1)

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2)$$

Following the procedure on page 196 in the notes, we have

set $\tilde{x} = \frac{x}{x_s}$ , $\tilde{y} = \frac{y}{x_s}$

$\tilde{p}_x = \frac{p_x}{p_s}$ , $\tilde{p}_y = \frac{p_y}{p_s}$

$\tilde{t} = \frac{t}{t_s}$ , $\tilde{H} = \frac{H}{E_s}$

$\tilde{V} = \frac{V}{E_s}$

$$\tilde{H} = \frac{p_s^2}{m E_s} \frac{\tilde{p}^2}{2} + \tilde{V}(\tilde{x} x_s, \tilde{y} x_s)$$

$$\tilde{V}(\tilde{x} x_s, \tilde{y} x_s) = \frac{1}{2} \frac{m x_s^2}{E_s} (\omega_x^2 \tilde{x}^2 + \omega_y^2 \tilde{y}^2)$$

$$= \frac{1}{2} \frac{m x_s^2}{E_s} \omega_x^2 \left( \tilde{x}^2 + \frac{\omega_y^2}{\omega_x^2} \tilde{y}^2 \right)$$

$$i \frac{\partial}{\partial \tilde{t}} = \frac{E_s t_s}{\hbar} \tilde{H}$$

$$[\tilde{x}, \tilde{p}_x] = [\tilde{y}, \tilde{p}_y] = i\left(\frac{\hbar}{x_s p_s}\right)$$

$$[\tilde{x}, \tilde{p}_y] = [\tilde{y}, \tilde{p}_x] = 0$$

$$x_s p_s = \hbar$$

$$E_s t_s = \hbar$$

$$m \omega_x^2 x_s^2 = E_s$$

$$p_s^2 = m E_s$$

So, since this is the same as before,

$$\boxed{\begin{aligned} x_s &= \sqrt{\frac{\hbar}{m\omega_x}} \\ p_s &= \sqrt{m\hbar\omega_x} \\ E_s &= \hbar\omega_x \\ t_s &= \frac{1}{\omega_x} \end{aligned}}$$

So the Hamiltonian becomes

$$\boxed{\begin{aligned} &H = \frac{1}{2}\left(p_x^2 + p_y^2 + x^2 + \tilde{\omega}_y^2 y^2\right) \\ &\text{Where} \quad \tilde{\omega}_y := \frac{\omega_y}{\omega_x} \end{aligned}}$$

2)

$$\partial_t \langle Q \rangle = -\frac{i}{\hbar} \langle [Q, H] \rangle$$

$\langle x \rangle$ :

$$\partial_t \langle x \rangle = -\frac{i}{\hbar} \langle [x, H] \rangle$$

$$\partial_t \langle x \rangle = -\frac{i}{\hbar} \langle [x, \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 ]$$

$$\partial_t \langle x \rangle = -\frac{i}{2m\hbar} \langle [x, p^2] \rangle$$

$$\partial_t \langle x \rangle = -\frac{i}{2m\hbar} 2i\hbar \langle p \rangle$$

$$\boxed{\partial_t \langle x \rangle = \frac{1}{m} \langle p \rangle}$$

$\langle p \rangle$ :

$$\partial_t \langle p \rangle = -\frac{i}{\hbar} \langle [p, H] \rangle$$

$$\partial_t \langle p \rangle = -\frac{i}{\hbar} \langle [p, \frac{1}{2} m\omega^2 x^2] \rangle$$

$$\partial_t \langle p \rangle = -\frac{im\omega^2}{2\hbar} \langle [p, x^2] \rangle$$

$$\partial_t \langle p \rangle = \frac{im\omega^2}{2\hbar} 2i\hbar \langle x \rangle$$

$$\boxed{\partial_t \langle p \rangle = -m\omega^2 \langle x \rangle}$$

$V_x$:

$$\partial_t V_x = \partial_t \langle x^2 \rangle - \partial_t \langle x \rangle^2$$

$$\partial_t \langle x^2 \rangle = -\frac{i}{\hbar} \langle [x^2, \frac{p^2}{2m}] \rangle$$

$$\partial_t \langle x^2 \rangle = \frac{-i}{2m\hbar} \langle [x^2, p^2] \rangle$$

$$[x^2, p^2] = 2i\hbar [x, p]_+$$

$$\partial_t \langle x^2 \rangle = \frac{1}{m} \langle [x, p]_+ \rangle$$

$$\partial_t \langle x \rangle^2 = \frac{2}{m} \langle x \rangle \langle p \rangle$$

So

$$\partial_t V_x = \frac{1}{m} \left( \langle [x, p]_+ \rangle - 2 \langle x \rangle \langle p \rangle \right)$$

$$\boxed{\partial_t V_x = \frac{2}{m} C_{xp}}$$

$V_p$ :

$$\partial_t V_p = \partial_t \langle p^2 \rangle - \partial_t \langle p \rangle^2$$

$$\partial_t V_p = \frac{-i}{\hbar} \langle [p^2, \frac{m}{2} \omega^2 x^2] \rangle - 2 \langle p \rangle \partial_t \langle p \rangle$$

$$\partial_t V_p = -\frac{i m \omega^2}{2\hbar} \langle [p^2, x^2] \rangle - 2 \langle p \rangle \partial_t \langle p \rangle$$

$$\partial_t V_p = \frac{i m \omega^2}{2\hbar} 2i\hbar \langle [x, p]_+ \rangle - 2 \langle p \rangle \partial_t \langle p \rangle$$

$$\partial_t V_p = -m \omega^2 \left( \langle [x, p]_+ \rangle - 2 \langle x \rangle \langle p \rangle \right)$$

