Exercise 1:

$$H = \frac{p_e^2}{2m_e} + \frac{p_n^2}{2m_n} - \frac{e^2}{4\pi\epsilon_0 r}$$

$$P = p_e + p_n \qquad p = \frac{m_n p_e - m_e p_n}{M}$$

$$M = m_e + m_n \qquad \mu = \frac{m_e m_n}{M}$$

$$\frac{p_e^2}{2m_e} + \frac{p_n^2}{2m_n}$$

$$\frac{P^2}{2M} + \frac{p^2}{2\mu} = \frac{p_e^2 + p_n^2 + p_e p_n + p_n p_e}{2M}$$

$$+ \frac{m_n^2 p_e^2 + m_e^2 p_n^2}{2\mu M^2} - \frac{m_e m_n p_e p_n}{2\mu M^2}$$

$$- \frac{m_e m_n p_n p_e}{2\mu M^2}$$

$$= \frac{p_e^2}{2M} + \frac{p_n^2}{2M} + \cancel{\frac{p_e p_n}{2M}} + \cancel{\frac{p_n p_e}{2M}}$$

$$+ \frac{m_n^2 p_e^2}{2\mu M^2} + \frac{m_e^2 p_n^2}{2\mu M}$$

$$- \cancel{\frac{p_e p_n}{2M}} - \cancel{\frac{p_n p_e}{2M}}$$

$$= \frac{p_e^2}{2M} + \frac{p_n^2}{2M} + \frac{m_n p_e^2}{2m_e M} + \frac{m_e p_n^2}{2m_n M}$$

$$= \frac{p_e^2}{2}\left(\frac{m_n}{m_e} \cdot \frac{1}{m_e+m_n} + \frac{1}{m_e+m_n}\right)$$

$$\frac{p_n^2}{2}\left(\frac{m_e}{m_n} \cdot \frac{1}{m_e+m_n} + \frac{1}{m_e+m_n}\right)$$

$$= \frac{p_e^2}{2m_e} + \frac{p_n^2}{2m_n}$$

Exercise 2:

In order for there to be a potential well, we need that, for all $\lambda \gg 0$, there is a minimum for

$$\frac{\lambda}{r^2} - \frac{\gamma}{r}$$

but $\frac{-2\lambda}{r^3} + \frac{\gamma}{r^2} = 0 \rightarrow r^2 = \frac{2\lambda}{\gamma}$

Which always exists since $\lambda > 0$, $\gamma > 0$.

Exercise 3:

$$\delta E_1 = \langle \psi_0 | V | \psi_0 \rangle ,$$

$$= \langle 1\,0\,0 | V | 1\,0\,0 \rangle$$

$$\langle r\,\theta\,\phi | 1\,0\,0 \rangle = R_{10}(r)\, Y_0^0(\Omega)$$

$$= \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

$$V(r) = \frac{\hbar c \alpha}{2a_n}\left(\frac{r^2}{a_n^2} - 3\right) + \frac{\hbar c \alpha}{r}$$

$$= \frac{\hbar c \alpha}{2a_n}\left(\frac{r^2}{a_n^2} - 3 + \frac{2a_n}{r}\right)$$

$$\delta E_1 = \frac{1}{\pi a_0^3} \int d\Omega \int_0^{a_n} \frac{\hbar c \alpha}{2a_n}\left(\frac{r^2}{a_n^2} - 3 + \frac{2a_n}{r}\right) e^{-2r/a_0} r^2 dr$$

$$= \frac{2\hbar c \alpha}{a_0^3 a_n} \int_0^{a_n} r^2\left(\frac{r^2}{a_n^2} - 3 + \frac{2a_n}{r}\right) e^{-2r/a_0} dr$$

