Now, all bells + whistles:

Forced, damped oscillator:

$$\ddot{x} + D\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$$

Separate into complex conjugate parts

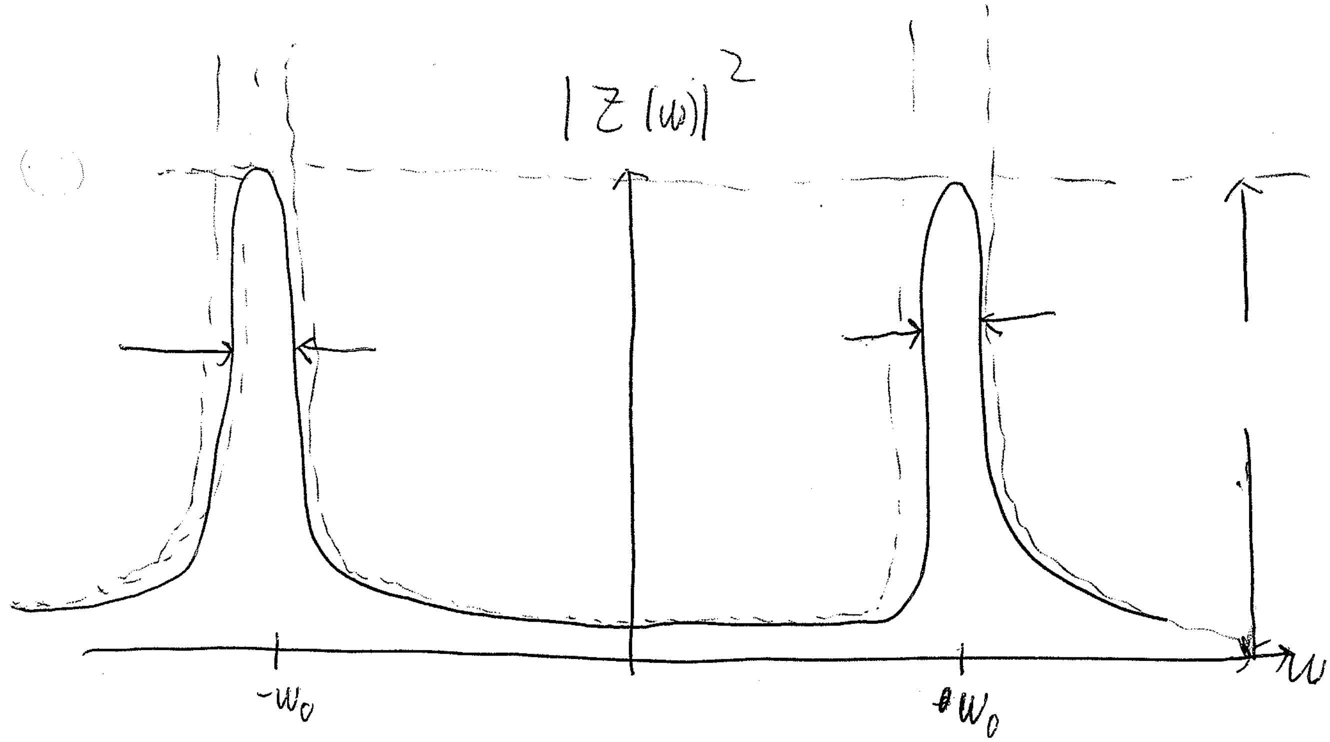
$$\ddot{x}_+ + D\dot{x}_+ + \omega_0^2 x_+ = \frac{F_0}{2m} e^{i\omega t}$$

Now, particular sol'n:

Note: $A(\omega)$ complex

$$A(\omega) \equiv Z(\omega)\frac{F_0}{2m}$$

$$Z(\omega) = \frac{1}{\omega_0^2 - \omega^2 + iD\omega}$$



for very small Damping ($D \ll \omega_0$)

