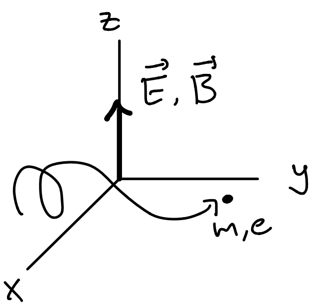
1) (a. 2.8)



$$\vec{E} = E\hat{z}$$

$$\vec{B} = B\hat{z}$$

a) The Lorentz force law says

$$\frac{d\vec{p}}{dt} = e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

where $\vec{p} = \gamma(v)\, m\vec{v}$ and $\gamma(v) = \dfrac{1}{\sqrt{1 - \frac{|v|^2}{c^2}}}$

So we have

$$\boxed{\begin{aligned} \dot{p}_x &= \frac{e}{c} v_y B \\ \dot{p}_y &= -\frac{e}{c} v_x B \\ \dot{p}_z &= eE \end{aligned}}$$

Therefore, $\dot{p}_z$ is independent of p_x and p_y (but not v_x and v_y)

We can combine $\dot{p}_x$ and $\dot{p}_y$ to get

$$\frac{\dot{p}_x}{v_y} + \frac{\dot{p}_y}{v_x} = \frac{e}{c}B - \frac{e}{c}B = 0$$

So

$$\frac{\dot{p}_x}{v_y} + \frac{\dot{p}_y}{v_x} = 0$$

$$\rightarrow \quad \dot{p}_x v_x + \dot{p}_y v_y = 0$$

multiplying both sides by $2\gamma(v)m$ gives

$$2\dot{p}_x \gamma(v) m v_x + 2\dot{p}_y \gamma(v) m v_y = 0$$

$$\rightarrow \quad 2\dot{p}_x p_x + 2\dot{p}_y p_y = 0$$

$$\rightarrow \quad \frac{d}{dt}\left(p_x^2 + p_y^2\right) = 0$$

Therefore, $p_x^2 + p_y^2 = \text{constant} =: p_\perp^2$

b)

For p_z, we have

$$\dot{p}_z = eE \qquad p_x^2 + p_y^2 = p_\perp^2$$

So

$$p_z(t) = eEt - p_z(0)$$

choosing $p_z(0) = 0$, we have

$$m\gamma(v)v_z = eEt$$

We can rewrite γ in terms of only v_z and $p_\perp^2$ as follows

$$\gamma = \frac{1}{\sqrt{1 - \frac{v_x^2 + v_y^2 + v_z^2}{c^2}}}$$

$$\gamma^2\left(1 - \frac{p_\perp^2}{m^2\gamma^2c^2} - \frac{v_z^2}{c^2}\right) = 1$$

since $v_x^2 + v_y^2 = \frac{p_x^2}{m^2\gamma^2} + \frac{p_y^2}{m^2\gamma^2} = \frac{p_\perp^2}{m^2\gamma^2}$

$$\gamma^2\left(1-\frac{V_z^2}{c^2}\right) = 1+\frac{P_\perp^2}{m^2c^2}$$

$$\gamma = \sqrt{\frac{1+\frac{P_\perp^2}{m^2c^2}}{1-\frac{V_z^2}{c^2}}}$$

$$\gamma = \sqrt{\frac{m^2c^2+P_\perp^2}{m^2c^2-m^2V_z^2}}$$

so we have

$$mV_z\sqrt{\frac{m^2c^2+P_\perp^2}{m^2c^2-m^2V_z^2}} = eEt$$

$$\frac{V_z^2}{c^2-V_z^2}\left(m^2c^2+P_\perp^2\right) = e^2E^2t^2$$

$$\rightarrow \frac{V_z^2}{c^2-V_z^2}\left(m^2c^2+P_\perp^2\right) = e^2E^2t^2$$

$$\dot{z}(t) = V_z = \frac{ceEt}{\sqrt{m^2c^2+P_\perp^2+e^2E^2t^2}}$$

$$z(t)-z(0) = \frac{1}{eE}\sqrt{c^4m^2+c^2P_\perp^2+c^2e^2E^2t^2}$$

With the appropriate choice of coordinate z, we can set $z(0)=0$ to arrive at

$$z(t) = \frac{1}{eE}\sqrt{c^4m^2 + c^2P_\perp^2 + c^2e^2E^2t^2}$$

Since we have set t such that $P_z(t=0)=0$, the initial kinetic energy is

$$T(t=0) = \sqrt{P_\perp^2c^2 + m^2c^4} =: T_0$$

$$\boxed{z(t) = \frac{1}{eE}\sqrt{T_0^2 + c^2e^2E^2t^2}}$$

c)

Let $T(t) = \sqrt{c^2 p^2 + m^2 c^4}$

be the time dependent kinetic energy. Using $z(t)$, we can rewrite $T(t)$ as follows

$$T(t) = \sqrt{c^2(p_\perp^2 + p_z^2) + m^2c^4}$$

$$T(t) = \sqrt{c^2(p_\perp^2 + e^2E^2t^2) + m^2c^4}$$

$$\boxed{T(t) = \sqrt{T_0^2 + c^2e^2E^2t^2}}$$

Now define φ by

$$\frac{d\varphi}{dt} := \frac{ceB}{T(t)}$$

Then

$$\boxed{\varphi(t) = \frac{B}{E} \operatorname{arctanh}\left(\frac{ceEt}{\sqrt{c^2e^2E^2t^2 + T_0^2}}\right)}$$

Let $\pi = p_x + i p_y$.