Exercise 4:

$$\delta E_1 = \frac{1}{\pi a_0^3} \int d\Omega \int_0^{a_n} r^2 \frac{\hbar c \alpha}{2a_n} \left(\frac{r^2}{a_n^2} - 3 + \frac{2a_n}{r} \right) e^{-2r/a_0} dr$$

$$\approx \frac{2\hbar c \alpha}{a_0^3 a_n} \int_0^{a_n} \left(\frac{r^4}{a_n^2} - 3r^2 + 2a_n r \right) dr$$

$$= \frac{2\hbar c \alpha}{a_0^3 a_n} \left(\frac{a_n^5}{5a_n^2} - a_n^3 + a_n^3 \right)$$

$$\delta E_1 = \frac{2\hbar c \alpha}{5 a_0^3 a_n} a_n^3 = \frac{2\hbar c \alpha a_n^2}{5 a_0^3}$$

DC Polarizability of Hydr. Atom

Exercise 1: since $V = e\mathcal{E}z$ and

$$\langle \psi_0 | V | \psi_0 \rangle = \iint d\Omega\, dr\, \psi_0^* \psi_0\, e\mathcal{E}z$$

is an integral of an odd function over the symmetric interval $[0, \pi]$, $\langle \psi_0 | V | \psi_0 \rangle = 0$

Exercise 2:

$$[z, f(p_z)] = i\hbar f'(p_z)$$

$$\rightarrow [z, p^2] = [z, p_z^2] = 2i\hbar p_z$$
$$= -2\hbar^2 \partial_z$$

Exercise 3:

$$[p^2, r]\psi = -\hbar^2 [\nabla^2, r]\psi$$
$$= -\hbar^2 \left(\nabla^2 (r\psi) - r\nabla^2 \psi \right)$$
$$= -\hbar^2 \left(2\nabla\psi \cdot \nabla r + \psi \nabla^2 r + \cancel{r\nabla^2\psi} - \cancel{r\nabla^2\psi} \right)$$
$$= -\hbar^2 \left(2\nabla\psi \cdot \nabla r + \psi \nabla^2 r \right)$$
$$= -\hbar^2 \left(2\partial_r + \frac{2}{r} \right)\psi$$
$$= -2\hbar^2 \left(\partial_r + \frac{1}{r} \right)$$

Exercise 4:

$$\langle \vec{r} | \left(\partial_r + \frac{1}{r}\right) z | \psi_0 \rangle$$

$$\partial_r (z | \psi_0 \rangle) = | \psi_0 \rangle \partial_r z + z \partial_r | \psi_0 \rangle$$

$$= -\frac{z}{a_0} | \psi_0 \rangle + | \psi_0 \rangle \partial_r z$$

$$= -\frac{z}{a_0} | \psi_0 \rangle + \frac{z}{r} | \psi_0 \rangle$$

$$\rightarrow \langle \vec{r} | \left(\partial_r + \frac{1}{r}\right) z | \psi_0 \rangle$$

$$= \langle \vec{r} | \left(\frac{z}{r} - \frac{z}{a_0} + \frac{z}{r} \right) | \psi_0 \rangle$$

$$= \langle \vec{r} | \left(\frac{2}{r} - \frac{1}{a_0} \right) z | \psi_0 \rangle$$

Exercise 5:

$$\langle \vec{r} | (r + 2a_0) \partial_z | \psi_0 \rangle$$

$$\partial_z | \psi_0 \rangle = \partial_r \psi_0 \, \partial_z r = -\frac{1}{a_0} \frac{z}{r} | \psi_0 \rangle$$

$$\langle \vec{r} | (r + 2a_0) \partial_z | \psi_0 \rangle$$

$$= \langle \vec{r} | (r + 2a_0) \left(-\frac{1}{a_0} \frac{z}{r} \right) | \psi_0 \rangle$$

$$= \langle \vec{r} | \left(-\frac{2}{r} - \frac{z}{a_0} \right) z | \psi_0 \rangle$$

