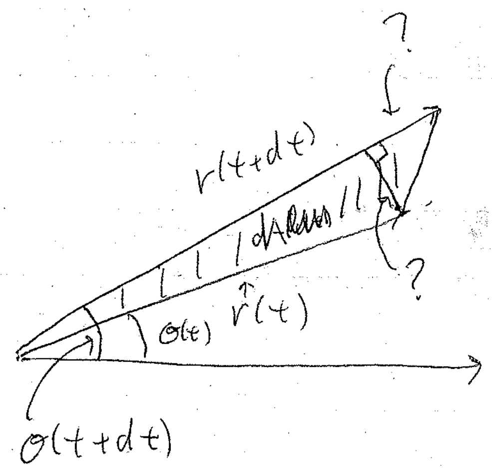
$$\mathcal{L} = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

$\mathcal{L}$ independent of θ

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \; ? \; = \text{constant}$$

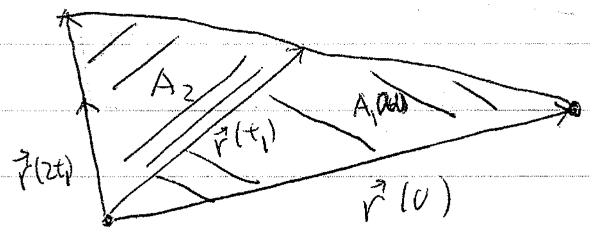
$? =$ "angular momentum" $\equiv L$

$\Rightarrow$ "Equal areas, Equal times" (Kepler's 1st (?) law of planetary motion)



$dA = ?$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{L}{2m} = \underline{\text{constant}}$$



A_1 = area swept out ~~out~~ between $t = 0$ and $t = t_1$

A_2 = " " " " $t = t_1$ and $t = 2t_1$

$\frac{dA}{dt}$ = constant $\Rightarrow$ $\boxed{A_2 = A_1}$ Equal areas, " times

---

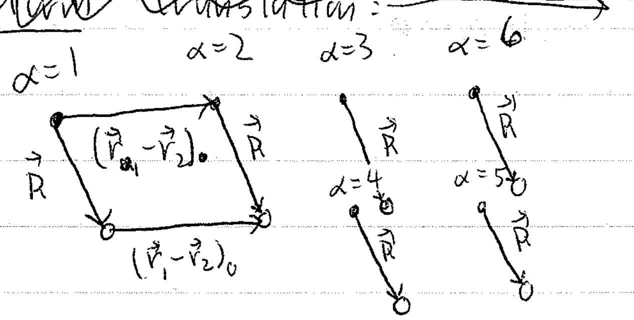
~~Now, system o~~

Now, system of many particles, α, no constraints

Pairwize interactions depending <u>only</u> on <u>relative</u>

separation $\vec{r}_\alpha - \vec{r}_B$, (α, B) label particles

~~Uniform translation:~~



Uniform translation: $\vec{r}_\alpha \to \vec{r}_\alpha + \vec{R}$, all α

$$\mathcal{L} = \frac{1}{2} \sum_\alpha m_\alpha |\dot{\vec{r}}_\alpha|^2 - \sum_{\alpha \neq \beta} U(\vec{r}_\alpha - \vec{r}_\beta)$$

$$\mathcal{L}(\{\vec{r}_\alpha, \dot{\vec{r}}_\alpha\}) = \mathcal{L}(\{\vec{r}_\alpha + \vec{R}, \dot{\vec{r}}_\alpha\})$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial R_x} = \frac{\partial \mathcal{L}}{\partial R_y} = \frac{\partial \mathcal{L}}{\partial R_z} = ?$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial R_x} = \sum_\alpha (?) = 0$$

$\Rightarrow$ Using EOM (E-LG equn), $\sum_\alpha \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_\alpha}\right) \underset{\uparrow}{=} \frac{d}{dt}\left(\sum_\alpha \frac{\partial \mathcal{L}}{\partial \dot{x}_\alpha}\right) = 0$

Why?

