Physics 623 Homework 5

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4.3.4. **Classical atom**

Consider a classical electron in a circular orbit in a Coulomb potential, for which the virial theorem yields $\overline{V} = 2E$.

a) Since the electron in a hydrogen atom is nonrelativistic, we can use the Larmor formula to describe the electromagnetic radiation of the electron as a point charge.

$$\mathscr{P} = \frac{2e^2}{3c^3}\dot{\boldsymbol{v}}^2 \tag{1}$$

We can use the equation of motion for the electron to rewrite (1) in terms of E.

$$\begin{aligned} m\dot{v} &= -\frac{e^2}{r^2} \\ \dot{v} &= -\frac{V^2}{me^2} \\ \dot{\boldsymbol{v}}^2 &= \frac{V^4}{m^2e^4} \\ \overline{\dot{\boldsymbol{v}}^2} &\approx \frac{\overline{V}^4}{m^2e^4} \\ \overline{\dot{\boldsymbol{v}}^2} &\approx \frac{16E^4}{m^2e^4} \end{aligned} \tag{2}$$

where we have approximated $\overline{V^4}$ with $\overline{V}^4$ since we may assume the potential is roughly constant over one period of the electron's orbit. Therefore, the average power radiated by the electron is

$$\begin{aligned} \mathscr{P} &= \frac{2e^2}{3c^3}\frac{16E^4}{m^2e^4} \\ \mathscr{P} &= \frac{32}{3e^2m^2c^3}E^4 \end{aligned} \tag{3}$$

b) We will show that the expected value of the potential diverges to $-\infty$ at finite time. We can rewrite (3) as a differential equation in E

$$-\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{32}{3e^2m^2c^3}E^4 \tag{4}$$

which we can then rewrite in terms of $\overline{V}$ using the virial theorem.

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}\overline{V}}{\mathrm{d}t} &= -\frac{1}{16}\frac{32}{3e^2m^2c^3}\overline{V}^4 \\ \frac{\mathrm{d}\overline{V}}{\mathrm{d}t} &= -\frac{4}{3e^2m^2c^3}\overline{V}^4 \\ \implies \overline{V}(t) &= \frac{1}{\left(\frac{4}{e^2m^2c^3}t + 1/\overline{V}_0^3\right)^{1/3}} \end{aligned} \tag{5}$$

We can assume that $\overline{V}_0 \approx -e^2/r_0$, where r_0 is the radius of the electron's orbit at $t = 0$. Then we have

$$\begin{aligned} \overline{V}(t) &= \frac{1}{\left(\frac{4}{e^2m^2c^3}t - r_0^3/e^6\right)^{1/3}} \\ \implies \lim_{t\to\alpha^-} \overline{V}(t) &= -\infty \end{aligned} \tag{6}$$

since $\frac{4}{e^2m^2c^3}t - r_0^3/e^6 \to 0^-$ as $t \to \alpha^-$, where $\alpha = \frac{r_0^3m^2c^3}{4e^4}$.

If the radius of the initial orbit is $r_0 = 10^{-8}$ cm then the time it takes for the electron to reach the nucleus is 105 ps $\approx 10^{-10}$ seconds.

4.3.5. **Absence of dipole radiation**

Consider a system of particles whose charge and current densities are

$$\begin{aligned} \rho(\boldsymbol{x}, t) &= \sum_j e_j\delta(\boldsymbol{x} - \boldsymbol{r}_j(t)) \\ \boldsymbol{j}(\boldsymbol{x}, t) &= \sum_j e_j\boldsymbol{v}_j(t)\delta(\boldsymbol{x} - \boldsymbol{r}_j(t)) \end{aligned} \tag{7}$$

respectively. Assume that $e_j/m_j = \mu$ for all j, where m_j is the mass of the j^{th} particle and μ is fixed. Assume also that there are no external forces or torques on the system. The electric and magnetic dipole moments of this system of charges are then given by

$$
\begin{aligned}
\boldsymbol{d}(t) &= \int d\boldsymbol{y}\; \boldsymbol{y}\rho(\boldsymbol{y},t) \\
\boldsymbol{m}(t) &= \frac{1}{2c}\int d\boldsymbol{y}\; \boldsymbol{y}\times \boldsymbol{j}(\boldsymbol{y},t) \\
\implies \boldsymbol{d}(t) &= \sum_j e_j \boldsymbol{r}_j(t) \\
\boldsymbol{m}(t) &= \frac{1}{2c}\sum_j e_j \boldsymbol{r}_j(t)\times \boldsymbol{v}_j(t)
\end{aligned}
\tag{8}
$$

