PHYS 632: Quantum Mechanics II (Winter 2021)
Homework 5
Assigned Monday, 8 February 2021
Due Monday, 15 February 2021

Problem 1. Show that under Hamiltonian evolution,

$$\partial_t \rho = -\frac{i}{\hbar}[H, \rho], \tag{1}$$

the purity $\mathrm{Tr}[\rho^2]$ is a constant of the motion.

Problem 2. For a single qubit/spin-$1/2$ system, it turns out that an arbitrary density operator may be written in terms of the Pauli matrices as

$$\rho = \frac{1}{2}\big(\mathcal{I}_2 + \mathbf{r} \cdot \boldsymbol{\sigma}\big), \tag{2}$$

where $|\mathbf{r}| \leq 1$ and $\mathcal{I}_2$ is the identity on the qubit Hilbert space.

(a) Show this by writing out the right-hand side as a 2×2 matrix, and showing that it parameterizes an arbitrary matrix with the correct properties to be a density matrix. (What are the properties?)

(b) By computing $\langle \boldsymbol{\sigma} \rangle$, show that $\mathbf{r}$ is in fact the Bloch vector.

(c) Show that the purity is related to the length of the Bloch vector by

$$\mathrm{Tr}[\rho^2] = \frac{1}{2}\big(1 + |\mathbf{r}|^2\big). \tag{3}$$

In particular, pure states have $|\mathbf{r}| = 1$, with mixed states $0 \leq |\mathbf{r}| < 1$ (so that $1/2 \leq \mathrm{Tr}[\rho^2] < 1$).

(d) What is the Bloch vector for the I-know-nothing state (proportional to the identity)?