Problem Set 5

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1.4.5

Consider the reals $\mathbb{R}$ with $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\rho(x, y) = |x - y|$.

Proposition: $\mathbb{R}$ with this definition of ρ makes $(\mathbb{R}, \rho)$ a metric space.

Proof. We will check each of the criteria in Definition 1 of section 4.5.

1. Let $x, y \in \mathbb{R}$. If $x \neq y$ then we can assume without loss of generality that $x > y$. Therefore,

$$\rho(x, y) = |x - y| = x - y > 0$$

since $x > y$. Suppose $x = y$. Then

$$\rho(x, y) = |x - y| = x - y = x - x = 0$$

Now suppose $\rho(x, y) = 0$. We can assume without loss of generality that $x \geqslant y$. Then we have that

$$\rho(x, y) = |x - y| = x - y = 0$$

But $x - y = 0$ implies that $x = y$. Therefore, for all $x, y \in \mathbb{R}$, $\rho(x, y) \geqslant 0$ and $\rho(x, y) = 0$ if and only if $x = y$.

2. Let $x, y \in \mathbb{R}$ and assume without loss of generality that $x \geqslant y$. Then $x - y \geqslant 0$ and $y - x \leqslant 0$ so

$$\rho(x, y) = |x - y| = x - y$$

and

$$\rho(y, x) = |y - x| = -(y - x) = x - y$$

Therefore, $\rho(x, y) = \rho(y, x)$ for all $x, y \in \mathbb{R}$.

3. First we observe that $\sqrt{(a-b)^2} = |a - b| = \rho(a, b)$ for each $a, b \in \mathbb{R}$. Let $x, y, z \in \mathbb{R}$ and set $\alpha = x - y$ and $\beta = y - z$. For all $\alpha, \beta \in \mathbb{R}$, we have the following:

$$\begin{aligned}(\alpha + \beta)^2 &= \alpha^2 + \beta^2 + 2\alpha\beta \\ &\leqslant |\alpha|^2 + |\beta|^2 + 2|\alpha||\beta| \\ &= (|\alpha| + |\beta|)^2\end{aligned}$$

Therefore,

$$\rho(x, z) = |x - z| = |\alpha + \beta| = \sqrt{(\alpha + \beta)^2} \leqslant |\alpha| + |\beta| = |x - y| + |y - z| = \rho(x, y) + \rho(y, z)$$

So the triangle inequality is satisfied.

Thus, the criteria of Definition 1 in Section 4.5 are satisfied so $(\mathbb{R}, \rho)$ forms a metric space. ■

1.4.6

Proposition a: A sequence in a metric space has at most one limit.

Proof. Let X be a metric space and let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in X. Assume that $a_n \to L_1$ and $a_n \to L_2$.
Let $\epsilon > 0$. Since $a_n \to L_1$ we can find N_1 such that $d(a_n, L_1) < \frac{\epsilon}{2}$, and since $a_n \to L_2$ we can find N_2 such that $d(a_n, L_2) < \frac{\epsilon}{2}$. Then for all $n > N_1 + N_2$ we have $d(a_n, L_1) < \frac{\epsilon}{2}$ and $d(a_n, L_2) < \frac{\epsilon}{2}$. By the triangle inequality, we have

$$d(L_1, L_2) \leqslant d(a_n, L_1) + d(L_2, a_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

But, for any $\alpha \in \mathbb{R}_{\geqslant 0}$, if $\alpha < \epsilon$ for all $\epsilon > 0$ then $\alpha = 0$. So we must have $d(L_1, L_2) = 0$. Since X is a metric space, $d(L_1, L_2) = 0$ implies that $L_1 = L_2$. Thus, for any sequence $\{a_n\}_{n\in\mathbb{N}}$ in a metric space, $\{a_n\}_{n\in\mathbb{N}}$ has at most one limit. ■

Proposition b: Every convergent sequence in a metric space is a Cauchy sequence.

