PHYS 633: Quantum Mechanics III (Spring 2021)

Course Notes: Time-Dependent Perturbations (Part I: Setup and Evolution Operator)

Reading for 26 April 2021

1 Preamble

And now for time-dependent perturbation theory. The formalism of course has some close connections to the time-dependent case, but of course being more general, there is more to it.

This is the start where we'll set up the basic framework, analogous to the resolvent operator (which we'll see again). More basic framework is coming next time, then we'll work with perturbed evolution in some simple cases.

Okay, this is a little long, but the Thursday reading will be correspondingly shorter.

2 Expansion of the Evolution Operator

Recall that, way back, we introduced the unitary time-evolution operator, which acts as a mapping between state vectors at different times:

$$|\psi(t)\rangle = U(t, t_0)\, |\psi(t_0)\rangle. \tag{1}$$

There we showed that the time-evolution operator satisfies the Schrödinger equation:

$$\partial_t U(t, t_0) = -\frac{i}{\hbar} H(t)\, U(t, t_0). \tag{2}$$

In this case, the Hamiltonian can depend on time, and in fact we will take $H(t)$ to be a combination of a static Hamiltonian H_0 (with known solutions) and a possibly time-dependent interaction (perturbation) potential:

$$H(t) = H_0 + V(t). \tag{3}$$

To simplify notation, we'll discard the bookkeeping parameter λ that we used in time-independent perturbation theory.

Now since we have the equation of motion

$$\partial_t U(t, t_0) = -\frac{i}{\hbar}\big[H_0 + V(t)\big] U(t, t_0) \tag{4}$$

for the evolution operator, we can define the evolution of the "base" evolution operator U_0 under H_0:

$$\partial_t U_0(t, t_0) = -\frac{i}{\hbar} H_0\, U_0(t, t_0), \tag{5}$$

Recall that since H_0 is time-independent, the base evolution operator has the explicit form

$$U_0(t, t_0) = \exp\left[-\frac{i}{\hbar} H_0 (t - t_0)\right]. \tag{6}$$

In the case of the full evolution operator U, we can write the solution of Eq. (4) in terms of an integral as

$$U(t,t_0) = U_0(t,t_0) - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, U_0(t,t_1)\, V(t_1)\, U(t_1,t_0). \qquad \text{(evolution operator, integral solution)} \quad (7)$$

That this is indeed the solution can be verified by checking the initial condition $U(t_0,t_0) = 1$ and then differentiating this expression.

Homework Problem #1. Check that Eq. (7) is indeed the solution to Eq. (4).

To develop a perturbation series, we can simply iterate the solution (7), with the result

$$\begin{aligned}
U(t,t_0) = U_0(t,t_0) &- \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, U_0(t,t_1)\, V(t_1)\, U_0(t_1,t_0) \\
&- \frac{1}{\hbar^2}\int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1\, U_0(t,t_2)\, V(t_2)\, U_0(t_2,t_1)\, V(t_1)\, U_0(t_1,t_0) \\
&+ \frac{i}{\hbar^3}\int_{t_0}^{t} dt_3 \int_{t_0}^{t_3} dt_2 \int_{t_0}^{t_2} dt_1\, U_0(t,t_3)\, V(t_3)\, U_0(t_3,t_2)\, V(t_2)\, U_0(t_2,t_1)\, V(t_1)\, U_0(t_1,t_0) \\
&+ \cdots .
\end{aligned} \qquad (8)$$

Note that in obtaining each of the successive terms from the previous one, we have renamed $t_k \longrightarrow t_{k+1}$ in order to maintain the ordering of the time variables (that is, t_1 is always the newest time variable from the iteration procedure). More formally, we can write the perturbation series for U as

$$U(t,t_0) = U_0(t,t_0) + \sum_{n=1}^{\infty} U_n(t,t_0), \qquad \text{(evolution-operator perturbation series)} \quad (9)$$

where the terms in the series are defined by

$$U_n(t,t_0) := \frac{1}{(i\hbar)^n} \int\limits_{t_0 \le t_1 \le \cdots \le t_n \le t} dt_n \cdots dt_1\, U_0(t,t_n)\, V(t_n)\, U_0(t_n,t_{n-1})\, V(t_{n-1}) \cdots V(t_2)\, U_0(t_2,t_1)\, V(t_1)\, U_0(t_1,t_0). \qquad \text{(evolution-operator expansion terms)} \quad (10)$$