$\sum_\alpha \frac{\partial \mathcal{L}}{\partial \dot{x}_\alpha} = ?$ $= P_x$: x-component of "total linear momentum" (or "center of mass momentum" $\vec{P}$)

$$\sum_\alpha \frac{\partial \mathcal{L}}{\partial \dot{y}} = ?$$

$$\sum_\alpha \frac{\partial \mathcal{L}}{\partial \dot{z}} = ?$$

$$\Rightarrow \vec{P} =$$

$$\vec{P} = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = \sum_\alpha m_\alpha \vec{v}_\alpha = \text{constant}$$

Define "center of mass position" = <u>weighted</u> average of all particle positions

$$\vec{R}_{cm} \equiv \frac{\sum_\alpha m_\alpha \vec{r}_\alpha}{\sum_\alpha m_\alpha} = \frac{\sum_\alpha m_\alpha \vec{r}_\alpha}{M_{total}}$$

$$\frac{d\vec{R}_{cm}}{dt} = ? = \text{constant} \equiv \vec{V}_c$$

$$\Rightarrow \boxed{\vec{R}_{cm}(t) = ?} \; *$$

⇒ In N-body problem, how many degrees of freedom <u>not</u> trivially determined by * ?

Conservation of <u>total</u> momentum

Now, conservation of total ∡ momentum:

Same Lagrangian

$$\mathcal{L} = \sum_\alpha \tfrac{1}{2} m_\alpha |\dot{\vec{r}}_\alpha|^2 - U(\{\vec{r}_\alpha - \vec{r}_\beta\})$$

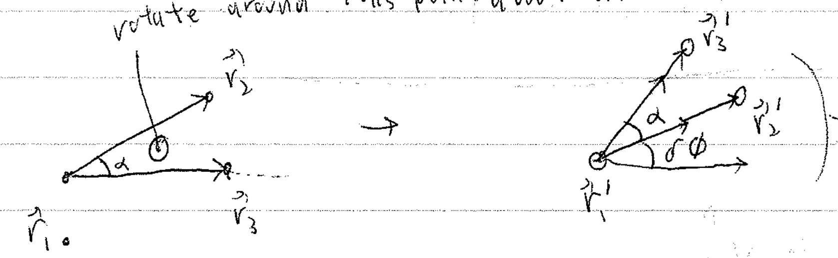
~~Additional~~ symmetry:

(3.19)

$\Rightarrow$ rotate everything together by same $\measuredangle$ about same axis

Additional symmetry: "Global rotation invariance"

~~"Uniform roto~~

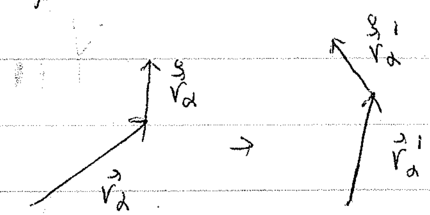


$\rightarrow$ "Uniform rotation"

$$U(\{\vec{r}_\alpha\}) = U(\{\vec{r}_\alpha\}_0)$$

$$\cancel{\Rightarrow \mathcal{L}(\{\vec{r}_\alpha, \dot{\vec{r}}_\alpha\}) = \mathcal{L}(\{\vec{r}_\alpha{}', \dot{\vec{r}}_\alpha{}'\})}$$

rotate velocities as well:



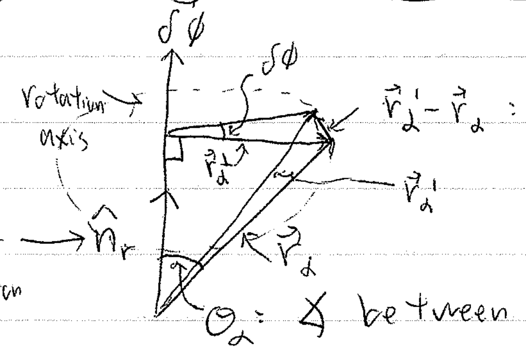
$$\cancel{\Rightarrow \mathcal{L}(\{\vec{r}_\alpha, \vec{r}_\alpha}$$

$$T(\{\dot{\vec{r}}_\alpha\}) = \sum_\alpha \frac{1}{2} m_\alpha |\dot{\vec{r}}_\alpha|^2$$

$$T(\{\dot{\vec{r}}_\alpha{}'\}) = \sum_\alpha \frac{1}{2} m_\alpha |\dot{\vec{r}}_\alpha{}'|^2 \quad \text{: Any change?}$$

$$\Rightarrow \mathcal{L}(\{\vec{r}_\alpha, \dot{\vec{r}}_\alpha\}) = \mathcal{L}(\{\vec{r}_\alpha{}', \dot{\vec{r}}_\alpha{}'\})$$