Proof. Let X be a metric space and let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $a_n \to L$. Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that, for all $n > N$, $d(a_n, L) < \frac{\epsilon}{2}$. If $m > N$, we have

$$d(a_n, a_m) \leqslant d(a_n, L) + d(a_m, L) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, if $a_n \to L$ then for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n, m > N$ then $d(a_n, a_m) < \epsilon$. Therefore, $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. ■

1.4.7

Let B be a K-vector space with null vector θ. Let $||...|| : B \to \mathbb{R}$ be a mapping such that

(i) $||x|| \geqslant 0 \ \forall x, y \in B$, and $||x|| = 0$ iff $x = \theta$

(ii) $||x + y|| \leqslant ||x|| + ||y|| \ \forall x, y \in B$

(iii) $||\lambda x|| = |\lambda| \cdot ||x|| \ \forall x \in B, \lambda \in K$

Define a mapping $d : B \times B \to \mathbb{R}$ by

$$d(x, y) := ||x - y|| \ \forall x, y \in B$$

Proposition a: d is a metric on B.

Proof. We again check the criteria in Definition 1 of section 4.5.

1. Let $x, y \in B$. Since $(x-y) \in B$, $d(x,y) = ||x-y|| \geqslant 0$, since $||\dots||$ is a norm on B. Since B is a vector space, $x - x = 0$ so
$$d(x,x) = ||x-x|| = ||0|| = 0$$
If $d(x,y) = 0$ then $||x-y|| = 0$ so $x - y = 0$, since $||\dots||$ is a norm.
2. Let $x, y \in B$. Since $||\dots||$ is a norm, if $b \in B$ then $||-b|| = |-1| \cdot ||b|| = ||b||$. So
$$d(x,y) = ||x-y|| = ||-(y-x)|| = |-1| \cdot ||y-x|| = d(y,x)$$
3. Let $x, y, z \in B$ and define $\alpha = x - z$ and $\beta = z - y$. Since $||\dots||$ satisfies the triangle inequality, we have
$$d(x,y) = ||x-y|| = ||\alpha+\beta|| \leqslant ||\alpha|| + ||\beta|| = ||x-z|| + ||z-y|| = d(x,z) + d(z,y)$$

Thus, $d(\cdot,\cdot)$ is a metric on B. ■

Define a function $|\dots| : \mathbb{C} \to \mathbb{R}_{\geqslant 0}$ such that, for $\lambda \in \mathbb{C}$, $|\lambda| = \sqrt{\lambda^* \lambda}$.

Proposition b: $\mathbb{R}$ and $\mathbb{C}$ with $|\dots|$, as defined above, are B-spaces.

Proof. Since $\mathbb{R}$ is a subspace of $\mathbb{C}$, we will prove the proposition for $\mathbb{C}$ which will show that it is also true for $\mathbb{R}$. First, we will show that $|\dots|$ satisfies the criteria for a norm.

(i) Let $\lambda \in \mathbb{C}$. Then $\lambda = a + ib$ for some $a, b \in \mathbb{R}$, and $\lambda^* = a - ib$. So
$$\lambda^* \lambda = (a+ib)(a-ib) = a^2 + b^2$$
and since $a, b \in \mathbb{R}$, $a^2 + b^2 \geqslant 0$. Therefore,
$$|\lambda| = \sqrt{\lambda^* \lambda} = \sqrt{a^2 + b^2} \geqslant 0$$
for each $\lambda \in \mathbb{C}$.
Suppose $|\lambda| = 0$. Then, if $\lambda = a + ib$, we have that $a^2 + b^2 = 0$. Therefore, $a = 0$ and $b = 0$, so $\lambda = 0$.
Now suppose $\lambda = 0$. Then $|\lambda| = \sqrt{0^2 + 0^2} = 0$. So (i) is satisfied.

