Physics 623 Midterm

Jeremy Welsh-Kavan

This one really made me do my research.

4.1.2. **Polaritons**

We consider a polarization field, $\boldsymbol{P}(\boldsymbol{x}, t)$, that determines the sources of the electromagnetic fields according to

$$\begin{aligned} \boldsymbol{j} &= \partial_t \boldsymbol{P} \\ \rho &= -\boldsymbol{\nabla} \cdot \boldsymbol{P} \end{aligned} \tag{1}$$

and which satisfies

$$(\partial_t^2 + \omega_0^2)\boldsymbol{P}(\boldsymbol{x}, t) = a^2 \boldsymbol{E}(\boldsymbol{x}, t) \tag{2}$$

a) We claim that Maxwell's equations combined with (2) have solutions given by both longitudinal, $(\boldsymbol{k} \parallel \boldsymbol{E}, \boldsymbol{P})$, and transverse, $(\boldsymbol{k} \perp \boldsymbol{E}, \boldsymbol{P})$, monochromatic plane waves.
From (1), Maxwell's equations become

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{M1}$$

$$\frac{1}{c}\partial_t \boldsymbol{B} + \boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{0} \tag{M2}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\rho = -4\pi \boldsymbol{\nabla} \cdot \boldsymbol{P} \tag{M3}$$

$$-\frac{1}{c}\partial_t \boldsymbol{E} + \boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} = \frac{4\pi}{c}\partial_t \boldsymbol{P} \tag{M4}$$

Ansatz: Let $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$ be monochromatic plane waves given by

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{x}, t) &= \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \\ \boldsymbol{B}(\boldsymbol{x}, t) &= \boldsymbol{B}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \end{aligned} \tag{3}$$

By (2), we can write $\boldsymbol{P}(\boldsymbol{x}, t)$ in terms of $\boldsymbol{E}(\boldsymbol{x}, t)$. Let $\boldsymbol{P}(\boldsymbol{x}, \tilde{\omega})$ be the time-domain Fourier transform of $\boldsymbol{P}(\boldsymbol{x}, t)$. Then we have

$$
\begin{aligned}
(\partial_t^2 + \omega_0^2)\boldsymbol{P}(\boldsymbol{x}, t) &= a^2 \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \\
(\omega_0^2 - \tilde{\omega}^2)\boldsymbol{P}(\boldsymbol{x}, \tilde{\omega}) &= 2\pi a^2 \boldsymbol{E}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \delta(\tilde{\omega} + \omega) \\
\boldsymbol{P}(\boldsymbol{x}, t) &= \int d\tilde{\omega} \frac{a^2 \boldsymbol{E}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \delta(\tilde{\omega} + \omega)}{(\omega_0^2 - \tilde{\omega}^2)} \\
\boldsymbol{P}(\boldsymbol{x}, t) &= \frac{a^2}{\omega_0^2 - \omega^2} \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \\
\boldsymbol{P}(\boldsymbol{x}, t) &= \frac{a^2}{\omega_0^2 - \omega^2} \boldsymbol{E}(\boldsymbol{x}, t)
\end{aligned} \tag{4}
$$

Define $k := |\boldsymbol{k}|$ and $\hat{\boldsymbol{n}} := \hat{\boldsymbol{k}} = \boldsymbol{k}/k$.

(i) Suppose $\boldsymbol{k} \perp \boldsymbol{E}, \boldsymbol{P}$. We will use the fact that, in a nonconducting medium, $\boldsymbol{B} = \sqrt{\epsilon(\omega)}\, \hat{\boldsymbol{n}} \times \boldsymbol{E}$, where $\epsilon(\omega)$ is the dielectric function such that $\boldsymbol{E} + 4\pi\boldsymbol{P} = \epsilon(\omega)\boldsymbol{E}$. [1] So we have $\epsilon(\omega) = 1 + 4\pi a^2/(\omega_0^2 - \omega^2)$.

