

# Physics 623 Midterm

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This one really made me do my research.

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## 4.1.2. Polaritons

We consider a polarization field,  $\mathbf{P}(\mathbf{x}, t)$ , that determines the sources of the electromagnetic fields according to

$$\begin{aligned} \mathbf{j} &= \partial_t \mathbf{P} \\ \rho &= -\nabla \cdot \mathbf{P} \end{aligned} \tag{1}$$

and which satisfies

$$(\partial_t^2 + \omega_0^2) \mathbf{P}(\mathbf{x}, t) = a^2 \mathbf{E}(\mathbf{x}, t) \tag{2}$$

- a) We claim that Maxwell's equations combined with (2) have solutions given by both longitudinal, ( $\mathbf{k} \parallel \mathbf{E}, \mathbf{P}$ ), and transverse, ( $\mathbf{k} \perp \mathbf{E}, \mathbf{P}$ ), monochromatic plane waves. From (1), Maxwell's equations become

$$\nabla \cdot \mathbf{B} = 0 \tag{M1}$$

$$\frac{1}{c} \partial_t \mathbf{B} + \nabla \times \mathbf{E} = \mathbf{0} \tag{M2}$$

$$\nabla \cdot \mathbf{E} = 4\pi \rho = -4\pi \nabla \cdot \mathbf{P} \tag{M3}$$

$$-\frac{1}{c} \partial_t \mathbf{E} + \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} = \frac{4\pi}{c} \partial_t \mathbf{P} \tag{M4}$$

*Ansatz:* Let  $\mathbf{E}(\mathbf{x}, t)$  and  $\mathbf{B}(\mathbf{x}, t)$  be monochromatic plane waves given by

$$\begin{aligned} \mathbf{E}(\mathbf{x}, t) &= \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \\ \mathbf{B}(\mathbf{x}, t) &= \mathbf{B}_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \end{aligned} \tag{3}$$

By (2), we can write  $\mathbf{P}(\mathbf{x}, t)$  in terms of  $\mathbf{E}(\mathbf{x}, t)$ . Let  $\mathbf{P}(\mathbf{x}, \tilde{\omega})$  be the time-domain Fourier transform of  $\mathbf{P}(\mathbf{x}, t)$ . Then we have

$$\begin{aligned}
(\partial_t^2 + \omega_0^2)\mathbf{P}(\mathbf{x}, t) &= a^2 \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \\
(\omega_0^2 - \tilde{\omega}^2)\mathbf{P}(\mathbf{x}, \tilde{\omega}) &= 2\pi a^2 \mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{x}} \delta(\tilde{\omega} + \omega) \\
\mathbf{P}(\mathbf{x}, t) &= \int d\tilde{\omega} \frac{a^2 \mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{x}} \delta(\tilde{\omega} + \omega)}{(\omega_0^2 - \tilde{\omega}^2)} \\
\mathbf{P}(\mathbf{x}, t) &= \frac{a^2}{\omega_0^2 - \omega^2} \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \\
\mathbf{P}(\mathbf{x}, t) &= \frac{a^2}{\omega_0^2 - \omega^2} \mathbf{E}(\mathbf{x}, t)
\end{aligned} \tag{4}$$

Define  $k := |\mathbf{k}|$  and  $\hat{\mathbf{n}} := \hat{\mathbf{k}} = \mathbf{k}/k$ .

- (i) Suppose  $\mathbf{k} \perp \mathbf{E}, \mathbf{P}$ . We will use the fact that, in a nonconducting medium,  $\mathbf{B} = \sqrt{\epsilon(\omega)} \hat{\mathbf{n}} \times \mathbf{E}$ , where  $\epsilon(\omega)$  is the dielectric function such that  $\mathbf{E} + 4\pi\mathbf{P} = \epsilon(\omega)\mathbf{E}$ .<sup>1</sup> So we have  $\epsilon(\omega) = 1 + 4\pi a^2/(\omega_0^2 - \omega^2)$ .

