Example: 1d motion:

$$\frac{1}{2}m\dot{x}^2 + U(x) = E \quad \text{(constant)}$$

This is a <u>1st</u> order differential eqn. (only involves $\dot{r}$, <u>not</u> $\ddot{r}$)

How to solve it?

1) Solve for $\dot{x}$:

2) $$\frac{dx}{dt} = \sqrt{\frac{2(E-U(x))}{m}}$$

$$\Rightarrow dt = \frac{dx}{\sqrt{\frac{2(E-U(x))}{m}}}$$

$$\Rightarrow \boxed{t = \int_{x_0}^{x} \frac{dx'}{\sqrt{\frac{2(E-U(x'))}{m}}}} \; *$$

$\Rightarrow$ If you can do this $\int$ over x, you can get $t(x)$. In principle, invert $\Rightarrow x(t)$

Question: In original ~~problem~~

equation of motion:

$$m\ddot{x} = -\frac{dU}{dx},$$

how many arbitrary constants? ~~[illegible]~~

i.e., how many initial conditions?

what are they?

Where have they gone in $*$?

Examples:

1) Constant force $\Rightarrow U(x) =$
F_0

$$\Rightarrow * \Rightarrow \quad t(x) = \int_{x_0}^{x} \frac{dx'}{\sqrt{\frac{2(E-U(x'))}{m}}} =$$

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~~Spring constant~~

2) Harmonic oscillator: $F(x) = -kx$

$$\Rightarrow U(x) = ?$$

$$\Rightarrow t(x) = \int_{x_0}^{x} \frac{dx'}{\sqrt{\frac{2(E-U(x))}{m}}} =$$

Substitution:

$$\sqrt{\frac{k}{2E}}\, x' = \sin\theta$$

$$\Rightarrow t(x) = \int_{\sin^{-1}\left(\sqrt{\frac{k}{2E}}x_0\right)}^{\sin^{-1}\left(\sqrt{\frac{k}{2E}}x\right)} \sqrt{\frac{m}{2E}}\sqrt{\frac{2E}{k}}\, d\theta$$

$$= \underbrace{\sqrt{\frac{m}{k}}}_{\equiv \frac{1}{\omega}} \left(\sin^{-1}\left(\sqrt{\frac{k}{2E}}x\right) - \underbrace{\sin^{-1}\left(\sqrt{\frac{k}{2E}}x_0\right)}_{\equiv \phi_0}\right)$$

define $x_{max} \equiv \sqrt{\frac{2E}{k}}$ (Why do I call this x_{max}?)

$$\Rightarrow \sin^{-1}\left(\frac{x}{x_m}\right) = \omega t + \phi_0 \Rightarrow x = x_m \sin(\omega t + \phi_0)$$

Simple harmonic motion

etc ....

~~Note: Because of~~

This also works for Central Force motion with effective potential:

$$\cancel{\dot{r} = \sqrt{2E - U_{eff}(r)}}$$

$$\frac{1}{2} m \dot{r}^2 + \underbrace{U_{eff}(r)}_{\frac{L^2}{2mr^2} + U(r)} = E$$

$$\Rightarrow \quad \dot{r} = \sqrt{\frac{2\left(E - \frac{L^2}{2mr^2} - U(r)\right)}{m}} \qquad (1)$$

$$\Rightarrow \quad \boxed{t = \int_{r_0}^{r} \frac{dr'}{\sqrt{\frac{2\left(E - \frac{L^2}{2mr^2} - U(r)\right)}{m}}}} \qquad (2)$$

But, even better: Can get shape of orbit using other conservation law.

Conservation of $\measuredangle$ momentum $\Rightarrow$?

$$\Rightarrow \quad d\theta = \frac{L}{mr^2}\,dt \quad \text{or} \quad dt = \frac{mr^2\,d\theta}{L}.$$

Use this in (1:

$$\Rightarrow \quad dr = \sqrt{\frac{2}{m}\left(E - \frac{L^2}{2mr^2} - U(r)\right)}\;dt$$

$$= \frac{mr^2}{L}\sqrt{\frac{2}{m}\left(E - \frac{L^2}{2mr^2} - U(r)\right)}\;d\theta$$

$$\Rightarrow \quad \boxed{\theta = \frac{L}{m}\int_{r_0}^{r} \frac{dr'}{r'^2\sqrt{\frac{2}{m}\left(E - \frac{L^2}{2mr'^2} - U(r')\right)}}}$$

