**PHYS 632: Quantum Mechanics II (Winter 2021)**
**Final Exam, 18 March 2021**

**Rules for this exam:** You should submit your exam solutions by 11:59 pm PDT on Thursday, 18 March.

You may consult any inanimate object while taking the exam (including computers, *Mathematica*, and information posted on the internet). However, you may **not** discuss any part of the exam with **any human by any means** (except the instructor) until after the last due date.

**Magic lifeline rule** is still in effect for this exam (2 lifeline questions). Please keep in mind that depending on when you email me, it could take me hours to respond. Clear, concise questions will help you to obtain, quick, useful responses.

**I have read and understand the rules.**

________________________________________ ⟵ **Sign your name here.**

**Problem 1.** (20 points) Consider the following entangled state of three spin-1/2 particles:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|+++\rangle - |---\rangle\Big). \tag{1}$$

We can assume the particles to be distinguishable and labeled as particles 1, 2, and 3 in the order they appear in the ket.

(a) (10 points) Show that this state is a simultaneous eigenstate of the operators $\sigma_{1x}\sigma_{2y}\sigma_{3y}$, $\sigma_{1y}\sigma_{2x}\sigma_{3y}$, $\sigma_{1y}\sigma_{2y}\sigma_{3x}$, and $\sigma_{1x}\sigma_{2x}\sigma_{3x}$, and find the eigenvalue in each case. (Note that $\sigma_{1\alpha}$ is the $\sigma_\alpha$ Bloch operator for particle 1, and so on for particles 2 and 3.)

(b) (10 points) Now suppose that you make joint measurements of the three particles in each of the four combinations specified in (a). Suppose also that you try to describe the measurement results in terms of a local hidden-variable theory. Let's use $A_\alpha$, $B_\beta$, and $C_\gamma$ to denote the respective hidden-variable counterparts to $\sigma_{1\alpha}$, $\sigma_{2\beta}$, and $\sigma_{3\gamma}$. Then write down what you should expect for the measurement combinations $A_xB_yC_y$, $A_yB_xC_y$, $A_yB_yC_x$, and $A_xB_xC_x$, in order for them to match quantum mechanics. By considering the product of the first three measurement combinations, show that you obtain a contradiction (so that again, quantum mechanics cannot be explained in terms of local hidden variables).

**Problem 2.** (20 points) Suppose that we have brainstormed a half-baked, modified teleportation scheme to clone a quantum state. Starting with three qubits in the state given in Eq. (1), Alice is to keep the first one, gives the second one to Bob, and gives the third to Casey. She does the same local operations to her shared qubit and qubit to teleport $|\psi\rangle_{\mathrm{T}}$ that she would in the standard teleportation protocol. Then she communicates the usual two classical bits of information to both Bob and Casey, and asks them to perform local unitary operations as appropriate in order to reconstruct $|\psi\rangle_{\mathrm{T}}$.

(a) (10 points) The no-cloning theorem says this procedure can't work as we envision. Analyze the procedure to show that it indeed doesn't work, and indicate why not.

(b) (10 points) However, show that Bob can apply a (local) unitary operation to his qubit, measures its state, and send the result to Casey, then Casey can use the information to receive the teleported state from Alice.

**(Don't forget the other side!)**

**Problem 3.** (30 points)
(a) (15 points) Consider a spin-1/2 particle in a quantum state given by the density operator $\rho$, and in particular in terms of the density matrix elements $\rho_{\pm\pm}$ and $\rho_{\pm\mp}$ in the representation of the eigenstates of $S_z$. Give the probability (in terms of density-matrix elements) for a measurement of $S_x$, $S_y$, or $S_z$ to give the result $+\hbar/2$ (i.e., do three separate calculations but stay in the same representation).

(b) (15 points) Repeat the calculation of (a) for a $j = 1$ particle to give the result $+\hbar$ under the same conditions (representation of $J_z$'s eigenstates, consider measurements along each axis).

**Problem 4.** (30 points)
(a) (15 points) The Wigner rotation matrix for a composite rotation through Euler angles $(\alpha, \beta, \gamma)$ is

$$\left[D^{(1/2)}_{m'm}(\alpha,\beta,\gamma)\right] = \begin{bmatrix} e^{-i(\alpha+\gamma)/2}\cos(\beta/2) & -e^{-i(\alpha-\gamma)/2}\sin(\beta/2) \\ e^{i(\alpha-\gamma)/2}\sin(\beta/2) & e^{i(\alpha+\gamma)/2}\cos(\beta/2) \end{bmatrix} \tag{2}$$

for the case $j = 1/2$. The point of this is, of course, that *any* rotation specified by a rotation vector $\boldsymbol{\zeta}$ can be specified in terms of the Euler angles.

Conversely, suppose that someone gives you a rotation in terms of the Euler angles $(\alpha, \beta, \gamma)$. Using the above form of the $j = 1/2$ rotation matrix, what is the equivalent rotation vector $\boldsymbol{\zeta}$ (i.e., the angle and axis of the equivalent, *single* rotation)?

(b) (15 points) In the case of $j = 1$, the Wigner matrix is

$$\left[D^{(1)}_{m'm}(\alpha,\beta,\gamma)\right] = \begin{bmatrix} e^{-i\alpha}\dfrac{1+\cos\beta}{2}e^{-i\gamma} & -e^{-i\alpha}\dfrac{\sin\beta}{\sqrt{2}} & e^{-i\alpha}\dfrac{1-\cos\beta}{2}e^{i\gamma} \\ \dfrac{\sin\beta}{\sqrt{2}}e^{-i\gamma} & \cos\beta & -\dfrac{\sin\beta}{\sqrt{2}}e^{i\gamma} \\ e^{i\alpha}\dfrac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\dfrac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\dfrac{1+\cos\beta}{2}e^{i\gamma} \end{bmatrix}. \tag{3}$$

Does the same equivalent single rotation still hold here? Prove your answer either way.