(M1)

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{B} &= i\boldsymbol{k} \cdot \boldsymbol{B} \\
&= ik\sqrt{\epsilon(\omega)}\hat{\boldsymbol{n}} \cdot (\hat{\boldsymbol{n}} \times \boldsymbol{E}) \\
&= 0
\end{aligned} \tag{5}
$$

(M2)

$$
\begin{aligned}
\frac{1}{c}\partial_t \boldsymbol{B} + \boldsymbol{\nabla} \times \boldsymbol{E} &= -i\omega \frac{\sqrt{\epsilon(\omega)}}{c} (\hat{\boldsymbol{n}} \times \boldsymbol{E}) + i\boldsymbol{k} \times \boldsymbol{E} \\
&= i\left(k - \frac{\omega\sqrt{\epsilon(\omega)}}{c}\right) \hat{\boldsymbol{n}} \times \boldsymbol{E} \\
&= 0
\end{aligned} \tag{6}
$$

provided $\omega = ck/\sqrt{\epsilon(\omega)}$.

(M3)

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{E} &= i\boldsymbol{k} \cdot \boldsymbol{E} \\
&= ik\hat{\boldsymbol{n}} \cdot \boldsymbol{E} \\
&= 0 \\
&= ik\hat{\boldsymbol{n}} \cdot \boldsymbol{P}
\end{aligned} \tag{7}
$$

since $\boldsymbol{P}$ is just a scaled copy of $\boldsymbol{E}$ and both are orthogonal to $\boldsymbol{k}$.

[1] p. 297, J.D. Jackson, *Classical Electrodynamics*, 3rd Edition.

(M4)

$$
\begin{aligned}
-\frac{1}{c}\partial_t \boldsymbol{E} + \boldsymbol{\nabla} \times \boldsymbol{B} &= \frac{i\omega}{c}\boldsymbol{E} + ik\sqrt{\epsilon(\omega)}\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \boldsymbol{E}) \\
&= i\left(\frac{\omega}{c} - k\sqrt{\epsilon(\omega)}\right)\boldsymbol{E} \\
&= \frac{4\pi}{c}\partial_t \boldsymbol{P} \\
&= -i\frac{4\pi\omega}{c}\frac{a^2}{\omega_0^2 - \omega^2}\boldsymbol{E}
\end{aligned}
\tag{8}
$$

again, provided $\omega = ck/\sqrt{\epsilon(\omega)}$.

Thus, Maxwell's equations have transverse monochromatic plane wave solutions in this paradigm. Additionally, the waves satisfy the frequency-wavenumber relation, $\omega = ck/\sqrt{\epsilon(\omega)}$.

(ii) Now suppose $\boldsymbol{k} \parallel \boldsymbol{E}, \boldsymbol{P}$. In this case, since $\boldsymbol{B} = \sqrt{\epsilon(\omega)}\ \hat{\boldsymbol{n}} \times \boldsymbol{E}$, we have $\boldsymbol{B} = \boldsymbol{0}$. So Maxwell's equations become

(M1)

$$
\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{9}
$$

(M2)

$$
\begin{aligned}
\frac{1}{c}\partial_t \boldsymbol{B} + \boldsymbol{\nabla} \times \boldsymbol{E} &= \boldsymbol{\nabla} \times \boldsymbol{E} \\
&= ik\hat{\boldsymbol{n}} \times \boldsymbol{E} \\
&= \boldsymbol{0}
\end{aligned}
\tag{10}
$$

since $\hat{\boldsymbol{n}} \parallel \boldsymbol{E}$.

(M3)

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{E} &= i\boldsymbol{k} \cdot \boldsymbol{E} \\
&= ik|\boldsymbol{E}| \\
&= -ik\frac{4\pi a^2}{\omega_0^2 - \omega^2}|\boldsymbol{E}| \\
&= -4\pi \boldsymbol{\nabla} \cdot \boldsymbol{P}
\end{aligned}
\tag{11}
$$

provided $\omega^2 = 4\pi a^2 + \omega_0^2$.

(M4)

$$
\begin{aligned}
-\frac{1}{c}\partial_t \boldsymbol{E} + \boldsymbol{\nabla} \times \boldsymbol{B} &= \frac{i\omega}{c}\boldsymbol{E} \\
&= -\frac{i\omega}{c}\frac{4\pi a^2}{\omega_0^2 - \omega^2}\boldsymbol{E} \\
&= \frac{4\pi}{c}\partial_t \boldsymbol{P}
\end{aligned}
\tag{12}
$$

again, provided $\omega^2 = 4\pi a^2 + \omega_0^2$.

