Reality check: as $r \to \infty$, is this result sensible?

$r \to \infty \Rightarrow \theta \to ?$

$\Rightarrow \phi' \to ?$

Which term in $*$ dominates?

$$p = \frac{L^2}{m\mu}$$

$$e = \sqrt{1 + \frac{2EL^2}{m\mu^2}} \Rightarrow e^2 - 1 = \frac{2EL^2}{m\mu^2}$$

$$\Rightarrow t \to \frac{p^{3/2}}{\sqrt{\frac{\mu}{m}}} \left(\frac{m\mu^2}{2EL^2}\right)^{\frac{3}{2}} \frac{2E}{\mu} r$$

$$= \sqrt{\frac{m}{\mu}} \frac{L^3}{(m\mu)^{3/2}} \frac{m^{\frac{3}{2}}\mu^3}{(2E)^{3/2}L^3} \frac{2E}{\mu} r = \sqrt{\frac{m}{2E}}\, r$$

Make sense? what's $\sqrt{\frac{m}{2E}}$?

Last word on central force laws:

Starting from conservation law:

$$\frac{1}{2}\dot{r}^2 = \frac{1}{m}\left(E - U(r) - \frac{h^2 m}{2r^2}\right) \quad ; \quad h \equiv \frac{L}{m}$$

Take a seeming step backwards: $\frac{d}{dt}$ (both sides)

Using $\frac{d}{dt} = \frac{h}{r^2}\frac{d}{d\theta}$ (cons. of ∡ momentum), $h =$

Get

$$\frac{h}{r^2}\frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right) = \frac{F_r(r)}{m} + \frac{h^2}{r^3}$$

Now, change of variables

$$u \equiv \frac{1}{r} \Rightarrow r \equiv \frac{1}{u}$$

$$\Rightarrow hu^2 \frac{d}{d\theta}\left(\qquad \right) = \frac{F_r(u)}{m} + h^2u^3$$

$$\Rightarrow \boxed{\frac{d^2u}{d\theta^2} = -\frac{F_r(u)}{h^2mu^2} - u}$$

Kepler problem:

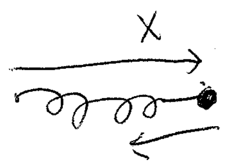
$$F_r(r) = -\frac{\mu}{r^2} \Rightarrow F_r(u) = -\mu u^2$$

$$\Rightarrow \boxed{\frac{d^2 u}{d\theta^2} = \frac{\mu}{h^2 m} - u} \quad ; \quad u = \frac{1}{r}$$

Solution:

Simple Harmonic motion + General theory of small oscillations + stability / instability

Simple 1 d mass on a spring:



$U(x) = \frac{1}{2} k x^2$

$F(x) = -kx$

Equation of motion:

$$m\ddot{x} = -kx \Rightarrow \boxed{\ddot{x} = -\frac{k}{m} x}$$

solution:

~~Solution for $x(0) = x_0$, $\dot{x}(0) = v_0$~~

Note: ω independent of amplitude

$\Rightarrow$ Period independent of amplitude (special feature of linear force)

Alternative expression:

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

~~Rela~~ Relation between $c_{1,2}$ and A, ϕ:

