Exercise 1:

$$\Delta p^2(t) = p^2(t+\Delta t) - p^2(t)$$

$$= \frac{1}{2}\Big[(p(t+\Delta t) + p(t))(p(t+\Delta t) - p(t)) + (p(t+\Delta t) - p(t))(p(t+\Delta t) + p(t)) \Big]$$

$$= \frac{1}{2}\Big[\Delta p(t)\,[p(t+\Delta t) + p(t)] + [p(t+\Delta t) + p(t)]\,\Delta p(t) \Big]$$

$$p(t+\Delta t) + p(t) = \Delta p(t) + 2p(t)$$

$$\rightarrow \Delta p^2(t) = \frac{1}{2}\Big[\Delta p(t)\,[\Delta p(t) + 2p(t)] + [\Delta p(t) + 2p(t)]\,\Delta p(t) \Big]$$

$$\Delta p^2(t) = \frac{1}{2}\Big[\Delta p(t)^2 + 2\Delta p(t)\,p(t) + \Delta p(t)^2 + 2p(t)\,\Delta p(t) \Big]$$

$$\Delta p^2(t) = \Delta p(t)\,p(t) + p(t)\,\Delta p(t) + \Delta p(t)^2$$

but $\dfrac{\Delta p(t)}{\Delta t} = F(t)$

$$\rightarrow \Delta p(t)^2 = F(t)^2 \Delta t^2$$

$$\rightarrow \Delta p^2(t) = \Delta p(t)\, p(t) + p(t)\, \Delta p(t) + F(t)^2 \Delta t^2$$

$$\rightarrow \Delta \langle p^2(t) \rangle = \Delta p\, p + p\, \Delta p + \Delta t^2 \langle F^2(t) \rangle$$

$$\begin{aligned}
\langle F^2(t) \rangle &= \frac{1}{2} \lim_{t' \to t} \langle [F(t'), F(t)]_+ \rangle \Delta t \\
&= \frac{1}{2} \lim_{\tau \to 0} \langle [F(\tau), F(0)]_+ \rangle \\
&\approx \frac{1}{2} \int_{-\infty}^{\infty} d\tau \, \langle [F(\tau), F(0)]_+ \rangle \\
&= \int_{-\infty}^{\infty} d\tau \, \langle [F(\tau), F(0)]_+ \rangle
\end{aligned}$$

$$\rightarrow \frac{d}{dt} \langle p^2 \rangle \approx \langle \dot{p}\, p + p\, \dot{p} \rangle + \int_0^{\infty} d\tau \, \langle [F(\tau), F(0)]_+ \rangle$$

$$\begin{aligned}
&\int_0^{\infty} d\tau \, \langle [F(\tau), F(0)]_+ \rangle \\
&= 2\hbar \int_0^{\infty} d\omega \, J(\omega) \coth\left(\frac{\hbar\omega}{2k_B T}\right) \delta(\omega) \\
&= \hbar \lim_{\omega \to 0} J(\omega) \coth\left(\frac{\hbar\omega}{2k_B T}\right) \\
&= \hbar M \gamma \lim_{\omega \to 0} \omega \coth\left(\frac{\hbar\omega}{2k_B T}\right) \\
&= 2 M \gamma k_B T
\end{aligned}$$

$$\boxed{\rightarrow \frac{d}{dt} \langle p^2 \rangle = -\langle p V'(q) + V'(q) p \rangle - 2\gamma \langle p^2 \rangle + 2M\gamma k_B T}$$

Exercise 2:

$$2\gamma V_p = 2m\gamma k_B T$$

$$\longrightarrow \quad V_p = m k_B T$$

$$V_p = m\gamma C_{qp}$$

$$\dot{V}_q = \frac{2 C_{qp}}{m} = \frac{2 V_p}{m^2 \gamma} = \frac{2 k_B T}{m \gamma}$$

$$\longrightarrow \quad \boxed{\dot{V}_q = \frac{2 k_B T}{m \gamma}}$$

Exercise 3:

$$V_q = \frac{\hbar \gamma}{\pi M} \int_0^{\infty} d\omega \frac{\omega}{(\omega_0^2 - \omega^2)^2 + (\gamma \omega)^2}$$

$$\omega_0^2 - \omega^2 \simeq (\omega_0 + \omega)(\omega_0 - \omega)$$

$$= 2\omega_0 (\omega_0 - \omega)$$

$$V_q = \frac{\hbar \gamma}{\pi M} \int_{-\infty}^{\infty} d\omega \frac{\omega_0}{4\omega_0^2 (\omega_0 - \omega)^2 + (\gamma \omega_0)^2}$$

$$V_q = \frac{\hbar \gamma}{4\pi \omega_0^2 M} \int_{-\infty}^{\infty} d\omega \frac{1}{(\omega_0 - \omega)^2 + (\gamma/2)^2}$$