Now, consider infinitesimal rotation ($\delta\phi \ll 1$)



$\vec{r}_\alpha{}' - \vec{r}_\alpha$: $|\vec{r}_\alpha{}' - \vec{r}_\alpha| = ?$

direction of $\vec{r}_\alpha{}' - \vec{r}_\alpha$?

unit vector along rotation axis $\rightarrow \hat{n}_r$

θ_α: $\measuredangle$ between $\vec{r}_\alpha$ and rotation axis

$|\vec{r}_\alpha' - \vec{r}_\alpha| = \delta\phi |\vec{r}_\alpha^{\perp}| = |\vec{r}_\alpha| \sin\theta \, \delta\phi = |\vec{r}_\alpha| |\vec{\delta\phi}| \sin\theta$

$\vec{r}_\alpha' - \vec{r}_\alpha$ perpendicular to $\vec{r}_\alpha$ and $\hat{n}_r$

$\Rightarrow \vec{r}_\alpha' - \vec{r}_\alpha$ " " " " $\vec{\delta\phi} \equiv \delta\phi \, \hat{n}$

Given 2 vectors $\vec{a}, \vec{b}$, what vector $\vec{c}$ has

1) ~~trans~~ $\vec{c} \perp \vec{a}, \vec{b}$

2) $|\vec{c}| = |\vec{a}||\vec{b}| \sin\theta_{ab}$?

Ans:

so, $\vec{r}_\alpha' - \vec{r}_\alpha = \vec{\delta\phi} \times \vec{r}_\alpha$, $\delta\phi \ll 1$

N.B.: Same $\vec{\delta\phi}$ for all α, if rotation uniform (="Global")

what about $\dot{\vec{r}}_\alpha' - \dot{\vec{r}}_\alpha$ (velocities?)

~~Sam~~

$\cancel{\vec{V}_\alpha - V}$

$$\dot{\vec{r}}_\alpha{}' - \dot{\vec{r}}_\alpha = \vec{\delta\phi} \times \dot{\vec{r}}_\alpha$$

So $\mathcal{L}' \equiv \mathcal{L}_{rot} \equiv \mathcal{L}(\{\vec{r}_\alpha', \dot{\vec{r}}_\alpha'\}) = \mathcal{L}(\{\vec{r}_\alpha, \dot{\vec{r}}_\alpha\})$

$$+ \sum_\alpha (? \qquad\qquad)$$

Equation of motion

$$\frac{\partial \mathcal{L}}{\partial \vec{r}_\alpha} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_\alpha} = \frac{d}{dt}(m\dot{\vec{r}}_\alpha) = \frac{d\vec{p}_\alpha}{dt} \equiv \dot{\vec{p}}_\alpha$$

$$\Rightarrow \cancel{\mathcal{L}_{rot}} \; \mathcal{L}' - \mathcal{L} = \sum_\alpha \left[\dot{\vec{p}}_\alpha \cdot (\vec{\delta\phi} \times \vec{r}_\alpha) + \vec{p}_\alpha \cdot (\vec{\delta\phi} \times \dot{\vec{r}}_\alpha)\right]$$

Now, use <u>cyclic</u> property: $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b})$

$$\mathcal{L}_{rot} - \mathcal{L}_{unrotated} = \sum_\alpha \vec{\delta\phi} \cdot \left[\cancel{\dot{\vec{p}}_\alpha \times \vec{r}_\alpha} \; \vec{r}_\alpha \times \dot{\vec{p}}_\alpha + \dot{\vec{r}}_\alpha \times \vec{p}_\alpha\right]$$

$$= \vec{\delta\phi} \cdot \sum_\alpha \left(\vec{r}_\alpha \times \dot{\vec{p}}_\alpha + \dot{\vec{r}}_\alpha \times \vec{p}_\alpha\right) = 0$$

