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Problem 1. Consider two observables Q and R, such that $[Q, R] = 0$. In this case the uncertainty principle says that $\sigma_Q \sigma_R \geq 0$. Is it *always* the case that $\sigma_Q \sigma_R = 0$? If yes, prove it; if not, give a counterexample.

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1)

Suppose $Q = R = S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

And let $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$ be a vector in the spin-z basis where $|a|^2 + |b|^2 = 1$.

We compute $\sigma_{S_y}\sigma_{S_y} = \sigma_{S_y}^2$

$$\sigma_{S_y}^2 = \langle S_y^2 \rangle - \langle S_y \rangle^2$$

$$= \frac{\hbar^2}{4}\left((a^* \ b^*) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} - \left((a^* \ b^*) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right)^2 \right)$$

$$= \frac{\hbar^2}{4}\left(1 + (ab^* - ba^*)^2 \right)$$

Which is non zero if, for example,

$a = b$.

This shows that while

$[Q, R] = 0$ implies $\sigma_Q \sigma_R \neq 0$,

$\sigma_Q \sigma_R$ is not always zero.

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Problem 2. Prove the generalized uncertainty relation

$$V_P V_Q \geq \frac{1}{4}\left|\langle [P,Q] \rangle\right|^2 + \frac{1}{4}\left|\langle [P,Q]_+ \rangle - 2\langle P \rangle \langle Q \rangle\right|^2, \tag{1}$$

where $[A,B]_+ := AB + BA$ is the **anticommutator**.

<u>Claim:</u> $V_P V_Q \geq \frac{1}{4} |\langle [P,Q] \rangle|^2$
$+ \frac{1}{4} |\langle [P,Q]_+ \rangle$
$- 2 \langle P \rangle \langle Q \rangle|^2$

P<u>roof:</u>

We know from class that for a pair of observables P, Q,

$\langle P^2 \rangle \langle Q^2 \rangle \geq |\langle PQ \rangle|^2$ ✓

and that this is preserved

✓ under $P \to P - \langle P \rangle$, $Q \to Q - \langle Q \rangle$.

So we have

✓ $V_P V_Q = \langle (P - \langle P \rangle)^2 \rangle \langle (Q - \langle Q \rangle)^2 \rangle$

$\geq |\langle (P - \langle P \rangle)(Q - \langle Q \rangle) \rangle|^2$

We proceed by rewriting the right hand side of the eq. above.

$$|\langle PQ - \langle P\rangle Q - P\langle Q\rangle + \langle P\rangle\langle Q\rangle\rangle|^2$$

$$= |\langle PQ\rangle - \langle P\rangle\langle Q\rangle|^2$$

(since $\langle\ \rangle$ is linear)

We now use the fact that

$$PQ = \tfrac{1}{2}[P,Q] + \tfrac{1}{2}[P,Q]_+ \checkmark$$

$$= |\tfrac{1}{2}\langle[P,Q]\rangle \checkmark + \tfrac{1}{2}(\langle[P,Q]_+\rangle - 2\langle P\rangle\langle Q\rangle)|^2$$

$$= \tfrac{1}{4}|\langle[P,Q]\rangle + \langle[P,Q]_+\rangle - 2\langle P\rangle\langle Q\rangle|^2 \checkmark$$

$$= \tfrac{1}{4}\left(\langle[P,Q]\rangle + \langle[P,Q]_+\rangle - 2\langle P\rangle\langle Q\rangle\right) \cdot \left(\langle[P,Q]\rangle^* + \langle[P,Q]_+\rangle^* - 2\langle P\rangle\langle Q\rangle\right)$$

since $\langle P\rangle, \langle Q\rangle \in \mathbb{R}$.

