

Physics 622 Homework 8

Jeremy Welsh-Kavan

My homework in LaTeX, as requested. I pray it pleases thee, exalted grader.

2.2.3 Electrostatics in d dimensions (continued)

b) We calculate and plot the potential φ and the field \mathbf{E} for $d = 2$, for the case of a homogeneously charged disk, $\rho(\mathbf{x}) = \rho_0 \Theta(r_0 - |\mathbf{x}|)$.

For $d = 2$, the \mathbf{E} field and the potential satisfy

$$\mathbf{E}(\mathbf{x}) = -\nabla\varphi(\mathbf{x}) \quad (1)$$

and

$$\nabla^2\varphi(\mathbf{x}) = -2\pi\rho(\mathbf{x}) \quad (2)$$

Since $\rho(\mathbf{x})$ is azimuthally symmetric, we know that $\varphi(\mathbf{x})$ will be azimuthally symmetric. In this case, we can rewrite (2) as

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \varphi(r) \right) = -2\pi\rho_0 \Theta(r_0 - r) \quad (3)$$

From this we can directly solve for $\varphi(r)$ by integrating (3) twice.

$$\begin{aligned} r \frac{\partial}{\partial r} \varphi(r) &= -2\pi\rho_0 \int dr \, r \Theta(r_0 - r) \\ r \frac{\partial}{\partial r} \varphi(r) &= -2\pi\rho_0 \begin{cases} \frac{r^2}{2}, & 0 \leq r \leq r_0 \\ \frac{r_0^2}{2}, & r > r_0 \end{cases} \\ \frac{\partial}{\partial r} \varphi(r) &= -\pi\rho_0 \left(r + \Theta(r - r_0) \left(\frac{r_0^2}{r} - r \right) \right) \\ \varphi(r) &= -\pi\rho_0 \left(\frac{r^2}{2} + \Theta(r - r_0) \left(\frac{r_0^2}{2} - \frac{r^2}{2} + r_0^2 \ln \left(\frac{r}{r_0} \right) \right) \right) \end{aligned} \quad (4)$$

or

$$\varphi(r) = -\pi\rho_0 \begin{cases} \frac{r^2}{2}, & 0 \leq r \leq r_0 \\ \frac{r_0^2}{2} + r_0^2 \ln \left(\frac{r}{r_0} \right), & r_0 < r \end{cases}$$

Finally, we can redefine $\varphi(r_0) = 0$ to get

$$\varphi(r) = \begin{cases} \pi\rho_0 \left(\frac{r_0^2}{2} - \frac{r^2}{2} \right), & 0 \leq r \leq r_0 \\ \pi\rho_0 r_0^2 \ln \left(\frac{r_0}{r} \right), & r_0 < r \end{cases} \quad (5)$$

We find the \mathbf{E} field by taking the negative gradient of (5).

$$\begin{aligned}
 \mathbf{E}(\mathbf{x}) &= -\nabla\varphi(\mathbf{x}) \\
 \mathbf{E}(\mathbf{x}) &= -\frac{\partial\varphi(r)}{\partial r}\hat{e}_r \\
 \mathbf{E}(\mathbf{x}) &= E(r)\hat{e}_r \\
 E(r) &= \begin{cases} \pi\rho_0 r, & 0 \leq r \leq r_0 \\ \pi\rho_0 \frac{r_0^2}{r}, & r_0 < r \end{cases}
 \end{aligned} \tag{6}$$