Solution in this ~~notation~~ form when

$x(0) = x_0$, $\dot{x}(0) = v_0$

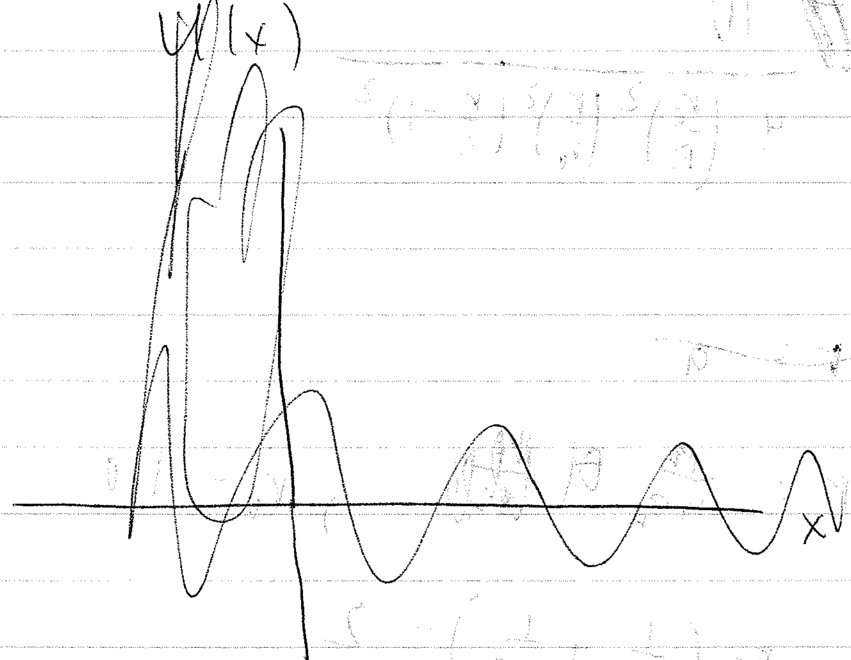
Finally, complex exponential solution:

~~Formally~~ General solution to

$\ddot{x} = \gamma x$, independent of sign of γ

$$x = z_1 e^{\sqrt{\gamma} t} + z_2 e^{-\sqrt{\gamma} t}$$

$\gamma > 0$: Unstable, z_1, z_2 real



$\gamma < 0$: stable, oscillating : $\gamma \equiv -\omega^2$

$\sqrt{\gamma} = i\,\omega$

Find z_1, z_2 s.t. $x(0) = x_0$, $\dot{x}(0) = v_0$; x_0, v_0 real

Best, when confronted with linear eqn's., to always assume exponential sol'n.

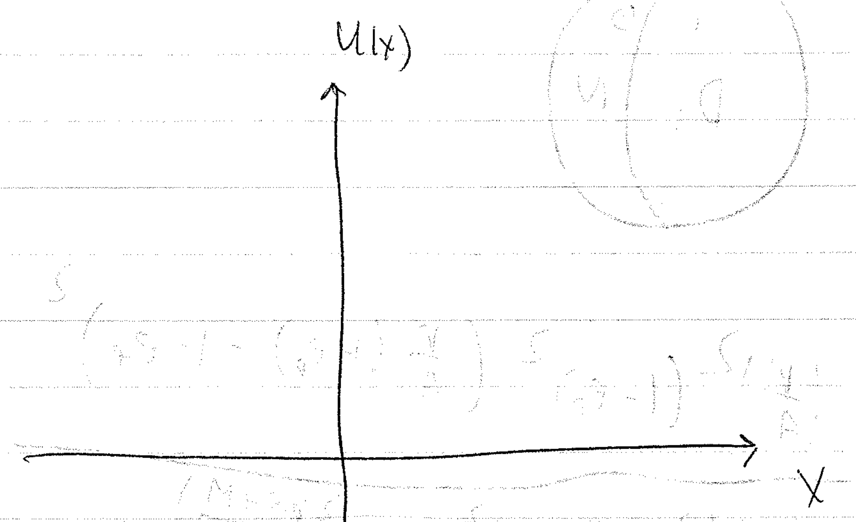
$x = e^{\gamma t}$. $\mathrm{Re}\,\gamma > 0 \Rightarrow$ unstable

