Physics 623 Homework 2

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3.1.1. **Electromagnetic waves and gauge invariance**

a) This seems trivial given that we know the field equations are unchanged by a gauge transformation but we calculate the effect of this gauge transformations on the field anyway.

We claim that the Lorenz gauge, $\frac{1}{c}\frac{\partial\varphi}{\partial t}+\nabla\cdot\mathbf{A}=0$, does not uniquely determine the potentials of an electromagnetic wave. In particular, if f is an arbitrary scalar solution to the wave equation, $\Box f=0$, then the transformation, $\mathbf{A}\to\mathbf{A}+\nabla f$, $\varphi\to\varphi-\frac{1}{c}\frac{\partial f}{\partial t}$, leaves both the wave equation for the 4-vector potential and the fields unchanged.
We know that the action from Axiom 3 is invariant under the gauge transformation $A^\mu\to A^\mu-\partial^\mu\chi$. Which is equivalent to the above transformation as follows

$$A^\mu-\partial^\mu\chi=(\varphi,\mathbf{A})-(\frac{1}{c}\partial_t\chi,-\nabla\chi) \tag{1}$$

Therefore, Maxwell's equations are also invariant under this gauge transformation. Setting $\chi=f$, we have

$$\begin{aligned}\Box A^\mu-\Box\partial^\mu f&=\Box A^\mu-\partial^\mu\Box f\\&=\Box A^\mu\end{aligned} \tag{2}$$

since $\Box f=0$. Therefore, if $\Box A^\mu=0$ then $\Box(A^\mu-\partial^\mu f)=0$.

For the $\mathbf{E}$ and $\mathbf{B}$ fields, we have that $\mathbf{E}=-\boldsymbol{\nabla}\varphi-\frac{1}{c}\partial_t\mathbf{A}$ and $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$. Therefore, under the transformation, $\mathbf{A}\to\mathbf{A}+\nabla f$, $\varphi\to\varphi-\frac{1}{c}\frac{\partial f}{\partial t}$, we have

$$\begin{aligned}\Box\mathbf{E}&=-\Box\boldsymbol{\nabla}\left(\varphi-\frac{1}{c}\partial_t f\right)-\Box\partial_t(\mathbf{A}+\nabla f)\\&=-\boldsymbol{\nabla}\left(\Box\varphi-\frac{1}{c}\partial_t\Box f\right)-\partial_t(\Box\mathbf{A}+\nabla\Box f)\\&=-\boldsymbol{\nabla}(\Box\varphi)-\partial_t(\Box\mathbf{A})\\&=-\Box(\boldsymbol{\nabla}\varphi+\frac{1}{c}\partial_t\mathbf{A})\\&=\Box\mathbf{E}\\\Box\mathbf{B}&=\Box(\boldsymbol{\nabla}\times(\mathbf{A}+\boldsymbol{\nabla}f))\\\Box\mathbf{B}&=\Box(\boldsymbol{\nabla}\times\mathbf{A}+\boldsymbol{\nabla}\times\boldsymbol{\nabla}f)\\\Box\mathbf{B}&=\Box(\boldsymbol{\nabla}\times\mathbf{A})\\&=\Box\mathbf{B}\end{aligned} \tag{3}$$

b) Given φ there exists f such that $\frac{1}{c}\partial_t f = \varphi$. Therefore, by a), we can always choose (transform) φ to $\varphi - \frac{1}{c}\partial_t f = 0$ such that the wave equations for the 4-vector potential and the fields are unchanged. Additionally, by a) we can make this choice in Lorenz gauge. In which case, $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$.

3.1.2. **Plane waves**

Consider the scalar field

$$\psi(\boldsymbol{x}, t) = \cos(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t) \tag{4}$$

and let $|\boldsymbol{k}| = k$.

a) We deduce a necessary and sufficient condition for ψ to be a solution of the wave equation.