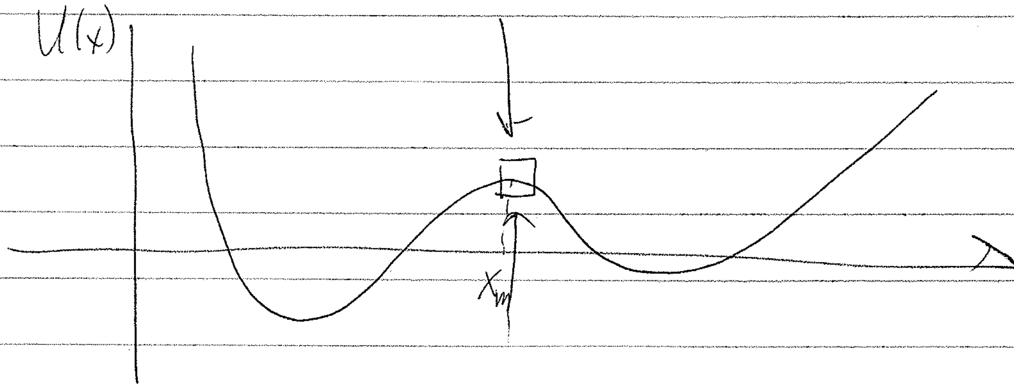
$$|Z(\omega \approx \omega_0)|^2 = \frac{1}{\quad} \quad : \text{Lorentzian}$$

Most ubiquitous function in physics

	oscillator	Particle physics (particle scattering)	Condensed matter physics
Peak position	ω_0 "natural frequency"	m : Particle mass	$q_0 = \frac{2\pi}{a}$, a = lattice constant
Peak width :	$\frac{1}{\tau}$: lifetime	$\frac{1}{\tau}$: lifetime	$\frac{1}{\xi}$: correlation length

Instability:

Suppose we expand around maximum of potential?



Now what are E.O.M. for small $x - x_m$?

Expand
$U(x) = \quad . \; . \; . \; .$

$\Rightarrow \ddot{x} =$

linear again; ~~try st~~ Solution?

$$X(t) = x_+ e^{\lambda t} + x_- e^{-\lambda t}$$

$$\lambda = \sqrt{\left.\frac{d^2U}{dx^2}\right|_{x=x_m}}$$

Sign?

$\Rightarrow$ + solution unstable!

Now: Beyond 1d motion.

Many variables.

In general, we work as follows

1) Write Lagrangian $\mathcal{L}(\{q_i, \dot{q}_i\})$

N variables
(N = # of d.o.f.)

2) Find EOM's (N of 'em)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}$$; 1 for each i

3) Solve for $\boxed{\ddot{q}_i = f_i(\{q_j, \dot{q}_j\}) \quad (1)}$

4) Despair! Can't solve 'em.

~~Approach~~

Overcoming despair:

1) Find fixed points: (FP)

$\{q_i^*\}$ s.t. $q_i = q_i^* =$ constants

solve (1)

What's $\mathring{q}_i$ ~~at~~ for such sol'n?

So, FP's at $\boxed{f(\{q_j^*, \mathring{q}_j = 0\}) = 0}$

Can solve these.

Now what?

2) <u>Expand</u> f in ~~small~~ $\mathring{q}_i$, $\delta q_i \equiv q_i(t) - q_i^*$

<u>Usually</u>, in energy conserving problems, this drops out

$f(\{q_j^* + \delta q_j, \mathring{q}_j\}) = ?$

$$f_i(\{q_j^* + \delta q_j, \mathring{q}_j\}) = \underbrace{f_i(\{q_j^*, 0\})}_{=?} + \sum_j \left(\frac{\partial f_i}{\partial q_j}\bigg|_{\{q_j = q_j^*\}} \delta q_j + \frac{\partial f_i}{\partial \mathring{q}_j}\bigg|_{q_j = q_j^*} \mathring{q}_j \right)$$

$$+ \underbrace{O((\delta q_j)^2, \mathring{q}_j^{\,2}, \delta q_j \mathring{q}_n)}_{\text{neglect for small pert.}}$$

<u>Note</u>: <u>Constants</u>

$\Rightarrow$ EOM (1) becomes

$$\boxed{(2) \quad \mathring{\mathring{q}}_i = \sum_j \left(\frac{\partial f_i}{\partial q_j}\bigg|_* \delta q_j + \frac{\partial f_i}{\partial \mathring{q}_j}\bigg|_* \mathring{q}_j \right)}$$

<u>N</u> <u>linear</u> <u>equations</u>!

How do we solve 'em?

