Physics 622 Homework 8

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My homework in LaTex, as requested. I pray it pleases thee, exalted grader.

2.2.3 Electrostatics in d dimensions (continued)

b) We calculate and plot the potential φ and the field $\mathbf{E}$ for $d=2$, for the case of a homogeneously charged disk, $\rho(\mathbf{x}) = \rho_0\Theta(r_0 - |\mathbf{x}|)$.
For $d=2$, the $\mathbf{E}$ field and the potential satisfy

$$\mathbf{E}(\mathbf{x}) = -\nabla\varphi(\mathbf{x}) \tag{1}$$

and

$$\nabla^2\varphi(\mathbf{x}) = -2\pi\rho(\mathbf{x}) \tag{2}$$

Since $\rho(\mathbf{x})$ is azimuthally symmetric, we know that $\varphi(\mathbf{x})$ will be azimuthally symmetric. In this case, we can rewrite (2) as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\varphi(r)\right) = -2\pi\rho_0\Theta(r_0 - r) \tag{3}$$

From this we can directly solve for $\varphi(r)$ by integrating (3) twice.

$$\begin{aligned}
r\frac{\partial}{\partial r}\varphi(r) &= -2\pi\rho_0\int dr\ r\Theta(r_0 - r) \\
r\frac{\partial}{\partial r}\varphi(r) &= -2\pi\rho_0\begin{cases}\frac{r^2}{2},\ 0 \le r \le r_0 \\ \frac{r_0^2}{2},\ r > r_0\end{cases} \\
\frac{\partial}{\partial r}\varphi(r) &= -\pi\rho_0\left(r + \Theta(r - r_0)(\frac{r_0^2}{r} - r)\right) \\
\varphi(r) &= -\pi\rho_0\left(\frac{r^2}{2} + \Theta(r - r_0)\left(\frac{r_0^2}{2} - \frac{r^2}{2} + r_0^2\ln\left(\frac{r}{r_0}\right)\right)\right) \\
&\text{or} \\
\varphi(r) &= -\pi\rho_0\begin{cases}\frac{r^2}{2},\ 0 \le r \le r_0 \\ \frac{r_0^2}{2} + r_0^2\ln\left(\frac{r}{r_0}\right),\ r_0 < r\end{cases}
\end{aligned} \tag{4}$$

Finally, we can redefine $\varphi(r_0) = 0$ to get

$$\varphi(r) = \begin{cases}\pi\rho_0\left(\frac{r_0^2}{2} - \frac{r^2}{2}\right), 0 \le r \le r_0 \\ \pi\rho_0 r_0^2\ln\left(\frac{r_0}{r}\right),\ r_0 < r\end{cases} \tag{5}$$

We find the **E** field by taking the negative gradient of (5).

$$
\begin{aligned}
\mathbf{E}(\mathbf{x}) &= -\nabla\varphi(\mathbf{x}) \\
\mathbf{E}(\mathbf{x}) &= -\frac{\partial\varphi(r)}{\partial r}\hat{e}_r \\
\mathbf{E}(\mathbf{x}) &= E(r)\hat{e}_r \\
E(r) &= \begin{cases} \pi\rho_0 r, 0 \le r \le r_0 \\ \pi\rho_0\frac{r_0^2}{r},\ r_0 < r \end{cases}
\end{aligned} \tag{6}
$$

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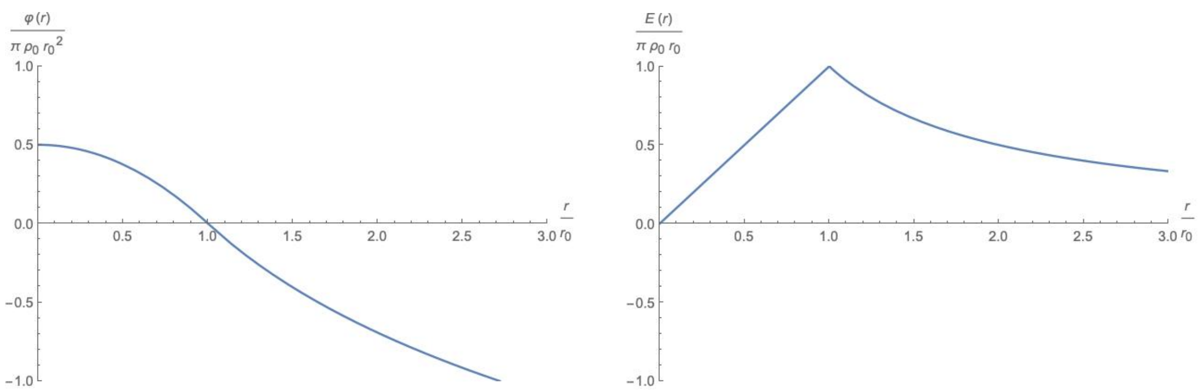

