PHYS 632: Quantum Mechanics II (Winter 2021)
Midterm Exam, 15–19 February 2020

Rules for this exam: You may take up to 24 hours after you download the exam to work on it, then you should upload your answers. Please plan to submit your exam by midnight on Friday.

You may consult any inanimate object while taking the exam (including computers, *Mathematica*, and information posted on the internet). However, you may **not** discuss any part of the exam with **any human by any means** (except the instructor) until after the last due date.

Magic lifeline rule is still in effect for this exam (2 lifeline questions).

I have read and understand the rules.

____ $\longleftarrow$ **Sign your name here.**

Problem 1. (20 points) A particle of spin $S = 1$ with $m = 1$ splits into two particles each with $S = 2$. If you make a measurement of S_z on one of the particles, compute the probability of observing the value $m = 0$ for that particle. (Assume the particles to be distinguishable.)

Problem 2. (20 points)
(a) (10 points) Consider a harmonic oscillator of mass m and frequency ω. Derive an explicit solution for the symmetrized covariance $C_{xp}(t)$ in terms of initial conditions $V_x(0)$, $V_p(0)$, and $C_{xp}(0)$.

(b) (10 points) Suppose that the oscillator is in a minimum-uncertainty state at $t = 0$, in the sense $V_x(0)V_p(0) = \hbar^2/4$, and suppose also that the oscillator is *not* in the ground state. What is the first time $t > 0$ when the oscillator is again in a minimum-uncertainty state? Give a (mathematical) argument to support your answer.

Problem 3. (30 points) Recall that an irreducible tensor operator $T_q^{(k)}$ was defined such that its components rotate in the same way as the angular momentum states $|k\ q\rangle$. Given this similarity, it shouldn't be surprising that there are other analogous expressions for $T_q^{(k)}$ vs. $|k\ q\rangle$.

(a) (15 points) Show, in analogy to $J_z|j\ m\rangle = \hbar m|j\ m\rangle$, that

$$[J_z, T_q^{(k)}] = \hbar q T_q^{(k)}. \tag{1}$$

(*Hint*: consider an *infinitesimal* rotation.)

(b) (15 points) Similarly, show that

$$[J_\pm, T_q^{(k)}] = T_{q\pm1}^{(k)}\hbar\sqrt{k(k+1) - q(q\pm1)}, \tag{2}$$

in analogy to the expressions for $J_\pm|j\ m\rangle$.

(c) (Extra Credit, 5 points) Finally, in analogy to $J^2|j\ m\rangle = \hbar^2 j(j+1)|j\ m\rangle$, show that

$$\sum_\alpha [J_\alpha, [J_\alpha, T_q^{(k)}]] = \hbar^2 k(k+1)T_q^{(k)}. \tag{3}$$

(Don't forget the other side!)

Problem 4. (30 points)
(a) (15 points) Consider a spin-1/2 particle, initially in the $|+\rangle$ state. Show that, *when operating on this particular initial state*, the following rotations are equivalent:

$$R(\pi\hat{x})|+\rangle = R(\hat{x}\pi/2)R(\pi\hat{y})R(\hat{x}\pi/2)|+\rangle. \tag{4}$$

(Equivalent, that is, modulo a global phase factor that we don't care about.)

(b) (15 points) Now suppose that you have the same particle in part (a), on which you want to perform the same rotation $R(\pi\hat{x})$. Suppose also that it has been a while since your Rotate-O-Matic® has been calibrated, and nowadays it effects rotations with a small error ϵ. That is, if $\tilde{R}(\boldsymbol{\zeta})$ denotes the *actual* rotation while the correct, *desired* rotation is $R(\boldsymbol{\zeta})$, then

$$\tilde{R}(\boldsymbol{\zeta}) = R[(1+\epsilon)\boldsymbol{\zeta}]. \tag{5}$$

Now show that using your (slightly) haywire Rotate-O-Matic®, the composite rotation $\tilde{R}(\hat{x}\pi/2)\tilde{R}(\pi\hat{y})\tilde{R}(\hat{x}\pi/2)$ is a better approximation to the desired rotation $R(\pi\hat{x})$ than is the simple attempt $\tilde{R}(\pi\hat{x})$.[1]

You need only work to first order in ϵ throughout the calculation, to show that the simple rotation's error is $O(\epsilon)$, while the composite rotation's error is $O(\epsilon^2)$.

[1] This kind of trick can be used to improve the quality of qubit manipulations in quantum computers, at the expense of speed and complexity of operations. For more details and a nice visualization of how the error cancels out in this example, see Lieven M.K. Vandersypen and Isaac L. Chuang, ''NMR Techniques for Quantum Control and Computation,'' *Reviews of Modern Physics* **76**, 1037 (2005) (doi: 10.1103/RevModPhys.76.1037) (arXiv: quant-ph/0404064), Fig. 24 for the vizualization.