Assume <u>all</u> q_i's have <u>same</u> exponential dependence on t:

$$q_i(t) = q_i e^{\lambda t} \quad \text{same } \lambda, \text{ all } t$$

q_i: N constants

$$\lambda^2 a_i = \sum_j M_{ij}(\lambda)\, a_j$$

$$M_{ij}(\lambda) = \left.\frac{\partial f_i}{\partial q_j}\right|_* + \lambda \left.\frac{\partial f_i}{\partial \dot{q}_j}\right|_*$$

Write in matrix form:

$$\boxed{\lambda^2 \underbrace{\vec{a}}_{N \text{ component vector}} = \underbrace{\underset{\sim}{M}(\lambda)}_{N \times N \text{ matrix}}\vec{a}}$$

$$\underset{\sim}{M} = \begin{array}{cc} & \overbrace{\hphantom{M_{11} \quad M_{12} \quad M_{13} \quad \cdots}}^{N \text{ columns}} \\ N \text{ rows} & \begin{pmatrix} M_{11} & M_{12} & M_{13} & \cdots \\ M_{21} & M_{22} & M_{23} & \cdots \\ M_{31} & & & \\ \vdots & \vdots & & \vdots \end{pmatrix} \end{array}$$

Eigenvalue problem:

Rewrite as

$$\left(\underset{\sim}{M}(\lambda) - \lambda^2 \underset{\sim}{I}\right)\vec{a} = 0$$

$$\underset{\sim}{I} = \text{identity matrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & & 1 & & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix}$$

Funny thing about eigenvalue problems:

It *looks* like you're solving for $\vec{a}$,

given eigenvalue λ

But: Really solving for λ.

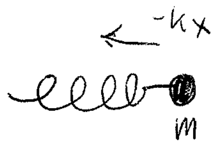
How so?

Only for *special* values of λ

the "eigenvalues" does * have

any solution for $\vec{a}$ other than $\vec{a} = 0$.

Why? Example: Ball on spring



$$\ddot{x} = -\frac{k}{m} x \Rightarrow \cancel{\ddot{x} + \omega_0^2 x = 0} \quad \ddot{x} + \omega_0^2 x = 0$$

$$\omega_0^2 = \frac{k}{m}$$

try $x(t) = x_0 e^{\lambda t}$

$$\Rightarrow \boxed{\underbrace{(\lambda^2 + \omega_0^2)}_{\equiv \underset{\sim}{M} + \lambda^2 \underset{\sim}{I}} x_0 = 0}$$

x_0: 1 component vector (i.e., number)

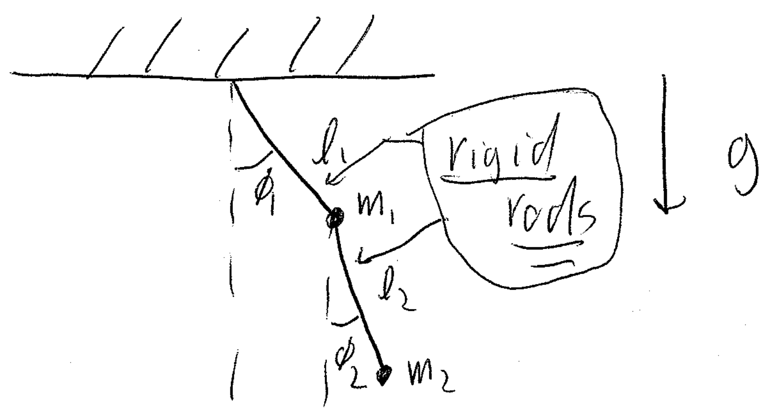
$\underset{\sim}{M}$, $\underset{\sim}{I}$ 1x1 matrices (i.e., numbers)

Solution for x_0 for arbitrary λ?

When can we have $x_0 \neq 0$?

$\Rightarrow$ This λ is eigenvalue for this problem.

2×2 example: Double pendulum:



Recall (heh, heh!) equations of motion:
(Pg. 2.10)

6.8

$$(m_1+m_2)\, l_1^2 \ddot{\phi}_1 + m_2 l_1 l_2 \ddot{\phi}_2 \cos(\phi_1-\phi_2) + \dot{\phi}_2^2 \sin(\phi_1-\phi_2)$$
$$+ (m_1+m_2)\, g\, l_1 \sin\phi_1 = 0$$

$$m_2 l_2^2 \ddot{\phi}_2 + m_2 l_1 l_2 \ddot{\phi}_1 \cos(\phi_1-\phi_2) - m_2 l_1 l_2 \dot{\phi}_1^2 \sin(\phi_1-\phi_2)$$
$$+ m_2 g\, l_2 \sin\phi_2 = 0$$

~~Linearize~~: Fixed points:

