Physics 614 Homework 4

Jeremy Welsh-Kavan

---

1. **Entropy of the photon gas**

(i) We can use the fact stated in the course notes that the free energy F is given by

$$\begin{aligned} F &= \frac{Vk_BT}{\pi^2c^3}\int_0^\infty d\omega\ \omega^2\ln\left(1-e^{-\beta\hbar\omega}\right) \\ F &= -\frac{Vk_B^4T^4}{\pi^2c^3}\frac{\pi^4}{45\hbar^3} \\ F &= -\frac{\pi^2Vk_B^4T^4}{45\hbar^3c^3} \end{aligned} \tag{1}$$

Then the entropy is given by

$$\begin{aligned} S &= -\frac{\partial F}{\partial T} \\ \frac{S}{k_B} &= \frac{4\pi^2Vk_B^3}{45\hbar^3c^3}T^3 \\ \frac{S}{k_B} &= \frac{8Vk_B}{\pi^2\hbar^3c^3\beta^3}\zeta(4) \end{aligned} \tag{2}$$

(ii) The density of states for a blackbody of volume V is $g(\omega) = V\omega^2/\pi^2c^3$. Therefore, the average number of particles is given by

$$\begin{aligned} \langle N\rangle &= \frac{V}{\pi^2c^3}\int_0^\infty d\omega\frac{\omega^2}{e^{\beta\hbar\omega}-1} \\ \langle N\rangle &= \frac{V}{\pi^2c^3}\int_0^\infty d\omega\frac{\omega^2}{e^{\beta\hbar\omega}-1} \\ \langle N\rangle &= \frac{2V\zeta(3)}{\pi^2\hbar^3c^3\beta^3} \end{aligned} \tag{3}$$

Therefore, we have

$$\frac{S}{\langle N\rangle k_B} = 4\frac{\zeta(4)}{\zeta(3)} \tag{4}$$

---

2. **Cosmic microwave background**

I got a little arrogant and thought I could do this one quickly. Instead I will be taking the proverbial "L" on this one.

---

3. **High-temperature limit of the phonon gas**

(i) Since we can find the heat capacity as a derivative of the average energy, we first find the average energy. Let $g(\omega)$ be the density of states for phonons in a 3D crystal and let N be the number of particles in the crystal. We can proceed as in the lecture notes. Define $\tilde{\omega}$ to be the solution to

$$\int_0^{\tilde{\omega}} d\omega \; g(\omega) = 3N \tag{5}$$

Then the average energy for the system can be written as

$$\langle E \rangle = \int_0^{\tilde{\omega}} d\omega \; \frac{\hbar\omega g(\omega)}{e^{\beta\hbar\omega} - 1} \tag{6}$$

From this we can find the heat capacity, C_V, by taking a derivative with respect to T.

$$\begin{aligned} C_V &= \frac{\partial \langle E \rangle}{\partial T} \\ C_V &= \frac{1}{k_B T^2} \int_0^{\tilde{\omega}} d\omega \; \frac{\hbar^2\omega^2 e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega} - 1\right)^2} g(\omega) \\ C_V &= k_B \int_0^{\tilde{\omega}} d\omega \; \frac{\beta^2\hbar^2\omega^2 e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega} - 1\right)^2} g(\omega) \end{aligned} \tag{7}$$

(ii) Expanding the fraction in (7) to second order in $\beta\hbar\omega$ allows us to approximate the integral in the high temperature limit, i.e. $\beta\hbar\omega \ll 1$.

$$
\begin{aligned}
\frac{(\beta\hbar\omega)^2 e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega}-1\right)^2} &\approx 1 - \frac{(\beta\hbar\omega)^2}{12} + \mathcal{O}\left((\beta\hbar\omega)^4\right) \\
\implies C_V &\approx k_B \int_0^{\tilde{\omega}} d\omega \; \left(1 - \frac{(\beta\hbar\omega)^2}{12}\right) g(\omega) \\
C_V &\approx 3Nk_B - \frac{1}{k_B T^2} \int_0^{\tilde{\omega}} d\omega \; \frac{\hbar^2\omega^2}{12} g(\omega) \\
C_V &\approx 3Nk_B \left(1 - \frac{1}{3Nk_B^2 T^2} \int_0^{\tilde{\omega}} d\omega \; \frac{\hbar^2\omega^2}{12} g(\omega)\right) \\
C_V &\approx 3Nk_B \left(1 + \frac{\alpha}{T^2}\right)
\end{aligned}
\tag{8}
$$

where $\alpha = -\frac{1}{36Nk_B^2} \int_0^{\tilde{\omega}} d\omega \; \hbar^2\omega^2 g(\omega)$.

(iii) Using the expressions for the density of states and ω_D defined using the Debye approximation in the lecture notes, we have

$$
\begin{aligned}
\alpha &= -\frac{1}{36Nk_B^2} \int_0^{\omega_D} d\omega \; \frac{3\hbar^2 V\omega^4}{2\pi^2 c_s^2} \\
\alpha &= -\frac{1}{36Nk_B^2} \frac{3\hbar^2 V}{10\pi^2 c_s^2} \left(\frac{6\pi^2 N}{V} c_s\right)^{5/3} \\
\alpha &= -\frac{1}{36k_B^2} \frac{3\hbar^2}{10\pi^2 c_s^2} \left(6\pi^2 c_s\right)^{5/3} \frac{N^{2/3}}{V^{2/3}} \\
\alpha &\propto \left(\frac{N}{V}\right)^{2/3}
\end{aligned}
\tag{9}
$$

♣