Exercise 1:

$$M^{(0)} = Tr(M_{\alpha\beta}) = M^{\alpha}{}_{\alpha}$$

$$(ABC)^{\alpha}{}_{\beta} = A^{\alpha}{}_{\delta} B^{\delta}{}_{\gamma} C^{\gamma}{}_{\beta}$$

$$
\begin{aligned}
Tr(ABC) &= (ABC)^{\alpha}{}_{\alpha} \\
&= A^{\alpha}{}_{\gamma} B^{\gamma}{}_{\lambda} C^{\lambda}{}_{\alpha} \\
&= C^{\lambda}{}_{\alpha} A^{\alpha}{}_{\gamma} B^{\gamma}{}_{\lambda} \\
&= (CAB)^{\lambda}{}_{\lambda} \\
&= Tr(CAB)
\end{aligned}
$$

Since cyclic permutations of ABC have adjacent indicies which get summed over,

$$Tr(ABC) = Tr(CAB) = Tr(BCA)$$

Exercise 2:

$$M_{\alpha\beta} = \frac{1}{3} M^{(0)} \delta_{\alpha\beta} + \frac{1}{4} M^{(1)}_{\mu} \epsilon_{\mu\alpha\beta} + M^{(2)}_{\alpha\beta}$$

$$M_{\alpha\beta} = \frac{1}{3} M_{\gamma\gamma} \delta_{\alpha\beta} + \frac{1}{4} \epsilon_{\mu\alpha\beta} \epsilon_{\mu\lambda\sigma} (M_{\lambda\sigma} - M_{\sigma\lambda}) + \frac{(M_{\alpha\beta} + M_{\beta\alpha})}{2} - \frac{1}{3} M_{\mu\mu} \delta_{\alpha\beta}$$

$$M_{\alpha\beta} = \frac{1}{4} \epsilon_{\mu\alpha\beta} \epsilon_{\mu\lambda\sigma} (M_{\lambda\sigma} - M_{\sigma\lambda}) + \frac{1}{2}(M_{\alpha\beta} + M_{\beta\alpha})$$

$$M_{\alpha\beta} = \frac{1}{4} (\delta_{\alpha\lambda}\delta_{\beta\sigma} - \delta_{\alpha\sigma}\delta_{\beta\lambda})(M_{\lambda\sigma} - M_{\sigma\lambda}) + \frac{1}{2}(M_{\alpha\beta} + M_{\beta\alpha})$$

$$M_{\alpha\beta} = \frac{1}{4} (\delta_{\alpha\lambda}\delta_{\beta\sigma} M_{\lambda\sigma} + \delta_{\alpha\sigma}\delta_{\beta\lambda} M_{\sigma\lambda} - \delta_{\alpha\sigma}\delta_{\beta\lambda} M_{\lambda\sigma} - \delta_{\alpha\lambda}\delta_{\beta\sigma} M_{\sigma\lambda}) + \frac{1}{2}(M_{\alpha\beta} + M_{\beta\alpha})$$

$$M_{\alpha\beta} = \frac{1}{4} (M_{\alpha\beta} + M_{\alpha\beta} - M_{\beta\alpha} - M_{\beta\alpha}) + \frac{1}{2}(M_{\alpha\beta} + M_{\beta\alpha})$$

$$\boxed{M_{\alpha\beta} = M_{\alpha\beta}}$$

Exercise 3:

$$R|\tilde{\alpha} j m\rangle = \sum_{q m'} (R T_q^{(K)} R^{\dagger}) R|\alpha' j' m'\rangle \langle j' m'; K q | j m\rangle$$

But $\quad R T_q^{(K)} R^{\dagger} = \sum_{q'=-K}^{K} T_{q'}^{(K)} d_{q'q}^{(K)}$

and $\quad R|j' m'\rangle = \sum_{m''} d_{m''m'}^{(j')} |j' m''\rangle$

$$R|\tilde{\alpha} j m\rangle =$$

$$\sum_{q q' m' m''} T_{q'}^{(K)} |\alpha' j' m''\rangle \langle j' m'; K q | j m\rangle d_{q'q}^{(K)} d_{m''m}^{(j')}$$

$$R|\tilde{\alpha} j m\rangle = \sum_{q' m''} T_{q'}^{(K)} |\alpha' j' m''\rangle$$

$$\times \sum_{q m'} \langle j' m''; K q' | R | j' m'; K q\rangle \langle j' m'; K q | j m\rangle$$

$$R|\tilde{\alpha} j m\rangle = \sum_{q' m''} T_{q'}^{(K)} |\alpha' j' m''\rangle \langle j' m''; K q' | R | j m\rangle$$

$$R|\tilde{\alpha} j m\rangle = \sum_{q' m'' m'} T_{q'}^{(K)} |\alpha' j' m''\rangle \langle j' m''; K q' | j m'\rangle \langle j m' | R | j m\rangle$$

$$R|\tilde{\alpha} j m\rangle = \sum_{q'm'm''} T_{q'}^{(k)} |\alpha' j' m''\rangle \langle j' m''; k q' | j m'\rangle$$

$$\times\ d^{(j)}_{m'm}$$

$$R|\tilde{\alpha} j m\rangle = \sum_{q'm'm''} T_{q'}^{(k)} |\alpha' j' m''\rangle \langle j m' | j' m''; k q'\rangle$$

$$\times\ d^{(j)}_{m'm}$$

$$R|\tilde{\alpha} j m\rangle = \sum_{m'} |\tilde{\alpha} j m'\rangle\, d^{(j)}_{m'm}$$