PHYS 631: Quantum Mechanics I (Fall 2020)
Homework 3
Assigned Monday, 12 October 2020
Due Monday, 19 October 2020

Problem 1. Show that

$$\int_{-\infty}^{\infty} dx\, \delta'(x-y)\, f(x) = -f'(y), \tag{1}$$

giving whatever suitable conditions on $f(x)$ that you require for this relation to hold. Also show that

$$\int_{-\infty}^{\infty} dx\, \delta'(y-x)\, f(x) = f'(y), \tag{2}$$

thereby showing that $\delta'(x)$ is odd, $\delta'(-x) = \delta'(x)$. Generalize these results to the kth derivative $\delta^{(k)}(x)$.

Problem 2. Using *only* the projection property

$$\delta(x-a)\, f(x) = \delta(x-a)\, f(a) \tag{3}$$

of the delta function, derive the scaling property of the delta function:

$$\delta(\alpha x) = \frac{1}{|\alpha|}\delta(x) \qquad (\alpha \in \mathbb{R},\ \alpha \neq 0). \tag{4}$$

That is, derive whatever other properties you need from Eq. (3).

Problem 3. In one dimension, where $[q,p] = i\hbar$, derive an expression for the operator $1/p$ in the position representation, subject to the boundary condition

$$\lim_{x\to-\infty} \langle x|\frac{1}{p}|\psi\rangle = 0 \tag{5}$$

for an arbitrary state vector $|\psi\rangle$.

Hint: remember that an operator is defined by its action on a state, so guess the form of the operator and show that it does the right thing and satisfies the boundary condition.

Problem 4. Consider the classical Lagrangian

$$L(x,\dot{x}) = \frac{m}{2} g(x)\, \dot{x}^2 - V(x). \tag{6}$$

The goal of this problem is to develop the corresponding quantum theory via canonical quantization—this is essentially a review of the entire quantization procedure, but with a twist to make it interesting.

(a) Show that the classical Hamiltonian is

$$H(x,p) = \frac{p^2}{2mg(x)} + V(x). \tag{7}$$

(b) Argue that, in quantizing this system, that the commutator should still be

$$[x,p] = i\hbar \tag{8}$$

as usual.

(c) The problem of course with simply promoting x and p to operators in the Hamiltonian is the ordering ambiguity in the kinetic-energy term. To resolve the ambiguity, consider the transformation to a new generalized coordinate q, defined by

$$\frac{dq}{dx} = \sqrt{g(x)}. \tag{9}$$

Rewrite the Lagrangian (6) in terms of q, write down the momentum p_q conjugate to q, and write down the Hamiltonian. You should find that the resulting Hamiltonian is readily carried over to quantum mechanics via canonical quantization.

(d) The coordinate transformation also induces some other changes in the structure of the Hilbert space. In terms of the q coordinate, the inner product has the usual form

$$\langle\psi_1|\psi_2\rangle = \int dq\, \psi_1^*(q)\, \psi_2(q). \tag{10}$$

Write down the analogous expression in terms of the original coordinate x. Show also that the identity in the position representation must now be

$$\int dx\, \sqrt{g(x)}\, |x\rangle\langle x| = 1. \tag{11}$$

(e) Derive a modified "orthonormality" expression for the position representation; that is, derive an expression for $\langle x|x'\rangle$. [Consider an arbitrary state $\psi(x) = \langle x|\psi\rangle$, and try inserting an identity.]

(f) Turning now the momentum operator, recall that it is defined (in part) by the standard commutator rule (8). Show that a modified form of the momentum operator, obtained by inserting factors of g,

$$p = \frac{\hbar}{i} g^{-\alpha}(x)\partial_x g^{\alpha}(x), \tag{12}$$

still satisfies this commutation rule.

(g) Furthermore, the momentum operator must be Hermitian with respect to the modified inner product that you derived in part (d). Show that this constraint implies that $\alpha = 1/4$.

(h) Starting with this form of the momentum operator, derive an expression for the momentum eigenstates, expressed in the position representation.

(i) Use your result from (h) to show that the identity expressed in the momentum representation is

$$\int dp\, |p\rangle\langle p| = 1. \tag{13}$$

Note that this implies the momentum orthonormality relation $\langle p|p'\rangle = \delta(p - p')$.

(j) Finally, show that the properly quantized and ordered Hamiltonian corresponding to the Lagrangian (6) is

$$H(x,p) = \frac{1}{2m}\left(\frac{1}{g^{1/4}}p\frac{1}{\sqrt{g}}p\frac{1}{g^{1/4}}\right) + V(x). \tag{14}$$

To do this, start with the operator ∂_q^2 and rewrite it in terms of the x coordinate. Use this result to infer how the p^2 term from the q-form Hamiltonian should carry over to the x-form Hamiltonian.

Problem 5. This is not really a problem, just something to think about for a future problem. You don't need to turn in a solution to this problem.

Below is an excerpt of a nice book by Leonard Mlodinow,[1] where Feynman muses about how little things you think up can be useful later.

> Staying playful, having fun, keeping a youthful outlook. It was clear to me that for Feynman, staying open to all the possibilities of nature, or life, was a key to both his creativity and his happiness.
>
> I asked him, ''Is it foolish to become mature?''
>
> He thought for a moment. He shrugged.
>
> I'm not sure. But an important part of the creative process is play. At least for some scientists. It is hard to maintain as you get older. You get less playful. But you shouldn't, of course.
>
> I have a large number of entertaining mathematical type of problems, little worlds of this kind that I play in and that I work in from time to time. For example, I first heard about calculus when I was in high school and I saw the formula for the derivative of a function. And the second derivative, and the third… Then I noticed a pattern that worked for the nth derivative, no matter what the integer n was—one, two, three, and so forth.
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> But then I asked, what about a ''half'' derivative? I wanted an operation that when you do it to a function gives you a new function, and if you do it *twice* you get the ordinary first derivative of the function. Do you know that operation? I invented it when I was in high school. But I didn't know how to calculate it in those days. I was only in high school, so I could only define it. I couldn't compute anything. And I didn't know how to do anything to check it or anything. I just defined it. Only later, when I was in the university, did I start over again. And I had a lot of fun with it. And found out that my original definition that I thought up in high school was right. It would work.
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> Then when I was in Los Alamos working on the atomic bomb, I saw some people doing a complicated equation. And I realized that the form they had corresponded to my half derivative. Well, I had invented a numerical operation for solving it, so I did it, and it worked. We checked it by doing it twice, which is just the ordinary derivative. So I did a nifty numerical method for solving their equation. Everything, well, not everything, but lots of stuff turns out to be useful. You just play it out.

For now, from what you know/are learning about quantum mechanics, can you guess what Feynman meant by a half-derivative, and how you might define it as a mathematical operator? (More specifically, obviously, than two half derivatives make a full derivative.) We'll revisit this in a coming homework problem (possibly in the far future)…

[1]Leonard Mlodinow, *Feynman's Rainbow: A Search for Beauty in Physics and in Life* (Warner, 2003), Chapter 11 (ISBN: 9780759570788).