$$= \frac{1}{4}\Big(|\langle [P,Q] \rangle|^2 + \langle [P,Q]_+ \rangle \langle [P,Q]_+ \rangle^*$$

$$+ \langle [P,Q]_+ \rangle \langle [P,Q] \rangle^*$$

$$+ \langle [P,Q]_+ \rangle^* \langle [P,Q] \rangle$$

$$- 2 \langle P \rangle \langle Q \rangle \langle [P,Q] \rangle^*$$

$$- 2 \langle P \rangle \langle Q \rangle \langle [P,Q]_+ \rangle^* \quad \leftarrow$$

$$+ 4 \langle P \rangle^2 \langle Q \rangle^2 \quad \leftarrow$$

$$- 2 \langle P \rangle \langle Q \rangle \langle [P,Q] \rangle$$

$$- 2 \langle P \rangle \langle Q \rangle \langle [P,Q]_+ \rangle \Big) \quad \leftarrow$$

Now since $[P,Q] = PQ - QP$

and $\langle PQ \rangle = \langle QP \rangle^*$,

we have $\langle [P,Q] \rangle = - \langle [P,Q] \rangle^*$

so the mess above simplifies

to:

$$= \frac{1}{4}\Big(|\langle [P,Q]\rangle|^2$$

$$+ \langle [P,Q]_+\rangle\langle [P,Q]_+\rangle^*$$

$$- 2\langle P\rangle\langle Q\rangle \langle [P,Q]_+\rangle^*$$

$$+ 4\langle P\rangle^2\langle Q\rangle^2$$

$$- 2\langle P\rangle\langle Q\rangle\langle [P,Q]_+\rangle \Big)$$

$$= \frac{1}{4}|\langle [P,Q]\rangle|^2$$

$$+ \frac{1}{4}\big(\langle [P,Q]_+\rangle - 2\langle P\rangle\langle Q\rangle\big)$$

$$\cdot \big(\langle [P,Q]_+\rangle - 2\langle P\rangle\langle Q\rangle\big)^*$$

$$= \frac{1}{4}|\langle [P,Q]\rangle|^2$$

$$+ \frac{1}{4}\left|\langle [P,Q]_+\rangle - 2\langle P\rangle\langle Q\rangle\right|^2$$

So

$$\boxed{V_P V_Q \geq \frac{1}{4}|\langle [P,Q]\rangle|^2 + \frac{1}{4}\left|\langle [P,Q]_+\rangle - 2\langle P\rangle\langle Q\rangle\right|^2}$$

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Problem 3. Any Hermitian operator P that satisfies $P^2 = P$ is called a **projection operator** or **projector**.

Suppose that P_1 and P_2 are projectors. Then show that the product $P_1 P_2$ is a projector if and only if $[P_1, P_2] = 0$.

Claim: If P_1 and P_2 are projectors, then $P_1 P_2$ is a projector if and only if $[P_1, P_2] = 0$

Proof:

(i) Suppose $P_1 P_2$ is a projector.

I'm not sure how to prove this implication and wikipedia says it is not necessarily true.

$P_1 P_2 = P_2 P_1$ via Hermiticity $\rightsquigarrow [P_1, P_2] = 0$

(ii) Now suppose $[P_1, P_2] = 0$

Then

$$P_1 P_2 = P_2 P_1$$

and

$$(P_1 P_2)^2 = (P_2 P_1)^2$$

$$(P_1 P_2)^2 = P_2 P_1 P_2 P_1 = P_1 P_2^2 P_1$$

but $P_1 = P_1^2$ and $P_2 = P_2^2$

and $P_1 P_2 = P_2 P_1$, so

$$(P_1 P_2)^2 = P_1 P_2$$

So $P_1 P_2$ is a projector. $\square$

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Problem 4. For the operator

$$A = \epsilon\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right) + \gamma|0\rangle\langle 1| + \gamma^*|1\rangle\langle 0| \qquad (2)$$

on the Hilbert space $\{|0\rangle, |1\rangle\}$, where $\epsilon \geq 0$ and $\gamma \in \mathbb{C}$, show that the eigenvectors may be written as

$$\begin{aligned} |+\rangle &= \sin\theta|0\rangle + e^{i\phi}\cos\theta|1\rangle \\ |-\rangle &= \cos\theta|0\rangle - e^{i\phi}\sin\theta|1\rangle, \end{aligned} \qquad (3)$$

where

$$\tan 2\theta = \frac{|\gamma|}{\epsilon} \qquad \left(0 \leq \theta < \frac{\pi}{2}\right). \qquad (4)$$

Also find the corresponding eigenvalues.