This process looks a lot like the procedure we used from the resolvent operator, where we started with the expression

$$G(z) = G_0(z) + G_0(z) V G(z) \qquad (11)$$

and iterated it to generate the Born expansion

$$G = G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + G_0 V G_0 V G_0 V G_0 + \cdots . \qquad (12)$$

Of course, this resemblance is not superficial, although the result here is more general (and correspondingly a bit more complex due to the iterated integrations involved). However, this connection will require some more development of the resolvent operator. We'll come back to this shortly; for the moment let's put the series (9) into a simpler and more standard form.

2.1 Interaction Picture

We'll be making use of the interaction picture for the rest of the term, so make sure you pick up the idea.

Before, we introduced the Schrödinger picture, where states carry the time dependence, and the Heisenberg picture, where the time dependence is associated with operators. Explicitly, the Schrödinger-picture state evolves as

$$\partial_t|\psi\rangle_{\rm S} = -\frac{i}{\hbar}H|\psi\rangle_{\rm S}, \tag{13}$$

while the operator $A_{\rm S}$ is static. The Heisenberg-picture state $|\psi\rangle_{\rm H}$ is static, but the

$$\partial_t A_{\rm H} = -\frac{i}{\hbar}\left[A_{\rm H}, H\right], \tag{14}$$

in the case where the evolution operator commutes with the Hamiltonian.

There is one more useful picture, the **interaction picture**, which is a hybrid of the Schrödinger and Heisenberg pictures. Again staying with the notation

$$H(t) = H_0 + V(t), \tag{15}$$

where V is the "interaction Hamiltonian," then the interaction picture is essentially the Schrödinger picture with respect to V, but the Heisenberg picture with respect to H_0. That is, the state vector carries the time dependence due to V, while the operators carry the time dependence due to H_0. The point is that the free time dependence due to H_0 should be relatively trivial, so it's handy to get it out of the way by burying it in the operators.

The transformation of the state vector to the interaction picture is

$$\boxed{|\psi\rangle_{\rm I} = U_0(0,t)|\psi\rangle_{\rm S} = e^{iH_0t/\hbar}|\psi\rangle_{\rm S}.} \tag{16}$$

(interaction-picture state)

The operator transforms according to

$$\boxed{A_{\rm I}(t) = U_0(0,t)A_{\rm S}U_0(t,0) = e^{iH_0t/\hbar}A_{\rm S}e^{-iH_0t/\hbar},} \tag{17}$$

(interaction-picture operator)

in order to produce the correct matrix elements with respect to the interaction-picture states. Then the background Hamiltonian causes the operator to evolve,

$$\boxed{\partial_t A_{\rm I} = -\frac{i}{\hbar}\left[A_{\rm I}, H_0\right],} \tag{18}$$

(interaction-picture evolution)

while the state evolves according to the interaction potential

$$\boxed{\partial_t|\psi\rangle_{\rm I} = -\frac{i}{\hbar}V_{\rm I}(t)|\psi\rangle_{\rm I},} \tag{19}$$

(interaction-picture evolution)

noting that the $V_{\rm I}$ here is the potential $V(t)$ transformed into the interaction picture as in Eq. (17). The interaction picture is useful in perturbation theory, where the evolution due to H_0 should already be known. It is thus convenient to bury this evolution in the operators, so that it is possible to focus on the perturbation operator V.

Exercise #1. Differentiate Eqs. (16) and (17) to verify the equations of motion (18) and (19).