$\mathrm{Re}\,\gamma < 0 \Rightarrow$ stable

$\mathrm{Im}\,\gamma \neq 0 \Rightarrow$ oscillations

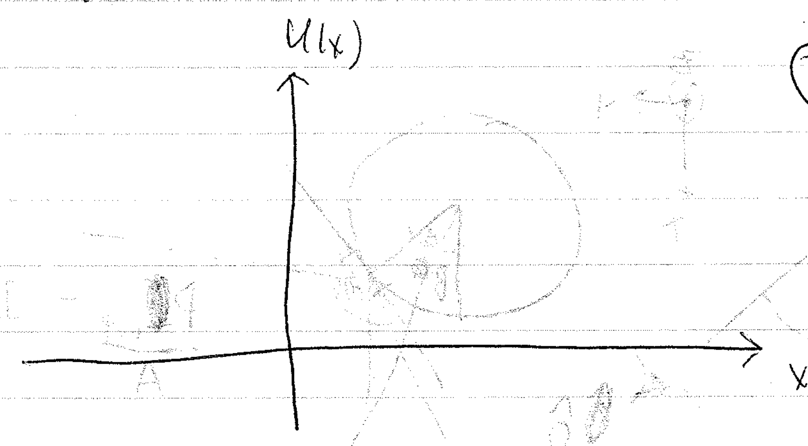
Why do we care so much about linear forces?

What's potential for ~~[illegible]~~ $F(x) = -kx$?



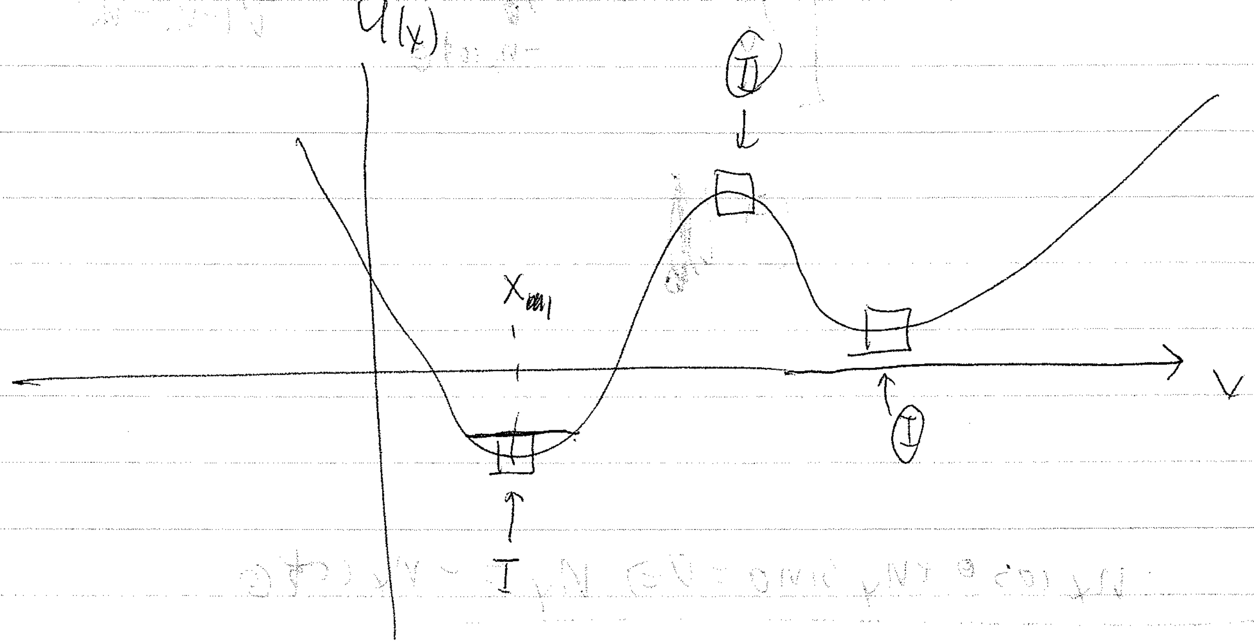
I

Potential for $F(x) = kx$ (unstable case)



II

Now, look at arbitrary potential



For motion near equilibrium positions of potential, potential is quadratic

$\Rightarrow$ linear eqns hold.

And these are most important points for motion.

What is eqn. for small oscillations about (I) ?

