Physics 614 Homework 5

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1. High-temperature limit of the Fermi gas

We claim that the equation of state for an ideal Fermi gas with spin s and incorporating the first quantum correction at high temperatures is

$$P = \rho k_B T \left(1 + \frac{\rho\lambda^3}{4\sqrt{2}(2s+1)}\right) \tag{1}$$

where $\rho = N/V$ and $\lambda = h/\sqrt{2\pi m k_B T}$. Recall that the grand potential, Φ, for the Fermi gas satisfies

$$\Phi = -k_B T \sum_\alpha \ln\left(1 + e^{-\beta(\epsilon_\alpha - \mu)}\right) = -PV \tag{2}$$

and that the density of states for a Fermi gas with spin s is given by

$$g(\epsilon) = (2s+1)\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2} \tag{3}$$

With equation (3), and $z = e^{\beta\mu}$ we can write equation (2) as an integral over ϵ

$$
\begin{aligned}
PV &= k_B T \int_0^\infty d\epsilon\; g(\epsilon) \ln\bigl(1 + z e^{-\beta\epsilon}\bigr) \\
P &= \frac{2s+1}{4\pi^2\beta} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty d\epsilon\; \epsilon^{1/2} \ln\bigl(1 + z e^{-\beta\epsilon}\bigr) \\
P &= \frac{2s+1}{4\pi^2\beta} \left(\frac{2m}{\beta\hbar^2}\right)^{3/2} \int_0^\infty dy\; y^{1/2} \ln\bigl(1 + z e^{-y}\bigr) \\
P &= -\frac{2s+1}{4\pi^2\beta} \left(\frac{2m}{\beta\hbar^2}\right)^{3/2} \frac{2}{3} \int_0^\infty dy\; \frac{-z e^{-y} y^{3/2}}{1 + z e^{-y}} \\
P &= \frac{2s+1}{4\pi^2\beta} \left(\frac{2m}{\beta\hbar^2}\right)^{3/2} \frac{2}{3} \int_0^\infty dy\; \frac{y^{3/2}}{z^{-1} e^{y} + 1} \\
P &= \frac{2s+1}{4\pi^2\beta} \left(\frac{2m}{\beta\hbar^2}\right)^{3/2} \frac{\sqrt{\pi}}{2} \bigl(-g_{5/2}(-z)\bigr) \\
P &= \frac{2s+1}{\beta} \left(\frac{m}{2\pi\beta\hbar^2}\right)^{3/2} \bigl(-g_{5/2}(-z)\bigr) \\
P &= \frac{2s+1}{\beta} \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \bigl(-g_{5/2}(-z)\bigr) \\
P &= -(2s+1) \frac{k_B T}{\lambda^3} g_{5/2}(-z)
\end{aligned}
\tag{4}
$$

We can also write the density, ρ, as

$$
\begin{aligned}
N &= \int_0^\infty d\epsilon\; \frac{g(\epsilon)}{z^{-1} e^{\beta\epsilon} + 1} \\
\rho &= (2s+1) \frac{1}{4\pi^2} \left(\frac{2m}{\beta\hbar^2}\right)^{3/2} \int_0^\infty d\epsilon\; \frac{y^{1/2}}{z^{-1} e^{y} + 1} \\
\rho &= (2s+1) \frac{1}{4\pi^2} \left(\frac{2m}{\beta\hbar^2}\right)^{3/2} \left(-\frac{\sqrt{\pi}}{2} g_{3/2}(-z)\right) \\
\rho\lambda^3 &= -(2s+1) g_{3/2}(-z)
\end{aligned}
\tag{5}
$$

Now, dividing equation (4) by equation (5), we have

$$
P = \rho k_B T \frac{g_{5/2}(-z)}{g_{3/2}(-z)} \tag{6}
$$

We can expand the quotient in equation (6) to lowest nontrivial order in z (using Mathematica out of laziness) around $z = 0$ since equation (5) requires that $z \to 0$ as $T \to \infty$. This yields

$$
P = \rho k_B T \left(1 + \frac{z}{4\sqrt{2}}\right) \tag{7}
$$

We can now expand equation (5) to first nontrivial order to solve for z, which gives

$$\begin{aligned} \rho\lambda^3 &= (2s+1)z \\ \implies z &= \frac{\rho\lambda^3}{2s+1} \end{aligned} \tag{8}$$

