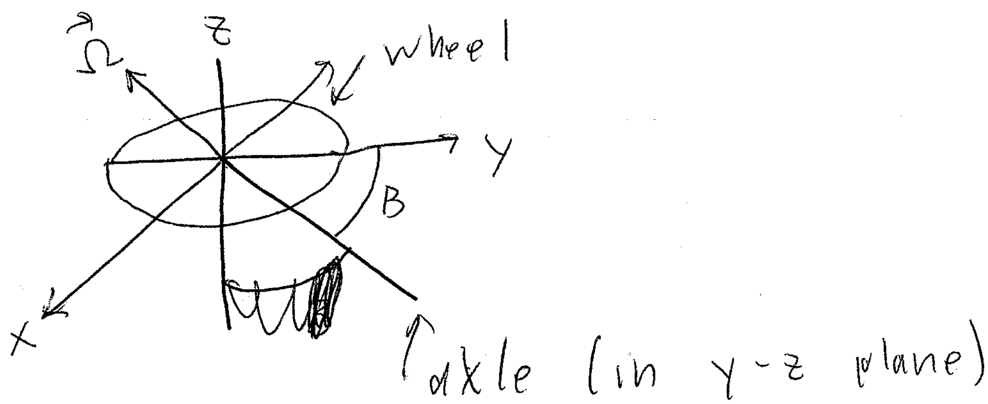
Problem Set # 4 Solutions

1) The moment of interia tensor is obviously easiest to calculate in the co-ordinate system illustrated below:



In which its only non-zero components are

$$(1.1)\quad I_{xx} = \rho \int dx\, dy\, (r^2 - x^2) = \rho \int dx\, dy\, y^2$$

circular co-ordinates

$$(1.2)\quad I_{yy} = \rho \int dx\, dy\, (r^2 - y^2) = \rho \int dx\, dy\, x^2$$

and

$$\boxed{(1.3)\quad I_{zz} = \rho \int dx\, dy\, r^2}$$

1) (cont) Clearly, by symmetry,

$$\boxed{(1.4)\quad I_{xx} = I_{yy}}$$

and, equally clearly,

$$\boxed{(1.5)\quad I_{xx} + I_{yy} = 2I_{xx} = I_{zz}}$$

(since $x^2+y^2=r^2$).

So we only need to calculate

$$\boxed{(1.6)\quad I_{zz} = \rho \int dx\,dy\; r^2 \underset{\substack{\uparrow \\ \text{polar ~~cylindrical~~} \\ \text{co-ordinates}}}{=} 2\pi\rho \int_0^R r^2 (r\,dr) = \frac{\pi\rho R^4}{2}}$$

Since $\rho A = \rho \pi R^2 = m$, $\boxed{\rho = \frac{m}{\pi R^2} \quad (1.7)}$

So, using (1.5), (1.6), and (1.7), I get

$$\boxed{\underset{\sim}{I} = \frac{mR^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad (1.8)}$$

1) cont) The energy is

$$\boxed{(1.9)\quad T = \frac{1}{2} I_{ij}\Omega_i \Omega_j = \frac{1}{2}\left(I_{xx}\Omega_x^2 + I_{yy}\Omega_y^2 + I_{zz}\Omega_z^2\right)}$$

Since $\vec{\Omega}$ is directed along the axle,

$$\boxed{(1.10)\quad \Omega_x = 0, \quad \Omega_z = \Omega \sin B, \quad \Omega_y = -\Omega \cos B}$$

Inserting these into (1.9) and using (1.8), I get

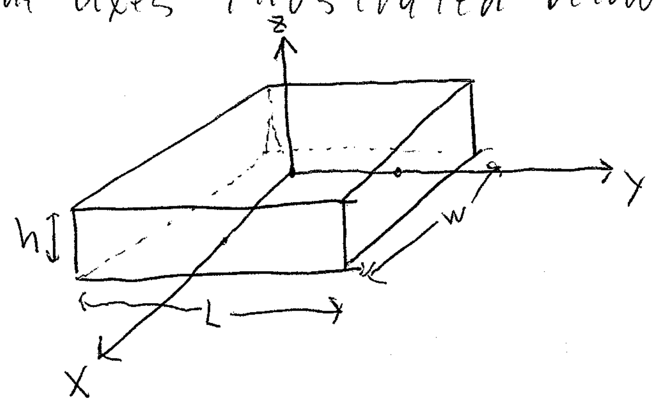
$$\boxed{\begin{aligned}(1.11)\quad T &= \frac{mR^2}{8}\Omega^2\left(2\sin^2 B + \cos^2 B\right)\\ &= \frac{mR^2}{8}\Omega^2\left(1+\sin^2 B\right)\end{aligned}}$$

which is clearly maximized when $B = \frac{\pi}{2}$ (i.e., when the axle is perpendicular to the ~~wheel~~ tire; that is, when the tire is properly aligned.)