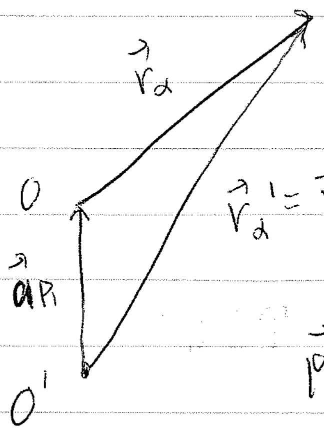
Global rotation invariance ⇒ This true for <u>any</u> $\vec{\delta\phi}$

$$\Rightarrow \sum_\alpha \left(\vec{r}_\alpha \times \dot{\vec{p}}_\alpha + \dot{\vec{r}}_\alpha \times \vec{p}_\alpha\right) = \sum_\alpha \frac{d}{dt}(\vec{r}_\alpha \times \vec{p}_\alpha) = 0$$

$$\Rightarrow \frac{d}{dt}\left(\sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha\right) = 0 \Rightarrow \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha \equiv \vec{L} = \text{constant}$$

$$\vec{L} \equiv \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha = \text{"total angular momentum"}$$

N. B. $\vec{L}$ depends on choice of origin:



$\vec{r}_\alpha{}' = \vec{r}_\alpha + \vec{R}$

same for all α, time independent

$\vec{p}_\alpha{}' = \vec{p}_\alpha + ?$

$$\vec{L}' = \sum_\alpha \vec{r}_\alpha{}' \times \vec{p}_\alpha = \sum_\alpha (\vec{r}_\alpha + \vec{R}) \times \vec{p}_\alpha = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha + \sum_\alpha \vec{R} \times \vec{p}_\alpha$$

$$= \vec{L} + \vec{R} \times \sum_\alpha \vec{p}_\alpha = \vec{L} + \vec{R} \times \vec{P}$$

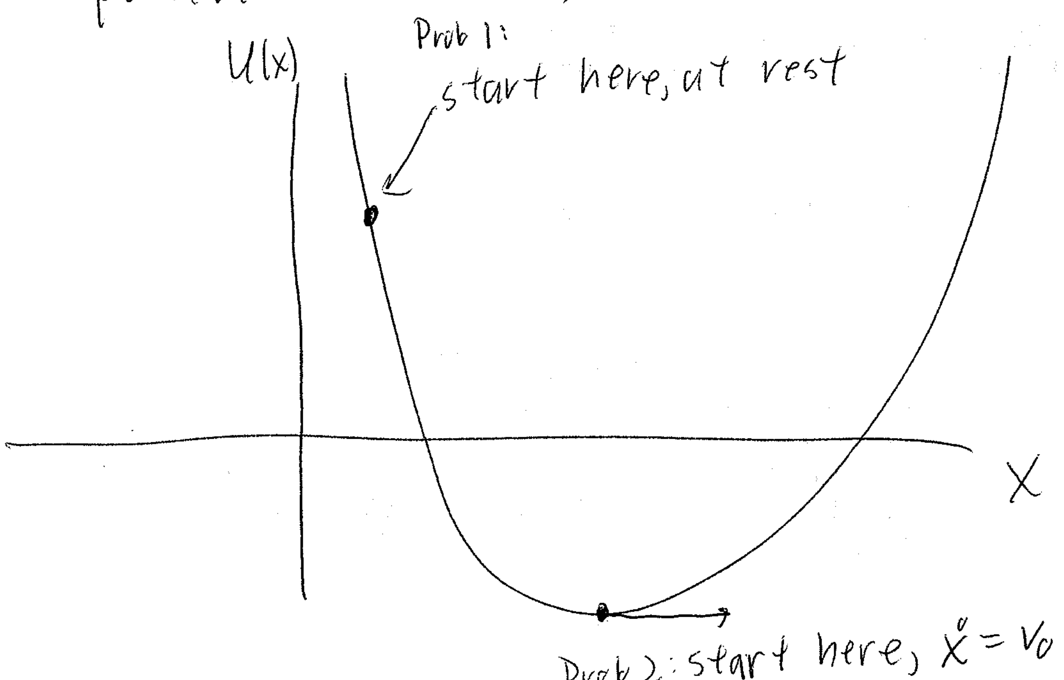
Since $\vec{L}$, $\vec{P}$ constants of motion,

$\vec{L}'$ also " " ", for all $\vec{R}$

So what do we do with all these conservation laws?