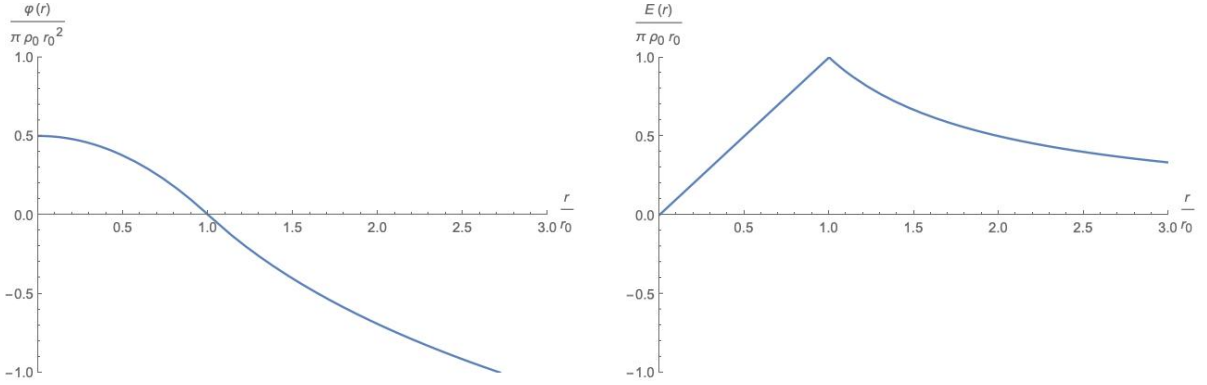


Figure 1: Plots of the potential, $\varphi(r)$, and the radial component of the electric field, $E(r)$, in convenient units.

c) We proceed as above, but for $d=1$. Here, $d=1$ and $\rho(x) = \rho_0\Theta(x_0^2/4 - x^2)$, which represents the charge density of a uniformly charged rod of length x_0 centered at the origin. In the $d=1$ case, $\mathbf{E}(\mathbf{x})$ and $\varphi(x)$ are both just scalar fields given by the equations

$$\begin{aligned}
 \frac{\partial E(x)}{\partial x} &= 2\rho(x) \\
 \frac{\partial^2 \varphi(x)}{\partial x^2} &= -2\rho(x)
 \end{aligned} \tag{7}$$

Solving for $E(x)$, we have

$$\begin{aligned}
 \frac{\partial E(x)}{\partial x} &= 2\rho_0\Theta\left(1 - \frac{4x^2}{x_0^2}\right) \\
 E(x) &= \begin{cases} -\rho_0 x_0, & x < -\frac{x_0}{2} \\ 2\rho_0 x, & -\frac{x_0}{2} \leq x \leq \frac{x_0}{2} \\ \rho_0 x_0, & \frac{x_0}{2} < x \end{cases}
 \end{aligned} \tag{8}$$

We can define $\varphi(\frac{x_0}{2}) = 0$, and since the potential is symmetric is the charge distribution is

symmetric, $\varphi(-\frac{x_0}{2}) = 0$ as well. To obtain $\varphi(x)$ we integrate (8), which yields

$$\varphi(x) = - \int_{x_0/2}^x dx E(x)$$

$$\varphi(x) = \begin{cases} \rho_0 x_0 x + \frac{\rho_0 x_0^2}{2}, & x < -\frac{x_0}{2} \\ \frac{\rho_0 x_0^2}{4} - \rho_0 x^2, & -\frac{x_0}{2} \leq x \leq \frac{x_0}{2} \\ -\rho_0 x_0 x + \frac{\rho_0 x_0^2}{2}, & \frac{x_0}{2} < x \end{cases} \quad (9)$$

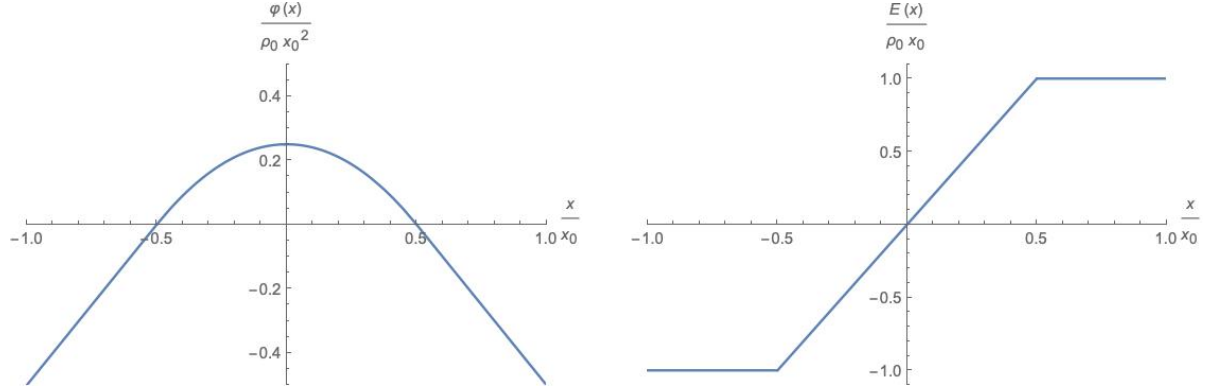


Figure 2: Plots of the potential, $\varphi(x)$, and the electric field, $E(x)$, in convenient units.

