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PHYS 631: Quantum Mechanics I (Fall 2020)
Homework 1
Assigned Tuesday, 29 September 2020
Due Monday, 6 October 2020

Note: Remember that you should submit your homework solutions via the course web site (submission link on the Homework page) before midnight on the due date. Single pdf file, keep it to a reasonable size if you scan it.

Problem 1. Consider the m-dimensional vector space $\mathbb{C}^m$ (i.e., the set of all complex m-tuples), and let $\phi : \mathbb{C}^m \longrightarrow \mathbb{C}^m$ be a linear transformation (operator). Show that ϕ can be represented by an $m \times m$ matrix; that is, show that there exists a matrix $\mathbf{A}$ such that $\mathbf{A} \cdot \mathbf{x} = \phi(\mathbf{x})$ for every vector $\mathbf{x} \in \mathbb{C}^m$.

Problem 2.
Let L and M be linear transformations on an inner-product space V. The **composition** of L with M is a defined by

$$(L \circ M)(\mathbf{x}) := L[M(\mathbf{x})], \tag{1}$$

where $\mathbf{x} \in V$. Using the axioms satisfied by linear transformations and inner-product spaces,

(a) show that $L \circ M$ is also a linear transformation. (In quantum mechanics, this means that the product AB of linear operators A and B is still a linear operator.)

(b) show that $(L \circ M)^\dagger = M^\dagger \circ L^\dagger$. [In quantum mechanics, this is the product-adjoint rule $(AB)^\dagger = B^\dagger A^\dagger$.]

Problem 3. Suppose that an operator Q has eigenvalues q_n and eigenvectors $|q_n\rangle$,

$$Q|q_n\rangle = q_n|q_n\rangle, \tag{2}$$

for $n \in \mathbb{Z}^+$. If the eigenvectors $|q_n\rangle$ form a complete set, prove that Q may always be written in the form

$$Q = \sum_{n=1}^{\infty} q_n |q_n\rangle\langle q_n|. \tag{3}$$

Note: to prove this equivalence you must consider the action of *both* expressions here on an *arbitrary* vector, not just an eigenvector.

Problem 1. Consider the m-dimensional vector space $\mathbb{C}^m$ (i.e., the set of all complex m-tuples), and let $\phi : \mathbb{C}^m \longrightarrow \mathbb{C}^m$ be a linear transformation (operator). Show that ϕ can be represented by an $m \times m$ matrix; that is, show that there exists a matrix $\mathbf{A}$ such that $\mathbf{A} \cdot \mathbf{x} = \phi(\mathbf{x})$ for every vector $\mathbf{x} \in \mathbb{C}^m$.

1)

Claim: If $\phi : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ is a linear operator, then there exists a matrix, A, such that $A \cdot x = \phi(x)$, for all $x \in \mathbb{C}^m$

Proof:

Let $\{\vec{e}_1, \ldots, \vec{e}_m\}$ be the standard basis for $\mathbb{C}^m$.

Define a matrix, A, whose columns are the vectors $\phi(\vec{e}_1), \ldots, \phi(\vec{e}_m)$:

$$A = [\phi(\vec{e}_1) \ \ldots \ \phi(\vec{e}_m)]$$

Then for each $n \in \{1, \ldots, m\}$

$$A\vec{e}_n = [\phi(\vec{e}_1) \ldots \phi(\vec{e}_m)]\,\vec{e}_n$$

But $\vec{e}_n = \begin{array}{c} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \}\, n \\ \}\, m-n \end{array}$

so

$$[\phi(\vec{e}_1) \ldots \phi(\vec{e}_m)]\,\vec{e}_n = \phi(\vec{e}_n)$$

So $A\vec{e}_n = \phi(\vec{e}_n)$ for each $\vec{e}_n$ in the basis.

Since each $\vec{x} \in \mathbb{C}^m$ is a linear combination of $\vec{e}_n$'s and ϕ is linear, this shows that there is a matrix, A, such that $A \cdot \vec{x} = \phi(\vec{x})$ for every $x \in \mathbb{C}^m$. $\square$

Problem 2.
Let L and M be linear transformations on an inner-product space V. The **composition** of L with M is a defined by

$$(L \circ M)(\mathbf{x}) := L[M(\mathbf{x})], \tag{1}$$

where $\mathbf{x} \in V$. Using the axioms satisfied by linear transformations and inner-product spaces,

(a) show that $L \circ M$ is also a linear transformation. (In quantum mechanics, this means that the product AB of linear operators A and B is still a linear operator.)