This Gives $\theta(r)$; invert $\Rightarrow r(\theta)$

Classic example: Kepler problem:

$$F(r) \propto \frac{1}{r^2} \Rightarrow U(r) = -\frac{\mu}{r}$$

$$\Rightarrow \quad \theta = \frac{L}{m}\int_{r_0}^{r} \frac{dr'}{r'^2\sqrt{\frac{2}{m}\left(E - \frac{L^2}{2mr'^2} + \frac{\mu}{r'}\right)}}$$

Obvious change of variables: $u = \frac{1}{r'} \Rightarrow du = -\frac{dr'}{r'^2}$

$$\Rightarrow \theta = \frac{L}{m}\int_{\frac{1}{r}}^{\frac{1}{r_0}} \frac{du}{\sqrt{\frac{2}{m}\big(\underbrace{E - \frac{L^2u^2}{2m} + \mu u}\big)}}$$

~~Shift u too write den. as $C_1 - C_2u^2$~~

~~Seek to write den~~

Shift to kill linear term in den:

$u = u' + \delta$, choose δ s.t. den $\propto \sqrt{1-ku'^2}$

(then trig sub obviously works)

$$\cancel{E - \frac{L^2u'^2}{2m} + \mu u' - \frac{L^2}{2m}{}^2}$$

$$= E - \frac{L^2}{2m}\left(u'^2 + 2u'\delta + \delta^2\right) + \mu u' + \mu\delta$$

$$\cancel{= E - u'\left[\frac{L^2\delta}{m} - \mu\right] +}$$

$$= \underbrace{E - \frac{L^2\delta^2}{2m} + \mu\delta}_{\substack{\text{constant pieces} \\ \text{|||} \\ E'}} + \underbrace{u'\left(\mu - \frac{L^2\delta}{m}\right)}_{\substack{u' \text{ pieces} \\ \text{choose} \\ = \\ 0}} - \underbrace{\frac{L^2u'^2}{2m}}_{u'^2 \text{ pieces}}$$

$$\Rightarrow \delta = \frac{m\mu}{L^2} \Rightarrow E' = E + \frac{m\mu^2}{2L^2}$$

$$\Rightarrow \quad \theta = \frac{L}{m} \int_{\frac{1}{r} - \frac{m\mu}{L^2}}^{\frac{1}{r_0} - \frac{m\mu}{L^2}} \frac{du'}{\sqrt{\underbrace{\frac{2}{m}\left(E'\right.}_{\text{pull this out of } \sqrt{\ }} \left. - \frac{L^2 u'^2}{2m}\right)}}$$

$$= \frac{L}{\sqrt{2E'm}} \int_{\frac{1}{r} - \frac{m\mu}{L^2}}^{\frac{1}{r_0} - \frac{m\mu}{L^2}} \frac{du'}{\sqrt{1 - \underbrace{\frac{L^2}{2E'm} u'^2}_{\cos^2\phi}}}$$

now, trig substitution

$$\Rightarrow \quad \cos\phi = \sqrt{\frac{L^2}{2E'm}}\, u' = \frac{L}{\sqrt{2E'm}}\, u'$$

$$\Rightarrow \quad -\sin\phi \, d\phi = \underbrace{\frac{L}{\sqrt{2E'm}}}_{\equiv \frac{p}{e}} du'$$

$$\Rightarrow \quad \theta = \int_{\underbrace{\cos^{-1}\left(\frac{p}{e}\left(\frac{1}{r_0} - \frac{m\mu}{L^2}\right)\right)}_{-\theta_0}}^{\cos^{-1}\left(\frac{p}{e}\left(\frac{1}{r} - \frac{m\mu}{L^2}\right)\right)} \qquad \left(\frac{\sin\phi \, d\phi}{\sqrt{1 - \cos^2\phi}} = d\phi\right)$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{p}{e}\left(\frac{1}{r} - \underbrace{\frac{m\mu}{L^2}}_{\frac{1}{p}}\right)\right) + \theta_0$$

$$\Rightarrow \quad \frac{p}{e}\left(\frac{1}{r} - \frac{1}{p}\right) = \cos(\theta - \theta_0) \Rightarrow \frac{1}{r} = \frac{1}{p}\left(1 + e\cos(\theta - \theta_0)\right)$$

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$$\boxed{r = \frac{p}{1+e\cos(\theta-\theta_0)}}$$ etc.

shape of Kepler orbit.