2.2.4 Helmholtz Equation

Our goal is to find the most general Fourier transformable solution, $\varphi(\mathbf{x})$, to the Helmholtz equation,

$$(\kappa^2 - \nabla^2) \varphi(\mathbf{x}) = 4\pi\rho(\mathbf{x}) \quad (10)$$

We can begin by finding the fundamental solution, $G(\mathbf{x})$, of the Helmholtz operator. That is, a function, $G(\mathbf{x})$ satisfying

$$(\kappa^2 - \nabla^2) G(\mathbf{x}) = -4\pi\delta(\mathbf{x}) \quad (11)$$

Taking the Fourier transform of both sides of (6) gives

$$\frac{1}{(\sqrt{2\pi})^3} \int d^3\mathbf{x} (\kappa^2 - \nabla^2) G(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} = -\frac{4\pi}{(\sqrt{2\pi})^3}$$

$$(\kappa^2 + \mathbf{k}^2) \hat{G}(\mathbf{k}) = -\sqrt{\frac{2}{\pi}}$$

$$\hat{G}(\mathbf{k}) = -\sqrt{\frac{2}{\pi}} \frac{1}{\mathbf{k}^2 + \kappa^2} \quad (12)$$

We now inverse Fourier transform the last line of (7) to get

$$G(\mathbf{x}) = -\sqrt{\frac{2}{\pi}} \frac{1}{(\sqrt{2\pi})^3} \int d^3\mathbf{k} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2 + \kappa^2} \quad (13)$$

which we can compute in spherical coordinates as follows:

$$\begin{aligned} G(\mathbf{x}) &= -\frac{1}{2\pi^2} \int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{e^{ir|\mathbf{x}|\cos\theta}}{r^2 + \kappa^2} r^2 \sin\theta d\theta d\phi dr \\ G(\mathbf{x}) &= -\frac{1}{\pi} \int_0^\infty \int_0^\pi \frac{e^{ir|\mathbf{x}|\cos\theta}}{r^2 + \kappa^2} r^2 \sin\theta d\theta dr \\ G(\mathbf{x}) &= -\frac{2}{\pi|\mathbf{x}|} \int_0^\infty \frac{r \sin(r|\mathbf{x}|)}{r^2 + \kappa^2} dr \\ G(\mathbf{x}) &= -\frac{e^{-\kappa|\mathbf{x}|}}{|\mathbf{x}|} \end{aligned} \quad (14)$$

Where the last integral in (9) can be computed with Mathematica or using the residue theorem. Note that this differs by a sign from the Green's function of the Laplacian that we found in 2.2.3 a) since the Laplacian carries a minus sign in the Helmholtz equation. With $G(\mathbf{x})$, we can find a general expression for $\varphi(\mathbf{x})$. From what we know about Green's functions, if $\varphi(\mathbf{x})$ satisfies (5), then we can express $\varphi(\mathbf{x})$ in terms of a convolution with $G(\mathbf{x})$. Just to convince ourselves that this makes sense, below is some hand-wavy algebra. Let $L = \kappa^2 - \nabla^2$. If $G(\mathbf{x})$ satisfies

$$L[G(\mathbf{x} - \mathbf{y})] = -4\pi\delta^3(\mathbf{x} - \mathbf{y}) \quad (15)$$

then we have

$$\begin{aligned} \int d^3\mathbf{y} L[G(\mathbf{x} - \mathbf{y})] \rho(\mathbf{y}) &= -4\pi \int d^3\mathbf{y} \delta^3(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}) \\ -L \left[\int d^3\mathbf{y} G(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}) \right] &= 4\pi \rho(\mathbf{x}) \end{aligned} \quad (16)$$

and since

$$L[\varphi(\mathbf{x})] = 4\pi\rho(\mathbf{x}) \quad (17)$$

we can conclude that

$$\varphi(\mathbf{x}) = - \int d^3\mathbf{y} G(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}) \quad (18)$$

So, with equation (9), we have

$$\varphi(\mathbf{x}) = \int d^3\mathbf{y} \frac{e^{-\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|} \rho(\mathbf{y}) \quad (19)$$

Which is Fourier transformable if $\rho(\mathbf{x})$ is Fourier Transformable.

