PHYS 622 Homework 9

Jeremy Welsh-Kavan

---

1) Consider the following ODE

$$(1-x^2)y'' - 2xy' + \lambda y = 0 \tag{1}$$

We claim that a necessary condition for a polynomial solution is $\lambda = n(n+1)$, $n = 0, 1, 2, ...$ To show this, suppose $p(x) = \sum_{n=0}^{\infty} a_n x^n$ is a polynomial solution to (1). That is, suppose there is N such that $a_N \neq 0$ and for $n > N$, $a_n = 0$. Then we have

$$\begin{aligned}
(1-x^2)\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - 2x\sum_{n=0}^{\infty} a_n n x^{n-1} + \lambda \sum_{n=0}^{\infty} a_n x^n &= 0 \\
(1-x^2)\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - 2x\sum_{n=0}^{\infty} a_n n x^{n-1} + \lambda \sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n n(n-1)x^n - 2\sum_{n=0}^{\infty} a_n n x^n + \lambda \sum_{n=0}^{\infty} a_n x^n &= 0 \qquad (2) \\
\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} \left(\lambda - n(n-1) - 2n\right) a_n x^n &= 0 \\
\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} \left(\lambda - n(n+1)\right) a_n x^n &= 0
\end{aligned}$$

Reindexing the first sum yields

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} \left(\lambda - n(n+1)\right) a_n x^n &= 0 \\
\implies a_{n+2} &= \frac{n(n+1)-\lambda}{(n+2)(n+1)} a_n
\end{aligned} \tag{3}$$

But, by assumption, the series truncates at $n = N$ so $a_{N+2} = 0$. So we must have $\lambda = N(N+1)$, since $a_N \neq 0$.
Therefore, if we are to have a polynomial solution to (1), we must have $\lambda = n(n+1)$ for $n = 0, 1, 2, ...$, where n is the degree of the polynomial.
Now suppose $a_0 \neq 0$ and $a_1 = 0$. By (2) and (3), if

$$a_{n+2} = \frac{n(n+1) - N(N+1)}{(n+2)(n+1)} a_n \tag{4}$$

then the polynomial, $p(x) = \sum_{n=0}^{N} a_n x^n$, satisfies (1) with $\lambda = N(N+1)$, $a_n \neq 0$ for even $n \leq N$, and $a_n = 0$ for $n > N$.

Similarly, suppose $a_0 = 0$ and $a_1 \neq 0$. Then the polynomial $p(x) = \sum_{n=0}^{N} a_n x^n$, satisfies (1) with $\lambda = N(N+1)$, $a_n \neq 0$ for odd $n \leq N$, and $a_n = 0$ for $n > N$.

Now suppose $a_0 \neq 0$ and $a_1 \neq 0$ and suppose there is N such that $a_N \neq 0$ and $a_n = 0$ for all $n > N$. Then, by (4), we have $a_N \neq 0$ and $a_{N-1} \neq 0$. But if $a_{N-1} \neq 0$ then (4) implies that $a_{N+1} \neq 0$. Which is a contradiction. Therefore, in order to have a polynomial solution, either $a_0 = 0$ or $a_1 = 0$.

Thus, $p(x) = \sum_{k=0}^{n} a_k x^k$ is a solution to (1) if and only if $a_0 = 0$ and $a_1 \neq 0$ and $\lambda = n(n+1)$ or $a_0 \neq 0$ and $a_1 = 0$ and $\lambda = n(n+1)$.

