1) $\epsilon_{\mu\alpha\beta}\epsilon_{\mu\sigma\tau} = \delta_{\alpha\sigma}\delta_{\beta\tau} - \delta_{\alpha\tau}\delta_{\beta\sigma}$

Let $a, b, c \in \mathbb{R}^3$, then

$$(b \times c)_i = \epsilon_{ijk} b_j c_k$$

$$(a \times (b \times c))_i = \epsilon_{ijk}\epsilon_{kln} a_j b_l c_n$$

$$= \epsilon_{kij}\epsilon_{kln} a_j b_l c_n$$

$$(b(a \cdot c) - c(a \cdot b))_i$$

$$= b_i a_k c_k - c_i a_l b_l$$

$$= a_j b_l c_n \delta_{jn}\delta_{il} - a_j b_l c_n \delta_{il}\delta_{jn}$$

$$= a_j b_l c_n (\delta_{jn}\delta_{il} - \delta_{il}\delta_{jn})$$

$$= a_j b_l c_n (\delta_{il}\delta_{jn} - \delta_{in}\delta_{jl})$$

and since a, b, c were arbitrary, we must have

$$\boxed{\epsilon_{kij}\epsilon_{kln} = \delta_{il}\delta_{jn} - \delta_{in}\delta_{jl}}$$

2)

We first recall the action of $J^2, J_z, J_\pm$ on $|j\,m\rangle$

$$J^2|j\,m\rangle = j(j+1)\hbar^2|j\,m\rangle$$

$$J_z|j\,m\rangle = m\hbar|j\,m\rangle$$

$$J_\pm|j\,m\rangle = \hbar\underbrace{\sqrt{j(j+1)-m(m\pm1)}}_{\lambda_{jm\pm}}|j\,m\pm1\rangle$$

$$J_\pm = J_x \pm iJ_y$$

$$J_x = \frac{1}{2}(J_+ + J_-)$$

$$J_y = \frac{1}{2i}(J_+ - J_-)$$

$$\langle J_x\rangle = \langle jm|J_x|jm\rangle$$

$$\langle J_x\rangle = \frac{1}{2}\langle jm|J_+|jm\rangle + \frac{1}{2}\langle jm|J_-|jm\rangle$$

$$\langle J_x\rangle = \frac{1}{2}(\lambda_{jm+}\langle jm|j\,m+1\rangle + \lambda_{jm-}\langle jm|j\,m-1\rangle$$

$$\boxed{\langle J_x\rangle = 0}$$

$$\langle J_x^2 \rangle = \langle j\, m | J_x^2 | j\, m \rangle$$

$$= \frac{\hbar^2}{4} \Big(\langle j\, m | J_+^2 + J_-^2 + J_+J_- + J_-J_+ | j\, m \rangle$$

$$= \frac{\hbar^2}{4} \Big(\langle j\, m | J_+J_- | j\, m \rangle + \langle j\, m | J_-J_+ | j\, m \rangle \Big)$$

$$= \frac{\hbar^2}{4} \Big(\sqrt{j(j+1) - m(m-1)}\sqrt{j(j+1) - (m-1)m}$$

$$+ \sqrt{j(j+1) - m(m+1)}\sqrt{j(j+1) - (m+1)m} \Big)$$

$$\boxed{\langle J_x^2 \rangle = \frac{\hbar^2}{2} \left(j^2 + j - m^2 \right)}$$

$$\sigma_{J_x} = \sqrt{\langle J_x^2 \rangle - \langle J_x \rangle^2} = \sqrt{\langle J_x^2 \rangle}$$

$$\boxed{\sigma_{J_x} = \frac{\hbar}{\sqrt{2}} \sqrt{j^2 + j - m^2}}$$

$$\langle J_y^2 \rangle = -\frac{1}{4}\langle jm| -J_+J_- |jm\rangle$$
$$-\frac{1}{4}\langle jm| -J_-J_+ |jm\rangle$$

$$= \langle J_x^2 \rangle$$

so

$$\sigma_{J_y} = \sigma_{J_x}$$

Therefore,

$$\boxed{\sigma_{J_x}\sigma_{J_y} = \frac{\hbar^2}{2}\left(j^2 + j - m^2\right)}$$

The uncertainty principle states that

$$\sigma_{J_x}\sigma_{J_y} \geq \frac{1}{2}\left|\langle [J_x, J_y] \rangle\right|$$

and $[J_x, J_y] = i\hbar J_z$

so

$$\sigma_{J_x}\sigma_{J_y} \geq \frac{\hbar}{2}\left|\langle J_z \rangle\right| = \frac{\hbar^2}{2}|m|$$

since $-j \le m \le j$,

$m^2 \le j^2$.

And since $j \geq 0$,

$|m| \le j$

So

$|m| + m^2 \le j + j^2$

and

$j^2 + j - m^2 \geq |m|$

Therefore,

$\sigma_{J_x} \sigma_{J_y} = \frac{\hbar^2}{2} (j^2 + j - m^2) \geq \frac{\hbar^2}{2} |m|$

so the uncertainty principle is satisfied.

3)

$$df(r,\theta,\phi) = d\vec{r} \cdot \nabla f(\vec{r})$$

in Spherical coordinates

$$d\vec{r} = dr + r\,d\phi + r^2 \sin\theta\, d\theta$$

and

$$df(r,\theta,\phi) = \partial_r f\, dr + \partial_\phi f\, d\phi + \partial_\theta f\, d\theta$$

So

$$(\nabla f \cdot \hat{r})\, dr = \partial_r f\, dr$$

$$(\nabla f \cdot \hat{\phi})\, r\, d\phi = \partial_\phi f\, d\phi$$

$$(\nabla f \cdot \hat{\theta})\, r^2 \sin\theta\, d\theta = \partial_\theta f\, d\theta$$

so, since $\hat{r}$, $\hat{\phi}$, and $\hat{\theta}$ form an orthogonal coordinate system, we must have

$$\boxed{\nabla = \partial_r \hat{r} + \frac{1}{r}\partial_\phi \hat{\phi} + \frac{1}{r^2 \sin\theta}\partial_\theta \hat{\theta}}$$