Exercise 1:

$$\langle x_\alpha(t)\rangle = \langle \psi(0)|\tilde{U}(0,t)\tilde{x}_\alpha(t)\tilde{U}(t,0)|\psi(0)\rangle$$

$$\langle x_\alpha(t)\rangle$$

$$= \langle \psi(0)|\left(1+\frac{i}{\hbar}\int_0^t dt_1 \tilde{V}(t_1)\right)\tilde{x}_\alpha(t)\left(1-\frac{i}{\hbar}\int_0^t dt_1\tilde{V}(t_1)\right)|\psi(0)\rangle$$

$$= \langle \psi(0)|\left(1+\frac{i}{\hbar}\int_0^t dt_1 \tilde{V}(t_1)\right)\left(\tilde{x}_\alpha(t)-\frac{i}{\hbar}\tilde{x}_\alpha(t)\int_0^t dt_1\tilde{V}(t_1)|\psi(0)\rangle\right.$$

$$= \langle \psi(0)|\left(\tilde{x}_\alpha(t)+\frac{i}{\hbar}\int_0^t dt_1\tilde{V}(t_1)\tilde{x}_\alpha(t)\right.$$

$$\left.-\frac{i}{\hbar}\tilde{x}_\alpha(t)\int_0^t dt_1\tilde{V}(t_1)\ |\psi(0)\rangle\right)$$

$$= \langle \tilde{x}_\alpha(t)\rangle + \frac{i}{\hbar}\int_0^t dt' \langle \psi(0)|\tilde{V}(t')\tilde{x}_\alpha(t)-\tilde{x}_\alpha(t)\tilde{V}(t')|\psi(0)\rangle$$

$$= \langle \tilde{x}_\alpha(t)\rangle + \frac{i}{\hbar}\int_0^t dt_1 \langle[\tilde{V}(t'),\ \tilde{x}_\alpha(t)]\rangle$$

$$= \langle \tilde{x}_\alpha(t)\rangle + \frac{i}{\hbar}\int_0^t dt_1 \langle[-\tilde{x}_\beta(t')F_\beta(t'),\tilde{x}_\alpha(t)]\rangle$$

$$= \langle \tilde{x}_\alpha(t)\rangle + \frac{i}{\hbar}\int_0^t \langle[\tilde{x}_\alpha(t),\ \tilde{x}_\beta(t')]\rangle F_\beta(t')$$

Exercise 2:

The delta function in the $\frac{1}{z-H}$ case does not represent anything physical since the Schrodinger equation does not have sources

Exercise 3:

With zero damping, the Imaginary part of the susceptibility goes to zero. The fact that χ_{ab} is a tensor reflects the fact that the polarization can point in a different direction to the driving force.

Exercise 4:

$$\chi(\omega) - \chi_0 = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega' \frac{\chi(\omega') - \chi_0}{\omega' - \omega - i0^+}$$

$$\chi(\omega) - \chi_0 = \frac{1}{2\pi i} \fint_{-\infty}^{\infty} \frac{\chi(\omega') - \chi_0}{\omega' - \omega}$$

$$+ \frac{1}{2\pi i} i\pi\, \delta(\omega' - \omega)(\chi(\omega') - \chi_0)$$

$$= \frac{1}{2\pi i} \fint_{-\infty}^{\infty} \frac{\chi(\omega') - \chi_0}{\omega' - \omega} + \frac{\chi(\omega) - \chi_0}{2}$$

$$\chi(\omega) - \chi_0 = \frac{-i}{\pi} \fint_{-\infty}^{\infty} \frac{\chi(\omega') - \chi_0}{\omega' - \omega}$$

$$R[\chi(\omega) - \chi_0] = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{Im[\chi(\omega) - \chi_0]}{\omega' - \omega}$$

$$Im[\chi(\omega) - \chi_0] = -\frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{Re[\chi(\omega) - \chi_0]}{\omega' - \omega}$$

Fluct. Diss.

Exercise 1:

$$\partial_\lambda \left(e^{\lambda H_0} [A, e^{-\lambda H_0}] \right) = \partial_t \left[e^{\lambda H_0} A e^{-\lambda H_0} - A \right]$$

$$= H_0 e^{\lambda H_0} A e^{-\lambda H_0} - e^{\lambda H_0} A e^{-\lambda H_0} H_0$$

$$= H_0 \tilde{A}(-i\hbar\lambda) - \tilde{A}(-i\hbar\lambda) H_0$$

$$= [H_0, \tilde{A}(-i\hbar\lambda)]$$

$$= -i\hbar \dot{\tilde{A}}(-i\hbar\lambda)$$

$$e^{\beta H_0} [A, e^{-\beta H_0}] = -i\hbar \int_0^\beta \dot{\tilde{A}}(-i\hbar\lambda) \, d\lambda$$

$$[A, \rho_0] = -i\hbar \rho_0 \int_0^\beta \dot{\tilde{A}}(-i\hbar\lambda) \, d\lambda$$