Taylor series:

$$U(x \approx x_1) = U(x_1) + \underbrace{\frac{dU}{dx}\Big|_{x_1}}_{= ?}(x-x_1) + \;?$$

$$\Rightarrow \ddot{x} = ?$$

$$\Rightarrow \omega = ?$$

This approach generalizes to radial motion + effective potentials:

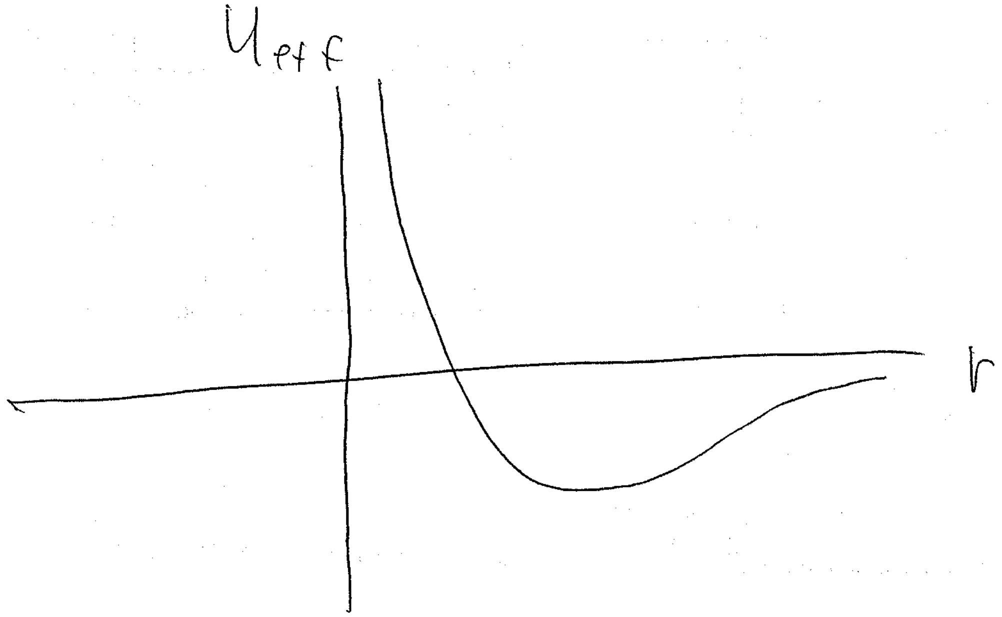
recall:

$$\frac{1}{2}\dot{r}^2 = \frac{1}{m}\left(E - U_{eff}(r)\right)$$

$$\Rightarrow \dot{r}\ddot{r} = -\frac{1}{m}\frac{dU_{eff}}{dr}\dot{r}$$

$$\Rightarrow \ddot{r} = -\frac{1}{m}\frac{dU_{eff}(r)}{dr}$$: Just like 1-d motion in pot'l U_{eff}!

So... $\rightarrow$ next page



where will stable small osc. occur?

What will their frequency be?

Example: Kepler problem:

$U_{eff}(r) =$

$\Rightarrow \dfrac{dU_{eff}}{dr} = \qquad\qquad = 0$ at $r = r_{min}$

$\Rightarrow r_{min} = ?$

radius of circular orbit? $r_{circle} =$

Now, small radial nudge. ~~Period of oscillatn~~

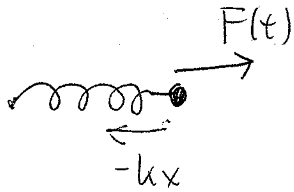
Frequency of oscillation

$\omega^2 =$

Familiar?

Start here:

Forced oscillations: and resonance



Equation of motion:

$$\ddot{x} + \omega_0^2 x = \frac{F(t)}{m}, \quad \omega_0^2 =$$

Oscillating force: $F(t) = F_0 \cos(\omega t)$, $\omega \neq \omega_0$

Sol'n of eqn. of motion?

Write eqn. as

$$(1) \quad \ddot{x} + \omega_0^2 x = \frac{F_0}{2m}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

Now, find sol'n of

$$\boxed{(2) \quad \ddot{x}_+ + \omega_0^2 x_+ = \frac{F_0}{2m} e^{i\omega t}}$$

and

$$\boxed{(3) \quad \ddot{x}_- + \omega_0^2 x_- = \frac{F_0}{2m} e^{-i\omega t}}$$

How can you write sol'n (x) of (1) i.t.o. sol'ns

$x_\pm$ of (2) + (3)?