Figure 1: Plots of the potential, $\varphi(r)$, and the radial component of the electric field, $E(r)$, in convenient units.

c) We proceed as above, but for d=1. Here, d=1 and $\rho(x) = \rho_0\Theta(x_0^2/4 - x^2)$, which represents the charge density of a uniformly charged rod of length x_0 centered at the origin. In the d=1 case, $\mathbf{E}(\mathbf{x})$ and $\varphi(x)$ are both just scalar fields given by the equations

$$
\begin{aligned}
\frac{\partial E(x)}{\partial x} &= 2\rho(x) \\
\frac{\partial^2\varphi(x)}{\partial x^2} &= -2\rho(x)
\end{aligned} \tag{7}
$$

Solving for $E(x)$, we have

$$
\begin{aligned}
\frac{\partial E(x)}{\partial x} &= 2\rho_0\Theta\left(1 - \frac{4x^2}{x_0^2}\right) \\
E(x) &= \begin{cases} -\rho_0 x_0,\ x < -\frac{x_0}{2} \\ 2\rho_0 x,\ \ -\frac{x_0}{2} \le x \le \frac{x_0}{2} \\ \rho_0 x_0,\ \frac{x_0}{2} < x \end{cases}
\end{aligned} \tag{8}
$$

We can define $\varphi(\frac{x_0}{2}) = 0$, and since the potential is symmetric is the charge distribution is symmetric, $\varphi(-\frac{x_0}{2}) = 0$ as well. To obtain $\varphi(x)$ we integrate (8), which yields

$$\varphi(x) = -\int_{x_0/2}^{x} dx E(x)$$
$$\varphi(x) = \begin{cases} \rho_0 x_0 x + \frac{\rho_0 x_0^2}{2}, \ x < -\frac{x_0}{2} \\ \frac{\rho_0 x_0^2}{4} - \rho_0 x^2, \ -\frac{x_0}{2} \le x \le \frac{x_0}{2} \\ -\rho_0 x_0 x + \frac{\rho_0 x_0^2}{2}, \ \frac{x_0}{2} < x \end{cases} \tag{9}$$

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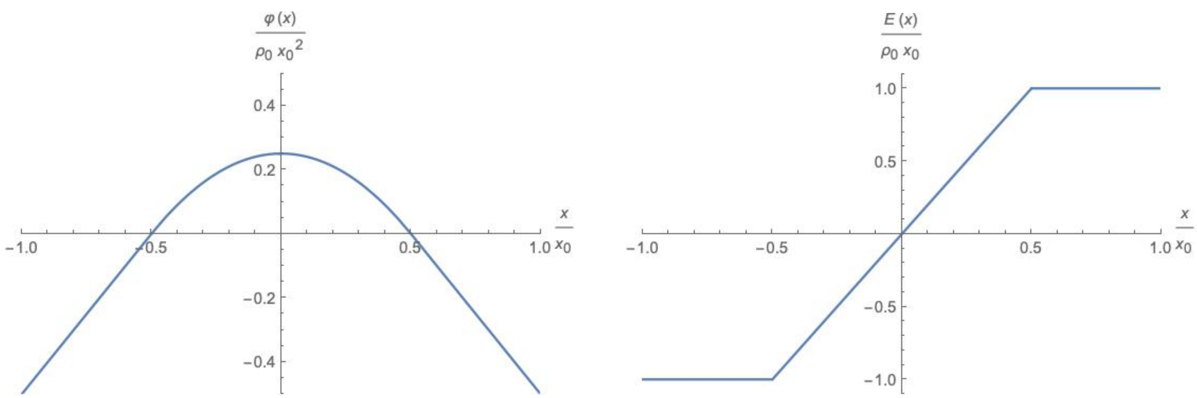

Figure 2: Plots of the potential, $\varphi(x)$, and the electric field, $E(x)$, in convenient units.