(M1)

$$\begin{aligned}
\nabla \cdot \mathbf{B} &= i\mathbf{k} \cdot \mathbf{B} \\
&= ik\sqrt{\epsilon(\omega)}\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} \times \mathbf{E}) \\
&= 0
\end{aligned} \tag{5}$$

(M2)

$$\begin{aligned}
\frac{1}{c}\partial_t \mathbf{B} + \nabla \times \mathbf{E} &= -i\omega \frac{\sqrt{\epsilon(\omega)}}{c} (\hat{\mathbf{n}} \times \mathbf{E}) + i\mathbf{k} \times \mathbf{E} \\
&= i \left( k - \frac{\omega \sqrt{\epsilon(\omega)}}{c} \right) \hat{\mathbf{n}} \times \mathbf{E} \\
&= 0
\end{aligned} \tag{6}$$

provided  $\omega = ck/\sqrt{\epsilon(\omega)}$ .

(M3)

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= i\mathbf{k} \cdot \mathbf{E} \\
&= ik\hat{\mathbf{n}} \cdot \mathbf{E} \\
&= 0 \\
&= ik\hat{\mathbf{n}} \cdot \mathbf{P}
\end{aligned} \tag{7}$$

since  $\mathbf{P}$  is just a scaled copy of  $\mathbf{E}$  and both are orthogonal to  $\mathbf{k}$ .

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<sup>1</sup>p. 297, J.D. Jackson, *Classical Electrodynamics*, 3rd Edition.

(M4)

$$\begin{aligned}
-\frac{1}{c}\partial_t \mathbf{E} + \nabla \times \mathbf{B} &= \frac{i\omega}{c}\mathbf{E} + ik\sqrt{\epsilon(\omega)}\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{E}) \\
&= i\left(\frac{\omega}{c} - k\sqrt{\epsilon(\omega)}\right)\mathbf{E} \\
&= \frac{4\pi}{c}\partial_t \mathbf{P} \\
&= -i\frac{4\pi\omega}{c}\frac{a^2}{\omega_0^2 - \omega^2}\mathbf{E}
\end{aligned} \tag{8}$$

again, provided  $\omega = ck/\sqrt{\epsilon(\omega)}$ .

Thus, Maxwell's equations have transverse monochromatic plane wave solutions in this paradigm. Additionally, the waves satisfy the frequency-wavenumber relation,  $\omega = ck/\sqrt{\epsilon(\omega)}$ .

- (ii) Now suppose  $\mathbf{k} \parallel \mathbf{E}, \mathbf{P}$ . In this case, since  $\mathbf{B} = \sqrt{\epsilon(\omega)}\hat{\mathbf{n}} \times \mathbf{E}$ , we have  $\mathbf{B} = \mathbf{0}$ . So Maxwell's equations become

(M1)

$$\nabla \cdot \mathbf{B} = 0 \tag{9}$$

(M2)

$$\begin{aligned}
\frac{1}{c}\partial_t \mathbf{B} + \nabla \times \mathbf{E} &= \nabla \times \mathbf{E} \\
&= ik\hat{\mathbf{n}} \times \mathbf{E} \\
&= \mathbf{0}
\end{aligned} \tag{10}$$

since  $\hat{\mathbf{n}} \parallel \mathbf{E}$ .

(M3)

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= i\mathbf{k} \cdot \mathbf{E} \\
&= ik|\mathbf{E}| \\
&= -ik\frac{4\pi a^2}{\omega_0^2 - \omega^2}|\mathbf{E}| \\
&= -4\pi\nabla \cdot \mathbf{P}
\end{aligned} \tag{11}$$

provided  $\omega^2 = 4\pi a^2 + \omega_0^2$ .

(M4)

$$\begin{aligned}
-\frac{1}{c}\partial_t \mathbf{E} + \nabla \times \mathbf{B} &= \frac{i\omega}{c} \mathbf{E} \\
&= -\frac{i\omega}{c} \frac{4\pi a^2}{\omega_0^2 - \omega^2} \mathbf{E} \\
&= \frac{4\pi}{c} \partial_t \mathbf{P}
\end{aligned} \tag{12}$$

again, provided  $\omega^2 = 4\pi a^2 + \omega_0^2$ .

Thus, Maxwell's equations have longitudinal plane wave solutions whose frequency satisfies  $\omega^2 = 4\pi a^2 + \omega_0^2$ .

