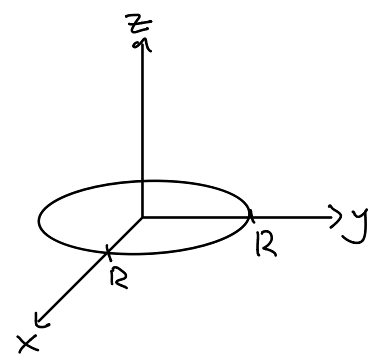
1) $$I_{ij} = \int \rho(\vec{r})\left(|\vec{r}|\,\delta_{ij} - \vec{r}_i\vec{r}_j\right)dV$$
$$\vec{r} = (x, y, z)$$

We first calculate the moment of inertia tensor for a disc in the xy plane centered on the z axis:



Assuming uniform density, the differential mass element is

$$dm = \rho\, dA = \rho\, r\,dr\,d\theta = \frac{m}{\pi R^2}\, r\,dr\,d\theta$$

$$I_{xx} = \int y^2 + z^2\, dm$$
$$I_{yy} = \int x^2 + z^2\, dm$$
$$I_{zz} = \int x^2 + y^2\, dm$$

$$I_{xy} = I_{yx} = -\int xy\,dm$$

$$I_{xz} = I_{zx} = -\int xz\,dm$$

$$I_{yz} = I_{zy} = -\int yz\,dm$$

Since $z=0$ on the disc,

$$I_{xx} = \int y^2\,dm$$

$$I_{yy} = \int x^2\,dm$$

$$I_{zz} = I_{yy} + I_{xx}$$

$$I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0$$

Additionally,

$$I_{xy} = \int xy\,dm = \int_0^R \int_0^{2\pi} r^3 \cos\theta \sin\theta \,dr\,d\theta = 0$$

Now we have

$$I_{xx} = \int y^2\,dm = \frac{m}{\pi R^2} \int_0^R \int_0^{2\pi} r^3 \sin^2\theta \,dr\,d\theta$$

$$= \frac{mR^2}{4}$$

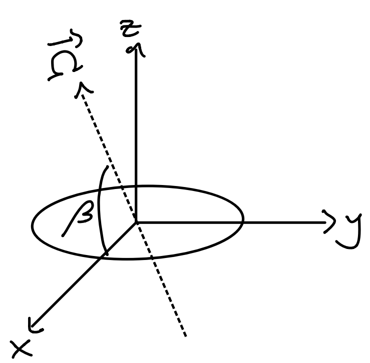
$$I_{yy} = \frac{m}{\pi R^2} \int_0^R \int_0^{2\pi} r^3 \cos^2\theta \,dr\,d\theta = \frac{ml^2}{4}$$

$$I_{zz} = I_{xx} + I_{yy} = \frac{mR^2}{2}$$

So

$$I = \frac{mR^2}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Now let the disc rotate about a vector $\vec{\Omega}$ which lies in the xz plane at an angle β with respect to the xy-plane



In components,

$$\vec{\Omega} = \Omega\begin{pmatrix} \cos(\beta) \\ 0 \\ \sin(\beta) \end{pmatrix}$$

The kinetic energy of the disc rotating about $\vec{\Omega}$ is

$$T = \frac{1}{2}\vec{\Omega}\cdot\left(I\vec{\Omega}\right)$$

$$T = \frac{m\Omega^2R^2}{8}\begin{pmatrix}\cos(\beta)\\0\\\sin(\beta)\end{pmatrix}\cdot\begin{pmatrix}\cos(\beta)\\0\\2\sin(\beta)\end{pmatrix}$$

$$T = \frac{m\Omega^2R^2}{8}\left(\cos^2(\beta)+2\sin^2(\beta)\right)$$

$$\boxed{T = \frac{m\Omega^2R^2}{8}\left(1+\sin^2(\beta)\right)}$$

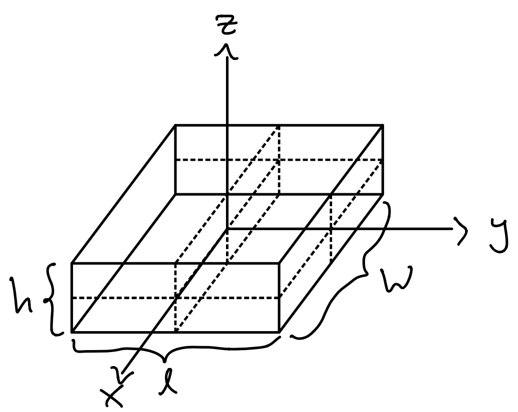
Which is maximized at

$$\boxed{\beta = \frac{\pi}{2}}$$

2)

a)

We choose a coordinate system whose origin is at the center of mass and whose axes puncture the center of each face of the brick.