---

2) We claim that

$$\left(\sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x} - m\frac{x}{\sqrt{1-x^2}}\right) P_l^m(x) = (l+m)(l-m+1)P_l^{m-1}(x) \tag{5}$$

By construction, $P_l(x)$ satisfies Legendre's differential equation.

$$(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_l(x) - 2x\frac{\mathrm{d}}{\mathrm{d}x}P_l(x) + l(l+1)P_l(x) = 0 \tag{6}$$

We can apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to (6) to get

$$(1-x^2)\frac{\mathrm{d}^3}{\mathrm{d}x^3}P_l(x) - 4x\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_l(x) + (l(l+1)-2)\frac{\mathrm{d}}{\mathrm{d}x}P_l(x) = 0 \tag{7}$$

Applying $\frac{\mathrm{d}}{\mathrm{d}x}$ again, we get

$$(1-x^2)\frac{\mathrm{d}^4}{\mathrm{d}x^4}P_l(x) - 6x\frac{\mathrm{d}^3}{\mathrm{d}x^3}P_l(x) + (l(l+1)-6)\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_l(x) = 0 \tag{8}$$

Continuing in this manner $m-1$ times, we find that

$$\begin{aligned}(1-x^2)\frac{\mathrm{d}^{m+1}}{\mathrm{d}x^{m+1}}P_l(x) - 2mx\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_l(x) + (l(l+1)-m(m-1))\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}P_l(x) = 0\\ (1-x^2)\frac{\mathrm{d}^{m+1}}{\mathrm{d}x^{m+1}}P_l(x) - 2mx\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_l(x) + (l+m)(l-m+1)\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}P_l(x) = 0\end{aligned} \tag{9}$$

Recall that the associated Legendre functions are related to the Legendre polynomials by

$$P_l^m(x) = (-1)^m(1-x^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_l(x) \tag{10}$$

Therefore, plugging (10) into (9) we have

$$
\begin{aligned}
&\left((1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\frac{P_l^m(x)}{(-1)^m(1-x^2)^{m/2}} - 2mx\frac{P_l^m(x)}{(-1)^m(1-x^2)^{m/2}}\right) \\
&\qquad + \frac{(l+m)(l-m+1)}{(-1)^{m-1}(1-x^2)^{(m-1)/2}}P_l^{m-1}(x) = 0 \\
&\left((1-x^2)\frac{1}{(1-x^2)^{m/2}}\frac{\mathrm{d}}{\mathrm{d}x}P_l^m(x) + (1-x^2)P_l^m(x)\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{(1-x^2)^{m/2}} - 2mx\frac{P_l^m(x)}{(1-x^2)^{m/2}}\right) \\
&\qquad - \frac{(l+m)(l-m+1)}{(1-x^2)^{(m-1)/2}}P_l^{m-1}(x) = 0 \\
&\left((1-x^2)\frac{1}{(1-x^2)^{m/2}}\frac{\mathrm{d}}{\mathrm{d}x}P_l^m(x) + P_l^m(x)\frac{mx(1-x^2)}{(1-x^2)^{m/2+1}} - 2mx\frac{P_l^m(x)}{(1-x^2)^{m/2}}\right) \\
&\qquad - \frac{(l+m)(l-m+1)}{(1-x^2)^{(m-1)/2}}P_l^{m-1}(x) = 0 \\
&\left((1-x^2)\frac{1}{(1-x^2)^{m/2}}\frac{\mathrm{d}}{\mathrm{d}x}P_l^m(x) + P_l^m(x)\frac{mx}{(1-x^2)^{m/2}} - 2mx\frac{P_l^m(x)}{(1-x^2)^{m/2}}\right) \\
&\qquad - \frac{(l+m)(l-m+1)}{(1-x^2)^{(m-1)/2}}P_l^{m-1}(x) = 0 \\
&\left((1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}P_l^m(x) - mxP_l^m(x)\right) - \sqrt{1-x^2}(l+m)(l-m+1)P_l^{m-1}(x) = 0 \\
&\implies \left(\sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x} - m\frac{x}{\sqrt{1-x^2}}\right)P_l^m(x) - (l+m)(l-m+1)P_l^{m-1}(x) = 0
\end{aligned}
\tag{11}
$$

---

3) Below we show some statements concerning the spherical harmonics.

