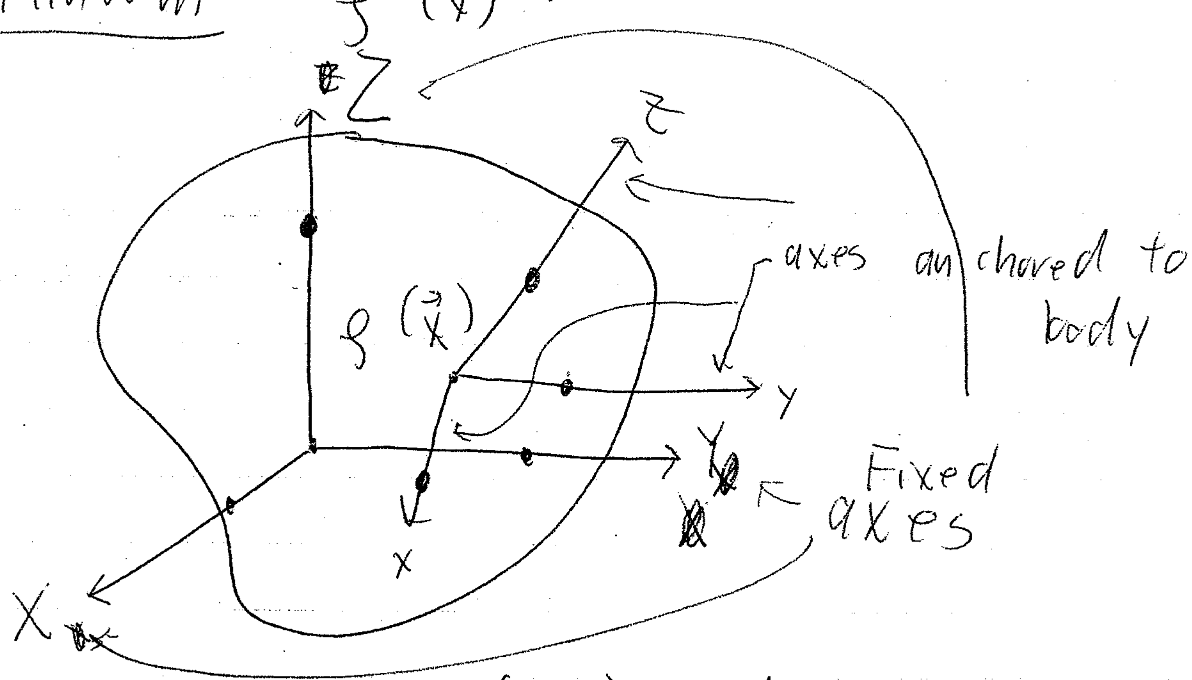
Rigid Body motion:

System with many particles ($\sim 6\times10^{23}$; often approximated by ∞; i.e., as a continuum $\rho(\vec{x})$:

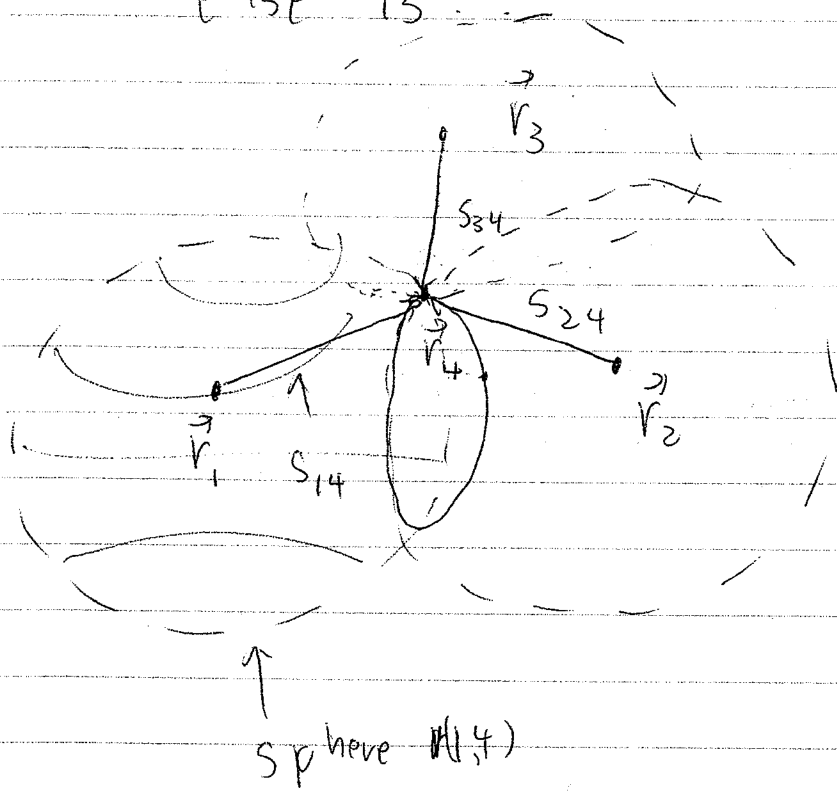


But very few (6) degrees of freedom. I'll show this in a moment.