Exercise 6:

$$\langle \vec{r} \,|\, H_0 (r + 2a_0) z \,|\, \psi_0 \rangle = \langle \vec{r} \,|\, (r + 2a_0) z H_0 \,|\, \psi_0 \rangle + \frac{2\hbar^2}{\mu a_0} \langle \vec{r} \,|\, z \,|\, \psi_0 \rangle$$

$$\frac{\mu a_0}{2\hbar^2} \Big(\langle p \,|\, H_0 (r + 2a_0) z \,|\, \psi_0 \rangle - \langle p \,|\, (r + 2a_0) z H_0 \,|\, \psi_0 \rangle \Big)$$

$$= \langle p \,|\, z \,|\, \psi_0 \rangle = z_{p0}$$

$$= \frac{\mu a_0}{2\hbar^2} \left(\langle p \,|\, (H_0 (r + 2a_0) z - (r + 2a_0) z H_0) \,|\, \psi_0 \rangle \right)$$

$$= \frac{\mu a_0}{2\hbar^2} \left(E_p \langle p \,|\, (r + 2a_0) z \,|\, \psi_0 \rangle - E_0 \langle p \,|\, (r + 2a_0) z \,|\, \psi_0 \rangle \right)$$

$$= \frac{\mu a_0}{2\hbar^2} \left(E_p - E_0 \langle p \,|\, (r + 2a_0) z \,|\, \psi_0 \rangle \right)$$

$\rightarrow$

$$\delta E_2 = e^2 \mathcal{E}^2 \sum_{p \neq 0} \frac{|z_{0p}|^2}{E_{0p}}$$

$$= -e^2 \mathcal{E}^2 \sum_{p \neq 0} \frac{\mu a_0}{2\hbar^2} \langle \psi_0 \,|\, z \,|\, p \rangle \langle p \,|\, (r + 2a_0) z \,|\, \psi_0 \rangle$$

$$= -\frac{e^2 \mathcal{E}^2 \mu a_0}{2\hbar^2} \left(\langle \psi_0 \,|\, z \left(1 - |\psi_0\rangle\langle\psi_0| \right) (r + 2a_0) z \,|\, \psi_0 \rangle \right)$$

$$\delta E_2 = -\frac{e^2 \mathcal{E}^2 \mu a_0}{2\hbar^2} \left[\langle \psi_0 \,|\, (r + 2a_0) z^2 \,|\, \psi_0 \rangle - \langle \psi_0 \,|\, z \,|\, \psi_0 \rangle \langle \psi_0 \,|\, (r + 2a_0) z \,|\, \psi_0 \rangle \right]$$

$$\delta E_2 = -\frac{e^2 \mathcal{E}^2 \mu a_0}{2\hbar^2} \langle \psi_0 \,|\, z^2 (r + 2a_0) \,|\, \psi_0 \rangle$$

$$\delta E_2 = -\frac{e^2 \mathcal{E}^2 \mu a_0}{6\hbar^2} \langle \psi_0 \,|\, r^2 (r + 2a_0) \,|\, \psi_0 \rangle$$

Exercise 7: Since ψ_{100} is spherically symmetric, $\hat{z}$ is the same as $\hat{x}, \hat{y}$ so $\langle \psi_{100} | \hat{z}^2 | \psi_{100} \rangle$

$= \frac{1}{3} \langle \psi_{100} | x^2 + y^2 + z^2 | \psi_{100} \rangle$

$= \frac{1}{3} \langle \psi_{100} | r^2 | \psi_{100} \rangle$

$$\langle \psi_{100} | r^2 (r + 2a_0) | \psi_{100} \rangle$$

$$= 4\pi \frac{1}{\pi a_0^3} \int_0^\infty dr\, r^2 e^{-2r/a_0} r^2 (r + 2a_0)$$

$$= \frac{27 a_0^3}{2}$$

$$\delta E_2 = - \frac{9 e^2 \mu a_0^4 \mathcal{E}^2}{4 \hbar^2} = -9\pi \epsilon_0 a_0^3 \mathcal{E}^2$$