(i) By definition,

$$
Y_l^m(\Omega) = \left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{\frac{1}{2}} e^{im\varphi}P_l^m(\eta) \tag{12}
$$

We complex conjugate both sides and substitute $P_l^{-m}(\eta)$, multiplying by the appropriate coefficient, to get

$$
\begin{aligned}
Y_l^m(\Omega)^* &= (-1)^m\left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{\frac{1}{2}} e^{-im\varphi}\frac{(l+m)!}{(l-m)!}P_l^{-m}(\eta) \\
Y_l^m(\Omega)^* &= (-1)^m\left[\frac{(2l+1)(l+m)!}{4\pi(l-m)!}\right]^{\frac{1}{2}} e^{-im\varphi}P_l^{-m}(\eta) \\
Y_l^m(\Omega)^* &= (-1)^m Y_l^{-m}(\Omega)
\end{aligned}
\tag{13}
$$

(ii) Next we find an expression for $\cos(\theta)Y_l^m(\Omega)$

$$\cos(\theta)Y_l^m(\Omega) = \left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{\frac{1}{2}} e^{im\varphi}\eta P_l^m(\eta) \tag{14}$$

We know, from the Wikipedia page on Associated Legendre Polynomials, that

$$\begin{aligned}
\eta P_l^m(\eta) &= \frac{l-m+1}{2l+1}P_{l+1}^m(\eta) + \frac{l+m}{2l+1}P_{l-1}^m(\eta) \\
\frac{e^{im\varphi}}{4\pi}\eta P_l^m(\eta) &= \frac{e^{im\varphi}}{4\pi(2l+1)}(l-m+1)P_{l+1}^m(\eta) + \frac{e^{im\varphi}}{4\pi(2l+1)}(l+m)P_{l-1}^m(\eta) \\
\left[\frac{(l+m)!}{(2l+1)(l-m)!}\right]^{\frac{1}{2}}\cos(\theta)Y_l^m(\Omega) &= \frac{(l-m+1)}{(2l+1)}\left[\frac{(l+m+1)!}{(2l+3)(l-m+1)!}\right]^{\frac{1}{2}} Y_{l+1}^m(\Omega) \\
&+ \frac{(l+m)}{(2l+1)}\left[\frac{(l+m-1)!}{(2l-1)(l-m-1)!}\right]^{\frac{1}{2}} Y_{l-1}^m(\Omega) \\
\cos(\theta)Y_l^m(\Omega) &= \frac{(l-m+1)}{\sqrt{2l+1}}\left[\frac{(l+m+1)!(l-m)!}{(2l+3)(l-m+1)!(l+m)!}\right]^{\frac{1}{2}} Y_{l+1}^m(\Omega) \\
&+ \frac{(l+m)}{\sqrt{2l+1}}\left[\frac{(l+m-1)!(l-m)!}{(2l-1)(l-m-1)!(l+m)!}\right]^{\frac{1}{2}} Y_{l-1}^m(\Omega) \\
\\
\cos(\theta)Y_l^m(\Omega) &= \frac{(l-m+1)}{\sqrt{2l+1}}\left[\frac{(l+m+1)}{(2l+3)(l-m+1)}\right]^{\frac{1}{2}} Y_{l+1}^m(\Omega) \\
&+ \frac{(l+m)}{\sqrt{2l+1}}\left[\frac{(l-m)}{(2l-1)(l+m)}\right]^{\frac{1}{2}} Y_{l-1}^m(\Omega) \\
\\
\cos(\theta)Y_l^m(\Omega) &= \left[\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}\right]^{\frac{1}{2}} Y_{l+1}^m(\Omega) \\
&+ \left[\frac{(l-m)(l+m)}{(2l+1)(2l-1)}\right]^{\frac{1}{2}} Y_{l-1}^m(\Omega)
\end{aligned} \tag{15}$$

(iii) We know that the associated Legendre functions satisfy

$$\sqrt{1-\eta^2}P_l^m(\eta) = \frac{1}{2l+1}\left[(l-m+1)(l-m+2)P_{l+1}^{m-1}(\eta) - (l+m-1)(l+m)P_{l-1}^{m-1}(\eta)\right] \tag{16}$$