Why? Rigid $\Rightarrow$ distances between all points in object constant.

$\Rightarrow$ If

$\Rightarrow$ If you know positions ~~d~~ $\vec{r}_i$, $i = 1, 2, 3$ of 3 points on object, you know where everything else is.



$\Rightarrow$ seems # of d.o.f. $= 9$ $(\vec{r}_1, \vec{r}_2, \vec{r}_3)$

But: 3 more constraints:

$$|\vec{r}_1 - \vec{r}_2| = s_{12}, \quad |\vec{r}_1 - \vec{r}_3| = s_{13}, \quad |\vec{r}_2 - \vec{r}_3| = s_{23}$$

$\Rightarrow$ 6 d.o.f. left.

~~What~~ How to parametrize the 6 d.o.f.?

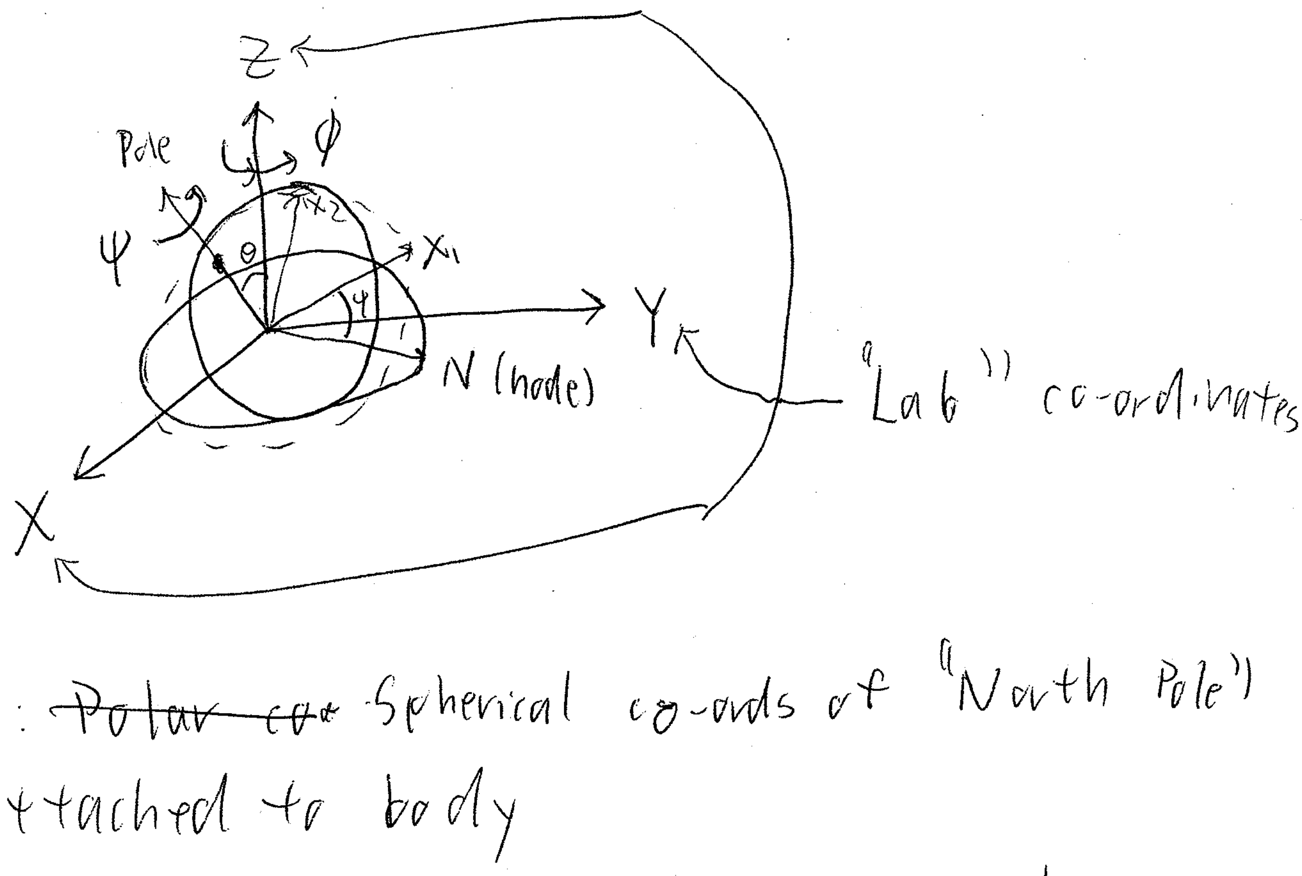
I.e., Choice of $\{q_i\}$'s, $i = 1 \to 6$.