$$dm = \rho\, dV = \frac{m}{hlw}\, dx\,dy\,dz$$

$$I_{ij} = \int \rho(\vec{r})\left(|\vec{r}|^2 \delta_{ij} - \vec{r}_i \vec{r}_j\right) dV$$

$$\vec{r} = (x, y, z)$$

Let $V := hlw$

$$I_{xx} = \frac{m}{V} \int_{-\frac{w}{2}}^{\frac{w}{2}} \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 + z^2 \, dz \, dy \, dz$$

$$= \frac{m}{V} w \left(\frac{h^3 l}{12} + \frac{h l^3}{12} \right)$$

$$I_{yy} = \frac{m l}{V} \left(\frac{w^3 h}{12} + \frac{h^3 w}{12} \right)$$

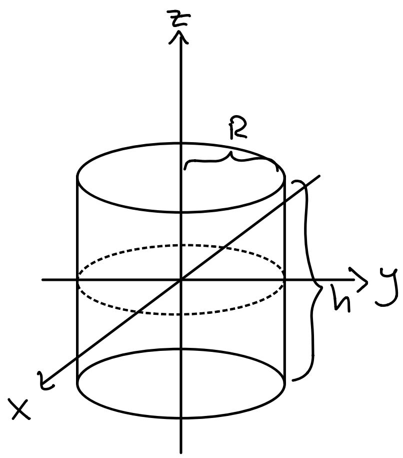
$$I_{zz} = \frac{m h}{V} \left(\frac{l^3 w}{12} + \frac{l w^3}{12} \right)$$

This is clearly the principal axis coordinate system so I is diagonal

$$\boxed{I = \frac{m}{12} \begin{bmatrix} h^2 + l^2 & 0 & 0 \\ 0 & h^2 + w^2 & 0 \\ 0 & 0 & w^2 + l^2 \end{bmatrix}}$$

b)

The coordinate system's origin will be at the center of mass and the z axis will run the length of the cylinder



$$dm = \rho\, dV = \frac{m}{\pi R^2 h}\, r\,dr\,d\theta\,dz$$

$$I_{xx} = \frac{m}{\pi R^2 h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^{2\pi} \int_0^R r^3 \sin^2\theta + r z^2 \, dr\,d\theta\,dz$$

$$= \frac{m}{12}\left(h^2 + 3R^2\right)$$

$$I_{yy} = \frac{m}{12}\left(h^2 + 3R^2\right)$$

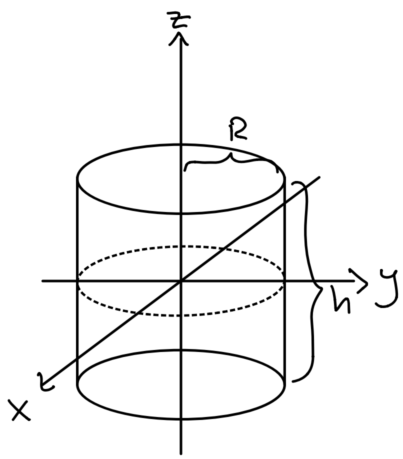
(by symmetry)

$$I_{zz} = \frac{m}{\pi R^2 h}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_0^{2\pi}\int_0^R r^3 \, dr\, d\theta\, dz$$

$$= \frac{mR^2}{2}$$

$$\boxed{I = \frac{mR^2}{2}\begin{bmatrix} \frac{1}{6}\left(3 + \frac{h^2}{R^2}\right) & 0 & 0 \\ 0 & \frac{1}{6}\left(3 + \frac{h^2}{R^2}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

c) We choose the same coordinate system as in the previous problem



In this problem, however,

$$dm = \rho \, dA = \frac{m}{2\pi R h} R \, d\theta \, dz$$

$$I_{xx} = \frac{m}{2\pi h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^{2\pi} R^2 \sin^2\theta + z^2 \, d\theta \, dz$$

$$= \frac{m}{12} \left(h^2 + 6R^2 \right)$$

$$I_{yy} = \frac{m}{12} \left(h^2 + 6R^2 \right)$$

(by symmetry)

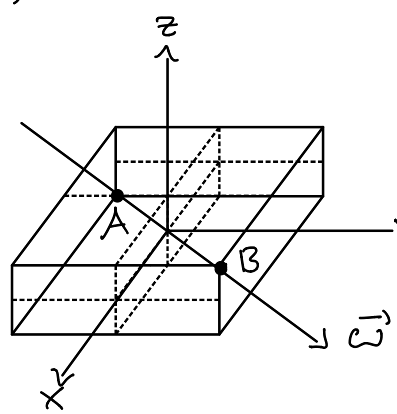
$$I_{zz} = \frac{m}{2\pi h}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{0}^{2\pi} R^2 \, d\theta \, dz$$

