Physics 614 Homework 2

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1. **Boson statistics: the more the merrier**

   Consider a bosonic system with two single-particle eigenstates ϕ_a and ϕ_b. Let $\psi_{i,j}(\boldsymbol{r}_1, ..., \boldsymbol{r}_{i+j})$ denote the symmetrized and normalized wavefunction for i particles in state a and j particles in state b.

   i. The symmetrized wave function will count all possible arrangements of $i+j$ particles in which i particles are in state a and j particles are in state b. To normalize this sum we have to divide by $\sqrt{i+j}$ so that

   $$\psi_{N,0}(\boldsymbol{r}_1, ..., \boldsymbol{r}_N) = \phi_a(\boldsymbol{r}_1)\phi_a(\boldsymbol{r}_2)\ldots\phi_a(\boldsymbol{r}_N) \tag{1}$$

   and

   $$\psi_{N,1}(\boldsymbol{r}_1, ..., \boldsymbol{r}_{N+1}) = \frac{1}{\sqrt{N+1}}\sum_{p=1}^{N+1}\phi_a(\boldsymbol{r}_1)\phi_a(\boldsymbol{r}_2)\ldots\phi_a(\boldsymbol{r}_{p-1})\phi_b(\boldsymbol{r}_p)\phi_a(\boldsymbol{r}_{p+1})\ldots\phi_a(\boldsymbol{r}_{N+1}) \tag{2}$$

   where the sum is over all states in which particle p is in state b. [1]

   ii. Now suppose the system begins in the state $\psi_{N,0}(\boldsymbol{r}_1, ..., \boldsymbol{r}_N)$ and a new particle is inserted in an equal superposition of states a and b: $\psi_{\text{new}}(\boldsymbol{r}) = \frac{1}{\sqrt{2}}(\phi_a(\boldsymbol{r}) + \phi_b(\boldsymbol{r}))$. Then the wave function for the system, $\psi_{\text{new}}(\boldsymbol{r}_1, ..., \boldsymbol{r}_{N+1})$, is given by symmetrizing the product state $\psi_{N,0}(\boldsymbol{r}_1, ..., \boldsymbol{r}_N)\psi_{\text{new}}(\boldsymbol{r})$.
   We can treat this new system as $N+1$ particles wherein one particle exists in the superposition above and the rest are in state a. Summing over all arrangements in which particle p is in the superposition gives

[1] Sorry I wasn't sure what else to show for this one.

$$
\begin{aligned}
\psi_{\text{new}}(\boldsymbol{r}_1, ..., \boldsymbol{r}_{N+1}) &= C_B \left[\sum_{p=1}^{N+1} \phi_a(\boldsymbol{r}_1)\dots\phi_a(\boldsymbol{r}_{p-1})\frac{1}{\sqrt{2}}\left(\phi_a(\boldsymbol{r}_p) + \phi_b(\boldsymbol{r}_p)\right)\phi_a(\boldsymbol{r}_{p+1})\dots\phi_a(\boldsymbol{r}_{N+1})\right] \\
\psi_{\text{new}}(\boldsymbol{r}_1, ..., \boldsymbol{r}_{N+1}) &= \frac{C_B}{\sqrt{2}} \left[\sum_{p=1}^{N+1} \phi_a(\boldsymbol{r}_1)\dots\phi_a(\boldsymbol{r}_{p-1})\phi_a(\boldsymbol{r}_p)\phi_a(\boldsymbol{r}_{p+1})\dots\phi_a(\boldsymbol{r}_{N+1})\right. \\
&\qquad \left. + \sum_{p=1}^{N+1} \phi_a(\boldsymbol{r}_1)\dots\phi_a(\boldsymbol{r}_{p-1})\phi_b(\boldsymbol{r}_p)\phi_a(\boldsymbol{r}_{p+1})\dots\phi_a(\boldsymbol{r}_{N+1})\right] \\
\psi_{\text{new}}(\boldsymbol{r}_1, ..., \boldsymbol{r}_{N+1}) &= \frac{C_B}{\sqrt{2}} \left[(N+1)\psi_{N+1,0}(\boldsymbol{r}_1, ..., \boldsymbol{r}_{N+1}) + \sqrt{N+1}\psi_{N,1}(\boldsymbol{r}_1, ..., \boldsymbol{r}_{N+1})\right]
\end{aligned} \tag{3}
$$

where C_B is an overall normalization.

iii. We compute the ratio of the probability of finding all particles in state a, $|\langle\psi_{N+1,0}|\psi_{\text{new}}\rangle|^2$, to the probability of finding a particle in state b, $|\langle\psi_{N,1}|\psi_{\text{new}}\rangle|^2$.