Note:

1) **Closed!**

(Return to same r when $\theta \to \theta + 2\pi$)

2) $\frac{p}{e} = \frac{L}{\sqrt{2E'm}}$, $E' = E + \frac{m\mu^2}{L^2}$, $p = \frac{L^2}{m\mu}$

$\Rightarrow$ "eccentricity" $e = \frac{\sqrt{2E'm}}{L}\, p = \frac{L}{m\mu}\sqrt{2m\left(E + \frac{m\mu^2}{2L^2}\right)}$

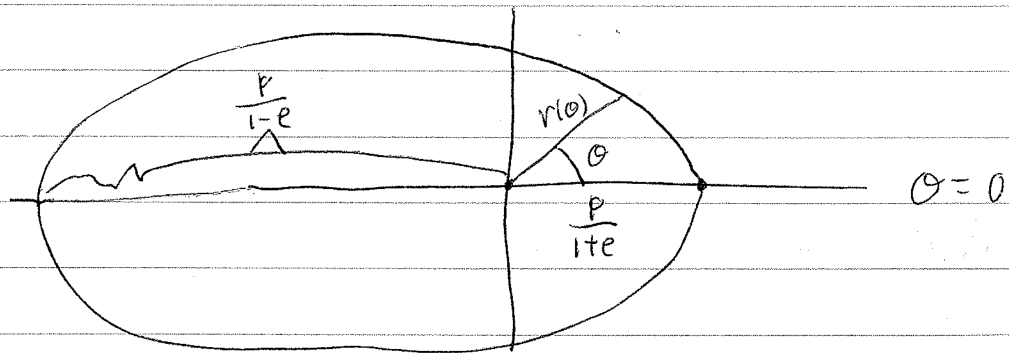
$$= \frac{L}{m\mu}\sqrt{\frac{m^2\mu^2}{L^2} + 2mE}$$

$$\Rightarrow \boxed{e = \sqrt{1 + \frac{2EL^2}{m\mu^2}}}$$

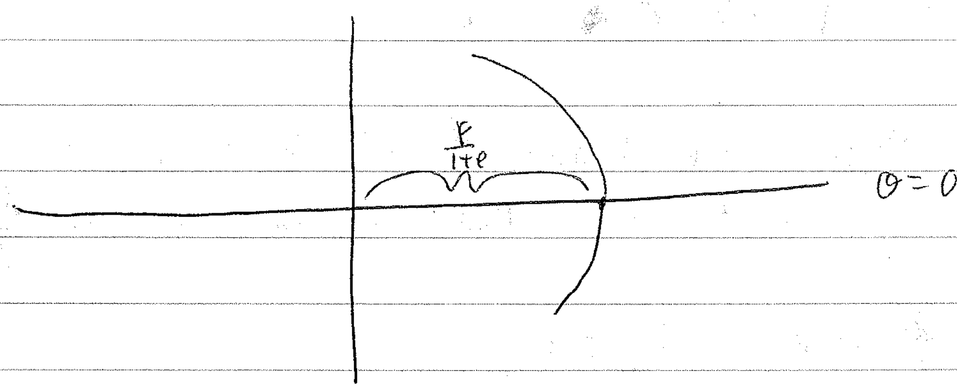
Note: $\underbrace{E<0 \Rightarrow e<1}_{\Downarrow \text{ closed}}$, $\underbrace{E>0 \Rightarrow e>1}_{\Downarrow \text{ escape}}$