2.2.4 Helmholtz Equation

Our goal is to find the most general Fourier transformable solution, $\varphi(\mathbf{x})$, to the Helmholtz equation,

$$\left(\kappa^2 - \nabla^2\right) \varphi(\mathbf{x}) = 4\pi\rho(\mathbf{x}) \tag{10}$$

We can begin by funding the fundamental solution, $G(\mathbf{x})$, of the Helmholtz operator. That is, a function, $G(\mathbf{x})$ satisfying

$$\left(\kappa^2 - \nabla^2\right) G(\mathbf{x}) = -4\pi\delta(\mathbf{x}) \tag{11}$$

Taking the Fourier transform of both sides of (6) gives

$$\frac{1}{(\sqrt{2\pi})^3} \int d^3\mathbf{x} \left(\kappa^2 - \nabla^2\right) G(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} = -\frac{4\pi}{(\sqrt{2\pi})^3}$$
$$(\kappa^2 + \mathbf{k}^2)\hat{G}(\mathbf{k}) = -\sqrt{\frac{2}{\pi}} \tag{12}$$
$$\hat{G}(\mathbf{k}) = -\sqrt{\frac{2}{\pi}} \frac{1}{\mathbf{k}^2 + \kappa^2}$$

We now inverse Fourier transform the last line of (7) to get

$$G(\mathbf{x}) = -\sqrt{\frac{2}{\pi}}\frac{1}{(\sqrt{2\pi})^3}\int d^3\mathbf{k}\,\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2+\kappa^2} \tag{13}$$

which we can compute in spherical coordinates as follows:

$$\begin{aligned}
G(\mathbf{x}) &= -\frac{1}{2\pi^2}\int_0^\infty\int_0^{2\pi}\int_0^\pi \frac{e^{ir|\mathbf{x}|\cos\theta}}{r^2+\kappa^2}r^2\sin\theta d\theta d\phi dr\\
G(\mathbf{x}) &= -\frac{1}{\pi}\int_0^\infty\int_0^\pi \frac{e^{ir|\mathbf{x}|\cos\theta}}{r^2+\kappa^2}r^2\sin\theta d\theta dr\\
G(\mathbf{x}) &= -\frac{2}{\pi|\mathbf{x}|}\int_0^\infty \frac{r\,\sin(r|\mathbf{x}|)}{r^2+\kappa^2}dr\\
G(\mathbf{x}) &= -\frac{e^{-\kappa|\mathbf{x}|}}{|\mathbf{x}|}
\end{aligned} \tag{14}$$

Where the last integral in (9) can be computed with Mathematica or using the residue theorem. Note that this differs by a sign from the Green's function of the Laplacian that we found in 2.2.3 a) since the Laplacian carries a minus sign in the Helmholtz equation. With $G(\mathbf{x})$,we can find a general expression for $\varphi(\mathbf{x})$. From what we know about Green's functions, if $\varphi(\mathbf{x})$ satisfies (5), then we can express $\varphi(\mathbf{x})$ in terms of a convolution with $G(\mathbf{x})$. Just to convince ourselves that this makes sense, below is some hand-wavy algebra. Let $L = \kappa^2 - \nabla^2$. If $G(\mathbf{x})$ satisfies

$$L\left[G(\mathbf{x}-\mathbf{y})\right] = -4\pi\delta^3(\mathbf{x}-\mathbf{y}) \tag{15}$$

then we have

$$\begin{aligned}
\int d^3\mathbf{y}L\left[G(\mathbf{x}-\mathbf{y})\right]\rho(\mathbf{y}) &= -4\pi\int d^3\mathbf{y}\delta^3(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})\\
-L\left[\int d^3\mathbf{y}G(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})\right] &= 4\pi\rho(\mathbf{x})
\end{aligned} \tag{16}$$

and since

$$L\left[\varphi(\mathbf{x})\right] = 4\pi\rho(\mathbf{x}) \tag{17}$$

we can conclude that

$$\varphi(\mathbf{x}) = -\int d^3\mathbf{y}G(\mathbf{x}-\mathbf{y})\rho(\mathbf{y}) \tag{18}$$

So, with equation (9), we have

$$\varphi(\mathbf{x}) = \int d^3\mathbf{y}\frac{e^{-\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\rho(\mathbf{y}) \tag{19}$$

Which is Fourier transformable if $\rho(\mathbf{x})$ is Fourier Transformable.