Linearize about (1):

~~Solution~~ Trial solution:

linearize

68a cheat sheet

$$(m_1+m_2) l_1^2 \ddot{\phi}_1 + m_2 l_1 l_2 \ddot{\phi}_2 + (m_1+m_2) g l_1 \phi_1 = 0$$

$$l_2^2 \ddot{\phi}_2 + l_1 l_2 \ddot{\phi}_1 + g l_2 \phi_2 = 0$$

$$\Rightarrow (m_1+m_2)\left[l_1^2 \lambda^2 + g l_1\right] a_1 + m_2 l_1 l_2 \lambda^2 a_2 = 0$$

$$l_1 l_2 \lambda^2 a_1 + (\lambda^2 l_2^2 + g l_2) a_2 = 0$$

Solve 2nd equation for a_1:

$$\boxed{a_1 = -\left(\frac{l_2}{l_1} + \frac{g}{l_1 \lambda^2}\right) a_2}$$

$$\Rightarrow (m_1+m_2)\left[l_1^2 \lambda^2 + g l_1\right] - m_2 l_1 l_2 \left(\frac{l_2}{l_1} + \frac{g}{l_1 \lambda^2}\right)$$

to it. So the fun and games will begin again as soon as Italy has remained stable enough to survive the introduction of the common currency in January.

We hope you're all doing well, that Cai's new job is as good as it sounded, and that the school year is off to a positive start. Kim, please give Nadia's hellos to everyone at Olum Center. We try to get her to send emails, but you know Nadia, so this is as good as it gets. She does talk about people there, so she's thinking of them and even talking about coming back there next year. Let us know how things are going.

Love,
Steve, Kim, Nadia

Steven Hecker
Labor Education and Research Center
1289 University of Oregon
Eugene, OR 97403, USA

Temporary address through March 1999:
via delle Alpi 11
00198 Roma, Italy
039/6/854/6320
shecker@oregon.uoregon.edu

(DP1) $\quad (m_1+m_2)\left[\ell_1^2\lambda^2 + g\ell_1\right] a_1 \quad + \quad m_2 \ell_1 \ell_2 \lambda^2 a_2 = 0$

$$\cancel{m_2\left[\ell_2^2\lambda^2 + g\ell_2\right]}$$

(DP2) $\quad m_2 \ell_1 \ell_2 \lambda^2 \, a_1 \quad + \quad m_2\left[\ell_2^2\lambda^2 + g\ell_2\right] a_2 = 0$

How can we write this in canonical form

$$\underset{\sim}{N}(\lambda) \equiv \underset{\sim}{M}(\lambda) - \lambda^2 \underset{\sim}{I} = 0 \quad ?$$

$$\underset{\sim}{N}(\lambda) = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

~~solutions~~ Solving DP1 + DP2 for a_1, a_2

6.10

$$\boxed{a_2 = -\frac{l_1 l_2 \lambda^2}{l_2^2 \lambda^2 + g l_2} a_1}$$

DP3

Stick into (1)

$$\Rightarrow a_1 \left[(m_1+m_2)(l_1^2 \lambda^2 + g l_1) - \frac{m_2 l_1^2 l_2^2 \lambda^4}{\lambda^2 l_2^2 + g l_2} \right] = 0$$

~~when can a_1 be $\neq 0$?~~

~~only when~~

Define $\frac{m_2}{m_1+m_2} \equiv \alpha$, $w_{1,2}^2 \equiv \frac{g}{l_{1,2}}$

$$\Rightarrow a_1 \left[\lambda^2 + w_1^2 - \frac{\alpha \lambda^4}{\lambda^2 + w_2^2} \right] = 0$$

When can a_1 be $\neq 0$?

(6.11

$$\frac{m_2}{m_1+m_2} \equiv \alpha < 1, \quad \frac{g}{l_1} \equiv w_1^2, \quad \frac{g}{l_2} \equiv w_2^2$$

$$\Rightarrow \quad \Rightarrow (\lambda^2+w_1^2) = \frac{\alpha \lambda^4}{\lambda^2+w_2^2}$$

$$\Rightarrow (\lambda^2+w_1^2)(\lambda^2+w_2^2) - \alpha\lambda^4 = 0$$

$$\Rightarrow (1-\alpha)\lambda^4 + (w_1^2+w_2^2)\lambda^2 + w_1^2 w_2^2 = 0$$

$$\Rightarrow \boxed{\lambda^2 = \frac{-(w_1^2+w_2^2) \pm \sqrt{(w_1^2+w_2^2)^2 - 4(1-\alpha)w_1^2 w_2^2}}{2(1-\alpha)}}$$

$$\alpha \to 0, \quad \lambda^2 \to -\frac{(w_1^2+w_2^2) \pm \sqrt{(w_1^2-w_2^2)^2}}{2} = \begin{cases} -w_1^2, & -\text{ root} \\ -w_2^2, & +\text{ root} \end{cases}$$