Multiplying on both sides by the appropriate factors, we have

$$\sin(\theta)e^{-i\varphi}Y_l^m(\Omega) = \frac{1}{2l+1}\left[\frac{(2l+1)(l-m)!}{(l+m)!}\right]^{\frac{1}{2}}\left[(l-m+1)(l-m+2)\left[\frac{(l+m)!}{(2l+3)(l-m+2)!}\right]^{\frac{1}{2}} Y_{l+1}^{m-1}(\Omega)\right.$$
$$\left. -(l+m-1)(l+m)\left[\frac{(l+m-2)!}{(2l-1)(l-m)!}\right]^{\frac{1}{2}} Y_{l-1}^{m-1}(\Omega)\right]$$
$$\sin(\theta)e^{-i\varphi}Y_l^m(\Omega) = \frac{1}{2l+1}\left[(l-m+1)(l-m+2)\left[\frac{(2l+1)(l-m)!}{(2l+3)(l-m+2)!}\right]^{\frac{1}{2}} Y_{l+1}^{m-1}(\Omega)\right.$$
$$\left. -(l+m-1)(l+m)\left[\frac{(2l+1)(l+m-2)!}{(2l-1)(l+m)!}\right]^{\frac{1}{2}} Y_{l-1}^{m-1}(\Omega)\right]$$
$$\sin(\theta)e^{-i\varphi}Y_l^m(\Omega) = \left[\frac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}\right]^{\frac{1}{2}} Y_{l+1}^{m-1}(\Omega) - \left[\frac{(l+m-1)(l+m)}{(2l+1)(2l-1)}\right]^{\frac{1}{2}} Y_{l-1}^{m-1}(\Omega) \tag{17}$$

The $\sin(\theta)e^{i\varphi}Y_l^m(\Omega)$ case follows from the fact that

$$\sqrt{1-\eta^2}P_l^m(\eta) = -\frac{1}{2l+1}\left[P_{l+1}^{m+1}(\eta) - P_{l-1}^{m+1}(\eta)\right] \tag{18}$$

After almost precisely the same algebra as in (17), we have

$$\sin(\theta)e^{\pm i\varphi}Y_l^m(\Omega) = \mp\left[\frac{(l\pm m+1)(l\pm m+2)}{(2l+1)(2l+3)}\right]^{\frac{1}{2}} Y_{l+1}^{m\pm1}(\Omega) \pm \left[\frac{(l\mp m-1)(l\mp m)}{(2l+1)(2l-1)}\right]^{\frac{1}{2}} Y_{l-1}^{m\pm1}(\Omega) \tag{19}$$

(iv) Define two differential operators, $\hat{L}_\mp$, by

$$\hat{L}_\mp = e^{\mp i\varphi}\left[\mp\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right] \tag{20}$$

We will compute $\hat{L}_{\mp} Y_l^m(\Omega)$.

$$
\begin{aligned}
\hat{L}_{-} Y_l^m(\Omega) &= e^{-i\varphi}\left[-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right]\left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{\frac{1}{2}} e^{im\varphi}P_l^m(\eta) \\
\hat{L}_{-} Y_l^m(\Omega) &= e^{-i\varphi}\left[\sqrt{1-\eta^2}\frac{\partial}{\partial\eta} + i\frac{\eta}{\sqrt{1-\eta^2}}\frac{\partial}{\partial\varphi}\right]\left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{\frac{1}{2}} e^{im\varphi}P_l^m(\eta) \\
\hat{L}_{-} Y_l^m(\Omega) &= e^{-i\varphi}\left[\sqrt{1-\eta^2}\frac{\partial}{\partial\eta} - m\frac{\eta}{\sqrt{1-\eta^2}}\right]\left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{\frac{1}{2}} e^{im\varphi}P_l^m(\eta) \\
\hat{L}_{-} Y_l^m(\Omega) &= e^{i(m-1)\varphi}\left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{\frac{1}{2}}\left[\sqrt{1-\eta^2}\frac{\partial}{\partial\eta} - m\frac{\eta}{\sqrt{1-\eta^2}}\right]P_l^m(\eta) \\
\hat{L}_{-} Y_l^m(\Omega) &= e^{i(m-1)\varphi}\left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{\frac{1}{2}}(l+m)(l-m+1)P_l^m(\eta) \\
\hat{L}_{-} Y_l^m(\Omega) &= e^{i(m-1)\varphi}\left[\frac{(2l+1)(l-(m-1))!}{4\pi(l+m-1)!}\right]^{\frac{1}{2}}\sqrt{(l+m)(l-m+1)}P_l^m(\eta) \\
\hat{L}_{-} Y_l^m(\Omega) &= \sqrt{(l+m)(l-m+1)}Y_l^{m-1}(\Omega)
\end{aligned}
\tag{21}
$$