2.3.1 Quadrupole moments

a) Let $\rho(\mathbf{y})$ be a localized charge density. To approximate the scalar potential, $\varphi(\mathbf{x})$, we first expand the denominator of Poisson's formula, omitting terms of order $\frac{1}{r^4}$ and greater.

$$\begin{aligned} \frac{1}{|\mathbf{x} - \mathbf{y}|} &= \frac{1}{\sqrt{r^2 - 2(\mathbf{x} \cdot \mathbf{y}) + \mathbf{y}^2}} \\ &= \frac{1}{r} \left(1 - \frac{2(\mathbf{x} \cdot \mathbf{y})}{r^2} + \frac{\mathbf{y}^2}{r^2} \right)^{-\frac{1}{2}} \\ &\approx \frac{1}{r} \left(1 + \frac{(\mathbf{x} \cdot \mathbf{y})}{r^2} - \frac{\mathbf{y}^2}{2r^2} + \frac{3}{2} \frac{(\mathbf{x} \cdot \mathbf{y})^2}{r^4} \right) \end{aligned} \quad (20)$$

Using Poisson's formula, we can construct an approximation for $\varphi(\mathbf{x})$.

$$\begin{aligned} \varphi(\mathbf{x}) &= \int d\mathbf{y} \frac{\rho(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \\ \varphi(\mathbf{x}) &\approx \int d\mathbf{y} \frac{\rho(\mathbf{y})}{r} \left(1 + \frac{(\mathbf{x} \cdot \mathbf{y})}{r^2} - \frac{\mathbf{y}^2}{2r^2} + \frac{3}{2} \frac{(\mathbf{x} \cdot \mathbf{y})^2}{r^4} \right) \\ &= \frac{Q}{r} + \frac{\mathbf{d} \cdot \mathbf{x}}{r^3} + \int d\mathbf{y} \frac{\rho(\mathbf{y})}{r} \left(\frac{3}{2r^4} \sum_{i,j} x_i x_j y_i y_j - \frac{1}{2r^2} \mathbf{y}^2 \right) \\ &= \frac{Q}{r} + \frac{\mathbf{d} \cdot \mathbf{x}}{r^3} + \frac{1}{r^5} \int d\mathbf{y} \rho(\mathbf{y}) \left(\frac{3}{2} \sum_{i,j} x_i x_j y_i y_j - \frac{r^2}{2} \mathbf{y}^2 \right) \\ &= \frac{Q}{r} + \frac{\mathbf{d} \cdot \mathbf{x}}{r^3} + \frac{1}{r^5} \int d\mathbf{y} \rho(\mathbf{y}) \left(\frac{3}{2} \sum_{i,j} x_i x_j y_i y_j - \frac{1}{2} \sum_j x_i x_i \mathbf{y}^2 \right) \\ &= \frac{Q}{r} + \frac{\mathbf{d} \cdot \mathbf{x}}{r^3} + \frac{1}{r^5} \sum_{i,j} x_i x_j \left[\frac{1}{2} \int d\mathbf{y} \rho(\mathbf{y}) (3y_i y_j - \delta_{i,j} \mathbf{y}^2) \right] \\ \varphi(\mathbf{x}) &\approx \frac{Q}{r} + \frac{\mathbf{d} \cdot \mathbf{x}}{r^3} + \frac{1}{r^5} \sum_{i,j} x_i x_j Q_{ij} \end{aligned} \quad (21)$$

where

$$Q_{ij} = \frac{1}{2} \int d\mathbf{y} \rho(\mathbf{y}) (3y_i y_j - \delta_{i,j} \mathbf{y}^2) \quad (22)$$

b) I didn't finish this one in time, which is a bummer because it doesn't seem difficult at all :(.