2.3.1 Quadrupole moments

a) Let $\rho(\mathbf{y})$ be a localized charge density. To approximate the scalar potential, $\varphi(\mathbf{x})$, we first expand the denominator of Poisson's formula, omitting terms of order $\frac{1}{r^4}$ and greater.

$$\begin{aligned}\frac{1}{|\mathbf{x}-\mathbf{y}|} &= \frac{1}{\sqrt{r^2 - 2(\mathbf{x}\cdot\mathbf{y}) + \mathbf{y}^2}} \\ &= \frac{1}{r}\left(1 - \frac{2(\mathbf{x}\cdot\mathbf{y})}{r^2} + \frac{\mathbf{y}^2}{r^2}\right)^{-\frac{1}{2}} \\ &\approx \frac{1}{r}\left(1 + \frac{(\mathbf{x}\cdot\mathbf{y})}{r^2} - \frac{\mathbf{y}^2}{2r^2} + \frac{3}{2}\frac{(\mathbf{x}\cdot\mathbf{y})^2}{r^4}\right)\end{aligned} \tag{20}$$

Using Poisson's formula, we can construct an approximation for $\varphi(\mathbf{x})$.

$$\begin{aligned}\varphi(\mathbf{x}) &= \int d\mathbf{y}\frac{\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \\ \varphi(\mathbf{x}) &\approx \int d\mathbf{y}\frac{\rho(\mathbf{y})}{r}\left(1 + \frac{(\mathbf{x}\cdot\mathbf{y})}{r^2} - \frac{\mathbf{y}^2}{2r^2} + \frac{3}{2}\frac{(\mathbf{x}\cdot\mathbf{y})^2}{r^4}\right) \\ &= \frac{Q}{r} + \frac{\mathbf{d}\cdot\mathbf{x}}{r^3} + \int d\mathbf{y}\frac{\rho(\mathbf{y})}{r}\left(\frac{3}{2r^4}\sum_{i,j} x_i x_j y_i y_j - \frac{1}{2r^2}\mathbf{y}^2\right) \\ &= \frac{Q}{r} + \frac{\mathbf{d}\cdot\mathbf{x}}{r^3} + \frac{1}{r^5}\int d\mathbf{y}\rho(\mathbf{y})\left(\frac{3}{2}\sum_{i,j} x_i x_j y_i y_j - \frac{r^2}{2}\mathbf{y}^2\right) \\ &= \frac{Q}{r} + \frac{\mathbf{d}\cdot\mathbf{x}}{r^3} + \frac{1}{r^5}\int d\mathbf{y}\rho(\mathbf{y})\left(\frac{3}{2}\sum_{i,j} x_i x_j y_i y_j - \frac{1}{2}\sum_{j} x_i x_i \mathbf{y}^2\right) \\ &= \frac{Q}{r} + \frac{\mathbf{d}\cdot\mathbf{x}}{r^3} + \frac{1}{r^5}\sum_{i,j} x_i x_j\left[\frac{1}{2}\int d\mathbf{y}\rho(\mathbf{y})\left(3y_i y_j - \delta_{i,j}\mathbf{y}^2\right)\right] \\ \varphi(\mathbf{x}) &\approx \frac{Q}{r} + \frac{\mathbf{d}\cdot\mathbf{x}}{r^3} + \frac{1}{r^5}\sum_{i,j} x_i x_j Q_{ij}\end{aligned} \tag{21}$$

where

$$Q_{ij} = \frac{1}{2}\int d\mathbf{y}\rho(\mathbf{y})\left(3y_i y_j - \delta_{i,j}\mathbf{y}^2\right) \tag{22}$$

b) I didn't finish this one in time, which is a bummer because it doesn't seem difficult at all :(.