We may rewrite (8) in terms of the masses of the particles and again in terms of the center of mass and angular momentum of the system

$$
\begin{aligned}
\boldsymbol{d}(t) &= \mu \sum_j m_j \boldsymbol{r}_j(t) \\
\boldsymbol{m}(t) &= \frac{\mu}{2c}\sum_j m_j \boldsymbol{r}_j(t)\times \boldsymbol{v}_j(t) \\
\implies \boldsymbol{d}(t) &= \mu M\, \boldsymbol{r}_{\mathrm{cm}}(t) \\
\boldsymbol{m}(t) &= \frac{\mu}{2c}\sum_j \boldsymbol{l}_j(t) = \frac{\mu}{2c}\boldsymbol{L}(t)
\end{aligned}
\tag{9}
$$

where M is the total mass of the system, $\boldsymbol{r}_{\mathrm{cm}}(t)$ is the center of mass of the system, and $\boldsymbol{L}(t)$ is the total angular momentum of the system. Since the total force and total torque on the system are both zero, we have

$$
\begin{aligned}
\ddot{\boldsymbol{d}}(t) &= \mu M\, \boldsymbol{a}_{\mathrm{cm}}(t) = \mu \boldsymbol{F}_{\mathrm{ext}} = \boldsymbol{0} \\
\dot{\boldsymbol{m}}(t) &= \frac{\mu}{2c}\dot{\boldsymbol{L}}(t) = \frac{\mu}{2c}\boldsymbol{\tau}_{\mathrm{ext}} = \boldsymbol{0}
\end{aligned}
\tag{10}
$$

Therefore, the total power due to electric and magnetic dipole radiation is

$$
\begin{aligned}
\mathscr{P} &= \frac{2}{3c^3}\left[\ddot{\boldsymbol{d}}^2 + \ddot{\boldsymbol{m}}^2\right] \\
&= 0
\end{aligned}
\tag{11}
$$

4.3.6. **Rotating dipole**

An electric dipole moment $\boldsymbol{d}$ rotates uniformly with angular velocity ω in a plane. We attempt to find the radiated power per solid angle, and the total radiated power, averaged over one rotational period. Since $\boldsymbol{d}$ rotates in the plane, we can let $\boldsymbol{d} = (d,0,0)$ at $t=0$ so that, as a function of time, $\boldsymbol{d}(t) = (d\cos\omega t, d\sin\omega t, 0)$. The power per solid angle, Ω, radiated by the dipole is given by

$$\frac{\mathrm{d}\mathscr{P}}{\mathrm{d}\Omega} = \frac{1}{4\pi c^3}\left(\hat{\boldsymbol{x}} \times \ddot{\boldsymbol{d}}\right)^2 \tag{12}$$

To compute the average power per solid angle radiated in one rotational period, we can simply compute

$$\overline{\frac{\mathrm{d}\mathscr{P}}{\mathrm{d}\Omega}} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt \frac{1}{4\pi c^3}\left(\hat{\boldsymbol{x}} \times \ddot{\boldsymbol{d}}\right)^2 \tag{13}$$

With $\hat{\boldsymbol{x}} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ and $\ddot{\boldsymbol{d}} = -\omega^2 \boldsymbol{d}(t)$, the power per solid angle, averaged over one rotational period, radiated by the dipole is

$$\begin{aligned}
\overline{\frac{\mathrm{d}\mathscr{P}}{\mathrm{d}\Omega}} &= \frac{\omega^4}{4\pi c^3}\frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\,(\hat{\boldsymbol{x}} \times \boldsymbol{d})^2 \\
&= \frac{\omega^4}{4\pi c^3}\frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt \frac{1}{4}(3 + \cos 2\theta - 2\cos(2(\phi - \omega t))\sin^2\theta) \\
&= \frac{\omega^4 d^2}{8\pi c^3}\left(1 + \cos^2\theta\right)
\end{aligned} \tag{14}$$

To get the total power radiated, averaged over one rotational period, we just integrate (14) over Ω. This yields

$$\begin{aligned}
\overline{\mathscr{P}} &= \int d\Omega \overline{\frac{\mathrm{d}\mathscr{P}}{\mathrm{d}\Omega}} \\
&= \frac{\omega^4 d^2}{8\pi c^3}\int_0^{2\pi}\int_0^{\pi} d\phi d\theta\,(1 + \cos^2\theta)\sin\theta \\
&= \frac{2\omega^4 d^2}{3c^3}
\end{aligned} \tag{15}$$

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