Exercise 1)

Prove $[a, a^+] = 1$

$$a = \frac{(x + ip)}{\sqrt{2}} \longrightarrow a^+ = \frac{(x - ip)}{\sqrt{2}}$$

$$[a, a^+] = aa^+ - a^+a$$

$$= \frac{1}{2}(x + ip)(x - ip) - \frac{1}{2}(x - ip)(x + ip)$$

$$= \frac{1}{2}\left(\cancel{x^2 + p^2} + ipx - ixp\right) - \frac{1}{2}\left(\cancel{x^2 + p^2} - ipx + ixp\right)$$

$$= \frac{1}{2}(-i[x, p]) - \frac{1}{2}(i[x, p])$$

$$= \frac{1}{2}(-ii) - \frac{1}{2}(i \cdot i)$$

$$= \frac{1}{2} - \frac{1}{2}(-1) = 1 \quad ✓$$

Exercise 2:

by induction:

$$(x - \partial_x)^n e^{-x^2/2} = (-1)^n e^{x^2/2} \partial_x^n e^{-x^2} \quad (*)$$

base cases $n=1$

$$(x - \partial_x) e^{-x^2/2} = x e^{-x^2/2} - \partial_x (e^{-x^2/2})$$

$$= x e^{-x^2/2} - \left(e^{-x^2/2} (-x) \right)$$

$$= 2x e^{-x^2/2} \quad \checkmark$$

Assume $(*)$ for all $k < n$

then

$$(x - \partial_x)^{n+1} e^{-x^2/2} = (-1)^{n+1} e^{x^2/2} \partial_x^{n+1} e^{-x^2}$$

$$(x - \partial_x)(x - \partial_x)^n e^{-x^2/2}$$

$$= (x - \partial_x)\left((-1)^n e^{x^2/2} \partial_x^n e^{-x^2} \right)$$

$$= x(-1)^n e^{x^2/2} \partial_x^n e^{-x^2}$$
$$- (-1)^n \partial_x \left(e^{x^2/2} \partial_x^n e^{-x^2} \right)$$

$$= x(-1)^n e^{x^2/2} \partial_x^n e^{-x^2}$$
$$- (-1)^n \left(x e^{x^2/2} \partial_x^n e^{-x^2} + e^{x^2/2} \partial_x^{n+1} e^{-x^2} \right)$$

$$= \cancel{x(-1)^n e^{x^2/2} \partial_x^n e^{-x^2}}$$
$$\cancel{- x(-1)^n e^{x^2/2} \partial_x^n e^{-x^2}}$$
$$- (-1)^n e^{x^2/2} \partial_x^{n+1} e^{-x^2}$$
$$= (-1)^{n+1} e^{x^2/2} \partial_x^{n+1} e^{-x^2} \quad \checkmark.$$

So $(n) \Rightarrow (n+1)$

Exercise 3)

We can write $a^\dagger$ as

$$a^\dagger = \sum_{n=0}^{\infty} \sqrt{n+1}\, |n+1\rangle\langle n|$$

Let $|\alpha\rangle$ be a vector

then we can write

$$|\alpha\rangle = \sum_{n=0}^{\infty} \langle n|\alpha\rangle |n\rangle$$

in the energy eigenbasis

so

$$a^\dagger|\alpha\rangle = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \langle k|\alpha\rangle \sqrt{n+1}\, \langle k|n\rangle |n+1\rangle$$

but $\langle k|n\rangle = \delta_{kn}$

$$a^\dagger|\alpha\rangle = \sum_{n=0}^{\infty} \langle n|\alpha\rangle \sqrt{n+1}\, |n+1\rangle$$

Now suppose $|\alpha\rangle$ is an

eigenvector of $a^\dagger$. Then

$$a^\dagger |\alpha\rangle = \alpha |\alpha\rangle$$

So

$$a^\dagger |\alpha\rangle = \sum_{n=0}^{\infty} \langle n|\alpha\rangle \sqrt{n+1}\, |n+1\rangle$$

$$= \alpha |\alpha\rangle = \alpha \sum_{k=0}^{\infty} \langle k|\alpha\rangle |k\rangle$$

but the expression of $|\alpha\rangle$ in the basis is unique. In particular when $k = n = 0$ we must have

$$\langle 0|\alpha\rangle |1\rangle = \alpha \langle 0|\alpha\rangle |0\rangle$$

but $\langle 1|0\rangle = 0$. So

we must have $\langle n|\alpha\rangle = 0$

for each n.