

Put (1) in (2); solve for $\dot{\phi}(\theta)$:

(7.60)

$$(3) \quad \dot{\phi} = \frac{L_z - L_3 \cos \theta}{I_1 \sin^2 \theta}$$

Now, put (1) and (3) into conservation of E (a):

$$(4) \quad E = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{(L_z - L_3 \cos \theta)^2}{2 I_1 \sin^2 \theta} + \frac{L_3^2}{2 I_3} + m g l \cos \theta$$
$$\equiv \frac{1}{2} I_1 \dot{\theta}^2 + U_{\text{eff}}(\theta)$$

where I've defined:

$$U_{\text{eff}}(\theta) = \frac{(L_z - L_3 \cos \theta)^2}{2 I_1 \sin^2 \theta} + \frac{L_3^2}{2 I_3} + m g l \cos \theta$$

Just like Energy for 1-D motion
in a potential, with $x \rightarrow 0$, $\dot{x} \rightarrow 0$.

\Rightarrow Can read off qualitative
features of motion from shape of
 $U_{\text{eff}}(x)$.

cheat
sheet

7:61g

$$U_e = \frac{(L_z - L_3 x)^2}{1 - x^2}$$

$$\partial_x U_e = \frac{2(L_z - L_3 x)(-L_3)}{(1 - x^2)^2} + \frac{(L_z - L_3 x)^2}{(1 - x^2)^2} (+2x) = 0$$

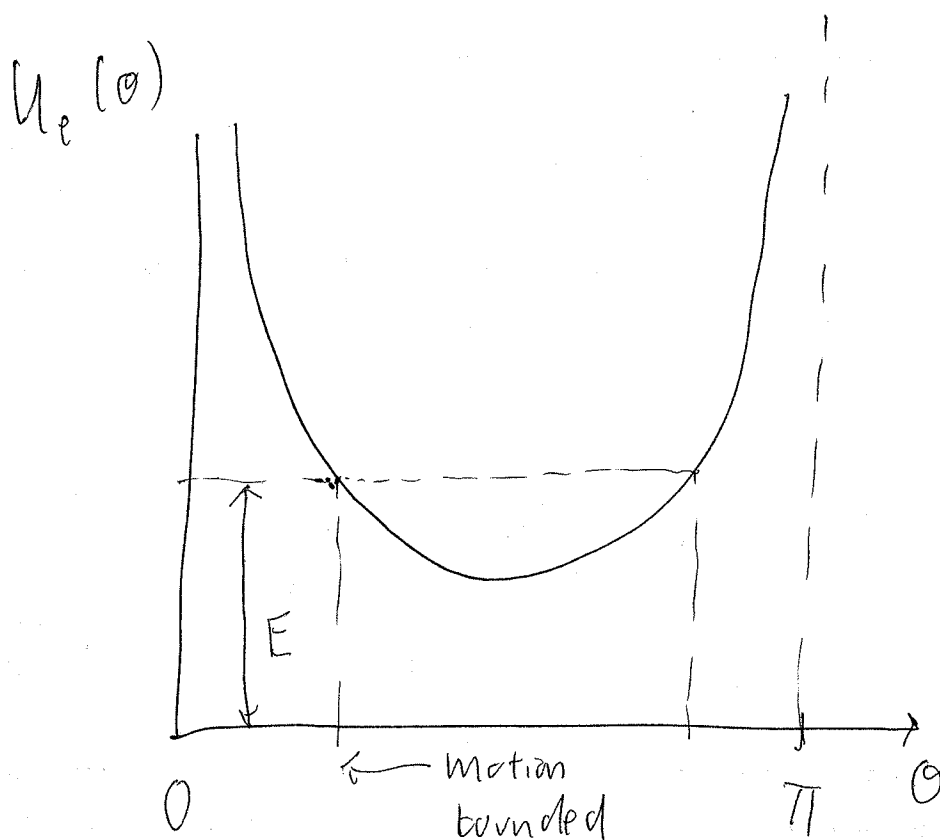
$$\Rightarrow \frac{(L_z - L_3 x)^2 (2x)}{(1 - x^2)^2} = \frac{2L_3 (L_z - L_3 x)}{1 - x^2}$$

$$\Rightarrow (L_z - \cancel{L_3 x})x = L_3 - \cancel{L_3 x^2}$$

$$\Rightarrow \boxed{x = \frac{L_3}{L_z}} \Rightarrow \text{only one minimum}$$

Hamiltonian dynamics

(7.62)



