PHYS 632: Quantum Mechanics II (Winter 2021)
Homework 5
Assigned Monday, 8 February 2021
Due Monday, 15 February 2021

Problem 1. Show that under Hamiltonian evolution,

$$\partial_t \rho = -\frac{i}{\hbar}[H, \rho], \tag{1}$$

the purity $\mathrm{Tr}[\rho^2]$ is a constant of the motion.

Problem 2. For a single qubit/spin-$1/2$ system, it turns out that an arbitrary density operator may be written in terms of the Pauli matrices as

$$\rho = \frac{1}{2}\big(\mathcal{I}_2 + \mathbf{r}\cdot\boldsymbol{\sigma}\big), \tag{2}$$

where $|\mathbf{r}| \leq 1$ and $\mathcal{I}_2$ is the identity on the qubit Hilbert space.

(a) Show this by writing out the right-hand side as a 2×2 matrix, and showing that it parameterizes an arbitrary matrix with the correct properties to be a density matrix. (What are the properties?)

(b) By computing $\langle\boldsymbol{\sigma}\rangle$, show that $\mathbf{r}$ is in fact the Bloch vector.

(c) Show that the purity is related to the length of the Bloch vector by

$$\mathrm{Tr}[\rho^2] = \frac{1}{2}\big(1 + |\mathbf{r}|^2\big). \tag{3}$$

In particular, pure states have $|\mathbf{r}| = 1$, with mixed states $0 \leq |\mathbf{r}| < 1$ (so that $1/2 \leq \mathrm{Tr}[\rho^2] < 1$).

(d) What is the Bloch vector for the I-know-nothing state (proportional to the identity)?

1)

$$\partial_t \rho = -\frac{i}{\hbar}[H, \rho]$$

$$\partial_t \mathrm{Tr}(\rho^2) = \mathrm{Tr}(\partial_t \rho^2)$$

$$\partial_t \mathrm{Tr}(\rho^2) = \mathrm{Tr}(\partial_t \rho \cdot \rho) + \mathrm{Tr}(\rho\, \partial_t \rho)$$

$$\partial_t \mathrm{Tr}(\rho^2) = \frac{-i}{\hbar}\Big(\mathrm{Tr}([H,\rho]\rho) + \mathrm{Tr}(\rho[H,\rho])\Big)$$

$$\partial_t \mathrm{Tr}(\rho^2) = \frac{-i}{\hbar}\Big(\mathrm{Tr}(H\rho^2 - \rho H\rho + \rho H\rho - \rho^2 H)$$

since Tr( ) is linear

$$\partial_t \mathrm{Tr}(\rho^2) = \frac{-i}{\hbar}\Big(\mathrm{Tr}(H\rho^2) - \mathrm{Tr}(\rho^2 H)\Big)$$

$$\partial_t \mathrm{Tr}(\rho^2) = \frac{-i}{\hbar}\Big(\mathrm{Tr}(H\rho^2) - \mathrm{Tr}(H\rho^2)$$

$$\partial_t \mathrm{Tr}(\rho^2) = 0$$

since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

2)

$$\rho = \frac{1}{2}(I_2 + \vec{r}\cdot\vec{\sigma})$$

$$, \quad |\vec{r}| \leq 1$$

a)

let $\vec{r} = (r_x, r_y, r_z)$

let $\rho = \frac{1}{2}(I_2 + \vec{r}\cdot\vec{\sigma})$

We claim ρ is a density operator

We will write out ρ explicitely and show that it satisfies the properties of a density operator.

$$\rho = \frac{1}{2}(I_2 + \vec{r}\cdot\vec{\sigma}) = \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix}$$

since $\rho = \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix} = \rho^T$

clearly ρ is Hermitian.

We also have

$$\mathrm{Tr}(\rho) = \frac{1}{2}(1 + r_z + 1 - r_z) = 1$$

so ρ satisfies the requirement that $\mathrm{Tr}(\rho) = 1$.