(ii) Let $\alpha, \beta \in \mathbb{C}$ and write $\alpha = a + ib$ and $\beta = c + id$, for some $a, b, c, d \in \mathbb{R}$. Since $a, b, c, d \in \mathbb{R}$, we have
$$\begin{aligned} 0 &\leqslant (bc - ad)^2 \\ (ac+bd)^2 &\leqslant (a^2+b^2)(c^2+d^2) \\ 2ac + 2bd &\leqslant 2\sqrt{(a^2+b^2)(c^2+d^2)} \\ a^2+b^2+c^2+d^2+2ac+2bd &\leqslant a^2+b^2+c^2+d^2+2\sqrt{(a^2+b^2)(c^2+d^2)} \\ |\alpha|^2 + |\beta|^2 + \alpha\beta^* + \beta\alpha^* &\leqslant |\alpha|^2 + |\beta|^2 + 2|\alpha||\beta| \\ |\alpha+\beta|^2 &\leqslant (|\alpha| + |\beta|)^2 \\ |\alpha+\beta| &\leqslant |\alpha| + |\beta| \end{aligned}$$
Therefore, (ii), the triangle inequality, is satisfied.

(iii) Let $\lambda, \xi \in \mathbb{C}$. Then

$$|\lambda\xi| = \sqrt{\lambda\xi(\lambda\xi)^*} = \sqrt{\lambda\lambda^*\xi\xi^*} = \sqrt{\lambda\lambda^*}\sqrt{\xi\xi)^*} = |\lambda||\xi|$$

So (iii) is satisfied.

Since $\mathbb{R} \subset \mathbb{C}$, all of the above hold for elements of $\mathbb{R}$. Since $\mathbb{R}$ is the completion of $\mathbb{Q}$, $\mathbb{R}$ is complete. $\mathbb{R}$ is the set of equivalence classes of Cauchy sequences in $\mathbb{Q}$ which get arbitrarily close to each other. This is the result of exercises 24 and 25 in chapter 3 Walter Rudin's *Principles of Mathematical Analysis*. ■

Now let B^* be the dual space of B, i.e., the space of linear functionals $l : B \to K$, and define a function on B^* by

$$||l|| := \sup_{||x||=1}\{|l(x)|\}$$

Proposition c: l is a norm on B^*

Proof. We will check the criteria above.

(i) Let $l \in B^*$. Since $|l(x)| \in \mathbb{R}$, $|l(x)| \geqslant 0$ for all $x \in B$.[1] By definition, $\sup_{||x||=1}\{|l(x)|\}$ is an upper bound for $\{|l(x)| : ||x|| = 1\}$. Therefore, since $|l(x)| \in \{|l(x)| : ||x|| = 1\}$, we have

$$\sup_{||x||=1}\{|l(x)|\} \geqslant |l(x)| \geqslant 0$$

Therefore, for each $l \in B^*$, $||l|| \geqslant 0$.
Suppose $l(x) = 0$ for each $x \in B$. Then $|l(x)| = 0$ for each $x \in B$ and

$$||l|| = \sup_{||x||=1}\{|l(x)|\} = \sup\{0\} = 0$$

Therefore, if $l = 0$ then $||l|| = 0$.
Now suppose $||l|| = 0$. Then $\sup_{||x||=1}\{|l(x)|\} = 0$. But since $|l(x)| \geqslant 0$ for each $x \in B$, we must have

$$0 = \sup_{||x||=1}\{|l(x)|\} \geqslant |l(x)| \geqslant 0$$

So $|l(x)| = 0$ for each $x \in B$, which implies that $l = 0$.[2] Therefore, $||l|| = 0$ if and only if $l = 0$.

(ii) **Lemma 1:** For two sets, A and B, of real numbers, if $A + B = \{a + b : a \in A \text{ and } b \in B\}$ then $\sup(A + B) = \sup A + \sup B$.