Frequently convenient choice:

$\vec{R}_{cm}$ (or $\vec{R}_{\text{some "center"}}$)

and

Euler $\measuredangle$'s: (θ, ϕ, ψ): ~~Rotation~~ Specify "orientation" of body:



(θ, ϕ): ~~Polar co~~ Spherical co-ords of "North Pole" attached to body

ψ: rotation $\measuredangle$ around this pole.

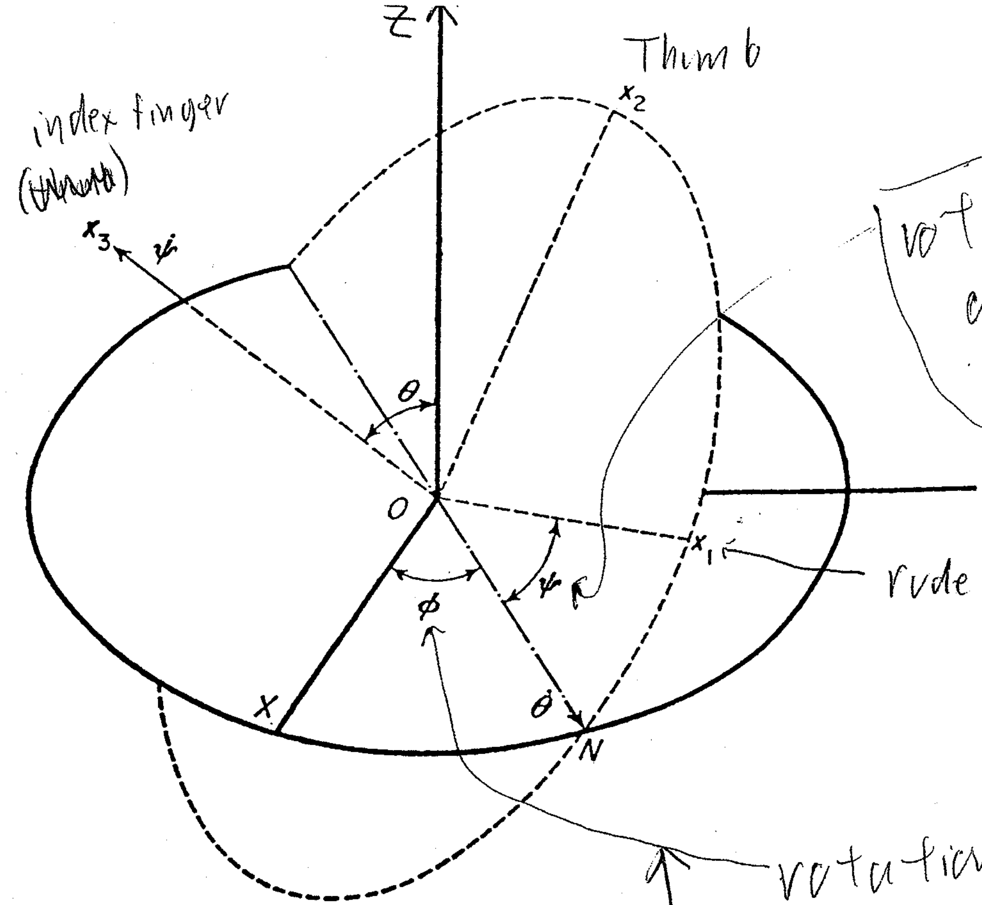
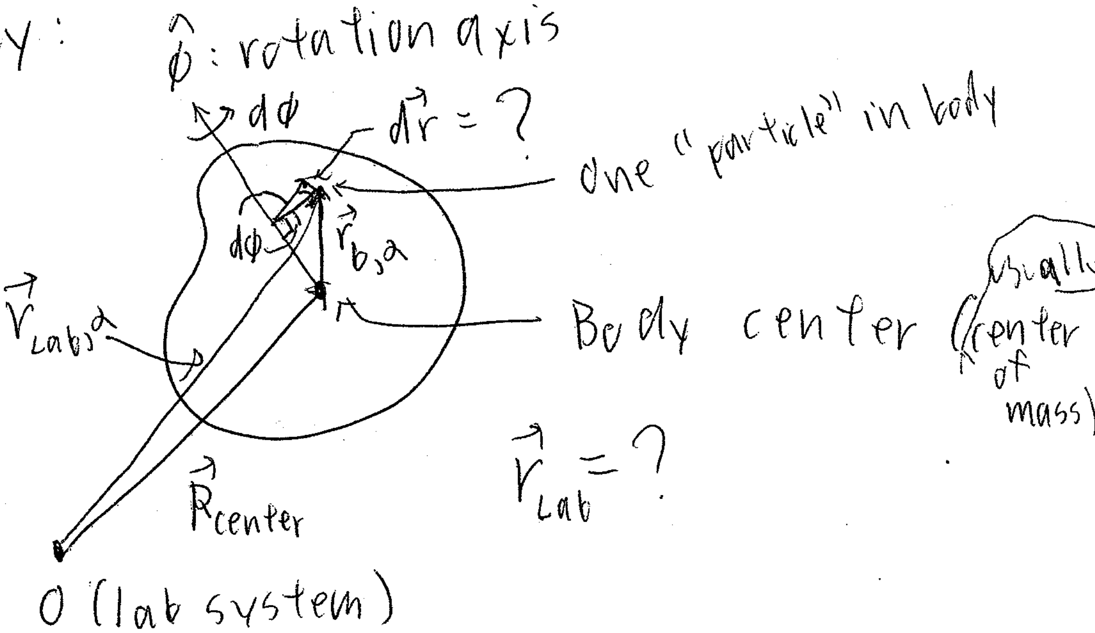