c) For a homogeneously charged ellipsoid, $(x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1$, with total charge q, we express the charge density in the ellipsoidal coordinates parameterized by

$$\begin{cases} y_1 = ar\cos(\phi)\sin(\theta) \\ y_2 = br\sin(\phi)\sin(\theta) \\ y_3 = cr\cos(\theta) \\ r \in [0,1]\,, \phi \in [0,2\pi)\,, \theta \in [0,\pi] \end{cases} \tag{23}$$

and

$$\rho(r,\theta,\phi) = \rho_0 = \frac{3q}{4\pi abc} \tag{24}$$

where q is the total charge. The quadrupole tensor is given by

$$\begin{aligned} Q_{ij} &= \frac{\rho_0}{2}\int_0^1\int_0^{2\pi}\int_0^{\pi} abcr^2\sin(\theta)d\theta d\phi dr \left(3y_iy_j - \delta_{i,j}\mathbf{y}^2\right) \\ Q &= \frac{2}{15}\pi abc\rho_0 \begin{pmatrix} 2a^2-b^2-c^2 & 0 & 0 \\ 0 & 2b^2-a^2-c^2 & 0 \\ 0 & 0 & 2c^2-a^2-b^2 \end{pmatrix} \end{aligned} \tag{25}$$

where we have used Mathematica to compute the integrals. By inspection, Q is traceless, but we calculate $\mathrm{tr}(Q)$ below for completeness.

$$\begin{aligned} \mathrm{tr}(Q) &= \frac{2}{15}\pi abc\rho_0\left(2a^2-b^2-c^2+2b^2-a^2-c^2+2c^2-a^2-b^2\right) \\ &= \frac{2}{15}\pi abc\rho_0\left(2a^2+2b^2+2c^2-2a^2-2b^2-2c^2\right) \\ &= 0 \end{aligned} \tag{26}$$

d) Let $\rho(r,\theta,z)$ be a localized charge density parameterized by the cylindrical coordinates and suppose $\rho(r,\theta,z)$ has the property that $\rho(r,\theta,z) = \rho(r,\theta+k\alpha,z)$ for $k\in\mathbb{Z}$ and $\alpha\in[-\pi,\pi]$. We claim that Q_{ij} has the form

$$Q = \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -2q \end{pmatrix} \tag{27}$$

We observe that the azimuthal symmetry of $\rho(r,\theta,z)$ implies that the coordinate system in which ρ has this property is the principle axis coordinate system. Therefore, Q_{ij} is diagonal in the current coordinate system. Additionally, since Q is traceless, we know that it has the form

$$Q = \begin{pmatrix} q_x & 0 & 0 \\ 0 & q_y & 0 \\ 0 & 0 & -q_x-q_y \end{pmatrix} \tag{28}$$

Therefore, we need only show that $q_x = q_y$. These components are given by

$$\begin{aligned} Q_{xx} &= \int_0^{z_0}\int_0^{r_0}\int_0^{2\pi} r\rho(r,\theta,z)(3r^2\cos^2(\theta)-r^2-z^2)d\theta dr dz \\ Q_{yy} &= \int_0^{z_0}\int_0^{r_0}\int_0^{2\pi} r\rho(r,\theta,z)(3r^2\sin^2(\theta)-r^2-z^2)d\theta dr dz \end{aligned} \tag{29}$$

Therefore, these components differ only by the following integral

$$\begin{aligned} Q_{xx}-Q_{yy} &= \int_0^{z_0}\int_0^{r_0}\int_0^{2\pi} 3r^3\rho(r,\theta,z)(\cos^2(\theta)-\sin^2(\theta))d\theta dr dz \\ Q_{xx}-Q_{yy} &= \int_0^{z_0}\int_0^{r_0}\int_0^{2\pi} 3r^3\rho(r,\theta,z)\cos(2\theta)d\theta dr dz \end{aligned} \tag{30}$$

Since ρ is necessarily 2π periodic, we know that $k\alpha = 2\pi$ for some minimum $k \in \mathbb{N}$. Therefore, $\rho(r, \theta, z) = \rho(r, \theta + n\frac{2\pi}{k}, z)$ for all $n \in \mathbb{N}$.

Note to the grader Ok so I didn't finish this one either. My best guess is that this periodicity makes the integral in (30) vanish but it's not obvious to me why that is at the moment.
