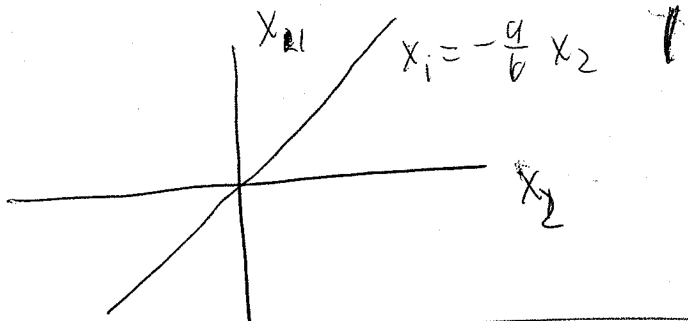
~~Back to our ex~~

When $\det \underset{\sim}{M} = 0$,

$$x_1 = -\frac{d}{c} x_2 \quad \cancel{= -\frac{b}{a} x_1} \quad \text{from (2)}$$

$$x_1 = -\frac{a}{b} x_2 \quad \text{from (1)}$$

but: if $\det \underset{\sim}{M} = 0$, these are <u>same</u> condition



Our example:

$$\det \underset{\sim}{M} = \begin{vmatrix} (m_1+m_2)[l_1^2\lambda^2 + l_1 g] & m_2 l_1 l_2 \lambda^2 \\ m_2 l_1 l_2 \lambda^2 & m_2[l_2^2\lambda^2 + g l_2] \end{vmatrix} = 0$$

$$\Rightarrow m_2(m_1+m_2)(l_1^2 \lambda^2 + l_1 g)(l_2^2\lambda^2 + l_2 g) - m_2^2 l_1^2 l_2^2 \lambda^4 = 0$$

Same as earlier condition (try it and see!)

Higher dimensions (bigger N)

$\underset{\sim}{M} \, \vec{x} = 0$ only has solution if

$\underset{\sim}{M}$: $N \times N$; $\vec{x}$: N-comp vector

$$\det \underset{\sim}{M} = \begin{vmatrix} M_{11} & M_{12} & \cdots & M_{1N} \\ M_{21} & M_{22} & \cdots & M_{2N} \\ M_{31} & & \ddots & \\ \vdots & & & \\ M_{N1} & & & \end{vmatrix} \equiv M_{11} \begin{vmatrix} M_{22} & \cdots & M_{2N} \\ M_{32} & \cdots & M_{3N} \\ \vdots & & \\ M_{N1} & \cdots & \end{vmatrix}$$

$$- M_{12} \begin{vmatrix} M_{21} & M_{23} & \\ M_{31} & M_{33} & \\ \vdots & \vdots & \end{vmatrix}$$

$$+ M_{13} \begin{vmatrix} M_{21} & M_{22} & M_{24} & \cdots \\ & & & \end{vmatrix}$$

$$- M_{14} \begin{vmatrix} \\ \end{vmatrix}$$

$$+ \cdots$$

By recursively applying this formula, can (with enough work) compute $\det \underset{\sim}{M}$ for any $N \times N$ matrix

Simplest ~~exan~~ extension: 3×3

$$\underset{\sim}{M} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\det \underset{\sim}{M} = \left| \begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array} \right.$$

aei

$-$ these

~~add~~ $+$ these

$= aei + bfg + cdh - bdi - afh - ceg$

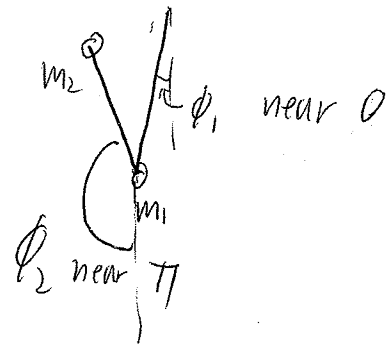
~~eigen~~ This leads to polynomial equation for eigenvalue λ. $\det \underset{\sim}{M}(\lambda) = 0$

In general, N d.o.f $\Rightarrow 2N$ λ_α's $(\alpha = 1, 2, \ldots 2N)$

In our example, $\lambda_1 = i\omega_+$, $\lambda_2 = -i\omega_+$, $\lambda_3 = i\omega_-$, $\lambda_4 = -i\omega_-$. In general, don't always come in pairs

Applying this approach to <u>unstable</u> fixed point:

$\phi_1 = 0, \ \phi_2 = \pi$



$\phi_1 \ll 1, \ \phi_2 = \pi + \delta\phi_2, \ \ \delta\phi_2 \ll 1$

Linearized E.O.M.'s :

$\det \underset{\sim}{M} = 0$

$$\Rightarrow \begin{vmatrix} (\lambda^2 + w_1^2) & -\alpha \frac{l_2}{l_1} \lambda^2 \\ -\frac{l_1}{l_2} \lambda^2 & (\lambda^2 - w_2^2) \end{vmatrix} = 0$$

$$\Rightarrow (\lambda^2 + w_1^2)(\lambda^2 - w_2^2) - \alpha \lambda^4 = 0$$

$$\Rightarrow \lambda^4 (1-\alpha) + \lambda^2 (w_1^2 - w_2^2) - w_1^2 w_2^2 = 0$$

$$\Rightarrow \boxed{\lambda^2 = \frac{w_2^2 - w_1^2 \pm \sqrt{(w_1^2 - w_2^2)^2 + 4(1-\alpha) w_1^2 w_2^2}}{2(1-\alpha)}}$$

stable or unstable?

$\lambda^2 > 0$ or < 0 ?

Note: In general, even in stable, 2×2 case, motion <u>not</u> simple.

In particular, quasiperiodic:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = a_1 \vec{\xi}_1 \cos(\omega_1 t + \phi_1) + b_1 \vec{\xi}_2 \cos(\omega_2 t + \phi_2)$$

In general, $\frac{\omega_2}{\omega_1}$ <u>irrational</u>.

$\Rightarrow$ Motion <u>never</u> repeats itself.

Example: $\phi_1 = \phi_2 = 0$

Start at $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = a_1 \vec{\xi}_1 + a_2 \xi_2$

Ever get back there? No!

why not?

When would $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = a \vec{\xi}_1 + b \vec{\xi}_2$?

$\cos(w_1 t) = 1 \Rightarrow w_1 t = 2\pi n_1$ $\quad n_1, n_2$ integer

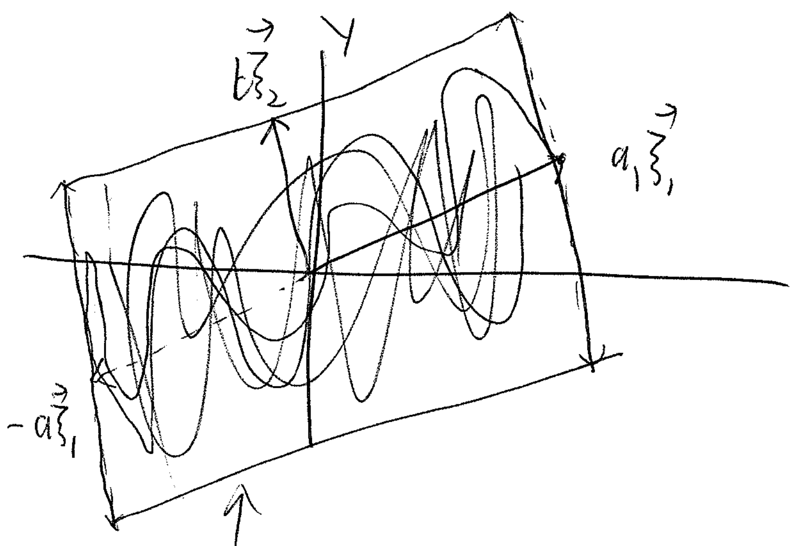
$\cos(w_2 t) = 1 \Rightarrow w_2 t = 2\pi n_2$

$$\Rightarrow \frac{w_1}{w_2} = \frac{n_1}{n_2} = \text{rational}$$

~~Contradicts original~~

Impossible for $\frac{w_1}{w_2}$ irrational. $\Rightarrow$ Never return.

So what <u>is</u> motion?



Eventually "fills in" whole parallelogram

~~General~~

Labor saving advice for linearized EOM: Don't expand EOM, expand $\mathcal{L}$.

In fact, don't even bother to derive full $\mathcal{L}$; just derive $\mathcal{L}$ valid for small $\{\delta q_i\}$.