Then the equations

$$\dot{p}_x = \frac{e}{c} v_y B$$

$$\dot{p}_y = -\frac{e}{c} v_x B$$

can be combined to get

$$\dot{\pi} = \dot{p}_x + i \dot{p}_y = -i \frac{eB}{c} (v_x + i v_y)$$

and since $p_x = m\gamma v_x$,

$$\dot{\pi} = -\frac{ieB}{\gamma m c} \pi$$

We also have $\dfrac{1}{\gamma m} = \dfrac{v_z}{p_z}$

$$\rightarrow \quad \frac{1}{\gamma m} = \frac{c}{\sqrt{m^2c^2 + p_\perp^2 + e^2E^2t^2}}$$

$$\frac{1}{\gamma m} = \frac{c^2}{\sqrt{T_0^2 + c^2e^2E^2t^2}}$$

$$\dot{\pi} = -\frac{iceB}{\sqrt{T_0^2 + c^2e^2E^2t^2}}\pi$$

$$\int_{\pi(0)}^{\pi(t)} \frac{\dot{\pi}}{\pi}\,dt = -iceB\int_0^t \frac{dt'}{\sqrt{T_0^2 + c^2e^2E^2t'^2}}$$

$$\ln\left(\frac{\pi(t)}{\pi(0)}\right) = -i\int_0^t \frac{d\varphi}{dt'}\,dt'$$

$$\pi(t) = \pi(0)\,e^{-i\varphi(t)}$$

but since $\pi(t) = p_x(t) + i\,p_y(t)$

and $|\pi(0)|^2 = p_\perp^2$, we have

$$p_x(t) = p_\perp \cos(\varphi(t))$$

$$p_y(t) = -p_\perp \sin(\varphi(t))$$

and

$$\frac{e}{c} v_y B = -p_\perp \sin(\varphi(t))\frac{d\varphi}{dt}$$

$$-\frac{e}{c} v_x B = -p_\perp \cos(\varphi(t))\frac{d\varphi}{dt}$$

So, since $v_y = \dot{y}$, $v_x = \dot{x}$,

$$\boxed{\begin{aligned} x(t) &= -\frac{cP_\perp}{eB}\sin(\varphi) \\ y(t) &= \frac{cP_\perp}{eB}\cos(\varphi) \end{aligned}}$$

If

$$\varphi(t) = \frac{B}{E}\operatorname{arctanh}\left(\frac{ceEt}{\sqrt{c^2e^2E^2t^2 + T_0^2}}\right)$$

Then

$$\frac{ceEt}{\sqrt{c^2e^2E^2t^2 + T_0^2}} = \tanh\left(\frac{E\varphi}{B}\right)$$

$$\frac{\dot{z}}{c} = \tanh\left(\frac{E\varphi}{B}\right)$$

$$\frac{\dot{z}}{T(t)}eB = \tanh\left(\frac{E\varphi}{B}\right)\frac{d\varphi}{dt}$$

$$\frac{\dot{z}}{z} = \frac{E}{B}\tanh\left(\frac{E\varphi}{B}\right)\frac{d\varphi}{dt}$$

$$\int_{z(0)}^{z(t)} \frac{\dot{z}}{z} \, dt' = \int_{0}^{\varphi(t)} \frac{E}{B} \tanh\left(\frac{E\varphi}{B}\right) \frac{d\varphi}{dt} \, dt'$$

$$\ln\left(\frac{z(t)}{z(0)}\right) = \ln\left(\cosh\left(\frac{E}{B}\varphi\right)\right)$$

$$z(0) = \frac{T_0}{eE}$$

$$\boxed{z(t) = \frac{T_0}{eE} \cosh\left(\frac{E}{B}\varphi\right)}$$

d)

The particle travels in a helical shape with increasing separation in the z-direction between cycles.

At large times, the particle travels very fast in the z-direction and therefore makes exponentially fewer rotations about the z-axis.

At small times, the orbit is approximately helical with radius proportional to $p_\perp$ and rate of verticle climb proportional to T_0.

If we plot $(x(t), y(t), \log(z(t)))$ we get very nearly a helix.

2) (1.1.1)

The dual field tensor is defined by

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\kappa}F_{\lambda\kappa}$$