Ex: prove Cauchy-Schwarz

claim: $\langle x, x\rangle \langle y, y\rangle \geq |\langle x, y\rangle|^2$

Proof:

let $z = y - \frac{\langle x, y\rangle}{\langle x, x\rangle} x$

Observe that

$$\langle z, z\rangle = \left\langle y - \frac{\langle x, y\rangle}{\langle x, x\rangle} x, \; y - \frac{\langle x, y\rangle}{\langle x, x\rangle} x \right\rangle$$

$$= \langle y, y\rangle + \left|\frac{\langle x, y\rangle}{\langle x, x\rangle}\right|^2 \langle x, x\rangle$$

$$- \left\langle y, \frac{\langle x, y\rangle}{\langle x, x\rangle} x \right\rangle$$

$$- \left\langle \frac{\langle x, y\rangle}{\langle x, x\rangle} x, y \right\rangle \geq 0$$

So $\langle y, y\rangle + \left|\frac{\langle x, y\rangle}{\langle x, x\rangle}\right|^2 \langle x, x\rangle$

$$- \frac{\langle x, y\rangle}{\langle x, x\rangle} \langle y, x\rangle - \left(\frac{\langle x, y\rangle}{\langle x, x\rangle}\right)^* \langle x, y\rangle \geq 0$$

$$\langle y, y\rangle + \left|\frac{\langle x, y\rangle}{\langle x, x\rangle}\right|^2 \langle x, x\rangle \geq$$

$$\frac{\langle x, y\rangle}{\langle x, x\rangle} \langle y, x\rangle + \left(\frac{\langle x, y\rangle}{\langle x, x\rangle}\right)^* \langle x, y\rangle$$

$$\rightarrow \langle y, y\rangle + \left|\frac{\langle x, y\rangle}{\langle x, x\rangle}\right|^2 \langle x, x\rangle \geq$$

$$\frac{\langle x, y\rangle \langle x, y\rangle^* + \langle x, y\rangle^* \langle x, y\rangle}{\langle x, x\rangle}$$

$\rightarrow$ multiplying both sides by $\langle x, x\rangle$,

$$\langle y, y\rangle \langle x, x\rangle + \left|\frac{\langle x, y\rangle}{\langle x, x\rangle}\right|^2 \langle x, x\rangle^2$$

$$\geq |\langle x, y\rangle|^2 + |\langle x, y\rangle|^2$$

since $\langle x, x\rangle \in \mathbb{R}$,

$$\langle y, y\rangle \langle x, x\rangle + |\langle x, y\rangle|^2 \geq 2|\langle x, y\rangle|^2$$

so $\langle y, y\rangle \langle x, x\rangle \geq |\langle x, y\rangle|^2$ $\square$