PHYS 631: Quantum Mechanics I (Fall 2020)
Homework 2
Assigned Monday, 5 October 2020
Due Monday, 12 October 2019, 11:59:59.$\bar{9}$ pm PDT

Problem 1. Consider two observables Q and R, such that $[Q, R] = 0$. In this case the uncertainty principle says that $\sigma_Q \sigma_R \geq 0$. Is it *always* the case that $\sigma_Q \sigma_R = 0$? If yes, prove it; if not, give a counterexample.

Problem 2. Prove the generalized uncertainty relation

$$V_P V_Q \geq \frac{1}{4}\Big|\big\langle[P, Q]\big\rangle\Big|^2 + \frac{1}{4}\Big|\big\langle[P, Q]_+\big\rangle - 2\langle P\rangle\langle Q\rangle\Big|^2, \tag{1}$$

where $[A, B]_+ := AB + BA$ is the **anticommutator**.

Problem 3. Any Hermitian operator P that satisfies $P^2 = P$ is called a **projection operator** or **projector**.

Suppose that P_1 and P_2 are projectors. Then show that the product $P_1 P_2$ is a projector if and only if $[P_1, P_2] = 0$.

Note for Problem 4. It's hard to overstate how important this problem is. We'll make extensive use of the result throughout the course, so try to work through the result carefully. While working through it, it's likely you'll end up with a messy expression without a good idea for what to do with it, at which point you should ask me for a hint.

Problem 4. For the operator

$$A = \epsilon\big(|1\rangle\langle 1| - |0\rangle\langle 0|\big) + \gamma|0\rangle\langle 1| + \gamma^*|1\rangle\langle 0| \tag{2}$$

on the Hilbert space $\{|0\rangle, |1\rangle\}$, where $\epsilon \geq 0$ and $\gamma \in \mathbb{C}$, show that the eigenvectors may be written as

$$\begin{aligned} |+\rangle &= \sin\theta|0\rangle + e^{i\phi}\cos\theta|1\rangle \\ |-\rangle &= \cos\theta|0\rangle - e^{i\phi}\sin\theta|1\rangle, \end{aligned} \tag{3}$$

where

$$\tan 2\theta = \frac{|\gamma|}{\epsilon} \qquad \left(0 \leq \theta < \frac{\pi}{2}\right). \tag{4}$$

Also find the corresponding eigenvalues.

Problem 5. Both the Poisson bracket and the quantum commutator satisfy the following properties (these aren't hard to verify; so if they aren't obvious to you, then go ahead and verify them).

1. antisymmetry: $[A, B] = -[B, A]$
2. null bracket with scalars: $[c, A] = 0$ ($c \in \mathbb{C}$)
3. associativity: $[A + B, C] = [A, B] + [B, C]$; $\qquad [A, B + C] = [A, B] + [A, C]$
4. product rules: $[AB, C] = A[B, C] + [A, C]B$; $\qquad [A, BC] = [A, B]C + B[A, C]$
5. Jacobi identity: $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

Suppose that we haven't yet settled on a quantum bracket $[A, B]$, but we wish to choose one that satisfies the above properties. Show that the *only* possible choice is the quantum commutator $[A, B] = AB - BA$ (up to an overall real factor).

Hint: starting with $[AA', BB']$ for arbitrary operators A, A', B, and B', apply the two product rules to write this expression as the sum of four brackets. Applying the product rules in either order gives two expressions; you should require that they be equivalent.