3) Shape of orbit:

a) $e < 0$; choose $\theta_0 = 0$: Ellipse

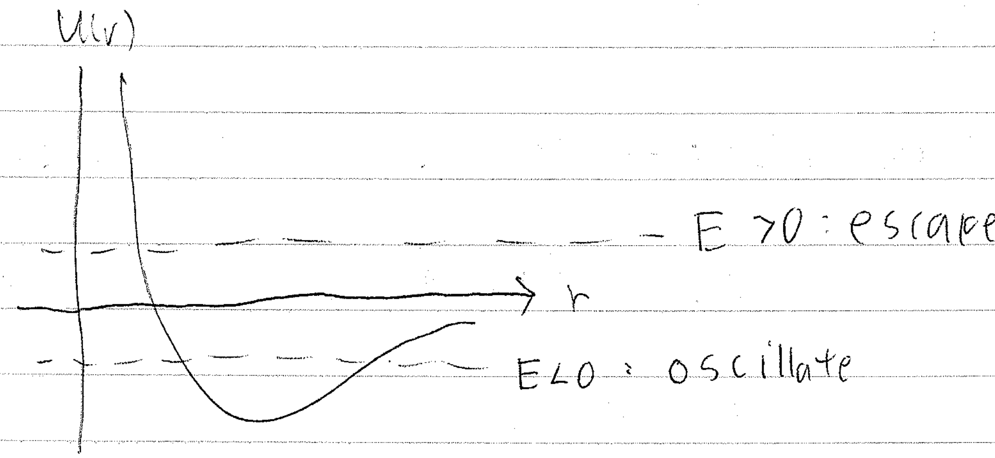


b) $e > 0$: hyperbola



Why change when E changes sign?

Effective potential:



Why did $\frac{1}{r^2}$ force work out so nicely?

Alternative approach: Equation of motion:

$$\vec{v} \equiv \dot{\vec{r}}$$

~~$\ddot{\vec r} = \frac{d}{dt}\dot{\vec r}$~~

Note: ~~I've~~ This is μ = earlier μ/m

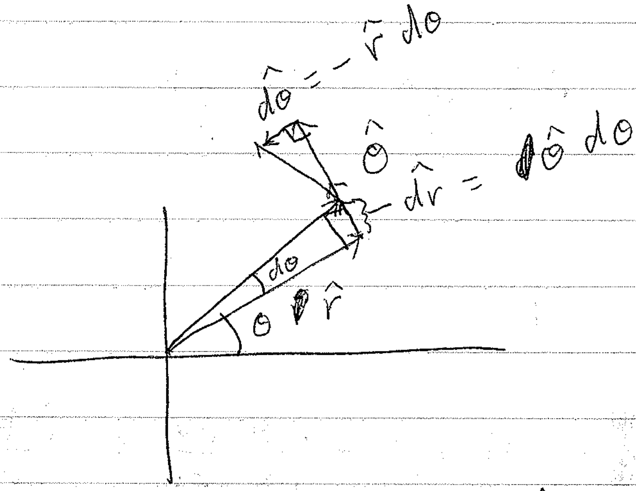
$$\ddot{\vec{r}} = \frac{d}{dt}\vec{v} = -\frac{\mu}{r^2}\hat{r}$$

Use ~~$dt =$~~ $mr^2\frac{d\theta}{dt} = L$, or $dt = \frac{r^2 d\theta}{h}$, $h \equiv \frac{L}{m}$

$$\Rightarrow \frac{d\vec{v}}{\left(\frac{r^2 d\theta}{h}\right)} = -\frac{\mu}{r^2}\hat{r} \Rightarrow \frac{h}{r^2}\frac{d\vec{v}}{d\theta} = -\frac{\mu}{r^2}\hat{r}$$

Note: r^2 cancel

$$\Rightarrow \quad \frac{d\vec{v}}{d\theta} = -\frac{\mu}{h}\hat{r}$$



$$\Rightarrow \quad \frac{d\hat{r}}{d\theta} = \hat{\theta}$$

$$\frac{d\hat{\theta}}{d\theta} = -\hat{r}$$

$$\Rightarrow \quad \frac{d\vec{v}}{d\theta} = \frac{\mu}{h}\left(\frac{d\hat{\theta}}{d\theta}\right)$$

$$\Rightarrow \quad \vec{V} = \vec{V}_0 + \frac{\mu}{h}\hat{\theta} \quad : A$$

"constant of integration"

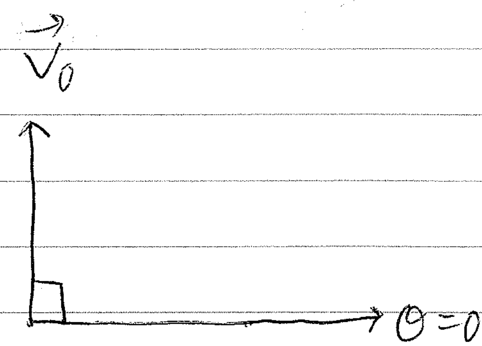
Note: This is already a broad hint that

orbit is closed: save $\vec{v}$ at same θ

Continue: let's compute $\mathring{r}$.