Make sense?

$$\alpha \neq 0, \quad \boxed{\lambda^2 = \frac{-(w_1^2+w_2^2) \pm \sqrt{(w_1^2-w_2^2)^2 + 4\alpha w_1^2 w_2^2}}{2(1-\alpha)}}$$

Always stable?

Yes! $\Rightarrow$ $\lambda = \pm i\omega$

$$\omega_{\pm}^2 = \frac{\omega_1^2 + \omega_2^2 \pm \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4\alpha\, \omega_1^2 \omega_2^2}}{2}$$

So, what's general solution for $\phi_1(t), \phi_2(t)$ near this Fixed Point?

How many independent parameters?

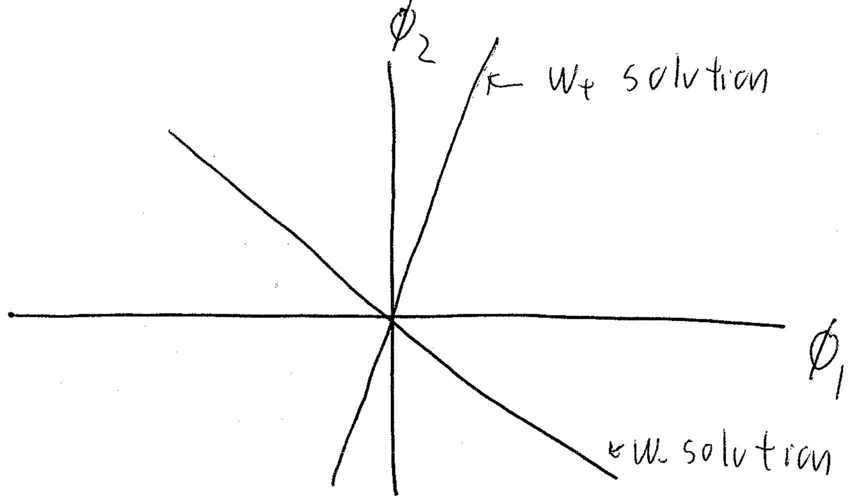
" " " initial conditions?

How do we reduce above to right #?

Answer: b P3

Determines direction of eigenvector $\vec{\xi}_{\pm}(\lambda_{\pm})$

Note: depends on eigenvalue



Simple numbers: $\frac{l_2}{l_1} = \frac{4}{3}$, $m_1 = m_2 \Rightarrow \alpha = \frac{1}{2}$

$$w_2^2 = \frac{g}{l_2} = \frac{3}{4}\frac{g}{l_1} = \frac{3}{4}w_1^2$$

$$\Rightarrow \lambda_{\pm}^2 = -\left(w_1^2 + \frac{3}{4}w_1^2\right) \pm \sqrt{\left(\frac{w_1^2}{4}\right)^2 + 2w_1^2\left(\frac{3}{4}w_1^2\right)}$$

$$= w_1^2\left[-\frac{7}{4} \pm \sqrt{\frac{1}{16} + \frac{3}{2}}\right] = w_1^2\left[-\frac{7}{4} \pm \frac{5}{4}\right]$$

$$\Rightarrow \boxed{\lambda_+^2 = -\frac{w_1^2}{2}, \quad \lambda_-^2 = -3w_1^2}$$

$\Rightarrow$ DP3 reads

$$a_2 = -\frac{\lambda^2}{\frac{l_2}{l_1}\lambda^2 + \frac{g}{l_1}} a_1 = \boxed{-\frac{\lambda^2}{\frac{4}{3}\lambda^2 + w_1^2} a_1 = a_2}$$

$$\lambda_+ : \quad a_{2+} = \frac{-\left(-\frac{w_1^2}{2}\right)}{\frac{4}{3}\left(-\frac{w_1^2}{2}\right) + w_1^2} a_{1+} = \boxed{\frac{3}{2} a_{1+} = a_{2+}}$$

$$\lambda_- : \quad a_{2-} = \frac{-(-3w_1^2)}{\frac{4}{3}(-3w_1^2) + w_1^2} a_{1-} = \boxed{-a_{1-} = a_{2-}}$$