Bigger $L \Rightarrow U_e \uparrow \Rightarrow$ tighter bounds on motion

\Rightarrow stable spinning (gyroscope effect)

Hamiltonian dynamics:

(8.1) ^{ab}

- Recall the invariant we found earlier;
rename it H ("Hamiltonian")

$$(1) \quad H(\{p_i, q_i\}) = \sum_i p_i \dot{q}_i - L \quad \left(p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \right) \text{ "conjugate momentum"}$$

Consider

Note: think of this as fn. of $\{p_i, q_i\}$, not $\{\dot{q}_i, q_i\}$

$$dH = \sum_i (p_i dq_i + q_i dp_i) - dL \quad (2)$$

What's dL ?

What's L a function of?

$$L(\{q_i, \dot{q}_i\}) \Rightarrow dL = \sum_i \left[\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \right]$$

Note: think of L as fn. of these

$$\text{But, by def'n, } \frac{\partial L}{\partial \dot{q}_i} = p_i.$$

$$\text{Furthermore, E-L eqn. } \Rightarrow \frac{\partial L}{\partial q_i} = ?$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = \frac{d}{dt} (p_i) = \dot{p}_i$$

$$\Rightarrow \boxed{d\mathcal{L} = \sum_i (p_i d\dot{q}_i + \dot{p}_i dq_i)} \quad (3)$$

Put into (2)

$$\Rightarrow dH = \sum_i (p_i d\dot{q}_i + \dot{q}_i dp_i - p_i d\dot{q}_i - \dot{p}_i dq_i)$$

$$\Rightarrow \boxed{dH = \sum_i [(-\dot{p}_i) dq_i + \dot{q}_i dp_i]} \quad (4)$$

But now: Suppose I ~~hadn't~~ hadn't told you what H was, but just said it was a fn. of $\{p_i, q_i\}$

$H(\{p_i, q_i\})$

$$\Rightarrow dH = ?$$

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Hamiltonian dynamics

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right) \quad (5)$$

Compare with (4) \Rightarrow

$$\left. \begin{aligned} \dot{p}_i &= - \frac{\partial H}{\partial q_i} \\ \dot{q}_i &= \frac{\partial H}{\partial p_i} \end{aligned} \right\} \text{"Hamilton's equations of motion"}$$

So, alternative (Hamiltonian)

prescription for getting EOM's:

1) Construct Lagrangian $L = T - U$

as before

2) Construct conjugate momenta

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \quad \text{for all co-ordinates } i$$

3) Construct $H = \sum_i p_i \dot{q}_i - L$

4) EOM's are:

$$\begin{aligned} \dot{p}_i &= - \frac{\partial H}{\partial q_i} \\ \dot{q}_i &= \frac{\partial H}{\partial p_i} \end{aligned}$$

Example: 1d motion in a potential

$U(x)$

Step 1) $L = T - U = ?$

$$2) \quad p = \frac{\partial L}{\partial \dot{x}} = m \dot{x}$$

$$3) \quad H = p \dot{x} - L = m \dot{x}^2 - \frac{1}{2} m \dot{x}^2 + U(x)$$

$$= \frac{1}{2} m \dot{x}^2 + U(x) \quad \text{total energy}$$

$$= \frac{p^2}{2m} + U(x) =$$

$$4) \quad \dot{p} = - \frac{\partial H}{\partial x} = - \frac{\partial U}{\partial x} \quad \checkmark$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad \checkmark$$

Back to general case:

(8.5)

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_i \left(\frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i \right)$$

$$\Rightarrow \frac{\partial H}{\partial t} + \sum_i \left(\frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

Using Hamilton's equations

$$= \frac{\partial H}{\partial t}$$

\Rightarrow If H does not depend explicitly on t , $\frac{\partial H}{\partial t} = 0$

then $\frac{dH}{dt} = 0 \Rightarrow H$ is conserved

conservation of "energy"