1) Bounds $\Rightarrow$ Qualitative description of motion

Example: 1d, unconstrained motion in potential $U(x)$:



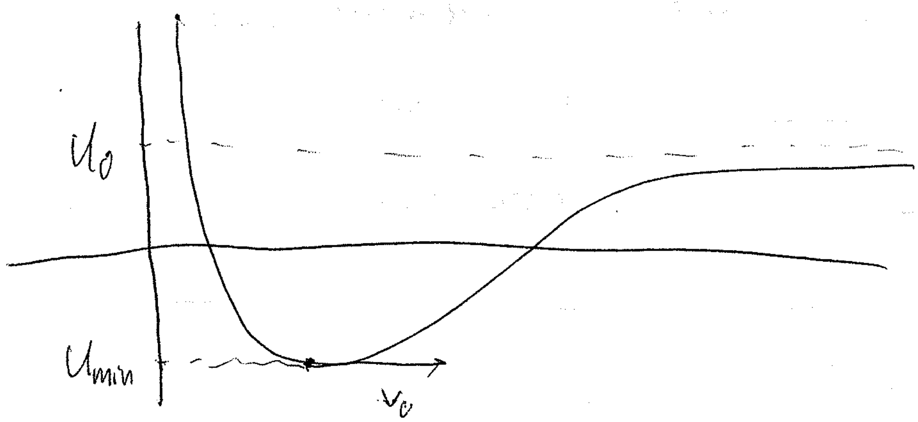
Conserved quantity:

What range of motion possible? Prob 1:

Prob 2:

~~Variand~~

Suppose pot'l bounded at ∞:

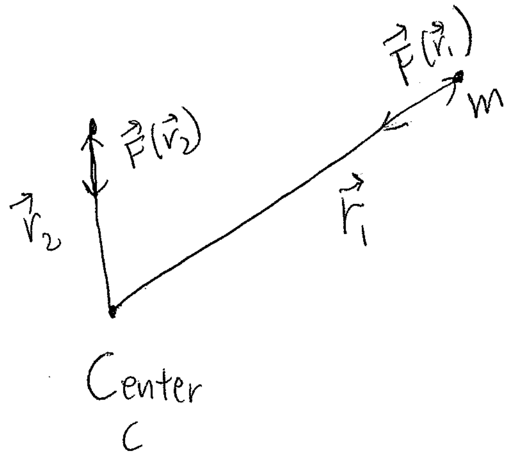


Escape velocity:

~~ww~~ Additional conservation laws $\Rightarrow$ Bounds even in higher dimensional cases

Example: (Extremely important one!):

Central Force motion:



$\vec{F}(\vec{r})$: 1) $|\vec{F}|$ depends only on $|\vec{r}|$

2) $\vec{F} \parallel \vec{r}$ or anti-parallel

Conserved quantities:

So, $\vec{L} = m\vec{r} \times \dot{\vec{r}} = \overrightarrow{\text{constant}}$

Angular momentum conserved

$\Rightarrow$ Orbit always in plane.

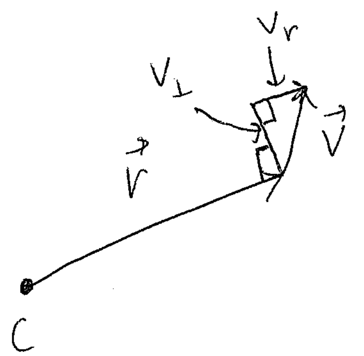
Why?

Which plane?