(b) show that $(L \circ M)^\dagger = M^\dagger \circ L^\dagger$. [In quantum mechanics, this is the product-adjoint rule $(AB)^\dagger = B^\dagger A^\dagger$.]

2)

a)

Claim: Let $L : V \longrightarrow V$ and $M : V \longrightarrow V$ be linear operators. Then, if

$(L \circ M)(x) := L(M(x))$,

$L \circ M$ is also a linear operator.

Proof: Since M is linear

$L \circ M(ax + by) = L(aM(x) + bM(y))$

And since L is linear,

$L \circ M(ax + by) = aL(M(x)) + bL(M(y))$

so $L \circ M(ax + by) = aL \circ M(x) + bL \circ M(y)$

$\square$

b) <u>Claim:</u> $(L \circ M)^\dagger = M^\dagger \circ L^\dagger$

<u>Proof:</u> By definition,

$$\langle L \circ M(x), y \rangle = \langle x, (L \circ M)^\dagger(y) \rangle$$

since $L \circ M(x) := L(M(x))$,

$$\langle L \circ M(x), y \rangle = \langle L(M(x)), y \rangle$$

But since

$$\langle L(M(x)), y \rangle = \langle M(x), L^\dagger(y) \rangle$$

and

$$\langle M(x), L^\dagger(y) \rangle = \langle x, M^\dagger(L^\dagger(y)) \rangle$$

We have

$$\langle x, (L \circ M)^\dagger(y) \rangle = \langle x, M^\dagger(L^\dagger(y)) \rangle$$

For all $x, y \in V$.

So $(L \circ M)^\dagger = M^\dagger \circ L^\dagger$ $\square$

Problem 3. Suppose that an operator Q has eigenvalues q_n and eigenvectors $|q_n\rangle$,

$$Q|q_n\rangle = q_n|q_n\rangle, \tag{2}$$

for $n \in \mathbb{Z}^+$. If the eigenvectors $|q_n\rangle$ form a complete set, prove that Q may always be written in the form

$$Q = \sum_{n=1}^{\infty} q_n |q_n\rangle\langle q_n|. \tag{3}$$

Note: to prove this equivalence you must consider the action of *both* expressions here on an *arbitrary* vector, not just an eigenvector.

Claim:

Let $Q: V \longrightarrow V$ be a linear operator with eigenvectors $\{|q_n\rangle\}_{n\in\mathbb{Z}^+}$ and eigenvalues $\{q_n\}_{n\in\mathbb{Z}^+}$. Assume that $\{|q_n\rangle\}_{n\in\mathbb{Z}^+}$ are orthonormal and complete.

Then $Q = \sum_{n=1}^{\infty} q_n |q_n\rangle\langle q_n|$.

Proof:

Let $|x\rangle \in V$. Since $\{|q_n\rangle\}_{n\in\mathbb{Z}^+}$ is a ccmplete set ane can write

$$|x\rangle = \sum_{n=1}^{\infty} \lambda_n |q_n\rangle$$

for some $\{\lambda_n\}_{n\in\mathbb{Z}^+}$

The action of Q on $|x\rangle$ is the following:

$$Q|x\rangle = Q \sum_{n=1}^{\infty} \lambda_n |q_n\rangle$$

$$= \sum_{n=1}^{\infty} \lambda_n Q|q_n\rangle$$

$$= \sum_{n=1}^{\infty} \lambda_n q_n |q_n\rangle$$

Similarly,

$$\sum_{n=1}^{\infty} q_n |q_n\rangle\langle q_n|x\rangle =$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \lambda_n q_n |q_n\rangle \langle q_n|q_m\rangle$$

But $\{|q_n\rangle\}_{n\in\mathbb{Z}^+}$ was orthonormal so $\langle q_n | q_m \rangle = \delta_{mn}$

Therefore,

$$\sum_{n=1}^{\infty} q_n |q_n\rangle\langle q_n | x\rangle = \sum_{n=1}^{\infty} \lambda_n q_n |q_n\rangle$$

But $\sum_{n=1}^{\infty} \lambda_n q_n |q_n\rangle = Q|x\rangle$.

And since $|x\rangle$ was arbitrary,

$$Q = \sum_{n=1}^{\infty} q_n |q_n\rangle\langle q_n| \qquad \square$$