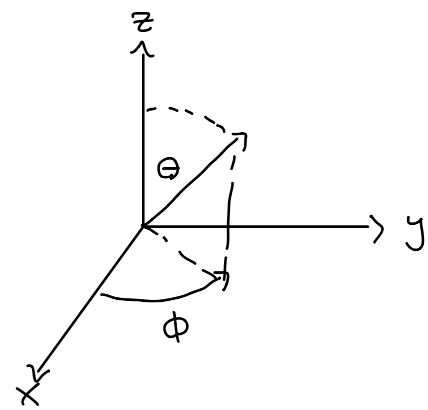
$$= mR^2$$

$$\boxed{I = mR^2 \begin{bmatrix} \frac{1}{12}\left(6 + \frac{h^2}{R^2}\right) & 0 & 0 \\ 0 & \frac{1}{12}\left(6 + \frac{h^2}{R^2}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

3)

a)





We first compute $\vec{\omega}$

$$\vec{\omega} = \omega \begin{pmatrix} \cos\phi \sin\theta \\ \sin\phi \sin\theta \\ \cos\theta \end{pmatrix}$$

where $\phi := \arctan\left(\frac{l}{W}\right)$

and $\theta := \arccos\left(\frac{h}{\sqrt{h^2+l^2+W^2}}\right)$

so

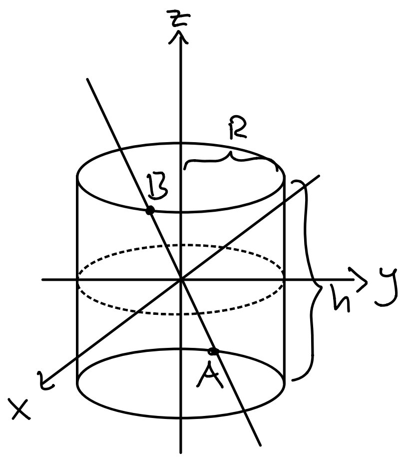
$$\vec{\omega} = \frac{\omega}{\sqrt{h^2+l^2+W^2}} \begin{pmatrix} W \\ l \\ h \end{pmatrix}$$

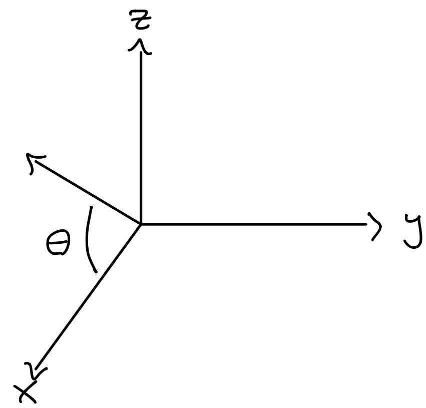
The kinetic energy is

$$T = \frac{1}{2}\vec{\omega}\cdot(I\vec{\omega})$$

$$\boxed{T = \frac{1}{12} m\,\omega^2 R^2 \left(\frac{w^2\ell^2 + h^2w^2 + \ell^2h^2}{w^2+h^2+\ell^2}\right)}$$

b)





$$\vec{\omega} = \omega \begin{pmatrix} \cos\theta \\ 0 \\ \sin\theta \end{pmatrix}$$

$$\theta = \arctan\left(\frac{h}{2R}\right)$$

$$\vec{\omega} = \frac{\omega}{\sqrt{1 + \frac{h^2}{4R^2}}} \begin{pmatrix} 1 \\ 0 \\ \frac{h}{2R} \end{pmatrix}$$

$$I = \frac{mR^2}{2} \begin{bmatrix} \frac{1}{6}\left(3 + \frac{h^2}{R^2}\right) & 0 & 0 \\ 0 & \frac{1}{6}\left(3 + \frac{h^2}{R^2}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T = \frac{1}{2}\vec{\omega} \cdot (I\vec{\omega})$$

$$T = \frac{1}{4}\frac{m\omega^2 R^2}{1 + \frac{h^2}{4R^2}}\left(\frac{1}{2} + \frac{5}{12}\frac{h^2}{R^2}\right)$$

$$\boxed{T = m\omega^2 R^2\left(\frac{1}{8} + \frac{7h^2}{24(h^2+4R^2)}\right)}$$

c)

This is essentially the same as b):

$$\vec{\omega} = \frac{\omega}{\sqrt{1 + \frac{h^2}{4R^2}}} \begin{pmatrix} 1 \\ 0 \\ \frac{h}{2R} \end{pmatrix}$$

$$I = mR^2 \begin{bmatrix} \frac{1}{12}\left(6 + \frac{h^2}{R^2}\right) & 0 & 0 \\ 0 & \frac{1}{12}\left(6 + \frac{h^2}{R^2}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T = \frac{1}{2} \vec{\omega} \cdot (I \vec{\omega})$$

$$\boxed{T = \frac{1}{12} m \omega^2 R^2 \left(3 + \frac{5h^2}{h^2 + 4R^2}\right)}$$

or

$$\boxed{T = \frac{1}{3} m \omega^2 R^2 \left(\frac{2h^2 + 3R^2}{h^2 + 4R^2}\right)}$$