~~Conse~~

~~Energy conserved~~

Go to that plane:



$\frac{1}{2}m|\vec{v}|^2 + U(r) = E = \text{constant}$

$|\vec{v}|^2 =$

$$\Rightarrow \frac{1}{2}m\dot{r}^2 + \underbrace{U(r) + \frac{L^2}{2mr^2}}_{\equiv\, U_{eff}(r)\text{: "Effective radial potential"}} = E = \text{constant}$$

Compare with E conservation for 1d motion

$r(t)$ in potential $U_{1d}(r) = U_{eff}(r)$

$\frac{1}{2} m \dot{r}^2 + U_{eff}(r) = E$: identical.

$\Rightarrow$ Radial distance $r(t)$ in central force problem evolves just like 1d position in ~~U_{1d} tr~~ potential $U_{1d}(r) = U_{eff}(r)$.

Using this:

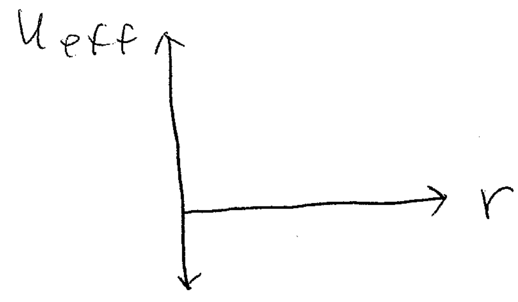
Examples:

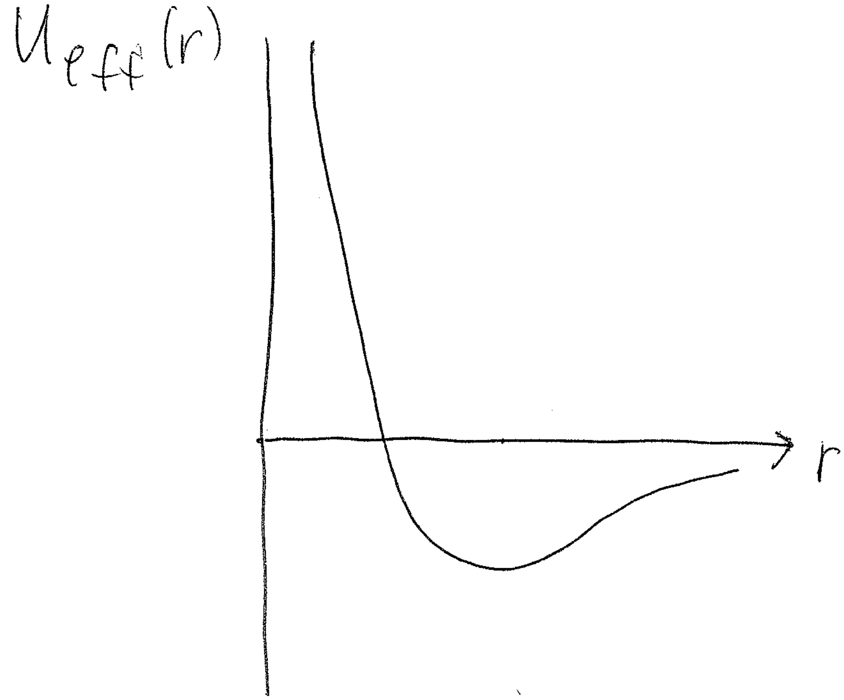
1) $\frac{1}{r^2}$ force (e.g., gravity, electrostatic force (holds atoms together, etc.)

Virtually ubiquitous in nature

$U_{eff}(r) = U(r) + \frac{L^2}{2mr^2} = ?$

~~Plot~~ Plot:





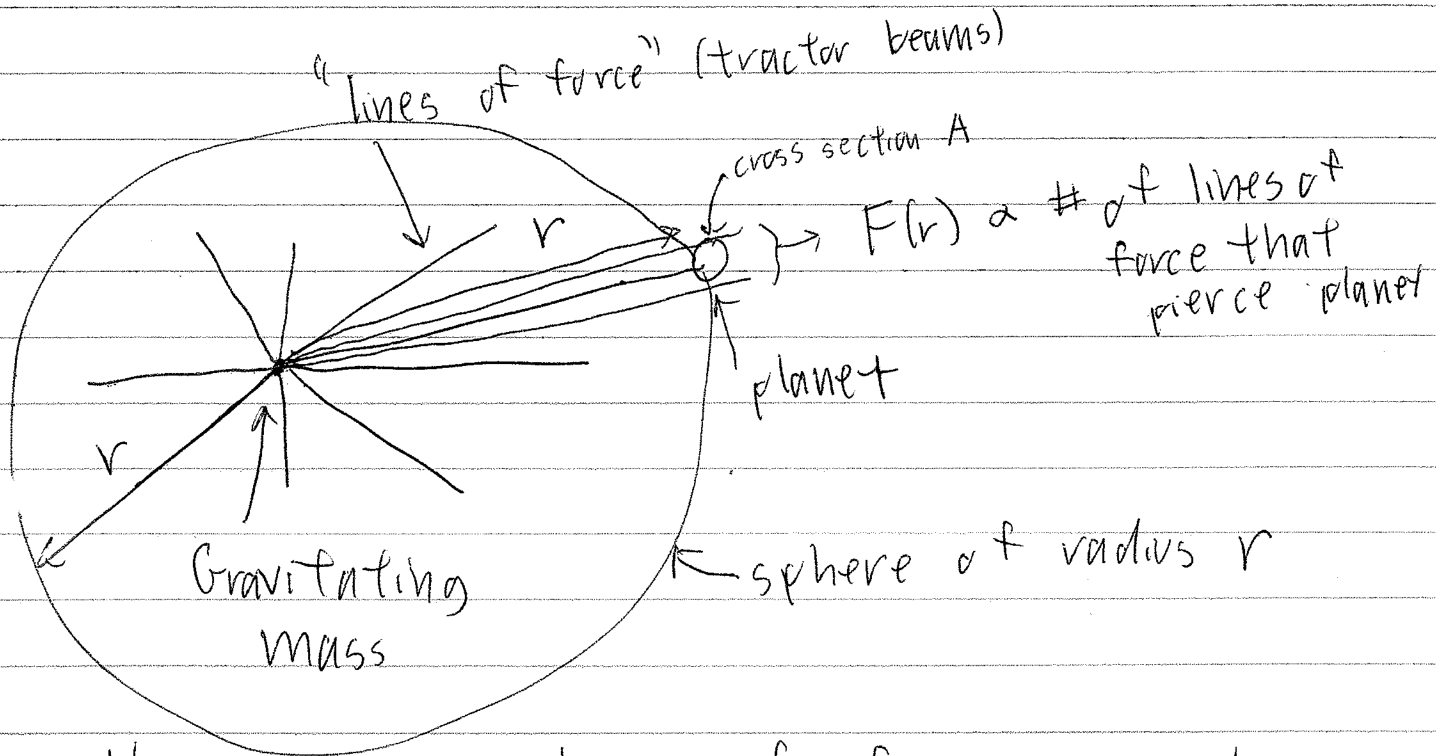
Radius of circular orbit?

Is this orbit stable against a small radial kick?

What, qualitatively, is motion?

Aside: Why is $\frac{1}{r^2}$ force so ubiquitous?

Answer: Science fiction "tractor beam"
+
geometry of 3 dimensional space



How many lines of force, pierce planet at r? Call this number n

Call N total # of lines of force coming out

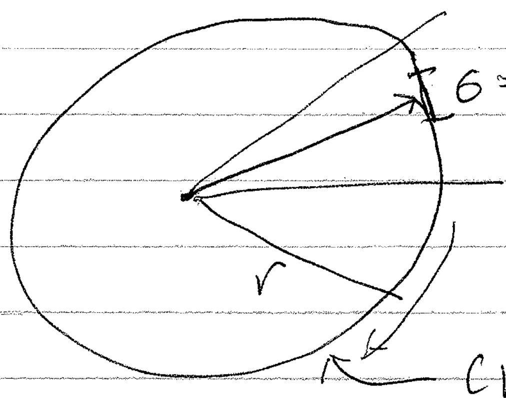
$n =$

$\Rightarrow F(r) = \frac{N}{4\pi r^2} A \propto \frac{1}{r^2}$

$\Rightarrow U(r) \propto \frac{1}{r}$

~~What~~

Suppose ~~world w~~ universe was 2-dimensional?