Thus, the equation of state for the ideal Fermi gas in the high temperature limit is

$$P = \rho k_B T \left(1 + \frac{\rho\lambda^3}{4\sqrt{2}(2s+1)}\right) \tag{9}$$

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2. Degenerate Fermi gas in 1D and 2D

i. To find the Fermi energy we first find the Fermi momentum which is given by counting the states in the volume of k space of radius k_F.

$$\begin{aligned} \frac{2k_F}{2\pi/L}(2s+1) &= N \\ k_F &= \frac{N}{L}\frac{\pi}{2s+1} \end{aligned} \tag{10}$$

So the Fermi energy, ϵ_F, is just

$$\begin{aligned} \epsilon_F &= \frac{\hbar^2 k^2}{2m} \\ \epsilon_F &= \frac{\hbar^2}{2m}\left[\frac{N}{L}\frac{\pi}{2s+1}\right]^2 \end{aligned} \tag{11}$$

We can compute the density of states to find the average internal energy. Set $g(\epsilon) = A\epsilon^{-1/2}$. Then

$$\begin{aligned} N &= \int_0^{\epsilon_F} d\epsilon\; g(\epsilon) \\ N &= 2A\epsilon_F^{1/2} \\ \implies g(\epsilon) &= \frac{N}{2}\left(\epsilon_F\, \epsilon\right)^{-1/2} \end{aligned} \tag{12}$$

With this, the average internal energy is just

$$\begin{aligned}
\langle E\rangle &= \int_0^{\epsilon_F} d\epsilon\ \epsilon\, g(\epsilon) \\
\langle E\rangle &= \frac{N}{2\epsilon_F^{1/2}} \int_0^{\epsilon_F} d\epsilon\ \epsilon^{1/2} \\
\langle E\rangle &= \frac{N}{3\epsilon_F^{1/2}} \epsilon_F^{3/2} \\
\langle E\rangle &= \frac{N}{3}\epsilon_F
\end{aligned} \tag{13}$$

In which case, the degeneracy pressure is

$$\begin{aligned}
P &= \frac{1}{L}\left(N\epsilon_F - \langle E\rangle\right) \\
P &= \frac{1}{L}\left(N\epsilon_F - \frac{N}{3}\epsilon_F\right) \\
P &= \frac{2}{3}\frac{N}{L}\epsilon_F
\end{aligned} \tag{14}$$

ii. In a two dimensional box of side length L, the Fermi momentum is

$$\begin{aligned}
\frac{\pi k_F^2}{(2\pi/L)^2}(2s+1) &= N \\
k_F^2 &= \frac{N}{L^2}\frac{4\pi}{2s+1}
\end{aligned} \tag{15}$$

and the Fermi energy is

$$\begin{aligned}
\epsilon_F &= \frac{\hbar^2 k^2}{2m} \\
\epsilon_F &= \frac{\hbar^2}{m}\frac{N}{L^2}\frac{2\pi}{2s+1}
\end{aligned} \tag{16}$$

Solving for the density of states, $g(\epsilon) = A$, we have

$$\begin{aligned}
N &= A\int_0^{\epsilon_F} d\epsilon \\
\implies g(\epsilon) &= \frac{N}{\epsilon_F}
\end{aligned} \tag{17}$$

Using this to find the average internal energy gives

$$\begin{aligned}\langle E\rangle &= \frac{N}{\epsilon_F}\int_0^{\epsilon_F} d\epsilon\ \epsilon\, g(\epsilon)\\ \langle E\rangle &= \frac{N}{2}\epsilon_F\end{aligned} \tag{18}$$

Finally, the degeneracy pressure is

$$\begin{aligned}P &= \frac{1}{L^2}\left(N\epsilon_F - \langle E\rangle\right)\\ P &= \frac{1}{2}\frac{N}{L^2}\epsilon_F\end{aligned} \tag{19}$$

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3. Ultrarelativistic degenerate Fermi gas