FIG. 47

Better pic from L+L.

~~Ultimately~~
Determining velocity $\vec{v}$ of any point on body:



$$d\vec{r}_{b,\alpha} = d\vec{\phi} \times \vec{r}_{b,\alpha}$$

$$\Rightarrow d\,\vec{r}_{\alpha} = d\,\vec{R}_c + d\vec{\phi} \times \vec{r}_{b,\alpha}$$

$$\Rightarrow \text{velocity } \vec{v}_\alpha = \underbrace{\frac{d\vec{R}_c}{dt}}_{\vec{V}_{CM}} + \underbrace{\frac{d\vec{\phi}}{dt}}_{\vec{\Omega}} \times \vec{r}_{b,\alpha}$$

$\vec{\Omega}$: "Angular velocity vector"

$\vec{V}_c, \vec{\Omega}$ <u>same</u> for all 6×10^{23} particles

$\Rightarrow$ Only 6 #'s needed to specify <u>all</u> $\vec{v}$'s $\Rightarrow$ " 6 #'s to get

Kinetic energy:

$$T = \frac{1}{2} \sum_\alpha m_\alpha |\vec{v}_\alpha|^2 = \frac{1}{2} \sum_\alpha m_\alpha |\vec{V}_{CM} + \vec{\Omega} \times \vec{r}_{b,\alpha}|^2$$

$$= \underbrace{\frac{1}{2}\left(\sum_\alpha m_\alpha |\vec{V}_c|^2\right)}_{\equiv \mathrm{I}} + \underbrace{\sum_\alpha m_\alpha \vec{V}_c \cdot (\vec{\Omega} \times \vec{r}_{b,\alpha})}_{\equiv \mathrm{II}} + \underbrace{\frac{1}{2} \sum_\alpha m_\alpha |\vec{\Omega} \times \vec{r}_{b,\alpha}|^2}_{\equiv \mathrm{III}}$$

I = ?

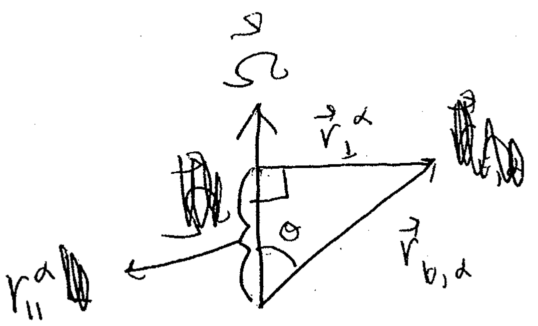
II = ?

III = ?

$$\mathrm{I} = \frac{1}{2}\Big(\underbrace{\sum_\alpha m_\alpha}_{M\text{ (total mass)}}\Big)|\vec{V}_c|^2 = \frac{1}{2}M|\vec{V}_c|^2 : \text{ K.E. if all mass was moving at } \vec{V}_c$$

$$\mathrm{II} = \vec{V}_c\cdot\Big[\vec{\Omega}\times\Big(\underbrace{\sum_\alpha m_\alpha \vec{r}_{b\alpha}}_{M(\vec{R}_{cm}-\vec{R}_c)}\Big)\Big] \quad (= 0 \text{ if } \vec{R}_c = \vec{R}_{cm})$$

$$\mathrm{III} = \frac{1}{2}\sum_\alpha m_\alpha |\vec{\Omega}\times\vec{r}_{b,\alpha}|^2$$



$$|\vec{\Omega}\times\vec{r}_{\alpha,b}|^2 = \Omega^2 |\vec{r}_{b,\alpha}|^2 \sin^2\theta$$

$$= \Omega^2 |\vec{r}_{\perp\alpha}|^2$$

write i.t.o. $|\vec{r}_{b,\alpha}|$, $r_{\parallel}^{\alpha}$:

$r_{\parallel}^{\alpha} =$

$$\Rightarrow |\vec{\Omega} \times \vec{r}_{\alpha,b}|^2 = \Omega^2 |\vec{r}_{\perp\alpha}|^2$$
$$= \Omega^2 \left(|\vec{r}_{b,\alpha}|^2 - \frac{(\vec{r}_{b,\alpha} \cdot \vec{\Omega})^2}{\Omega^2} \right)$$
$$= \Omega^2 |\vec{r}_{b,\alpha}|^2 - (\vec{r}_{b,\alpha} \cdot \vec{\Omega})^2$$

Write out in Cartesian components $r^{\alpha}_{b,i}$, $i = x, y, z$

$$|\vec{r}_{b,\alpha}|^2 = \underbrace{r^{\alpha}_{b,i} \, r^{\alpha}_{b,i}}, \quad \vec{r}_{b\alpha} \cdot \vec{\Omega} =$$

Einstein summation convention: repeated index $\Rightarrow$ sum over i

$$\Rightarrow r_i \, r_i \equiv \sum_{i=x,y,z} r_i^2 = r_x^2 + r_y^2 + r_z^2 = |\vec{r}|^2$$

$$a_i \, b_i \equiv \sum_i a_i b_i = a_x b_x + a_y b_y + a_z b_z$$?

$$\Rightarrow \sum_\alpha m_\alpha |\vec{\Omega} \times \vec{r}_{b,\alpha}|^2 = \Omega_i \, \Omega_j \, \underbrace{I_{ij}}$$

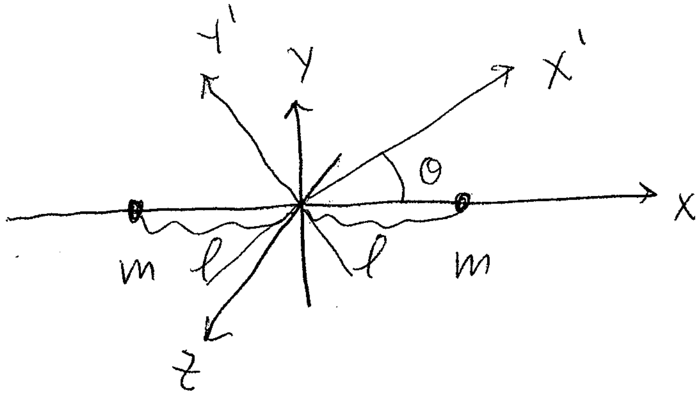
$$I_{ij} \equiv \sum_\alpha m_\alpha \left(|\vec{r}_{b,\alpha}|^2 \delta_{ij} - r^\alpha_i r^\alpha_j \right) \equiv$$ "Moment of inertia tensor"

"Tensor" $\equiv$ Matrix whose indices are directions of real space

$\Downarrow$

Changes when you change co-ordinates

Example: Dumbell (Not me)



$$\underset{\sim}{I} = \begin{array}{cc} & \begin{array}{ccc} x & \quad y \quad & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix} \end{array}$$

~~Diagonal~~ elements = ~~Moment of inertia ab~~

$\underbrace{I_{ii}}_{\text{no sum}}$

New co-ords:

Principal axes = choice of co-ordinates such that $\underset{\sim}{I}$ diagonal:

$$\underset{\sim}{I} = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$$

How to find?

Easy way: symmetry: Choose axes such that mass distribution is symmetric

Not always obvious, can't always find one.

Brute force way, <u>always</u> works:

Find eigenvectors + eigenvalues of $\underset{\sim}{I}$.

I.e., Given

$$\underset{\sim}{I} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

none $= 0$

Find directions $\hat{x}_\alpha$, ~~wth~~ $\alpha = 1, 2, 3$ s.t.

$\underset{\sim}{I}\,\hat{x}_\alpha = \lambda_\alpha \hat{x}_\alpha$: λ_α : Eigenvalue of I, $\alpha = 1, 2, 3$

$\hat{x}_\alpha$: Eigen direction.

Note: for <u>symmetric</u> matrices, ~~$\hat{x}_\alpha \cdot \hat{x}_B = 0$, α~~

$\hat{x}_\alpha \perp \hat{x}_B$, $\alpha \neq B$

Proof:

$$\hat{X}_\alpha \cdot \hat{X}_B = \frac{1}{\lambda_B}\hat{X}_\alpha \cdot (\lambda_B \hat{X}_B) = \frac{1}{\lambda_B}\hat{X}_\alpha \cdot \underset{\sim}{I}\,\hat{X}_B$$

$$= \frac{1}{\lambda_B} X_{\alpha i}\, I_{ij}\, X_{Bj} \Rightarrow \boxed{I_{ij} X_{\alpha i} X_{Bj} = \lambda_B\, \hat{X}_\alpha \cdot \hat{X}_B} \quad (\mathrm{I})$$

$$\hat{X}_B \cdot \hat{X}_\alpha = \frac{1}{\lambda_\alpha}\hat{X}_B \cdot (\lambda_\alpha \hat{X}_\alpha) = \frac{1}{\lambda_\alpha}\hat{X}_B \cdot \underset{\sim}{I}\,\hat{X}_\alpha$$

$$= \frac{1}{\lambda_\alpha} I_{ij}\, X_{Bi}\, X_{\alpha j}$$

$$= \frac{1}{\lambda_\alpha} I_{ji}\, X_{\alpha j}\, X_{Bi}$$

can switch, 'cause multiplication is commutat.

