Physics 614 Final Exam

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1. Phonons in a Wigner crystal

Consider a $2D$ Wigner crystal whose phonon vibration frequencies, ω, obey the dispersion relation $\omega(\boldsymbol{k}) = \alpha\sqrt{k}$, where $k = |\boldsymbol{k}|$ and $\alpha > 0$ is a constant. Keeping in mind the fact that in $2D$, vibrations can have only 2 polarizations and that the integration over $\boldsymbol{k}$ has only a factor of kdk, the density of states is given by

$$\sum_{\text{polariz.}} \sum_{\boldsymbol{k}} \rightarrow \frac{2V}{(2\pi)^2} \int d\boldsymbol{k} \rightarrow \frac{V}{\pi} \int kdk \rightarrow \frac{2V}{\pi\alpha^4} \int \omega^3 d\omega$$

$$\implies g(\omega) = \frac{2V}{\pi\alpha^4}\omega^3 \tag{1}$$

Now, the total energy in the crystal is given by

$$\begin{aligned}
\langle E \rangle &= \int_0^{\omega_D} d\omega \, \frac{\hbar\omega \, g(\omega)}{e^{\beta\hbar\omega} - 1} \\
&= \frac{2V}{\pi\alpha^4} \int_0^{\omega_D} d\omega \, \frac{\omega^3}{e^{\beta\hbar\omega} - 1} \\
&= \frac{2V}{\pi\alpha^4\hbar^4\beta^4} \int_0^{x_D} dx \, \frac{x^3}{e^x - 1} \\
&= \frac{2Vk_B^4}{\pi\alpha^4\hbar^4} T^4 \int_0^{T_D/T} dx \, \frac{x^3}{e^x - 1}
\end{aligned} \tag{2}$$

For low temperatures, $T \ll T_D$, we can replace T_D/T with ∞, yielding,

$$\langle E \rangle = \frac{2Vk_B^4}{\pi\alpha^4\hbar^4} T^4 \int_0^{\infty} dx \, \frac{x^3}{e^x - 1} \tag{3}$$

But since $C_V = \partial \langle E \rangle / \partial T$, we have that $C_V \propto T^3$. Which is not what we were asked to show, but I'm not sure what went wrong.

2. Ultrarelativistic Bose-Einstein condensation

We consider an ideal gas of N ultra-relativistic spinless bosons in a volume V, with energy momentum relation $\epsilon(\mathbf{k}) = \hbar k c$, where $k = |\mathbf{k}|$.

We claim that this system exhibits Bose-Einstein condensation at a finite temperature. We first find that the density of states for this system is

$$\sum_{\mathbf{k}} \rightarrow \frac{V}{(2\pi)^3} \int d\mathbf{k} \rightarrow \frac{V}{2\pi^2} \int k^2 dk \rightarrow \frac{V}{2\hbar^3 c^3 \pi^2} \int \epsilon^2 d\epsilon$$
$$\implies g(\epsilon) = \frac{V}{2\pi^2 \hbar^3 c^3} \epsilon^2 \tag{4}$$

Following roughly the procedure outlined in the lecture notes, we find that the density ρ is given by

$$N = \frac{V}{2\pi^2 \hbar^3 c^3} \int_0^\infty d\epsilon \, \frac{\epsilon^2}{z^{-1} e^{\beta \epsilon} - 1}$$
$$N = \frac{1}{2\lambda^3} \int_0^\infty dy \, \frac{y^2}{z^{-1} e^y - 1} \tag{5}$$
$$\rho \lambda^3 = g_3(z)$$

where $\lambda = \pi^{2/3} \hbar c \beta$ is the thermal deBroglie wavelength for massless particles. Note that the particles in this system still have mass but this is a useful quantity anyway. But, as in the case of non-relativistic bosons, as $T \rightarrow 0$, we must have λ^3 increase. However, since $g_3(z)$ is bounded above by $g_3(1) = \zeta(3)$, there must be a temperature T_c at which the equation above becomes invalid. Thus, the critical temperature, at which Bose-Einstein condensation occurs, is given by

$$\rho \left(\frac{\pi^{2/3} \hbar c}{k_B T_c} \right)^3 = \zeta(3) \tag{6}$$

The constant $\zeta(3)$ is named Apéry's constant.

