1)

$$H = \begin{bmatrix} E_0 & \hbar\Omega/2 \\ \hbar\Omega/2 & E_0 \end{bmatrix}$$

$$|\psi\rangle = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

a) $|\psi\rangle$ must satisfy the Shrödinger equation so

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H|\psi\rangle$$

$$\rightarrow i\hbar \begin{bmatrix} \dot{c}_0 \\ \dot{c}_1 \end{bmatrix} = \begin{bmatrix} E_0 & \hbar\Omega/2 \\ \hbar\Omega/2 & E_0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

$$\boxed{\begin{aligned} \rightarrow \dot{c}_0 &= -\frac{i}{\hbar} E_0 c_0 - i\frac{\Omega}{2} c_1 \\ \dot{c}_1 &= -i\Omega c_0 - \frac{i}{\hbar} E_0 c_1 \end{aligned}} \quad (1)$$

b)

Let $\tilde{c}_0 = c_0 e^{iE_0 t/\hbar}$

$\tilde{c}_1 = c_1 e^{iE_0 t/\hbar}$

So $c_0 = \tilde{c}_0 e^{-iE_0 t/\hbar}$

$c_1 = \tilde{c}_1 e^{-iE_0 t/\hbar}$

and $\dot{c}_0 = \dot{\tilde{c}}_0 e^{-iE_0 t/\hbar} - \frac{iE_0}{\hbar} \tilde{c}_0 e^{-iE_0 t/\hbar}$

$\dot{c}_1 = \dot{\tilde{c}}_1 e^{-iE_0 t/\hbar} - i\frac{E_0}{\hbar} \tilde{c}_1 e^{-iE_0 t/\hbar}$

Plugging these into (1) and multiplying by $e^{iE_0 t/\hbar}$ gives

$$\dot{\tilde{c}}_0 = -\frac{i\Omega}{2} \tilde{c}_1$$

$$\dot{\tilde{c}}_1 = -\frac{i\Omega}{2} \tilde{c}_0$$

Which we can rewrite as

$$i\hbar \begin{bmatrix} \dot{\tilde{c}}_0 \\ \dot{\tilde{c}}_1 \end{bmatrix} = \begin{bmatrix} 0 & \hbar\frac{\Omega}{2} \\ \hbar\frac{\Omega}{2} & 0 \end{bmatrix} \begin{bmatrix} \tilde{c}_0 \\ \tilde{c}_1 \end{bmatrix}$$

c)

Setting $E_0 = 0$ and differentiating gives

$$\dot{c}_0 = -i\frac{\Omega}{2} c_1$$

$$\dot{c}_1 = -i\frac{\Omega}{2} c_0$$

$$\rightarrow \quad \ddot{c}_0 = -i\frac{\Omega}{2}\dot{c}_1 = -i\frac{\Omega}{2}\left(-i\frac{\Omega}{2}\right) c_0$$

$$\ddot{c}_1 = -i\frac{\Omega}{2}\dot{c}_0 = -i\frac{\Omega}{2}\left(-i\frac{\Omega}{2}\right) c_1$$

$$\rightarrow \quad \boxed{\begin{aligned} \ddot{c}_0 &= -\left(\frac{\Omega}{2}\right)^2 c_0 \\ \ddot{c}_1 &= -\left(\frac{\Omega}{2}\right)^2 c_1 \end{aligned}}$$

d)

$$C_0(t) = A_0 \sin\left(\frac{\Omega}{2} t\right) + B_0 \cos\left(\frac{\Omega}{2} t\right)$$

$$C_1(t) = A_1 \sin\left(\frac{\Omega}{2} t\right) + B_1 \cos\left(\frac{\Omega}{2} t\right)$$

$t=0$ implies

$$C_0(0) = B_0 \quad , \quad C_1(0) = B_1$$

$$\dot{C}_0(0) = A_0 \frac{\Omega}{2} \quad , \quad \dot{C}_1(0) = A_1 \frac{\Omega}{2}$$

and since

$$\dot{C}_0(0) = -\frac{i\Omega}{2} C_1(0)$$

$$\dot{C}_1(0) = -\frac{i\Omega}{2} C_0(0)$$

we have

$$A_0 = -i C_1(0)$$

$$A_1 = -i C_0(0)$$

So

$$\boxed{\begin{aligned} C_0(t) &= C_0(0)\cos\left(\frac{\Omega}{2} t\right) - i C_1(0) \sin\left(\frac{\Omega}{2} t\right) \\ C_1(t) &= C_1(0)\cos\left(\frac{\Omega}{2} t\right) - i C_0(0) \sin\left(\frac{\Omega}{2} t\right) \end{aligned}}$$

e)