2) a) The center of mass is clearly at the center of the brick (i.e., a distance $\frac{L}{2}$ from both the left and right walls, $\frac{W}{2}$ from front + back, and $h/2$ from top and bottom. If I choose my axes to be perpendicular to the faces, it is clear by symmetry that the off-diagonal components of $\underset{\sim}{I}$ vanish. This leaves the task of computing the diagonal terms.

~~(2.1) $I_{xx} = \int_{-L}$~~

Using the axes illustrated below:



I get

2a) cont)

$$\boxed{(2.1) \quad I_{xx} = \rho \int_{-\frac{w}{2}}^{\frac{w}{2}} dx \int_{-\frac{L}{2}}^{\frac{L}{2}} dy \int_{-\frac{h}{2}}^{\frac{h}{2}} dz \left(y^2+z^2\right)}$$

Doing the integrals in sequence:

$$\boxed{(2.2) \quad \int_{-\frac{h}{2}}^{\frac{h}{2}} dz \left(y^2+z^2\right) = y^2 h + \frac{2}{3}\left(\frac{h}{2}\right)^3 = h\left(y^2 + \frac{h^2}{12}\right)}$$

$$(2.3) \quad \int_{-\frac{L}{2}}^{\frac{L}{2}} dy \; h\left(y^2+\frac{h^2}{12}\right) = \frac{2h}{3}\left(\frac{L}{2}\right)^3 + L\frac{h^3}{12}$$

$$= \frac{hL(L^2+h^2)}{12}$$

This being independent of x, the x integral just gives w times this, and we get

$$\boxed{(2.4) \quad I_{xx} = \frac{\rho w h L (h^2+L^2)}{12}}$$

Since $\rho V = \rho L w h = m$, $\boxed{\rho = \frac{m}{Lwh} \quad (2.5)}$

2a) (cont) Using this in (2.4), I get

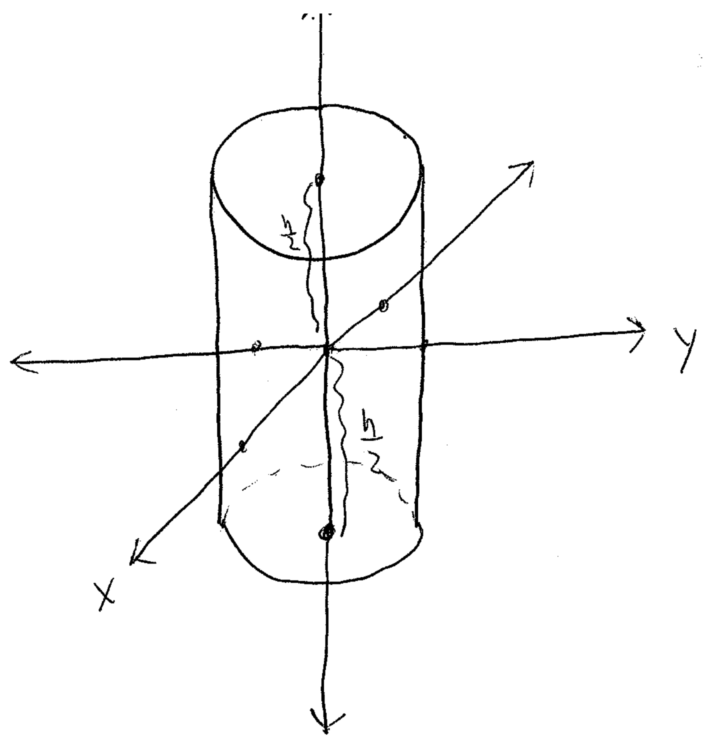
$$\boxed{(2.6) \quad I_{xx} = \frac{m(L^2+h^2)}{12}}$$

The other moments of inertia follow by identical calculations; or, equivalently, by simple permutatio of L, h, and w:

$$\boxed{(2.7) \quad I_{yy} = \frac{m(h^2+w^2)}{12}, \quad I_{zz} = \frac{m(L^2+w^2)}{12}}$$

2b) I'll take my z-axis to be that of the cylinder; and my x and y axes to be any ~~pair in~~ perpendicular pair in the plane perpendicular to z. ~~M~~ The center of mass is on the z-axis halfway up the cylinder (see figure →)

2 b) cont)



~~working in cylindri~~

Clearly, by symmetry, I again have a diagonal moment of inertia tensor. Furthermore, by cylindrical symmetry, $I_{xx} = I_{yy}$.