The $\hat{L}_{+}$ case follows in the same manner. Thus, we have

$$
\hat{L}_{\mp} Y_l^m(\Omega) = \sqrt{(l\pm m)(l\mp m+1)}Y_l^{m\mp 1}(\Omega) \tag{22}
$$

---

4) Consider the electric field generated by a charge density $\rho(\mathbf{y})$ that vanishes inside a sphere of radius r_0.

a) We claim that if $\rho(\mathbf{y}) = \rho(-\mathbf{y})$ then $\mathbf{E}(\mathbf{0}) = \mathbf{0}$. The electric field due to the charge density is given by

$$
\mathbf{E}(\mathbf{x}) = \int d\mathbf{y}\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\rho(\mathbf{y}) \tag{23}
$$

Therefore,

$$
\begin{aligned}
\mathbf{E}(\mathbf{0}) &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dy_1 dy_2 dy_3 \frac{(y_1, y_2, y_3)}{|\mathbf{y}|^3}\rho(\mathbf{y}) \\
\text{let } \mathbf{u} = \textbf{-y} & \\
\mathbf{E}(\mathbf{0}) &= -\int_{\infty}^{-\infty}\int_{\infty}^{-\infty}\int_{\infty}^{-\infty} du_1 du_2 du_3 \frac{(u_1, u_2, u_3)}{|\mathbf{u}|^3}\rho(-\mathbf{u}) \\
\mathbf{E}(\mathbf{0}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} du_1 du_2 du_3 \frac{(u_1, u_2, u_3)}{|\mathbf{u}|^3}\rho(-\mathbf{u}) \\
\mathbf{E}(\mathbf{0}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} du_1 du_2 du_3 \frac{(u_1, u_2, u_3)}{|\mathbf{u}|^3}\rho(\mathbf{u}) \\
\mathbf{E}(\mathbf{0}) &= -\mathbf{E}(\mathbf{0}) \\
\implies \mathbf{E}(\mathbf{0}) &= \mathbf{0}
\end{aligned}
\tag{24}
$$

b) The field gradient tensor is given by

$$
\begin{aligned}
\varphi(\mathbf{x})_{ij} &= \frac{\partial^2 \varphi(\mathbf{x})}{\partial x_i \partial x_j} \\
\varphi(\mathbf{x})_{ij} &= \int d\mathbf{y}\rho(\mathbf{y})\frac{3(x_i - y_i)(x_j - y_j)}{|\mathbf{x}-\mathbf{y}|^5},\ \text{if } i \neq j \\
\varphi(\mathbf{x})_{ij} &= \int d\mathbf{y}\rho(\mathbf{y})\frac{3(y_i - x_i)^2 - |\mathbf{x}-\mathbf{y}|^2}{|\mathbf{x}-\mathbf{y}|^5},\ \text{if } i = j \\
\varphi(\mathbf{0})_{ij} &= \int d\mathbf{y}\rho(\mathbf{y})\frac{3y_i y_j}{|\mathbf{y}|^5},\ \text{if } i \neq j \\
\varphi(\mathbf{0})_{ij} &= \int d\mathbf{y}\rho(\mathbf{y})\frac{3y_i^2 - |\mathbf{y}|^2}{|\mathbf{y}|^5},\ \text{if } i = j \\
\varphi(\mathbf{0})_{ij} &= \int_{r_0}^{\infty}\int_0^{\pi}\int_0^{2\pi} r^2\sin(\theta)d\phi d\theta dr \rho(r,\theta,\phi)\frac{3y_i y_j}{r^5},\ \text{if } i \neq j \\
\varphi(\mathbf{0})_{ij} &= \int_{r_0}^{\infty}\int_0^{\pi}\int_0^{2\pi} r^2\sin(\theta)d\phi d\theta dr \rho(r,\theta,\phi)\frac{3y_i^2 - r^2}{r^5},\ \text{if } i = j
\end{aligned}
\tag{25}
$$