$$
\begin{aligned}
|\langle\psi_{N+1,0}|\psi_{\text{new}}\rangle|^2 &= \frac{C_B^2}{2}(N+1)^2 \\
|\langle\psi_{N,1}|\psi_{\text{new}}\rangle|^2 &= \frac{C_B^2}{2}(N+1) \\
\implies \frac{|\langle\psi_{N+1,0}|\psi_{\text{new}}\rangle|^2}{|\langle\psi_{N,1}|\psi_{\text{new}}\rangle|^2} &= N+1
\end{aligned} \tag{4}
$$

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2. **Density of states**

i. For a one dimensional box of length L, the energy eigenstates are simply

$$
\psi(x) = \frac{1}{\sqrt{L}}e^{ik_n x} \tag{5}
$$

each with energy $E_k = \frac{\hbar^2 k_n^2}{2m} = \frac{4n^2\pi^2\hbar^2}{2mL^2}$, where $k_n = 2\pi n/L$. Following Tong's notes, we have

$$
\begin{aligned}
\sum_n \approx \int dn = \frac{L}{2\pi}\int_{-\infty}^{\infty} dk &= \frac{L}{2\pi}\int_{-\infty}^{\infty} d\epsilon \frac{\mathrm{d}k}{\mathrm{d}\epsilon} \\
&= \frac{L}{2\pi}\int_{-\infty}^{\infty} d\epsilon \frac{m}{\hbar^2}\left(\frac{2m}{\hbar^2}\epsilon\right)^{-1/2} \\
&= \int_{-\infty}^{\infty} d\epsilon g(\epsilon)
\end{aligned} \tag{6}
$$

where $g(\epsilon) = \frac{mL}{2\pi\hbar^2}\left(\frac{2m\epsilon}{\hbar^2}\right)^{-1/2}$.

ii. The wave function for the 2 dimensional case is similarly

$$\psi(\boldsymbol{x}) = \frac{1}{L}e^{i\boldsymbol{k}_n\boldsymbol{x}} \tag{7}$$

with eigenenergies $E_k = \frac{\hbar^2\boldsymbol{k}_n^2}{2m}$, $\boldsymbol{k}_n = 2\pi/L(n_x, n_y)$. So we again have

$$\begin{aligned}\sum_{n_x,n_y} = \int dn_x dn_y = \frac{L^2}{2\pi}\int_{-\infty}^{\infty} kdk = \frac{L^2}{2\pi}\int_{-\infty}^{\infty}\frac{m}{\hbar^2}d\epsilon \\ = \int_{-\infty}^{\infty} g(\epsilon)d\epsilon\end{aligned} \tag{8}$$

where $g(\epsilon) = \frac{mL^2}{2\pi\hbar^2}$. In the second equality in (8) we have integrated over the angle θ in polar coordinates.

3. **"Maxwell-Boltzmann" ideal gas**

i. We can write the grand potential for Maxwell-Boltzmann particles in the plane-wave basis, $\Phi = -k_BT\ln(\mathcal{Q}) = -k_BT\sum_{\boldsymbol{k}} e^{-\beta(\epsilon_{\boldsymbol{k}}-\mu)}$, as an integral using the density of states, $g(\epsilon) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2}$.

$$\begin{aligned}
\Phi &= -k_BT\sum_{\boldsymbol{k}} e^{-\beta(\epsilon_{\boldsymbol{k}}-\mu)} \\
\Phi &\approx -k_BTe^{\beta\mu}\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^{\infty}\epsilon^{1/2}e^{-\beta\epsilon}d\epsilon \\
\Phi &\approx -k_BTe^{\beta\mu}\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{1}{\beta^{3/2}}\int_0^{\infty}x^{1/2}e^{-x}dx \\
\Phi &\approx -k_BTe^{\beta\mu}\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{1}{\beta^{3/2}}\Gamma\left(\frac{3}{2}\right) \\
\Phi &\approx -k_BTe^{\beta\mu}\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{\sqrt{\pi}}{2\beta^{3/2}} \\
\Phi &\approx -k_BTe^{\beta\mu}V\left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2} \\
\Phi &\approx -k_BTe^{\beta\mu}V\frac{1}{\lambda^3}
\end{aligned} \tag{9}$$

where $\lambda = \sqrt{\frac{2\pi\hbar^2\beta}{m}}$ is the thermal deBroglie wavelength.

ii. The expected value of N is given by summing the average occupancies over α. But this is proportional to Φ. So, using $\Phi = -PV$, the equation of state for the Maxwell-Boltzmann ideal gas is $PV = k_B T \langle N \rangle$.

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