Finally, note that we can think of the evolution operator $U_{\text{I}}(t, t_0)$ in the interaction picture, where the "free" evolution due to H_0 has been removed. Explicitly,

$$\begin{aligned}|\psi(t)\rangle_{\text{I}} &= U_0(0,t)|\psi(t)\rangle_{\text{S}} \\ &= U_0(0,t)\,U_{\text{S}}(t,t_0)|\psi(t_0)\rangle_{\text{S}} \\ &= U_0(0,t)\,U_{\text{S}}(t,t_0)\,U_0(t_0,0)|\psi(t_0)\rangle_{\text{I}},\end{aligned} \tag{20}$$

and thus we can identify the total operator acting on $|\psi(t_0)\rangle_{\text{I}}$ as $U_{\text{I}}(t, t_0)$,

$$U_{\text{I}}(t,t_0) := U_0(0,t)\,U_{\text{S}}(t,t_0)\,U_0(t_0,0), \qquad \text{(interaction-picture evolution operator)} \tag{21}$$

so that

$$|\psi(t)\rangle_{\text{I}} = U_{\text{I}}(t,t_0)|\psi(t_0)\rangle_{\text{I}}. \tag{22}$$

Here, $U_{\text{S}}(t, t_0)$ is the Schrödinger-picture evolution operator.

2.2 Dyson Series

Now let's reconsider the perturbation series (9) in the interaction picture. We will also use tildes to denote operators in the interaction picture ($A_{\text{I}} \equiv \tilde{A}$), while operators with no tilde are by default in the Schrödinger picture ($A_{\text{S}} \equiv A$). Then we can transform the evolution operator as

$$\tilde{U}(t,t_0) = U_0(0,t)\,U(t,t_0)\,U_0(t_0,0), \tag{23}$$

while the base evolution operator obviously does nothing:

$$\tilde{U}_0(t,t_0) = U_0(0,t)\,U_0(t,t_0)\,U_0(t_0,0) = 1. \tag{24}$$

Transforming the entire series (9), using

$$\tilde{V}(t_k) = U_0(0,t_k)\,V(t_k)\,U_0(t_k,0) \tag{25}$$

and remembering $U_0(t_k, t_{k-1}) = U_0(t_k, 0)\, U_0(0, t_{k-1})$, thus leads to

$$\tilde{U}(t,t_0) = 1 + \sum_{n=1}^{\infty} \frac{1}{(i\hbar)^n} \int\limits_{t_0\le t_1\le\cdots\le t_n\le t} dt_n \cdots dt_1\, \tilde{V}(t_n)\cdots\tilde{V}(t_1). \qquad \text{(Dyson series)} \tag{26}$$

This interaction-representation series is called the **Dyson series**,[1] and this is obviously a more compact form than the Schrödinger-picture counterpart.

Going on to introduce some formal notation, we can introduce the **chronological operator** $\mathscr{T}$ to write

$$\tilde{U}_n(t,t_0) = \frac{1}{(i\hbar)^n n!} \int_{t_0}^{t} dt_n \cdots dt_1\, \mathscr{T}\big[\tilde{V}(t_n)\cdots\tilde{V}(t_1)\big]. \tag{27}$$

[1]F. J. Dyson, "The Radiation Theories of Tomonaga, Schwinger, and Feynman," *Physical Review* **75**, 486 (1949) (doi: 10.1103/PhysRev.75.486)

That is, the chronological operator reorders its argument such that the times are in increasing order from right to left. Because of the action of the chronological operator, we don't need the time-ordering restriction on the integration limits. But then we are summing over redundant parts of the integrand, corresponding to the $n!$ different orderings of the potential—hence, the new factor of $1/n!$. Writing out the series explicitly as

$$\tilde{U}(t,t_0) = \mathscr{T} \sum_{n=0}^{\infty} \frac{1}{(i\hbar)^n n!} \int_{t_0}^{t} dt_n \cdots dt_1 \, \tilde{V}(t_n) \cdots \tilde{V}(t_1), \tag{28}$$

the sum has the form of the series expansion of the exponential function. Recognizing that the n integrations in each term are independent, we can rewrite this series formally as

$$\boxed{\tilde{U}(t,t_0) = \mathscr{T} \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt' \, \tilde{V}(t')\right].}$$

(evolution operator, interaction picture) (29)

This is the general form of the evolution operator in the interaction picture. Though common, this expression should be understood to be a shorthand for the full Dyson series, as the proper interpretation of $\mathscr{T}$ acting on the exponential function.