4)

First, we identify $\{|0\rangle, |1\rangle\}$ with $\left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\}$ since it makes A easier to deal with. (In my opinion)

So

$$A = \begin{pmatrix} \epsilon & \gamma^* \\ \gamma & -\epsilon \end{pmatrix}$$

Which has eigenvalues given by the solutions, λ, to the equation:

$$\lambda^2 - \epsilon^2 - |\gamma|^2 = 0$$

So $\lambda = \pm\sqrt{\epsilon^2 + |\gamma|^2}$ ✓

Then for some $\vec{a}, \vec{b} \in \text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$

we have

$$A\vec{a} = \sqrt{\epsilon^2 + |\gamma|^2}\,\vec{a}$$

and

$$A\vec{b} = -\sqrt{\epsilon^2 + |\gamma|^2}\,\vec{b}$$

letting $\vec{a} = \begin{pmatrix}a\\1\end{pmatrix}$, $\vec{b} = \begin{pmatrix}1\\-a^*\end{pmatrix}$, $a \in \mathbb{C}$

<u>Note:</u> It is clear that A is Hermitian so we know that $\langle \vec{a}, \vec{b} \rangle = 0$, and that we are free to scale eigenvectors

To compute a we solve the following equation:

$$\begin{pmatrix}\epsilon & \gamma^* \\ \gamma & -\epsilon\end{pmatrix}\begin{pmatrix}a\\1\end{pmatrix} = \sqrt{\epsilon^2 + |\gamma|^2}\begin{pmatrix}a\\1\end{pmatrix}$$

This yields

$$a = \frac{-\gamma}{\epsilon - \sqrt{\epsilon^2 + |\gamma|^2}}$$

so the eigenvectors of A are

$$\left\{ \begin{pmatrix} \frac{-\gamma}{\epsilon - \sqrt{\epsilon^2+|\gamma|^2}} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{\gamma^*}{\epsilon - \sqrt{\epsilon^2+|\gamma|^2}} \end{pmatrix} \right\}$$

with eigen values $\sqrt{\epsilon^2+|\gamma|^2}$, $-\sqrt{\epsilon^2+|\gamma|^2}$ respectively.

Since we are free to scale $\vec{a}$ and $\vec{b}$, we will use the new eigenbasis:

$$\left\{ \begin{pmatrix} e^{i\phi} \\ \frac{-\epsilon + \sqrt{\epsilon^2+|\gamma|^2}}{|\gamma|} \end{pmatrix}, \begin{pmatrix} \frac{-\epsilon + \sqrt{\epsilon^2+|\gamma|^2}}{|\gamma|} \\ -e^{-i\phi} \end{pmatrix} \right\}$$

Where $\phi = \text{Arg}(\gamma)$ ✓

Now let $x = \frac{-\epsilon + \sqrt{\epsilon^2 + |\gamma|^2}}{|\gamma|}$

the eigenvectors become

$$\left\{ \begin{pmatrix} e^{i\phi} \\ x \end{pmatrix}, \begin{pmatrix} x \\ -e^{-i\phi} \end{pmatrix} \right\}$$

we make one more rescaling

$$\left\{ \begin{pmatrix} \frac{e^{i\phi}}{\sqrt{1+x^2}} \\ \frac{x}{\sqrt{1+x^2}} \end{pmatrix}, \begin{pmatrix} \frac{-x}{\sqrt{1+x^2}} e^{i\phi} \\ \frac{1}{\sqrt{1+x^2}} \end{pmatrix} \right\}$$

Now, using the substitution,

$\tan(2\theta) = \frac{|\gamma|}{\epsilon}$, we will

write $\vec{a}$ and $\vec{b}$ in terms

of θ.