Proof. Let $a \in A$ and $b \in B$. Since $\sup A$ is an upper bound for A and $\sup B$ is an upper bound for B, we have

$$a \leqslant \sup A \text{ and } b \leqslant \sup B$$

$$a + b \leqslant \sup A + \sup B$$

[1]this is true whether K is the real or complex numbers
[2]This follows from that fact that the absolute value is a norm on K

for each $a \in A$ and $b \in B$. So $\sup A + \sup B$ is an upper bound for $A + B$. Let $\epsilon > 0$. Then there exists $a \in A$ and $b \in B$ such that

$$\sup A - \frac{\epsilon}{2} \leqslant a \text{ and } \sup B - \frac{\epsilon}{2} \leqslant b$$

Therefore, for all $\epsilon > 0$ there exists $a + b \in A + B$ such that

$$\sup A + \sup B - \epsilon \leqslant a + b$$

So $\sup A + \sup B$ is the least upper bound for $A + B$. ■

Lemma 2: If $A \subset \mathbb{R}$, $\lambda \in \mathbb{R}$, and $\lambda \geqslant 0$, then $\lambda \cdot \sup A = \sup\{\lambda a : a \in A\}$.

Proof. Define $\lambda A = \{\lambda a : a \in A\}$. Since $\sup A$ is an upper bound for A, for each $a \in A$ we have

$$a \leqslant \sup A$$

and since $\lambda \geqslant 0$,

$$\lambda a \leqslant \lambda \cdot \sup A$$

So $\lambda \cdot \sup A$ is an upper bound for A. Let $\epsilon > 0$. Since $\sup A$ is the least upper bound for λA, there exists $a \in A$ such that

$$\sup A - \frac{\epsilon}{\lambda} \leqslant a$$

But since $\lambda \geqslant 0$,

$$\lambda \cdot \sup A - \epsilon \leqslant \lambda a$$

Thus, for each $\epsilon > 0$ there exists $\lambda a \in \lambda A$ such that

$$\lambda \cdot \sup A - \epsilon \leqslant \lambda a$$

Therefore, $\lambda \cdot \sup A = \sup(\lambda A)$ ■

Let $k, l \in B^*$. Since $|\dots|$ is a norm on K, we have

$$|k(x) + l(x)| \leqslant |k(x)| + |l(x)|$$

for each $x \in B$. In particular, if $x \in B$ and $||x|| = 1$ then $|k(x)| + |l(x)|$ must be an upper bound for the set $\{|k(x) + l(x)| : ||x|| = 1\}$. Which means

$$\sup_{||x||=1}\{|k(x) + l(x)|\} \leqslant |k(x)| + |l(x)|$$

And

$$|k(x)| + |l(x)| \leqslant \sup_{||x||=1}\{|k(x)| + |l(x)|\}$$

for each $x \in B$ such that $||x|| = 1$. Therefore, we have

$$\sup_{||x||=1}\{|k(x) + l(x)|\} \leqslant \sup_{||x||=1}\{|k(x)| + |l(x)|\}$$

By Lemma 1, we know that

$$\sup_{||x||=1}\{|k(x)| + |l(x)|\} = \sup_{||x||=1}\{|k(x)|\} + \sup_{||x||=1}\{|l(x)|\}$$

Finally, we arrange the sequence of inequalities above to arrive at

$$||k+l|| = \sup_{||x||=1}\{|k(x)+l(x)|\} \leqslant \sup_{||x||=1}\{|k(x)|\} + \sup_{||x||=1}\{|l(x)|\} = ||k|| + ||l||$$

Therefore, for each $k, l \in B^*$,

$$||k+l|| \leqslant ||k|| + ||l||$$

so the triangle inequality is satisfied.

(iii) Let $l \in B^*$ and $\lambda \in K$. By definition,

$$||\lambda l|| = \sup_{||x||=1}\{|\lambda l(x)|\}$$

Since $|\lambda l(x)| = |\lambda| \cdot |l(x)|$, by Lemma 2 we have

$$||\lambda l|| = \sup_{||x||=1}\{|\lambda| \cdot |l(x)|\} = |\lambda| \cdot \sup_{||x||=1}\{|l(x)|\} = |\lambda| \cdot ||l||$$

So condition (iii) is satisfied.