How?

Co-ordinate choice:



$\cancel{\frac{dr}{dt}}$

$$\frac{dr}{dt} = V_0 \sin\theta$$

$$\Rightarrow \quad \frac{h}{r^2}\frac{dr}{d\theta} = V_0 \sin\theta$$

$$\Rightarrow \quad h\left(\frac{1}{r_0} - \frac{1}{r}\right) = -V_0 \cos\theta$$

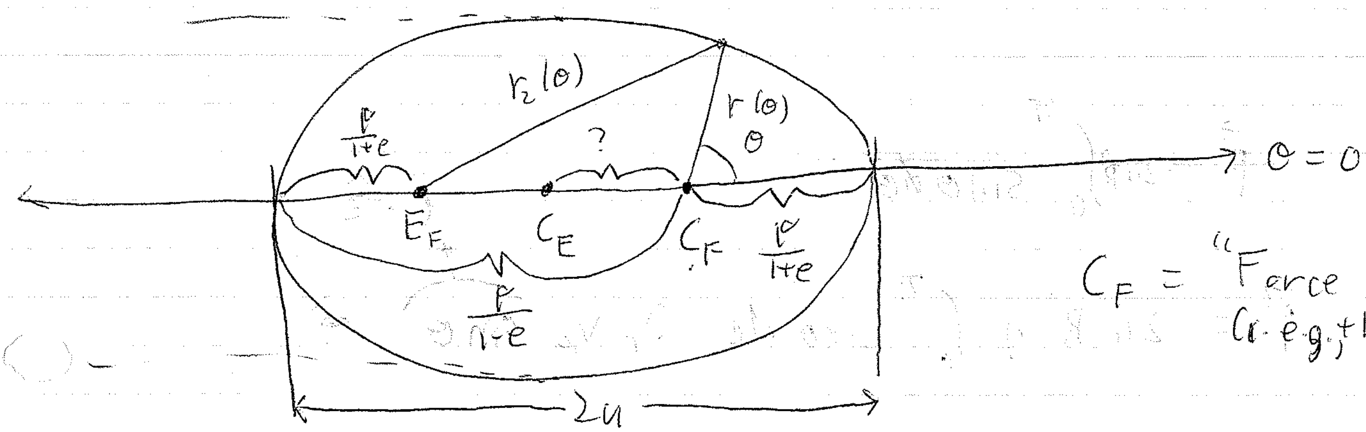
$$\Rightarrow \quad \frac{1}{r} = \frac{1}{r_0} + \frac{V_0}{h}\cos\theta$$

$$\Rightarrow \quad r = \frac{r_0}{1 + e\cos\theta}, \quad e = \frac{V_0 r_0}{h}$$

Some simple geometrical properties of ellipse:

1) Focus construction: $\theta_0 = 0$

$$r = \frac{p}{1+e\cos\theta}\ , \quad 2p \equiv \text{"latus rectum"}$$



C_F = "Force center" (r.e.g., the sun)

C_E = "Center of ellipse" (empty)

E_F = "empty focus"

$a \equiv$ "semi-major axis" = ?

