Physics 614 Homework 8

Jeremy Welsh-Kavan

---

1. :(

2. **Fully-connected Ising model**

We consider an Ising model on a "complete graph", whose Hamiltonian is given by

$$H = -\frac{J}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j - B\sum_{i=1}^{N}\sigma_i \tag{1}$$

i. Let $S = \sum_{i=1}^{N}\sigma_i$ be the total spin of the system. Then H can be rewritten in terms of S as follows:

$$\begin{aligned} H &= -\frac{J}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i\sigma_j - B\sum_{i=1}^{N}\sigma_i \\ &= -\frac{J}{2N}\left(\sum_{i=1}^{N}\sigma_i\right)^2 - BS \\ &= -\frac{JS^2}{2N} - BS \end{aligned} \tag{2}$$

ii. We claim that

$$e^{\frac{\beta J}{2N}S^2} = \sqrt{\frac{\beta JN}{2\pi}}\int_{-\infty}^{\infty} dm\, e^{-\frac{1}{2}\beta JNm^2 + \beta JSm} \tag{3}$$

To show this, we can simply manipulate the integral of a normalized Gaussian

$$\begin{aligned} 1 = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\frac{x^2}{\sigma^2} + \frac{\mu}{\sigma}x - \frac{\mu^2}{2\sigma^2}} \\ 1 &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\mu^2}{2\sigma^2}}\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\frac{x^2}{\sigma^2} + \frac{\mu}{\sigma}x} \\ \implies e^{\frac{\mu^2}{2\sigma^2}} &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\frac{x^2}{\sigma^2} + \frac{\mu}{\sigma}x} \end{aligned} \tag{4}$$

Now substituting $\sigma = 1/\sqrt{\beta J N}$ and $\mu = S\sqrt{\beta J/N}$, we have

$$e^{\frac{\beta J}{2N}S^2} = \sqrt{\frac{\beta J N}{2\pi}} \int_{-\infty}^{\infty} dm\, e^{-\frac{1}{2}\beta J N m^2 + \beta J S m} \tag{5}$$

In terms of the total spin, S, the partition function can be written as

$$\mathcal{Z} = \sum_{\{S\}} e^{\beta \frac{JS^2}{2N} + \beta B S} \tag{6}$$

with $\{S\}$ the set of all possible configurations that yield the total spin S. We can use formula for the integral above to write $\mathcal{Z}$ as

$$\begin{aligned}
\mathcal{Z} &= \sum_{\{S\}} e^{\beta \frac{JS^2}{2N} + \beta B S} \\
\mathcal{Z} &= \sqrt{\frac{\beta J N}{2\pi}} \sum_{\{S\}} e^{\beta B S} \int_{-\infty}^{\infty} dm\, e^{-\frac{1}{2}\beta J N m^2 + \beta J S m} \\
\mathcal{Z} &= \sqrt{\frac{\beta J N}{2\pi}} \int_{-\infty}^{\infty} dm\, e^{-\frac{1}{2}\beta J N m^2} \sum_{\{S\}} e^{\beta S (B + J m)}
\end{aligned} \tag{7}$$

To evaluate the sum, we expand the exponential as a power series and S as a finite sum.

$$\begin{aligned}
\sum_{\{S\}} e^{\beta S(B+Jm)} &= \sum_{\{S\}} \sum_{n=0}^{\infty} \frac{(\beta S(B+Jm))^n}{n!} \\
&= \sum_{\{S\}} \sum_{n=0}^{\infty} \frac{(\beta(B+Jm))^n}{n!} \left(\sum_{i=1}^{N} \sigma_i\right)^n \\
&= \sum_{n=0}^{\infty} \frac{(\beta(B+Jm))^n}{n!} \sum_{\{S\}} \left(\sum_{i=1}^{N} \sigma_i\right)^n
\end{aligned} \tag{8}$$

where the last step is justified since only σ_i depends on the configuration $\{S\}$. We can rewrite the last sum as follows

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{a^n}{n!} \sum_{\{S\}} \left(\sum_{i=1}^{N} \sigma_i \right)^n &= \sum_{n=0}^{\infty} \frac{a^n}{n!} \sum_{\{S\}} \sum_{j_1 \dots j_n}^{N} \sigma_{j_1} \dots \sigma_{j_n} \\
&= \left(\sum_{n=0}^{\infty} \frac{a^n}{n!} + \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \right)^N \\
&= (2\cosh(a))^N \\
&= (2\cosh(\beta(B + Jm)))^N
\end{aligned} \tag{9}$$

where the simplification in the sum has been left unjustified out of desperation. Thus, we may write the partition function as

$$\begin{aligned}
\mathcal{Z} &= \sqrt{\frac{\beta JN}{2\pi}} \int_{-\infty}^{\infty} dm\, e^{-\frac{1}{2}\beta JNm^2} \left(2\cosh(\beta(B + Jm))\right)^N \\
&= \sqrt{\frac{\beta JN}{2\pi}} \int_{-\infty}^{\infty} dm\, e^{-\frac{1}{2}\beta JNm^2 + N\ln(2\cosh(\beta(B+Jm)))} \\
&= \sqrt{\frac{\beta JN}{2\pi}} \int_{-\infty}^{\infty} dm\, e^{-N\beta f(m)}
\end{aligned} \tag{10}$$

where $f(m) = Jm^2/2 - k_B T \ln(2\cosh(\beta(B + Jm)))$.

Jeez. Yet another "L" taken... Thanks anyway for grading our homework this year!

3. :(

♣