How far do we have to go with $\mathcal{L}$ expansion to get linear EOM's?

$$\mathcal{L}(\{q_j, \dot{q}_j\}) = \mathcal{L}(\{q_j^* + \delta q_j, \delta\dot{q}_j\})$$

$$=$$

$$= \mathcal{L}(\{q_j^*, 0\}) + \sum_j \left(\frac{\partial \mathcal{L}}{\partial q_j}\bigg|_* \delta q_j + \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\bigg|_* \delta \dot{q}_j \right)$$

$$+ \frac{1}{2} \sum_{j,k} \left(\frac{\partial^2 \mathcal{L}}{\partial q_j \partial q_k}\bigg|_* \delta q_j \, \delta q_k + \frac{\partial^2 \mathcal{L}}{\partial q_j \partial \dot{q}_k}\bigg|_* \delta q_j \, \delta \dot{q}_k \right.$$

$$\left. + \frac{\partial^2 \mathcal{L}}{\partial \dot{q}_j \partial \dot{q}_k}\bigg|_* \delta \dot{q}_j \, \delta \dot{q}_k \right) + O(\delta q^3)$$

---

(4) EOM:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}$$

~~LHS =~~

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial \delta \dot{q}_i} =$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\partial \mathcal{L}}{\partial \delta q_i} =$$

6.26

$\Rightarrow$ EOM is

$$\cancel{\sum_j \frac{\partial^2 \mathcal{L}}{\partial \dot q_i \partial \dot q_j}\Big|_*}$$

$$\sum_j \frac{\partial^2 \mathcal{L}}{\partial \dot q_i \partial \dot q_j}\,\delta \ddot q_j = \overbrace{\frac{\partial \mathcal{L}}{\partial q_j}\Big|_*}^{0?} + \frac{1}{2}\sum_j \frac{\partial^2 \mathcal{L}}{\partial q_i \partial \dot q_j}\,\delta \dot q_j + \sum_j \frac{\partial^2 \mathcal{L}}{\partial q_i \partial q_j}\,\delta q_j + O(\delta q^2)$$

What's this?
" special property of fixed point?

So, only need $\mathcal{L}$ to $O(\delta q^2)$.

$\Rightarrow$ If you can guess where fixed points are, just calculate Lagrangian near there.

(6.27

Examples:

1) Single particle, 1d motion in potential

$$\mathcal{L} =$$

$$\sum_j \frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \partial \dot{q}_j} = \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^2} =$$

$$\frac{\partial \mathcal{L}}{\partial q_j^*} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial x} = \qquad = 0$$

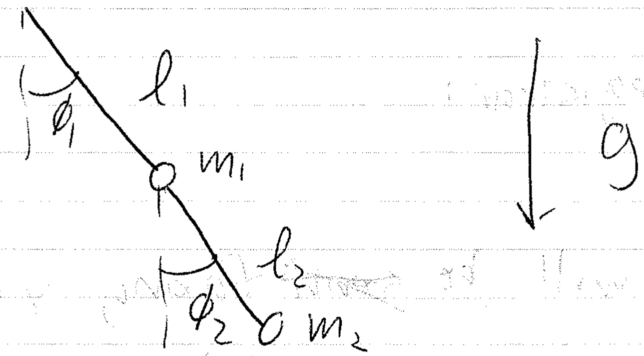
Familiar?

$$\frac{\partial^2 \mathcal{L}}{\partial q_i \partial \dot{q}_j} = ?$$

$$\frac{\partial^2 \mathcal{L}}{\partial q_i \partial q_j} = ?$$

$\Rightarrow$ ~~Eq~~ linearized EOM:

2nd example: Double pendulum:



Let's just <u>derive</u> $\mathcal{L}$ to $O(\delta q^2, \delta \dot{q}^2)$

$v_1^2 =$ + higher order terms

$v_2^2 =$ + h.o.t.

$U(\phi_1, \phi_2) =$

$\approx$

$\Rightarrow \mathcal{L} = ?$

$$\mathcal{L} = \frac{1}{2}\Big(m_1 l_1^2 \dot{\phi}_1^2 + m_2\left(l_1\dot{\phi}_1 + l_2\dot{\phi}_2\right)^2 - \frac{1}{2}m_1 g l_1 \phi_1^2$$
$$- \frac{1}{2}m_2 g\left(l_1\phi_1^2 + l_2\phi_2^2\right)\Big)$$

$\Rightarrow$ EOM's:

$\cancel{m_1 l_1^2 \ddot{\phi}_1 +}$

Same as we found by linearizing full expression earlier. Much easier.

Very general approach: Used in

- Condensed Matter Physics
- Birds
- High Energy Physics
- Solid State

⋮