Implementing Analytical Solutions

Matthew Kafker, Jeremy Welsh-Kavan

1 The Diffusion Equation

1.1 The 1D Boundary Value problem.

To implement the analytical solution to the 1D diffusion equation with fixed boundary conditions, we shall simply cite the result from Asmar (p. 138) [1].
The solution of the 1D boundary value problem for the diffusion equation

$$\begin{aligned} \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad & 0 < x < L,\ t > 0 \\ u(0,t) = u(L,t) = 0 \quad & \text{for all } t > 0 \\ u(x,0) = f(x) \quad & \text{for} 0 < x < L \end{aligned} \tag{1}$$

can be written as the following Fourier series

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t} \sin\Big(\frac{n\pi x}{L}\Big) \tag{2}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\Big(\frac{n\pi x}{L}\Big) dx \quad \text{and} \quad \lambda_n = \frac{cn\pi}{L} \tag{3}$$

Our analytical solution will have 3 components: the 1-dimensional arrays $[x]$ and $[t]$, which together define the extent and resolution of our system in space and time, and the 2-dimensional array $[u(x,t)]$. The function defining our initial condition, $[f(x)]$, is also a 1D array. Next, we choose a maximal value for n. This choice is usually motivated by the desired degree of accuracy in the solution. This is defined as an array, $[n]$, containing an ordered range of consecutive natural numbers.

We can define arrays for $[e^{-\lambda_n^2 t}]$ and $[\sin\big(\frac{n\pi x}{L}\big)]$ with shapes (t,n) and (n,x), respectively. This is most easily done by applying a vectorized sin() and exp() to arrays $[t]^T \cdot [\lambda_n^2]$ and $[n]^T \cdot [\pi x/L]$, where $\cdot$ indicates a matrix product. For brevity, we will refer to these as $[\exp(t,n)]$ and $[\sin(n,x)]$

[1] Asmar, Nakhlé H., and Nakhlé H. Asmar. Partial Differential Equations with Fourier Series and Boundary Value Problems. 2nd ed. Upper Saddle River, N. j: Pearson Prentice Hall, 2005. Print.

To compute $[\lambda_n]$ is very simple. For a given function, the procedure for computing $[b_n]$ may differ. The accuracy of the numerical integration will depend on the method chosen. However, for an arbitrary 1D array, $[f(x)]$, there is no reason to expect higher accuracy than the trapezoid rule, since we have no a priori knowledge of the behavior of $f(x)$ at points not contained in $[x]$.

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