if $\underset{\sim}{I}$ symmetric

$$= \frac{1}{\lambda_\alpha} I_{ij}\, X_{\alpha i}\, X_{Bj}$$

rename dummy indices

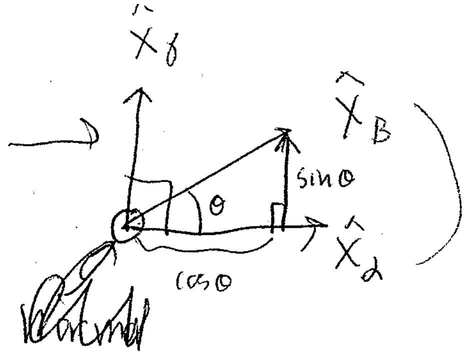
$$\Rightarrow \boxed{I_{ij} X_{\alpha i} X_{Bj} = \lambda_\alpha\, \hat{X}_B \cdot \hat{X}_\alpha = \lambda_\alpha\, \hat{X}_\alpha \cdot \hat{X}_B} \quad (\mathrm{II})$$

Subtract (I) from (II)

$$\Rightarrow (\lambda_\alpha - \lambda_B)\, \hat{X}_\alpha \cdot \hat{X}_B = 0 \Rightarrow \hat{X}_\alpha \cdot \hat{X}_B = 0 \text{ \underline{unless} } \lambda_\alpha = \lambda_B$$

If $\lambda_\alpha = \lambda_B$, find new vector $\hat{x}_\gamma$, in $\hat{x}_\alpha, \hat{x}_B$ plane, $\perp$ to $\hat{x}_\alpha$

also in Blackboard plane



Both in blackboard plane

$$\hat{x}_\gamma = a\hat{x}_B + b\hat{x}_\alpha \quad , \quad a = ? \quad b = ?$$

$$\Rightarrow \underset{\sim}{I}\,\hat{x}_\gamma = a\,\underset{\sim}{I}\,\hat{x}_B + b\,\underset{\sim}{I}\,\hat{x}_\alpha = a\lambda_B\hat{x}_B + b\lambda_\alpha\hat{x}_\alpha$$

$$\underset{\substack{\uparrow \\ \lambda_B = \lambda_\alpha}}{=} a\lambda_\alpha\hat{x}_B + b\lambda_\alpha\hat{x}_\alpha = \lambda_\alpha(a\hat{x}_B + b\hat{x}_\alpha)$$

$$= \lambda_\alpha\hat{x}_\gamma$$

$\Rightarrow$ $\hat{x}_\gamma$ is also an eigenvector, <u>and</u> its orthogonal to $\hat{x}_\alpha$

$\Rightarrow (\hat{x}_\alpha, \hat{x}_\gamma, \hat{x}_\alpha \times \hat{x}_\gamma)$ are principal axes

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~~In this~~ In general orthog. syst. $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$

In this co-ord system, $\hat{x}_1 = (1,0,0)$, $\hat{x}_2 = (0,1,0)$, $\hat{x}_3 = (0,0,1)$

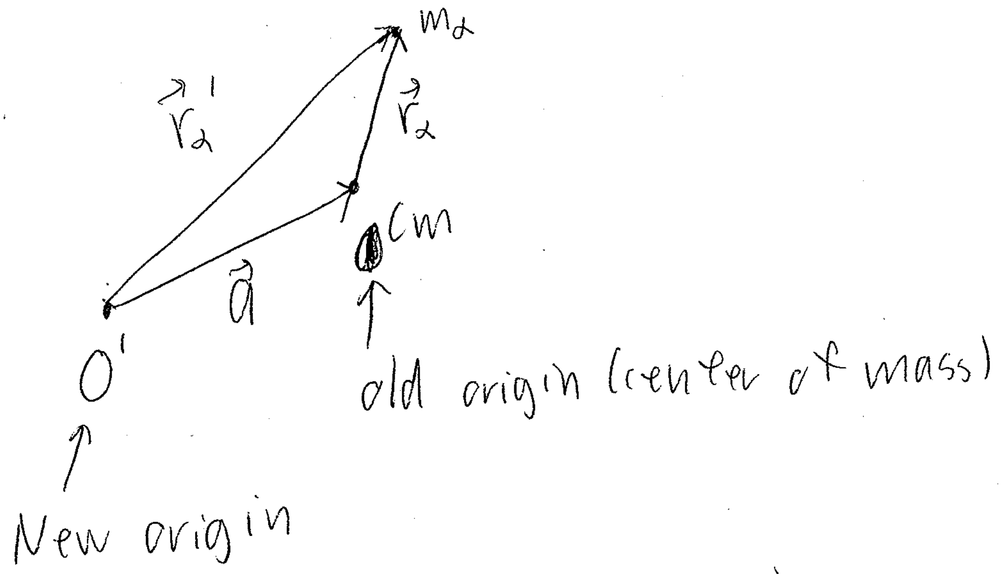
$$\underset{\sim}{I}\,\hat{x}_1 = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} I_1 \\ 0 \\ 0 \end{pmatrix} = \lambda_1\hat{x}_1 = \begin{pmatrix} \lambda_1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow I_1 = \lambda_1$$

