Exercise 1:

$$M^{(0)} = Tr(M_{\alpha\beta}) = M^{\alpha}{}_{\alpha}$$

$$(ABC)^{\alpha}{}_{\beta} = A^{\alpha}{}_{\delta} B^{\delta}{}_{\gamma} C^{\gamma}{}_{\beta}$$

$$Tr(ABC) = (ABC)^{\alpha}{}_{\alpha}$$

$$= A^{\alpha}{}_{\gamma} B^{\gamma}{}_{\lambda} C^{\lambda}{}_{\alpha}$$

$$= C^{\lambda}{}_{\alpha} A^{\alpha}{}_{\gamma} B^{\gamma}{}_{\lambda}$$

$$= (CAB)^{\lambda}{}_{\lambda}$$

$$= Tr(CAB)$$

Since cyclic permutations of ABC have adjacent indicies which get summed over,

$$Tr(ABC) = Tr(CAB) = Tr(BCA)$$

Exercise 2:

$$M_{\alpha\beta} = \frac{1}{3} M^{(0)} \delta_{\alpha\beta} + \frac{1}{4} M^{(1)}_{\mu} \epsilon_{\mu\alpha\beta} + M^{(2)}_{\alpha\beta}$$

$$M_{\alpha\beta} = \frac{1}{3} M_{\gamma\gamma} \delta_{\alpha\beta} + \frac{1}{4} \epsilon_{\mu\alpha\beta} \epsilon_{\mu\lambda\sigma} (M_{\lambda\sigma} - M_{\sigma\lambda})$$
$$+ \frac{(M_{\alpha\beta} + M_{\beta\alpha})}{2} - \frac{1}{3} M_{\mu\mu} \delta_{\alpha\beta}$$

$$M_{\alpha\beta} = \frac{1}{4} \epsilon_{\mu\alpha\beta} \epsilon_{\mu\lambda\sigma} (M_{\lambda\sigma} - M_{\sigma\lambda})$$
$$+ \frac{1}{2} (M_{\alpha\beta} + M_{\beta\alpha})$$

$$M_{\alpha\beta} = \frac{1}{4} (\delta_{\alpha\lambda} \delta_{\beta\sigma} - \delta_{\alpha\sigma} \delta_{\beta\lambda})(M_{\lambda\sigma} - M_{\sigma\lambda})$$
$$+ \frac{1}{2}(M_{\alpha\beta} + M_{\beta\alpha})$$

$$M_{\alpha\beta} = \frac{1}{4} \big(\delta_{\alpha\lambda} \delta_{\beta\sigma} M_{\lambda\sigma} + \delta_{\alpha\sigma} \delta_{\beta\lambda} M_{\sigma\lambda}$$
$$- \delta_{\alpha\sigma} \delta_{\beta\lambda} M_{\lambda\sigma} - \delta_{\alpha\lambda} \delta_{\beta\sigma} M_{\sigma\lambda} \big)$$
$$+ \frac{1}{2} (M_{\alpha\beta} + M_{\beta\alpha})$$

$$M_{\alpha\beta} = \frac{1}{4} (M_{\alpha\beta} + M_{\alpha\beta} - M_{\beta\alpha} - M_{\beta\alpha})$$
$$+ \frac{1}{2} (M_{\alpha\beta} + M_{\beta\alpha})$$

$$\boxed{M_{\alpha\beta} = M_{\alpha\beta}}$$

Exercise 3:

$$R|\tilde{\alpha} j m\rangle = \sum_{q m'} (R T_q^{(k)} R^\dagger) R|\alpha' j' m'\rangle \langle j' m'; k q | j m\rangle$$

But $\quad R T_q^{(k)} R^\dagger = \sum_{q'=-k}^{k} T_{q'}^{(k)} d_{q'q}^{(k)}$

and $\quad R|j' m'\rangle = \sum_{m''} d_{m''m'}^{(j')} |j' m''\rangle$

$$R|\tilde{\alpha} j m\rangle =$$

$$\sum_{q q' m' m''} T_{q'}^{(k)} |\alpha' j' m''\rangle \langle j' m'; k q | j m\rangle d_{q'q}^{(k)} d_{m''m}^{(j')}$$

$$R|\tilde{\alpha} j m\rangle = \sum_{q' m''} T_{q'}^{(k)} |\alpha' j' m''\rangle$$

$$\times \sum_{q m'} \langle j' m''; k q' | R | j' m'; k q\rangle \langle j' m'; k q | j m\rangle$$

$$R|\tilde{\alpha} j m\rangle = \sum_{q' m''} T_{q'}^{(k)} |\alpha' j' m''\rangle \langle j' m''; k q' | R | j m\rangle$$

$$R|\tilde{\alpha} j m\rangle = \sum_{q' m'' m'} T_{q'}^{(k)} |\alpha' j' m''\rangle \langle j' m''; k q' | j m'\rangle \langle j m' | R | j m\rangle$$

$$R|\tilde{\alpha} j m\rangle = \sum_{q'm'm''} T_{q'}^{(k)} |\alpha' j' m''\rangle \langle j' m''; k q' | j m'\rangle$$

$$\times\, d^{(j)}_{m'm}$$

$$R|\tilde{\alpha} j m\rangle = \sum_{q'm'm''} T_{q'}^{(k)} |\alpha' j' m''\rangle \langle j m' | j' m''; k q'\rangle$$

$$\times\, d^{(j)}_{m'm}$$

$$R|\tilde{\alpha} j m\rangle = \sum_{m'} |\tilde{\alpha} j m'\rangle\, d^{(j)}_{m'm}$$

$$\langle \sigma_x \rangle = c_+^* c_- + c_+ c_-^*$$

$$\langle \sigma_y \rangle = -i c_+^* c_- + i c_+ c_-^*$$

$$\langle \sigma_z \rangle = |c_+|^2 - |c_-|^2$$

$$\langle \vec{\sigma} \rangle = \langle \sigma_x \rangle \hat{x} + \langle \sigma_y \rangle \hat{y} + \langle \sigma_z \rangle \hat{z}$$

$$|\langle \vec{\sigma} \rangle|^2 = \Big(|c_+^* c_- + c_+ c_-^*|^2 + |-i c_+^* c_- + i c_+ c_-^*|^2 + (|c_+|^2 - |c_-|^2)^2 \Big)^{\frac{1}{2}}$$

$$|\langle \vec{\sigma} \rangle|^2 = \Big(2|c_+|^2|c_-|^2 + \cancel{c_+^2 c_-^{*2}} + \cancel{c_+^{*2} c_-^2} + \cancel{2|c_+|^2}|c_-|^2 - \cancel{c_+^{*2} c_-^2} - \cancel{c_+^2 c_-^{*2}} + |c_+|^4 - 2\cancel{|c_+^2|}|c_-|^2 + |c_-|^4 \Big)^{\frac{1}{2}}$$

$$|\langle \vec{\sigma} \rangle|^2 = \left(|c_+|^4 + 2|c_+|^2|c_-|^2 + |c_-|^4 \right)^{\frac{1}{2}}$$

$$|\langle \vec{\sigma} \rangle|^2 = |c_+|^2 + |c_-|^2 = 1$$