PHYS 614 – Spring 2021 – Midterm

Allotted time – 2 hours; Due – Wednesday, May 5, 2021 at 11:59 pm on Canvas

Instructions:

- Please take this exam in a contiguous two-hour window, as you would an in-class exam. The two-hour period includes time for reading the questions — please do not look at the questions until after you have begun the two-hour timer.
- You are required to work on this exam on your own — no discussions allowed.
- Please submit your exam online as a PDF upload as you have been for the problem sets. If your exam is handwritten, please use a scanner app on your smartphone (such as Microsoft Lens or Scanner Pro); it vastly improves legibility. Time taken for scanning and uploading does not count towards your two-hour limit, but please do this immediately after taking the exam. You should not edit your answers after the two-hour period is over.
- You may refer to all lecture materials, your own notes, and the online books and notes mentioned in the syllabus. You can use any results from the lecture notes or problem sets directly without having to prove them again. The answers do not rely on any results from the readings, but if you use any results from outside the lectures and problem sets, please cite them.
- The questions were designed to be answered without recourse to symbolic manipulation software such as `Mathematica`. A list of potentially useful mathematical facts is provided further down on this page. If you absolutely must look up a property of a mathematical function online or using `Mathematica`, you must state what you used, and some points may be taken off .
- The questions are intended to be straightforward; they're not "trick" questions. If a statement seems unclear, choose the most straightforward, least convoluted interpretation. If you're worried that your interpretation was wrong, write down what you have assumed and continue with your answer under the assumption.
- Before you start, read all the questions. If you absolutely need clarification on something, check your Canvas announcements first in case it's been addressed already. If not, send me an email at jpaulose@uoregon.edu. I'll monitor my email as closely as I can, but I might not be able to respond right away depending on the time you are taking the exam. Please get started assuming the most straightforward interpretation (see previous point), and don't wait for me to respond.

Potentially useful mathematical facts:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

1. Bosonic canonical ensemble (10 points)

A quantum-mechanical system has a single-particle Hamiltonian with two energy eigenstates $|\phi_1\rangle$ and $|\phi_2\rangle$ whose wavefunctions in the position representation are given by the functions $\phi_1(\mathbf{r})$ and $\phi_2(\mathbf{r})$ respectively ($\mathbf{r}$ is the particle position). The corresponding energies are ε_1 and ε_2 respectively. Assume that $\varepsilon_1 \neq \varepsilon_2$.

The system is occupied by two identical bosons, which do not interact with one another.

i. Describe, in words, the energy eigenstates $|\Psi_\alpha\rangle$ of the two-particle system in terms of the occupancies of the single-particle eigenstates. Write down their energies, and the wavefunctions $\Psi_\alpha(\mathbf{r}_1, \mathbf{r}_2)$, in terms of the corresponding single-particle quantities. Here, α is an index which distinguishes the unique energy eigenstates of the two-particle system.

ii. Find the density matrix of the two-particle system for the mixed state corresponding to the canonical ensemble when the system is equilibrated at temperature T. You can write your answer in terms of the $|\Psi_\alpha\rangle$, or as an actual matrix (if the latter, label which states the rows and columns correspond to).

iii. Evaluate the probability of finding the the state $|\phi_1\rangle$ occupied when the system is in equilibrium at temperature T.

2. Entropy of the Bose-Einstein condensate (10 points)

Starting from the thermodynamic relation $E = TS - PV + \mu N$, where the symbols have their usual meanings, find the entropy of a 3D ideal Bose gas of N particles confined to a volume V, in the condensate phase (i.e. for temperature $T < T_\text{c}$ where T_c is the condensate temperature). You only need to include contributions that are extensive, i.e. proportional to particle number N or volume V.

3. Black-body radiation in 2D (10 points)

In a two-dimensional world, what would be the spectral energy density $u(\omega)$ of a black body in equilibrium at temperature T? The spectral energy density is defined analogously to the 3D case: $u(\omega)\, d\omega$ is the average energy stored in photons in the frequency range $(\omega, \omega + d\omega)$, per unit area of the black body. All relevant properties of the photons (the energy-momentum relation, speed of light, Planck's constant, etc.) are unchanged in the 2D world.