$F(r) = \frac{\sigma}{\text{Perimeter}} \propto \frac{1}{r^x}$, $x =$

So, in $d = 3$, $F(r) \propto \frac{1}{r^2}$

" " ", $F(r) \propto \frac{1}{r}$

What about arbitrary

d (i.e., 4, 5, etc)?

4,9

Clearly, $F(r) \propto \frac{1}{r^{d-1}}$

$\cancel{U(r)}$

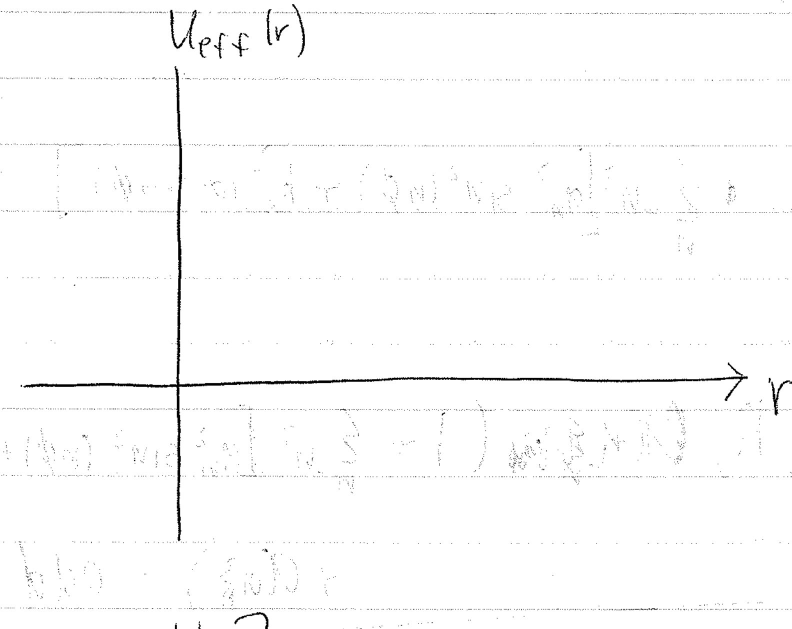
$\Rightarrow \quad U(r) \propto \, ?$

$$\Rightarrow \quad U_{eff}(r) = \frac{L^2}{2mr^2} + U(r) = \frac{L^2}{2mr^2} - \frac{C}{r^{d-2}}$$

Example: $d = 4$

$U_{eff}(r) =$

Plot:



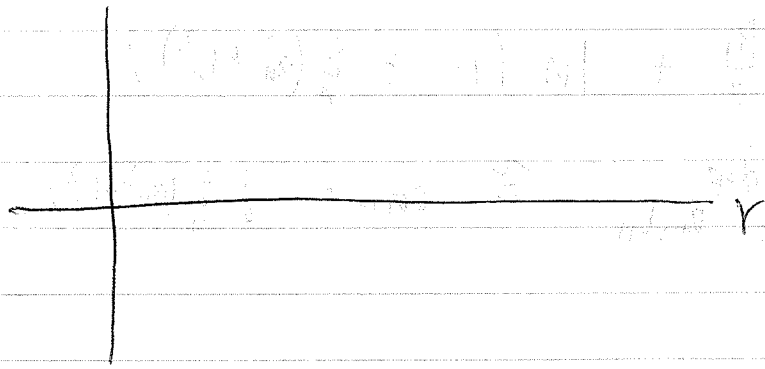
Orbits stable?

4.10

$d > 4$:

$$U_{eff}(r) = \frac{L^2}{2mr^2} - \frac{C}{r^{d-2}}$$

Plot:



orbits stable?

General conclusion: if $U(r) \propto \frac{1}{r^\gamma}$, $\gamma \geq 2$,

orbits are unstable.

$\Rightarrow$ $F(r) \propto \frac{1}{r^\gamma}$, $\gamma \geq 3$, orbits unstable

---

This is all nice + qualitative,

what about quantitative?

If we know E = constant, how does this

help us solve equations of motion?