- b) We claim that, in the long-wavelength limit, transverse waves are photon-like. As  $k \rightarrow 0$ ,  $\omega$  must also go to zero since the wave velocity may not exceed  $c$ . Therefore,  $\lim_{k \rightarrow 0} \omega(k) = 0$ . In this limit, we can assume  $\omega^2 \approx 0$ . In this case, the frequency-wavenumber relation for transverse waves becomes

$$\begin{aligned}
\omega &= \frac{ck}{\sqrt{\epsilon(\omega)}} \\
\Rightarrow \omega(k \rightarrow 0) &= \frac{ck}{\sqrt{1 + \frac{4\pi a^2}{\omega_0^2}}} \\
\omega(k \rightarrow 0) &= \frac{ck}{n}
\end{aligned} \tag{13}$$

where  $n = \sqrt{1 + 4\pi a^2/\omega_0^2}$  is the index of refraction.

- c) Let  $\omega_- = \omega_0$  and let  $\omega_+ = \sqrt{\omega_0^2 + 4\pi a^2}$ . We claim that there can be no wave propagation in the frequency band  $\omega_- < \omega < \omega_+$ . Since we assume  $n \in \mathbb{R}$ , in order for electromagnetic waves to propagate, we must have  $\epsilon(\omega) > 0$ . With this requirement we have

$$\begin{aligned}
\epsilon(\omega) &> 0 \\
1 + \frac{4\pi a^2}{\omega_0^2 - \omega^2} &> 0 \\
\omega_0^2 - \omega^2 + 4\pi a^2 &> 0, \text{ if } \omega_0 > \omega \\
\omega_0^2 - \omega^2 + 4\pi a^2 &< 0, \text{ if } \omega_0 < \omega \\
\Rightarrow \omega_0^2 + 4\pi a^2 &> \omega^2, \text{ if } \omega_0 > \omega \\
\omega_0^2 + 4\pi a^2 &< \omega^2, \text{ if } \omega_0 < \omega
\end{aligned} \tag{14}$$

But since  $\omega_0^2 + 4\pi a^2 > \omega_0^2$ , in order for waves to propagate, we have either  $\omega_0 > \omega$  or  $\sqrt{\omega_0^2 + 4\pi a^2} < \omega$ . Therefore, no wave propagation is possible in the range  $\omega_0 < \omega < \sqrt{\omega_0^2 + 4\pi a^2}$ . Moreover, we have

$$\begin{aligned}\frac{\omega_+^2}{\omega_-^2} &= \frac{\omega_0^2 + 4\pi a^2}{\omega_0^2} \\ &= 1 + \frac{4\pi a^2}{\omega_0^2} \\ &= \epsilon(\omega = 0)\end{aligned}\tag{15}$$

- d) The frequency-wavenumber relation for this system is, in general,  $\omega(k) = ck/\sqrt{\epsilon(\omega(k))}$ . A plot of this relation is shown below.

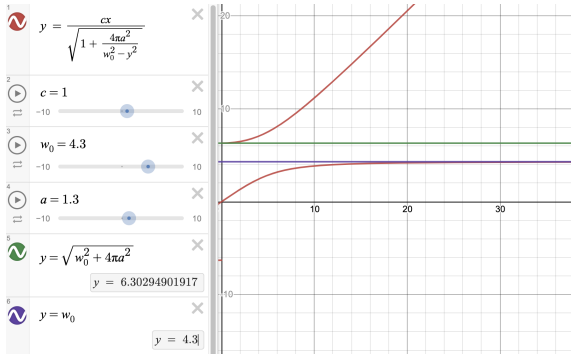


Figure 1: Sorry this isn't bigger and for the overall quality. The y-axis shows  $\omega$  and the x-axis is  $k$ .

In the  $k \rightarrow 0$  limit, and for  $\omega > \sqrt{\omega_0^2 + 4\pi a^2}$ , we have  $\omega \rightarrow \sqrt{\omega_0^2 + 4\pi a^2}$ . For  $\omega < \omega_0$ , we have  $\omega \rightarrow ck/\sqrt{\epsilon(0)}$ .

In the  $k \rightarrow \infty$  limit, and for  $\omega > \sqrt{\omega_0^2 + 4\pi a^2}$ , we have  $\omega \rightarrow ck$ . For  $\omega < \omega_0$ , we have  $\omega \rightarrow \omega_0$ .

## References

- (1) J. Schwinger et al., *Classical Electrodynamics*.
- (2) J.D. Jackson, *Classical Electrodynamics*.