We consider an ideal gas of N ultrarelativistic fermions of spin s in a volume V in $3D$, which have the energy-momentum relation $\epsilon(\boldsymbol{k}) = \hbar k c$, where c is the speed of light, $\boldsymbol{k}$ is the wavevector indexing the plane-wave energy eigenstate, and $k = |\boldsymbol{k}|$. And we work in the $T \to 0$ limit.

i. We first compute the density of states, $g(\epsilon)$ for this system.

$$\begin{gathered}\sum_{\boldsymbol{k}}(2s+1) \to \frac{V(2s+1)}{(2\pi)^3}\int d\boldsymbol{k} \to \frac{V(2s+1)}{2\pi^2}\int k^2 dk \to \frac{V(2s+1)}{2\pi^2\hbar^3c^3}\int \epsilon^2 d\epsilon\\ \implies g(\epsilon) = \frac{V(2s+1)}{2\pi^2\hbar^3c^3}\epsilon^2\end{gathered} \tag{20}$$

The Fermi energy is then given by

$$\begin{aligned}N &= \int_0^{\epsilon_F} d\epsilon\ g(\epsilon)\\ N &= \frac{V(2s+1)}{6\pi^2\hbar^3c^3}\epsilon_F^3\\ \implies \epsilon_F &= \left(\frac{6\pi^2 N\hbar^3c^3}{V(2s+1)}\right)^{1/3}\end{aligned} \tag{21}$$

which allows us to rewrite the density of states as $g(\epsilon) = 3N\epsilon^2/\epsilon_F^3$

ii. The average internal energy, $\langle E \rangle$, is given simply in terms of ϵ_F by

$$\begin{aligned}
\langle E \rangle &= \int_0^{\epsilon_F} d\epsilon \; \epsilon \, g(\epsilon) \\
\langle E \rangle &= \frac{3N}{\epsilon_F^3} \int_0^{\epsilon_F} d\epsilon \; \epsilon^3 \\
\langle E \rangle &= \frac{3}{4} N \epsilon_F
\end{aligned} \tag{22}$$

The average internal energy of the ultrarelativistic Fermi gas is higher than the nonrelativistic gas. In the relativistic case, $g(\epsilon) \sim \epsilon^2$ while in the nonrelativistic case, $g(\epsilon) \sim \epsilon^{1/2}$. Therefore, for $\epsilon < \epsilon_F$, the distribution of states for the relativistic case will be more heavily skewed towards ϵ_F. Whereas, for the nonrelativistic case, there are a larger number of states near $\epsilon = 0$.

iii. The degeneracy pressure is then

$$\begin{aligned}
P &= \frac{1}{V} \left(N\epsilon_F - \langle E \rangle \right) \\
P &= \frac{1}{V} \left(N\epsilon_F - \frac{3}{4} N \epsilon_F \right) \\
P &= \frac{1}{4} \frac{N}{V} \epsilon_F
\end{aligned} \tag{23}$$

which we may also write as

$$\begin{aligned}
P &= \frac{1}{V} \left(\frac{4}{3} \langle E \rangle - \langle E \rangle \right) \\
P &= \frac{1}{3} \frac{\langle E \rangle}{V}
\end{aligned} \tag{24}$$

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4. Low-temperature Fermi gas from Sommerfeld expansion

i. For the ideal Fermi gas in $3D$, $g(\epsilon) = A\epsilon^{1/2}$. First obverse that

$$\begin{aligned}
N &= \int_0^{\infty} d\epsilon \; \frac{g(\epsilon)}{e^{\beta(\epsilon - \mu)} + 1} \\
\frac{N}{A} &= \int_0^{\infty} d\epsilon \; \frac{\epsilon^{1/2}}{e^{\beta(\epsilon - \mu)} + 1}
\end{aligned} \tag{25}$$

And recall that $A = 3N/2\epsilon_F^{3/2}$. So we have

$$\begin{aligned} \frac{N}{A} &= \int_0^\infty d\epsilon \, \frac{\epsilon^{1/2}}{e^{\beta(\epsilon-\mu)}+1} \\ \implies \frac{2}{3}\epsilon_F^{3/2} &= \int_0^\infty d\epsilon \, \frac{\epsilon^{1/2}}{e^{\beta(\epsilon-\mu)}+1} \end{aligned} \tag{26}$$