Exercise 2:

$$\langle \tilde{x}(t_1)\tilde{y}(t_2)\rangle = \mathrm{Tr}(\tilde{x}(t_1)\tilde{y}(t_2)\rho_0)$$

$$= \mathrm{Tr}(\tilde{x}(t_1)e^{iH_0t_2/\hbar}y\,e^{-iH_0t_2/\hbar}\rho_0)$$

$$= \mathrm{Tr}(e^{iH_0t_1/\hbar}x\,e^{-iH_0t_1/\hbar}e^{iH_0t_2/\hbar}y\,e^{-iH_0t_2/\hbar}\rho_0)$$

$$= \mathrm{Tr}(e^{iH_0(t_1-t_2)/\hbar}x\,e^{-iH_0(t_1-t_2)/\hbar}\rho_0)$$

$$= \langle \tilde{x}(t_1-t_2)\,y(0)\rangle$$

Exercise 3:

$$\chi_{\mu\nu} = i\omega \mathrm{Tr} \int_0^\beta d\lambda\, e^{\hbar\omega\lambda} \int_{i\hbar\lambda}^{\infty + i\hbar\lambda} d\tau\, \rho_0\, \tilde{x}_\nu(-\tau)\, \tilde{x}_\mu(0)\, e^{i\omega\tau}$$

$$\chi_{\mu\nu} - \chi^*_{\nu\mu}(\omega) = i\omega \int_0^\beta d\lambda\, e^{\hbar\omega\lambda} \int_{i\hbar\lambda}^{\infty + i\hbar\lambda} d\tau \langle \tilde{x}_\nu(-\tau)\, \tilde{x}_\mu(0) \rangle e^{i\omega\tau}$$

$$+ i\omega \int_0^\beta d\lambda\, e^{\hbar\omega\lambda} \int_{-i\hbar\lambda}^{\infty - i\hbar\lambda} d\tau \langle \tilde{x}_\nu(\tau)\, \tilde{x}_\mu(0) \rangle e^{-i\omega\tau}$$

$$\chi_{\mu\nu} - \chi^*_{\nu\mu} = i\omega \int_0^\beta d\lambda\, e^{\hbar\omega\lambda} \int_{-\infty + i\hbar\lambda}^{\infty + i\hbar\lambda} d\tau \langle \tilde{x}_\nu(-\tau)\, \tilde{x}_\mu(0) \rangle e^{i\omega\tau}$$

$$\chi_{\mu\nu}(\omega) - \chi^*_{\nu\mu} = i\omega \left(e^{\beta\hbar\omega} - 1\right) \int_{-\infty}^{\infty} d\tau \langle \tilde{x}_\nu(0)\, \tilde{x}_\mu(\tau) \rangle e^{i\omega\tau}$$

$$\rightarrow \frac{\chi_{\mu\nu}(\omega) - \chi^*_{\nu\mu}(\omega)}{e^{\beta\hbar\omega} - 1} = \frac{1}{\hbar} \int_{-\infty}^{\infty} d\tau \langle \tilde{x}_\nu(0)\, \tilde{x}_\mu(\tau) \rangle e^{i\omega\tau}$$

Exercise 4:

$$j_\mu(\vec{r},\omega) = \sigma_{\mu\nu}(\vec{r},\omega)\, E_\nu(\vec{r},\omega)$$

$$E(t) = -\partial_t A(t)$$

$$\rightarrow j_\mu(\vec{r},\omega) = i\omega\, \sigma_{\mu\nu}(\vec{r},\omega) A_\nu(\vec{r},\omega)$$

$$V j_\mu(\vec{r},\omega) = i\omega V \sigma_{\mu\nu}(\vec{r},\omega)\, A_\nu(\vec{r},\omega)$$

$$\langle j_\mu(\vec{r}) j_\nu(r)\rangle = \frac{\hbar}{2\pi V}\int_{-\infty}^{\infty} d\omega\, \omega\, \mathrm{Re}(\sigma_{\mu\nu}) \coth\left(\frac{\beta\hbar\omega}{2}\right)$$

$$I = ja \quad , \quad G = \sigma \frac{a}{l}$$

$$\rightarrow \left\langle \frac{I^2}{a^2} \right\rangle = \frac{\hbar l}{2\pi V a}\int_{-\infty}^{\infty} d\omega\, \omega\, \mathrm{Re}[G(\omega)] \coth\left(\frac{\beta\hbar\omega}{2}\right)$$

$$\rightarrow \langle I^2 \rangle = \frac{\hbar}{2\pi}\int_{-\infty}^{\infty} d\omega\, \omega\, \mathrm{Re}[G(\omega)] \coth\left(\frac{\beta\hbar\omega}{2}\right)$$

$$\langle I^2 \rangle \approx \frac{\hbar}{\pi R}\int_0^{\infty} d\omega\, \omega \coth\left(\frac{\beta\hbar\omega}{2}\right)$$

$$\rightarrow \langle V^2 \rangle \approx \frac{\hbar R}{\pi}\int_0^{e} d\omega\, \omega \coth\left(\frac{\beta\hbar\omega}{2}\right)$$

for $\beta = (k_B T)^{-1}$ small, $\coth x \approx \frac{1}{x}$

$$\rightarrow \langle V^2 \rangle = \frac{\hbar R}{\pi} \frac{2}{\beta \hbar} \int_0^\infty d\omega$$

$$\langle V^2 \rangle = \frac{2 R k_B T}{\pi} \int_{\Delta\omega} d\omega$$

$$\langle V^2 \rangle = 4 R k_B T \Delta\nu$$

For $\beta \rightarrow \infty$, $\coth x \approx 1$

$$\rightarrow \langle V^2 \rangle \approx \frac{\hbar R}{\pi} \int_0^\infty d\omega\, \omega$$

$$\langle V^2 \rangle \approx \frac{\hbar R}{\pi} \int_{\Delta\omega} d\omega\, \omega$$

$$\langle V^2 \rangle = \frac{\hbar R}{\pi} \frac{(2\pi \Delta\nu)^2}{2}$$

$$\langle V^2 \rangle = 2\pi \hbar R (\Delta\nu)^2$$