By symmetry, the current coordinate system is the principle axis coordinate system. Therefore. $\varphi(\mathbf{0})_{ij}$ is diagonal and we have $\varphi(\mathbf{0})_{ij} = 0$ for $i \neq j$. We can parameterize $\varphi(\mathbf{0})_{ij}$ as

$$
\varphi(\mathbf{0})_{ij} = \begin{pmatrix} \varphi_1 & 0 & 0 \\ 0 & \varphi_2 & 0 \\ 0 & 0 & \varphi_3 \end{pmatrix}
\tag{26}
$$

We first show that $\varphi_1 = \varphi_2$. Observe that $\varphi_1 - \varphi_2$ is proportional to the following integral

$$\begin{aligned}\varphi_1 - \varphi_2 &\propto \int_0^{2\pi} d\phi \rho(r,\theta,\phi)(\cos^2(\phi) - \sin^2(\phi)) \\ &= \int_0^{2\pi} d\phi \rho(r,\theta,\phi)\cos(2\phi) \\ &= \int_0^{2\pi} d\phi \rho(r,\theta,\phi+\alpha)\cos(2\phi) \\ &= \int_\alpha^{2\pi+\alpha} d\phi \rho(r,\theta,\phi)\cos(2\phi - 2\alpha) \\ &= \int_0^{2\pi} d\phi \rho(r,\theta,\phi)(\cos(2\phi)\cos(2\alpha) + \sin(2\phi)\sin(2\alpha))\end{aligned} \tag{27}$$

But the second term in the last line is proportional to $\varphi(\mathbf{0})_{xy} = 0$ so

$$\begin{gathered}\int_0^{2\pi} d\phi \rho(r,\theta,\phi)(\cos(2\phi) = \cos(2\alpha)\int_0^{2\pi} d\phi \rho(r,\theta,\phi)(\cos(2\phi) \\ \implies \varphi_1 - \varphi_2 \propto 0\end{gathered} \tag{28}$$

So we may write $\varphi(\mathbf{0})_{ij}$ as

$$\varphi(\mathbf{0})_{ij} = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & \varphi_3 \end{pmatrix} \tag{29}$$

We note that $\varphi(\mathbf{0})_{ij}$ is traceless, since $3y_1^2 - r^2 + 3y_1^2 - r^2 + 3y_3^2 - r^2 = 3r^2 - 3r^2 = 0$. Therefore, we can write $\varphi(\mathbf{0})_{ij}$ as

$$\varphi(\mathbf{0})_{ij} = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & -2\varphi \end{pmatrix} \tag{30}$$

c) Now suppose $\rho(\mathbf{y})$ has cubic symmetry so that $\rho(r,\theta,\phi) = \rho(r,\theta \pm \pi/2,\phi) = \rho(r,\theta,\phi \pm \pi/2)$. We claim that $\varphi(\mathbf{0})_{ij}$ vanishes in this case. From part b), we know that $\varphi(\mathbf{0})_{ij}$ is of the form (23). Therefore, we need only show that $\varphi = 0$. If $\rho(\mathbf{y})$ has cubic symmetry then we must have $\varphi(\mathbf{0})_{xx} = \varphi(\mathbf{0})_{yy} = \varphi(\mathbf{0})_{zz}$. But this implies that $\varphi = -2\varphi$. So we must have $\varphi = 0$. Therefore, $\varphi(\mathbf{0})_{ij} = 0$.

$$\clubsuit \tag{31}$$