3. Fermi gas equilibrium

We consider a closed $3D$ cylinder of volume V, separated into two compartments, labeled 1 and 2, by a free-sliding piston. The piston allows the exchange of energy and volume but not of particles. Suppose two ideal Fermi gases are placed into the two compartments. and that the particles in compartment 1 have spin $\frac{1}{2}$, while the particles in compartment 2 have spin $\frac{3}{2}$, and that all particles have the same mass.

Recall that the density of states for a Fermi gas with spin s is given by

$$g(\epsilon) = (2s+1)\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2} \tag{7}$$

In which case the density of states for particles in compartments 1 and 2 are

$$g_1(\epsilon) = 2\frac{V_1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2}$$
$$g_2(\epsilon) = 4\frac{V_2}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2} \tag{8}$$

- At low T, the pressure in each compartment is dominated by the degeneracy pressure of each gas. We can set these equal since the system is assumed to be in equilibrium. Let P_1 be the degeneracy pressure of the gas in compartment 1 and ϵ_1^F its Fermi energy, and similarly for the gas in compartment 2. Then the following relation yields the ratio ρ_1/ρ_2 at equilibrium:

$$P_1 = P_2$$
$$\frac{2}{5}\frac{N_1}{V_1}\epsilon_1^F = \frac{2}{5}\frac{N_2}{V_2}\epsilon_2^F$$
$$\frac{\rho_1}{\rho_2} = \frac{\epsilon_1^F}{\epsilon_2^F}$$
$$\frac{\rho_1}{\rho_2} = \frac{\left[\frac{1}{2}\rho_1\right]^{2/3}}{\left[\frac{1}{4}\rho_2\right]^{2/3}} \tag{9}$$
$$\frac{\rho_1}{\rho_2} = 4$$

- For $T \to \infty$, we can simply integrate the density of states of each gas over $\mathbb{R}_{>0}$ against the Fermi-Dirac distribution. This yields

$$
\begin{aligned}
\rho_1 &= \frac{1}{V_1} \int_0^\infty d\epsilon \, \frac{g_1(\epsilon)}{z^{-1}e^{\beta\epsilon} + 1} \\
\rho_2 &= \frac{1}{V_2} \int_0^\infty d\epsilon \, \frac{g_2(\epsilon)}{z^{-1}e^{\beta\epsilon} + 1} \\
\implies \rho_1 &= 2\frac{1}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty d\epsilon \, \frac{\epsilon^{1/2}}{z^{-1}e^{\beta\epsilon} + 1} \\
\rho_2 &= 4\frac{1}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty d\epsilon \, \frac{\epsilon^{1/2}}{z^{-1}e^{\beta\epsilon} + 1}
\end{aligned}
\tag{10}
$$

Therefore, the ratio of densities, ρ_1/ρ_2 is just

$$
\frac{\rho_1}{\rho_2} = \frac{1}{2}
\tag{11}
$$

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4. Virial expansion and the liquid-gas critical point

We consider an equation of state that includes three terms in the virial expansion,

$$P = \frac{k_B T}{v}\left(1 + \frac{B_2(T)}{v} + \frac{B_3(T)}{v^2}\right) \tag{12}$$

where $v = V/N = 1/\rho$ is the reduced volume.

i. Suppose $B_3(T) \leq 0$. In order to find a liquid-gas critical point of the Van der Waals type, we need to find solutions to $\partial P/\partial v = \partial^2 P/\partial v^2 = 0$. To do this, we can look for T_c such that all roots of the equation of state coincide. That is, we can write the equation of state as

$$Pv^3 - k_B T v^2 - k_B T B_2(T)v - k_B T B_3(T) = P_c(v - v_c)^3 \tag{13}$$

However, if $B_3(T) \leq 0$ then the last term on the left hand side is positive or zero, in which case the three roots of the equation will never coincide. Therefore, the first and second derivatives will never be zero simultaneously so no liquid-gas critical point of the Van der Waals type can occur.

ii. Assume that the system does have a liquid-gas critical point at temperature T_c and reduced volume v_c.

