PHYS 631: Quantum Mechanics I (2020)
Exercises 12 October 2020 (Monday, Week 3)
Due Monday, 19 October 2020

Fixing The Momentum Operator

Introduction

In the representation of the position operator $\mathbf{q}$, the momentum operator $\mathbf{p}$ is defined by the commutator

$$[q_\alpha, p_\beta] = i\hbar\delta_{\alpha\beta}. \tag{1}$$

This looks a lot like the commutator $[q, \partial_q] = -1$, but because any function $f(q)$ commutes with q, the best we can do is to say that

$$\mathbf{p} = \frac{\hbar}{i}\nabla + \mathbf{f}(\mathbf{q}), \tag{2}$$

or in components,

$$p_\alpha = \frac{\hbar}{i}\partial_\alpha + f_\alpha(\mathbf{q}), \tag{3}$$

for some vector function $\mathbf{f}(\mathbf{q})$. Let's fix that, because nobody wants to go carrying around an extra $\mathbf{f}(\mathbf{q})$ everywhere.

Step 1: Reduce it to a scalar

Exercise 1. Show that

$$[f_\alpha(\mathbf{q}) - p_\alpha, f_\beta(\mathbf{q}) - p_\beta] = 0. \tag{4}$$

Exercise 2. Noting that we can multiply out the commutator

$$[f_\alpha(\mathbf{q}) - p_\alpha, f_\beta(\mathbf{q}) - p_\beta] = [f_\alpha(\mathbf{q}), f_\beta(\mathbf{q})] - [p_\alpha, f_\beta(\mathbf{q})] - [f_\alpha(\mathbf{q}), p_\beta] + [p_\alpha, p_\beta], \tag{5}$$

and using the standard commutator [which follows from Eq. (1)]

$$[f_\alpha(\mathbf{q}), p_\beta] = i\hbar\partial_\beta f_\alpha, \tag{6}$$

show that

$$\partial_\beta f_\alpha = \partial_\alpha f_\beta. \tag{7}$$

What you have shown is that $\mathbf{f}$ is the gradient of some *scalar* function, say $F(\mathbf{q})$:

$$f_\alpha(\mathbf{q}) = \partial_\alpha F. \tag{8}$$

This is the same reasoning leading to $\mathbf{E} = -\nabla V$ in electrostatics.

Step 2: Transform it away

Consider an *alternate* position representation, defined by

$$|\mathbf{x}\rangle = e^{i\varphi(\mathbf{q})}|\mathbf{q}\rangle, \tag{9}$$

where the position eigenvalues $\mathbf{x} = \mathbf{q}$ are equivalent between representations. The states are equivalent except for some space-dependent phase $\varphi(\mathbf{q})$ to be determined. The idea is to compare the derivative operators $\partial/\partial q_\alpha$ and $\partial/\partial x_\alpha$ in the two representations. To relate the derivative operators in the two representations, note that both representations are equally valid position representations, so the matrix element of the derivative operator should be the same expressed either way:

$$\langle x'_\alpha|\frac{\partial}{\partial q_\alpha}|x_\alpha\rangle = \langle q'_\alpha|\frac{\partial}{\partial q_\alpha}|q_\alpha\rangle. \tag{10}$$

(Note: no summation over α is implied here.)

Exercise 3. Using the transformation rule (9) between the position representations, show that

$$\langle x'_\alpha|\frac{\partial}{\partial x_\alpha}|x_\alpha\rangle = \langle q'_\alpha|e^{-i\varphi(q_\alpha)}\frac{\partial}{\partial x_\alpha}e^{i\varphi(q_\alpha)}|q_\alpha\rangle. \tag{11}$$

(Still the case that no summation over α is implied here, and note the $\phi(q_\alpha)$ in the leftmost phase factor, rather than $\phi(q'_\alpha)$.)

Comparing Eqs. (10) and (11), which express the same matrix element in the two position representations, we can conclude

$$\frac{\partial}{\partial q_\alpha} = e^{-i\varphi}\frac{\partial}{\partial x_\alpha}e^{i\varphi}. \tag{12}$$

Exercise 4. Show that this can be rewritten as

$$\frac{\partial}{\partial q_\alpha} = \frac{\partial}{\partial x_\alpha} + i\frac{\partial\varphi}{\partial x_\alpha}. \tag{13}$$

In terms of the momentum operator in the $\mathbf{x}$-representation, you have obtained

$$p_\alpha = -i\hbar\frac{\partial}{\partial x_\alpha} + \frac{\partial F(\mathbf{q})}{\partial x_\alpha} + \hbar\frac{\partial\varphi}{\partial x_\alpha}. \tag{14}$$

By choosing $\hbar\varphi(\mathbf{x}) = -F(\mathbf{x}) + \text{const}$ (where the constant corresponds to an irrelevant global phase), the momentum operator takes the standard form

$$p_\alpha = -i\hbar\frac{\partial}{\partial x_\alpha}. \tag{15}$$

Thus, even if there is some extra function tacked onto the momentum operator, we can always eliminate it by making an appropriate phase choice.