Furthermore,

$$(2.8)\quad I_{xx} + I_{yy} = 2I_{xx} = \rho \int dV \left[\overbrace{x^2+y^2+z^2-x^2}^{I_{xx}} + \underbrace{x^2+y^2+z^2-y^2}_{I_{yy}} \right]$$

$$= \rho \int dV \left[x^2+y^2+2z^2 \right] \underset{\substack{\uparrow \\ \text{cylindrical} \\ \text{co-ordinates}}}{=} 2\pi\rho \int_0^R r\,dr \int_{-\frac{h}{2}}^{\frac{h}{2}} dz \left(r^2 + 2z^2\right)$$

Using $\boxed{\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 dz = \frac{2}{3}\left(\frac{h}{2}\right)^3 = \frac{h^3}{12} \quad (2.9)}$

~~and~~ $\boxed{\int_0^R r\,dr = \frac{R^2}{2} \quad (2.10)}$, $\boxed{\int_{-\frac{h}{2}}^{\frac{h}{2}} dz = h \quad (2.11)}$

and

$$\boxed{\int_0^R r^3 dr = \frac{R^4}{4} \quad (2.12)}$$

~~I get~~ and dividing (2.8) by 2, I get

$$\boxed{(2.13) \quad I_{xx} = I_{yy} = \pi\rho\left[\frac{R^2h^3}{12} + \frac{hR^4}{4}\right]}$$

~~which~~ Finding ρ using

$\rho V = \rho \pi R^2 h = m$, I get $\boxed{\rho = \frac{m}{\pi R^2 h} \quad (2.14)}$

which turns (2.13) into

$$\boxed{(2.14) \quad I_{xx} = I_{yy} = \frac{m}{12}\left[h^2 + 3R^2\right]}$$

For I_{zz}, I get

$$(15)\, I_{zz} = \rho \int dV \left[x^2+y^2+z^2-z^2\right] = 2\pi\rho \int_0^R r\,dr \int_{-\frac{h}{2}}^{\frac{h}{2}} dz\, r^2 = \frac{\pi\rho R^4 h}{2}$$

$$= mR^2$$

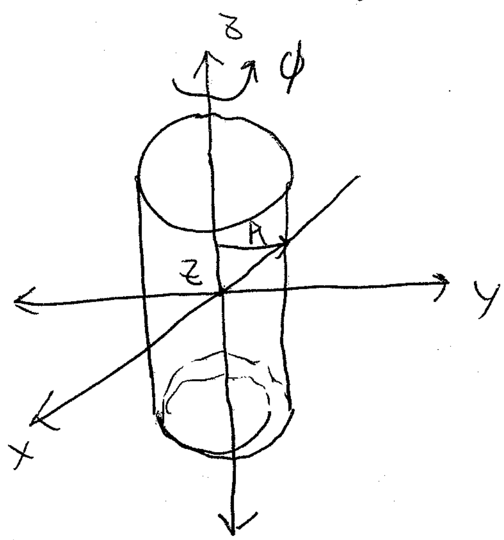
2c) If all of the mass lies on the outer wall, I now have to integrate over the area of the surface:

$$\boxed{(2.16)\ I_{ij} = \rho_s \int dA \left(r^2 \delta_{ij} - r_i r_j\right)}$$

where ~~the~~ $\boxed{\rho_s = \frac{m}{A_{surface}} = \frac{m}{2\pi R h} \quad (2.17)}$

is the mass per unit area on the surface.

Choosing my axes as in the previous problem, I have, using z and ϕ as co-ordinates on the surface (see fig:)



2c) cont)

$$\boxed{(2.18) \quad r_i = z\,\delta_{iz} + R\left(\cos\phi\,\delta_{ix} + \sin\phi\,\delta_{iy}\right)}$$

Using this and (2.17) in (2.16), I get

$$(2.19) \quad I_{xx} = \frac{m}{2\pi R h}\int_0^{2\pi} R\,d\phi \int_{-\frac{h}{2}}^{\frac{h}{2}} dz\left(R^2 + z^2 - R^2\cos^2\phi\right)$$

$$= \frac{m}{2\pi R h}\int_0^{2\pi} R\,d\phi \int_{-\frac{h}{2}}^{\frac{h}{2}} dz\left(R^2\sin^2\phi + z^2\right)$$

Where I've used the fact that the element of area on the surface of the cylinder is $R\,d\phi\,dz$.

