Put (1) in (2); solve for $\dot{\phi}(\theta)$:

$$(3)\quad \boxed{\dot{\phi} = \frac{L_z - L_3\cos\theta}{I_1 \sin^2\theta}}$$

Now, put (1) and (3) into conservation of E (θ):

$$(4)\quad \boxed{\begin{aligned} E &= \frac{1}{2}I_1\dot{\theta}^2 + \frac{(L_z - L_3\cos\theta)^2}{2I_1\sin^2\theta} + \frac{L_3^2}{2I_3} + mgl\cos\theta \\ &\equiv \frac{1}{2}I_1\dot{\theta}^2 + U_{eff}(\theta) \end{aligned}}$$

where I've defined:

$$\boxed{U_{eff}(\theta) = \frac{(L_z - L_3\cos\theta)^2}{2I_1\sin^2\theta} + \frac{L_3^2}{2I_3} + mgl\cos\theta}$$

Just like Energy for 1-d motion in a potential, with $x \to \theta$, $\dot{x} \to \dot{\theta}$.

$\Rightarrow$ Can read off qualitative features of motion from shape of $U_{eff}(\theta)$.

cheat sheet

7:61a

$$U_e = \frac{(L_z - L_3 x)^2}{1 - x^2}$$

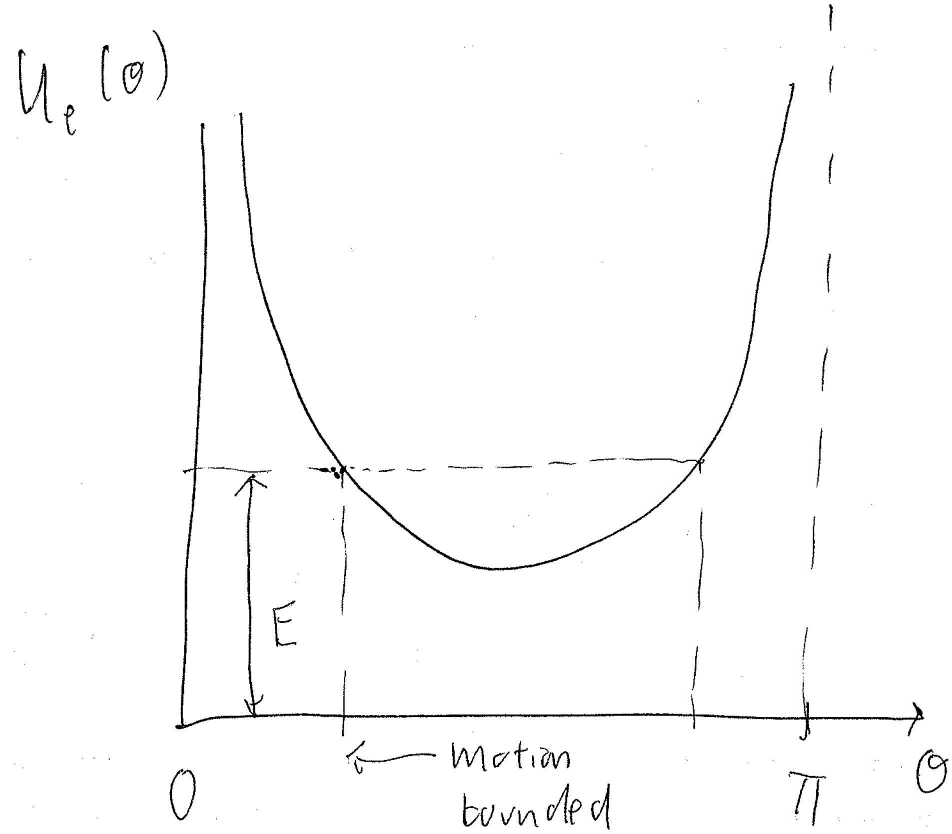
$$\partial_x U_e = \frac{2(L_z - L_3 x)(-L_3)}{(1 - x^2)} + \frac{(L_z - L_3 x)^2}{(1 - x^2)^2}(+2x) = 0$$

$$\Rightarrow \frac{(L_z - L_3 x)^{\cancel{2}}(\cancel{2}x)}{(1 - x^2)^{\cancel{2}}} = \cancel{2} L_3 \frac{\cancel{(L_z - L_3 x)}}{\cancel{1 - x^2}}$$

$$\Rightarrow (L_z - \cancel{L_3 x})\, x = L_3 - \cancel{L_3 x^2}$$

$$\Rightarrow \boxed{x = \frac{L_3}{L_z}} \Rightarrow \text{only one minimum}$$

Hamiltonian dynamics



Bigger $L \Rightarrow U_e \uparrow \Rightarrow$ tighter bounds on motion

$\Rightarrow$ stable spinning (gyroscope effect)