3 Connection to the Resolvent Operator

Now let's return to the development of the perturbation series for $U(t,t_0)$ in Eqs. (9) and (10), and how its structure parallels the Born series

$$G = G_0 + G_0VG_0 + G_0VG_0VG_0 + G_0VG_0VG_0VG_0 + \cdots . \tag{30}$$

These series are equivalent in the correct light, and we'll learn quite a bit in the process of exploring this connection.

3.1 Energy-Space Green Functions

Given a time-independent Hamiltonian H, consider the following function, defined by the integral expression

$$G^+(E) := -\lim_{\delta\to 0^+} \frac{i}{\hbar} \int_0^{\infty} d\tau \, e^{i(E-H)\tau/\hbar} e^{-\tau\delta/\hbar}, \tag{31}$$

where the δ exponential factor guarantees the convergence of the integral.

Exercise #2. Carry out the integral to show

$$G^+(E) = G(E + i0^+). \tag{32}$$

Similarly, we can define the function

$$G^-(E) := \lim_{\delta\to 0^+} \frac{i}{\hbar} \int_{-\infty}^{0} d\tau \, e^{i(E-H)\tau/\hbar} e^{+\tau\delta/\hbar}. \tag{33}$$

Exercise #3. Carry out this integral too, showing

$$G^-(E) = G(E - i0^+). \tag{34}$$

For reasons we will see, $G^+(E)$ is called the **retarded Green operator**, in energy (frequency) space, while $G^-(E)$ is called the **advanced Green operator** in energy space. We have thus shown that both Green functions are related to the resolvent via

$$G^\pm(E) = G(E \pm i0^+) = \frac{1}{E - H \pm i0^+}.$$

(retarded and advanced Green operators) (35)

That is, they are essentially the resolvent along the line displaced infinitesimally above and below the real axis—a location riddled with poles and maybe a branch cut, if you recall. Note that $G^\pm(E)$ are indeed operators, although it is also common to call them Green *functions*; however, it's a little more appropriate to reserve the term "Green function" for matrix elements of the corresponding operator, such as $G^\pm(x, x'; E) = \langle x|G^\pm(E)|x'\rangle$ in the position representation.

3.2 Time-Dependent Green Functions and Propagators

Now note that the definition (31) can be rewritten

$$G^+(E) = -\frac{i}{\hbar}\int_{-\infty}^{\infty} d\tau \, e^{i(E+i0^+)\tau/\hbar}\, U(\tau, 0)\, \Theta(\tau), \tag{36}$$

where $\Theta(\tau)$ is the Heaviside step function and $U(\tau, 0) = e^{-iH\tau/\hbar}$ is the unitary time-evolution operator from time 0 to τ for evolution under (the time-independent) Hamiltonian H. That is, $G^+(E)$ is (within a specific normalization convention) the Fourier transform of "half" of the time-evolution operator, $U(\tau, 0)\,\Theta(\tau)$. Similarly, from the definition (33), we see that

$$G^-(E) = \frac{1}{i\hbar}\int_{-\infty}^{\infty} d\tau \, e^{i(E-i0^+)\tau/\hbar}\, [-U(\tau, 0)\, \Theta(-\tau)], \tag{37}$$

so that up to a minus sign, $G^-(E)$ is the Fourier transform of the "other half" of the time-evolution operator, $U(\tau, 0)\,\Theta(-\tau)$. Thus, defining the time-dependent retarded and advanced Green operators ($+$ for retarded, $-$ for advanced)

$$G^\pm(t, t_0) := \pm U(t, t_0)\, \Theta[\pm(t - t_0)],$$

(retarded and advanced Green operators) (38)

these are related by the above Green operators by a Fourier transform:

$$G^\pm(E) = \frac{1}{i\hbar}\int_{-\infty}^{\infty} d\tau \, e^{i[E+(\mathrm{sgn}\,\tau)i0^+]\tau/\hbar}\, G^\pm(\tau, 0).$$

(Green-operator Fourier transform) (39)

In the notation we use here, we'll just use the argument of the Green operator to determine whether it is a time-dependent or energy-space Green operator. Note that since we are dealing with time-independent systems, $G^\pm(t, t_0)$ only depends on $t - t_0$.

