PHYS 631: Quantum Mechanics I (Fall 2020)
Exercises 29 September 2020 (Week 1)
Due Monday, 6 October 2020

Note: These are the **exercises** to go along with the first lecture. We'll work on them during class; you should write up your solutions after class and submit them electronically by midnight on the due date. There is only one class this week, but remember that you'll submit the exercises *together for the entire week, as a single pdf file* in future weeks. For the submission link, see the Notes page on the course web site.

You'll turn in your **homework problems** separately, details on your first homework set. You should do the exercises *before* the homework problems, as they're more fundamental (and easier); it's even better if you wrap up the exercises within a day of the class meeting.

Exercise 1. Prove the Cauchy–Schwarz inequality

$$\langle x, x\rangle\langle y, y\rangle \geq \big|\langle x, y\rangle\big|^2, \tag{1}$$

starting by defining

$$z := y - \frac{\langle x, y\rangle}{\langle x, x\rangle}x \tag{2}$$

and observing that $\langle z, z\rangle \geq 0$ for any inner product. Note that you must handle the cases where either x or y is the null vector.

Exercise 2.
(a) The space of *continuous*, square-integrable functions ($\|f\| < \infty$) is **not** a Hilbert space. Find a counterexample that shows this (i.e., find a Cauchy sequence whose limit is not in this space).

(b) The space of square-integrable functions (not necessarily continuous) **is** a Hilbert space.

So consider this: Let $G_n(x)$ be a normalized Gaussian function, centered at $x = 0$ and of standard deviation (width) $1/n$. Then as $n \longrightarrow \infty$, $G_n(x)$ has the "limit" $\delta(x)$, which isn't a function in the Hilbert space. Why isn't this a contradiction? [Give a *specific* argument in terms of the $G_n(x)$, in enough detail that you can convince the average person on the street that you actually understand your answer.]

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Ex 2.

a)

Consider the sequence of functions,

$$f_n(x) = \frac{e^{-x^2}}{1 + e^{-nx}}$$

since $0 \leq f_n^2(x) \leq e^{-2x^2}$ for all $n \in \mathbb{N}$,

we have that $\int_{-\infty}^{\infty} f_n^2(x)\,dx$

since $\int_{-\infty}^{\infty} e^{-2x^2}\,dx$ converges.

So $f_n(x) \in \{f : \|f\| < \infty\}$

Observe,

$$\lim_{n\to\infty} \lim_{m\to\infty} \left| \frac{e^{-x^2}}{1 + e^{-nx}} - \frac{e^{-x^2}}{1 + e^{-mx}} \right|$$

$$= e^{-x^2} \lim_{m\to\infty} \lim_{n\to\infty} \left| \frac{1}{1+e^{-nx}} - \frac{1}{1+e^{-mx}} \right|$$

$$= e^{-x^2} \left| 1 - 1 \right|$$

$$= 0 \quad , \quad \text{for all } x,$$

So $f_n(x)$ is Cauchy.

However, let $f(x) = \lim_{n\to\infty} f_n(x)$.

Then $\lim_{x\to 0^+} f(x) = 1$ but

$\lim_{x\to 0^-} f(x) = 0$

So $f(x)$ is not continuous.

Therefore, the space of continuous square integrable functions is not a Hilbert space.

b) Let $G_n(x) = \frac{1}{\frac{1}{n}\sqrt{2\pi}} e^{-\frac{x^2}{2(\frac{1}{n})^2}}$

since $G_n(x)$ is a normalized Gaussian $\|G_n(x)\| < \infty$ for all n.

consider the limit,

$\lim_{n\to\infty} G_n(0)$, since

$$\lim_{n\to\infty} G_n(0) = \lim_{n\to\infty} \frac{n}{\sqrt{2\pi}} = \infty,$$

$G_n(x)$ is not bounded as $n \to \infty$ at $x=0$, so it cannot be Cauchy