i) Suppose $\psi(\boldsymbol{x}, t)$ satisfies $\Box\psi = 0$. Then

$$\begin{aligned} 0 &= \Box\psi(\boldsymbol{x}, t) \\ \implies \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} &= \nabla^2\psi \\ -\frac{\omega^2}{c^2}\cos(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t) &= -|\boldsymbol{k}|^2\cos(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t) \\ \implies \frac{\omega^2}{k^2} &= c^2 \end{aligned} \tag{5}$$

ii) Now suppose $\omega^2 = c^2k^2$. Then

$$\begin{aligned} \nabla^2\psi(\boldsymbol{x}, t) &= -|\boldsymbol{k}|^2\cos(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t) \\ &= -k^2\cos(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t) \\ \frac{1}{c^2}\frac{\partial^2\psi(\boldsymbol{x}, t)}{\partial t^2} &= -\frac{\omega^2}{c^2}\cos(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t) \\ &= -\frac{\omega^2}{c^2}\cos(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t) \\ &= -k^2\cos(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t) \\ &= \nabla^2\psi(\boldsymbol{x}, t) \end{aligned} \tag{6}$$

Thus, $\omega^2 = c^2k^2$ is a necessary and sufficient condition for (4) to be a solution to the wave equation.

b) Let D be a Lorentz boost into the coordinate system defined by $x_\mu = D^\nu{}_\mu x'_\nu$. Let $\psi'(\boldsymbol{x}', t')$ be the same function, (4), of the transformed coordinates. Define $k^\mu = (\frac{\omega}{c}, \boldsymbol{k})$, which we emphasize is not *necessarily* a Minkowski vector.
Since $\psi(\boldsymbol{x}, t)$ is a Minkowski scalar, it ought to be invariant under the transformation, D, so we must have $\psi(\boldsymbol{x}, t) = \psi'(\boldsymbol{x}', t')$. If it is to be invariant for all $(ct, \boldsymbol{x})$, then we must have $\boldsymbol{k}' \cdot \boldsymbol{x}' - \omega' t' = \boldsymbol{k} \cdot \boldsymbol{x} - \omega t$.
We can rewrite this as $x'_\nu k'^\nu = x_\mu k^\mu$ and differentiate both sides with respect to x'_ν to get $k'_\nu = \frac{\partial x_\mu}{\partial x'_\nu} k_\mu$. But $\frac{\partial x_\mu}{\partial x'_\nu} = D^\nu{}_\mu$, so $k'^\nu = D^\nu{}_\mu k^\mu$. Therefore, k^μ transforms as a Minkowski vector.

Suppose D is a Lorentz boost in the x-direction to a frame with velocity v with respect to the original frame, and that $\boldsymbol{k} = k\hat{x}$. Then, with $\beta = v/c$ and $\gamma = 1/\sqrt{1 - v^2/c^2}$, we have

$$
\begin{aligned}
\omega' t' - k' x' = x'_\nu k'^\nu &= (D^{-1})^\mu{}_\nu x_\mu D^\nu{}_\alpha k^\alpha \\
x'_\nu k'^\nu &= ((D^{-1})^\mu{}_0 x_\mu)(D^0{}_0 k^0 + D^0{}_1 k^1) + ((D^{-1})^\mu{}_1 x_\mu)(D^1{}_0 k^0 + D^1{}_1 k^1) \\
x'_\nu k'^\nu &= ((D^{-1})^0{}_0 x_0 + (D^{-1})^1{}_0 x_1)(D^0{}_0 k^0 + D^0{}_1 k^1) \\
&\quad + ((D^{-1})^0{}_1 x_0 + (D^{-1})^1{}_1 x_1)(D^1{}_0 k^0 + D^1{}_1 k^1) \\
x'_\nu k'^\nu &= (\gamma ct - \beta\gamma x)(\gamma \frac{\omega}{c} - \beta\gamma k) + (\beta\gamma ct - \gamma x)(\gamma k - \beta\gamma \frac{\omega}{c}) \\
x'_\nu k'^\nu &= \gamma^2(\beta^2 - 1)(kx - \omega t) \\
x'_\nu k'^\nu &= \omega t - kx \\
x'_\nu k'^\nu &= x_\mu k^\mu
\end{aligned}
\tag{7}
$$

3.1.3. **Spherical waves**

Consider the wave equation

$$
\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) f(\boldsymbol{x}, t) = 0 \tag{8}
$$

