PHYS 631: Quantum Mechanics I (Fall 2020)
Midterm Exam, 8-12 November 2020

Rules for this exam: You may take up to 8 hours after you download the exam to work on it, then you should upload your answers right away.

You may consult any inanimate object while taking the exam. However, you may **not** discuss any part of the exam with **any human by any means** (except the instructor) until after the last due date.

"Magic lifeline rule." You may ask me questions that go beyond normal clarifications (like, I'm stuck, can you give me a hint on this problem? or, here is a snippet of my work on this problem, am I on the right track?) You may of course ask clarification questions without limit (within reason). **You may use the lifeline option up to two times** for this exam; just clarify that you are using it when you ask the question.

The rationale for the lifeline rule is that on an exam, I often see people getting on the wrong track or getting hung up on some particular wording. As a result, they end up with a wrong solution even though with a minor prompt they could easily produce the *correct* solution. I intend for this to be a way to reduce these "spurious" missed exam questions, because I don't believe them to be good indicators of your knowledge and skill.

I have read and understand the rules.

____ ⟵ **Sign your name here.**

(If you are LaTeXing your solutions, you can instead type out this statement with your name at the beginning of your solutions in lieu of scanning your signature.)

Problem 1. (20 points) Carry out the limit in the expression

$$L(x) := \lim_{a\to 0^+} \Big[(a-1)\delta(a^2x + a^3) + 2\delta(a^2x) - (a+1)\delta(a^2x - a^3)\Big]. \tag{1}$$

Problem 2. (25 points) Let U be a unitary operator. Then show that

$$A = i\frac{1+U}{1-U} \tag{2}$$

is Hermitian, provided that 1 is not an eigenvalue of U.

(Don't forget the other side!)

Problem 3. (30 points total)
Consider the potential-energy operator V, which in the position representation has the diagonal form $\langle x|V|x'\rangle = V(x)\,\delta(x-x')$. Further, suppose that $dV/dx \equiv V'(x) > 0$ everywhere.

(a) (10 points) Show that the state $|V(x)\rangle$, which in the position representation is defined by

$$\langle x|V(x')\rangle := \big[V'(x')\big]^{-1/2}\delta(x-x'), \tag{3}$$

is an eigenvector of V, and find the eigenvalue.

(b) (10 points) Since these are eigenstates of a Hermitian operator, they ought to form an orthonormal basis. Demonstrate the orthonormality explicitly by computing $\langle V(x)|V(x')\rangle$, and explain how your result makes sense.

For this part it may be useful to remember the composition property of the delta function

$$\delta[f(x)] = \sum_{x_0 \in g^{-1}(0)} \frac{\delta(x-x_0)}{|f'(x_0)|}, \tag{4}$$

where the sum is over all (simple) roots x_0 of f.

(c) (10 points) Work out the probability density for the potential energy V, and demonstrate that your result makes sense.

Problem 4. (25 points total)
Let Q be a Heisenberg-picture operator corresponding to an observable, with no explicit time dependence ($\partial Q/\partial t = 0$).

(a) (10 points) Show that the expectation value of $\dot{Q}$ vanishes for an energy eigenstate, assuming a discrete, nondegenerate energy spectrum.

(b) (15 points) Suppose we relax the assumption of a discrete, nondegenerate energy spectrum. Consider a free particle, $H = p^2/2m$, and let Q be the position operator x. First, show that

$$\langle \dot{x} \rangle = \frac{1}{m}\langle p \rangle. \tag{5}$$

For momentum eigenstates (which are also energy eigenstates), this is clearly nonvanishing for any $p \neq 0$. On the other hand, you should be able to apply your argument from (a) to this case, and conclude that the expectation value should always vanish (so go ahead and do this). What gives?