Hamiltonian dynamics:

Recall the invariant we ~~found~~ earlier;

rename it H ("Hamiltonian")

(1) $$\boxed{H_{(\{p_i, q_i\})} \sum_i p_i \dot{q}_i - \mathcal{L}}$$ $\left(p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$ "conjugate momentum"

Note: think of this as fn. of $\{p_i, q_i\}$, not $\{q_i, \dot{q}_i\}$

Consider

$$\boxed{dH = \sum_i (p_i \, d\dot{q}_i + \dot{q}_i \, dp_i) - d\mathcal{L}} \quad (2)$$

What's $d\mathcal{L}$?

What's $\mathcal{L}$ a function of?

$$\mathcal{L}(\{q_i, \dot{q}_i\}) \Rightarrow d\mathcal{L} = \sum_i \left[\frac{\partial \mathcal{L}}{\partial q_i} dq_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i\right]$$

Note: think of $\mathcal{L}$ as fn. of these

But, by def'n, $\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i$.

Furthermore, E-LG eqn. $\Rightarrow \frac{\partial \mathcal{L}}{\partial q_i} = ?$

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{d}{dt}\left(p_i\right) = \dot{p}_i$$

$$\Rightarrow \boxed{d\mathcal{L} = \sum_i \left(p_i\, d\dot{q}_i + \dot{p}_i\, dq_i\right)} \quad (3)$$

Put into (2)

$$\Rightarrow dH = \sum_i \left(\cancel{p_i\, d\dot{q}_i} + \dot{q}_i\, dp_i - \cancel{p_i\, d\dot{q}_i} - \dot{p}_i\, dq_i\right)$$

$$\Rightarrow \boxed{dH = \sum_i \left[(-\dot{p}_i)\, dq_i + \dot{q}_i\, dp_i\right]} \quad (4)$$

But now: Suppose I hadn't told you what H was, but just said it was a fn. of $\{p_i, q_i\}$

$H(\{p_i, q_i\})$

$\Rightarrow dH = ?$

~~Hamiltonian dynamics~~

$$\boxed{dH = \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right)} \quad (5)$$

Compare with (4) $\Rightarrow$

$$\boxed{\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ \dot{q}_i &= \frac{\partial H}{\partial p_i} \end{aligned} \quad \text{"Hamilton's equations of motion"}}$$

So, alternative (Hamiltonian) prescription for getting EOM's:

1) Construct Lagrangian $\mathcal{L} = T - U$ as before

2) Construct conjugate momenta

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad \text{for all co-ordinates } i$$

3) Construct $H = \sum_i p_i \dot{q}_i - \mathcal{L}$

4) EOM's are:

$$\boxed{\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ \dot{q}_i &= \frac{\partial H}{\partial p_i} \end{aligned}}$$

Example: 1d motion in a potential $U(x)$

Step 1) $\mathcal{L} = T - U = ?$

2) $p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$

3) $H = p\dot{x} - \mathcal{L} = m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + U(x)$

$= \frac{1}{2}m\dot{x}^2 + U(x)$ ~~a~~ total ~~energy~~

$= \frac{p^2}{2m} + U(x) =$

4) ~~a~~ $\dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x}$ ✓

$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$ ✓

Back to general case:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_i \left(\frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i \right)$$

$$= \frac{\partial H}{\partial t} + \sum_i \cancelto{0}{\left(\frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_i} \right)}$$

Using Hamilton's equations

$$= \frac{\partial H}{\partial t}$$

$\Rightarrow$ If H does not depend explicitly on t, ~~dH~~ ($\Rightarrow \frac{\partial H}{\partial t} = 0$)

then $\frac{dH}{dt} = 0 \Rightarrow H$ is conserved

Conservation of "energy"