If $c_0(t=0) = 1$ then

$$|C_0(0)|^2 + |C_1(0)|^2 = 1$$

so $C_1(0) = 0$

Therefore,

$$P_0(t) = |C_0(t)|^2$$

$$P_0(t) = \cos\left(\frac{\Omega}{2} t\right)^2$$

$$\boxed{P_0(t) = \frac{1}{2} + \frac{1}{2}\cos(\Omega t)}$$

Which oscillates between 1 and 0 with frequency Ω

2)

$$H = \frac{\hbar}{2}\begin{bmatrix} -\Delta & \Omega \\ \Omega & \Delta \end{bmatrix}$$

a)

$$i\hbar|\dot{\psi}\rangle = H|\psi\rangle$$

$$\psi = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

gives

$$\boxed{\begin{aligned} \dot{c}_0 &= i\frac{\Delta}{2}c_0 - i\frac{\Omega}{2}c_1 \\ \dot{c}_1 &= -i\frac{\Omega}{2}c_0 - i\frac{\Delta}{2}c_1 \end{aligned}}$$

b)

$$\ddot{c}_0 = \frac{i\Delta}{2}\dot{c}_0 - \frac{i\Omega}{2}\dot{c}_1$$

$$\ddot{c}_0 = \frac{i\Delta}{2}\left(\frac{i\Delta}{2}c_0 - \frac{i\Omega}{2}c_1\right)$$

$$-\frac{i\Omega}{2}\left(-\frac{i\Omega}{2}c_0 - \frac{i\Delta}{2}c_1\right)$$

$$\ddot{c}_0 = -\left(\frac{\Delta}{2}\right)^2 c_0 + \frac{\Delta\Omega}{4}c_1$$

$$-\left(\frac{\Omega}{2}\right)^2 c_0 - \frac{\Delta\Omega}{4}c_1$$

$$\ddot{c}_1 = -\frac{i\Omega}{2}\left(\frac{i\Delta}{2}c_0 - \frac{i\Omega}{2}c_1\right)$$

$$-\frac{i\Delta}{2}\left(-\frac{i\Omega}{2}c_0 - \frac{i\Delta}{2}c_1\right)$$

$$\ddot{c}_1 = \frac{\Omega\Delta}{4}c_0 - \left(\frac{\Omega}{2}\right)^2 c_1 - \frac{\Delta\Omega}{4}c_0$$

$$-\left(\frac{\Delta}{2}\right)^2 c_1$$

$$\longrightarrow \boxed{\begin{aligned}\ddot{c}_0 &= -\left(\left(\frac{\Delta}{2}\right)^2 + \left(\frac{\Omega}{2}\right)^2\right)c_0 \\ \ddot{c}_1 &= -\left(\left(\frac{\Omega}{2}\right)^2 + \left(\frac{\Delta}{2}\right)^2\right)c_1\end{aligned}} \qquad (2^{\cdot})$$

c)

As with problem 1,

$$\boxed{\begin{aligned} c_0(t) &= c_0(0)\cos\left(\tfrac{1}{2}\tilde{\Omega}t\right) - i c_1(0)\sin\left(\tfrac{1}{2}\tilde{\Omega}t\right) \\ c_1(t) &= c_1(0)\cos\left(\tfrac{1}{2}\tilde{\Omega}t\right) - i c_0(0)\sin\left(\tfrac{1}{2}\tilde{\Omega}t\right) \\ &\text{where } \tilde{\Omega} = \sqrt{\Delta^2 + \Omega^2} \end{aligned}}$$

d) If $c_0(0) = 1$ then $c_1(0) = 0$

so

$$P_0(t) = |c_0(t)|^2$$

$$P_0(t) = \cos^2\left(\tfrac{1}{2}\tilde{\Omega}t\right)$$

$$\boxed{\begin{aligned} P_0(t) &= \tfrac{1}{2} + \tfrac{1}{2}\cos(\tilde{\Omega}t), \\ \tilde{\Omega} &= \sqrt{\Delta^2 + \Omega^2} \end{aligned}}$$

3)

Let

$$P_R(\max) = 4\cos^2\theta \sin^2\theta$$

$$\tan(2\theta) = \frac{|\Omega|}{\Delta} \quad , \left(0 \le \theta < \frac{\pi}{2}\right)$$

Then we have

$$\tan(2\theta) = \frac{|\Omega|}{\Delta}$$

$$P_R(\max) = 4\cos^2\theta \sin^2\theta = \sin^2(2\theta)$$

$$P_R(\max) = \sin^2\left(\arctan\left(\frac{|\Omega|}{\Delta}\right)\right)$$

$$= \frac{\left(\frac{|\Omega|}{\Delta}\right)^2}{\left(\frac{|\Omega|^2}{\Delta^2} + 1\right)}$$

$$\boxed{P_R(\max) = \frac{\Omega^2}{\Omega^2 + \Delta^2}}$$