Density operator

Exercise 1:

If $\rho = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ then,

a) if ρ is to be a density matrix we must have

$A + D = 1$

$A, D \in \mathbb{R}$ (since $\rho = \rho^{+}$)

and $A \geq 0$, $D \geq 0$

since ρ must be positive semidefinite

b) If ρ is to be a pure density matrix then $Tr(\rho^2) = 1$

So we must have

$A^2 + D^2 = 1$

Exercise 2)

$$\rho = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

a) if ρ is a density matrix then $A + D = 1$,

$A, D \in \mathbb{R}$

and $C = B^*$ since ρ must be Hermitian.

We also must have

$$\lambda_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4\det(\rho)} \geq 0$$

so $\det\rho = AD - |B|^2 \geq 0$

b) If ρ is to be a pure density matrix then we must also have

$$\mathrm{Tr}(\rho^2) = 1$$

$$\rightarrow \mathrm{Tr}\begin{pmatrix} A^2 + |B|^2 & B \\ B^* & D^2 + |B|^2 \end{pmatrix}$$

$$= A^2 + D^2 + 2|B|^2 = 1$$

Exercise 1:

$$H_{int} = -\int d^3r \, \vec{J} \cdot \vec{A}$$

$$\tilde{\vec{J}}(\vec{k}) = \int d^3r \, \vec{J}(\vec{r}) \, e^{-i\vec{k}\cdot\vec{r}}$$

$$\tilde{\vec{A}}(\vec{k}) = \int d^3r \, \vec{A}(\vec{r}) \, e^{-i\vec{k}\cdot\vec{r}}$$

$$\vec{J}(\vec{r}) = \frac{1}{2\pi} \int d^3k \, \tilde{\vec{J}}(\vec{k}) \, e^{i\vec{k}\cdot\vec{r}}$$

$$\vec{A}(\vec{r}) = \frac{1}{2\pi} \int d^3k \, \tilde{\vec{A}}(\vec{k}) \, e^{i\vec{k}\cdot\vec{r}}$$

$$H_{int} = \frac{-1}{(2\pi)^2} \int d^3r \int d^3k \, \tilde{\vec{J}}(\vec{k}) \, e^{i\vec{k}\cdot\vec{r}} \cdot \int d^3k' \, \tilde{\vec{A}}(\vec{k}') \, e^{i\vec{k}'\cdot\vec{r}}$$

$$H_{int} = \frac{-1}{(2\pi)^2} \int d^3k \int d^3k' \int d^3r \, \tilde{\vec{J}}(\vec{k}) \cdot \tilde{\vec{A}}(\vec{k}') \, e^{i(\vec{k}+\vec{k}')\cdot\vec{r}}$$

$$H_{int} = \frac{-1}{(2\pi)^2} \int d^3k \, d^3k' \, \tilde{\vec{J}}(\vec{k}) \cdot \tilde{\vec{A}}(\vec{k}') \int d^3r \, e^{i(\vec{k}+\vec{k}')\cdot\vec{r}}$$

$$H_{int} = \frac{-1}{2\pi} \int d^3k \, d^3k' \, \tilde{\vec{J}}(\vec{k}) \, \tilde{\vec{A}}(\vec{k}') \, \delta^3(\vec{k}+\vec{k}')$$

$$H_{int} = \frac{-1}{2\pi} \int d^3k' \, \tilde{\vec{J}}(-\vec{k}') \, \tilde{\vec{A}}(\vec{k}')$$

Exercise 2:

$$A_q = \hat{e}_q \cdot \vec{A}$$

$$(A^*)_q = \hat{e}_q \cdot \vec{A}^*$$

$$(A^*)_\pm = -(\hat{e}_\mp)^* \cdot \vec{A}^*$$

$$(A^*)_\pm = -(\hat{e}_\mp \cdot \vec{A})^*$$

$$(A^*)_\pm = (-1)^{\pm 1}(\vec{A}_{-q})^*$$

$$(A^*)_0 = \hat{e}_0^* \cdot A^* = A_0^*$$

$$\rightarrow (A^*)_q = (-1)^q A_{-q}^*$$

Exercise 3:

The tensor elements in (31) correspond to the integral of $j_{q''}(\vec{r})$.

The state $|\alpha' j' m'\rangle$ corresponds to $Y_{\ell}^{m''}(\hat{r})$.

The CG coefficients correspond to each other.

Exercise 4:

$$\nabla \cdot (r_{m'} \vec{j}\,) = \vec{j} \cdot \nabla r_{m'} + r_{m'} \nabla \cdot \vec{j}$$

and $$\vec{j} \cdot \nabla r_{m'} = j_{m'}$$

So $$j_{m'} = \nabla (r_{m'} \vec{j}\,) - r_{m'} \nabla \cdot \vec{j}$$

$$\mathcal{J}_{m'}^{(l'=1)}(l=0) = \frac{1}{\sqrt{4\pi}} \int d^3r\, j_{m'}(\vec{r}\,)$$

$$\mathcal{J}_{m'}^{(l'=1)}(l=0) = \frac{1}{\sqrt{4\pi}} \int d^3r\, (\vec{j} \cdot \nabla r_{m'})$$

$$\mathcal{J}_{m'}^{(l'=1)}(l=0) = -\frac{1}{\sqrt{4\pi}} \int d^3r\, (r_{m'} \cdot \nabla \vec{j}\,)$$

since $\nabla \cdot (r_{m'} \vec{j}\,) = 0$ at ∞.

Exercise 5:

For an octupole $l' = 3$
so J' may change by at most 3.

We also know that since
$m_J = m_J' + m'$, m_J may change
by at most l'.

Since $l' = 3$ is odd, $m_J = m_J' = 0$
and $J \neq J'$.

Exercise 6:

If $l' = 3$ then the parity of the initial and final states differ by -1 so $\pi_j \pi_{j'} = -1$.