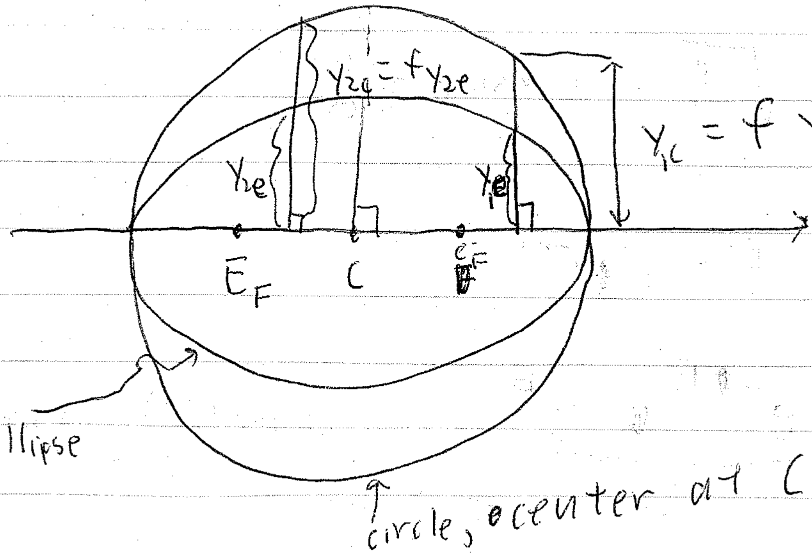
$$\boxed{a = -\frac{\mu}{E} : \text{depends \underline{only} on } E, \text{ \underline{not} on } L}$$

Focus construction

$r + r_2 = \text{constant} = ?$

String + thumbtacks

2) Ellipse = squashed circle



f = constant = ?

$\Rightarrow A \equiv$ Area of ellipse = ?

Period of orbit $\equiv T = \dfrac{A}{\left(\frac{L}{2m}\right)} = \dfrac{2m}{L} A = \pi a^{3/2} \sqrt{\dfrac{m}{\mu}}$

Why?

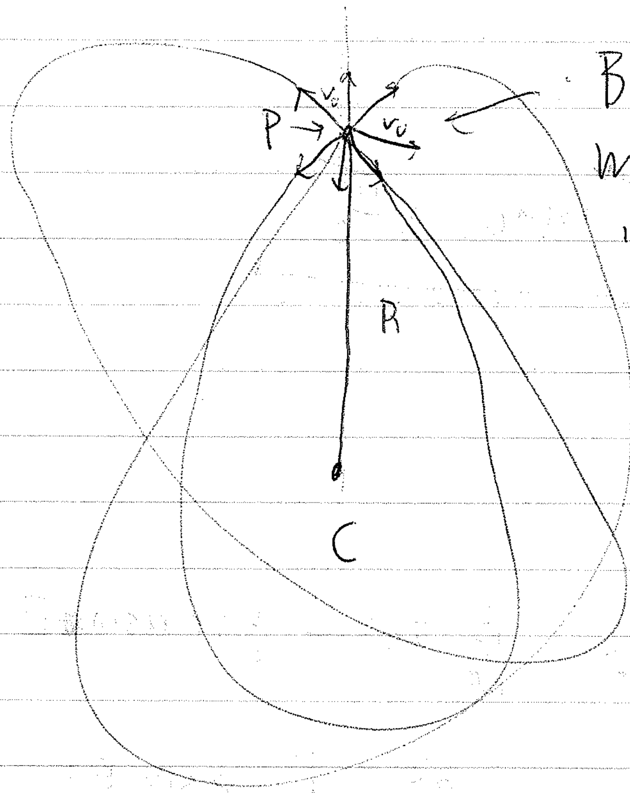
$\boxed{T \propto a^{3/2} : \text{Kepler's 2nd law}}$

Note: <u>T independent of L</u>

depends <u>only</u> on a

$\Rightarrow$ " " " E,

$\Rightarrow$ ∞'ly repeating firecracker:



Boom!
many particles fly out in all direction, all with same speed v_0

$\Rightarrow$ All have same E = ?

