PHYS 631: Quantum Mechanics I (2020)

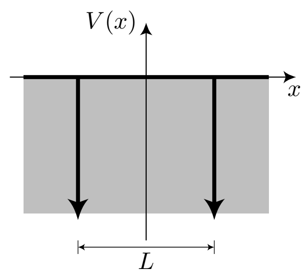
Notes: Double-Well Tunneling Exercise

This is a review of some of the key steps that we just covered in class.

Setup of the Problem

Double-delta well:

$$V(x) = -\beta\,\delta(x - L/2) - \beta\,\delta(x + L/2). \tag{1}$$



Ansatz for piecewise solution:

$$\psi_E(x) = \begin{cases} a_{\mathrm{I}} e^{\kappa x} & (x < -L/2) \\ a_{\mathrm{II}} e^{-\kappa x} + b_{\mathrm{II}} e^{\kappa x} & (-L/2 < x < L/2) \\ a_{\mathrm{III}} e^{-\kappa x} & (x > L/2), \end{cases} \tag{2}$$

where

$$\kappa = \frac{\sqrt{2m|E|}}{\hbar}, \tag{3}$$

and $E < 0$ for a bound state.

Taking Stock

Problem 1. How many undetermined quantities are there? What are the constraint equations to be solved? What does the relative count here imply about the states (continuous vs. discrete spectrum).

Even Bound State

Problem 2.
(a) What are the constraints on the undetermined coefficients under the assumption of an even solution?

(b) Write out the continuity condition for $\psi(x = -L/2)$,

$$\psi_E(-L/2 + 0^+) = \psi_E(-L/2 - 0^+), \tag{4}$$

in terms of the wave function *ansatz* (2).

(c) Write out the "kink" condition for the wave function at $x = -L/2$,

$$\psi'_E(-L/2 + 0^+) = \psi'_E(-L/2 - 0^+) - \frac{2m\beta}{\hbar^2}\psi_E(-L/2), \tag{5}$$

in terms of the wave function *ansatz* (2). Make sure to be clear on which side of the delta function is which in this equation!

(d) Dividing the result of (c) by the result from (b) to derive the condition

$$\kappa\left(\tanh\frac{\kappa L}{2} + 1\right) = \frac{2m\beta}{\hbar^2}. \tag{6}$$

for the even bound states.

Odd Bound State

Problem 3. Repeat this procedure to derive the condition

$$\kappa\left(\coth\frac{\kappa L}{2} + 1\right) = \frac{2m\beta}{\hbar^2}. \tag{7}$$

for the odd bound states.

Existence of Bound-State Solutions

Problem 4.
(a) Argue that for $\beta > 0$, there is exactly one even bound state.

(b) Argue that for $\beta > 0$, there is at most one odd bound state, which exists provided

$$L \geq \frac{\hbar^2}{m\beta}. \tag{8}$$

For smaller distances, intuitively the (tunneling) interaction between the wells is large enough to push the odd-parity energy above $E = 0$.