Using the integral (2.9) and $\int_0^{2\pi}\sin^2\phi\,d\phi = \pi$, I get

$$\boxed{(2.20) \quad I_{xx} = \frac{m}{2\pi R h}\left[\pi h R^3 + \frac{2\pi h^3 R}{12}\right] = \frac{m\left(6R^2 + h^2\right)}{12} = I_{yy}}$$

the last equality following by symmetry.

2c) (cont) For I_{zz}, I have

$$(2.21) \quad I_{zz} \doteq \frac{m}{2\pi R h} \int_0^{2\pi} R\, d\phi \int_{-\frac{h}{2}}^{\frac{h}{2}} dz\, R^2 = mR^2$$

unsurprizingly, since all of the mass is at the same distance R from the z-axis.

3) In all cases, I just need to find $\vec{\Omega} \doteq (\Omega_x, \Omega_y, \Omega_z)$ in the system of principal axes for which $\underset{\sim}{I}$ ~~computed~~ was diagonal. Then I have for the kinetic energy

$$(3.1) \quad T = \frac{1}{2} I_{ij} \Omega_i \Omega_j = \frac{1}{2} \left[I_{xx} \Omega_x^2 + I_{yy} \Omega_y^2 + I_{zz} \Omega_z^2 \right]$$

For problem a, $\vec{\Omega}$ is parallel to the body diagonal vector $\vec{B}$:

3a) cont)

$$\boxed{(3.2)\quad \vec{B} = (w, L, h)}$$

So,
$$\boxed{\vec{\Omega} = \Omega \frac{\vec{B}}{|\vec{B}|} = \frac{(w\Omega, L\Omega, h\Omega)}{\sqrt{w^2+L^2+h^2}} \quad (3.3)}$$

Using this in (3.1), I get

$$\boxed{(3.4)\quad T = \frac{1}{2}\Omega^2\left(\frac{I_{xx}w^2 + I_{yy}L^2 + I_{zz}h^2}{w^2+L^2+h^2}\right)}$$

Inserting I_{xx}, I_{yy}, and I_{zz} from problem 2, eqns. (2.6), (2.7)

I get

$$\boxed{(3.5)\quad T = \frac{m\Omega^2}{24}\left(\frac{w^2(L^2+h^2)+L^2(h^2+w^2)+h^2(L^2+w^2)}{w^2+L^2+h^2}\right)}$$

3b) Taking the "body diagonal" points to lie in the y-z plane (no loss of generality here, thanks to the cylindrical symmetry)

3b) cont) I have

$$\boxed{(3.6) \quad \vec{B} = (\cancel{2R}\,0,\ 2R,\ h)}$$

So

$$\boxed{(3.7) \quad \vec{\Omega} = \Omega \frac{\vec{B}}{|\vec{B}|} = \frac{\Omega(0,\ 2R,\ h)}{\sqrt{4R^2+h^2}}}$$

Hence, from (3.1)

$$\cancel{(3.8) \quad T = \frac{\Omega^2}{2} I_{yy} M}$$

$$\boxed{(3.8) \quad T = \frac{\Omega^2}{2}\left[\frac{4R^2 I_{yy} + h^2 I_{zz}}{4R^2+h^2}\right]}$$

Using the ~~earlier~~ results (2.14), (2.15)

for I_{yy} and I_{zz}, I get, after some algebra,

$$\cancel{(3.9) \quad T = \frac{m\Omega^2}{6} \quad R^2(h^2+3R^2)+}$$

$$\boxed{(3.9) \quad T = \frac{mR^2\Omega^2}{12}\left(\frac{5h^2+R^2}{4R^2+h^2}\right)}$$

3c) Everything in this calculation is identical to 3b, except that the moment of inertia tensor is different. So now, I need to use (2.20) and (2.21) for I_{yy} and I_{zz} in (3.8). This gives, after gathering terms:

~~(3.10)~~ $\cancel{T = \frac{mR^2\Omega^2}{3} \, 3R^2+2h^2}$

$$\boxed{(3.10) \quad T = \frac{mR^2\Omega^2}{3} \frac{(3R^2+2h^2)}{(4R^2+h^2)}}$$

4)