c) For a homogeneously charged ellipsoid, $(x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1$, with total charge q , we express the charge density in the ellipsoidal coordinates parameterized by

$$\begin{cases} y_1 = \arccos(\phi) \sin(\theta) \\ y_2 = b \sin(\phi) \sin(\theta) \\ y_3 = c \cos(\theta) \\ r \in [0, 1], \phi \in [0, 2\pi], \theta \in [0, \pi] \end{cases} \quad (23)$$

and

$$\rho(r, \theta, \phi) = \rho_0 = \frac{3q}{4\pi abc} \quad (24)$$

where q is the total charge. The quadrupole tensor is given by

$$Q_{ij} = \frac{\rho_0}{2} \int_0^1 \int_0^{2\pi} \int_0^\pi abc r^2 \sin(\theta) d\theta d\phi dr (3y_i y_j - \delta_{i,j} \mathbf{y}^2) \quad (25)$$

$$Q = \frac{2}{15} \pi abc \rho_0 \begin{pmatrix} 2a^2 - b^2 - c^2 & 0 & 0 \\ 0 & 2b^2 - a^2 - c^2 & 0 \\ 0 & 0 & 2c^2 - a^2 - b^2 \end{pmatrix}$$

where we have used Mathematica to compute the integrals. By inspection, Q is traceless, but we calculate $\text{tr}(Q)$ below for completeness.

$$\begin{aligned} \text{tr}(Q) &= \frac{2}{15} \pi abc \rho_0 (2a^2 - b^2 - c^2 + 2b^2 - a^2 - c^2 + 2c^2 - a^2 - b^2) \\ &= \frac{2}{15} \pi abc \rho_0 (2a^2 + 2b^2 + 2c^2 - 2a^2 - 2b^2 - 2c^2) \\ &= 0 \end{aligned} \quad (26)$$

d) Let $\rho(r, \theta, z)$ be a localized charge density parameterized by the cylindrical coordinates and suppose $\rho(r, \theta, z)$ has the property that $\rho(r, \theta, z) = \rho(r, \theta + k\alpha, z)$ for $k \in \mathbb{Z}$ and $\alpha \in [-\pi, \pi]$. We claim that Q_{ij} has the form

$$Q = \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -2q \end{pmatrix} \quad (27)$$

We observe that the azimuthal symmetry of $\rho(r, \theta, z)$ implies that the coordinate system in which ρ has this property is the principle axis coordinate system. Therefore, Q_{ij} is diagonal in the current coordinate system. Additionally, since Q is traceless, we know that it has the form

$$Q = \begin{pmatrix} q_x & 0 & 0 \\ 0 & q_y & 0 \\ 0 & 0 & -q_x - q_y \end{pmatrix} \quad (28)$$

Therefore, we need only show that $q_x = q_y$. These components are given by

$$\begin{aligned} Q_{xx} &= \int_0^{z_0} \int_0^{r_0} \int_0^{2\pi} r \rho(r, \theta, z) (3r^2 \cos^2(\theta) - r^2 - z^2) d\theta dr dz \\ Q_{yy} &= \int_0^{z_0} \int_0^{r_0} \int_0^{2\pi} r \rho(r, \theta, z) (3r^2 \sin^2(\theta) - r^2 - z^2) d\theta dr dz \end{aligned} \quad (29)$$

Therefore, these components differ only by the following integral

$$\begin{aligned} Q_{xx} - Q_{yy} &= \int_0^{z_0} \int_0^{r_0} \int_0^{2\pi} 3r^3 \rho(r, \theta, z) (\cos^2(\theta) - \sin^2(\theta)) d\theta dr dz \\ Q_{xx} - Q_{yy} &= \int_0^{z_0} \int_0^{r_0} \int_0^{2\pi} 3r^3 \rho(r, \theta, z) \cos(2\theta) d\theta dr dz \end{aligned} \quad (30)$$

Since ρ is necessarily 2π periodic, we know that $k\alpha = 2\pi$ for some minimum $k \in \mathbb{N}$. Therefore, $\rho(r, \theta, z) = \rho(r, \theta + n\frac{2\pi}{k}, z)$ for all $n \in \mathbb{N}$.

Note to the grader Ok so I didn't finish this one either. My best guess is that this periodicity makes the integral in (30) vanish but it's not obvious to me why that is at the moment.