So what <u>are</u> sol'ns of (2)+(3)?

Simple guess: everything in (2) has <u>same</u> dependence on t.

$$\Rightarrow \quad x_+ = z_+ e^{i\omega t}$$

$$\Rightarrow \quad \ddot{x}_+ = -\omega^2 x_+ = -\omega^2 z_+ e^{i\omega t}$$

$$\Rightarrow (\omega_0^2 - \omega^2) z_+ \cancel{e^{i\omega t}} = \frac{F_0}{2m} \cancel{e^{i\omega t}}$$

Beauty of linear eqns: exponential sol'n always works, 'cause der. $\propto$ fn.
$\Rightarrow$ exponential factors <u>always</u> cancel

$$\Rightarrow \quad z_+ = \frac{F_0}{2m(\omega_0^2 - \omega^2)}$$

Ditto for x_-, z_-

So $$x(t) = x_+ + x_- = \frac{F_0}{m(\omega_0^2-\omega^2)} \left(\frac{e^{i\omega t} + e^{-i\omega t}}{2} \right)$$

$$= \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

$$\equiv A(\omega) \cos(\omega t)$$

To Xerox

$$Z_\pm(w) = \frac{F_0}{2w} e^{iwt} \equiv Z(w) F_\pm(t)$$

$$\boxed{Z(w) = \frac{1}{m(w_0^2 - w^2)}} \equiv \text{"Frequency dependent mechanical impedance"}$$

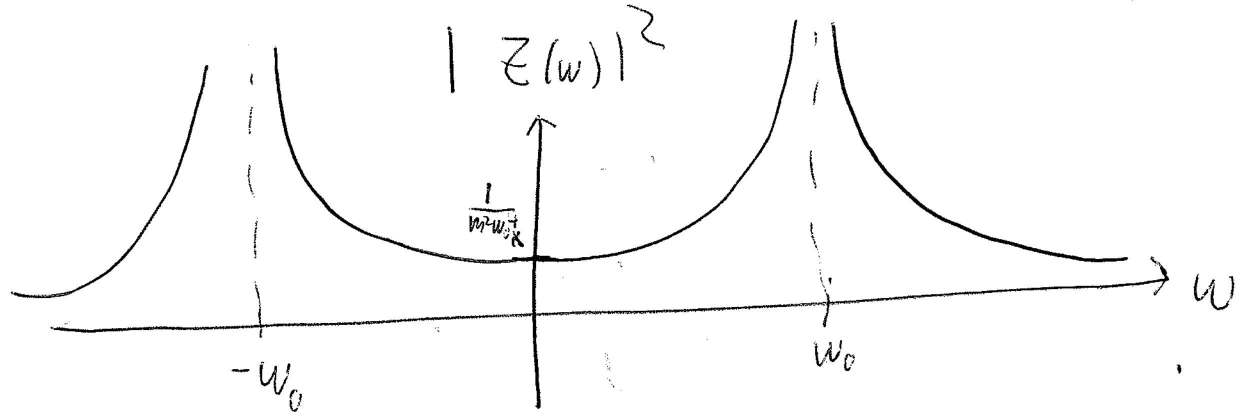
General definition:

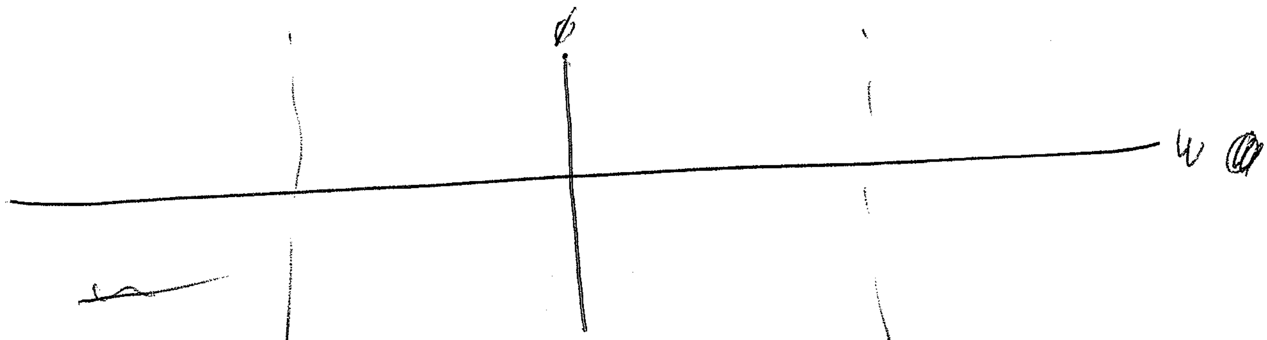
$$F(t) = F_0 e^{iwt} \Rightarrow x(t) = Z(w) F(t)$$

In general, complex

$$\Rightarrow Z(w) = |Z(w)| e^{i\phi}$$

For simple, undamped harmonic oscillator





What happens when $\omega = \omega_0$?

"steady state" response $\to \infty$

$\Rightarrow$ try growing, oscillating sol.'n:

$$x(t) = A t e^{i\omega_0 t}$$

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{2m} e^{i\omega_0 t}$$

(Add complex conjugate sol'n later)

$\Rightarrow$

5.13

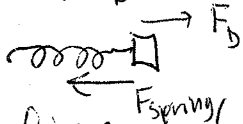
So, general solution for forced harmonic oscillator:

$$x(t) = \frac{F_0}{m(\omega_0^2-\omega^2)}\cos(\omega t) + ?$$

Sol'n for $x(0) = x_0$, $\dot{x}(0) = v_0$

Now, for only time in course, add damping:

$F_{drag} = -mD\dot{x}$: D = damping coefficient



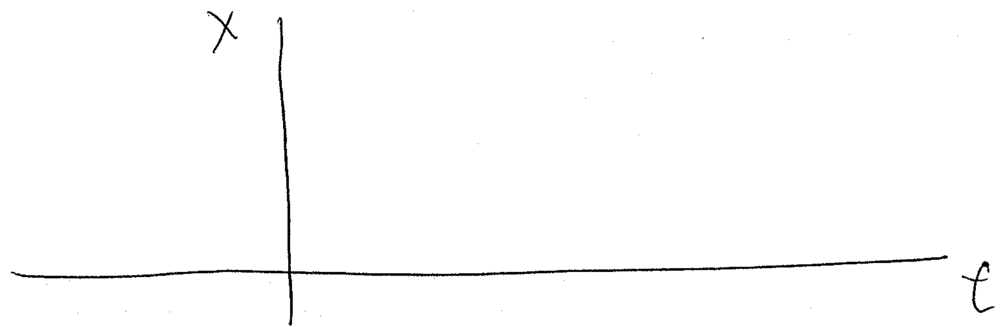
Equation of motion: (Unforced)

Solution:

$$\Rightarrow \gamma = -\frac{D \pm \sqrt{D^2 - 4\omega_0^2}}{2}$$: Pure Real, Pure imaginary, or complex?

If complex, $\Rightarrow$ "Underdamped". For what D, ω_0 does this case apply?

Behavior?



Note that $D \uparrow \Rightarrow$ decays faster for this underdamped case.

Now, suppose $D > 2\omega_0 \Rightarrow$ 2 real roots

"overdamped" case

~~$x(t) = a e^{-|\gamma_+|t} + b$~~

$$\gamma = \frac{-D \pm \sqrt{D^2 - 4\omega_0^2}}{2} : >, <, \text{ or } = 0 ?$$

$$\Rightarrow x(t) = a e^{\gamma_+ t} + b e^{\gamma_- t} = a e^{-|\gamma_+| t} + b e^{-|\gamma_-| t}$$

What happens to γ_+ as $D \uparrow$?

" " " γ_+ for $D \gg \omega_0$?

Now, decay rate of "slow solution" $\downarrow$ as $D \uparrow$.

Why?