi}_1^2 + a_{22}\dot{\phi}_2^2 + 2a_{12}\dot{\phi}_1\dot{\phi}_2$$

So,

$$T = \sum_{j,l} a_{jl}(\{q_i\})\,\dot{q}_j \dot{q}_l$$

$$\mathcal{L} = T - U = \sum_{j,l} a_{jl}\,\dot{q}_j \dot{q}_l - U(\{q_i\})$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}_i} =$$

$$\frac{\partial \mathcal{L}}{\partial \mathring{q}_i} = \sum_{j,l} a_{jl}\left(\mathring{q}_j \frac{\partial \mathring{q}_l}{\partial \mathring{q}_i} + \mathring{q}_l \frac{\partial \mathring{q}_j}{\partial \mathring{q}_i}\right)$$

$$\frac{\partial \mathring{q}_l}{\partial \mathring{q}_i} = \begin{cases} 0, & l \neq i \\ 1, & l = 1 \end{cases} \equiv \underbrace{\delta_{il}}_{\text{Kronecker delta}}$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \mathring{q}_i} = \sum_j \mathring{q}_j \Big(\underbrace{\sum_l a_{jl}\delta_{il}}_{=?}\Big) + \sum_l \mathring{q}_l \Big(\underbrace{\sum_j a_{jl}\delta_{ij}}_{=?}\Big)$$

General rule: $\sum_l a_{jl}\delta_{il} = a_{ji}$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \mathring{q}_i} = \underbrace{\sum_j a_{ji}\mathring{q}_j}_{\substack{= \sum_j a_{ij}\mathring{q}_j \\ \text{(symmetry)}}} + \underbrace{\sum_l a_{il}\mathring{q}_l}_{\substack{= \sum_j a_{ij}\mathring{q}_j \\ \text{(rename dummy index)}}} = 2\sum_j a_{ij}\mathring{q}_j$$

$$\Rightarrow \sum_i \mathring{q}_i \frac{\partial \mathcal{L}}{\partial \mathring{q}_i} = 2\sum_{i,j} a_{ij}\mathring{q}_i\mathring{q}_j = ?$$

So, 1st $\int$ of motion

$$= 2T - (T-U) = T+U = E$$

True whenever constraints independent of time

Other conserved quantities: generalized momentum

~~An~~ easy ones to derive

* $\boxed{\text{Suppose } \mathcal{L}(\{q_j, \dot{q}_j\}) \text{ is independent of one particular } q_i.}$ What's EOM for that q_i?

~~So~~

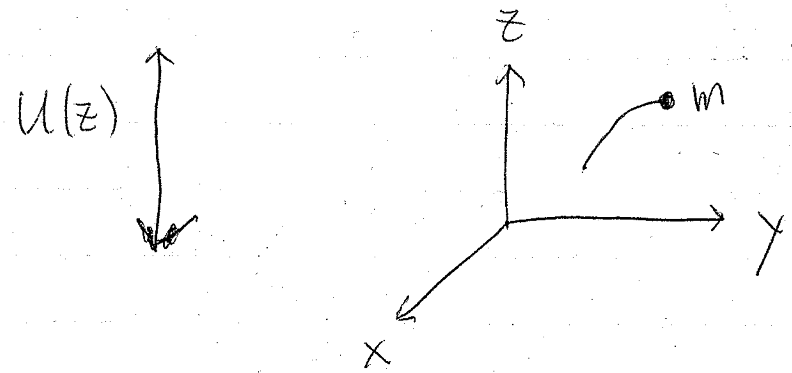
So,

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{constant}$$

$p_i \equiv$ ~~the~~ "generalized momentum conjugate to q_i"

Examples: 1) Linear momentum

a) Single particle in 3d, (i.e., $\vec{r} = (x, y, z)$) in potential that depends only on z



Co-ordinates: $q_i = (x, y, z)$

What's $\mathcal{L}$?

$T = ?$

$\Rightarrow \mathcal{L} =$

$U = ?$

$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial x} = ? \quad , \quad \frac{\partial \mathcal{L}}{\partial y} = ?$

$\Rightarrow \quad p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = ?$

$p_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} = ?$

What are these?

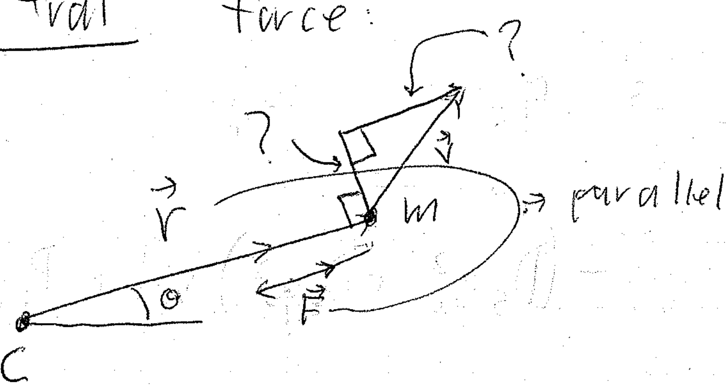
- <u>linear</u> momentum conserved in directions
- U is independent of
  Why? Force in those directions?

$\Rightarrow \quad x(t) = ?$

$\quad\quad y(t) = ?$

2nd example: <u>Angular</u> momentum

<u>Central</u> force:



$\vec{F}(\vec{r})$ conservative

$\Rightarrow U(|\vec{r}| \equiv r, \theta) =$

$T(r, \theta, \dot{r}, \dot{\theta}) = \frac{1}{2} m |\vec{v}|^2 =$

$\Rightarrow \quad \mathcal{L} = T - U =$