$$\boxed{\partial_t V_p = -2m\omega^2 C_{xp}}$$

$C_{xp}$ :

$$\partial_t C_{xp} = \partial_t \frac{\langle xp \rangle}{2} + \partial_t \frac{\langle px \rangle}{2} - \partial_t \left( \langle x \rangle \langle p \rangle \right)$$

$$\partial_t \langle xp \rangle = -\frac{i}{\hbar} \langle [xp, H] \rangle$$

$$\partial_t \langle xp \rangle = \frac{-i}{\hbar} \langle [xp, \frac{p^2}{2m} + \frac{1}{2} m \omega x^2] \rangle$$

$$\partial_t \langle xp \rangle = -\frac{i}{\hbar}\Big( \frac{1}{2m} \langle [xp, p^2]$$

$$+ \frac{1}{2} m\omega^2 \langle [xp, x^2] \rangle \Big)$$

$$= -\frac{i}{\hbar}\Big( \frac{1}{2m} \langle [x, p^2] p \rangle$$

$$+ \frac{1}{2} m\omega^2 \langle x [p, x^2] \rangle \Big)$$

$$= -\frac{i}{\hbar}\Big( \frac{1}{2m} 2i\hbar \langle p^2 \rangle$$

$$- \frac{1}{2} m\omega^2 \, 2i\hbar \langle x^2 \rangle \Big)$$

$$= \frac{1}{m} \langle p^2 \rangle - m\omega^2 \langle x^2 \rangle$$

$$\partial_t \langle px \rangle = -\frac{i}{\hbar}\Big( \frac{1}{2m} \langle [px, p^2] \rangle$$

$$+ \frac{1}{2} m\omega^2 \langle [px, x^2] \rangle \Big)$$

$$= -\frac{i}{\hbar}\Big( \frac{1}{2m} 2i\hbar \langle p^2 \rangle$$

$$- \frac{1}{2} m\omega^2 \, 2i\hbar \langle x^2 \rangle$$

$$= \frac{1}{m} \langle p^2 \rangle - m\omega^2 \langle x^2 \rangle$$

$$\partial_t \langle x \rangle \langle p \rangle = \frac{1}{m} \langle p \rangle^2 - m\omega^2 \langle x \rangle^2$$

$$\partial_t C_{xp} = \frac{1}{m} \langle p^2 \rangle - m\omega^2 \langle x^2 \rangle - \frac{1}{m} \langle p \rangle^2 + m\omega^2 \langle x \rangle^2$$

$$\boxed{\partial_t C_{xp} = \frac{1}{m} V_p - m\omega^2 V_x}$$

3)

We can use the same equations of motion for a QHO where $\omega = 0$

$$\dot{V}_x = \frac{2}{m} C_{xp}$$

$$\dot{V}_p = 0 \longrightarrow V_p \text{ is constant}$$

$$\dot{C}_{xp} = \frac{1}{m} V_p$$

$$\longrightarrow \ddot{V}_x = \frac{2}{m^2} V_p$$

Since the wave packet is initially at minimum uncertainty, we have

$$V_x(0) V_p(0) - C_{xp}(0)^2 = V_x(0) V_p(0) = \frac{\hbar^2}{4}$$

and since the wave packet is Gaussian, $V_x(0) = \sigma^2$.

So

$$\ddot{V}_x = \frac{2}{m^2} V_p = \frac{\hbar^2}{2m^2\sigma^2}$$

Additionally, since

$$V_x(0)V_p(0) - C_{xp}(0)^2 = V_x(0)V_p(0),$$

we have

$C_{xp}(0) = 0$. So, since

$$\dot{V}_x = \frac{2}{m} C_{xp},$$ we have

$$\dot{V}_x(0) = 0$$

Thus,

$$\ddot{V}_x(0) = \frac{\hbar^2}{2m^2\sigma^2}, \quad V_x(0) = \sigma^2, \quad \dot{V}_x(0) = 0$$

Therefore,

$$\boxed{\begin{aligned} V_x &= \frac{\hbar^2}{4m^2\sigma^2} t^2 + \dot{V}_x(0)t + V_x(0) \\ &= \frac{\hbar^2}{4m^2\sigma^2} t^2 + \sigma^2 \\ \longrightarrow \quad \sigma_x(t) &= \sqrt{V_x} = \sigma\sqrt{1 + \frac{\hbar^2 t^2}{4m^2\sigma^2}} \end{aligned}}$$

4)

We will show that

$$\frac{d}{dt}\left(V_x V_p - C_{xp}^2\right) = 0$$

$$\frac{d}{dt}\left(V_x V_p - C_{xp}^2\right) = \dot{V}_x V_p + V_x \dot{V}_p - 2 C_{xp} \dot{C}_{xp}$$

From problem 1 we have

$$\dot{V}_x = \frac{2}{m} C_{xp}$$

$$\dot{V}_p = -2m\omega^2 C_{xp}$$

$$\dot{C}_{xp} = \frac{1}{m} V_p - m\omega^2 V_x$$

So

$$\boxed{\begin{aligned} &\frac{d}{dt}\left(V_x V_p - C_{xp}^2\right) \\ &= \frac{2}{m} C_{xp} V_p - 2m\omega^2 C_{xp} V_x \\ &\quad - 2 C_{xp}\left(\frac{1}{m} V_p - m\omega^2 V_x\right) \\ &= 0 \end{aligned}}$$

So $\frac{d}{dt}(V_x V_p - C x p^2) = 0$, therefore

$V_x V_p - C x p^2$ is constant.