Finally, we know that a density operator must be positive semidefinite. The eigenvalues of ρ are given by

$$\det(\lambda I - \rho) = 0$$

$$\rightarrow \lambda^2 - \lambda + \frac{1}{4} - \frac{r_x^2}{4} - \frac{r_y^2}{4} - \frac{r_z^2}{4} = 0$$

$$\lambda^2 - \lambda + \frac{1}{4}(1 - |\vec{r}|^2) = 0$$

$$\rightarrow \lambda_\pm = \frac{1 \pm \sqrt{1 - (1 - |\vec{r}|^2)}}{2}$$

$$\lambda_\pm = \frac{1}{2} \pm \frac{1}{2}|\vec{r}|$$

and since $|\vec{r}| \leq 1$, $\lambda_\pm \geqslant 0$

so ρ is positive semidefinite.

b)

$$\langle \vec{\sigma} \rangle = \mathrm{Tr}(\vec{\sigma}\rho)$$

$$\langle \vec{\sigma} \rangle = \mathrm{Tr}(\sigma_x \rho)\hat{x} + \mathrm{Tr}(\sigma_y \rho)\hat{y} + \mathrm{Tr}(\sigma_z \rho)\hat{z}$$

$$\sigma_x \rho = \frac{1}{2}\begin{pmatrix} r_x + i r_y & 1 - r_z \\ 1 + r_z & r_x - i r_y \end{pmatrix}$$

$$\sigma_y \rho = -\frac{i}{2}\begin{pmatrix} r_x + i r_y & 1 - r_z \\ -(1+r_z) & -(r_x - i r_y) \end{pmatrix}$$

$$\sigma_z \rho = \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - i r_y \\ -r_x - i r_y & r_z - 1 \end{pmatrix}$$

$$\langle \vec{\sigma} \rangle = r_x \hat{x} + r_y \hat{y} + r_z \hat{z} = \vec{r}$$

So $\vec{r}$ is the Bloch vector.

c)

$$\rho^2 = \frac{1}{4}\left(I_2 + (\vec{r}\cdot\vec{\sigma})^2 + 2\,\vec{r}\cdot\vec{\sigma}\right)$$

$$Tr(\rho^2) = \frac{1}{4}\left(Tr(I_2) + Tr((\vec{r}\cdot\vec{\sigma})^2) + 2\,Tr(\vec{r}\cdot\vec{\sigma})\right)$$

Since the Pauli matrices are traceless,

$$Tr(\vec{r}\cdot\vec{\sigma}) = r_x Tr(\sigma_x) + r_y Tr(\sigma_y) + r_z Tr(\sigma_z)$$
$$= 0 .$$

Since the Pauli matrices anti-commute, and $[\sigma_\alpha, \sigma_\beta] = 2\delta_{\alpha\beta} I_2$

$$(\vec{r}\cdot\vec{\sigma})^2 = (r_x^2 + r_y^2 + r_z^2)\, I_2$$

$$(\vec{r}\cdot\vec{\sigma})^2 = |\vec{r}|^2 I_2$$

So

$$Tr(\rho^2) = \frac{1}{4}\left(Tr(I_2) + |\vec{r}|^2 Tr(I_2)\right)$$

$$Tr(\rho^2) = \frac{1}{4}\left(2 + 2|\vec{r}|^2\right)$$

$$\boxed{Tr(\rho^2) = \frac{1}{2}\left(1 + |\vec{r}|^2\right)}$$

d)

If $\rho = \frac{1}{2}(I_2 + \vec{r} \cdot \vec{\sigma})$

but $\rho = \frac{1}{2} I_2$ (it has to be otherwise $tr(\rho) \neq 1$), then the Bloch vector,

$$\boxed{\vec{r} = \langle \vec{\sigma} \rangle = 0 .}$$

This makes sense because it we're given uniform distribution of states then the expected value of $\langle \vec{\sigma} \rangle$ should be zero