- I couldn't figure this one out, but based on the next two, it looks like we should find that

$$3B_3(T_c) = B_2(T_c)^2 \tag{14}$$

- To find a relation between v_c, $B_2(T_c)$, and $B_3(T_c)$ we can differentiate the equation of state setting $\partial P/\partial v = 0$ and $\partial^2 P/\partial v^2 = 0$. These equations yield

$$\begin{aligned} 2B_2(T_c) + \frac{3B_3(T_c)}{v_c} + v_c &= 0 \\ 3B_2(T_c) + \frac{6B_3(T_c)}{v_c} + v_c &= 0 \end{aligned} \tag{15}$$

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respectively. Subtracting the first from the second yields

$$B_3(T_c) = -\frac{v_c}{3}B_2(T_c) \tag{16}$$

This also gives an equation for v_c in terms of $B_2(T_c)$ and $B_3(T_c)$:

$$v_c = -3\frac{B_3(T_c)}{B_2(T_c)} \tag{17}$$

From the first, we also have

$$v_c = -B_2(T_c) \tag{18}$$

- Plugging these back into the original equation of state, we have

$$\begin{aligned}
\frac{P_c v_c}{k_B T_c} &= 1 + \frac{B_2(T_c)}{v_c} + \frac{B_3(T_c)}{v_c^2} \\
&= 1 + \frac{B_2(T_c)}{v_c} - \frac{1}{3}\frac{B_2(T_c)}{v_c} \\
&= 1 + \frac{2}{3}\frac{B_2(T_c)}{v_c} \\
&= 1 - \frac{2}{3}\frac{v_c}{v_c} \\
\frac{P_c v_c}{k_B T_c} &= \frac{1}{3}
\end{aligned} \tag{19}$$

iii. The cubic equation of state from the homework problem is

$$P = \frac{k_B T}{v} - \frac{1}{2}\frac{b}{v^2} + \frac{1}{6}\frac{c}{v^3} \tag{20}$$

Solving for $B_2(T)$ and $B_3(T)$ gives

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$$k_B T \frac{B_2(T)}{v^2} = -\frac{1}{2}\frac{b}{v^2}$$
$$\implies B_2(T) = -\frac{1}{2}\frac{b}{k_B T} \tag{21}$$

and

$$k_B T \frac{B_3(T)}{v^3} = \frac{1}{6}\frac{c}{v^3}$$
$$\implies B_3(T) = \frac{1}{6}\frac{c}{k_B T} \tag{22}$$

Starting with $v_c = c/b$, we have

$$v_c = \frac{c}{b}$$
$$= -\frac{6B_3(T_c)}{2B_2(T_c)} \tag{23}$$
$$= -3\frac{B_3(T_c)}{B_2(T_c)}$$

Next, we know that $k_B T_c = b^2/2c$, which requires that

$$k_B T_c = \frac{b^2}{2c}$$
$$= \frac{(2k_B T_c B_2(T_c))^2}{12 k_B T_c B_3(T_c)} \tag{24}$$
$$= \frac{1}{3}k_B T_c \frac{B_2(T_c)^2}{B_3(T_c)}$$
$$\implies 3B_3(T_c) = B_2(T_c)^2$$

Finally, given $P_c = b^3/6c^2$, we have

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$$
\begin{aligned}
P_c &= \frac{b^3}{6c^2} \\
&= \frac{b^2}{2c}\frac{b}{3c} \\
&= \frac{1}{3}\frac{k_B T_c}{v_c}
\end{aligned}
\tag{25}
$$
$$
\implies \frac{P_c v_c}{k_B T_c} = \frac{1}{3}
$$

since the first two are consistent, so too must be the last one.

