PHYS 632: Quantum Mechanics II (Winter 2021)
Exercises 4 January 2021 (Monday, Week 1)
Due Monday, 11 January 2021

Exercise 1. A classic and simple example of the saddle-point approximation is the derivation of Stirling's approximation for $n!$. To set this up, recall the integral representation for the gamma function,

$$\Gamma(x) = \int_0^\infty dt\, t^{x-1}\, e^{-t}, \tag{1}$$

and since $n! = \Gamma(n+1)$, we have

$$n! = \int_0^\infty dt\, t^n\, e^{-t} = \int_0^\infty dt\, e^{-t+n\log t}. \tag{2}$$

(a) Now the idea is to approximate the integrand by a Gaussian factor, which is valid because the integrand becomes sharply peaked as n becomes large. To do this, write the integrand as $e^{f(t)}$, and expand $f(t)$ to second order in t about the maximum to write

$$n! \approx e^{-n+n\log n} \int_0^\infty dt\, e^{-(t-n)^2/2n}. \tag{3}$$

(b) Now to finish the integration, since the integrand is sharply peaked far away from $t = 0$, we can extend the lower integration limit so that

$$n! \approx e^{-n+n\log n} \int_{-\infty}^\infty dt\, e^{-(t-n)^2/2n}. \tag{4}$$

Now carry out the integral to find Stirling's approximation,

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n. \tag{5}$$

a)

$$e^{-t+n\log t}$$

$$f(t) = n\log t - t$$

$$f(n) = n\log n - n$$

$$f'(n) = 0$$

$$f''(n) = -\frac{1}{n}$$

$$f(t) \approx n\log n - n - \frac{1}{2n}(t-n)^2$$

$$\rightarrow \quad n! \approx e^{-n+n\log n} \int_0^{\infty} dt\, e^{-(t-n)^2/2n}$$

b) $$n! \approx e^{-n+n\log n} \int_{-\infty}^{\infty} e^{-(t-n)^2/2n}\, dt$$

$$= e^{-n+n\log n} \sqrt{2n\pi}$$

$$\rightarrow \quad n! \approx \sqrt{2\pi n}\, e^{-n} n^n$$

from $$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$