From the substitution and

the fact that

$$\frac{2\tan(\theta)}{1-\tan^2(\theta)} = \tan(2\theta),$$

We have

$$\theta = \arctan\left(\frac{-\epsilon + \sqrt{\epsilon^2 + |\gamma|^2}}{|\gamma|} \right)$$

$$\theta = \arctan(x)$$

since $$\frac{1}{\sqrt{1+x^2}} = \cos(\arctan(x))$$

and $$\frac{x}{\sqrt{1+x^2}} = \sin(\arctan(x))$$

the eigenvectors of A become

$$\left\{ \begin{pmatrix} e^{i\phi}\cos(\theta) \\ \sin(\theta) \end{pmatrix}, \begin{pmatrix} -e^{i\phi}\sin(\theta) \\ \cos(\theta) \end{pmatrix} \right\}$$

with eigenvalues $\sqrt{\epsilon^2 + |\gamma|^2}$, $-\sqrt{\epsilon^2 + |\gamma|^2}$

which is what was to be shown.

In Dirac notation, $\vec{a} = |+\rangle$, $\vec{b} = |-\rangle$

$$\{ \sin(\theta)|0\rangle + e^{i\phi}\cos(\theta)|1\rangle,$$
$$\cos(\theta)|0\rangle - e^{i\phi}\sin(\theta)|1\rangle \}$$

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Problem 5. Both the Poisson bracket and the quantum commutator satisfy the following properties (these aren't hard to verify; so if they aren't obvious to you, then go ahead and verify them).

1. antisymmetry: $[A, B] = -[B, A]$
2. null bracket with scalars: $[c, A] = 0$ $(c \in \mathbb{C})$
3. associativity: $[A + B, C] = [A, B] + [B, C]$; $\quad [A, B + C] = [A, B] + [A, C]$
4. product rules: $[AB, C] = A[B, C] + [A, C]B$; $\quad [A, BC] = [A, B]C + B[A, C]$
5. Jacobi identity: $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

Suppose that we haven't yet settled on a quantum bracket $[A, B]$, but we wish to choose one that satisfies the above properties. Show that the *only* possible choice is the quantum commutator $[A, B] = AB - BA$ (up to an overall real factor).

Hint: starting with $[AA', BB']$ for arbitrary operators A, A', B, and B', apply the two product rules to write this expression as the sum of four brackets. Applying the product rules in either order gives two expressions; you should require that they be equivalent.

5)

We will show that we must have $[A, B] = AB - BA$ for two arbitrary operators, A, B, If $[\,,\,]$ is to satisfy the relations above.

Let A, A', B, B' be arbitrary operators and consider $[AA', BB']$

By applying the first and then the second product rule, we have

$[AA', BB']$

$$= A[A', BB'] + [A, BB']A'$$

$$= A([A', B]B' + B[A', B']) \qquad (1)$$

$$+ ([A, B]B' + B[A, B'])A' \checkmark$$

Applying the second and then the first yields

$[AA', BB']$

$$= [AA', B]B' + B[AA', B']$$

$$= (A[A', B] + [A, B]A')B' \qquad (2)$$

$$+ B(A[A', B'] + [A, B']A') \checkmark$$

But (1) must equal (2) so

$$(A[A', B] + [A, B]A')B'$$
$$+ B(A[A', B'] + [A, B']A')$$
$$= A([A', B]B' + B[A', B'])$$
$$+ ([A, B]B' + B[A, B'])A'$$

simplifying a little...

$$A[A', B]B' + [A, B]A'B'$$
$$+ BA[A', B'] + B[A, B']A'$$
$$- A[A', B]B' - AB[A', B']$$
$$- [A, B]B'A' - B[A, B']A' = 0$$

cancelling some terms, we have

$$[A, B](A'B' - B'A')$$
$$= (AB - BA)[A', B'] \quad (*)$$

since this must be true for

all operators A, A', B, B',

we can fix A', B' and then each side of $(*)$ only depends on A and B. Since the action of each side is the same for all test functions, f, we must have

$$[A, B] = AB - BA$$

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