We wish to find the most general, spherically symmetric, solution to (8) of the form $f(\boldsymbol{x}, t) = u(r, t)/r$. In this case, (8) reduces to

$$
\begin{aligned}
\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right)\frac{u(r,t)}{r} &= 0\\
\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{u(r,t)}{r} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{u(r,t)}{r}\right) &= 0\\
\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{u(r,t)}{r} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2(\frac{\partial_r u(r,t)}{r} - \frac{u(r,t)}{r^2})\right) &= 0\\
\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{u(r,t)}{r} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\partial_r u(r,t) - u(r,t)\right)) &= 0\\
\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{u(r,t)}{r} - \frac{1}{r^2}\left(r\partial_r^2 u(r,t) + \partial_r u(r,t) - \partial_r u(r,t))\right) &= 0\\
\frac{1}{r}\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \partial_r^2\right)u(r,t) &= 0
\end{aligned}
\tag{9}
$$

Since $1/r \neq 0$, d'Alembert's solution solves (9). Therefore, the most general solution to (8), of the form $f(\boldsymbol{x}, t) = u(r,t)/r$, is

$$
f(\boldsymbol{x}, t) = \frac{u_1(r - ct)}{r} + \frac{u_2(r + ct)}{r} \tag{10}
$$

where $u_1(x)$ and $u_2(x)$ are any twice differentiable functions defined on $\mathbb{R}$.

3.1.4. **Cosmological redshift**

a) The frequency shift due to the nonrelativistic Doppler effect is given by

$$
\frac{\omega}{\omega_0} = 1 - \frac{v}{c} \tag{11}
$$

Since $\lambda_0/\lambda = \omega/\omega_0$, by Hubble's observation, we have

$$
\begin{aligned}
1 - \left(1 - \frac{v}{c}\right) &= \frac{Hr}{c}\\
v &= Hr
\end{aligned}
\tag{12}
$$

b) If the galaxy travels at a constant velocity v then the time it takes to reach r is $r/v = 1/H \approx 1.4 \times 10^{10}$ years.

c) The problem with Hubble's original estimate of $\approx 1.8 \times 10^9$ is that this is significantly younger than the age of the Earth. There are fossils of cyanobacteria that are 3.5 billion years old. [1]

3.2.1. **General solution of the wave equation**

By corollary (1) in § 2.2, one can rewrite $\hat{f}(\boldsymbol{k}, t) = a^0_{\boldsymbol{k}} \cos(\omega_{\boldsymbol{k}} t) + \frac{\dot{a}^0_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}}} \sin(\omega_{\boldsymbol{k}} t)$ as

$$
\begin{aligned}
f(\boldsymbol{x}, t) &= \frac{1}{(2\pi)^3} \int d\boldsymbol{k} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \left[f^+_{\boldsymbol{k}} e^{-i\omega_{\boldsymbol{k}} t} + f^-_{-\boldsymbol{k}} e^{i\omega_{\boldsymbol{k}} t} \right] \\
f(\boldsymbol{x}, t) &= \frac{1}{(2\pi)^3} \int d\boldsymbol{k} \left[f^+_{\boldsymbol{k}} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega_{\boldsymbol{k}} t)} + f^-_{-\boldsymbol{k}} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} + \omega_{\boldsymbol{k}} t)} \right]
\end{aligned} \tag{13}
$$

In 1 dimension, and substituting $\omega_k = ck$, (11) has the form

$$
\begin{aligned}
f(x, t) &= \frac{1}{2\pi} \int dk \left[f^+_k e^{i(kx - \omega_k t)} + f^-_{-k} e^{i(kx + \omega_k t)} \right] \\
f(x, t) &= \frac{1}{2\pi} \int dk \left[f^+_k e^{ik(x - ct)} + f^-_{-k} e^{ik(x + ct)} \right]
\end{aligned} \tag{14}
$$

which is a solution to § 2.2 ($*$) of the same form as d'Alembert's solution.
Now let $f(x, t) = u_1(x - ct) + u_2(x + ct)$ where $u_1(x), u_2(x)$ are twice differentiable functions on $\mathbb{R}$, as in d'Alembert's solution. Consider the Fourier transforms of $u_1(x)$ and $u_2(x)$,

$$
\begin{aligned}
\hat{u}_1(k) &:= \int dx \ u_1(x) e^{-ikx} \\
\hat{u}_2(k) &:= \int dx \ u_2(x) e^{-ikx}
\end{aligned} \tag{15}
$$

We can rewrite d'Alembert's solution in terms of the Fourier transforms in (13) as

$$
\begin{aligned}
f(x, t) &= u_1(x - ct) + u_2(x + ct) \\
f(x, t) &= \frac{1}{2\pi} \int dk \ \hat{u}_1(k) e^{ik(x - ct)} + \frac{1}{2\pi} \int dk \ \hat{u}_2(k) e^{ik(x + ct)}
\end{aligned} \tag{16}
$$

which has the same form as in Corollary 1 of § 2.2 under the identification, $\hat{u}_1(k) = f^+_k$, $\hat{u}_2(k) = f^-_{-k}$.