Now to explain the terminology of advanced and retarded Green operators. First, recall that the time-evolution operator satisfies the Schrödinger equation,

$$i\hbar\partial_t U(t, t_0) = HU(t, t_0). \tag{40}$$

We can use this relation and $\partial_\tau \Theta(\tau) = \delta(\tau)$ to differentiate the Green functions:

$$\begin{aligned} i\hbar\partial_t G^{\pm}(t,t_0) &= \pm i\hbar\partial_t \Big[U(t,t_0)\Theta[\pm(t-t_0)]\Big] \\ &= \pm HU(t,t_0)\Theta[\pm(t-t_0)] \pm i\hbar U(t,t_0)\big[\pm\delta(t-t_0)\big] \\ &= HG^{\pm}(t,t_0) + i\hbar\delta(t-t_0). \end{aligned} \tag{41}$$

In the last step, we used $U(t_0,t_0) = 1$.

Exercise #4. Work through the differentiation here in enough detail that you understand it.

Thus, we have shown that

$$(i\hbar\partial_t - H)\, G^{\pm}(t,t_0) = i\hbar\delta(t-t_0)$$

(Green operators for the Schrödinger equation) (42)

(noting that this equation is also valid if the underlying Hamiltonian is explicitly time-dependent). Remember that the solution to a differential equation with a delta-function "driving" term *defines* a Green function (or impulse-response function) in general. Thus, $G^{\pm}(t,t_0)$ is the solution to the Schrödinger equation, driven by a delta-function impulse at $t = t_0$. In particular, $G^{+}(t,t_0)$ is the "retarded" Green operator, because the "source" is in the past, and the response follows after the impulse, $t > t_0$. Similarly, $G^{-}(t,t_0)$ is the "advanced" Green operator, because the source is in the future, and the response comes *before* the impulse, $t < t_0$. Note, however, that there is no kick in the form of an added *potential*; it's an impulse that normally wouldn't show up in the Schrödinger equation. The impulse effectively creates a wave function from nothing at time t_0 in the retarded case, and annihilates a wave function into nothing in the advanced case. Both Green operators obey the same equation, but correspond to different boundary conditions as $t \longrightarrow \pm\infty$.

Inverting the Fourier-transform relation (39) gives

$$G^{\pm}(\tau,0) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} dE\, e^{-iE\tau/\hbar}\, G^{\pm}(E).$$

(Green-function inverse Fourier transform) (43)

(Reading off the Fourier transform is the easiest way to infer this equation, but a contour integral coming up next time is also a good way to do this. Don't worry, this will be a future homework problem.) In particular, the case of $G^{+}(\tau,0)$ is important, as it gives the time-evolution operator for evolving the system forward from $t = 0$ to τ:

$$U(\tau,0) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} dE\, e^{-iE\tau/\hbar}\, G^{+}(E) \qquad (\tau > 0).$$

(Green-function Fourier relation) (44)

This relation is particularly useful in that it shows that matrix elements of the evolution operator,

$$K(\beta,t;\alpha,t_0) := \langle\beta|U(t,t_0)|\alpha\rangle,$$

(45) (propagator)

collectively called the **propagator**, can be computed from matrix elements of the retarded Green function. The term "propagator" is commonly used to refer specifically to matrix elements in the position representation as

$$K(x,t;x_0,t_0) := \langle x|U(t,t_0)|x_0\rangle,$$

(propagator, position representation) (46)

although it is of course useful to think of this concept more generally. The propagator gives the transition amplitude, or the probability amplitude for the system to be in state β at time t, given that it was in state

α at the (earlier) time t_0. Note that in the case of forward propagation $t > t_0$, the propagator can also be regarded as the collection of matrix elements of of the retarded Green operator $G^+(t, t_0)$ in view of the definition (38). In this way we can also talk about a retarded propagator K^+ and advanced propagator K^-, in exactly the same way as the Green operators (because they're really the same thing).