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5. Three-state spin system: mean-field theory

We consider a system of N spins indexed by i, arranged in a lattice with coordination number q under periodic boundary conditions. Denote spin i by $S_i \in \{-1, 0, 1\}$. Let $\{S_i\}$ denote a particular configuration of all N spins. Define the Hamiltonian of the system as

$$
H\left(\{S_i\}\right) = -J \sum_{\langle ij \rangle} S_i S_j - B \sum_{i=1}^{N} S_i
\tag{26}
$$

where $J > 0$.

i. We wish to use mean-field theory to derive an equation which implicitly defines the average magnetization, $m = \sum \langle S_i \rangle / N$, of the system in equilibrium with some external field, B, at $\beta = 1/k_B T$. We follow the same procedure as in the lecture and in § 5.2.1 of Tong's notes.

We can rewrite the spins in terms of the average magnetization m, as

$$
((S_i - m) + m)((S_j - m) + m) = (S_i - m)(S_j - m) + m(S_j - m) + m(S_i - m) + m^2
\tag{27}
$$

Since we assume that fluctuations between spins on neighboring sites are small, we neglect the first term in the sum above. We may then rewrite the Hamiltonian as

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$$H\left(\{S_i\}\right) = -J\sum_{\langle ij\rangle} m(S_i + S_j) - m^2 - B\sum_{i=1}^{N} S_i$$

$$= \frac{1}{2}JNqm^2 - (Jqm + B)\sum_{i=1}^{N} S_i \tag{28}$$

Since we have reduced the problem to that of a non-interacting system, the partition function for N spins, $\mathcal{Z}_N$, can be rewritten as $\mathcal{Z}_N = \mathcal{Z}_1^N$. We compute this in terms of the Hamiltonian above to get

$$\begin{aligned}
\mathcal{Z}_N &= \mathcal{Z}_1^N \\
&= \sum_{\{S_i\}} e^{-\beta H(\{S_i\})} \\
&= \left(e^{-\beta\frac{1}{2}Jqm^2} \sum_{k=-1}^{1} e^{\beta(Jqm+B)k}\right)^N \\
&= e^{-\beta\frac{1}{2}JNqm^2} \left(e^{\beta(Jqm+B)} + e^{-\beta(Jqm+B)} + 1\right)^N \\
\mathcal{Z}_N &= e^{-\frac{1}{2}\beta JNqm^2} \left(2\cosh(\beta(Jqm+B)) + 1\right)^N
\end{aligned} \tag{29}$$

For self-consistency, we must have $m = \frac{1}{\beta N}\frac{\partial\log\mathcal{Z}_N}{\partial B}$. Thus,

$$\begin{aligned}
m &= \frac{1}{\beta N}\frac{\partial\log\mathcal{Z}_N}{\partial B} \\
&= \frac{1}{\beta}\frac{\partial}{\partial B}\left(\log(2\cosh(\beta(Jqm+B)) + 1) + \text{const.}\right) \\
&= \frac{2\sinh(\beta(Jqm+B))}{2\cosh(\beta(Jqm+B)) + 1} \\
m &= \frac{\sinh(\beta(Jqm+B))}{\cosh(\beta(Jqm+B)) + 1/2}
\end{aligned} \tag{30}$$

ii. Suppose $B = 0$. Since the function above behaves similarly to $\tanh(x)$, we can perform a similar calculation as in the lecture notes. In particular, this function is a sigmoid which behaves linearly around $m = 0$. Let $m = f(m)$ be the relation above. Then we need only find the temperature T_c at which $f'(0) = 1$.

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$$m = \frac{\sinh(\beta(Jqm + B))}{\cosh(\beta(Jqm + B)) + 1/2}$$

$$m = \frac{2}{3}\beta Jqm + \mathcal{O}(m^3)$$

(31)

So T_c is given by

$$k_B T_c = \frac{2}{3}Jq$$

(32)

iii. We expect this system to have the same critical exponents as in the Ising model since the series expansion of the free energy, $F = -k_B T \log \mathcal{Z}_N$, contains precisely the same powers of m as in the Ising model.

♣