Thus, Maxwell's equations have longitudinal plane wave solutions whose frequency satisfies $\omega^2 = 4\pi a^2 + \omega_0^2$.

b) We claim that, in the long-wavelength limit, transverse waves are photon-like. As $k \to 0$, ω must also go to zero since the wave velocity may not exceed c. Therefore, $\lim_{k\to 0}\omega(k) = 0$. In this limit, we can assume $\omega^2 \approx 0$. In this case, the frequency-wavenumber relation for transverse waves becomes

$$
\begin{aligned}
\omega &= \frac{ck}{\sqrt{\epsilon(\omega)}} \\
\implies \omega(k \to 0) &= \frac{ck}{\sqrt{1 + \frac{4\pi a^2}{\omega_0^2}}} \\
\omega(k \to 0) &= \frac{ck}{n}
\end{aligned}
\tag{13}
$$

where $n = \sqrt{1 + 4\pi a^2/\omega_0^2}$ is the index of refraction.

c) Let $\omega_- = \omega_0$ and let $\omega_+ = \sqrt{\omega_0^2 + 4\pi a^2}$. We claim that there can be no wave propagation in the frequency band $\omega_- < \omega < \omega_+$. Since we assume $n \in \mathbb{R}$, in order for electromagnetic waves to propagate, we must have $\epsilon(\omega) > 0$. With this requirement we have

$$
\begin{aligned}
\epsilon(\omega) &> 0 \\
1 + \frac{4\pi a^2}{\omega_0^2 - \omega^2} &> 0 \\
\omega_0^2 - \omega^2 + 4\pi a^2 &> 0 \text{ , if } \omega_0 > \omega \\
\omega_0^2 - \omega^2 + 4\pi a^2 &< 0 \text{ , if } \omega_0 < \omega \\
\implies \omega_0^2 + 4\pi a^2 &> \omega^2 \text{ , if } \omega_0 > \omega \\
\omega_0^2 + 4\pi a^2 &< \omega^2 \text{ , if } \omega_0 < \omega
\end{aligned}
\tag{14}
$$

But since $\omega_0^2 + 4\pi a^2 > \omega_0^2$, in order for waves to propagate, we have either $\omega_0 > \omega$ or $\sqrt{\omega_0^2 + 4\pi a^2} < \omega$. Therefore, no wave propagation is possible in the range $\omega_0 < \omega < \sqrt{\omega_0^2 + 4\pi a^2}$. Moreover, we have

$$\begin{aligned} \frac{\omega_+^2}{\omega_-^2} &= \frac{\omega_0^2 + 4\pi a^2}{\omega_0^2} \\ &= 1 + \frac{4\pi a^2}{\omega_0^2} \\ &= \epsilon(\omega = 0) \end{aligned} \tag{15}$$

d) The frequency-wavenumber relation for this system is, in general, $\omega(k) = ck/\sqrt{\epsilon(\omega(k))}$. A plot of this relation is shown below.

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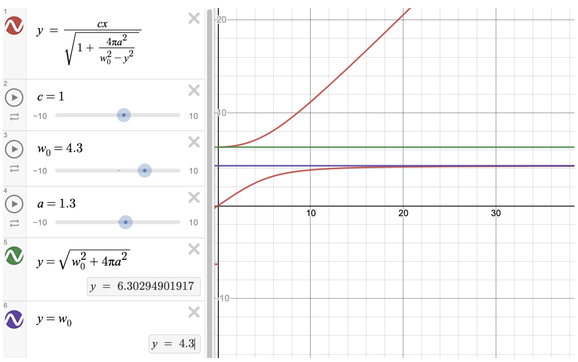

Figure 1: Sorry this isn't bigger and for the overall quality. The y-axis shows ω and the x-axis is k.

In the $k \to 0$ limit, and for $\omega > \sqrt{\omega_0^2 + 4\pi a^2}$, we have $\omega \to \sqrt{\omega_0^2 + 4\pi a^2}$. For $\omega < \omega_0$, we have $\omega \to ck/\sqrt{\epsilon(0)}$.

In the $k \to \infty$ limit, and for $\omega > \sqrt{\omega_0^2 + 4\pi a^2}$, we have $\omega \to ck$. For $\omega < \omega_0$, we have $\omega \to \omega_0$.

References

(1) J. Schwinger et al., *Classical Electrodynamics*.

(2) J.D. Jackson, *Classical Electrodynamics*.

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