3.2.1 Free-Particle Green Function

Study this derivation in enough detail that you understand it. The test will be in the following homework problem, to generalize the calculation to one and two spatial dimensions.

As a (relatively) simple example of a Green function, let's compute the retarded (causal) Green function for the free particle with $H = p^2/2m$. Then the Green operator is

$$G^+(E) = \frac{1}{E - p^2/2m + i0^+}, \tag{47}$$

and we can proceed to compute the Green function in the position representation by inserting a momentum identity:

$$\begin{aligned} G^+(\mathbf{x}, \mathbf{x}_0; E) &= \langle \mathbf{x}| \frac{1}{E - p^2/2m + i0^+} |\mathbf{x}_0\rangle \\ &= \int d^3p \, \langle \mathbf{x}| \frac{1}{E - p^2/2m + i0^+} |\mathbf{p}\rangle\langle \mathbf{p}|\mathbf{x}_0\rangle \\ &= \frac{1}{(2\pi\hbar)^3} \int d^3p \, \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)/\hbar}}{E - p^2/2m + i0^+}. \end{aligned} \tag{48}$$

Note that in the last equality the p^2 went from an operator to an eigenvalue, and then we used $\langle x|p\rangle = e^{ikx}/\sqrt{2\pi\hbar}$ in each dimension. Now setting

$$\mathbf{r} := \mathbf{x} - \mathbf{x}_0, \tag{49}$$

we can write out the integral in spherical coordinates as

$$G^+(\mathbf{x}, \mathbf{x}_0; E) = \frac{1}{4\pi^2\hbar^3} \int_0^\infty dp \, p^2 \int_0^\pi d\theta \, \sin\theta \, \frac{e^{ipr\cos\theta/\hbar}}{E - p^2/2m + i0^+}, \tag{50}$$

where the angular integral is

$$\int_0^\pi d\theta \, \sin\theta \, e^{ipr\cos\theta/\hbar} = \int_{-1}^1 d\mu \, e^{ipr\mu/\hbar} = 2\,\mathrm{sinc}(pr/\hbar), \tag{51}$$

and $\mathrm{sinc}(x) := \sin(x)/x$ [i.e., basically $j_0(x)$]. Then

$$\begin{aligned} G^+(\mathbf{x}, \mathbf{x}_0; E) &= \frac{1}{2\pi^2\hbar^3} \int_0^\infty dp \, p^2 \, \frac{\mathrm{sinc}(pr/\hbar)}{E - p^2/2m + i0^+} \\ &= \frac{1}{2\pi^2 r^3} \int_0^\infty dz \, \frac{z^2 \,\mathrm{sinc}\, z}{E - \frac{\hbar^2}{2mr^2} z^2 + i0^+}, \end{aligned} \tag{52}$$

where

$$z := \frac{pr}{\hbar}. \tag{53}$$

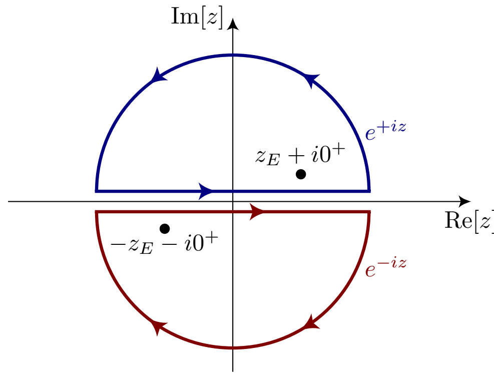
Also defining

$$z_E^2 := \frac{2mr^2 E}{\hbar^2}, \tag{54}$$

we find

$$
\begin{aligned}
G^{+}(\mathbf{x}, \mathbf{x}_0; E) &= \frac{m}{\pi^2 \hbar^2 r} \int_0^\infty dz\, \frac{z \sin z}{z_E^2 - z^2 + i0^+} \\
&= \frac{m}{4i\pi^2 \hbar^2 r} \int_{-\infty}^\infty dz\, \frac{z\big(e^{-iz} - e^{iz}\big)}{[z - (z_E + i0^+)][z + (z_E + i0^+)]}.
\end{aligned}
\tag{55}
$$