$\Rightarrow$ " " " a

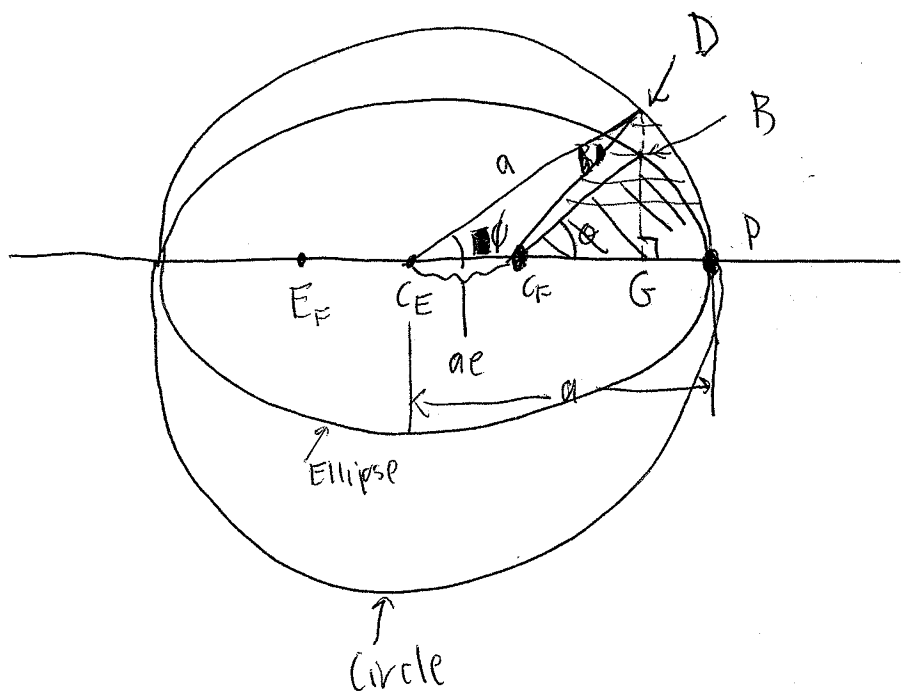
$\Rightarrow$ " " " T

$\Rightarrow$ after T, all return to point P of original explosion (Don't be there then)

Ditto at $2T, 3T, \cdots NT$

How to determine time of flight for fractions of complete orbit:

$\frac{1}{2} h t = A(t)$



To find time to get from P to B, need $A_{C_F PB}$.

But, since circle is just a vertically stretched (with constant stretch factor) version of ellipse

$$\frac{A^{(circle)}_{C_F DP}}{A_{cirle}} = \frac{A_{C_F BP}}{A_{ellipse}} = \frac{t_{PB}}{T}$$

So, what's $A_{C_F D P}$?

1st, what's ϕ?

What's $A^{circle}_{C_E D P}$?

what's $A^{triangle}_{C_E D C_F}$?

$\Rightarrow$ What's $A^{circle}_{C_F D P}$?

$$\cancel{\Rightarrow t = T \frac{(E - e\sin E)}{2\pi}}$$

$$\Rightarrow t = T\,\frac{(\phi - e\sin\phi)}{2\pi}\,, \qquad \boxed{\sin\phi = \frac{p\sin\theta}{\sqrt{1-e^2}\,(1+e\cos\theta)\,a} = \frac{(1-e^2)^{1/2}\sin\theta}{(1+e\cos\theta)}}$$

$$T = \frac{2\pi}{\sqrt{(\mu/m)}}\,a^{3/2} = \frac{2\pi\,p^{3/2}}{\sqrt{\left(\frac{\mu}{m}\right)}}\,(1-e^2)^{-3/2}$$

Example: Time from perihelion to $r = a$?

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4.28

Ex

Escape orbits, $e > 1$: use same formulae

$$\sin\phi = \frac{i \sin\theta \sqrt{e^2-1}}{1+e\cos\theta}$$

Look for imaginary sol'n

$$\phi = i\phi', \quad \phi' \text{ real}$$

$$\Rightarrow \sin\phi = \frac{e^{i\phi}-e^{-i\phi}}{2i} = \frac{e^{-\phi'}-e^{\phi'}}{2i} = -\frac{\sinh\phi'}{i} = i\sinh\phi'$$

$$\Rightarrow \boxed{\sinh\phi' = \frac{\sin\theta\sqrt{e^2-1}}{1+e\cos\theta}}$$

$$t = \frac{p^{3/2}}{\sqrt{\mu/m}}(1-e^2)^{-3/2}(i\phi' - e i \sinh\phi')$$

$$= \frac{p^{3/2}}{\sqrt{\mu/m}}(e^2-1)^{-3/2}\, i(i\phi' - e i\sinh\phi')$$

$$\boxed{t = \frac{p^{3/2}}{\sqrt{\mu/m}}(e^2-1)^{-3/2}(e\sinh\phi' - \phi')} \quad \boxed{*}$$