unegiven

In general Orthog. syst. of princ. axes,

$$\underset{\sim}{I} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

So, $\underset{\sim}{I}$ depends on choice of axis <u>directions</u>. It also depends on choice of <u>origin</u>. Easiest ~~the~~ origin: center of mass



$$I_{ij} = \sum_\alpha m_\alpha \left(|\vec{r}_\alpha|^2 \delta_{ij} - r_i^\alpha r_j^\alpha \right)$$

$$I_{ij}' = \sum_\alpha m_\alpha \left(|\vec{r}_\alpha'|^2 \delta_{ij} - r_i'^\alpha r_j'^\alpha \right)$$

7.1 3c

cf) $r'^{\alpha}_i = r^{\alpha}_i + a_i$

$$\Rightarrow \quad r'^{\alpha}_i \, r'^{\alpha}_j = r^{\alpha}_i r^{\alpha}_j + a_i r^{\alpha}_j + a_j r^{\alpha}_i + a_i a_j$$

$$|\vec{r}_\alpha{}'|^2 = |\vec{r}_\alpha|^2 + 2\vec{a}\cdot\vec{r}_\alpha + |\vec{a}|^2$$

$$\Rightarrow \quad I_{ij}' = I_{ij} + \overbrace{\sum_\alpha m_\alpha \, \vec{a}\cdot\vec{r}_\alpha}^{\mathrm{I}} + \overbrace{\sum_\alpha m_\alpha |\vec{a}|^2 \delta_{ij}}^{\mathrm{II}}$$

$$- \underbrace{\sum_\alpha m_\alpha a_i r^{\alpha}_j}_{\mathrm{III}} - \underbrace{\sum_\alpha m_\alpha r^{\alpha}_i a_j}_{\mathrm{IV}} - \sum_\alpha m_\alpha a_i a_j$$

$\mathrm{I} =$

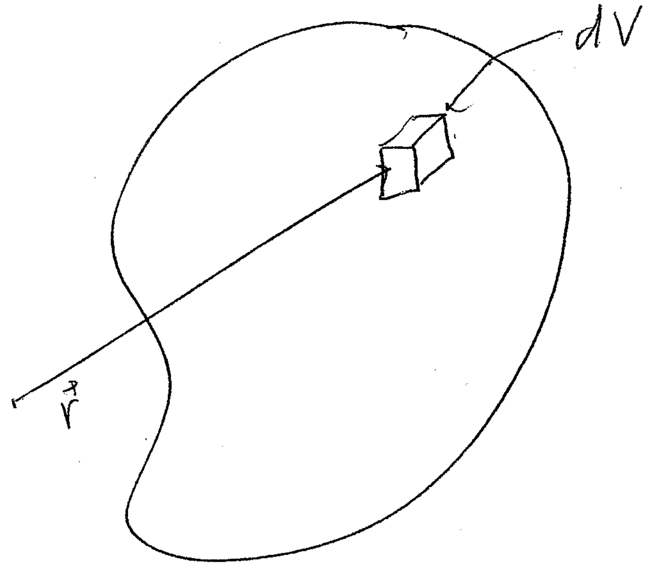
$\mathrm{II} =$

$\mathrm{III} =$

$\mathrm{IV} =$

$$\Rightarrow \boxed{I_{ij}' = I_{ij} + M_T \left(|\vec{a}|^2 \delta_{ij} - a_i a_j \right)}$$

Continuous media:



$dm = \rho\, dV$

$$\Rightarrow I_{ij} = \int dV\, \rho(\vec{r})\,(r^2 \delta_{ij} - r_i r_j)$$

So,

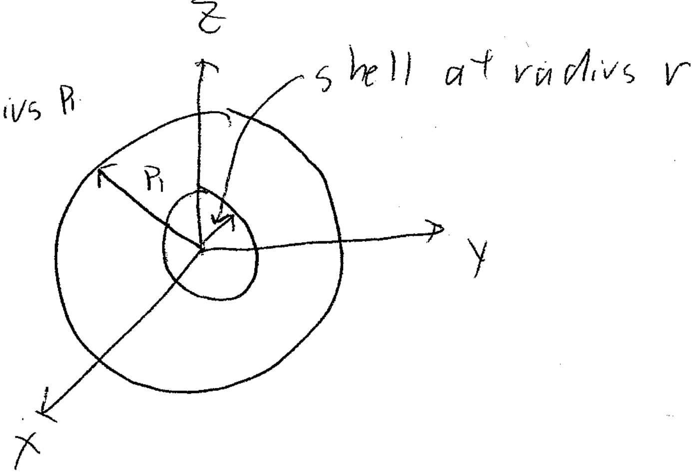
$$T = \underbrace{\tfrac{1}{2} M |\vec{V}_c|^2}_{\text{"Center of mass K.E."}} + \underbrace{M\, \vec{V}_{cm} \cdot \vec{\Omega} \times (\vec{R}_{cm} - \vec{R}_c)}_{\text{"Center of mass rotational K.E."}} + \tfrac{1}{2} \underbrace{I_{ij}\, \Omega_i \Omega_j}_{\text{rotational K.E.}}$$

Examples:

Uniform density

1) Sphere:

mass m, radius R



$$\cancel{I_{xx} = \rho \int}$$

$$I_{ij} = \int d^3r \; \rho\left(|\vec{r}|^2 \delta_{ij} - r_i r_j\right)$$

$$= \rho \int_0^R r^2 dr \int \underbrace{d\Omega}_{\int \text{over } \measuredangle\text{'s}} \left(r^2 \delta_{ij} - r_i r_j\right)$$

$$\int d\Omega \, r_i r_j = \begin{cases} & , \; i \neq j \\ & , \; i = j \end{cases}$$

$$\sum_{i=j} \int d\Omega \, r_i r_j =$$

7.16

$$\Rightarrow I_{ij} = \frac{8\pi}{3}\rho\,\delta_{ij}\int_0^R r^4\,dr = \frac{8\pi}{3}\rho\,\frac{\delta_{ij}R^5}{5} = \frac{2}{5}\left(\frac{4\pi\rho R^3}{3}\right)R^2$$

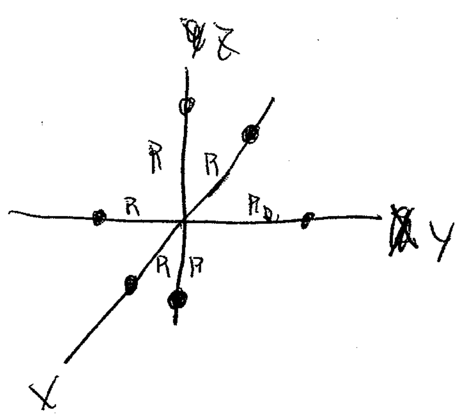
What's ρ i.t.o. m, R?

$$\Rightarrow I_{ij} = \frac{2}{5}mR^2\,\delta_{ij} = \begin{pmatrix} \frac{2mR^2}{5} & 0 & 0 \\ 0 & \frac{2mR^2}{5} & 0 \\ 0 & 0 & \frac{2mR^2}{5} \end{pmatrix}$$

Principal axes?

Isotropy of I_{ij} reflects Isotropy of mass distribution.

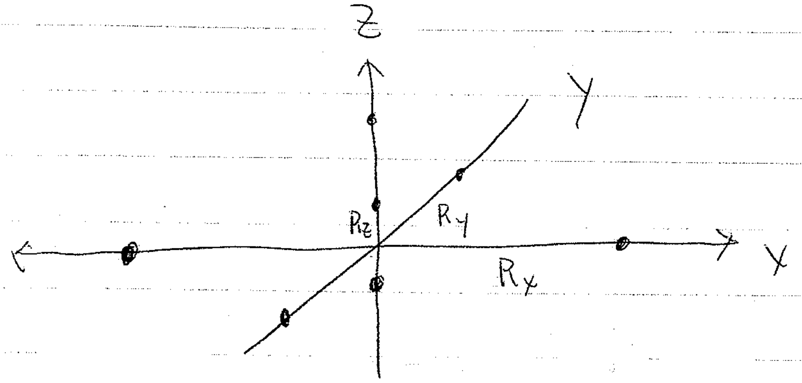
Note: ~~Insens~~ Not completely sensitive: Jack



$I_{ij} = ?$

Anisotropic distribution but isotropic I_{ij}

Now, asymmetric Jack:



$R_z < R_y < R_x$

$$I_{ij} = \begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix}$$

Note: For principal axes, smaller moment of inertia $\Rightarrow$ mass distribution closer to that axis.