Note that in the last step, $(z_E + i0^+)^2 = z_E^2 + i0^+$, counting only the "linear" terms in $i0^+$, and discarding the second-order term $(i0^+)^2$. To carry out the integral, we have to handle the $e^{\pm iz}$ components as separate contour integrals: e^{-iz} damps the contour going around the lower half-plane, while e^{+iz} damps the contour going around the upper half-plane. The contours are shown below; each contour encloses one pole.



Now applying the Cauchy integral formula in each case, we obtain

$$
\begin{aligned}
G^{+}(\mathbf{x}, \mathbf{x}_0; E) &= \frac{m}{4i\pi^2 \hbar^2 r} (2\pi i) \left[-\frac{e^{iz_E}}{2} - \frac{e^{iz_E}}{2} \right] \\
&= -\frac{m}{2\pi \hbar^2 r}\, e^{iz_E}.
\end{aligned}
\tag{56}
$$

Putting in the expression for z_E, we arrive at

$$
\boxed{G^{+}(\mathbf{x}, \mathbf{x}_0; E) = -\frac{m}{2\pi\hbar^2} \frac{e^{ip_E r/\hbar}}{r},} \tag{57}
$$

(free-particle Green function)

where $p_E := \sqrt{2mE}$. (Quick units check: the resolvent should have units of 1/(energy), and $G^{+}(\mathbf{x}, \mathbf{x}_0; E)$ should thus have units of $(\text{energy})^{-1}(\text{length})^{-3}$, which is true here.) This Green function has the form of a spherical wave; indeed, it is the Green function for the Helmholtz equation $(\nabla^2 + k^2)\psi = 0$, for which the solution is the radiation from a point source e^{ikr}/r. Here the boundary condition for the retarded Green function is more clear, since it corresponds to outgoing radiation at large r. In the calculation of the advanced Green function $G^{-}(\mathbf{x}, \mathbf{x}_0; E)$, the poles hop over the real axis, and the net effect is that the functional form is an *ingoing* solution e^{-ikr}/r, where the wave is "absorbed" by the source at $r = 0$. Which brings up another point: We discussed the Green functions as impulse-response functions for the Schrödinger equation, but note that in the position representation there is *another* sense of impulse response in that, in the absence of the resolvent, the inner product $\langle \mathbf{x} | \mathbf{x_0} \rangle = \delta^3(\mathbf{r})$ defines a spatial impulse (i.e., the radiation source).

Homework Problem #2. Work out the retarded Green function $G^+(E)$ for the free particle

(a) in one (spatial) dimension. Be careful to ensure that your result is a function of $|x - x_0|$.

Note that the setup of the expression in terms of a momentum integral will be the same as in three dimensions, the difference is in the evaluation of the integral.

(b) in two (spatial) dimensions. There is a trick to doing the integral here. Working in polar coordinates, you should get a Bessel function after carrying out the angular integral. But it's not straightforward to see how to proceed with setting up a contour integral along the whole real axis. If you use the integral representation

$$J_0(x) = \frac{2}{\pi} \int_1^\infty du \frac{\sin ux}{\sqrt{u^2 - 1}}, \qquad (x > 0), \tag{58}$$

you should be able to proceed with the momentum integral. To carry out the final integral, use the integral representation of the Hankel function

$$H_0(x) = -\frac{2i}{\pi} \int_1^\infty du \frac{e^{iux}}{\sqrt{u^2 - 1}}, \qquad (x > 0). \tag{59}$$