Thus, since conditions (i),(ii), and (iii) are satisfied, l is a norm on B^*. ■

1.4.8

Proposition a: The norm, $||x|| = \sqrt{(x,x)}$, in 4.7 Definition 1 is a norm in the sense of 4.6 Definition 1.

Proof. We will check the criteria of 4.6 Definition 1. Let H be a linear space over $\mathbb{C}$ with null vector 0.

(i) Let $x \in H$. Then $||x|| = \sqrt{(x,x)}$. By (ii) in 4.7 Definition 1, $(x,x) \geqslant 0$. Therefore, $||x|| \geqslant 0$. Since taking square roots preserves positive semidefiniteness, by (ii) in 4.7 Definition 1, $||\ldots||$ on H satisfies (i) in 4.6 Definition 1.

(ii) Let $\alpha, \beta \in H$. By 4.7 Lemma 1, we have

$$|(\alpha,\beta)|^2 \leqslant (\alpha,\alpha)(\beta,\beta)$$
$$|(\alpha,\beta)| \leqslant ||\alpha||||\beta||$$

We use the fact that if $z \in \mathbb{C}$,

$$Re(z) \leqslant |z|$$

So we have

$$\begin{aligned}
2\mathfrak{Re}((\alpha,\beta)) &\leqslant 2||\alpha||||\beta|| \\
||\alpha||^2 + ||\beta||^2 + 2\mathfrak{Re}((\alpha,\beta)) &\leqslant ||\alpha||^2 + ||\beta||^2 + 2||\alpha||||\beta|| \\
(\alpha,\alpha) + (\beta,\beta) + (\alpha,\beta) + (\beta,\alpha) &\leqslant (||\alpha|| + ||\beta||)^2 \\
(\alpha+\beta, \alpha+\beta) &\leqslant (||\alpha|| + ||\beta||)^2 \\
||\alpha+\beta||^2 &\leqslant (||\alpha|| + ||\beta||)^2 \\
||\alpha+\beta|| &\leqslant ||\alpha|| + ||\beta||
\end{aligned}$$

Therefore, (ii) of 4.6 Definition 1 is satisfied.

(iii) Let $x \in H$ and $\lambda \in \mathbb{C}$. Then by (iv) in 4.7 Definition 1, we have

$$||\lambda x|| = \sqrt{\lambda(x, \lambda x)} = \sqrt{\lambda\lambda^*(x, x)} = |\lambda|\sqrt{(x, x)} = |\lambda|||x||$$

Therefore, (iii) of 4.6 Definition 1 is satisfied.

Thus, the norm in 4.7 Definition 1 is a norm in the sense of 4.6 Definition 1. ■

Proposition b: The mappings, l, defined in 4.7 Definition 4 are linear forms in the sense of 4.3 Definition 1 a).

Proof. Let $y \in H$ and let $l : H \to \mathbb{C}$ be a mapping such that $l(x) := (y, x)$ for all $x \in H$.

(i) Let $x, z \in H$. Then, by (i) and (iii) of 4.7 Definition 1, we have

$$l(x + z) = (y, x + z) = (x + z, y)^* = (x, y)^* + (z, y)^* = (y, x) + (y, z) = l(x) + l(z)$$

So (i) of 4.3 Definition 1 a) is satisfied.

(ii) Let $x \in H$ and $\lambda \in \mathbb{C}$. Then, by (iv) of 4.7 Definition 1, we have

$$l(\lambda x) = (y, \lambda x) = (\lambda x, y)^* = (\lambda^*(x, y))^* = \lambda(x, y)^* = \lambda l(x)$$

So (ii) of 4.3 Definition 1 a) is satisfied.

Therefore, $l : H \to \mathbb{C}$ is a linear form in the sense of 4.3 Definition 1 a). ■