Now, using the Sommerfield expansion, we can rewrite equation (26) as follows:

$$\begin{aligned} \frac{2}{3}\epsilon_F^{3/2} &= \int_0^\infty d\epsilon \, \frac{\epsilon^{1/2}}{e^{\beta(\epsilon-\mu)}+1} \\ \frac{2}{3}\epsilon_F^{3/2} &\approx \int_0^\mu d\epsilon \, \epsilon^{1/2} + \frac{\pi^2}{6}(k_B T)^2 \frac{1}{2\sqrt{\mu}} \end{aligned} \tag{27}$$

Solving for μ, we have

$$\begin{aligned} \frac{2}{3}\epsilon_F^{3/2} &\approx \frac{2}{3}\mu^{3/2} + \frac{\pi^2}{6}(k_B T)^2 \frac{1}{2\sqrt{\mu}} \\ \mu^{3/2} &\approx \epsilon_F^{3/2} - \frac{\pi^2}{8}(k_B T)^2 \frac{1}{\sqrt{\mu}} \end{aligned} \tag{28}$$

In the low T limit, $\mu \approx \epsilon_F$, so we can rewrite equation (28) as

$$\begin{aligned} \mu^{3/2} &\approx \epsilon_F^{3/2} - \frac{\pi^2}{8}(k_B T)^2 \frac{1}{\sqrt{\epsilon_F}} \\ \mu^{3/2} &\approx \epsilon_F^{3/2}\left[1 - \frac{\pi^2}{8}(k_B T)^2 \frac{1}{\epsilon_F^2}\right] \\ \mu &\approx \epsilon_F \left[1 - \frac{\pi^2}{8}\left(\frac{T}{T_F}\right)^2\right]^{2/3} \\ \mu &\approx \epsilon_F \left[1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right] \end{aligned} \tag{29}$$

via the binomial expansion.

ii. We can perform a similar calculation to approximate $\langle E \rangle$:

$$
\begin{aligned}
\langle E\rangle &= \int_0^\infty d\epsilon\ \frac{\epsilon\, g(\epsilon)}{e^{\beta(\epsilon-\mu)}+1}\\
\langle E\rangle &= A\int_0^\infty d\epsilon\ \frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)}+1}\\
\langle E\rangle &\approx A\int_0^\mu d\epsilon\ \epsilon^{3/2}+\frac{\pi^2}{6}(k_BT)^2(g(\mu)\mu)'\\
\langle E\rangle &\approx A\frac{2}{5}\mu^{5/2}+A\frac{\pi^2}{4}(k_BT)^2\mu^{1/2}\\
\langle E\rangle &\approx A\frac{2}{5}\left(\mu^{5/2}+\frac{5\pi^2}{8}(k_BT)^2\mu^{1/2}\right)\\
\langle E\rangle &\approx A\frac{2}{5}\left(\epsilon_F^{5/2}\left(1-\frac{5\pi^2}{24}\left(\frac{T}{T_F}\right)^2\right)+\frac{5\pi^2}{8}(k_BT)^2\epsilon_F^{1/2}\left(1-\frac{\pi^2}{24}\left(\frac{T}{T_F}\right)^2\right)\right)\\
\langle E\rangle &\approx A\frac{2}{5}\epsilon_F^{5/2}\left(1-\frac{5\pi^2}{24}\left(\frac{T}{T_F}\right)^2+\frac{5\pi^2}{8}\left(\frac{T}{T_F}\right)^2\left(1-\frac{\pi^2}{24}\left(\frac{T}{T_F}\right)^2\right)\right)\\
\langle E\rangle &\approx A\frac{2}{5}\epsilon_F^{5/2}\left(1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2+\mathcal{O}\left(\left(\frac{T}{T_F}\right)^4\right)\right)\\
\langle E\rangle &\approx \frac{2}{5}\epsilon_F^2\, g(\epsilon_F)\left(1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right)
\end{aligned}
\tag{30}
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