General solution:

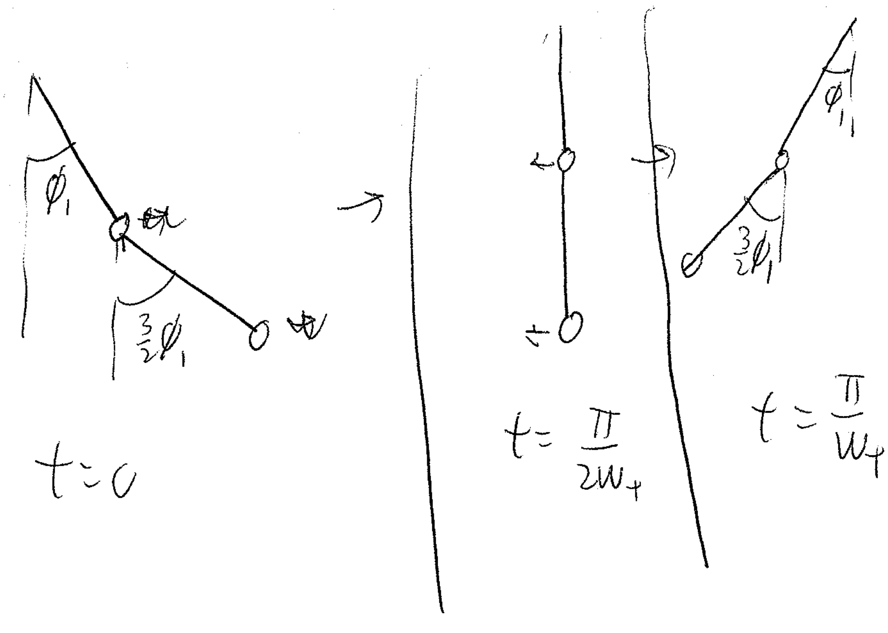
$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = a_+ \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} e^{i w_+ t} + b_+ \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i w_- t} + c_+ \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} e^{-i w_+ t}$$

$$+ d_+ \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-i w_- t}$$

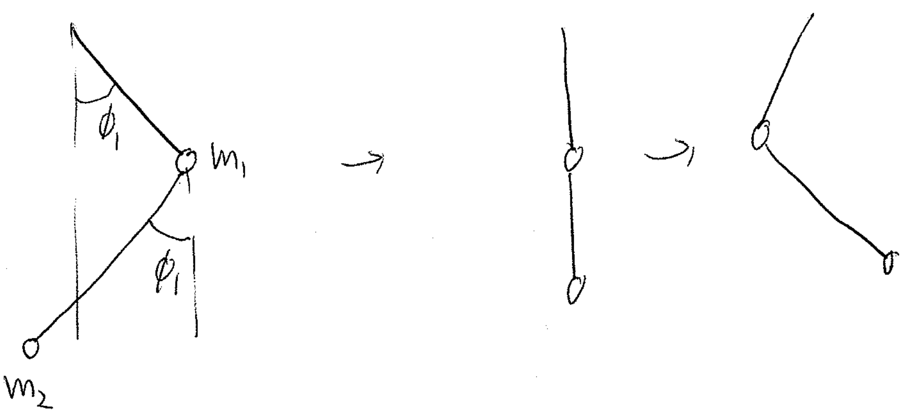
$$= \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} a_+ \cos(w_+ t + \phi_+) + b_+ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(w_- t + \phi_-)$$

What's motion look like?

ω_+:



ω_-:



"Normal modes": Motion with oscillation (or, more generally, oscillation, decay, or growth) at 1 particular frequency

Always has particular eigenvector associated with it.

(Characteristic form of motion)

6.1
6

General linear algebraic formalism for this:

When does $\underset{\sim}{M}\,\vec{x} = 0$ have a non-zero solution for $\vec{x}$?

1st, 2×2 case: $\underset{\sim}{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$a x_1 + b x_2 = 0$$

$$c x_1 + d x_2 = 0$$

Solve (2):

Stick into (1):

~~Only~~ non-zero solution exists iff ~~a~~

$$\det \underset{\sim}{M} \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 0$$

ad

bc