PHYS 631: Quantum Mechanics I (Fall 2020)
Homework 4
Assigned Monday, 19 October 2020
Due Monday, 26 October 2020

Problem 1. A rite of passage in the training of every physicist is ‘‘completing the square’’ for the first time. You'll have to do this essentially every time you work with a Gaussian state in quantum mechanics, so let's just get this out of the way once and for all. That is, start with the integral formula

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}} \qquad (a \in \mathbb{C},\ \mathrm{Re}[a] > 0). \tag{1}$$

(Do you remember the trick for proving this? If not, ask, look it up, figure it out, etc., as it's a classic.) Use this formula to prove the more general formula

$$\int_{-\infty}^{\infty} e^{-(\alpha x^2+\beta x+\gamma)}\,dx = \sqrt{\frac{\pi}{\alpha}}\,\exp\!\left(\frac{\beta^2}{4\alpha} - \gamma\right) \qquad (\alpha, \beta, \gamma \in \mathbb{C},\ \mathrm{Re}[\alpha] > 0). \tag{2}$$

Problem 2. In class, we showed that the propagator for the free particle can be written

$$K(x,t;x_0,t_0) = \frac{1}{2\pi\hbar}\int dp\, e^{-ip^2(t-t_0)/2m\hbar} e^{ip(x-x_0)/\hbar}. \tag{3}$$

Carry out the integral to show that

$$K(x,t;x_0,t_0) = \sqrt{\frac{m}{i2\pi\hbar(t-t_0)}}\,\exp\!\left[\frac{im(x-x_0)^2}{2\hbar(t-t_0)}\right]. \tag{4}$$

But be careful! Note that the integral in Eq. (3) is *not* of the form (2), nor is the integral even well defined. Carefully (and *explicitly*) regularize the integral and then carry it out to prove the result.

Introduction to Problems 3 and 4. These are classic problems in quantum mechanics, and they are among the most labor-intensive problems you'll work through in this term/class. You may feel a strong temptation to plow through it the ‘‘brute force’’ way, ending up with a solution that looks like the entire discipline of calculus vomited all over your homework.

This *is* a valid way to solve a problem.

It is, however, the amateur-hour approach to problem-solving. I want you to start solving problems like *pro*.

There is an important pedagogical principle that is widely appreciated, though somehow not so much in physics:

Practice does *not* make *perfect*, it makes *permanent*.

Example: Say you're a rock star learning to play a new song (Warren Zevon's ‘‘Werewolves of London’’ will do for an example). If you start making a mistake at some point in the song, and then you keep making the same mistake every time you practice, you won't improve. Not only will you just ingrain the bad habit, you'll make it much harder correct the issue later. The superior approach is to focus on the problem area, play through the section so *slowly* that you can play it *perfectly*, and repeat until you're good at it. The slow-motion approach to learning translates to just about anything: perfecting your tennis swing, improving fluency in a new language, whatever.

This principle works in physics, too. Solving problems will eventually make you a better problem-solver. Solving problems *with an awareness of what you're doing*, especially regarding the efficiency and elegance of your solutions, will get you to a high level much more quickly. And after you level up, not only will everyone be amazed by your newfound physics bad-assery, but it's also a lot *less work* to be efficient.

I want you to try out this approach on the following two problems, focusing on improving your problem-solving technique. Specifically:

- Make use of prior results from this problem set to minimize your workload.
- Stay organized by breaking the problem into smaller, more manageable calculations (the results of Problems 1 and 2 are good examples of these first two principles).
- There are particular forms of the result, and some more or less conventional definitions for symbols here (τ_0, ζ, etc.). Try to be aware of why these are sensible choices (e.g., try to sort out their physical significance). Also, make use of definitions as soon as you can to minimize the complexity your solution.

Problem 3. Consider the free quantum particle with initial state

$$\psi(x, t = 0) = \frac{1}{(2\pi)^{1/4}\sigma^{1/2}}\, e^{-x^2/4\sigma^2}\, e^{ikx}. \tag{5}$$

(a) Show that the solution for all t may be written compactly as

$$\psi(x, t) = \frac{1}{(2\pi)^{1/4}}\sqrt{\frac{\zeta(0)}{\sigma\zeta(t)}}\, \exp\left(\frac{i\tau_0(x - vt)^2}{4\sigma^2\zeta(t)}\right) e^{ik(x-vt/2)}, \tag{6}$$

where

$$\zeta(t) := t - i\tau_0, \qquad \tau_0 := \frac{2m\sigma^2}{\hbar}, \qquad v := \frac{\hbar k}{m}. \tag{7}$$

(b) This expression (6) is elegant, but because of the complex functions in odd places, it may be a bit hard to get a sense of what is going on. To get a better handle on the Gaussian free particle, eliminate $\zeta(t)$ in Eq. (6) to obtain the equivalent expression

$$\psi(x, t) = \frac{1}{(2\pi)^{1/4}\sqrt{\sigma_x(t)}} \exp\left[-\frac{(x - vt)^2}{4\sigma_x^2(t)}\right] \exp\left[i\frac{t(x - vt)^2}{4\tau_0\sigma_x^2(t)}\right] \exp\left[ik\left(x - \frac{vt}{2}\right) - \frac{i}{2}\tan^{-1}\left(\frac{t}{\tau_0}\right)\right], \tag{8}$$

where

$$\sigma_x(t) = \sigma\sqrt{1 + \frac{t^2}{\tau_0^2}} = \sigma\sqrt{1 + \frac{\hbar^2 t^2}{4m^2\sigma^4}}. \tag{9}$$

(c) Now, to make sure we're all on the same page, what is the position uncertainty as a function of time for the Gaussian free particle? (You can read it off of the solution, but justify your reading.) What are physical interpretations of τ_0 and v?

Also, try your hand at giving a physical interpretation to each of the factors in Eq. (8). At minimum you should be able to handle the prefactor and first exponential factor, but give the last two factors a try too.

Actually, I'll do one of the least obvious for you—the arctangent part of the last factor is called a **Gouy phase** (rhymes with "chewy maze") in optics, and it is a slow phase variation associated with the focusing of the wave packet (for $t < 0$) and the dispersal of the wave packet (for $t > 0$). Actually, it has the form of a slight effective *increase* in energy, especially around $t = 0$.

Problem 4. For the time-dependent Gaussian state in Problem 3, show explicitly that, in terms of the canonical operators p and x, it is consistent with:

(a) the standard uncertainty relation

$$V_P V_Q \geq \frac{1}{4}\Big|\big\langle [P, Q] \big\rangle\Big|^2, \tag{10}$$

which in this case reduces to

$$V_x V_p \geq \frac{\hbar^2}{4}. \tag{11}$$

(b) the generalized (stronger) uncertainty relation

$$V_P V_Q \geq \frac{1}{4}\Big|\big\langle [P, Q] \big\rangle\Big|^2 + \frac{1}{4}\Big|\big\langle [P, Q]_+ \big\rangle - 2\langle P \rangle \langle Q \rangle\Big|^2. \tag{12}$$

which may be rewritten

$$V_x V_p - C_{xp}^2 \geq \frac{\hbar^2}{4} \tag{13}$$

in terms of the **symmetrized covariance**

$$C_{xp} := \frac{1}{2}\big\langle [x, p]_+ \big\rangle - \langle x \rangle \langle p \rangle\,. \tag{14}$$

In fact, you should find that the equality is satisfied for the Gaussian state, something that turns out to be unique to Gaussian wave functions.