[1] https://ucmp.berkeley.edu/bacteria/cyanofr.html

4.1.1. **Wave equations for the electromagnetic fields**
Maxwell's equations say

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{B}&=0\\ \frac{1}{c}\partial_t\boldsymbol{B}+\boldsymbol{\nabla}\times\boldsymbol{E}&=0\\ \boldsymbol{\nabla}\cdot\boldsymbol{E}&=4\pi\rho\\ -\frac{1}{c}\partial_t\boldsymbol{E}+\boldsymbol{\nabla}\times\boldsymbol{B}&=\frac{4\pi}{c}\boldsymbol{j}\end{aligned}\tag{17}$$

We claim that

$$\Box\boldsymbol{E}=-4\pi\left(\boldsymbol{\nabla}\rho+\frac{1}{c^2}\partial_t\boldsymbol{j}\right)\tag{18}$$

and

$$\Box\boldsymbol{B}=\frac{4\pi}{c}\boldsymbol{\nabla}\times\boldsymbol{j}\tag{19}$$

To show (16), we can subtract the gradient of (M3) from the time derivative of (M4).

$$\begin{aligned}\frac{1}{c^2}\partial_t^2\boldsymbol{E}-\frac{1}{c}\partial_t(\boldsymbol{\nabla}\times\boldsymbol{B})-\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{E})&=-4\pi\boldsymbol{\nabla}\rho-\frac{4\pi}{c^2}\partial_t\boldsymbol{j}\\ \frac{1}{c^2}\partial_t^2\boldsymbol{E}-\frac{1}{c}\partial_t(\boldsymbol{\nabla}\times\boldsymbol{B})-\nabla^2\boldsymbol{E}-\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times E)&=-4\pi\boldsymbol{\nabla}\rho-\frac{4\pi}{c^2}\partial_t\boldsymbol{j}\\ \Box\boldsymbol{E}-\boldsymbol{\nabla}\times\left(\frac{1}{c}\partial_t\boldsymbol{B}+\boldsymbol{\nabla}\times E\right)&=-4\pi\left(\boldsymbol{\nabla}\rho+\frac{1}{c^2}\partial_t\boldsymbol{j}\right)\\ \Box\boldsymbol{E}-\boldsymbol{\nabla}\times(\boldsymbol{0})&=-4\pi\left(\boldsymbol{\nabla}\rho+\frac{1}{c^2}\partial_t\boldsymbol{j}\right)\\ \Box\boldsymbol{E}&=-4\pi\left(\boldsymbol{\nabla}\rho+\frac{1}{c^2}\partial_t\boldsymbol{j}\right)\end{aligned}\tag{20}$$

To show (17), we subtract the negative curl of (M4) from the time derivative of (M2).

$$\begin{aligned}\frac{1}{c^2}\partial_t^2\boldsymbol{B}+\frac{1}{c}\partial_t\boldsymbol{\nabla}\times\boldsymbol{E}-\frac{1}{c}\partial_t\boldsymbol{\nabla}\times\boldsymbol{E}+\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B})&=\frac{4\pi}{c}\boldsymbol{\nabla}\times\boldsymbol{j}\\ \frac{1}{c^2}\partial_t^2\boldsymbol{B}+\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B})&=\frac{4\pi}{c}\boldsymbol{\nabla}\times\boldsymbol{j}\\ \frac{1}{c^2}\partial_t^2\boldsymbol{B}+\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot B)-\nabla^2\boldsymbol{B}&=\frac{4\pi}{c}\boldsymbol{\nabla}\times\boldsymbol{j}\\ \Box\boldsymbol{B}&=\frac{4\pi}{c}\boldsymbol{\nabla}\times\boldsymbol{j}\end{aligned}\tag{21}$$

since $\mathbf{\nabla} \cdot B = 0$.

♣